Abstract

In section 7 of [1] it was indicated how a principal 2-bundle is reconstructed from any one of its cocycles as the pullback of the 2-groupoid incarnation of the universal principal 2-bundle. Here are details of the proof.

Contents

Ambient context. For definiteness, let all of the following be in the ambient category of manifolds.

Let G be a strict 2-group. Write $\mathbf{B}G$ for the corresponding one-object 2-groupoid.

0.1 Principal 2-bundles.

Definition 1 (semistrict principal 2-bundles) *A semistrict principal G-bundle over a space X is a groupoid P equipped with a surjection $P \to X$ and equipped with a strict G-action $P \times G \to P$ such that there is a regular epimorphism $\pi : Y \to X$ and a strictly G-equivariant equivalence of categories*

$$t : \pi^* P \xrightarrow{\simeq} Y \times G \; .$$

Principal G-bundles over X form a category $G\mathrm{Bund}(X)$ in the obvious way.

0.2 Cocycles

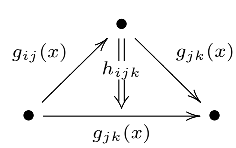
For $\pi : Y \to X$ a regular epimorphism, let $\mathcal{P}_0^Y(X)$ be the corresponding Cech 2-groupoid: its space of objects is Y, its 1-morphisms are freely generated from $Y \times_X Y$ and its 2-morphisms are generated from triangles in $Y \times_X Y \times_X Y$ modulo tetrahedra.

Definition 2 (G-cocycle) *A G-cocycle relative to Y is a strict 2-functor*

$$g : \mathcal{P}_0^Y(X) \longrightarrow \mathbf{B}G \; .$$

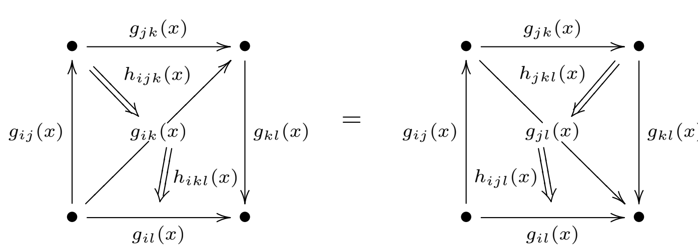
A G cocycle with respect to a cover by open subsets U_i, i.e. for $Y = \sqcup_i U_i$ is a collection of 2-cells

$\forall (x, i, j, k)$:



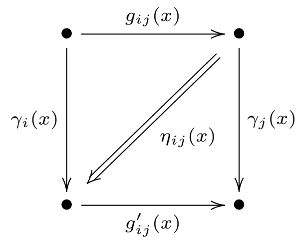
in $\mathbf{B}G$ satisfying the cocycle condition

$\forall (x, i, j, k, l)$:



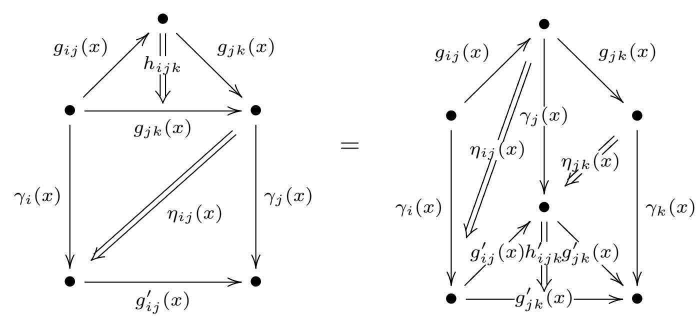
Coboundaries, i.e. morphisms of cocycles, are collections of 2-cells

$\forall(x,i,j)$:



which satisfy

$\forall(x,i,j,k)$:



