Monadic and Higher-Order Structure

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$$
\begin{tikzcd}
& B \arrow[dr, "r"] \arrow[d, phantom, "\dashv"] & \\
A \arrow[ur, "\ell"] \arrow[rr, "j"'] & & E
\end{tikzcd}
$$

This dissertation is submitted for the degree of Doctor of Philosophy

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted before for any degree or other qualification except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the Degree Committee for the Faculty of Computer Science and Technology.

Collaborations Chapter 4 and Sections 5.3, 5.4 and 6.1.1 are the result of work done in collaboration with Dylan McDermott.

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Abstract

Simple type theories, ubiquitous in the study of programming language theory, augment algebraic theories with higher-order, variable-binding structure. This motivates the definition of *higher-order algebraic theories* to capture this structure, permitting the study of simple type theories in a categorical setting analogous to that of algebraic theories. The theory of higher-order algebraic theories is in one sense much richer than that of algebraic theories, as we may stratify the former according to their order: for instance, the first-order algebraic theories are precisely the classical algebraic theories, the second-order algebraic theories permit operators to abstract over operators, the third-order algebraic theories permit operators to abstract over operators that themselves abstract over operators, and so on. We study the structure of the category of $(n+1)^{\text{th}}$-order algebraic theories, demonstrating that it may be viewed as a construction on the category of n^{th}-order algebraic theories, facilitating an inductive construction of the category of higher-order algebraic theories. In turn, this description leads naturally to a monad–theory correspondence for higher-order algebraic theories, subsuming the classical monad–theory correspondence, and providing a new, monadic understanding of higher-order structure.

In proving the monad–theory correspondence for higher-order algebraic theories, we are led to reconsider the traditional perspective on the classical monad–theory correspondence. In doing so, we reveal a new understanding of the relationship between algebraic theories and monads that clarifies the nature of the correspondence. The crucial insight follows from the consideration of relative monads, which are shown to act as an intermediary in the correspondence. To support our proposal that this be viewed as the correct perspective of the monad–theory correspondence, we show how the same proof may be carried out in a formal 2-categorical setting. The classical monad–theory correspondence, as well as those in the literature for enriched and internal categories, then follow as corollaries of a general theory.

Acknowledgements

First, I must thank my supervisor, Marcelo Fiore. I decided to pursue a PhD in theoretical computer science, rather than continue into compiler development as I had originally intended, due to Marcelo, who introduced me to categorical abstract syntax in the final year of my undergraduate degree and agreed to supervise my master's dissertation. That project, in which I formalised various aspects of Marcelo's programme of abstract syntax in a proof assistant, gave me my first real taste of category theory, and I discovered that I found the subject entirely fascinating. I decided I would apply for a PhD, and Marcelo graciously accepted to supervise me. While the topic of my thesis has changed dramatically since I began the PhD, the original motivation for my research being merely alluded to herein, Marcelo's influence on the way I think about categorical logic will nonetheless be clearly visible throughout.

Special thanks are due to Dylan McDermott, with whom much of the last two years has been spent in collaboration. Indeed, the nucleus for this entire thesis (namely, the question of whether there existed a monad–theory correspondence for second-order algebraic theories) arose during discussion together, whilst Dylan was visiting Cambridge in late 2019. My understanding of the topics of this thesis have been greatly influenced by our conversations.

I am very grateful for Martin Hyland and Richard Garner, who kindly agreed to examine this thesis. Their insights and suggestions have been greatly beneficial to the thesis.

I am exceedingly thankful for my friends in the mathematical community. To name but a few: Shaun Steenkamp, with whom I have been studying since the start of our undergraduate degrees, and in whom, most serendipitously, I have found someone with similar interests both in mathematics and other areas of life; and in addition Dima Szamozvancev, Adam Ó Conghaile, and Andrej Ivašković, my PhD cohort in the Programming, Logic, and Semantics group; Vikraman Choudhury, Younesse Kaddar, and Christian Williams, with whom I spent many enjoyable evenings at a blackboard, endeavouring to understand one categorical idea or another; Philip Saville and Hugo Paquet, who have always been ready to answer any of my mathematical questions and lend advice about the mechanisms of academia; Ivan Di Liberti, who has always been happy to work through a vexing problem together and to whom, along with Fosco Loregian, I owe my introduction to formal category theory; Gabriele Lobbia, who has been a great support in the final months of my writing up; and Tim Ophelders, who has provided me with an inspiring example of a computer scientist for a long time.

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More widely, I have found the category theory community to be most friendly and helpful, and one in which I am glad to be part. In particular, I am eternally grateful to those members of the community who have graciously given their time to help those who are less experienced, by answering questions, lending intuition, and engaging in wider discussions. In particular, I would like to thank Michael Barr, John Bourke, Marta Bunge, Tom Hirschowitz, Anders Kock, Joachim Kock, Roald Koudenburg, Steve Lack, Mike Shulman, Todd Trimble, Taichi Uemura, Richard Wood, and Gavin Wraith, along with the numerous contributors to the nLab and MathOverflow, which have become invaluable resources for category theorists.

Outside of academia, I am thankful for my friends in Cambridge, particularly Jonny and Jordan, for countless stimulating non-mathematical discussions and distractions from the stresses of the PhD.

Finally, I thank my family for their constant support, especially over these past four years.

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The diagrams in this thesis were created using **quiver**, a tool for typesetting commutative diagrams that I developed during my PhD.

Contents

Chapter 1

Introduction

This thesis is a study of two questions, which may at first seem tangentially related. To begin, we shall therefore start with the context, and explain the process by which one might come to be interested in these questions.

Algebra may be viewed broadly as the study of structure. *Universal algebra* considers very simple structures: objects equipped with finitary operators, subject to universally-quantified axioms. By this, we mean we fix a collection S of objects (which are traditionally called *sorts* or *types*), and specify *operators* between them: formally, pairs $((A_1, \ldots, A_n), B)$, denoted $A_1, \ldots, A_n \to B$, which we understand conceptually to represent some transformation from *terms* $a_1, \ldots, a_n$ having sorts $A_1, \ldots, A_n$, to a term having sort B, where terms are build inductively from variables and the constants and operators of the algebra. Here "object" and "transformation" are intended entirely abstractly: it is not important to give them a precise meaning. For instance, addition $+ \colon \mathbb{N}, \mathbb{N} \to \mathbb{N}$ and multiplication of natural numbers $\times \colon \mathbb{N}, \mathbb{N} \to \mathbb{N}$ are binary operators on natural numbers. The natural numbers themselves are the terms having sort (or type) $\mathbb{N}$, and are built inductively from the constant 0 and the successor operator: in other words, every natural number is defined by a finite sum $(\cdots(0+1)+\cdots)+1$. It is not necessary for the sorts of an operator to be homogeneous: for example, if $\mathbb{N}^\star$ is a sort representing lists of natural numbers, then the length of a list $|{-}| \colon \mathbb{N}^\star \to \mathbb{N}$ is a unary operator whose input sort (called the *arity*) is distinct from its output (called the *coarity*). A *universally-quantified axiom* is a rule that imposes a constraint on the properties of the operators. For instance, addition of natural numbers satisfies the *associativity law*, which says that for all natural numbers ℓ, m, n, the following equality holds:

$$(\ell + m) + n = \ell + (m + n)$$

It is *universally-quantified* because the law holds for *all* natural numbers.

Although universal algebra is simple, it is powerful, and a great many structures of interest may be expressed in the language of universal algebra. However, there are also various structures of interest, particularly within computer science, that cannot be.

Algebraic simple type theory is a generalisation of universal algebra in two orthogonal directions. The first is through the introduction of *type operators*, which permit the description of structure on sorts, rather than structure on terms. For instance, the process of taking a sort A and producing the sort $A^\star$ of *lists of* A is a unary type operator. The second generalisation is through the introduction of *binding operators*. A binding operator is one that involves a term that quantifies over a variable in some way. For instance, the differential operator $\partial x. f(x)$ takes a single input f, the function to differentiate, which is parameterised by x, the variable with respect to which f is differentiated. Algebraic simple type theories are prevalent in theoretical computer science, since they can be used to represent and study simple programming languages.

Universal algebra is a relatively old field, dating back to the 1930s [Bir35], and there are now many tools and techniques with which to study universal algebraic structures. Algebraic simple type theory is much more recent: though examples of simple type theories date back to the 1940s [Chu40], there have until very recently been no satisfying formalisms by which to study them, and the definition we employ[1] was given only in 2020 [AF20] (cf. [Fio17a]), though the augmentation of universal algebra with binding operators was described some time earlier [CP07; FH10]). Consequently, compared to universal algebra, we have far fewer techniques for studying algebraic simple type theories. Given their prevalence in computer science, there is strong motivation to develop tools by which we might prove theorems about classes of simple type theories, facilitating the elimination of the repetitive proofs that are, at present, commonplace in the study of programming languages. This is essentially the high-level starting point for this thesis.

One particularly valuable tool in the study of universal algebra is category theory. There exist two distinct, but strongly related, approaches to categorical universal algebra: the first is through the study of *algebraic theories*, and the second is through *monads*. We shall defer the definition of both, so as not to drive away the reader unfamiliar with category theory. It suffices to note that both approaches are useful for reasoning about universal algebraic structures, and have distinct strengths and weaknesses. Given the power of both approaches, we might wonder whether similar (but appropriately generalised) approaches exist for algebraic simple type theory. Recall that simple type theories generalise universal algebras in two directions: through type constructors, and through binding operators. Rather than attempt to formulate both generalisations at once in the languages of algebraic theories and of monads, it is simpler to attempt to generalise in each of the two directions separately, and then afterwards to combine them. In fact, the generalisation of algebraic theories to the setting of binding operators has already been initiated by Fiore and Mahmoud [FM10], who introduced a notion of *second-order algebraic theory*. A natural starting point for such an investigation, therefore, is to determine whether there is an analogous generalisation of monad.

While algebraic theories and monads were developed independently, Linton later established the two concepts to be equivalent [Lin66b]: we refer to this relationship as the *classical monad–theory correspondence*. If there were a generalisation of monads to binding universal algebra, we would therefore hope for them to be equivalent to second-order algebraic theories. Our first motivating question may thus be phrased as follows.

Question 1. *Is there a monad–theory correspondence for second-order algebraic theories?*

To attempt to develop a monad–theory correspondence for second-order algebraic theories, it seems prudent to study the precise relationship between algebraic theories and monads – for instance, how to associate a monad to an algebraic theory, and vice versa – with the hope of then modifying the relationship appropriately. In this way, we would hope to formulate a notion of "second-order monad" that corresponds to second-order algebraic theories, essentially by definition. At first, such a task seems straightforward: for one thing, it is relatively straightforward to give a concrete proof of the classical correspondence; and for another, there have been many, more abstract, reformulations of the correspondence in the years since, which one would expect to give a more abstract, conceptual understanding, aiding the formulation of the appropriate generalisation for second-order algebraic theories. However, in the first case, it is not always the case that being able to prove a theorem is enough to ensure that one understands it: this is particularly keenly felt in the case of the classical monad–theory correspondence. In the second case, perhaps surprisingly, the subsequent generalisations of the correspondence seem more mysterious still, establishing monad–theory correspondences in a wide variety of settings, and yet employing definitions that are often unmotivated by the authors. This makes finding the appropriate setting in which to frame second-order algebraic theories a difficult task indeed, as they do not obviously fit into any of the available frameworks. Thus, to answer our first question, we are motivated to first answer another.

Question 2. *Why is there a correspondence between algebraic theories and monads?*

By this, we emphasise that it is not enough simply to find a proof: plenty exist already in the literature. Instead, we wish to find a conceptual explanation for its existence, to the extent that the monad–theory correspondence should appear *inevitable* – a trivial consequence of the axioms of *algebraic theory* and *monad*. While such a deep understanding may seem inessential to answer our first question, in practice it will transpire to be invaluable.

[1] We should note that the definition of algebraic simple type theory to which we refer was introduced by the author together with Marcelo Fiore [AF20], and so there is naturally some bias in what we view here as the appropriate formalism for simple type theories. However, given that we have not found other attempts to give a precise definition of simple type theory (that captures at least the motivating example of simply-typed λ-calculus); the naturality of our definition; and the strength of our examples, we believe that our choice is well justified.

To explain the process by which we may approach the second question, we shall need to be a little more precise about the form of the classical monad–theory correspondence, for which the language of category theory is unavoidable. We define an *algebraic theory* to be a finite-coproduct-preserving bijective-on-objects functor from the free category with strict finite coproducts on a single object; a morphism of algebraic theories is a morphism of coslices. We define a *finitary monad on* **Set** to be a monad on **Set** that preserves sifted colimits[2]; a morphism of finitary monads on **Set** is simply a monad morphism. Since **Set** is freely generated under sifted colimits from finite sets, a finitary monad on **Set** in some sense carries redundant information, because the action of the monad on the sifted colimits is entirely determined by the action on finite sets. Tellingly, the finite sets too are freely generated, this time under finite coproducts of the singleton set. We therefore see the same structure appearing (up to equivalence) in the definitions both of algebraic theory, and of finitary monad: namely, the category freely generated under finite coproducts of a single object.

Since a finitary monad on **Set** is determined by its action on the finite sets, we might look for a notion that captures this structure: that is, a monad whose action is determined by a subcategory of its codomain. In doing so, we are led inevitably to *relative monads* [ACU10]. A monad relative to the inclusion of finite sets into all sets is equivalent to a finitary monad on **Set**. To recover the monad–theory correspondence then, it suffices to relate algebraic theories and *relative* monads. The relationship is established in two steps: the first by a characterisation of Kleisli inclusions for relative monads in terms of bijective-on-objects left- relative adjoint functors, and the second by a (relative) adjoint functor theorem, relating adjointness and colimit-preservation properties. This leads to a new and remarkably simple proof of the monad–theory correspondence. Furthermore, this proof is highly amenable to generalisation: in fact, we can recover the majority of monad–theory correspondences in the literature using the same techniques. Chapter 3 is dedicated to the exposition and proof of the monad–theory correspondence from this perspective.

With the results of Chapter 3, it is possible to return to our first question and, while a correspondence with monads for second-order algebraic theories is not immediate, it is now tractable. However, it is first necessary to understand the structure of **Law**[3], the category of algebraic theories. In particular, just as the category of sets is generated by sifted colimits of finite sets, we desire a similar characterisation of **Law**: while finite sets act as the arities of operators in universal algebra, finite algebraic theories act as the arities of binding operators, and so we expect the category of finite algebraic theories to play the same role that the category of finite sets did for the classical monad–theory correspondence. Though **Law** is known folklorically to be *locally strongly finitely presentable*, meaning that it is generated under sifted colimits in this way, a concrete description of the category of finite algebraic theories is absent. In Chapter 4, we give a new proof of this property that makes the subcategory of finite objects explicit, in doing so providing a new universal property of the category of algebraic theories.

Local strong finite presentability paves the way for a monad–theory correspondence, which we subsequently obtain: our general monad–theory correspondence makes this relatively straightforward. However, the correspondence uncovers a new mystery. To elaborate, we must first recall the notions of *models* for an algebraic theory, and the *algebras* for a monad. An algebraic theory is an abstract specification of an algebraic structure, such as a *magma*: a structure with a single binary operator. A *model* for the algebraic theory of magmas is a specific instance of such a structure, for instance the natural numbers equipped with addition, or the truth values equipped with disjunction. Similarly, a monad is an abstract specification, while an *algebra* for a monad is a specific instance of that specified structure. The classical monad–theory correspondence actually relates more than just algebraic theories and monads: it states that the models for an algebraic theory coincide with the algebras for the corresponding monad, and vice versa. We might therefore hope that the same holds for second-order algebraic theories. However, there is a subtlety. Models of algebraic theories are by definition certain structure-preserving functors (in this case, the structure being finite coproduct structure). However, the algebras for a monad corresponding to a second-order algebraic theory correspond to functors that do not preserve all relevant structure. They therefore appear weaker than the appropriate definition of model for a second-order algebraic theory. In the final section of Chapter 4, we explain this disparity, and shed light on the meaning of *algebra* and *model* in categorical logic.

[2] Sifted colimits are those that commute with finite products in **Set**.

[3] Named so after Lawvere, who introduced algebraic theories [Law63].

One might wonder, since there is a good theory of (first-order) algebraic theories and second-order algebraic theories, whether there might be a coherent story for *third-order* algebraic theories, however they might be defined. This is a little more difficult to justify from the perspective of algebraic type theory than second-order algebraic theories, since third-order operators are operators that quantify over *metavariables*: variables which themselves may quantify over variables. Such operators are uncommon in type theory, but do make an appearance in programming languages with *control operators* [Gri89], like continuations. With this motivation in mind, it becomes pragmatic to generalise to n^{th}-order algebraic theories for $n \in \mathbb{N} + \{\omega\}$, and develop a theory specialising to the known results when $n \in \{1, 2\}$. This is in fact the approach we take. This has an unexpected benefit beyond mere generalisation: in particular, we show that the relationship between algebras and models described earlier becomes particularly satisfying when $n = \omega$.

At this point, it may appear that we have answered our motivating questions, and may move on to studying the structure associated to type constructors in our investigation into algebraic simple type theories. However, there is good reason not to be entirely satisfied with our answer to the second question. For one, there are several monad–theory correspondences in the literature that are not captured by our generalisation in Chapter 3. For us to truly claim to have understood the monad–theory correspondence, it stands to reason that we should be able to recover *all* known monad–theory correspondences. For another, there are aspects of our development that seem dissatisfyingly tied to the nature of categories, functors, and natural transformations. To truly understand a categorical phenomenon, it is not enough to understand it solely from the perspective of categories: we must instead move to the world of 2-categories, where the properties of categories, functors, and natural transformations may be neatly axiomatised, forcing us to relinquish any reliance on the phenomena that hold incidentally in the well-behaved setting of categories. A prime example of an aspect that remains mysterious is the characterisation of Kleisli inclusions in terms of bijective-on-objects functors: it is not clear what the appropriate generalisation of being *bijective-on-objects* should be in an arbitrary 2-category.

In the final chapters of this thesis, we therefore continue to investigate our second question. Here, our motivations depart from our desire to develop the theory of algebraic simple type theories: we are motivated purely by the pursuit of mathematical enlightenment. Our hope is to axiomatise those properties of a 2-category that permit us to carry out monad–theory correspondences in such a way that recovers the classical monad–theory correspondence, and its divers generalisations. Since the concept of *relative monad* is integral to our understanding, we must first develop the theory of relative monads within a 2-category, following in the footsteps of Street in the seminal study of monads within a 2-category [Str72b; Str72a]. This is the focus of Chapter 5, where we define relative monads in a 2-category (technically, a *proarrow equipment* in the sense of Wood [Woo82]) and study their Kleisli and Eilenberg–Moore constructions, as well as the process of *rerooting* a relative monad: comparing relative monads relative to different 1-cells. In particular, the theory of this chapter allows us to characterise when relative monads might be extended to monads with the same algebras, which is crucial for the monad–theory correspondence. In the subsequent Chapter 6, we characterise Kleisli inclusions for relative monads in the presence of a bicategorical factorisation system satisfying a property we call *resoluteness*: such a factorisation system equips the 2-category with a formal theory of monads that is very much like that of the 2-category of categories. The combination of these two chapters provides us with the ingredients for a *formal monad–theory correspondence*, explicating at last the true nature of the classical monad–theory correspondence.

In Chapter 7, the final chapter of this thesis, we justify our abstract analysis by recovering the monad–theory correspondences of Lucyshyn-Wright [Luc16] and Bourke and Garner [BG19] for enriched categories, and the monad–theory correspondence of Johnstone and Wraith [JW78] for internal categories, thereby subsuming every (1-dimensional) correspondence appearing in the literature. To do so, we show how the various notions of *theory* appearing therein may be characterised by relative adjointness properties, which clarifies several aspects of the previous developments.

To give context for our results, we also provide in Chapter 7 a survey of the monad–theory correspondences that have appeared up to this point. The historical context proves particularly illuminating. For instance, it reveals that our perspective on algebraic theories, as relative adjoint functors, was independently observed by Diers [Die74], who also subsequently established a correspondence with a notion of relative monad [Die75]. Sadly, Diers's work was entirely overlooked, and this perspective was lost until now. Fortunately for this thesis, there is little overlap between our work and that of Diers's beyond our definition of j-*theory*, particularly as Diers works with a different, less general, definition of relative monad (which we relate to that of Altenkirch, Chapman and Uustalu [ACU10] in Chapter 7). To the best of our knowledge, our survey is a complete reference for the extant development of monad–theory correspondences since Linton [Lin66c]; we view the preservation of the mathematical record an important responsibility, and have taken care to present as holistic a picture of this line of research as we were able.

Chapter 2

Preliminaries

We briefly review the notation and terminology we use throughout the thesis.

2.1 Conventions

We shall use $f \mathbin{;} g$ for diagrammatic composition order, and juxtaposition gf for traditional composition order, unless stated otherwise.

2.2 1-categories

Definition 2.2.1. A *shape* is a small category. Given a class of shapes Φ, a category A is Φ-*(co)complete* when every diagram of a shape in Φ in A has a (co)limit. A functor from a Φ-(co)complete category is Φ-*(co)continuous* if it preserves (co)limits of shapes in Φ. We denote by $[A, B]_\Phi$ the full subcategory of the functor category $[A, B]$ spanned by Φ-continuous[1] functors.

Definition 2.2.2. For a category A, denote by $\mathcal{P}A$ the small-presheaf construction on A, i.e. the full subcategory of $[A^{\mathrm{op}}, \mathbf{Set}]$ spanned by small colimits of representables, and by よ$_A \colon A \to \mathcal{P}A$ the Yoneda embedding. In the case that A admits Ψ-colimits, for Ψ a class of shapes, denote by $\mathcal{P}_\Psi A$ the full subcategory of $\mathcal{P}A$ spanned by Ψ-continuous presheaves (cf. [Kel82, §5.7]).

2.3 2-categories

For the definition of bicategory, we refer to [Bén67, Definition 1.1]. Every bicategory is biequivalent to a 2-category [MP85, §2], and so we shall generally work in the setting of 2-categories for simplicity. Similarly, when drawing pasting diagrams in bicategories, we shall occasionally leave the structural isomorphisms implicit for ease of comprehension.

Definition 2.3.1. We denote by **CAT** the 2-category of locally small categories, functors, and natural transformations; and by **Cat** the locally full sub-2-category spanned by the small categories. We denote by **Prof** the bicategory of small categories, profunctors, and natural transformations.

Definition 2.3.2. A bicategory $\mathcal{K}$ is *locally* ___ if, for all objects $X, Y \in \mathcal{K}$, the hom-category $\mathcal{K}[X, Y]$ is ___.

Definition 2.3.3. A 1-cell $f \colon A \to B$ in a bicategory is *representably fully faithful* if, for all objects $X \in \mathcal{K}$, the functor $\mathcal{K}[X, f] \colon \mathcal{K}[X, A] \to \mathcal{K}[X, B]$ induced by postcomposition by f is fully faithful.

[1] We shall not require notation for the full subcategory of the functor category $[A, B]$ spanned by Φ-*co*continuous functors.

Definition 2.3.4. A 1-cell $u \colon B \to E$ in a bicategory satisfies *invertible-path lifting* (resp. is a *discrete isofibration*) if, for each span $B \xleftarrow{b} X \xrightarrow{e} E$ and invertible 2-cell $\chi \colon b \mathbin{;} u \cong e$, there is a (unique) pair $b' \colon X \to B$ and $\chi' \colon b' \mathbin{;} u \cong e$ such that $\chi = \chi' \mathbin{;} u$.

$$
\begin{tikzcd}
X \arrow[r, "b"] \arrow[dr, "e"'] \arrow[dr, phantom, bend left=20, "\cong"] & B \arrow[d, "u"] \\
& E
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\cong"] \arrow[dr, bend right, "e"'] & B \arrow[d, "u"] \\
& E
\end{tikzcd}
$$

2.3.1 Extensions and lifts

Definition 2.3.5. A bicategory $\mathcal{K}$ is respectively *left-coclosed*, *left-closed*, *right-coclosed*, or *right-closed* if equipped with each of the following operations[2]:

$\rhd \colon \mathcal{K}[A,B] \times \mathcal{K}[A,C]^{\mathrm{op}} \to \mathcal{K}[B,C]$	Left extension
$\blacktriangleright \colon \mathcal{K}[A,B]^{\mathrm{op}} \times \mathcal{K}[A,C] \to \mathcal{K}[B,C]$	Right extension
$\lhd \colon \mathcal{K}[A,C]^{\mathrm{op}} \times \mathcal{K}[B,C] \to \mathcal{K}[A,B]$	Left lifting
$\blacktriangleleft \colon \mathcal{K}[A,C] \times \mathcal{K}[B,C]^{\mathrm{op}} \to \mathcal{K}[A,B]$	Right lifting

satisfying

$$
\begin{aligned}
\mathcal{K}[A,B](f \lhd h, g) \cong \mathcal{K}[A,C](f, g \mathbin{;} h) \cong \mathcal{K}[B,C](g \rhd f, h) \\
\mathcal{K}[A,B](g, f \blacktriangleleft h) \cong \mathcal{K}[A,C](g \mathbin{;} h, f) \cong \mathcal{K}[B,C](h, g \blacktriangleright f)
\end{aligned}
$$

natural in 1-cells $f \colon A \to C$, $g \colon A \to B$, $h \colon B \to C$ in $\mathcal{K}$. A 2-category $\mathcal{K}$ is *left-biclosed* if it is left-coclosed and left-closed, *right-biclosed* if it is right-coclosed and right-closed, *coclosed* if it is left- and right-coclosed, *closed* if it is left- and right-closed, and *biclosed* if it is left- and right-biclosed.