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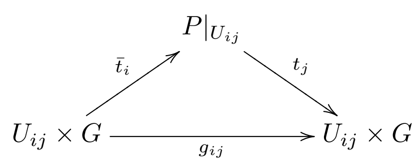
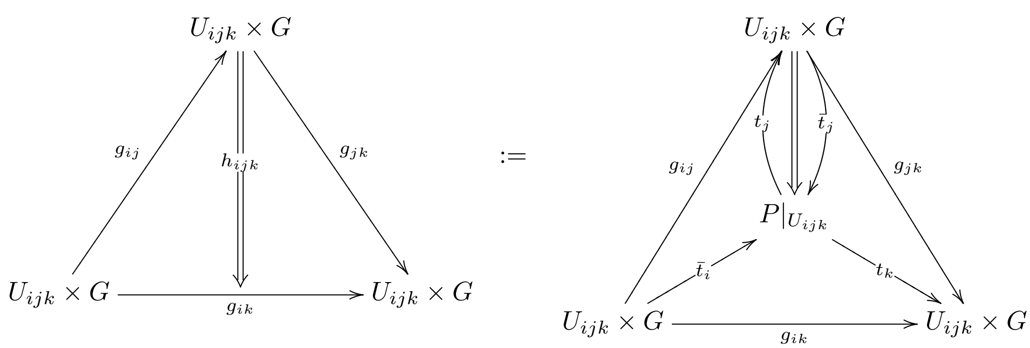
0.3 Cocycles from principal bundles

Given a principal G-bundle, we obtain from it a cocycle by first choosing a local tivialization $t:\ \pi^*P \xrightarrow{\simeq} Y \times X$ and then defining the components g_{ij} and h_{ijk} by



and



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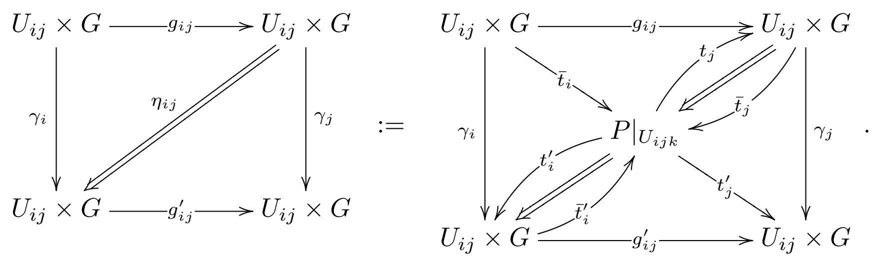
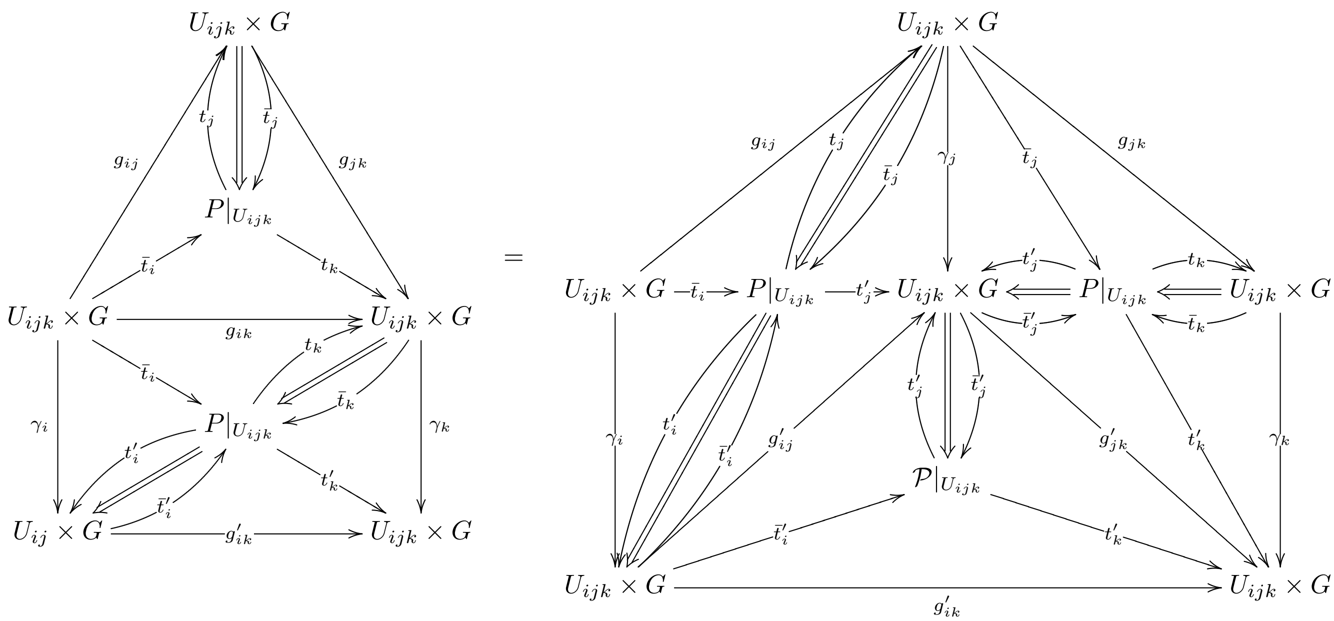
It is immediate that this satisfies the cocycle condition. To see that two different choices t, t' of local trivializations yield cohomologous cocycles set