We may present these operations diagrammatically as follows, where η denotes the unit of an adjunction, and ϵ denotes the counit.

$$
\begin{tikzcd}
& B \arrow[dr, dashed, "g \rhd f"] \arrow[d, phantom, "\eta\Uparrow"] & \\
A \arrow[ur, "g"] \arrow[rr, "f"'] & {} & C
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dr, dashed, "g \blacktriangleright f"] \arrow[d, phantom, "\epsilon\Downarrow"] & \\
A \arrow[ur, "g"] \arrow[rr, "f"'] & {} & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
& B \arrow[dr, "h"] \arrow[d, phantom, "\eta\Uparrow"] & \\
A \arrow[ur, dashed, "f \lhd h"] \arrow[rr, "f"'] & {} & C
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dr, "h"] \arrow[d, phantom, "\epsilon\Downarrow"] & \\
A \arrow[ur, dashed, "f \blacktriangleleft h"] \arrow[rr, "f"'] & {} & C
\end{tikzcd}
$$

Note that a left extension in $\mathcal{K}^{\mathrm{co}}$ is a right extension; a left extension in $\mathcal{K}^{\mathrm{op}}$ is a left lifting; and a left extension in $\mathcal{K}^{\mathrm{co\,op}}$ is a right lifting. In practice, $\mathcal{K}$ may only have some extensions and lifts, in which case we use the same notation, but treat the operators as partially-defined.

[2] We use the mnemonic symbols of May and Sigurdsson [MS06, Definition 16.3.1] for left extensions and lifts, rather than the common notation lan and lift. In their setting everything is symmetric, so that left (co)closure coincides with right (co)closure, necessitating the use only of two symbols; we choose to use the corresponding filled symbols for right extensions and lifts.

There is a helpful mnemonic to remember to which operation each symbol corresponds. Observe that, in each case, the operator produces a 1-cell from the domain or codomain of the left operand to the domain or codomain of the right operand. The triangle points in the direction of the domain or codomain: thereby, $g \rhd f$ is a 1-cell from the codomain of g to the codomain of f, whereas $f \lhd g$ is a 1-cell from the domain of f to the domain of g. For left versus right, we have that $\rhd$ is empty, whilst $\blacktriangleright$ is full, which is analogous to the intuition that initial objects are often uninhabited and terminal objects are often inhabited (more whimsically, the shape $\rhd$ is homotopic to the shape 0, whilst the shape $\blacktriangleright$ is homotopic to the shape 1).

By convention, $(-) \rhd (-)$ will bind tightly on the left and on the right, so that $f \,;\, g \rhd h \,;\, i \cong f \,;\, (g \rhd h) \,;\, i$.

One helpful intuition for these concepts is given by observing that, for a one-object 2-category viewed as a monoidal category, extensions and lifts correspond to (co)closed structure, i.e. internal (co)homs.

Definition 2.3.6. Given 1-cells $f\colon A \to C$ and $h\colon B \to C$, a left lift $f \lhd h$ is *absolute* if it is preserved by precomposition by any 1-cell, i.e. given any 1-cell $a\colon X \to A$, we have that $a \,;\, (f \lhd h) \cong (a \,;\, f) \lhd h$.

$$\begin{tikzcd}
 & & B \arrow[dr, "h"] \arrow[d, phantom, "\eta\Uparrow"] & \\
X \arrow[r, "a"'] \arrow[urr, bend left, "(a;f)\lhd h"] & A \arrow[ur, "f\lhd h"] \arrow[rr, "f"'] \arrow[u, phantom, "\cong"] & {} & C
\end{tikzcd}$$

Absolute right lifts and extensions are defined analogously.

We recall finally that it is possible to characterise right adjoints in a bicategory in terms of absolute left extensions.

Lemma 2.3.7 ([SW78, Proposition 2])**.** *Let $\ell \dashv r$ be an adjunction in a bicategory $\mathcal{K}$. Then $r \cong \ell \rhd 1$, and this extension is absolute.*

Chapter 3

The classical monad–theory correspondence

There are two traditional approaches to algebra in category theory: the first through *algebraic theories*, which are often considered fundamentally *syntactic*, determining algebraic structures by their operations and equations (whilst remaining free from presentations); and the second through *monads*, which are often considered fundamentally *semantic*, determining algebraic structures by the categories of varieties they present. Whilst each has a distinct flavour, it has long been known that these two approaches are inseparable: the category of infinitary algebraic theories is equivalent to the category of monads on **Set**, and the models for any given algebraic theory are precisely the algebras for the induced monad [Lin66c; Lin66b; Lin69a].

Since one may consider monads on *any* category, it is natural to wonder whether this correspondence between theories and monads carries over to settings other than sets and, following this line of inquiry, the classical monad–theory correspondence has been generalised repeatedly since its inception (cf. [Lin69a; Dub70; Die75; BD80; Pow99; NP09; Mel10; LR11; BMW12; Luc16; BG19]). However, despite these divers approaches, the monad–theory correspondence remains somewhat enigmatic. Establishing the most general correspondences appears to require significant mathematical sophistication and, as such, it is not clear why one should *expect* these correspondences to hold at all. In this chapter, we will give a new conceptual understanding of the monad–theory correspondence, demonstrating that the correspondence is, in essence, an inevitable consequence of the definitions of *monad* and *theory*.

To realise this desire, it is helpful to introduce an intermediate notion – that of a *relative monad* [ACU10] – to bridge the gap between theories and monads. Intuitively, just as monads are monoids in categories of endofunctors, relative monads are monoids in arbitrary functor categories. Many of the properties and constructions of monads carry over to the relative setting. In particular, every relative monad is induced canonically by two relative adjunctions [Ulm68; ACU10]: the Kleisli and Eilenberg–Moore resolutions. The central insight of the perspective we present here, from which the entire monad–theory correspondence will be shown to follow, is that theories correspond precisely to relative monads, in the following manner.

Idea. *Algebraic theories are precisely the Kleisli inclusions of relative monads.*

Having proven this characterisation, a result which makes use of little more than the Yoneda lemma and an elementary observation regarding Kleisli categories, the path to a monad–theory correspondence is direct. Since the Kleisli relative adjunction is initial amongst those inducing the given relative monad, such inclusions are in one-to-one correspondence with relative monads themselves, establishing an equivalence between the category of theories and the category of relative monads. Furthermore, in well-behaved situations, relative monads are equivalently given by monads preserving certain colimits, and from this the monad–theory correspondence may be concluded.

From the perspective of relative monads, many of the definitions and constructions that have arisen in the study of theories, such as the definition of the categories of models in terms of a pullback [Lin69a; Die74; Mel10;

LR11; BG19], and the nervousness conditions associated to theories and monads [BG19], are also illuminated.

For simplicity, in this section we work in the setting of unenriched categories. The reader versed in enriched category theory will observe that the proofs carry over with essentially no modification to the setting of enriched categories. We will later recover an enriched monad–theory correspondence through a general 2-categorical framework in Chapters 5 to 7.

3.1 The relative monad–theory correspondence

We begin by introducing the notion of *relative monad*, which generalises the notion of monad from a structured endofunctor to an arbitrary functor with structure.

Definition 3.1.1 ([ACU10; ACU15]). Let $j \colon A \to E$ be a functor. A *j-relative monad* (or simply *j-monad*) $(t, \eta, (-)^\dagger)$ comprises

1. for each object $a \in A$, an object $ta \in E$ and morphism $\eta_a \colon ja \to ta$;
2. for each morphism $f \colon ja \to tb$, a morphism $f^\dagger \colon ta \to tb$,

satisfying the following laws, for all $a \in A$, $f \colon ja \to tb$, $g \colon jb \to tc$:

$$\eta_a{}^\dagger = 1_{ta} \qquad f^\dagger \eta_a = f \qquad g^\dagger f^\dagger = (g^\dagger f)^\dagger \tag{3.1}$$

A *morphism* $\tau \colon (t, \eta, (-)^\dagger) \to (t', \eta', (-)^\ddagger)$ of j-monads consists of, for each object $a \in A$, a morphism $\tau_a \colon ta \to t'a$ satisfying the following laws, for all $a \in A$, $f \colon ja \to tb$:

$$\tau_a \eta_a = \eta'_a \qquad (\tau_b f)^\ddagger \tau_a = \tau_b f^\dagger \tag{3.2}$$

j-monads and their morphisms form a category $\mathbf{RMnd}(j)$.

It follows from the definitions that, for every j-monad $(t, \eta, (-)^\dagger)$, t canonically forms a functor, and η and $\dagger$ natural transformations; and every j-monad morphism τ canonically forms a natural transformation between the induced functors [ACU15, §2.1].

Example 3.1.2. A monad relative to an identity functor is equivalently a (non-relative) monad. For every functor $j \colon A \to E$, the functor j itself is canonically equipped with the structure of a j-monad.

In the non-relative setting, there is a strong relationship between monads and adjunctions: every adjunction induces a monad, and, conversely, every monad arises from an adjunction. This fact carries over to the relative setting, which we now recall.

Definition 3.1.3 ([Ulm68, Definition 2.2]). Let $j \colon A \to E$ be a functor. A functor $\ell \colon A \to B$ is *left-j-relative adjoint* (or simply *left-j-adjoint*) to a functor $r \colon B \to E$, denoted $\ell \mathrel{{}_j\!\dashv} r$, if there is a natural isomorphism $B(\ell -, -) \cong E(j-, r-)$. We call such a situation a *j-relative adjunction* (or simply *j-adjunction*), and indicate it diagrammatically by the following.

$$\begin{tikzcd} & B \arrow[dr, "r"] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "\ell"] \arrow[rr, "j"'] & {} & E \end{tikzcd}$$

We call B the *apex* of the relative adjunction.

Proposition 3.1.4 ([ACU15, Theorem 2.10]). *Let $j \colon A \to E$ be a functor. Every j-adjunction $\ell \mathrel{{}_j\!\dashv} r$ induces a j-monad with underlying functor $\ell \mathbin{;} r$.*

A converse is given by the Kleisli and Eilenberg–Moore constructions for relative monads (though we shall only have need of the Kleisli construction in this chapter).

Definition 3.1.5 ([ACU10, §2.3])**.** Let $j\colon A \to E$ be a functor and let $T = (t, (-)^\dagger, \eta)$ be a j-monad. The *Kleisli category of* T, denoted $\mathbf{Kl}(T)$, has the same objects as A and hom-classes $\mathbf{Kl}(T)(a, b) = E(ja, tb)$. The identity on an object $a \in A$ is given by η_a; and the composite of morphisms $f\colon a \to b$ and $g\colon b \to c$ in $\mathbf{Kl}(T)$ is given by $g^\dagger f$ in E. The *Kleisli inclusion* $k_T\colon A \to \mathbf{Kl}(T)$ is defined as the identity-on-objects functor sending $f\colon a \to b$ in A to $\eta_b j(f)\colon ja \to tb$ in E. The *forgetful Kleisli functor* $v_T\colon \mathbf{Kl}(T) \to E$ is the functor sending $a \mapsto ta$ and $f\colon ja \to tb$ in E to $f^\dagger\colon ta \to tb$ in E.

Definition 3.1.6 ([ACU10, §2.3])**.** Let $j\colon A \to E$ be a functor and let $T = (t, (-)^\dagger, \eta)$ be a j-monad. An *algebra* for T (or simply T*-algebra*) comprises an object $x \in E$ and, for each morphism $f\colon ja \to x$, a morphism $f^\ddagger\colon ta \to x$, satisfying the following laws, for all $a \in A$, $f\colon ja \to x$, $g\colon ja \to tb$, $h\colon jb \to x$:

$$f^\ddagger \eta_A = f \qquad\qquad (h^\ddagger g)^\ddagger = h^\ddagger g^\dagger \tag{3.3}$$

A *homomorphism* of T-algebras from $(x, \ddagger)$ to $(x', \ddagger')$ is a morphism $w\colon x \to x'$ in E such that, for all $f\colon ja \to x$:

$$(wf)^{\ddagger'} = wf^\ddagger \tag{3.4}$$

The *Eilenberg–Moore category of* T, denoted $\mathbf{EM}(T)$, has as objects T-algebras and as morphisms T-algebra homomorphisms. The *free functor* $f_T\colon A \to \mathbf{EM}(T)$ is defined by t on objects and $(\eta-)^\dagger$ on morphisms. The *forgetful functor* $u_T\colon \mathbf{EM}(T) \to E$ is the functor forgetting the algebra structure of each object.

Definition 3.1.7. Let T be a relative monad. A relative adjunction inducing T is called a *resolution* of T. A *morphism* of resolutions is a functor between their apices rendering commutative the triangles formed by the left-j-adjoints and right-j-adjoints respectively. Resolutions of T and their morphisms form a category $\mathbf{Res}(T)$.

$$\begin{tikzcd}
 & B \arrow[dd, dashed] \arrow[dr, "r"] & \\
A \arrow[ur, "\ell"] \arrow[dr, "\ell'"'] & & E \\
 & B' \arrow[ur, "r'"'] &
\end{tikzcd}$$

Proposition 3.1.8 ([ACU10, Theorem 3])**.** *Let $j\colon A \to E$ be a functor. Given a j-monad $T = (t, (-)^\dagger, \eta)$, the Kleisli inclusion k_T is left-j-adjoint to v_T. The relative adjunction induces T. Furthermore, it is initial amongst resolutions of T.*

Proposition 3.1.9 ([ACU10, Theorem 3])**.** *Let $j\colon A \to E$ be a functor. Given a j-monad $T = (t, (-)^\dagger, \eta)$, the free functor f_T associated to the Eilenberg–Moore construction is left-j-adjoint to u_T. The relative adjunction induces T. Furthermore, it is terminal amongst resolutions of T.*

The universal property of the Kleisli construction exhibits Kleisli resolutions as being in bijection with relative monads. Since, as a rule, we are interested in categories rather than sets, and equivalences rather than bijections, we might wonder whether there is a similar relationship for relative monad morphisms. This is indeed the case, at least supposing that j is dense.

Lemma 3.1.10. *Let $j\colon A \to E$ be a dense functor and let $T = (t, (-)^\dagger, \eta)$ and $T' = (t, (-)^\ddagger, \eta')$ be j-monads. There is a bijection between the following.*

1. *j-monad morphisms $\tau\colon T \to T'$.*
2. *Functors $k\colon \mathbf{Kl}(T) \to \mathbf{Kl}(T')$ such that $k_T \mathbin{;} k = k_{T'}$.*

Proof. Given a j-monad morphism $\tau\colon t \Rightarrow t'$, we form an identity-on-objects functor $k\colon \mathbf{Kl}(T) \to \mathbf{Kl}(T')$ by postcomposition by τ. The law $k_T \mathbin{;} k = k_{T'}$ follows from the relative monad morphism law $\eta \mathbin{;} \tau = \eta'$. For the other direction, observe that, since j is dense, $\mathcal{P}E(E(j-, t-), E(j-, t'-)) \cong [A, E](t, t')$, so that, given a functor $k\colon \mathbf{Kl}(T) \to \mathbf{Kl}(T')$, we have a natural transformation $t \Rightarrow t'$. This is a relative monad morphism, the unit law being satisfied since $k_T \mathbin{;} k = k_{T'}$ and the extension law following from functoriality of k. $\square$

For non-relative monads, similar statements first appear in [Mar66, Theorem 1] and [Pum70, Satz 6], though it is also implicit in the work of Linton (cf. [Lin69a, Lemma 10.2]).

Since we have a bijection between Kleisli resolutions and relative monads, and functors between Kleisli categories and relative monad morphisms, there is an isomorphism of categories, between the category of j-monads, and the category of Kleisli resolutions and functors therebetween. However, this isomorphism is not quite satisfactory, because the objects of the category of Kleisli resolutions are defined with respect to relative monads, making the relationship somewhat tautological. We would therefore like to find a way to characterise those relative adjunctions that are relatively opmonadic without referring to a particular relative monad (namely, the relative monad they induce). The following result allows us to do so.

Proposition 3.1.11 (Relative opmonadicity)**.** *Let $j\colon A \to E$ be a functor and consider a j-adjunction as below, inducing a j-monad T.*

$$\begin{tikzcd} & B \arrow[dr, "r"] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "\ell"] \arrow[rr, "j"'] & {} & E \end{tikzcd}$$

ℓ is isomorphic to the Kleisli inclusion k_T in the coslice category $A/\mathbf{CAT}$ if and only if ℓ is bijective-on-objects.

Proof. If ℓ is isomorphic to a Kleisli inclusion, it is bijective-on-objects, since k_T is identity-on-objects by definition. Conversely, assume ℓ is bijective-on-objects. By initiality of the Kleisli adjunction, and recalling that there is a (bijective-on-objects, fully faithful)-factorisation system on **CAT**, there is a fully faithful functor $\mathbf{Kl}(T) \to B$ factoring the j-adjunction since $\mathbf{Kl}(T)(x, y) \cong E(jx, ty) \cong B(\ell x, \ell y)$. By the 2-out-of-3 property, this functor is also bijective-on-objects and hence an isomorphism of categories. □

For non-relative monads, this observation first appears as [Sch69, §1.0], though again it is also implicit in the work of Linton; for relative monads, the observation appears to be new. The aspect that is perhaps surprising in the generalisation to relative monads is that the characterisation of relative opmonadicity does not depend upon j in any way: therefore, if ℓ forms a left j-adjoint and a left j'-adjoint, inducing distinct relative monads, the two relative monads necessarily have the same Kleisli category.

Remark 3.1.12. The choice between considering identity-on-objects, bijective-on-objects, or essentially-surjective-on-objects functors is trifling, and makes no essential difference to the monad–theory correspondence, save for the strictness of the correspondences. We will follow the convention that theories are bijective-on-objects functors for convenience, but note that our proofs carry through with minor modifications for identity-on-objects or essentially-surjective-on-objects functors.

Since, for dense j, right j-adjoints are uniquely determined by their left j-adjoints [Ulm68, (2.4)], Kleisli resolutions are thus determined entirely by bijective-on-objects functors that are left- j-adjoint. We define *j-theories* to capture precisely this notion.

Definition 3.1.13. Let $j\colon A \to E$ be a dense functor. The category $\mathbf{Th}(j)$ of *j-theories* is the subcategory of $A/\mathbf{CAT}$ spanned by bijective-on-objects left- j-adjoint functors[1].

It thus follows that j-theories are essentially the same as j-monads.

Theorem 3.1.14. *Let $j\colon A \to E$ be a dense functor. There is an equivalence of categories*

$$\mathbf{Th}(j) \simeq \mathbf{RMnd}(j)$$

between the category of j-theories, and the category of j-monads.

Proof. By Lemma 3.1.10, there is an isomorphism of categories between the category of j-monads and the category of Kleisli j-adjunctions. A bijective-on-objects left- j-adjoint functor is a j-relative Kleisli inclusion up to isomorphism by Proposition 3.1.11, from which the result follows. □

[1] Our definition of *j-theory* coincides with what [Die74, Définition 4.1.0] calls an *algebraic j-theory*, except that we do not require j to be fully faithful.

Remark 3.1.15. We might be tempted to wonder whether the above may be strictified into an isomorphism of categories. The situation is quite subtle. Certainly, if we are looking to obtain an isomorphism of categories, we must take j-theories to be identity-on-objects functors, because Kleisli inclusions for relative monads are identity-on-objects. Unfortunately, this is not quite enough. The problem is that our definition of *j-theory* is nonconstructive: while density of j ensures that right relative adjoints are unique up to isomorphism, it does not give us a canonical right relative adjoint for each left- j-adjoint functor. The relative monad associated to a left- j-adjoint functor is therefore determined only up to isomorphism. To address this shortcoming, one might wish to define a j-theory to be an identity-on-objects functor *equipped with* a right j-adjoint. In this case, we obtain an isomorphism of categories. We satisfy ourselves with an equivalence here to simplify exposition; save for some additional bookkeeping, there is no difference between the approaches.

Remark 3.1.16. Throughout, we made the assumption that j is dense. The reader may wonder whether this is truly necessary, or whether the assumption may be relaxed. The answer is that, without density, the fundamental theorems fail to hold in general. For instance, when j is not dense, right j-adjoints are not uniquely determined (cf. [Ulm68, (2.5)]). The failure of uniqueness may be rectified in line with the previous remark by equipping right j-adjoints as structure, so this objection perhaps seems a minor one; a greater problem is that Lemma 3.1.10 does not hold without density of j: if j is not assumed dense, then morphisms of j-theories are no longer determined by functors between Kleisli categories. One could relax the notion of morphism, but at this point, the resemblance to algebraic theories becomes tenuous. As will be evidenced throughout the later chapters, density of j is essentially necessary to have a good theory of j-relative monads more generally.

3.2 Cocompletion-relative monads

We shall now relate our notion of j-theory to the classical notion of algebraic theory [Law63]. The first step towards doing so is the following observation: while being left adjoint is a relatively strong condition for a functor, being a left adjoint *relative to the Yoneda embedding* is a very weak condition. In particular, every functor from a small category A is a left よ_A-relative adjoint, a fact which follows directly from the Yoneda lemma.

Definition 3.2.1 ([BF99, Definition 1.1])**.** A functor $f\colon A \to B$ is *admissible* when, for all $b \in B$, the hom-class $B(f-, b)$ is a small functor (that is, a small colimit of representables). In this case, we denote by $N_f\colon B \to \mathcal{P}A = B(f-, -)$ the *nerve* of f.

Proposition 3.2.2. *Let $f\colon A \to B$ be an admissible functor. There is a よ_A-relative adjunction*

$$\begin{tikzcd} & B \arrow[dr, "N_f"] & \\ A \arrow[ur, "f"] \arrow[rr, "\text{よ}_A"'] & \arrow[u, phantom, "\dashv"] & \mathcal{P}A \end{tikzcd}$$

Proof. We have

$$\begin{aligned} B(f-, =) &\cong \mathcal{P}A(\text{よ}_A-, B(f-, =)) && \text{(Yoneda lemma)} \\ &= \mathcal{P}A(\text{よ}_A-, N_f-) && \text{(definition of } N_f\text{)} \end{aligned}$$

so that $f \;{}_{\text{よ}_A}\!\dashv N_f$. □

More abstractly, this proposition is a consequence of the fact that the Yoneda embedding $\text{よ}_A\colon A \to \mathcal{P}A$ exhibits $\mathcal{P}A$ as the completion of A under small colimits. Assuming that $f\colon A \to B$ preserves certain colimits, we obtain a stronger result. First observe that if A has Ψ-colimits, for some class Ψ of shapes, then the Yoneda embedding restricts to a functor $\text{よ}_A^\Psi\colon A \to \mathcal{P}_\Psi A$ because the hom-functor preserves limits in its first argument.

Proposition 3.2.3. *Let Ψ be a class of shapes, and let $f\colon A \to B$ be an admissible Ψ-cocontinuous functor. Then the relative adjunction $f \,{}_{\text{よ}_A}\!\dashv N_f$ restricts to a relative adjunction.*

$$\begin{tikzcd} & B \arrow[dr, "N_f"] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "f"] \arrow[rr, "\text{よ}_A^\Psi"'] & {} & \mathcal{P}_\Psi A \end{tikzcd}$$

Proof. If f is Ψ-cocontinuous, then each $N_f(b)$ is Ψ-continuous for the same reason that the Yoneda embedding is continuous: namely, by the universal properties of (co)limits. □

Recall that a *(finitary, monosorted) algebraic theory* is an identity-on-objects functor $\mathbb{F}1 \to B$ strictly preserving finite coproducts [Law63, §II.1], where we denote by $\mathbb{F}\colon \mathbf{Cat} \to \mathbf{Cat}$ the cocompletion of a small category under strict finite coproducts (cf. [Bén85, (3.5)]). It therefore follows from Proposition 3.2.3 that every algebraic theory is left-adjoint relative to the inclusion $\mathbb{F}1 \simeq \mathbf{FinSet} \hookrightarrow \mathbf{Set} \simeq [\mathbb{F}1^{\mathrm{op}}, \mathbf{Set}]_{\mathbb{F}}$. It so happens that the converse is also true: every such left relative adjoint preserves coproducts.

Proposition 3.2.4. *Let $j\colon A \to E$ be an admissible functor and let $\ell \,{}_j\!\dashv r$ be a j-relative adjunction. Then ℓ is admissible and, for any class Φ of shapes, if j is Φ-cocontinuous, then so is ℓ.*

Proof. First, since $\ell \,{}_j\!\dashv r$, if j is admissible, i.e. $E(j-, e)$ is a small presheaf for all $e \in E$, then so is ℓ, since for all $b \in B$, we have $B(\ell-, b) \cong E(j-, rb)$, which is a small presheaf by assumption.

For colimit preservation, observe that:

$$\begin{aligned} B(\ell(\mathrm{colim}_\phi x), y) &\cong E(j(\mathrm{colim}_\phi x), ry) && (\ell \,{}_j\!\dashv r) \\ &\cong E(\mathrm{colim}_\phi (jx), ry) && (j \text{ is } \Phi\text{-cocontinuous}) \\ &\cong \lim_\phi E(jx, ry) && (\text{universal property of colimits}) \\ &\cong \lim_\phi B(\ell x, y) && (\ell \,{}_j\!\dashv r) \\ &\cong B(\mathrm{colim}_\phi (\ell x), y) && (\text{universal property of colimits}) \end{aligned}$$

□

The colimit-preservation property in the above first appears as [Ulm68, Theorem 2.13]. Observe that, when j is the identity, we recover the well-known fact that left adjoints preserve all colimits[2].