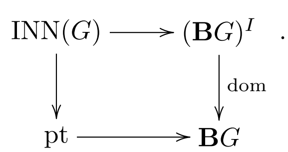


That this indeed is a coboundary between $(\{g_{ij}\}, \{h_{ijk}\})$ follows from



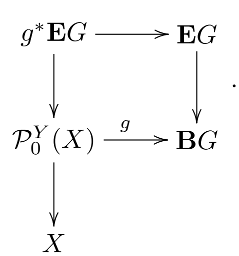
0.4 Principal bundles from cocycles

Recall from [1] the 2-groupoid $\mathrm{INN}(G)$ defined as the pullback



To amplify its role as the universal G-2-bundle write $\mathbf{E}G := \mathrm{INN}(G)$ here. Notice the canonical projection $\mathrm{codom} : \mathbf{E}G \to \mathbf{B}G$.

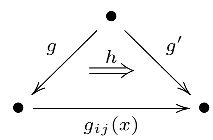
Given a cocycle $g : \mathcal{P}_0^Y(X) \to \mathbf{B}G$, denote by $g^*\mathbf{E}G$ the (strict) pullback



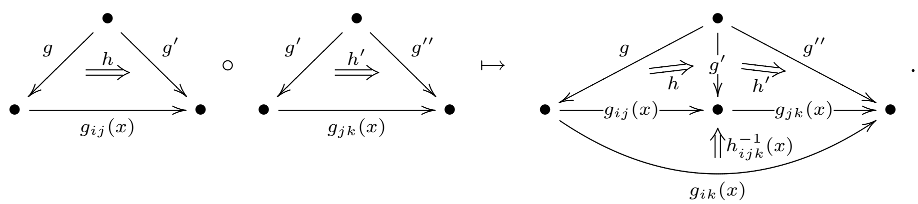
Let $g^*\mathbf{E}G/_\sim$ be the result of dividing out 2-isomorphisms.

Proposition 1 *The bundle* $(g^*\mathbf{E}G/_\sim) \to X$ *is a principal* G*-bundle whose local trivialization yields a cocycle cohomologous to* g*.*

Proof. The morphisms of $(g^*\mathbf{E}G/\sim)$ are are 2-cells

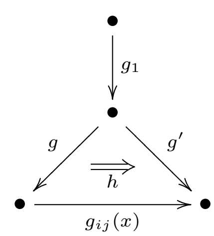


in $\mathbf{B}G$ with source $((x,i),g)$ and target $((x,j),g')$. Notice that g' is fixed by the rest of the data. Composition of such morphisms in $(g^*\mathbf{E}G/_\sim)$ is horizontal pasting of these triangles followed by composition with the cocycle triangle:



That this composition is associative is precisely equivalent to the cocycle condition. This defines $g^*\mathbf{E}G/\sim$ as a groupoid. The G-action on it is by precomposition with the corresponding 1-cells

$$((x,i,j),h,g)\cdot g_1 \;:=$$



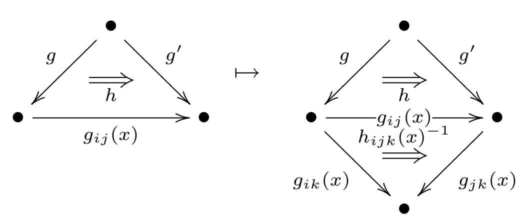
with the obvious action of morphisms. The projection down to X is the obvious one.