The final step is to establish that the *relative monad*–theory correspondence of the previous section extends to a *monad*–theory correspondence. In fact, this follows directly from [ACU15, Theorem 4.8] by considering the fixed points of the adjunction therein, but we will give a more conceptual proof. Given a class Φ of shapes, we denote by (Φ, μ, η) the 2-monad on $\mathbf{CAT}$ for the cocompletion of locally-small categories under Φ-colimits [KL00] and by $\mathbf{CAT}_\Phi$ and $\mathbf{CAT}^\Phi$ its Kleisli and Eilenberg–Moore bicategories respectively.

Proposition 3.2.5. *Let Φ be a class of shapes and let A be a category. There is an equivalence of categories*

$$\mathbf{RMnd}(\eta_A) \simeq \mathbf{Mnd}_\Phi(\Phi A)$$

between the category of η_A-monads and the category of Φ-cocontinuous monads on ΦA, commuting with the process of taking algebras.

Proof. η_A-relative monads and monads are both monoids in appropriate monoidal categories, the former by [ACU15, Theorem 3.5, Theorem 4.4] since η_A is well-behaved, and the latter by definition:

$$\mathbf{RMnd}(\eta_A) \cong \mathbf{Mon}(\mathbf{CAT}_\Phi(A, A)) \tag{3.5}$$

$$\mathbf{Mnd}_\Phi(\Phi A) = \mathbf{Mon}(\mathbf{CAT}^\Phi(\Phi A, \Phi A)) \tag{3.6}$$

The equivalence then follows from full faithfulness of the embedding of the Kleisli bicategory into the Eilenberg–Moore bicategory. That this commutes with taking algebras follows from [ACU15, §4.4]. □

[2] In general, r will not preserve limits. However, when j is dense, r preserves all limits by the usual argument.

The monad–theory correspondence now follows.

Theorem 3.2.6. *Let Ψ and Φ be classes of shapes, and let A be a Ψ-cocomplete category. If $\Phi A \simeq \mathcal{P}_\Psi A$, then there is an equivalence of categories*

$$\mathbf{Th}(\Psi, A) \simeq \mathbf{Mnd}_\Phi(\Phi A)$$

between the full subcategory of $A/\mathbf{CAT}$ spanned by Ψ-continuous admissible identity-on-objects functors, and the category of Φ-cocontinuous monads on ΦA.

Proof. We have an equivalence $\mathbf{Th}(よ^\Psi_A) \simeq \mathbf{Mnd}_\Phi(\Phi A)$ from Theorem 3.1.14 and Proposition 3.2.5 given the equivalence $\Phi A \simeq \mathcal{P}_\Psi A$, and by Proposition 3.2.3 and Proposition 3.2.4, an admissible functor preserves Ψ-limits if and only if it is a left $よ^\Psi_A$-relative adjoint, so that $\mathbf{Th}(\Psi, A) \simeq \mathbf{Th}(よ^\Psi_A)$. □

The condition relating Ψ and Φ in the correspondence above may appear unexpected. It is related to the notion of *soundness* [Adá+02, Definition 2.2] for a class of shapes [DV14, Proposition 3.8]. In our case, it is required to express the limit-preservation condition arising for nerves in Proposition 3.2.3 in terms of a cocompletion property necessary to invoke Proposition 3.2.5. The condition thus acts as a bridge between the characterisations of relative adjointness and monadicity.

In our motivating example, we may take Ψ to be the class of finite discrete categories (representing finite coproducts), and Φ to be the class of sifted categories (representing sifted colimits). In this case, soundness is satisfied: we have an equivalence $\mathbf{Sind}(-) \simeq [(-)^{\mathrm{op}}, \mathbf{Set}]_{\mathbb{F}}$ for small categories with finite coproducts [AR01, Corollary 2.8]. Therefore, we recover the classical monad–theory correspondence.

Corollary 3.2.7. *There is an equivalence of categories*

$$\mathbf{Law}(1) \simeq \mathbf{Mnd}_{\mathit{sf}}(\mathbf{Set})$$

between the category of finitary monosorted algebraic theories, and the category of sifted-cocontinuous monads on **Set**.

Proof. Let $A = \mathbb{F}$, Ψ be the class of finite discrete categories, and Φ be the class of sifted categories. The equivalence follows directly from Theorem 3.2.6, since Ψ is sound and $\mathbf{Sind}(\mathbb{F}) \simeq \mathbf{Set}$; the admissibility condition is trivial because $\mathbb{F}$ is small. In this case, $j = \mathbf{FinSet} \hookrightarrow \mathbf{Set}$ (equivalently $j = \mathbb{F}1 \hookrightarrow \mathbf{Sind}(\mathbb{F}1) \simeq \mathcal{P}1$). □

Remark 3.2.8. Finitary algebraic theories are often considered to correspond to *finitary* (i.e. filtered-cocontinuous), rather than strongly-finitary (i.e. sifted-cocontinuous), monads on **Set**. However, an endofunctor on **Set** preserves sifted colimits if and only if it preserves filtered colimits, and so these classes of monads coincide. Morally speaking, the sifted colimits are the appropriate class to consider, because algebraic theories are defined through finite coproduct structure, rather than finite colimit structure.

Various similar correspondences are easily captured. For instance, technically speaking, the *classical monad–theory correspondence* refers to the correspondence between *infinitary* monosorted algebraic theories and *arbitrary* monads on **Set** [Lin66c; Lin66b], which we recover by taking $A = \mathbf{Set}$, Ψ to be the class of small discrete categories, and Φ to be the empty class, i.e. $j = 1_{\mathbf{Set}}$. (The admissibility condition is hidden in [Lin69a, Lemma 10.2], as Linton defines infinitary algebraic theories to be left-adjoint functors from **Set**, rather than functors satisfying a coproduct-preservation property.) Alternatively, we may take $j = \mathbb{F}S \hookrightarrow \mathbf{Sind}(\mathbb{F}S) \simeq \mathcal{P}S$ for a set of sorts S to recover the monad correspondence for finitary multisorted algebraic theories (or $j = 1_{\mathbf{Set}^S}$ for the infinitary variant); for any cardinal κ, we can take $j = \mathbf{Set}_{\leq\kappa} \hookrightarrow \mathbf{Set}$ to obtain a monad correspondence for theories with at most κ-ary operations and equations (cf. [Die74, Exemple 4.1.1]).

In Chapter 7, we will explore monad–theory correspondences arising in more general situations than those for which soundness applies. In particular, while here we have established a monad–theory correspondence only for those j-monads for which j is *well-behaved* [ACU15, Definition 4.1]), it is possible to relax this assumption in some settings.

Remark 3.2.9. It has become customary to define algebraic theories in terms of *product* structure[3], rather than coproduct structure. However, for our purposes, it is convenient to take Lawvere's original definition as primary. The two perspectives are equivalent via the duality involution on **Cat**. However, our monad–theory correspondence justifies Lawvere's perspective: strictly speaking, algebraic theories defined with product structure correspond to comonads on $\mathbf{Set}^{\mathrm{op}}$.

[3] In fact, it is common to define algebraic theories simply to be categories with finite products (cf. [AHS04, Definition 1.1]). This is entirely dissatisfactory from the perspective of the monad–theory correspondence, as in doing so one forgets the structure of the Kleisli inclusion that is essential in obtaining the correspondence.

Chapter 4

Higher-order algebraic theories

4.1 Introduction

Algebraic theories were introduced by Lawvere to provide a categorical, presentation-free axiomatisation of universal algebraic structure [Law63]. Shortly thereafter, Linton proved that algebraic theories are equivalent to monads on the category of sets [Lin69a]. Consequently, we may view algebra through three lenses: the equational logic of Birkhoff [Bir35]; algebraic theories; or monads. Despite the richness of universal algebra, there are many structures throughout mathematics that are not captured thereby and, since their introduction, many generalisations and variations of algebraic theories have arisen: for instance, many-sorted algebraic theories [Bén68], enriched algebraic theories [Dub70], and essentially algebraic theories [Fre72]. One such variation is the notion of second-order algebraic theory, introduced by Fiore and Mahmoud [FM10] to capture the variable-binding structure encountered in simple type theories such as the λ-calculus [Chu40]. Second-order algebraic theories generalise algebraic theories by permitting variable-binding operators such as differential operators ∂, logical quantifiers $\exists$ and $\forall$, and abstraction operators λ. Fiore and Mahmoud established a correspondence between second-order algebraic theories and the second-order equational logic of Fiore and Hur [FH10], and later further established a correspondence with a class of abstract clones equipped with algebraic structure [Mah11; FM14]. However, Fiore and Mahmoud did not pursue a monadic perspective. The first main contribution of the current chapter is to establish a monad–theory correspondence for second-order algebraic theories subsuming that of the first-order setting.

Having generalised algebraic theories through the consideration of second-order operators, which, intuitively, are operators whose operands are first-order operators, it is natural to ask whether we might do the same for operators of order three or higher. Our second main contribution is to generalise first- and second-order algebraic theories to n^{th}-order algebraic theories for arbitrary natural numbers $n \in \mathbb{N}$, as well as to ω-order algebraic theories, which capture structure whose operators have unbounded order. Furthermore, we follow Bénabou in working throughout with S-sorted theories, for a fixed set S, in particular generalising the monosorted setting of Fiore and Mahmoud for $n = 2$. When $n = \omega$, we recover a categorical, presentation-free axiomatisation of the λ-theories of Lambek and Scott [LS88]. Considering n^{th}-order algebraic theories directly, rather than working only with ω-order algebraic theories, is illuminating, as the structure of the category of $(n+1)^{\text{th}}$-order algebraic theories is naturally determined by the category of n^{th}-order algebraic theories: in particular, we show that the category of $(n+1)^{\text{th}}$-order algebraic theories is given by a category of monads on the category of n^{th}-order algebraic theories. It is natural to identify the category of 0^{th}-order algebraic theories with the category of sets and so, when $n = 0$, we recover the classical monad–theory correspondence.

Our third main contribution is to exhibit a universal characterisation of the category of n^{th}-order algebraic theories as a locally strongly finitely presentable category whose subcategory of strongly finitely presentable objects is the free cartesian category on a repeatedly-exponentiable object.

While in this chapter we pursue a purely abstract categorical understanding of higher-order structure, we view higher-order algebraic theories as an important practical tool, particularly for the study of simple type theories. As such, we deem it useful also to provide a syntactic perspective, and define a notion of presentation and equational logic for higher-order algebraic theories. This work may therefore be taken as the starting point for a systematic treatment of higher-order algebra in the spirit of that for first-order algebra (cf. [ARV10]), which we hope will serve to motivate further understanding and application.

4.1.1 Related work

While our notion of first- and second-order algebraic theories are standard [Law63; FM10], n^{th}-order algebraic theories for $n \notin \{1, 2\}$ do not appear to have previously been studied. However, structures similar to our ω-order algebraic theories do appear in the literature.

Presentations Syntactic presentations of ω-order algebraic theories have appeared frequently in the literature; this is essentially the notion of a presentation of the simply-typed λ-calculus. For instance, see [Poi86; LS88; Mei92; Mei95; Cro93; Joh02], all of which are equivalent to our definition (though there is some variation in the precise formulations). Homomorphisms of presentations are rarely treated.

Cartesian-closed categories It is common in categorical algebra to take the term *algebraic theory* to refer simply to a cartesian category, rather than the stricter functorial structure imposed by Lawvere [Law63] (which are then called *monosorted* or *one-sorted algebraic theories*). In the same vein, cartesian-closed categories have often been proposed for higher-order algebra: for instance, see [Poi86; LS88; Cro93; Joh02], and [Hay85, §3] for a variant involving semi-cartesian-closed categories. Though this simplifies many aspects of the theory, there are shortcomings to this approach. In particular, the monad–theory correspondence is lost when the functorial structure is forgotten; similarly, the elegant characterisation of the category of algebraic theories as a locally strongly finitely presentable category only holds in the setting in which the sorts are fixed. For this reason, we work with algebraic theories à la Lawvere [Law63] and Bénabou [Bén68].

Unityped and non-extensional ω-order algebraic theories The higher-order structure we consider is interpreted by exponentiable structure in a cartesian category. From a type-theoretic perspective, this corresponds to the binding structure of the extensional simply-typed λ-calculus [Chu40], where both β- and η-rules are present for type formers. One may also consider a notion of theory corresponding to the unityped λ-calculus [Chu36]: categorically, this amounts to identifying the generating object X of an algebraic theory with its own exponential X^X [Sco80]. For instance, see the *algebraic theories of type λ-$\beta\eta$* of Obtułowicz [Obt77]; the *Church algebraic theories* of Obtułowicz and Wiweger [OW82, §5.4]; the *semi-cartesian-closed algebraic theories* of Hayashi [Hay85, §2.3.2]; and the *λ-theories* of Hyland [Hyl17, Definition 3.1]. The latter two of these notions correspond more specifically to the *intensional* unityped λ-calculus, where the η-rule is absent. However, in practice, the unityped λ-calculus is too degenerate to capture most of the examples of interest.

4.1.2 Notation

Throughout, $n \in \mathbb{N}_\omega$ is taken to range over the extended natural numbers $(\mathbb{N}+\{\omega\}, \leq)$, except where indicated otherwise. We use the term *higher-order algebraic theory* to refer to an n^{th}-order algebraic theory for arbitrary n. The definitions and proofs for the case $n = 0$ are deferred to Section 4.8.

4.2 Perspectives

There are several perspectives from which higher-order structure may be viewed, each of which is distinctly elucidating. We give an overview to provide intuition for the following development; each of these perspectives has appeared separately in the literature, but, to our knowledge, the connection between them has not previously been explicated, which makes it difficult for a non-expert to develop a holistic picture of the subject.

4.2.1 Higher-order natural deduction

Universal algebra, or more precisely its associated first-order equational logic, may be seen as a natural deduction system in which there are two judgements, one for the well-formedness of terms, and one for their equality. The operators of an algebra take a sequence of terms, the *operands*, in doing so forming a new term. One may present an operator syntactically by an inference rule of the following form:

$$\frac{\vdash t_1 \qquad \cdots \qquad \vdash t_n}{\vdash \mathsf{f}(t_1, \ldots, t_n)} \tag{4.1}$$

We read this inference rule as "given well-formed terms t_1 through to t_n, we may form a new well-formed term $\mathsf{f}(t_1, \ldots, t_n)$"; we think of f as being an operator that we apply to the operands t_1 through to t_n. One may consider variations on first-order equational logic by modifying the structure of these inference rules. For example, associating a sort (or type) to each term leads to the notion of multisorted algebraic theory [Bén68; BL70], in which inference rules may only be applied if the operands have the appropriate types. For instance, the action of a monoid may be presented by an inference rule of the following form:

$$\frac{\vdash m : \mathsf{M} \qquad \vdash x : \mathsf{X}}{\vdash \mathsf{act}(m, x) : \mathsf{X}} \tag{4.2}$$

We read this inference rule as "given a well-formed term m of type M and a well-formed term x of type X, we may form a new well-formed term $\mathsf{act}(m, x)$ of type X". In fact, in the absence of ill-formed terms (which arise only when one considers concrete syntax, formed through string concatenation from basic symbols), we may drop "well-formed" and simply talk about unqualified "terms".

Higher-order equational logic arises when one considers operators that may themselves take operators, rather than terms, as their operands. A second-order operator may therefore be presented by an inference rule whose premisses are themselves (first-order) inference rules. For instance, consider the following second-order inference rule:

$$\frac{\left(\dfrac{\vdash t_{1\,1} \qquad \cdots \qquad \vdash t_{1\,m_1}}{\vdash f_1(t_{1\,1}, \ldots, t_{1\,m_1})}\right) \qquad \cdots \qquad \left(\dfrac{\vdash t_{n\,1} \qquad \cdots \qquad \vdash t_{n\,m_n}}{\vdash f_n(t_{n\,1}, \ldots, t_{n\,m_n})}\right)}{\vdash \mathsf{g}(f_1, \ldots, f_n)} \tag{4.3}$$

We read this inference rule as "given (derivable) inference rules that take terms $t_{i\,1}$ through to $t_{i\,n_i}$, thereby forming terms $f_i(t_{i\,1}, \ldots, t_{i\,n_i})$, for $1 \leq i \leq n$, we may form a term $\mathsf{g}(f_1, \ldots, f_n)$"; we think of g as being an operator that we apply to the inference rules f_1 through to f_n. Note that when we say "inference rule" for f_i, we really mean any possible derivation of a term given terms $t_{i\,1}$ through to $t_{i\,n_i}$: we permit the composition of inference rules by grafting conclusions of one inference rule to a premiss of another, to form open derivations of terms.

Similarly, we may consider third-order operators, which take second-order operators as operands, and so on for arbitrary $n \in \mathbb{N}$. Note that $(n+1)^{\text{th}}$-order operators with no n^{th}-order operands are equivalently n^{th}-order operators: in this way, $(n+1)^{\text{th}}$-order operators strictly subsume n^{th}-order operators. From this perspective, 0^{th}-order operators are equivalently constants. We may define an ω-order operator to be a k^{th}-order operator for some $k \in \mathbb{N}$, so that, for an infinite family of ω-order operators, there may not be a natural number bounding the order of the operators.

These higher-order operators may be motivated by their use in metatheoretic reasoning: by ascending to $(n+1)^{\text{th}}$-order operators, it is possible to perform operations on n^{th}-order operators. For example, we can describe a second-order operator that formally adds an inverse to any unary first-order operator:

$$\frac{\left(\dfrac{\vdash x}{\vdash f(x)}\right) \quad \vdash t}{\vdash \mathsf{inv}(f, t)} \qquad \frac{\left(\dfrac{\vdash x}{\vdash f(x)}\right) \quad \vdash t}{\vdash \mathsf{inv}(f, f(t)) \equiv t} \qquad \frac{\left(\dfrac{\vdash x}{\vdash f(x)}\right) \quad \vdash t}{\vdash f(\mathsf{inv}(f, t)) \equiv t} \tag{4.4}$$

Here, $\mathsf{inv}(f, t)$ should be interpreted as $f^{-1}(t)$. In practice, as evidenced by even the simple second-order operator above, higher-order equational logic quickly becomes unwieldy when presented recursively in this nested natural deductive style, but it nevertheless provides a useful intuition. This approach to higher-order reasoning was explored by Schroeder-Heister [Sch84] from a purely syntactic perspective.

4.2.2 Equational logics with metavariables

First-order operators are usually defined as symbols taking terms as operands: for any compatible choice of operand terms, we may form a new term, which may be considered the application of that operator. However, there is another choice: we may instead define operators as symbols parameterised by a context of variables, such as in the following inference rule:

$$\frac{}{x_1, \ldots, x_n \vdash \mathsf{f}} \tag{4.5}$$

We read this inference rule as "we may form a term f in any context with n variables"; and think of f as being some term containing free variables (this is called an *open term*): to apply the operator f, we substitute each of the variables x_1 through to x_n by terms t_1 through to t_n, as in the following:

$$\frac{\vdash t_1 \quad \cdots \quad \vdash t_n}{\vdash \mathsf{f}[{}^{t_1}/_{x_1}] \cdots [{}^{t_n}/_{x_n}]} \tag{4.6}$$

Observe that this inference rule has the same form as (4.1): the difference is simply in whether we form a compound term $\mathsf{f}(t_1, \ldots, t_n)$ or substitute for free variables in an open term $\mathsf{f}[{}^{t_1}/_{x_1}] \cdots [{}^{t_n}/_{x_n}]$. Note also that, in this style, the variables by which the term f is parameterised do not appear in the term itself; this information is implicitly recorded in the context. In an appropriate, formal sense, this perspective is equivalent to that of Section 4.2.1; one may consider premisses $\vdash t_i$ in empty contexts to correspond to variables x_i in the context of the conclusion, and vice versa. It is natural to then ask whether there is an analogue, in terms of variables and substitution, for the higher-order operators of Section 4.2.1. It turns out that there is such an analogue: one may present second-order operators as terms in *metavariable contexts* [Acz78; Ham04; FH10]. Formally, metavariables are variables that are themselves parameterised by variables: we can instantiate any metavariable by providing terms for each of its parameterising variables, akin to the application of (4.1) or substitution of (4.6). For example, the metavariable context below has n metavariables, each of which is parameterised by m_i variables.

$$(x_{1\,1}, \ldots, x_{1\,m_1})x_1, \ldots, (x_{n\,1}, \ldots, x_{n\,m_n})x_n \tag{4.7}$$

A second-order operator may be defined, similarly to (4.5), as a symbol parameterised by a context of metavariables, rather than simply variables, such as in the following inference rule:

$$\frac{}{(x_{1\,1}, \ldots, x_{1\,m_1})x_1, \ldots, (x_{n\,1}, \ldots, x_{n\,m_n})x_n \vdash \mathsf{g}} \tag{4.8}$$

We read this inference rule as "we may form a term g in any context with n metavariables, the i^{th} of which is parameterised by m_i variables"; and think of g as some term containing free metavariables. Just as variables have an associated notion of substitution, metavariables have an associated notion of *meta-substitution* [Ham04; Fio08; FH10]: in particular, while variables x may be substituted by terms t; so may metavariables $(x_1, \ldots, x_n)x$ be substituted by open terms $x_1, \ldots, x_n \vdash f$. This allows us to apply a second-order operator as in (4.8), by meta-substituting each of the metavariables $(x_{1\,1}, \ldots, x_{1\,m_1})x_1$ through $(x_{n\,1}, \ldots, x_{n\,m_n})x_n$ by open terms $x_{1\,1}, \ldots, x_{1\,m_1} \vdash f_1$ through to $x_{n\,1}, \ldots, x_{n\,m_n} \vdash f_n$, as below (we use the same notation for substitution and meta-substitution):

$$\frac{x_{1\,1}, \ldots, x_{1\,m_1} \vdash f_1 \quad \cdots \quad x_{n\,1}, \ldots, x_{n\,m_n} \vdash f_n}{\vdash \mathsf{g}[{}^{f_1}/_{x_1}] \cdots [{}^{f_n}/_{x_n}]} \tag{4.9}$$

Observe that this inference rule has the same form as (4.3), under the relationship between the first-order operators exhibited by (4.1) and (4.6). As in the first-order setting, these two perspectives on second-order operators are equivalent. We may similarly describe third-order operators by way of meta-metavariables, meta-metasubstitution, and so on. In theory, we could combine the two perspectives, introducing metavariable contexts to the formalism of Section 4.2.1, but we gain no extra expressivity by doing so.

The perspective of equational logic with metavariables is well-suited to describing axiom schemata, which are typically formalised nonfinitarily by infinite families of axioms (requiring a suitably expressive metatheory): metavariables permit axiom schemata to be described finitarily, with each axiom induced by a schema arising from a higher-order operator by meta-substitution (cf. [FH13, §1]). In this sense, the notion of metavariable described here aligns with that of the traditional notion in mathematical logic. This is the perspective taken by Fiore and Hur [FH10] in the setting of second-order equational logic (cf. [Ham04; Fio08]). In their setting, contexts contain both metavariables and variables. However, just as first-order operators are equivalent to second-order operators with no first-order operands, so variables are equivalent to nullary metavariables and there is no loss in generality to consider solely contexts of metavariables.

4.2.3 Higher-order logical frameworks

Logical frameworks are deductive systems whose reasoning is formally expressed via a type theory, which plays the role of the metatheory. In particular, those type theories possessing function types form the metatheories for higher-order logical frameworks. Traditionally, the study of categorical theories (such as algebraic theories, essentially algebraic theories, geometric theories, and so forth) has been distinct from the study of logical frameworks, but the objects of interest in both fields are the same, albeit in different dress. For example, (multisorted) universal algebra can equivalently be viewed as the logical framework corresponding to the *simply-typed pairing calculus*: the fragment of the simply-typed λ-calculus with products but without function types (cf. [Cro93, Chapter 3]); this view lends itself as a useful bridge between the approaches of categorical algebra and programming language theory.

Following this observation, there is an evident candidate for the metatheory associated to higher-order equational logic: namely, the simply-typed λ-calculus [Chu40]. Metavariables may be represented in the simply-typed λ-calculus by variables of function types, while meta-substitution is given by the ordinary substitution of λ-terms. In fact, it is common in computer science to use the simply-typed λ-calculus to represent variable-binding operators, treating the λ-abstraction operator as a canonical variable-binding operator through which all others may be defined: this is essentially the motivating idea behind higher-order universal algebra [Mei95; Mei92; Poi86], and higher-order abstract syntax[1] [PE88]. However, one could argue that this practice was formally justified only once the binding structure of the simply-typed λ-calculus was proven to be universal, in the sense of being equivalent to generic algebraic binding structure by Fiore and Mahmoud [FM10; Mah11; FM14]. Following their development, we may in good conscience present n^{th}-order operators as operators with restricted order within the simply-typed λ-calculus. For instance, we may present a second-order operator by a (sorted) function constant, such as the following.

$$\frac{}{\vdash \mathsf{g} : (\mathsf{U}^{n_1} \to \mathsf{U}) \times \cdots \times (\mathsf{U}^{n_n} \to \mathsf{U}) \to \mathsf{U}} \tag{4.10}$$

Here, g is thought of as an operator taking functions as operands, and is equivalent to (4.8) by uncurrying. Given terms $\vdash f_1 : \mathsf{U}^{n_1} \to \mathsf{U}$ through to $\vdash f_n : \mathsf{U}^{n_n} \to \mathsf{U}$, corresponding to open terms by uncurrying, we may form a new term $\mathsf{g}(f_1, \ldots, f_n)$ using the application operation of the simply-typed λ-calculus:

$$\frac{\vdash f_1 : \mathsf{U}^{n_1} \to \mathsf{U} \qquad \cdots \qquad \vdash f_n : \mathsf{U}^{n_n} \to \mathsf{U}}{\vdash \mathsf{g}(f_1, \ldots, f_n) : \mathsf{U}} \tag{4.11}$$

Note that, though we distinguish informally between the operators defined using the simply-typed λ-calculus and the operators of the simply-typed λ-calculus itself (such as λ-abstraction and application), there is no formal difference between the two from this perspective; this is analogous to the formalism of algebraic theories, in which the structural operations are not distinguished amongst the algebraic operations.