Now notice that there are naturally given local trivialization functors

$$t_k : g^*\mathbf{E}G/_\sim|_{U_k} \to G \times U_k$$

given by

t_k :

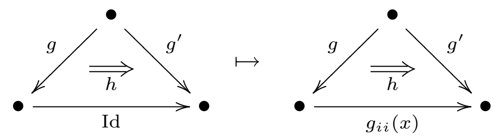


with the right hand side regarded as a 1-morphism in G. Again, it is the cocycle law which makes this assignment functorial. Accordingly, take

$$\bar{t}_k : G \times U_k \to (g^*\mathbf{E}G/_\sim)|_{U_k}$$

to be given by

$\bar{t}_k$:

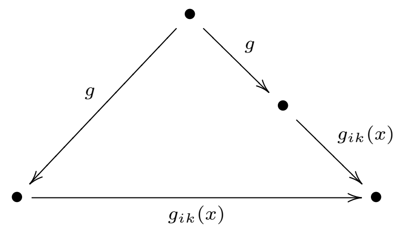


(recalling that we are working with $g_{ii}(x) = \mathrm{Id}$). It is immediate that $t_k \circ \bar{t}_k = \mathrm{Id}$ and that the natural transformation

$$\mathrm{Id} \Rightarrow \bar{t}_k \circ t_k$$

has the component function

$((x,i),g) \mapsto$



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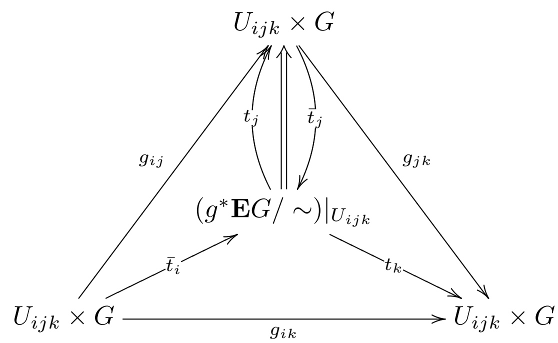
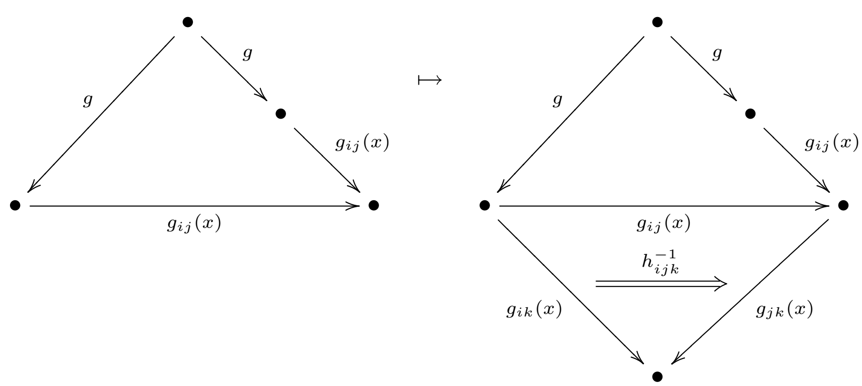
Using this in the law for horizontal composition of natural transformations, it follows that the natural transformation



defined this way has the component map

$((x,i,j,k),g) \mapsto ((x,i),g) \mapsto$



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which says precisely that it does reproduce the cocycle c that we started with. □

0.5 Classification of principal 2-bundles.

The colimit over all G-cocycle 2-categories is G-cohomology:

Definition 3

$$H(X, \mathbf{B}G) := \mathrm{colim}_Y 2\mathrm{Funct}(\mathcal{P}_0^Y(X), \mathbf{B}G)\,.$$

Hence we get

Theorem 1 *The semistrict principal G-2-bundles from definition 1 are classified by G-cohomology:*

$$H(X, \mathbf{B}G) \simeq G\mathrm{Bund}(X).$$

References

[1] David Roberts, Urs Schreiber, *The inner automorphism 3-group of a strict 2-group*, Journal of Homotopy and Related Structures, Vol. 3(2008), No. 1, pp. 193-244, [arXiv:0708.1741v2]