The presentation of higher-order equational logic by the simply-typed λ-calculus is the one we choose to use throughout this chapter, as the syntax is particularly elegant and will be the most familiar to those accustomed with categorical logic and type theory.

[1] We note that the metalogic of [PE88] is also polymorphic, but reserve the term *higher-order abstract syntax* for the fragment restricted to the simply-typed λ-calculus, following the tradition of second-order abstract syntax [FPT99].

4.2.4 Simply-typed λ-calculi

We consider the simply-typed λ-calculus in Section 4.2.3 as a logical framework for higher-order deduction. However, extensions of the simply-typed λ-calculus are often instead studied for the purpose of defining programming languages qua simple type theories. While, in a logical framework, the primitive type and term operators have philosophical import – for instance, product types correspond to conjunction, and function types to implication – in a programming language, they are concrete syntactic devices, and their meaning is defined through their behaviour. Practically, these perspectives are similar, but the distinction between taking the simply-typed λ-calculus as a metatheory, or as a programming language, is conceptually important (indeed, the relationship between these perspectives forms the essence of the Curry–Howard correspondence [How80]).

In practice, the difference between this perspective and that of Section 4.2.3 manifests itself in whether the purpose of considering the simply-typed λ-calculus is to study the calculus itself, or whether the purpose is to define higher-order structures within the calculus viewed as a metatheory.

4.3 Presentations and translations

We begin the development by describing an equational logic for higher-order algebraic theories, based on the perspective of the simply-typed λ-calculus as a logical framework for higher-order equational logic. This allows us to present various examples in Section 4.4, motivating the study of higher-order structures, and gives a concrete syntactic intuition for several of the constructions that follow.

As we are interested in n^{th}-order structure, rather than simply the ω-order structure that is present in the classical simply-typed λ-calculus, we must restrict the calculus so that the order of each type is limited, intuitively by forbidding the construction of arbitrarily-nested function types. We start by giving examples of the order of several types to aid intuition. Below, B is some base type.

Order	Types			
0	1			
1	B	$\mathsf{B} \times \mathsf{B}$	$(\mathsf{B} \times \mathsf{B}) \times \mathsf{B}$	$\mathsf{B} \times (\mathsf{B} \times \mathsf{B})$
2	$\mathsf{B} \to \mathsf{B}$	$\mathsf{B} \times \mathsf{B} \to \mathsf{B}$	$\mathsf{B} \to \mathsf{B} \times \mathsf{B}$	$\mathsf{B} \to \mathsf{B} \to \mathsf{B}$
3	$(\mathsf{B} \to \mathsf{B}) \to \mathsf{B}$	$(\mathsf{B} \times \mathsf{B} \to \mathsf{B}) \to \mathsf{B}$	$(\mathsf{B} \to \mathsf{B}) \to (\mathsf{B} \to \mathsf{B}) \to \mathsf{B}$	

Remark 4.3.1. Throughout, we use *order* to refer to the order of operators, and by extension their types and calculi. For us, the *second-order λ-calculus* refers to a simply-typed λ-calculus the types of whose function arguments may be at most first-order, rather than the polymorphic λ-calculus, which has also gone by that name.

The full *n^{th}-order simply-typed λ-calculus* is presented as a deductive system in Figure 4.1, parameterised by a set of sorts (*base types*) S. For those familiar with the simply-typed λ-calculus, the only differences are the definition of order (ord), and the restricted function-type formation ($\to$-FORM) and λ-abstraction ($\to$-INTRO) rules.

$$\mathrm{ord}(1) \stackrel{\text{def}}{=} 0$$

$$\mathrm{ord}(\mathsf{B}) \stackrel{\text{def}}{=} 1 \qquad (\mathsf{B} \in S)$$

$$\mathrm{ord}(X \times Y) \stackrel{\text{def}}{=} \max(\mathrm{ord}(X), \mathrm{ord}(Y))$$

$$\mathrm{ord}(X \to Y) \stackrel{\text{def}}{=} \max(\mathrm{ord}(X) + 1, \mathrm{ord}(Y))$$

$$\frac{}{\cdot\ \mathsf{ctx}}\ \textsc{empty} \qquad \frac{\Gamma\ \mathsf{ctx} \quad X\ \mathsf{ty}}{\Gamma, x : X\ \mathsf{ctx}}\ \textsc{ext} \qquad \frac{}{\Gamma, x : X, \Delta \vdash x : X}\ \textsc{var}$$

$$\frac{}{\mathsf{B}\ \mathsf{ty}}\ (\mathsf{B} \in S)\ \textsc{base}$$

$$\frac{}{1\ \mathsf{ty}}\ 1\text{-}\textsc{form} \qquad \frac{X\ \mathsf{ty} \quad Y\ \mathsf{ty}}{X \times Y\ \mathsf{ty}}\ \times\text{-}\textsc{form}$$

$$\frac{X\ \mathsf{ty} \quad \mathsf{ord}(X) < n \quad Y\ \mathsf{ty}}{X \to Y\ \mathsf{ty}}\ \to\text{-}\textsc{form}$$

$$\frac{}{\Gamma \vdash \langle\rangle : 1}\ 1\text{-}\textsc{intro} \qquad \frac{\Gamma \vdash u : 1}{\Gamma \vdash u \equiv \langle\rangle : 1}\ 1\text{-}\eta$$

$$\frac{\Gamma \vdash a : X \quad \Gamma \vdash b : Y}{\Gamma \vdash \langle a, b\rangle : X \times Y}\ \times\text{-}\textsc{intro}$$

$$\frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \pi_1(p) : X}\ \times\text{-}\textsc{elim}_1 \qquad \frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \pi_2(p) : Y}\ \times\text{-}\textsc{elim}_2$$

$$\frac{\Gamma \vdash a : X \quad \Gamma \vdash b : Y}{\Gamma \vdash \pi_1\langle a, b\rangle \equiv a : X}\ \times\text{-}\beta_1 \qquad \frac{\Gamma \vdash a : X \quad \Gamma \vdash b : Y}{\Gamma \vdash \pi_2\langle a, b\rangle \equiv b : Y}\ \times\text{-}\beta_2$$

$$\frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \langle \pi_1 p, \pi_2 p\rangle \equiv p : X \times Y}\ \times\text{-}\eta$$

$$\frac{\Gamma, x : X \vdash t : Y \quad \mathsf{ord}(X) < n}{\Gamma \vdash \lambda x : X.\, t : X \to Y}\ \to\text{-}\textsc{intro} \qquad \frac{\Gamma \vdash f : X \to Y \quad \Gamma \vdash a : X}{\Gamma \vdash f\ a : Y}\ \to\text{-}\textsc{elim}$$

$$\frac{\Gamma, x : X \vdash t : Y \quad \Gamma \vdash a : X}{\Gamma \vdash (\lambda x : X.\, t)\ a \equiv t[{}^{a}/_{x}] : Y}\ \to\text{-}\beta \qquad \frac{\Gamma \vdash f : X \to Y}{\Gamma \vdash \lambda x : X.\, f\ x \equiv f : X \to Y}\ \to\text{-}\eta$$

Rules are given up to α-equivalence.
$t[{}^{a}/_{x}]$ denotes the substitution of the term a for the free variable x in the term t.
Context extension is defined inductively by repeated variable extension.

Figure 4.1: The n^{th}-order simply-typed λ-calculus on S (for $n > 0$).

When $n = 1$, we recover the *simply-typed pairing calculus*, the fragment of the simply-typed λ-calculus with product types, but not function types. When $n = \omega$, we recover the classical simply-typed λ-calculus. It is well known that the categorical semantics of the simply-typed pairing calculus (equivalently, equational logic) is given by cartesian categories [Law63]; while the categorical semantics of the simply-typed λ-calculus is given by cartesian-closed categories [Lam80; LS88], and so the less syntactically-inclined reader should understand the n^{th}-order simply-typed λ-calculus to correspond to some class of cartesian categories with a limited number of exponentials. A precise categorical treatment for $n \in \mathbb{N}^+_\omega$ is deferred till Section 4.5.

Remark 4.3.2. In this chapter only, we will consider algebraic theories to be defined with respect to cartesian structure, rather than cocartesian structure, to aid comparison with traditional approaches to the categorical semantics of type theories.

Remark 4.3.3. Presenting the equational logics of higher-order algebraic theories as order-limited λ-calculi leads to several simplifications over previous approaches. For example, the meta-substitution operation of

Fiore [Fio08] is given in our framework by the substitution of a second-order variable by a λ-abstraction. The near-semiring compatibility structure between substitution and meta-substitution observed by Fiore [Fio16] then follows directly from the associativity of substitution. On the other hand, our approach requires us to reason about terms up to $\beta\eta$-equivalence; in contrast, the second-order equational logic of Fiore and Hur [FH10], which may be viewed as the $\beta\eta$-normal fragment of the second-order simply-typed λ-calculus, requires no consideration of $\beta\eta$-equivalence, at the expense of possessing multiple forms of variable and substitution.

We now describe presentations for higher-order algebraic theories using the n^{th}-order simply-typed λ-calculus; our definitions are analogous to the standard definitions from universal algebra [Bir35] and present no surprises. For the remainder of this section, we consider only $n > 0$; we discuss the $n = 0$ case in Section 4.8.

Definition 4.3.4. Denote by $\Lambda_n(S)$ the set of types of the n^{th}-order simply-typed λ-calculus, generated from the set S of sorts according to the rules of Figure 4.1.

Definition 4.3.5. An *S-sorted n^{th}-order signature* consists of a set O of operators and a function $|{-}| : O \to \Lambda_n(S) \times S$. Given an operator $\mathsf{o} \in O$ for which $|\mathsf{o}| = (X, \mathsf{B})$, we call X the *arity* of o, and B the *coarity* of o. Denote by $\Lambda_{(O,|-|)}$ the family of terms (indexed by their context and type) generated by extending Figure 4.1 by the following axiom schema:

$$\frac{\Gamma \vdash t : X}{\Gamma \vdash \mathsf{o}(t) : \mathsf{B}}\ \textsc{o-op} \quad (|\mathsf{o} \in O| = (X, \mathsf{B}))$$

Definition 4.3.6. An *S-sorted n^{th}-order presentation* Σ consists of an S-sorted n^{th}-order signature $(O, |{-}|)$ and a set $E \subseteq \sum_{(X,\mathsf{B}) \in \Lambda_n(S) \times S} \Lambda_{(O,|-|)}(X, \mathsf{B})^2$ of *equations*. Denote by Λ_Σ the family of terms (indexed by their context and type) generated by extending $\Lambda_{(O,|-|)}$ by the following axiom schema:

$$\frac{\Gamma \vdash t : X}{\Gamma \vdash l[^t/_x] \equiv r[^t/_x] : \mathsf{B}}\ (l, r)\text{-}\textsc{eq} \quad ((X, \mathsf{B}, l, r) \in E)$$

We denote by $Q_\Sigma \colon \Lambda_{(O,|-|)} \twoheadrightarrow \Lambda_\Sigma$ the quotient of $\Lambda_{(O,|-|)}$ by the equations of Σ.

There are two natural notions of morphism between presentations: the first, which we call *transliterations*, are homomorphisms between signatures, mapping operators in one presentation to operators in another; the second, which we call *translations* following Fiore and Mahmoud [FM10], instead map operators in one presentation to *terms* in another. Morphisms of presentations are frequently elided in treatments of categorical logic, but are useful practically; we give several examples in Section 4.4.

Definition 4.3.7. Let $\Sigma = (O, |{-}|, E)$ and $\Sigma' = (O', |{-}|', E')$ be S-sorted n^{th}-order presentations. A *transliteration* from Σ to Σ' consists of a function $f : O \to O'$ such that $|f(\mathsf{o})|' = |\mathsf{o}|$ for all $\mathsf{o} \in O$; and such that, for all $(X, \mathsf{B}, l, r) \in E$, we have that $Q_\Sigma(l) = Q_\Sigma(r)$ implies $Q_{\Sigma'}(f(l)) = Q_{\Sigma'}(f(r))$, where f extends congruently from operators to terms in the usual manner.

S-sorted n^{th}-order presentations and transliterations form a category $\mathbf{Pres}_n^{\text{lit}}(S)$, with composition and identities inherited from **Set**.

Definition 4.3.8. Let $\Sigma = (O, |{-}|, E)$ and $\Sigma' = (O', |{-}|', E')$ be S-sorted n^{th}-order presentations. A *translation* from Σ to Σ' consists of a function $f : \prod_{\mathsf{o} \in O} \Lambda_{(O',|-|')}(|\mathsf{o}|)$, such that, for all $(X, \mathsf{B}, l, r) \in E$, we have that $Q_\Sigma(l) = Q_\Sigma(r)$ implies $Q_{\Sigma'}(f(l)) = Q_{\Sigma'}(f(r))$, where f extends congruently from operators to terms in the usual manner.

S-sorted n^{th}-order presentations and translations form a category $\mathbf{Pres}_n(S)$, with identities given by inclusions, and compositions $g \circ f$ given by composing f with the congruent extension of g to terms.

4.4 Examples

We give a range of examples of presentations and translations for higher-order algebraic theories.

Example 4.4.1. The *unityped λ-calculus* [Chu36] is a second-order algebraic theory presented by a single sort U together with the following operators and equations.

$$\frac{\Gamma \vdash f : \mathsf{U} \qquad \Gamma \vdash x : \mathsf{U}}{\Gamma \vdash \mathsf{app}(f, x) : \mathsf{U}}\ \mathsf{U}\text{-}\textsc{intro} \qquad \frac{\Gamma \vdash f : \mathsf{U} \to \mathsf{U}}{\Gamma \vdash \mathsf{abs}(f) : \mathsf{U}}\ \mathsf{U}\text{-}\textsc{elim}$$

$$\frac{\Gamma \vdash f : \mathsf{U} \to \mathsf{U} \qquad \Gamma \vdash u : \mathsf{U}}{\Gamma \vdash \mathsf{app}(\mathsf{abs}(f), u) \equiv f\, u : \mathsf{U}}\ \mathsf{U}\text{-}\beta$$

The unityped λ-calculus is called *extensional* when equipped with the U-η rule.

$$\frac{\Gamma \vdash f : \mathsf{U}}{\Gamma \vdash \mathsf{abs}(\lambda x : \mathsf{U}.\, \mathsf{app}(f, x)) \equiv f : \mathsf{U}}\ \mathsf{U}\text{-}\eta$$

The *continuation-passing style transform* forms a second-order translation from the unityped λ-calculus to itself [Mah11, Example 6.2(3)].

Example 4.4.2. The *simply-typed λ-calculus* on a set of base types S is an $\Lambda_\omega(S)$-sorted second-order algebraic theory, presented by the usual rules for the simply-typed λ-calculus (e.g. those for $n = \omega$ in Figure 4.1). Note that this example demonstrates that we may express arbitrary higher-order structure in a second-order algebraic theory, but only given an infinite set of sorts.

Example 4.4.3. The natural numbers with addition and multiplication form a monosorted first-order algebraic theory: *the theory of arithmetic*. There is a second-order translation from the theory of arithmetic to the unityped λ-calculus given by Church encoding [Mah11, Example 6.2(2)].

Example 4.4.4. For all $n \in \mathbb{N}$, n^{th}-order logic is an $(n+1)^{\text{th}}$-order algebraic theory. Higher-order logic is an ω-order algebraic theory. Analogously, Hilbert's ϵ-calculus is a second-order algebraic theory, for which the choice operator ϵ is second-order (cf. [EO10b]).

Example 4.4.5. Staton's *parameterised algebraic theories* [Sta13a; Sta13b] are $\{\mathsf{P}, \mathsf{T}\}$-sorted[2] second-order algebraic theories whose binding operands have arity $\mathsf{P}^n \to \mathsf{T}$ for $n \in \mathbb{N}^+$ and whose operations with coarity P are monosorted. Consequently, examples of parameterised algebraic theories, such as Fiore and Staton's theory of jumping [FS14], and the equational theory of the Beta-Bernoulli process [Sta+18], are also examples of second-order algebraic theories.

Example 4.4.6. Context-free expressions, which extend regular expressions with a least fixed-point operator μ [KY19], form a monosorted second-order algebraic theory.

Example 4.4.7. Plotkin's axiomatisation of partial differentiation [Plo20] is a monosorted second-order algebraic theory axiomatising the operation of evaluating a derivative at a point.

$$\frac{\Gamma \vdash f : \mathbb{R} \to \mathbb{R} \qquad \Gamma \vdash x_0 : \mathbb{R}}{\Gamma \vdash \partial f(x_0) : \mathbb{R}}$$

Example 4.4.8. The simply-typed λ-calculus with sum types extends Example 4.4.2 with coprojection and case-splitting operations.

$$\frac{\Gamma \vdash a : \mathsf{A}}{\Gamma \vdash \mathsf{inl}(a) : \mathsf{A} + \mathsf{B}}\ +\text{-}\textsc{intro}_1 \qquad \frac{\Gamma \vdash b : \mathsf{B}}{\Gamma \vdash \mathsf{inr}(b) : \mathsf{A} + \mathsf{B}}\ +\text{-}\textsc{intro}_2$$

$$\frac{\Gamma \vdash s : \mathsf{A} + \mathsf{B} \qquad \Gamma \vdash f : \mathsf{A} \to \mathsf{C} \qquad \Gamma \vdash g : \mathsf{B} \to \mathsf{C}}{\Gamma \vdash \mathsf{case}(s, f, g) : \mathsf{C}}\ +\text{-}\textsc{elim}$$

[2] Multisorted parameterised algebraic theories, as introduced in [Sta13b], may be represented similarly.

$$\frac{\Gamma \vdash a : \mathsf{A} \qquad \Gamma \vdash f : \mathsf{A} \to \mathsf{C} \qquad \Gamma \vdash g : \mathsf{B} \to \mathsf{C}}{\Gamma \vdash \mathsf{case}(\mathsf{inl}(a), f, g) \equiv f\, a : \mathsf{C}}\ +\text{-}\beta_1$$

$$\frac{\Gamma \vdash b : \mathsf{B} \qquad \Gamma \vdash f : \mathsf{A} \to \mathsf{C} \qquad \Gamma \vdash g : \mathsf{B} \to \mathsf{C}}{\Gamma \vdash \mathsf{case}(\mathsf{inr}(b), f, g) \equiv g\, b : \mathsf{C}}\ +\text{-}\beta_2$$

$$\frac{\Gamma \vdash s : \mathsf{A} + \mathsf{B} \qquad \Gamma \vdash h : \mathsf{A} + \mathsf{B} \to \mathsf{C}}{\Gamma \vdash \mathsf{case}(s, h\,\mathsf{inl}(a), h\,\mathsf{inr}(b)) \equiv h\, s : \mathsf{C}}\ +\text{-}\eta$$

Example 4.4.9. Type-theoretic *control operators* are presented by two sorts $\{\mathsf{A}, \mathsf{Z}\}$ together with third-order operators, subject to various equations, typically forcing Z to be uninhabited. Examples include Felleisen and Friedman's control operator [FF86; Gri89] and call/cc [Hof95], given respectively by the following inference rules.

$$\frac{\Gamma \vdash f : (\mathsf{A} \to \mathsf{Z}) \to \mathsf{Z}}{\Gamma \vdash \mathcal{C}(f) : \mathsf{A}} \qquad\qquad \frac{\Gamma \vdash f : (\mathsf{A} \to \mathsf{Z}) \to \mathsf{A}}{\Gamma \vdash \mathsf{call/cc}(f) : \mathsf{A}}$$

There is a third-order translation from call/cc to $\mathcal{C}$ (described in [EO10a] as a monad morphism from the selection monad to the continuation monad) that maps call/cc to the term $f : (\mathsf{A} \to \mathsf{Z}) \to \mathsf{A} \vdash \mathcal{C}(\lambda g.\ g\ (f\ g)) : \mathsf{A}$.

We remark that the axiomatisation of control operators by higher-order operators suggests it may be fruitful to study computational effects such as continuations from the perspective of higher-order algebraic theories (cf. [Fio]), as opposed to large algebraic theories such as in [Hyl+07]. Higher-order algebraic theories are much more well-behaved than large algebraic theories in general: for instance, unlike large algebraic theories, higher-order algebraic theories are closed under small colimits (Corollary 4.6.9).

4.5 Higher-order algebraic theories

Algebraic theories provide a presentation-free axiomatisation of universal algebraic structure [Law63]: any given algebraic structure may be presented in many different ways, and it is often useful to work with a single invariant structure, rather than an individual presentation. Higher-order algebraic theories serve the same function for the higher-order algebraic structure of Section 4.3. However, while the semantic, categorical perspective is typically more conducive to proofs, the syntactic perspective often offers helpful intuition and we will frequently refer to it when introducing new concepts.

We recall some standard definitions to establish terminology. We use *strict* to mean that canonical isomorphisms are identities.

Definition 4.5.1. A category (resp. functor) is *cartesian* if it admits (resp. preserves) finite products. Cartesian categories, cartesian functors, and natural transformations[3] form a 2-category **Cart**. A cartesian category (resp. functor) is *strict* if it is so as monoidal category (resp. functor). A category (resp. functor) is *cartesian-closed* if it is cartesian and, for every object X, the functor $X \times (-)$ admits a (resp. preserves the) right adjoint.

As suggested in Section 4.3, the categorical structure of the n^{th}-order simply-typed λ-calculus lies between that of cartesian categories and cartesian-closed categories. In an algebraic theory, the structure of interest – namely, the finite products – is global, since we may take the product of any pair of objects; the same is true in an ω-order algebraic theory, where every object is exponentiable. However, in n^{th}-order algebraic theories in general, only some objects are distinguished by exponentiability. Consequently, to define n^{th}-order structure requires a little more complexity than first- or ω-order structure, as we must impose structure only on the generating sorts, rather than on every object.

Definition 4.5.2. An object X in a cartesian category is *exponentiable* if $X \times (-)$ has a right adjoint $(-)^X$. A fully faithful functor $I \colon \mathscr{C} \hookrightarrow \mathscr{D}$ is *pointwise exponentiable*[4] if, for every object $X \in \mathscr{C}$, the object IX is exponentiable in $\mathscr{D}$. A subcategory is *exponentiable* if its inclusion functor is pointwise exponentiable.

[3] Natural transformations are automatically monoidal for cartesian monoidal structure.
[4] To be distinguished from exponentiability of I in the functor category $[\mathscr{C}, \mathscr{D}]$.

Definition 4.5.3. We define *tetration* for an object X in a cartesian category inductively, whenever the requisite powers exist; intuitively, tetration is iterated exponentiation.

$$X \uparrow\uparrow 0 \stackrel{\text{def}}{=} 1 \qquad\qquad X \uparrow\uparrow (n+1) \stackrel{\text{def}}{=} X^{X \uparrow\uparrow n}$$

An object X is n*-tetrable* if, for all $0 \leq i \leq n$, the object $X \uparrow\uparrow i$ is exponentiable. An object is ω*-tetrable* if it is n-tetrable for all $n \in \mathbb{N}$. A fully faithful functor $I \colon \mathscr{C} \hookrightarrow \mathscr{D}$ between cartesian categories is *pointwise* n*-tetrable* if, for every object $X \in \mathscr{C}$, the object IX is n-tetrable in $\mathscr{D}$. A subcategory is n*-tetrable* if its inclusion functor is pointwise n-tetrable.

It follows that every object in a cartesian category is 0-tetrable, and is 1-tetrable if and only if it is exponentiable. In a cartesian category, the terminal object is ω-tetrable, as is every object in a cartesian-closed category. A product $\Pi_{i \in I} X_i$ is n-tetrable if and only if each multiplicand X_i is n-tetrable.

Definition 4.5.4. A cartesian functor $F \colon \mathscr{C} \to \mathscr{D}$ *preserves* n*-tetrable objects* if, for each n-tetrable object $X \in \mathscr{C}$ and $0 \leq i \leq n$, the canonical map $F(X \uparrow\uparrow i) \to FX \uparrow\uparrow i$ is invertible.

For the purposes of this chapter, we shall take an S-sorted algebraic theory to be a strict cartesian functor $L \colon \mathbb{L}(S) \to \mathscr{L}$, where $\mathbb{L}(S)$ (the *theory of equality*) is the free strict cartesian category on S. This is dual to our convention elsewhere in this thesis (where algebraic theories are cocartesian functors from $\mathbb{F} \cong \mathbb{L}^{\mathrm{op}}$), but facilitates more convenient comparison with the work of Fiore and Mahmoud [FM10]. We now introduce the analogue of $\mathbb{L}(S)$ in the n^{th}-order setting. $\mathbb{L}_0(S)$ is necessarily defined separately, in Section 4.8.

Definition 4.5.5. $\mathbb{L}_{n+1}(S)$ is the free n-tetrable supercategory of S. That is, $\mathbb{L}_{n+1}(S)$ is the (essentially unique) strict cartesian category for which there exists a fully faithful n-tetrable subcategory $F \colon S \hookrightarrow \mathbb{L}_{n+1}(S)$ such that, given any category $\mathscr{C}$ for which S is a strict n-tetrable subcategory, there is a unique strict cartesian functor $\mathbb{L}_{n+1}(S) \to \mathscr{C}$ strictly preserving n-tetrable objects and making the following triangle commute.

$$\begin{tikzcd}
\mathbb{L}_{n+1}(S) \arrow[r, dashed] & \mathscr{C} \\
S \arrow[u] \arrow[ur, hook, "F"'] &
\end{tikzcd}$$

While defining $\mathbb{L}_{n+1}(S)$ up to isomorphism by this universal property will be sufficient for the later development, we give an explicit construction to prove its existence.

Definition 4.5.6. Denote by $\mathsf{Tree}(S) = \mu X.\, X^2 + S$ the set of binary trees whose leaves are labelled by elements of S. The *left-width* of a binary tree is defined as the maximum number of left-steps from its root to any leaf, explicitly by the following function $\ell \colon \mathsf{Tree}(S) \to \mathbb{N}$, where ι denotes coprojection.

$$\ell(\iota_1(l, r)) = \max(1 + \ell(l), \ell(r)) \qquad\qquad \ell(\iota_2(\mathsf{B})) = 1 \quad (\mathsf{B} \in S)$$

Denote by $\mathsf{Tree}_n(S)$ the restriction of $\mathsf{Tree}(S)$ to those trees t such that $\ell(t) \leq n$, and by $\mathsf{Col}_n(S) = (\mathsf{Tree}_n(S))^\star$ the set of ordered lists of such trees[5].

There is a canonical injective function $\jmath_n \colon \mathsf{Tree}_n(S) \to \Lambda_n(S)$ mapping S-labelled trees to types of the n^{th}-order simply-typed λ-calculus,

$$\jmath_n(\iota_1(l, r)) = \jmath_n(l) \to \jmath_n(r) \qquad\qquad \jmath_n(\iota_2(\mathsf{B})) = \mathsf{B} \quad (\mathsf{B} \in S)$$

which extends to a function $\jmath_n \colon \mathsf{Col}_n(S) \to \Lambda_n(S)$,

$$\jmath_n([t_1, \ldots, t_n]) = \prod_{1 \leq i \leq n} \jmath_n(t_i)$$

(where by convention we take $\prod$ to associate to the left).

[5] Col is short for "colonnade", a row of trees.

Definition 4.5.7. We denote by $\Lambda_n(S)$ the classifying category of the n^{th}-order simply-typed λ-calculus[6]. Explicitly, $\Lambda_n(S)$ has

- objects, the types of the n^{th}-order simply-typed λ-calculus on S;
- morphisms $X \to Y$, the ($\equiv$-equivalence classes of) terms $x : X \vdash t : Y$;
- identity morphisms $X \to X$, variable projections $x : X \vdash x : X$;
- the composition $X \to Y \to Z$ of terms $x : X \vdash s : Y$ and $y : Y \vdash t : Z$ being the substitution $x : X \vdash t[{}^{s}/_{y}] : Z$.

Proposition 4.5.8. *Let $n \geq 1$. $\mathbb{L}_n(S)$ is isomorphic to the full subcategory of $\Lambda_n(S)$ on $\mathrm{Col}_n(S)$.*

Proof. That $\Lambda_n(S)$ has the correct universal property is clear from the usual classifying category construction for the simply-typed λ-calculus (cf. [Cro93, Chapter 4]). The function $\jmath$ is essentially surjective, assigning to each type of the n^{th}-order simply-typed λ-calculus a canonical normal form by eliminating the isomorphisms induced by associativity and unitality of the finite products, the unit and zero for exponentiation, currying, and distributivity of exponentials over finite products. This permits a strict choice of finite products and exponentials in $\Lambda_n(S)$. The full image of $\jmath$ therefore defines a category $\mathbb{L}_n(S)$ satisfying the universal property of Definition 4.5.5. $\square$

Note that, while we take the morphisms of $\mathbb{L}_n(S)$ to be equivalence classes of terms quotiented by the β- and η-laws of Figure 4.1, we could equivalently take $\beta\eta$-normal forms, which are canonical inhabitants of each equivalence class. In the case $n = 2$, this aligns more closely with the approach of Fiore and Mahmoud, and is essentially the philosophy of *lambda-free logical frameworks* in the sense of [Ada08], where terms are given through parameterisation and instantiation, rather than abstraction and application.

Given this syntactic construction of $\mathbb{L}_n(S)$, we can concretely relate tetrability in a cartesian category to order in the n^{th}-order simply-typed λ-calculus. Tetrability is inverse to order: a type in the n^{th}-order algebraic theory has order $1 \leq k \leq n$ if it is $(n-k)$-tetrable as an object of $\mathbb{L}_n(S)$; the base types of the n^{th}-order simply-typed λ-calculus, having order 1, are therefore $(n-1)$-tetrable in $\mathbb{L}_n(S)$.

Proposition 4.5.9. *$\mathbb{L}_1(S)$ is the free cartesian category on S, while $\mathbb{L}_\omega(S)$ is the free cartesian-closed category on S.*

Proof. The first statement is trivial, since every object in a cartesian category is 0-tetrable. For the second, observe that every object in $\mathbb{L}_\omega(S)$ is exponentiable, exhibiting $\mathbb{L}_\omega(S)$ as cartesian-closed, from which the universal property follows. $\square$

Proposition 4.5.10. *The canonical injective-on-objects functor $I_n \colon \mathbb{L}_n(S) \to \mathbb{L}_{n+1}(S)$ induced by Definition 4.5.5 is fully faithful, and hence exhibits $\mathbb{L}_n(S)$ as an exponentiable subcategory of $\mathbb{L}_{n+1}(S)$.*

Proof. By Proposition 4.5.8, we can view the hom-sets $\mathbb{L}_n(S)(\Gamma, \Delta)$ and $\mathbb{L}_{n+1}(S)(\Gamma, \Delta)$, for $\Gamma, \Delta \in \mathrm{Col}_n(S)$ as the sets of terms $\prod_i \Lambda_n(S)(\Gamma, \mathsf{B}_i)$ and $\prod_i \Lambda_{n+1}(S)(\Gamma, \mathsf{B}_i)$ respectively. Because the simply-typed λ-calculus is strongly normalising (see, e.g. [GTL89, Chapter 6]), we know that the terms in any context are equivalently given by $\beta\eta$-normal form terms $\Gamma \vdash t_i : \mathsf{B}_i$, which are identical in the n^{th}-order and $(n+1)^{\text{th}}$-order simply-typed λ-calculi. $\square$

We are now ready to present the main definition. In the following we take $n > 0$; 0^{th}-order algebraic theories are defined in Section 4.8.

Definition 4.5.11. An *S-sorted n^{th}-order algebraic theory* is a strict cartesian identity-on-objects functor $L \colon \mathbb{L}_n(S) \to \mathscr{L}$ strictly preserving n-tetrable objects. Given n^{th}-order algebraic theories $L \colon \mathbb{L}_n(S) \to \mathscr{L}$ and $L' \colon \mathbb{L}_n(S) \to \mathscr{L}'$, a *map* from L to L' is a functor $F \colon \mathscr{L} \to \mathscr{L}'$ such that $FL = L'$. n^{th}-order algebraic theories and their maps form a category $\mathbf{Law}_n(S)$.

[6]Note that, while we are technically overloading the meaning of $\Lambda_n(S)$ here, the notation is consistent with the previous usage in Section 4.3, given the convention to use the same symbol for a category and its underlying object-set.

In particular, the category $\mathbf{Law}_1(1)$ is the classical category of monosorted algebraic theories [Law63]; $\mathbf{Law}_1(S)$ is the category of S-sorted algebraic theories [Bén68]; and $\mathbf{Law}_2(1)$ is the category of monosorted second-order algebraic theories as defined by Fiore and Mahmoud [FM10]. Note that maps of $(n+1)^{\text{th}}$-order algebraic theories are necessarily strict cartesian identity-on-objects functors that strictly preserve n-tetrable objects.

4.5.1 Equivalence of theories and presentations

To justify our definition of presentation and translation in Definition 4.3.6, we prove that the categories of n^{th}-order algebraic theories and of presentations for n^{th}-order algebraic theories are equivalent.

Lemma 4.5.12. *There is a reflection of categories*

$$\begin{tikzcd} \mathbf{Pres}_n^{\text{lit}}(S) \arrow[r, shift left, "\Lambda_{(-)}"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_n(S) \arrow[l, shift left, hook, "\Pi_{(-)}"] \end{tikzcd}$$

Proof. The functor $\Lambda_{(-)} \colon \mathbf{Pres}_n^{\text{lit}}(S) \to \mathbf{Law}_n(S)$ is given by the classifying category construction for each presentation, as in Definition 4.5.7, which is easily seen to be functorial. The functor $\Pi_{(-)} : \mathbf{Law}_n(S) \to \mathbf{Pres}_n^{\text{lit}}(S)$ sends a theory $L \colon \mathbb{L}_n(S) \to \mathscr{L}$ to the presentation $(\coprod_{(\Gamma,\mathsf{B}) \in \Lambda_n(S) \times S} \mathscr{L}(\Gamma, \mathsf{B}), \pi, E_L)$, where the set of equations E_L is given by identifying the formal projections, compositions, and evaluations in $\mathscr{L}$ with the corresponding variable projections, substitutions, and applications in the equational logic; and sends each map $F \colon \mathscr{L} \to \mathscr{L}'$ to the transliteration specified by the function mapping $(\Gamma, A, t) \mapsto (\Gamma, A, Ft)$. That this preserves the equations in Π_L follows from functoriality of F.

Consider a presentation $\Sigma \in \mathbf{Pres}_n^{\text{lit}}(S)$ and theory $L \in \mathbf{Law}_n(S)$. A map $F \colon \Lambda_\Sigma \to L$ is specified entirely by the action of F on the operators of Σ, as the action on the derived terms is forced by functoriality and structure-preservation of F; this then exactly coincides with the data of a transliteration $f : \Sigma \to \Pi_L$. It follows that $\mathbf{Law}_n(S)(\Lambda_\Sigma, L) \cong \mathbf{Pres}_n^{\text{lit}}(S)(\Sigma, \Pi_L)$, which is easily seen to be natural in Σ and L.

For full faithfulness of $\Pi_{(-)}$, observe that the counit of the adjunction is a natural isomorphism, since the closure under composition and identities of a theory corresponds to closure under substitution and variable projections in the corresponding presentation. Hence closure under derived operators is idempotent for presentations induced by theories. $\square$

From this, we may deduce the equivalence of presentations and theories, observing that translations (Definition 4.3.8) are the Kleisli morphisms for the monad induced by Lemma 4.5.12.

Corollary 4.5.13. *There is an equivalence of categories*

$$\mathbf{Pres}_n(S) \simeq \mathbf{Law}_n(S)$$

Proof. Translations, the morphisms of $\mathbf{Pres}_n(S)$, may be seen to correspond to Kleisli morphisms for the monad $\Pi_{(-)}\Lambda_{(-)}$ induced by the reflection of Lemma 4.5.12, essentially by definition, so that $\mathbf{Pres}_n(S) \simeq \mathbf{Kl}(\Pi_{(-)}\Lambda_{(-)})$. Since the adjunction $\Lambda_{(-)} \dashv \Pi_{(-)}$ is idempotent, it follows that $\mathbf{Law}_n(S)$ is equivalent to both the Kleisli category and Eilenberg–Moore category for the induced monad, and hence to $\mathbf{Pres}_n(S)$. $\square$

Remark 4.5.14. As noted, the idempotence of the adjunction in Lemma 4.5.12 implies that $\mathbf{Pres}_n(S)$ is both (1) the Kleisli category and (2) the Eilenberg–Moore category for the induced monad. This justifies the two alternate practices in categorical logic of defining theories syntactically either as (1) sets of operators and equations on the derived terms (as we do in Definition 4.3.6); or (2) sets of terms closed under the deductive structure of the metatheory (for instance, as in [LS88, §I.10]). The observation that $\mathbf{Pres}_1(S)$ is the Kleisli category for a monad on $\mathbf{Pres}_1^{\text{lit}}(S)$ also appears in [VT10, Proposition 5.11].

4.6 Local strong finite presentability

In this section, we prove that the category $\mathbf{Law}_n(S)$ of n^{th}-order algebraic theories is locally strongly finitely presentable, which will be instrumental in establishing our monad–theory correspondence in Section 4.7. Locally strongly finitely presentable categories are analogous to the better-known locally finitely presentable categories, but for which finite products (and sifted colimits), rather than finite limits (and filtered colimits), are primary [AR01]. By the duality theorem of [ALR03], locally strongly finitely presentable categories are precisely the categories of models of multisorted algebraic theories and we will frequently make use of this characterisation.

We recall the definitions of sifted colimits and locally strongly finitely presentable categories.

Definition 4.6.1 ([AR01])**.** A category $\mathscr{C}$ is *sifted* if colimits of diagrams of shape $\mathscr{C}$ commute with finite products in **Set**. A category (resp. functor) is *sifted-cocomplete* (resp. sifted-cocontinuous) if it admits (resp. preserves) all sifted colimits (i.e. colimits indexed by sifted categories). We denote by $よ^{sf}_{\mathscr{C}} \colon \mathscr{C} \hookrightarrow \mathbf{Sind}(\mathscr{C})$ the cocompletion of a small category $\mathscr{C}$ under sifted colimits.

We note for future reference that, since the sifted-cocompletion of a category $\mathscr{C}$ embeds fully faithfully into the free cocompletion of $\mathscr{C}$, we have a corresponding Yoneda lemma:

$$\mathbf{Sind}(\mathscr{C})(よ^{sf}_{\mathscr{C}}X, P) \cong \widehat{\mathscr{C}}(よ_{\mathscr{C}}X, P) \cong PX \tag{4.12}$$

Definition 4.6.2 ([AR01; LR11])**.** A category is *locally strongly finitely presentable* if it is equivalent to $\mathbf{Sind}(\mathscr{C})$ for some small cocartesian category $\mathscr{C}$.

Definition 4.6.3 ([AR01])**.** An object X of a category $\mathscr{C}$ is *strongly finitely presentable* if its co-Yoneda embedding $\mathscr{C}(X, -)$ is sifted-cocontinuous. We denote by $\mathscr{C}_{sf}$ the full subcategory of $\mathscr{C}$ spanned by the strongly finitely presentable objects.

For every small cocartesian category $\mathscr{C}$, the objects of $\mathscr{C}$ are strongly finitely presentable in $\mathbf{Sind}(\mathscr{C})$.

To exhibit $\mathbf{Law}_n(S)$ as locally strongly finitely presentable, we shall give an explicit characterisation of the small cocartesian category that presents $\mathbf{Law}_n(S)$.

Proposition 4.6.4. *There is an equivalence of categories*

$$\mathbf{Cart}(\mathbb{L}_{n+1}(S), \mathbf{Set}) \simeq \mathbf{Law}_n(S)$$

Proof. Up to isomorphism, a cartesian functor $F \colon \mathbb{L}_{n+1}(S) \to \mathbf{Set}$ is determined by a set $F(\mathsf{B}^X)$ for each $\mathsf{B} \in S$ and $X \in \mathbb{L}_n(S)$ and a function $Ff \colon F(\mathsf{B}_1^{X_1}) \times \cdots \times F(\mathsf{B}_n^{X_n}) \to F(\mathsf{B}^X)$ for each morphism $f \colon \mathsf{B}_1^{X_1} \times \cdots \times \mathsf{B}_n^{X_n} \to \mathsf{B}^X$ in $\mathbb{L}_{n+1}(S)$.

We define a cartesian category $\mathscr{F}$ as follows. The objects of $\mathscr{F}$ are those of $\mathbb{L}_n(S)$. The hom-sets are given by $\mathscr{F}(X, Y) \stackrel{\text{def}}{=} F(Y^X)$. Identities are given by $1 \cong F1 \xrightarrow{F(\lambda 1_X)} F(X^X)$. Composition is given by internal composition, $F(Y^X) \times F(Z^Y) \cong F(Y^X \times Z^Y) \xrightarrow{F(\circ)} F(Z^X)$. This forms a category, with unitality and associativity of composition following from that of internal composition. For every morphism $f \colon X \to Y$ in $\mathbb{L}_n(S)$, we have a corresponding element $1 \cong F1 \xrightarrow{\lambda f} F(Y^X)$, hence a morphism $X \to Y$ in $\mathscr{F}$. This defines an identity-on-objects functor $\mathbb{L}_n(S) \to \mathscr{F}$. It remains to show that $\mathscr{F}$ is cartesian and has the required exponentiable objects (given by those of $\mathbb{L}_n(S)$), as this structure will trivially be preserved by the identity-on-objects functor. For the former property, observe that $F((Y_1 \times Y_2)^X) \cong F(Y_1^X \times Y_2^X) \cong F(Y_1^X) \times F(Y_2^X)$ natural in $X, Y_1, Y_2 \in \mathbb{L}_n(S)$; while for the latter $F(Y^{X_1 \times X_2}) \cong F((Y^{X_2})^{X_1})$ natural in $X_1, X_2, Y \in \mathbb{L}_n(S)$. Thus the induced identity-on-objects functor $\mathbb{L}_n(S) \to \mathscr{F}$ is an n^{th}-order algebraic theory.

Conversely, every n^{th}-order algebraic theory $L \colon \mathbb{L}_n(S) \to \mathscr{L}$ defines a cartesian functor $F \colon \mathbb{L}_{n+1}(S) \to \mathbf{Set}$ by defining $F(Y^X) \stackrel{\text{def}}{=} \lceil \mathscr{L} \rceil (1, Y^X)$, where $\lceil - \rceil \colon \mathbf{Law}_n(S) \simeq \mathbf{Pres}_n \hookrightarrow \mathbf{Pres}_{n+1} \simeq \mathbf{Law}_{n+1}(S)$ is given by interpreting L as an $(n+1)^{\text{th}}$-order algebraic theory (cf. Corollary 4.6.10). It is straightforward to see these processes are mutually inverse up to isomorphism, and are functorial in the obvious manner. □

In fact, this characterisation is sufficient to establish that $\mathbb{L}_{n+1}(S)^{\mathrm{op}}$ presents $\mathbf{Law}_n(S)$ as locally strongly finitely presentable, as shown by the following.

Theorem 4.6.5. *There are equivalences of categories*

$$\mathbf{Law}_n(S) \simeq \mathbf{Cart}(\mathbb{L}_{n+1}(S), \mathbf{Set}) \simeq \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})$$

establishing $\mathbf{Law}_n(S)$ *to be locally strongly finitely presentable.*

Proof. The first equivalence is Proposition 4.6.4. The second equivalence follows by [AR01, Corollary 2.8], since $\mathbb{L}_{n+1}(S)$ is small and cartesian. □

Remark 4.6.6. Theorem 4.6.5 may be derived, in the special case where $n = 1$ and $S = 1$, from [Mah11, Proposition 7.8] by specialising to the trivial second-order presentation and observing that abstract clones are equivalently algebraic theories. However, the observation that this result gives a precise characterisation of the strongly finitely presentable objects of $\mathbf{Law}_1(1)$ does not appear ibid.

Remark 4.6.7. Theorem 4.6.5, for $n = \omega$ and $S = 1$, is an analogue (in the simply-typed, extensional setting, rather than the unityped, intensional setting) of what Hyland [Hyl17, Theorem 4.11] calls the *Fundamental Theorem of the λ-Calculus*. There, Hyland's *λ-theories* play the role of ω-order algebraic theories, Hyland's Λ the role of $\mathbb{L}_\omega(S)$, and Hyland's *Λ-algebras* the role of cartesian functors $\mathbb{L}_\omega(S) \to \mathbf{Set}$ (which are identified with the *term algebras* for $\mathbb{L}_\omega(S)$ in Theorem 4.9.11).

Remark 4.6.8. Theorem 4.6.5, for $n = 1$, bears resemblance to the main result of [Uem20], in which it is proven that the category of generalised algebraic theories, a dependently-sorted analogue of algebraic theories, is the cocompletion under filtered colimits of the free finitely complete category on an exponentiable morphism. Uemura's result may be seen as a dependently-sorted analogue of our result.

Corollary 4.6.9. $\mathbf{Law}_n(S)$ *is locally finitely presentable, cocomplete, and complete.*

Proof. It follows from the duality theorem for varieties [ALR03] that every locally strongly finitely presentable category is locally finitely presentable. Every locally (strongly) finitely presentable category is cocomplete and complete. □

From the perspective of the equational logic, cocompleteness of the category of n^{th}-order algebraic theories amounts to the ability to combine presentations by conjoining operators and equations. Completeness is more subtle, as there is usually not a convenient syntactic description of a limit of higher-order algebraic theories.

It follows from Theorem 4.6.5 that there is an embedding

$$J_n \colon \mathbb{L}_{n+1}(S)^{\mathrm{op}} \hookrightarrow \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}}) \simeq \mathbf{Law}_n(S)$$

It is helpful to spend a little time to understand this functor concretely. Intuitively, each object $X \in \mathbb{L}_{n+1}(S)$ induces an n^{th}-order algebraic theory by freely extending the theory of equality $\mathbb{L}_n(S)$ by a constant of type X. For instance, the object $(\mathsf{U} \times \mathsf{U} \to \mathsf{U}) \times ((\mathsf{U} \to \mathsf{U}) \to \mathsf{U})$ of $\mathbb{L}_3(\{\mathsf{U}\})$ induces an equation-free second-order algebraic theory with two operators, corresponding to the abstraction and application operators of the unityped λ-calculus as in Example 4.4.1. Observe that the n^{th}-order algebraic theories induced in this way will have a finite number of operators, since $\mathbb{L}_{n+1}(S)$ has only finite products, and will have no nontrivial equations. This is to be expected, since $\mathbb{L}_{n+1}(S)^{\mathrm{op}}$ is a category of strongly finitely presentable objects in $\mathbf{Law}_n(S)$, which are conceptually understood in general to correspond to finite free objects.

The following corollary of Theorem 4.6.5 establishes a strong relationship between the categories of n^{th}-order and $(n+1)^{\mathrm{th}}$-order algebraic theories that will be seen to play an important role in the monad–theory correspondence.

Corollary 4.6.10. *There is a coreflection of categories*

$$\begin{tikzcd} \mathbf{Law}_n(S) \arrow[r, hook, shift left, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n+1}(S) \arrow[l, shift left, "\lfloor - \rfloor"] \end{tikzcd}$$

Proof. The inclusion functor $I_{n+1}\colon \mathbb{L}_{n+1}(S) \hookrightarrow \mathbb{L}_{n+2}(S)$ induces an algebraic functor $\mathbf{Law}_{n+1}(S) \to \mathbf{Law}_n(S)$. Since algebraic functors are precisely the sifted-cocontinuous right adjoints, this establishes an adjunction between $\mathbf{Law}_n(S)$ and $\mathbf{Law}_{n+1}(S)$. Concretely, the left adjoint is given by $\mathbf{Sind}({I_{n+1}}^{\mathrm{op}})$, and the right adjoint by $\mathbf{Cart}(I_{n+1}, \mathbf{Set})$. Since the Yoneda embedding is fully faithful, **Sind** preserves fully faithful functors, exhibiting the adjunction as a coreflection. □

As a notational convenience, for an n^{th}-order algebraic theory $L\colon \mathbb{L}_n(S) \to \mathscr{L}$, we shall often denote by $\lceil \mathscr{L} \rceil$ the codomain of the functor $\lceil L \rceil$, and by $I_{\mathscr{L}}\colon \mathscr{L} \to \lceil \mathscr{L} \rceil$ the fully faithful inclusion.

Syntactically, the left adjoint of the coreflection can be viewed as interpreting an n^{th}-order presentation as an $(n+1)^{\text{th}}$-order presentation that has no $(n+1)^{\text{th}}$-order operations or equations. Explicitly, using the equivalence between theories and presentations, it is given by $\mathbf{Law}_n(S) \simeq \mathbf{Pres}_n(S) \hookrightarrow \mathbf{Pres}_{n+1}(S) \simeq \mathbf{Law}_{n+1}(S)$. The right adjoint can be viewed as discarding all induced $(n+1)^{\text{th}}$-order operators: categorically, this means sending a theory $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ to the identity-on-objects part of the (identity-on-objects, fully faithful)-factorisation of $\mathbb{L}_n(S) \xhookrightarrow{I_n} \mathbb{L}_{n+1}(S) \xrightarrow{L} \mathscr{L}$. (The first-order algebraic theory associated to a second-order algebraic theory L via $\lfloor - \rfloor$ was called the *underlying algebraic theory of* L in [FM10]. Cf. the *truncation* of an algebraic theory discussed in [Hyl14b, §3.3].)

The same reasoning establishes a chain of coreflections,

$$\mathbf{Law}_0(S) \rightleftarrows \cdots \rightleftarrows \mathbf{Law}_n(S) \rightleftarrows \mathbf{Law}_{n+1}(S) \rightleftarrows \cdots \rightleftarrows \mathbf{Law}_\omega(S)$$

permitting us freely extend or restrict a higher-order theory to any order.

Proposition 4.6.11. $\mathbf{Law}_\omega(S)$ *is the limit of the* ω*-chain*

$$\mathbf{Law}_0(S) \longleftarrow \cdots \longleftarrow \mathbf{Law}_n(S) \longleftarrow \mathbf{Law}_{n+1}(S) \longleftarrow \cdots$$

Proof. Observe that $\mathbb{L}_\omega(S)$ is the colimit of the ω-cochain $\mathbb{L}_0(S) \xhookrightarrow{i_0} \mathbb{L}_1(S) \xhookrightarrow{i_1} \cdots$. The result then follows, since $\mathbf{Cart}(-, \mathbf{Set})$ sends colimits in **Cart** to limits in **Cat**. □

Remark 4.6.12. Note, however, that $\mathbf{Law}_\omega(S)$ is not the colimit of the coreflections: the colimit is instead given by the subcategory of $\mathbf{Law}_\omega(S)$ for which, for each object L, there exists a $k_L \in \mathbb{N}$ such that L is a ${k_L}^{\text{th}}$–order algebraic theory, meaning each theory therein has bounded order.

We take the opportunity to note briefly that the coreflection facilitates a concrete description of the equivalence functor $\mathbf{Law}_n(S) \to \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})$, which will be useful later.

Lemma 4.6.13. *The functor* $\mathbf{Law}_n(S) \to \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})$ *induced by the equivalence of Theorem 4.6.5 is given concretely by the co-Yoneda embedding* $(L\colon \mathbb{L}_n(S) \to \mathscr{L}) \mapsto \lceil \mathscr{L} \rceil(1, \lceil L \rceil -)$.

Proof. For any pair of objects $X, Y \in \mathscr{L}$ (with X not necessarily exponentiable), we may form the exponential Y^X in $\lceil \mathscr{L} \rceil$. Therefore, the hom-set $\mathscr{L}(X, Y)$ is equivalently given by $\lceil \mathscr{L} \rceil(1, Y^X)$. The result then follows directly from the definition of the functor $\mathbf{Law}_n(S) \to \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})$ in Proposition 4.6.4. □

4.7 A monad–theory correspondence

With local strong finite presentability established, we are ready to prove the existence of a monad–theory correspondence for higher-order algebraic theories. However, we shall not do so directly: instead, higher-order algebraic theories will first be shown to correspond to *relative* monads. This aligns with the philosophy of Chapter 3 that *theories*, qua structured identity-on-objects functors, are precisely relative monads. In well-behaved situations, relative monads are equivalent to monads preserving certain colimits, and in this way we achieve the monad–theory correspondence. In our setting, we consider monads relative to the inclusion of a small cocartesian category into its sifted-cocompletion, so that the relative monad may be viewed as taking "small" inputs (namely, the strongly finitely presentable objects), to "possible large" outputs. Syntactically, we restrict to such relative monads because the operators of a higher-order algebraic theory are finitary, i.e. have only a finite number of free variables, which is reflected categorically by the consideration only of finite products, rather than κ-ary or small products.

The central thesis of Chapter 3 is that monad–theory correspondences arise generally as correspondences between Kleisli adjunctions, relative monads, and monads. As a consequence of Theorem 3.1.14, we can immediately deduce that the strict cartesian identity-on-objects functors from $\mathbb{L}_n(S)$ (i.e. n^{th}-order algebraic theories without the condition imposing preservation of exponentiable objects) are equivalently given by monads relative to the inclusion $J_n \colon \mathbb{L}_{n+1}(S)^{\text{op}} \hookrightarrow \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\text{op}}) \simeq \mathbf{Law}_n(S)$, with each strict cartesian identity-on-objects functor $\mathbb{L}_n(S) \to \mathscr{L}$ being the opposite of the Kleisli inclusion for the corresponding relative monad. To obtain a correspondence for higher-order algebraic theories, we must restrict this equivalence of categories to the functors preserving the appropriate exponentiable objects. To that end, we introduce the following definitions.

Definition 4.7.1. Let $I \colon \mathscr{C} \hookrightarrow \mathscr{D}$ be a coexponentiable subcategory and let $J \colon \mathscr{D} \to \mathscr{E}$ be a cocartesian functor. A J-relative monad (T, η) is called *$+$-linear* if the canonical strength $[\eta, T] \colon JI(-) + T(-) \to T(I- + (-))$ is invertible; a monad (T, η) on $\mathscr{E}$ is called *$+$-linear* if the canonical strength $[\eta, T] \colon JI(-) + T(-) \to T(JI- + (-))$ is invertible. We denote by $\mathbf{RMnd}_+(J) \hookrightarrow \mathbf{RMnd}(J)$ the full subcategory spanned by $+$-linear relative monads; and by $\mathbf{Mnd}_+(\mathscr{E}) \hookrightarrow \mathbf{Mnd}(\mathscr{E})$ the full subcategory spanned by $+$-linear monads.

Note that our notion of strength for a relative monad is more general than that of [Uus10], and might naturally be called an *I-relative strength* for a J-relative monad, where the notion ibid. is recovered by taking $I = 1_{\mathscr{D}}$; and similarly for monads. Our terminology is inspired by [BR19], where *linear monads* are defined to be those monads with invertible strengths.

The definition of $+$-linearity is motivated by the following result.

Lemma 4.7.2. *Let $I \colon \mathscr{C} \to \mathscr{D}$ be a coexponentiable subcategory and let $J \colon \mathscr{D} \to \mathscr{E}$ be a dense cocartesian functor that preserves coexponentiable objects in the image of I. The Kleisli embedding for a J-relative monad T preserves the coexponentiable objects in $\mathscr{D}$ if and only if T is $+$-linear.*

Proof. For X a coexponentiable object in a cocartesian category, denote by $(-)_X \dashv X + (-)$ the coexponential. We have the following chain of isomorphisms, natural in $X, Z \in \mathscr{D}$ and $Y \in \mathscr{C}$,

$$\begin{aligned}
\mathbf{Kl}(T)(X_{IY}, Z) &= \mathscr{E}(J(X_{IY}), TZ) \\
&\cong \mathscr{E}((JX)_{JIY}, TZ) \\
&\cong \mathscr{E}(JX, JIY + TZ) \\
&\cong \mathscr{E}(JX, T(IY + Z)) && (*) \\
&= \mathbf{Kl}(T)(JX, IY + Z)
\end{aligned}$$

where $(*)$ follows if and only if the canonical $+$-strength for T is invertible. □

To apply this result in the context of higher-order algebraic theories, we require the following.

Lemma 4.7.3. $J_n \colon \mathbb{L}_{n+1}(S)^{\text{op}} \hookrightarrow \mathbf{Law}_n(S)$ *preserves coexponentials.*

Proof. We have the following, natural in $X, Y \in \mathbb{L}_n(S)$ and $L \in \mathbf{Law}_n(S)$.

$$\begin{aligned}
\mathbf{Law}_n(S)(J_n(Y^X), L) &\cong \lceil \mathscr{L} \rceil(1, \lceil L \rceil(Y^X)) \\
&\cong \lceil \mathscr{L} \rceil(1, \lceil L \rceil Y^{\lceil L \rceil X}) \\
&\cong \lceil \mathscr{L} \rceil(\lceil L \rceil X, \lceil L \rceil Y) \\
&\cong \lceil \mathscr{L} \rceil(1, (\lceil J_n X \rceil + \lceil L \rceil) Y) \\
&\cong \lceil \mathscr{L} \rceil(1, \lceil J_n X + L \rceil(Y)) \\
&\cong \mathbf{Law}_n(S)(J_n Y, J_n X + L)
\end{aligned}$$

□

Consequently, we have the following correspondence between higher-order algebraic theories and $+$-linear relative monads, where the coexponentiable subcategory is given by ${I_n}^{\mathrm{op}} \colon \mathbb{L}_n(S)^{\mathrm{op}} \hookrightarrow \mathbb{L}_{n+1}(S)^{\mathrm{op}}$ for $n > 0$, and by $0 \colon 1^{\mathrm{op}} \hookrightarrow \mathbb{L}_1(S)^{\mathrm{op}}$ for $n = 0$.

Theorem 4.7.4. *Let $J = よ^{sf}_{\mathbb{L}_{n+1}(S)^{\mathrm{op}}} \colon \mathbb{L}_{n+1}(S)^{\mathrm{op}} \hookrightarrow \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})$. There is an isomorphism of categories*

$$\mathbf{Law}_{n+1}(S) \cong \mathbf{RMnd}_{+}(J)$$

Proof. By Lemma 4.7.2, a Kleisli embedding for a relative monad T (i.e. a J-theory) preserves coexponentiable objects if and only if T is $+$-linear, since J is dense and preserves coexponentials (Lemma 4.7.3). $(n+1)^{\mathrm{th}}$-order algebraic theories are precisely the duals of J-theories satisfying this preservation condition, and, by composing the equivalence of Theorem 4.6.5, the isomorphism of Theorem 3.1.14 thus restricts as stated. □

The instantiation of Theorem 4.7.4 for $n = 0$ and $S = 1$ establishes that monosorted first-order algebraic theories are in bijection with ($\mathbf{FinSet} \hookrightarrow \mathbf{Set}$)-relative monads. This observation is typically regarded as folklore, though it follows from the developments in [FPT99] and [ACU15]; a direct proof appears in [Voe16]. Using the results of Chapter 3, this immediately implies a monad correspondence, since J exhibits a cocompletion under a class of shapes (namely, the sifted categories).

Definition 4.7.5. Denote by $\mathbf{Mnd}_{sf}(\mathscr{C})$ the category of sifted-cocontinuous (also called *strongly finitary* [LR11]) monads on a sifted-cocomplete category $\mathscr{C}$, i.e. those monads whose underlying endofunctors preserve sifted colimits.

Theorem 4.7.6. *There is an equivalence of categories*

$$\mathbf{Law}_{n+1}(S) \simeq \mathbf{Mnd}_{+,sf}(\mathbf{Law}_n(S))$$

Proof. Follows directly from Theorem 4.7.4 and Theorem 3.2.6, taking $\Phi = \mathbf{Sind}$. □

Remark 4.7.7. When $n = 1$, the $+$-linearity condition trivialises, and we recover the classical monad–theory correspondence: $\mathbf{Law}_1(S) \simeq \mathbf{Mnd}_{sf}(\mathbf{Set}^S)$.

This monad–theory correspondence makes precise the sense in which $(n+1)^{\mathrm{th}}$-order binding structure is algebraic over n^{th}-order binding structure: let us remark that this phenomenon is not entirely surprising, since the arities of $(n+1)^{\mathrm{th}}$-order algebraic theories are taken from the objects of n^{th}-order algebraic theories, which suggests a relationship along these lines.

4.7.1 Simple slices

The advantage of the abstract approach of the previous section to the monad–theory correspondence is its simplicity. However, it is less helpful in gaining an intuition for the relationship between higher-order algebraic theories and monads, since we obtain the monads on $\mathbf{Law}_n(S)$ from monads on $\mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})$ by transferring them across the equivalence of Theorem 4.6.5. We shall now give a more concrete understanding of the correspondence in terms of the *simple slice* construction.

Definition 4.7.8 ([Jac99, Definition 1.3.1])**.** Let $\mathscr{C}$ be a cartesian category and let $X \in \mathscr{C}$ be an object. The *simple slice over* X, denoted $\mathscr{C} /\!\!/ X$, is the Kleisli category of the comonad $(-) \times X$.

Simple slices have proven useful in categorical treatments of simple type theory, where they are occasional called *polynomial categories* [LS88, §I.5]. For $\mathscr{C}$ a cartesian category viewed as the classifying category for a simple type theory, the simple slice $\mathscr{C} /\!\!/ X$ corresponds syntactically to extending each context of $\mathscr{C}$ by a variable of type X; equivalently, to adding a new constant of type X to the type theory. The following proposition makes this intuition precise.

Proposition 4.7.9 ([LS88, Proposition I.7.1])**.** *Let $\mathscr{C}$ be a cartesian category and let $X \in \mathscr{C}$ be an object. The category $\mathscr{C} /\!\!/ X$ is the free extension of $\mathscr{C}$ by a single morphism $1 \to X$.*

Given a higher-order algebraic theory $L\colon \mathbb{L}_n(S) \to \mathscr{L}$, every simple slice over $\mathscr{L}$ inherits the theory structure. In this way, the simple slices over a higher-order algebraic theory may be assembled into a functor.

Lemma 4.7.10. *Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory. The simple slices of $\mathscr{L}$ form a functor $L/\!\!/(-)\colon \mathscr{L}^{\mathrm{op}} \to L/\mathbf{Law}_{n+1}(S)$.*

Proof. For each object $X \in \mathscr{L}^{\mathrm{op}}$, the simple slice category $\mathscr{L}/\!\!/X$ inherits an identity-on-objects functor $L/\!\!/X\colon \mathbb{L}_{n+1}(S) \to \mathscr{L} \to \mathscr{L}/\!\!/X$ by composing L with the inclusion $\mathscr{L} \to \mathscr{L}/\!\!/X$ given by the universal property of Proposition 4.7.9. This functor strictly preserves finite products and exponentiable objects [LS88, Proposition I.7.1], and is hence an $(n+1)^{\mathrm{th}}$-order algebraic theory, which is equipped with a canonical inclusion $\mathscr{L} \to \mathscr{L}/\!\!/X$. This process is easily seen to be contravariantly functorial in X. □

The relationship between simple slices and higher-order algebraic theories as concerns the monad–theory correspondence is given as follows. Recall that the *nerve* of a functor $F\colon \mathscr{C} \to \mathscr{D}$ with small domain is given by the restricted Yoneda embedding $N_F\colon \mathscr{D} \to \widehat{\mathscr{C}} = \mathscr{D}(F-,-)$. From the development in Chapter 3, the relative monad corresponding to an $(n+1)^{\mathrm{th}}$-order algebraic theory $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is induced by the following relative adjunction.

$$\begin{tikzcd}
 & \mathscr{L}^{\mathrm{op}} \arrow[dr, "N_{L^{\mathrm{op}}}"] \arrow[d, phantom, "\dashv"] & \\
\mathbb{L}_{n+1}(S)^{\mathrm{op}} \arrow[ur, "L^{\mathrm{op}}"] \arrow[rr, hook, "よ^{sf}_{\mathbb{L}_{n+1}(S)^{\mathrm{op}}}"'] & & \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})
\end{tikzcd}$$

We may use the simple slice construction, along with the extension–restriction adjunction $\lceil - \rceil \dashv \lfloor - \rfloor$ of Corollary 4.6.10, to give a concrete interpretation of this relative adjunction. Denote by $U\colon L/\mathbf{Law}_{n+1}(S) \to \mathbf{Law}_{n+1}(S)$ the forgetful functor from the coslice category.

Lemma 4.7.11. *Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory. The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd}
\mathscr{L}^{\mathrm{op}} \arrow[d, "{I_{\mathscr{L}}}^{\mathrm{op}}"'] \arrow[ddr, bend left, "U(L/\!\!/(-))"] & \\
\lceil \mathscr{L} \rceil^{\mathrm{op}} \arrow[d, "N_{\lceil L \rceil^{\mathrm{op}}}"'] & \\
\mathbf{Sind}(\mathbb{L}_{n+2}(S)^{\mathrm{op}}) & \mathbf{Law}_{n+1}(S) \arrow[l, "\simeq"]
\end{tikzcd}$$

Proof. For all $X \in \mathscr{L}$, we have the following:

$$\begin{aligned}
(\simeq) \circ U(L/\!\!/X) &\cong \lceil \mathscr{L}/\!\!/X \rceil(1, \lceil L \rceil -) && \text{(Lemma 4.6.13)} \\
&= \lceil \mathscr{L} \rceil(1 \times X, \lceil L \rceil -) && \text{(Definition 4.7.8)} \\
&\cong \lceil \mathscr{L} \rceil(X, \lceil L \rceil -) \\
&= \lceil \mathscr{L} \rceil^{\mathrm{op}}(\lceil L \rceil^{\mathrm{op}} -, X) \\
&= (N_{\lceil L \rceil^{\mathrm{op}}} \circ {I_{\mathscr{L}}}^{\mathrm{op}})(X)
\end{aligned}$$

which is functorial in X. □

We shall also need the following, which relates the simple slice construction to coproducts of higher-order algebraic theories.

Lemma 4.7.12. *Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory. The following diagram commutes up to natural isomorphism.*

$$
\begin{tikzcd}
\mathbb{L}_{n+1}(S)^{\mathrm{op}} \arrow[r, "J_n"] \arrow[ddr, phantom, ""] \arrow[dr, "U(1_{\mathbb{L}_{n+1}(S)} /\!\!/ (-))"] \arrow[dd, "L^{\mathrm{op}}"'] & \mathbf{Law}_n(S) \arrow[d, "\lceil - \rceil"] \\
& \mathbf{Law}_{n+1}(S) \arrow[d, "L + (-)"] \\
\mathscr{L}^{\mathrm{op}} \arrow[r, "L /\!\!/ (-)"'] & L/\mathbf{Law}_{n+1}(S)
\end{tikzcd}
$$

Proof. Commutativity of the triangle follows from Proposition 4.7.9 and the explicit description of J_n given in Section 4.6. Commutativity of the square follows again from Proposition 4.7.9 and the definition of colimits of categories, as both functors freely adjoint a constant to L. □

Using this relationship, we can deduce that the simple slice construction is fully faithful, which is the final ingredient necessary to give the promised concrete understanding.

Lemma 4.7.13. *Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory. The simple slice functor $L /\!\!/ (-)\colon \mathscr{L}^{\mathrm{op}} \to L/\mathbf{Law}_{n+1}(S)$ is fully faithful.*

Proof. For all $X, Y \in \mathbb{L}_{n+1}(S)$, we have:

$$
\begin{aligned}
L/\mathbf{Law}_{n+1}(S)(L /\!\!/ X, L /\!\!/ Y) &\cong L/\mathbf{Law}_{n+1}(S)(L + \lceil J_n X \rceil, L /\!\!/ Y) && \text{(Lemma 4.7.12)} \\
&\cong \mathbf{Law}_{n+1}(S)(\lceil J_n X \rceil, U(L /\!\!/ Y)) && (L + (-) \dashv U) \\
&\cong \mathbf{Law}_{n+1}(S)(J_{n+1} X, U(L /\!\!/ Y)) && \\
&\cong \mathbf{Sind}(\mathbb{L}_{n+2}(S)^{\mathrm{op}})(よ^{sf} X, (N_{\lceil L \rceil^{\mathrm{op}}} \circ I_{\mathscr{L}}{}^{\mathrm{op}}) Y) && \text{(Lemma 4.7.11)} \\
&\cong (N_{\lceil L \rceil^{\mathrm{op}}} \circ I_{\mathscr{L}}{}^{\mathrm{op}}) Y(X) && \text{(Yoneda lemma)} \\
&\cong N_L{}^{\mathrm{op}} Y(X) && (I_{\mathscr{L}} \text{ is fully faithful}) \\
&= \mathscr{L}^{\mathrm{op}}(X, Y)
\end{aligned}
$$

□

We may now give a more conceptual characterisation of the monad induced by a higher-order algebraic theory. While it is known classically that free term algebras for first-order algebraic theories (equivalently, free algebras for finitary monads on **Set**) may be alternatively described by first-order algebraic theories with adjoined constants (cf. [Law63, Proposition V.1.1]), by making use of the coreflections between higher-order algebraic theories, we may present this result in a particularly elegant fashion.

Theorem 4.7.14. *Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory. The underlying functor of the relative monad T_L corresponding to L is given by $\lfloor U(L + \lceil J_n(-) \rceil) \rfloor$. Consequently, the monad $\hat{T}_L$ on $\mathbf{Law}_n(S)$ corresponding to L is induced by the following adjunction:*

$$
\begin{tikzcd}
\mathbf{Law}_n(S) \arrow[r, hook, shift left, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n+1}(S) \arrow[l, shift left, "\lfloor - \rfloor"] \arrow[r, shift left, "L + (-)"] \arrow[r, phantom, "\perp"] & L/\mathbf{Law}_{n+1}(S) \arrow[l, shift left, "U"]
\end{tikzcd}
\tag{4.13}
$$

Suppressing the forgetful functor U, the underlying endofunctor of $\hat{T}_L$ is hence given by

$$
\lfloor L + \lceil - \rceil \rfloor \tag{4.14}
$$

Proof. Observe that the monad induced by the adjunction (4.13) is sifted-cocontinuous, as left adjoints preserve colimits; U preserves sifted colimits because they are in particular connected colimits, which are preserved by the forgetful functor from a coslice category; and $\lfloor - \rfloor$ preserves sifted colimits because it is algebraic. Therefore, to show that the adjunction induces $\hat{T}_L$, it suffices to show that precomposing $J_n \colon \mathbb{L}_{n+1}(S)^{\mathrm{op}} \hookrightarrow \mathbf{Law}_n(S)$ induces the relative monad T_L. To do so, consider the functor $L/\!\!/(-) \colon \mathscr{L}^{\mathrm{op}} \to L/\mathbf{Law}_{n+1}(S)$,

$$\begin{tikzcd} \mathscr{L}^{\mathrm{op}} \arrow[r, dashed, "L/\!\!/(-)"] & L/\mathbf{Law}_{n+1}(S) \\ \mathbb{L}_{n+1}(S)^{\mathrm{op}} \arrow[u, "L^{\mathrm{op}}"] \arrow[ur, "L+\lceil J_n(-)\rceil"'] & \end{tikzcd}$$

which makes the diagram above commute up to natural isomorphism by Lemma 4.7.12. Since $L/\!\!/(-)$ is fully faithful by Lemma 4.7.13, L^{op} is exhibited as the Kleisli inclusion for the relative monad induced by precomposing (4.13) by J_n, from which the conclusion follows, since T_L is induced by the Kleisli relative adjunction $L^{\mathrm{op}} {}_{よ^{sf}}\!\dashv N_{L^{\mathrm{op}}}$. □

In the first-order setting, we may understand this characterisation conceptually as meaning that the monad induced by an algebraic theory L takes a set of constants (which are the terms of a 0^{th}-order algebraic theory, as will be expounded in Section 4.8) and closes them under the operations of L. More generally, the monad induced by an $(n+1)^{\text{th}}$-order algebraic theory L takes an n^{th}-order algebraic theory L' and adjoins the operators of L' to those of L to produce a new $(n+1)^{\text{th}}$-order algebraic theory $L + \lceil L' \rceil$, and then extracts the closed terms (equivalently the open terms of the underlying n^{th}-order algebraic theory) to produce a new n^{th}-order algebraic theory $\lfloor L + \lceil L' \rceil \rfloor$.

In this light, higher-order algebraic theories may be viewed as (particularly well-behaved) monad transformers [LHJ95]. For instance, for a second-order algebraic theory L, the underlying endofunctor of the induced monad on $\mathbf{Law}_1(S)$ corresponds to a functor that, informally, takes a monad and freely adds the structure of L by taking its coproduct with L:

$$\mathbf{Mnd}_{sf}(\mathbf{Set}^S) \simeq \mathbf{Law}_1(S) \xrightarrow{\lfloor L + \lceil - \rceil \rfloor} \mathbf{Law}_1(S) \simeq \mathbf{Mnd}_{sf}(\mathbf{Set}^S)$$

The form of the monad in Theorem 4.7.14 facilitates its expression as a coend, which reduces to the well-known coend characterisation in the first-order setting (cf. [HP07, Proposition 4.1]).

Corollary 4.7.15. *Let $L \colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory. The underlying endofunctor of the monad T_L on $\mathbf{Law}_n(S)$ induced by L is given by the coend*

$$T_L(L')(X,Y) \cong \int^{\Gamma \in \mathbb{L}_{n+1}(S)} \lceil \mathscr{L}' \rceil(1,\Gamma) \times \mathscr{L}(\Gamma, Y^X) \tag{4.15}$$

Proof. First, observe that

$$\begin{aligned} T_L\Gamma(-,-) &\cong \lfloor L/\!\!/\Gamma \rfloor(-,-) && \text{(Theorem 4.7.14)} \\ &\cong \lfloor \mathscr{L} \rfloor(\Gamma \times (-), -) && \text{(Lemma 4.7.10)} \\ &\cong \mathscr{L}(\Gamma \times (-), -) && (*) \\ &\cong \mathscr{L}(\Gamma, (-)^{(-)}) \end{aligned}$$

where $(*)$ follows since $\lfloor \mathscr{L} \rfloor$ is given by the full image of $\mathscr{L}$ on $\mathbb{L}_n(S)$.

The monad associated to the relative monad T_L is given by extending the underlying functor T_L along $J = よ^{sf}_{\mathbb{L}_{n+1}(S)^{\mathrm{op}}}$. Since $\mathbf{Law}_n(S)$ is cocomplete, we may evaluate $J \rhd T_L$ as a coend (see, e.g. [Kel82, (4.25)]). Using the Yoneda lemma together with Lemma 4.6.13, we have:

$$\begin{aligned} (よ^{sf}_{\mathbb{L}_{n+1}(S)^{\mathrm{op}}} \rhd T_L)(L') &\cong \int^{\Gamma \in \mathbb{L}_{n+1}(S)} \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})(よ^{sf}\Gamma, \lceil \mathscr{L}' \rceil(1, \lceil L' \rceil -)) \times T_L\Gamma(-,-) \\ &\cong \int^{\Gamma \in \mathbb{L}_{n+1}(S)} \lceil \mathscr{L}' \rceil(1, \lceil L' \rceil \Gamma) \times \mathscr{L}(\Gamma, (-)^{(-)}) \end{aligned}$$

□

Monads for binding signatures Observe that to each S-sorted second-order algebraic theory we may associate a monad on $\mathbf{Set}^S$ by $\lfloor - \rfloor : \mathbf{Law}_2(S) \to \mathbf{Law}_1(S) \simeq \mathbf{Mnd}_{sf}(\mathbf{Set}^S)$. Conceptually, the induced monad describes the closed term structure of the second-order theory, which is equivalent to the open term structure of the underlying first-order theory; the second-order structure is therefore forgotten. Several existing approaches to generating monads from signatures of variable-binding operators may be viewed as variations on this theme. For instance, the motivation of [Ahr+19] is to provide a monadic approach to the work of Fiore and Hur [FH10] (in the monosorted setting). However, there the authors only consider monads on **Set** and thus lose the higher-order structure. Our work may therefore be seen to provide a more principled and complete realisation of this intention.

Similarly, Matthes and Uustalu [MU04] consider binding signatures as pointed endofunctors on endofunctor categories $[\mathscr{C}, \mathscr{C}]$, which induce monads on $\mathscr{C}$ (for instance, taking $\mathscr{C} = \mathbf{Set}$ to recover monosorted second-order signatures).

4.8 Zeroth-order algebraic theories

We have established that, for $n > 1$, the category of S-sorted $(n+1)^{\text{th}}$-order algebraic theories is given by taking $+$-linear, sifted-cocontinuous monads on the category of S-sorted n^{th}-order algebraic theories. It is well known that the category of S-sorted first-order algebraic theories is given by taking sifted-cocontinuous monads on the category of S-indexed sets (Corollary 3.2.7). As the $+$-linearity condition is by definition trivial for $n = 1$, it is natural to ask to what extent the category of S-indexed sets may be viewed as the category of S-sorted 0^{th}*-order* algebraic theories. This would eliminate the seemingly arbitrary base case for an inductive definition of $\mathbf{Law}_n(S)$ in terms of $+$-linear sifted-cocontinuous monads. The purpose of this section is to demonstrate that it is entirely natural to take S-indexed sets as the definition of S-sorted 0^{th}-order algebraic theories, which, intuitively, represent *theories of constants*.

Definition 4.8.1. $\mathbb{L}_0(S)$ is the *free nullary completion* of S, giving by freely adjoining a terminal object to the discrete category S.

Definition 4.8.2. An S*-sorted* 0^{th}*-order algebraic theory* is an identity-on-objects functor $L \colon \mathbb{L}_0(S) \to \mathscr{L}$ strictly preserving the terminal object, such that every morphism in $\mathscr{L}$ is *constant* (that is, factors through the terminal object). Given 0^{th}-order algebraic theories $L \colon \mathbb{L}_0(S) \to \mathscr{L}$ and $L' \colon \mathbb{L}_0(S) \to \mathscr{L}'$, a *map* from L to L' is a functor $F \colon \mathscr{L} \to \mathscr{L}'$ such that $FL = L'$. 0^{th}-order algebraic theories and their maps form a category $\mathbf{Law}_0(S)$.

This requirement that each morphism in a 0^{th}-order algebraic theory be constant is perhaps surprising. This is necessary because the operations of a 0^{th}-order algebraic theory must have trivial domain. We may view this definition from the operadic perspective, where this restriction is more naturally expressed.

Remark 4.8.3. 0^{th}-order algebraic theories are precisely T-multicategories (cf. [Lei04]), for T the terminal monad on **Set**. The object-set of the multicategory is given by the set of sorts S.

It is trivial to define presentations for 0^{th}-order algebraic theories; we omit the unilluminating details.
Definition 4.8.2 is justified by the following proposition.

Proposition 4.8.4. *There is an isomorphism of categories:*

$$\mathbf{Law}_0(S) \cong \mathbf{Set}^S$$

Proof. Consider the co-Yoneda embedding of the terminal object in any 0^{th}-order algebraic theory. □

Note that this definition is consistent with the results throughout the chapter, in which we set $n = 0$: for instance, there is an equivalence $\mathbf{Law}_0(S) \simeq \mathbf{Sind}(\mathbb{L}_1(S)^{\text{op}}) \simeq \mathbf{Cart}(\mathbb{L}_1(S), \mathbf{Set})$. This result allows us to exhibit an inductive construction of the category of n^{th}-order algebraic theories from the category of 0^{th}-order algebraic theories.

Corollary 4.8.5. *There is an equivalence of categories*

$$\mathbf{Law}_n(S) \simeq \mathbf{Mnd}_{+,sf}{}^n(\mathbf{Law}_0(S))$$

Conceptually, we may view the process of taking sifted-cocontinuous $+$-linear monads as parameterising the objects of a category, viewed as theories, by finitary contexts. Iterating this construction increases the order of parameterisation, thus moving from first-order algebraic structure, to second-order, and so on. One might suppose that a similar process is possible for other notions of theory: for instance, that higher-order linear structure might be given by iteratively taking analytic monads [Joy86; Web04]. We leave the investigation of such matters to future work.

Remark 4.8.6. The reader familiar with 2-category theory should note that, since in this chapter $\mathbf{Mnd}(-)$ denotes the category of monads *on a category*, rather than the 2-category of monads *in a 2-category* as in [Str72a], the objects of $\mathbf{Mnd}(\mathbf{Mnd}(\mathscr{C}))$, for some category $\mathscr{C}$, are not distributive laws as in [Str72a, §6] or [Che11], but rather well-behaved monad transformers.

4.9 Models and algebras

Up to this point, we have entirely omitted the consideration of models of higher-order algebraic theories. There is good reason for doing so, as the situation is more subtle in the higher-order setting than the first-order setting. In this final section, we discuss models of higher-order algebraic theories and their relationship to algebras for the corresponding monads.

We may model the structure of a first-order algebraic theory $L\colon \mathbb{L}_1(S) \to \mathscr{L}$ in an arbitrary cartesian category $\mathscr{C}$ by considering cartesian functors $\mathscr{L} \to \mathscr{C}$; natural transformations, which automatically commute with the cartesian structure, then describe homomorphisms between models. The algebras for the monad corresponding to the first-order algebraic theory L are equivalent to models of L in **Set**: in this sense, models subsume algebras.

Moving to higher-order algebraic theories, the situation is more subtle. While the obvious definition of model (namely a cartesian functor preserving the appropriate exponentiable objects) is perfectly satisfactory, the appropriate definition of homomorphism in the presence of higher-order structure is unclear. Furthermore, while the algebras for a monad T_L are, in particular, models for L when $n = 1$, the same is not true for $n > 1$. This motivates us to distinguish between *models*, which are structure-preserving functors, and *term algebras*, which are structures corresponding to the algebras of the induced monad. We argue that this distinction gives conceptual justification for the seeming mismatch in the first-order setting between the ability to take models for a theory in any cartesian category, and the ability to consider algebras for a monad only in the category of sets.

Finally, we introduce a strict notion of model, given by coslices in $\mathbf{Law}_n(S)$, and show that there is a strong relationship between strict models and term algebras for a theory.

4.9.1 Models

Following Lawvere's programme of functorial semantics [Law63], we expect the correct notion of model to be given by a structure-preserving functor into a category with suitable structure.

Definition 4.9.1. Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory and let $\mathscr{C}$ be a cartesian category. A *model of L in $\mathscr{C}$* is a cartesian functor $\mathscr{L} \to \mathscr{C}$ that preserves n-tetrable objects.[7]

Example 4.9.2. Models of the unityped λ-calculus (Example 4.4.1) (with U-η) in a cartesian category $\mathscr{C}$ are equivalently (extensional) reflexive objects in $\mathscr{C}$, i.e. exponentiable objects U equipped with an isomorphism $U \cong U^U$ [Sco80]. Note that the only reflexive object in **Set** is the terminal object: preservation of exponentials is typically a strong requirement.

[7] We are implicitly assuming that $\mathscr{C}$ has enough exponentiable objects.

Example 4.9.3. A model of the simply-typed λ-calculus with base types S (Example 4.4.2) in a cartesian-closed category $\mathscr{C}$ is equivalently an interpretation of the base types as objects of $\mathscr{C}$.

There are difficulties in defining homomorphisms for higher-order theories. For instance, consider the monosorted second-order algebraic theory $L_{\mathsf{abs}} \colon \mathbb{L}_2(\{X\}) \to \mathscr{L}_{\mathsf{abs}}$ generated by the second-order operator $\mathsf{abs} \colon X^X \to X$. Suppose we have models $F, G \colon \mathscr{L}_{\mathsf{abs}} \to \mathscr{C}$, for some cartesian-closed category $\mathscr{C}$. Generalising from the first-order setting, we might hope that a homomorphism $m \colon F \to G$ is given by a morphism $m_X \colon FX \to GX$ for the generating object $X \in \mathbb{L}_2(\{X\})$, along with a condition requiring $F(\mathsf{abs})$ to cohere with $G(\mathsf{abs})$ in a suitable sense. However, consider the following diagram.

$$\begin{tikzcd} FX^{FX} \arrow[r, dashed, "?"] \arrow[d, "F(\mathsf{abs})"'] & GX^{GX} \arrow[d, "G(\mathsf{abs})"] \\ FX \arrow[r, "m_X"'] & GX \end{tikzcd}$$

In general, there does not exist a morphism $FX^{FX} \to GX^{GX}$. Fundamentally, this is a problem with contravariance: a morphism $m_X \colon FX \to GX$ defines a morphism $(-)^{m_X} \colon (-)^{GX} \to (-)^{FX}$, which is in the wrong direction to express the coherence condition. There are various approaches to addressing this problem, but all have severe shortcomings.

- We could define a homomorphism to be a natural transformation $F \Rightarrow G$, as in the first-order setting. This is the approach of Meinke [Mei92]. A natural transformation defines morphisms $FX^{FX} \to GX^{GX}$, which allows the coherence condition to be expressed. However, the data for a homomorphism in this case is no longer determined by the action of the homomorphism on the sorts, and is presented by an infinitude of morphisms, one for each element of $\mathrm{Col}_2(S)$.
- We could define a homomorphism to be a natural isomorphism $F \overset{\cong}{\Longrightarrow} G$. This is the typical approach in categorical algebra and logic [BKP89; Cro93; Joh02]. In contrast to natural transformations, natural isomorphisms are determined by their action on the sorts. However, this is clearly a restriction of the notion of homomorphism: in the case $n = 1$, we do not recover homomorphisms of algebraic structures.
- We could require a homomorphism $m \colon F \to G$ to define morphisms $m_X \colon FX \to GX$ and $m_X^\circ \colon GX \to FX$. There is then a canonical morphism $(m_X)^{m_X^\circ} \colon FX^{FX} \to GX^{GX}$. This is the usual approach in the literature on recursive domain equations (cf. [SP82]). Requiring m_X to be inverse to m_X° recovers the approach via natural isomorphisms. However, it has the same shortcoming: restricting to $n = 1$ does not recover the usual notion of homomorphism.
- We could change the coherence condition for a homomorphism: for instance, we could instead require that the following diagram commute.

$$\begin{tikzcd} & FX^{GX} \arrow[dl, "FX^{(m_X)}"'] \arrow[dr, "(m_X)^{GX}"] & \\ FX^{FX} \arrow[d, "F(\mathsf{abs})"'] & & GX^{GX} \arrow[d, "G(\mathsf{abs})"] \\ FX \arrow[rr, "m_X"'] & & GX \end{tikzcd}$$

  However, unlike in the first-order setting, imposing such a condition for operators is not enough to impose the condition also for terms. Thus, such a coherence condition requires an infinitude of equations, one for each morphism of $\mathscr{L}_{\mathsf{abs}}$, to be satisfied. Even if we decided this were acceptable, there are further problems. Readers may notice this condition is reminiscent of the definition of dinatural transformations [DS70], which are known not to compose in general. Although our direction of composition is different here (horizontal, rather than vertical), the same problem arises: the composite of two homomorphisms may no longer be a homomorphism unless very strong conditions are imposed on the theory or on the models.

We have been led to conclude that there is likely no fully satisfactory notion of homomorphism of higher-order structures, by which we mean a definition such that at least the following conditions hold: the action on the sorts determines the action on all objects; there are a finite number of coherence conditions for each operator; and when $n = 1$, we recover the traditional notion of homomorphism. In this light, there is no appropriate category of models for a higher-order algebraic theory in a cartesian-closed category. Consequently, we restrict our consideration henceforth to a stricter notion of model.

4.9.2 Strict models

Consider an S-sorted n^{th}-order algebraic theory $L \colon \mathbb{L}_n(S) \to \mathscr{L}$. From a syntactic perspective, we may model L in any other S-sorted n^{th}-order algebraic theory $L' \colon \mathbb{L}_n(S) \to \mathscr{L}'$ by giving an interpretation of each of the operations of L by terms in L', satisfying the equations of L: this is precisely a coslice of L in $\mathbf{Law}_n(S)$ and is, in particular, a model of L in $\mathscr{L}'$. Homomorphisms are trivial in this setting, because everything must commute strictly. This motivates the following definition.

Definition 4.9.4. Let $L \colon \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. The category of *strict models for* L is given by the coslice category $L/\mathbf{Law}_n(S)$.

The category of strict models is well-behaved. To prove this, we prove a more general result. The analogue for locally presentable categories appears as [AR94, Proposition 1.57] (cf. [Osm21, Theorem 2.12]), but there does not appear to be a prior reference for the result in the context of locally strongly finitely presentable categories.

Theorem 4.9.5. *Let $\mathscr{C}$ be a locally strongly finitely presentable category and let $X \in \mathscr{C}$. The coslice category $X/\mathscr{C}$ is also locally strongly finitely presentable.*

Proof. For the purpose of this proof, a *term algebra* for a first-order algebraic theory is a model in the traditional sense, i.e. a cartesian functor into **Set**; a more detailed discussion will take place in Section 4.9.3. By the duality theorem for varieties [ALR03, Theorem 4.1], $\mathscr{C}$ is equivalent to L-**TmAlg** for some first-order algebraic theory $L \colon \mathbb{L}_1(S) \to \mathscr{L}$ such that $\mathscr{L} \simeq {\mathscr{C}_{\text{sf}}}^{\text{op}}$, so that X is equivalently given by some term algebra $A \in L$-**TmAlg**. Every term algebra is the quotient of some free algebra: that is, A specifies a set $A(\mathsf{B})$ of B-sorted constants for each $\mathsf{B} \in S$, with the action on other objects being fixed. The action of A on morphisms identifies terms in $L + \sum_{\mathsf{B} \in S} \lceil A(\mathsf{B}) \rceil$, which is the free term algebra for L on $\sum_{\mathsf{B} \in S} A(\mathsf{B})$. Since $\mathbf{Law}_1(S)$ has coequalisers, this defines a new S-sorted first-order algebraic theory, which we call L_A. The action of coslices under A specifies only how the constants of A are mapped into constants of other algebras, with this being preserved by the homomorphisms. This is precisely the data of a term algebra for L_A. Therefore $X/\mathscr{C} \simeq L_A$-**TmAlg** and, since L_A-**TmAlg** is locally strongly finitely presentable, so is $X/\mathscr{C}$. □

Corollary 4.9.6. *Let $L \colon \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. The category of strict models $L/\mathbf{Law}_n(S)$ is locally strongly finitely presentable, hence cocomplete and complete.*

Intuitively, the algebraic theory presenting $L/\mathbf{Law}_n(S)$ is the extension of $\mathbb{L}_{n+1}(S)$ by the operators and equations of L. From this result, we may deduce an analogue for strict models of the theorem that algebraic functors have left adjoints. Note that the classical theorem is given for models with a fixed codomain (namely, **Set**), whereas we allow the codomain to vary.

Proposition 4.9.7. *Let $F \colon L \to L'$ be a map of S-sorted n^{th}-order algebraic theories. The functor $L'/\mathbf{Law}_n(S) \to L/\mathbf{Law}_n(S)$ induced by precomposition by F has a left adjoint (in fact, it is algebraic).*

Proof. Since $\mathbf{Law}_n(S)$ has finite colimits by Corollary 4.6.9, the left adjoint is given by the pushout functor $F + (-) \colon L/\mathbf{Law}_n(S) \to L'/\mathbf{Law}_n(S)$; this follows directly by the universal property of pushouts. (For readers acquainted with the categorical semantics of dependent type theories, we note the dual statement is more familiar in that setting.)

For the stronger statement, observe that $F \colon L \to L'$ induces a map of $(n+1)^{\text{th}}$-order algebraic theories $(L/\mathbf{Law}_n(S))_{sf} \to (L'/\mathbf{Law}_n(S))_{sf}$. The induced algebraic functor is given by the functor induced by precomposition by F, from which the result follows. □

4.9.3 Term algebras

While strict models give a reasonable interpretation of model for a theory L, it is unclear how they relate to the algebras for the induced monad T_L. Ideally, we should hope for some relationship between the two. To shed light on this situation, we introduce the notion of term algebra.

For a first-order algebraic theory $L\colon \mathbb{L}_1(S) \to \mathscr{L}$, the terms in the empty context form a model $\mathscr{L}(1,-)\colon \mathscr{L} \to \mathbf{Set}$. For higher-order theories, this is no longer the case, since $\mathscr{L}(1,-)$ does not preserve exponentials. However, the structure formed by the closed terms of a theory is nevertheless important, and so we define a separate notion, that of *term algebra*, to capture it: terms in the empty context of a theory form the prototypical example.

Term algebras intuitively describe the substitution structure of a theory. Before giving the definition, we shall consider the unityped λ-calculus (Example 4.4.1) as an exemplar. The closed terms of the theory $L_\lambda\colon \mathbb{L}_2(\{\mathsf{U}\}) \to \mathscr{L}_\lambda$ are given by the hom-sets $\mathscr{L}_\lambda(1, \mathsf{U}^{\mathsf{U}^n}) \cong \mathrm{Lam}_n$, where Lam_n is the set of open unityped λ-calculus terms with at most $n \in \mathbb{N}$ free variables, up to $\beta\eta$-equivalence. The sets Lam_n are equipped with canonical substitution structure, and so we may assemble them into a category $\mathscr{L}_{\mathrm{Lam}}$ with objects $\mathsf{U}^n \in \mathbb{L}_1(\{\mathsf{U}\})$ and hom-sets $\mathscr{L}_{\mathrm{Lam}}(\mathsf{U}^n, \mathsf{U}^m) = {\mathrm{Lam}_n}^m$, where identities and composition are given by the variables and substitution respectively. This construction forms a first-order algebraic theory $L_{\mathrm{Lam}}: \mathbb{L}_1(\{\mathsf{U}\}) \to \mathscr{L}_{\mathrm{Lam}}$: in fact, it is precisely the first-order algebraic theory $\lfloor L_\lambda \rfloor$. Furthermore, the sets are equipped with interpretations of the λ-abstraction and application operators presented by L_λ, of the following form.

$$[\![\mathrm{abs}]\!]\colon \mathrm{Lam}_{n+1} \to \mathrm{Lam}_n \qquad [\![\mathrm{app}]\!]\colon \mathrm{Lam}_n \times \mathrm{Lam}_n \to \mathrm{Lam}_n \qquad (n \in \mathbb{N})$$

The first-order algebraic theory L_{Lam} induces a functor

$$\mathbb{L}_2(S) \xrightarrow{\lceil L_{\mathrm{Lam}} \rceil} \lceil \mathscr{L}_{\mathrm{Lam}} \rceil \xrightarrow{\lceil \mathscr{L}_{\mathrm{Lam}} \rceil(1,-)} \mathbf{Set}$$

The functions $[\![\mathrm{abs}]\!]$ and $[\![\mathrm{app}]\!]$ provide exactly the structure required to extend this functor to a functor $\mathscr{L}_\lambda \to \mathbf{Set}$. In general, the closed terms for an $(n+1)^{\text{th}}$-order algebraic theory $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ form an n^{th}-order algebraic theory $L' = \lfloor L \rfloor$, with $\mathscr{L}'(X, Y) = \mathscr{L}(1, Y^X)$, and the induced functor $\lceil \mathscr{L}' \rceil(1, \lceil L' \rceil(-))\colon \mathbb{L}_{n+1}(S) \to \mathbf{Set}$ extends to a functor $\mathscr{L} \to \mathbf{Set}$. The extension interprets the operators of L as functions on closed terms. Term algebras axiomatise this situation, which describes the substitution structure of the closed terms of a theory.

We follow Linton [Lin69a] in defining the category of term algebras as a pullback.

Definition 4.9.8. Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory. The category L-**TmAlg** of *term algebras for* L is defined (up to isomorphism) by the following pullback in **CAT**:

$$\begin{tikzcd}
L\text{-}\mathbf{TmAlg} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & {[\mathscr{L}, \mathbf{Set}]} \arrow[d, "{[L, \mathbf{Set}]}"] \\
\mathbf{Law}_n(S) \arrow[r, "\simeq"'] & \mathbf{Cart}(\mathbb{L}_{n+1}(S), \mathbf{Set}) \arrow[r, hook] & {[\mathbb{L}_{n+1}(S), \mathbf{Set}]}
\end{tikzcd}$$

Example 4.9.9. A term algebra for the second-order algebraic theory L_λ of the unityped λ-calculus (Example 4.4.1) consists of a first-order algebraic theory $L : \mathbb{L}_1(S) \to \mathscr{L}$ and two families of functions, natural in $X \in \mathscr{L}$,

$$[\![\mathrm{abs}]\!]_X : \mathscr{L}(X \times \mathsf{U}, \mathsf{U}) \to \mathscr{L}(X, \mathsf{U}) \qquad [\![\mathrm{app}]\!]_X : \mathscr{L}(X, \mathsf{U}) \times \mathscr{L}(X, \mathsf{U}) \to \mathscr{L}(X, \mathsf{U})$$

satisfying two equations, corresponding to the β- and η-laws respectively:

$$[\![\mathrm{app}]\!]_X([\![\mathrm{abs}]\!]_X(f), g) = f \circ \langle 1_X, g \rangle \qquad [\![\mathrm{abs}]\!]_X([\![\mathrm{app}]\!]_{X \times \mathsf{U}}(f \circ \pi_X, \pi_\mathsf{U})) = f$$

If we omit the η-law, we recover the λ*-theories*, or *algebraic theories equipped with semi-closed structure*, of [Hyl17, Definition 3.1]. The first-order term algebras for the underlying first-order algebraic theory of the initial second-order term algebra for L_λ are precisely Λ*-algebras* [Hyl17, Definition 4.1].

We may relate term algebras of an $(n+1)^{\text{th}}$-order algebraic theory $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ to algebras for the induced monad $\hat{T_L}$ using the following characterisation theorem for the algebras of a relative monad.

Theorem 4.9.10 (Corollary 5.5.6 and Remark 5.5.7)**.** *Let $J\colon \mathscr{A} \to \mathscr{E}$ be a dense functor with small domain. Let $T\colon \mathscr{A} \to \mathscr{E}$ be a J-relative monad. The Eilenberg–Moore category for T is given by the following (pseudo)pullback in* **CAT**.

$$\begin{tikzcd} T\text{-}\mathbf{Alg} \arrow[r] \arrow[d, "U_T"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[{\mathscr{E}_T}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "{[{K_T}^{\mathrm{op}}, \mathbf{Set}]}"] \\ \mathscr{E} \arrow[r, "N_J"'] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

Specialising this theorem to our setting, we can show that the category of term algebras for L is equivalent to the category of algebras for both T_L and $\hat{T_L}$, and is furthermore given by the category of cartesian functors $\mathscr{L} \to \mathbf{Set}$, a situation familiar from classical categorical algebra [Law63].

Theorem 4.9.11. *Let $L\colon \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory. Denote by $T_L\colon \mathbb{L}_n(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ the induced relative monad, and by $\hat{T_L}\colon \mathbf{Law}_n(S) \to \mathbf{Law}_n(S)$ the induced monad. There are equivalences of categories*

$$L\text{-}\mathbf{TmAlg} \simeq T_L\text{-}\mathbf{Alg} \simeq \hat{T_L}\text{-}\mathbf{Alg} \simeq \mathbf{Cart}(\mathscr{L}, \mathbf{Set})$$

Proof. The following forms a pullback in **CAT**.

$$\begin{tikzcd} \mathbf{Cart}(\mathscr{L}, \mathbf{Set}) \arrow[r, hook] \arrow[d, "{\mathbf{Cart}(L, \mathbf{Set})}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathscr{L}, \mathbf{Set}]} \arrow[d, "{[L, \mathbf{Set}]}"] \\ \mathbf{Cart}(\mathbb{L}_{n+1}(S), \mathbf{Set}) \arrow[r, hook] & {[\mathbb{L}_{n+1}(S), \mathbf{Set}]} \end{tikzcd}$$

Under the identification of L^{op} as the Kleisli inclusion for T and $\mathbf{Cart}(\mathbb{L}_{n+1}(S), \mathbf{Set}) \simeq \mathbf{Sind}(\mathbb{L}_{n+1}(S)^{\mathrm{op}})$, this pullback exhibits that of Theorem 4.9.10 up to equivalence, exhibiting $T_L\text{-}\mathbf{Alg} \simeq \mathbf{Cart}(\mathscr{L}, \mathbf{Set})$. By Proposition 3.2.5, we furthermore have $T_L\text{-}\mathbf{Alg} \simeq \hat{T_L}\text{-}\mathbf{Alg}$. Finally, the above pullback is equivalent to that of Definition 4.9.8 by precomposing the equivalence $\mathbf{Law}_n(S) \simeq \mathbf{Cart}(\mathbb{L}_{n+1}(S), \mathbf{Set})$, from which the result follows. □

We may now observe that our prototypical example is captured.

Proposition 4.9.12. *Let $L\colon \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. Up to the equivalence of Theorem 4.9.11, the hom-functor $\mathscr{L}(1, -) : \mathscr{L} \to \mathbf{Set}$ is the initial term algebra.*

Proof. Let $A\colon \mathscr{L} \to \mathbf{Set}$ be a cartesian functor. We have

$$\begin{aligned} \mathbf{Cart}(\mathscr{L}, \mathbf{Set})(\mathscr{L}(1, -), A) &= [\mathscr{L}, \mathbf{Set}](\mathscr{L}(1, -), A) \\ &\cong A1 && \text{(Yoneda lemma)} \\ &\cong 1 && \text{(A is cartesian)} \end{aligned}$$

Hence there is a unique morphism from $\mathscr{L}(1, -)$ to any term algebra for L, exhibiting it as initial. □

The initial term algebra for an n^{th}-order algebraic theory $L\colon \mathbb{L}_n(S) \to \mathscr{L}$ is therefore given by the sets of closed terms in L. It should be emphasised that it is because terms form a set that **Set** is distinguished amongst the cartesian categories as the canonical category in which to take models: categorically, this is because **Set** is the base of enrichment.

We may give a syntactic intuition for term algebras viewed as cartesian functors. Let $A\colon \mathscr{L} \to \mathbf{Set}$ be a cartesian functor. Each object $\Gamma \in \mathscr{L}$ may be considered a context $x_1 : \mathsf{B}_1, \ldots, x_k : \mathsf{B}_k$ in $\Lambda_n(S)$, and in this light one may think of the set $A(\Gamma)$ as the set of k-tuples of terms that may be substituted for the variables x_1 through x_k. To provide a substitute for the entire context Γ is to provide a substitute for each variable x_i, and it is this that necessitates A be cartesian. Functoriality of A ensures that the substitutes are closed under the operators of L. Consequently, the substitution structure captured by Definition 4.9.8 is equivalently captured by cartesian functors into **Set**. In general, the substitutes in $A(\Gamma)$ are not terms in L; however, Theorem 4.9.13 will give us a way in which a term algebra can always be seen as being given by the sets of closed terms of *some* theory, which justifies our nomenclature.

4.9.4 Relation between strict models and term algebras

We may now relate the concepts of strict models and term algebras, which both interpret the structure of a theory in a certain sense. Our final result of this chapter establishes a strong connection between the two. In particular, in the ω-order setting, the two notions coincide.

Theorem 4.9.13. *Let $L \colon \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. There are algebraic adjunctions of categories*

$$\begin{tikzcd} L\text{-}\mathbf{TmAlg} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & L/\mathbf{Law}_n(S) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \lceil L \rceil\text{-}\mathbf{TmAlg} \arrow[l, shift left] \end{tikzcd}$$

factorising the coreflective algebraic adjunction induced by $\mathscr{L} \hookrightarrow \lceil \mathscr{L} \rceil$.

Furthermore, when $n = \omega$, these adjunctions exhibit adjoint equivalences

$$L\text{-}\mathbf{TmAlg} \simeq L/\mathbf{Law}_\omega(S) \simeq \lceil L \rceil\text{-}\mathbf{TmAlg}$$

Proof. $L/\mathbf{Law}_n(S)$ is locally strongly finitely presentable by Corollary 4.9.6. We can describe its category of strongly finitely presentable objects explicitly: it is given by adjoining to $\mathbb{L}_{n+1}(S)$ constants $1 \to Y^X$, for each $X, Y \in \mathbb{L}_n(S)$, specified by the set $\mathscr{L}(X, Y)$, and then quotienting by the equations induced by composition and identities in $\mathscr{L}$. Therefore, there is a fully faithful strict cartesian injective-on-objects functor $\mathscr{L} \to (L/\mathbf{Law}_n(S))_{\mathrm{sf}}$ strictly preserving n-tetrable objects. However, note that $(L/\mathbf{Law}_n(S))_{\mathrm{sf}}$ is not an object of $\mathbf{Law}_{n+1}(S)$, because the sorts S are only n-tetrable: the process of adjoining constants takes place in **Cart** and thus fails to preserve exponentiable structure. Therefore, there is a strict cartesian identity-on-objects functor $(L/\mathbf{Law}_n(S))_{\mathrm{sf}} \to \lceil L \rceil$, which is faithful but not full. Together, these functors provide a factorisation of the inclusion $\mathscr{L} \hookrightarrow \lceil \mathscr{L} \rceil$, and hence induce the required adjunctions.

When $n = \omega$, there is an equivalence $\mathscr{L} \simeq \lceil \mathscr{L} \rceil$ since every object is exponentiable in $\mathscr{L}$. Hence $(L/\mathbf{Law}_n(S))_{\mathrm{sf}} \simeq \lceil \mathscr{L} \rceil$, since the structural operations exhibiting exponentiability in $\lceil \mathscr{L} \rceil$ are present in $\mathscr{L}$, so that $(L/\mathbf{Law}_n(S))_{\mathrm{sf}}$ is an object of $\mathbf{Law}_\omega(S)$. □

In particular, Theorem 4.6.5 is recovered from Theorem 4.9.13 when $n = \omega$ by taking L to be the initial ω-order algebraic theory, i.e. the identity functor on $\mathbb{L}_\omega(S)$.

Chapter 5

The formal theory of relative monads

In this chapter, we develop the beginnings of the formal theory of relative monads, which is integral to the understanding of the formal monad–theory correspondence. By *formal theory*, we mean the study of structures not in the setting of categories, functors, and natural transformations, but in an arbitrary 2-category with enough structure to define the objects of interest. Monads are a prototypical example of a structure that may be studied formally, originally being defined as structured endofunctors of a categories [God58, §A.3], then later being generalised to structured 1-cells on an object of a bicategory [Bén67, Definition 5.4.1] and studied extensively in that setting [Str72b; Str72a]. Though there is a great deal to be said regarding the formal theory of relative monads – most of the theory of monads extending to the relative setting, in addition to the new phenomena arising in the context of relativity – we shall content ourselves here with developing enough material to prove a general monad–theory correspondence, thereby recovering our motivating examples in Chapter 7. We have consequently been forced to omit some important aspects of the story; we hope to provide a more comprehensive treatment in the future.

Before we begin, we should note that an approach to the formal theory of relative monads was recently proposed by Lobbia [Lob20]. Unfortunately, the theory developed there fails to capture an important family of examples, namely that of relative monads in $\mathcal{V}$-$\mathbf{CAT}$, for $\mathcal{V}$ a monoidal category[1]. The problem is analogous to that of defining pointwise extensions or full faithfulness in a 2-category. In [Str74b, §3], Street gave a definition of pointwise extensions involving comma objects, which in particular recovers pointwise extensions in $\mathbf{CAT}$, but which is too strong for various 2-categories of enriched categories (cf. the introduction ibid.). For similar reasons, representable full faithfulness of 1-cells in $\mathcal{V}$-$\mathbf{CAT}$ is typically weaker than $\mathcal{V}$-enriched full faithfulness (cf. [SW78, Proposition 9]). Concretely, the issue in both cases is that 2-cells between generalised elements do not correspond in general to hom-objects. Therefore, definitions involving global quantification over generalised elements (such as comma objects, representability, or the operators of [Lob20, Definition 1.2]) are inherently flawed. The solution, appreciated first by Street and Walters [SW78] in their seminal study of Yoneda structures, is to move to a context for formal category theory, such as Yoneda structures, proarrow equipments [Woo82; Woo85], or lax idempotent pseudomonads [Koc95; Zöb76; BF99]. These settings permit the internalisation of definitions involving hom-objects in such a way as to recover the appropriate definitions for enriched categories and other category-like structures. Therefore, to give an appropriate definition of relative monad in a 2-category, it will be necessary to work in such a setting. We should say that, while Lobbia's setting is not appropriate for our purposes, the ideas presented there are nevertheless valuable in the formulation of a formal theory of relative monads, and we will take inspiration from several of the definitions therein.

For any 2-category $\mathcal{K}$, we may define the 2-category $\mathbf{Mnd}(\mathcal{K})$ of monads in $\mathcal{K}$ [Str72a]. Monads are particularly special structures, in that they are entirely *diagrammatic* notions: their data can be specified by a 2-functor from a 2-category freely generated by the axioms of a monad. Relative monads, however, are not: their data includes an *extension operator*, which is a transformation between hom-objects. It is this data that

[1] We thank Dylan McDermott for pointing this out to us.

necessitates the use of a context for formal category theory. In this thesis, we choose proarrow equipments as our means by which to carry out formal category theory, as proarrow equipments are particularly general. However, this means that we are technically unable to talk of "relative monads in a 2-category": we must instead talk of "relative monads in a proarrow equipment" (though it is usually harmless to elide the distinction). In concrete examples, this is unproblematic, since there is often a canonical proarrow equipment structure on a given 2-category[2] (given in the setting of enriched categories, for instance, by the inclusion $\mathcal{V}$-**Cat** $\hookrightarrow$ $\mathcal{V}$-**Prof**). However, it does introduce a subtlety not present in the study of monads, which raises interesting questions: for instance, while it is known that distributive laws between monads in $\mathcal{K}$ arise as the objects of the 2-category[3] $\mathbf{Mnd}(\mathbf{Mnd}(\mathcal{K}))$ [Str72a, §6; Bur73, §2], it is not clear whether something similar is true for relative monads, as this would require the 2-category of relative monads in a proarrow equipment to be equipped with a proarrow equipment structure.

We shall begin in Section 5.1 by briefly introducing the use of proarrow equipments (we refer to Wood [Woo82; Woo85] for more comprehensive introductions), before defining relative adjunctions and relative monads in a proarrow equipment in Section 5.2. We focus in particular on morphisms between relative adjunctions, since these will be crucial to establishing the formal monad–theory correspondence. In Section 5.3, we introduce Eilenberg–Moore objects for relative monads, which mostly follows the treatment of Lobbia [Lob20, §4]. Eilenberg–Moore objects are necessary to relate relative monads and monads in Section 5.4. Finally, in Section 5.5, we study how categories of j-monads may be embedded in categories of $(j\mathbin{;}j')$-monads via postcomposition, from which we derive several useful representation results.

5.1 Proarrow equipments

A *proarrow equipment* is, in effect, a calculus of homs. While a 2-category captures 2-dimensional equational reasoning, a proarrow equipment captures reasoning about hom-objects. For instance, recall that an adjunction can be defined equivalently as either a pair of functors $\ell\colon A \to B$ and $r\colon B \to A$ along with natural transformations $\eta\colon 1_A \Rightarrow \ell\mathbin{;}r$ and $\epsilon\colon r\mathbin{;}\ell \Rightarrow 1_B$ satisfying the triangle identities; or as a pair of functors $\ell\colon A \to B$ and $r\colon B \to A$ along with an isomorphism $B(\ell a, b) \cong A(a, rb)$ natural in $a \in A$ and $b \in B$. The former definition can be internalised in any 2-category, but the latter cannot be expressed in an arbitrary 2-category, because there is no way to express the natural isomorphism of hom-sets. A proarrow equipment facilitates the expression of the latter definition (after which both definitions can be proven equivalent, as in the traditional setting). These kinds of operations on hom-sets (and, more generally, hom-objects) are commonplace in category theory, and so this is a useful abstraction to capture traditional category theoretic arguments in a general setting.

Formally, a proarrow equipment axiomatises the inclusion of the 2-category **Cat** of small categories, functors, and natural transformations into the bicategory **Prof** of small categories, profunctors, and natural transformations. Conceptually, a profunctor $A \nrightarrow B$, i.e. a functor $B^{\mathrm{op}} \times A \to \mathbf{Set}$, can be seen as a "generalised hom-set", sending the pair (b, a) to a set of "(hetero)morphisms" from b to a. In particular, for every category A, the hom-functor $A(-,-)$ is an endoprofunctor $A \nrightarrow A$: this is the identity profunctor on A. Composition of profunctors is defined explicitly using coends, but abstractly it may be thought of as composition of heteromorphisms. Every functor $f\colon A \to B$ induces a profunctor in two ways: the profunctor $B(1, f)\colon A \nrightarrow B$ maps $(b, a) \mapsto B(b, fa)$, while the profunctor $B(f, 1)\colon B \nrightarrow A$ maps $(a, b) \mapsto B(fa, b)$. These profunctors turn out to be adjoint in **Prof**: we have $B(1, f) \dashv B(f, 1)$. Therefore, there are two identity-on-objects pseudofunctors, $(-)_*\colon \mathbf{Cat} \to \mathbf{Prof}$ and $(-)^*\colon \mathbf{Cat}^{\mathrm{co\,op}} \to \mathbf{Prof}$, related by an adjointness property. Both are locally fully faithful, meaning that the 2-cells $B(1, f) \Rightarrow B(1, g)$ are in natural bijection with the 2-cells $f \Rightarrow g$, as are the 2-cells $B(f, 1) \Rightarrow B(g, 1)$. Remarkably, it turns out that this structure is sufficient to perform a great deal of category theory in a formal context.

Definition 5.1.1 ([Woo85, §0]). A *proarrow equipment* comprises

[2]In fact, this statement can be made precise in a suitable sense: every proarrow equipment satisfying a certain natural exactness property is canonically determined by the codiscrete cofibrations in the domain 2-category [RW88].

[3]Here, we are distinguishing between the 2-category of monads *in a 2-category*, and the 1-category of monads *on a 1-category* as in Corollary 4.8.5.

1. a 2-category $\mathcal{K}$ and a bicategory $\mathcal{N}$[4];
2. a locally fully faithful pseudofunctor $(-)_* \colon \mathcal{K} \to \mathcal{N}$;
3. for every 1-cell f in $\mathcal{K}$, a right adjoint $f_* \dashv f^*$ in $\mathcal{N}$.

We call 1-cells of the form f_* *representable* and 1-cells of the form f^* *corepresentable.*

Much of the value of a context for formal category theory comes from the ability to transfer intuition from (enriched) category theory to the formal context with little modification. To aid this intuition, it is invaluable to have suggestive notation. Though we assume $\mathcal{N}$ is a bicategory, rather than a 2-category, in general, we shall suppress the coherence isomorphisms as a notational convenience. We shall denote non-diagrammatic composition in $\mathcal{N}$ by $\odot$, which in many ways acts like the tensor product of a monoidal category in enriched category theory. Given a cospan $X \xrightarrow{g} B \xleftarrow{f} Y$ in $\mathcal{K}$, we shall write $B(f, g)$ for $f^* \odot g_* \colon X \to Y$ in $\mathcal{N}$: in analogy with enriched category theory, it plays the role of a "hom-cell" (or hom-object in the case of monoidal enrichment) from f to g. We shall also often simply write 1 for the identity 1-cell on an object. Note that this means we have

$$A(1,1) \cong 1_A \qquad\qquad B(fg, hi) \cong C(g, 1) \odot B(f, h) \odot A(1, i) \tag{5.1}$$

For each 1-cell $f \colon A \to B$ in $\mathcal{K}$, we shall write $\overline{f} \colon A(1,1) \Rightarrow B(f,f)$ for the unit of the adjunction $B(1,f) \dashv B(f,1)$ in $\mathcal{N}$: conceptually, this is postcomposition by f. For convenience, we shall often also write $\overline{f} \colon A(a,a) \Rightarrow B(a\,;f, a\,;f)$ for the appropriate whiskering. Dually, we shall write

$$\mu_f \colon B(1,f) \odot B(f,1) \Rightarrow B(1,1)$$

for the counit: conceptually, this is composition along f. We shall write

$$\mu_{f,g,h} \colon A(f,g) \odot A(g,h) \Rightarrow A(f,h)$$

for the appropriate whiskering of μ_g, though we shall often elide the outer variables, and make explicit only the 1-cell along which we are composing, writing μ_g also for the whiskering. We note that one of the triangle laws gives us a cancellation law for composition: namely, that the following operation of postcomposing a 1-cell, and then composing along it, is idempotent (i.e. isomorphic to the identity).

$$C(f, ghi) \cong C(f, gh) \odot A(1, i) \xrightarrow{C(f,gh)\odot\overline{h}} C(f, gh) \odot B(h, hi) \xrightarrow{C(f,g)\odot\mu_h\odot B(1,hi)} C(f, ghi) \tag{5.2}$$

The unit allows us to recover the notion of full faithfulness.

Definition 5.1.2. A 1-cell $f \colon A \to B$ in $\mathcal{K}$ is *fully faithful* if $\bar{f}$ is invertible.

Given a 2-cell $\varphi \colon f \Rightarrow g$ in $\mathcal{K}$, we shall write $\varphi^\sharp \colon A(1,1) \Rightarrow B(f,g)$ for its transpose, the 2-cell in $\mathcal{N}$ given by $\overline{f}\,; B(f,\varphi)$; in other words, by the pasting:

$$\begin{tikzcd}
A \arrow[rr, bend left=50, "A(1,1)"] \arrow[rr, bend right=50, "B(f,g)"'] \arrow[r, bend left=20, "B(1,f)"] \arrow[r, bend right=20, "B(1,g)"'] \arrow[r, phantom, "\Downarrow B(1,\varphi)"] & B \arrow[r, "B(f,1)"'] \arrow[u, phantom, "\Downarrow\overline{f}"] \arrow[d, phantom, "\cong"] & A
\end{tikzcd}$$

Conversely, given a 2-cell $\psi \colon A(1,1) \Rightarrow B(f,g)$, we shall write $\psi^\flat \colon f \Rightarrow g$ for its transpose, the 2-cell in $\mathcal{N}$

[4] The codomain of a proarrow equipment is often denoted $\mathcal{M}$ for (bi)*modules*, but we choose a different symbol, as we shall use $\mathcal{M}$ to denote the right-class of a factorisation system in Chapter 6.

given via local full faithfulness by $(B(1,f) \odot \psi)\ ;\ (\mu_f \odot B(1,g))$; in other words, by the pasting:

$$
\begin{tikzcd}[column sep=large]
A \arrow[r, "{B(1,g)}"'] \arrow[rr, bend left=40, "{A(1,1)}"] \arrow[rrr, bend left=60, "{B(1,f)}"] \arrow[rrr, bend right=60, "{B(1,g)}"'] \arrow[rr, phantom, bend left=20, "\Downarrow\psi"] & B \arrow[r, "{B(f,1)}"] \arrow[rr, bend right=40, "{B(1,1)}"'] \arrow[rr, phantom, bend right=15, "\Downarrow\mu_f"] & A \arrow[r, "{B(1,f)}"] & B
\end{tikzcd}
$$

These transposition operations are inverse to one another, via the triangle identities, establishing that 2-cells in $\mathcal{K}$ are the same as "internal 2-cells" in $\mathcal{N}$.

As expected, $B(f,g)$ is appropriately functorial in f and g. The covariant functoriality is induced trivially by $B(1,-)$. Given a 2-cell $\varphi \colon f \Rightarrow g$ in $\mathcal{K}$, we may construct a 2-cell $B(\varphi,1) \colon B(g,1) \Rightarrow B(f,1)$ in $\mathcal{N}$ by

$$B(g,1) \cong B(1,1) \odot B(g,1) \xrightarrow{\overline{f} \odot B(g,1)} B(f,f) \odot B(g,1) \xrightarrow{B(f,\varphi) \odot B(g,1)} B(f,g) \odot B(g,1) \xrightarrow{\mu_g} B(f,1) \tag{5.3}$$

in other words, by the pasting:

$$
\begin{tikzcd}[column sep=large]
B \arrow[r, "{B(g,1)}"] \arrow[rrr, bend left=50, "{B(g,1)}"] \arrow[rr, bend right=40, "{B(1,1)}"'] \arrow[rrr, bend right=60, "{B(f,1)}"'] \arrow[r, phantom, bend right=20, "\mu_g\Downarrow"'] & A \arrow[r, bend left, "{B(1,f)}"] \arrow[r, bend right, "{B(1,g)}"'] \arrow[r, phantom, "\Downarrow B(1,\varphi)"] \arrow[rr, bend left=40, "{A(1,1)}"] \arrow[rr, phantom, bend left=20, "\Downarrow\overline{f}"] & B \arrow[r, "{B(f,1)}"'] & A
\end{tikzcd}
$$

While this may appear to be a significant amount of notation to introduce, in practice it quickly becomes natural to reason about proarrow equipments with this notation, using existing intuition about (enriched) category theory.

Pointwise extensions

While left and right extensions can be defined in any 2-category, in practice the useful extensions of (enriched) functors are the pointwise ones (cf. [Kel82, p. 65]), which satisfy a stronger universal property with respect to profunctors. We shall have cause to employ pointwise extensions, and to that end, we recall the definition of pointwise left extension in a proarrow equipment, in terms of a right lift (Definition 2.3.5) in $\mathcal{N}$.

Definition 5.1.3 ([Woo82, §3]). Let $j \colon A \to E$ and $\ell \colon A \to B$ be 1-cells in $\mathcal{K}$. A 1-cell $j \rhd \ell \colon E \to B$ is the *pointwise left extension of ℓ along j* when there is an isomorphism $B(j \rhd \ell, 1) \cong B(\ell, 1) \blacktriangleleft E(j,1)$ in $\mathcal{N}$.

In particular, pointwise left extensions are left extensions in the sense of Definition 2.3.5. (The converse does not generally hold.)

Lemma 5.1.4. *Let $j \colon A \to E$ and $\ell \colon A \to B$ be 1-cells in $\mathcal{K}$. If a pointwise left extension $j \rhd \ell$ exists, then it is furthermore a left extension $j \rhd \ell$.*

Proof. We have natural bijections of 2-cells

$$
\begin{array}{cl}
j \rhd \ell \Rightarrow \cdot & \\
\hline\hline
B(1, j \rhd \ell) \Rightarrow B(1, \cdot) & ((-)_* \text{ is locally fully faithful}) \\
\hline\hline
B(\cdot, 1) \Rightarrow B(j \rhd \ell, 1) & (\text{taking mates}) \\
\hline\hline
B(\cdot, 1) \Rightarrow B(\ell, 1) \blacktriangleleft E(j, 1) & (\text{pointwise left extension}) \\
\hline\hline
E(j, 1) \odot B(\cdot, 1) \Rightarrow B(\ell, 1) & (\text{right lift}) \\
\hline\hline
B(1, \ell) \Rightarrow B(1, j \mathbin{;} \cdot) & (\text{taking mates}) \\
\hline\hline
\ell \Rightarrow j \mathbin{;} \cdot & ((-)_* \text{ is locally fully faithful})
\end{array}
$$

witnessing the universal property of the left extension $j \rhd \ell$. □

We observe the following properties regarding the interaction between right lifts and right and left adjoints.

Lemma 5.1.5. *Let $p \colon E \to Y$ and $q \colon X \to Y$ be 1-cells in $\mathcal{N}$ and let $g \colon A \to E$ be a 1-cell in $\mathcal{K}$. Then supposing that $q \blacktriangleleft p \colon X \to E$ exists, we have*

$$E(g, 1) \odot (q \blacktriangleleft p) \cong q \blacktriangleleft p(1, g)$$

Proof. Let $r \colon X \to A$ be a 1-cell in $\mathcal{N}$. We have

$$
\begin{array}{cl}
r \Rightarrow E(g, 1) \odot (q \blacktriangleleft p) & \\
\hline\hline
E(1, g) \odot r \Rightarrow q \blacktriangleleft p & (\text{transposing}) \\
\hline\hline
p(1, g) \odot r \Rightarrow q & (\text{right lift}) \\
\hline\hline
r \Rightarrow q \blacktriangleleft p(1, g) & (\text{right lift})
\end{array}
$$

□

Lemma 5.1.6. *Let $p \colon X \to Y$ and $q \colon E \to Y$ be 1-cells in $\mathcal{N}$ and let $h \colon Z \to E$ be a 1-cell in $\mathcal{K}$. Then supposing that $q \blacktriangleleft p \colon E \to X$ exists, we have*

$$(q \blacktriangleleft p) \odot E(1, h) \cong (q \odot E(1, h)) \blacktriangleleft p$$

Proof. Let $r \colon Z \to X$ be a 1-cell in $\mathcal{N}$. We have

$$
\begin{array}{cl}
r \Rightarrow (q \odot E(1, h)) \blacktriangleleft p & \\
\hline\hline
p \odot r \Rightarrow q \odot E(1, h) & (\text{right lift}) \\
\hline\hline
p \odot r \Rightarrow q \odot (E(h, 1) \blacktriangleright 1) & (\text{Lemma 2.3.7}) \\
\hline\hline
p \odot r \Rightarrow E(h, 1) \blacktriangleright q & (\text{absolute extension}) \\
\hline\hline
p \odot r \odot E(h, 1) \to q & (\text{right extension}) \\
\hline\hline
r \odot E(h, 1) \to q \blacktriangleleft p & (\text{right lift}) \\
\hline\hline
r \odot (E(1, h) \rhd 1) \to q \blacktriangleleft p & (\text{Lemma 2.3.7}) \\
\hline\hline
E(1, h) \rhd r \to q \blacktriangleleft p & (\text{absolute extension}) \\
\hline\hline
r \to (q \blacktriangleleft p) \odot E(1, h) & (\text{left extension})
\end{array}
$$

□

As a corollary, we may observe that 2-cells $g \,;(f \rhd h) \Rightarrow h$ are in bijection with 2-cells $E(f,g) \odot E(h,i)$. (In particular, this implies the representability property of [MT08, Definition 5, Proposition 2].)

Corollary 5.1.7. *Let $f, g, h, i \colon A \to E$ be 1-cells in $\mathcal{K}$. If a pointwise left extension $j \rhd \ell$ exists, then there is a natural isomorphism of 2-cells*

$$\frac{g \,;(f \rhd h) \Rightarrow i}{\overline{E(f,g) \Rightarrow E(h,i)}}$$

Proof. We have

$$\begin{aligned}
E(g \,;(f \rhd h), i) &\cong E(g \,;(f \rhd h), 1) \odot E(1,i) \\
&\cong E(g,1) \odot E(f \rhd h, 1) \odot E(1,i) \\
&\cong E(g,1) \odot (E(h,1) \blacktriangleleft E(f,1)) \odot E(1,i) && \text{(pointwise extension)} \\
&\cong (E(h,1) \blacktriangleleft E(f,g)) \odot E(1,i) && \text{(Lemma 5.1.5)} \\
&\cong (E(h,1) \odot E(1,i)) \blacktriangleleft E(f,g) && \text{(Lemma 5.1.6)} \\
&\cong E(h,i) \blacktriangleleft E(f,g)
\end{aligned}$$

and so

$$\begin{array}{c}
g \,;(f \rhd h) \Rightarrow i \\ \hline\hline
E(1, g \,;(f \rhd h)) \Rightarrow E(1,i) \\ \hline\hline
A(1,1) \Rightarrow E(g \,;(f \rhd h), i) \\ \hline\hline
A(1,1) \Rightarrow E(h,i) \blacktriangleleft E(f,g) \\ \hline\hline
E(f,g) \Rightarrow E(h,i)
\end{array}$$

□

Finally, the notion of pointwise left extension allows us to recover the notion of density.

Definition 5.1.8. A 1-cell $f \colon A \to B$ in $\mathcal{K}$ is *dense* if the pointwise left extension $f \rhd f \colon B \to B$ exists and the canonical 2-cell $f \rhd f \Rightarrow 1_B$, given by applying

$$\begin{array}{cl}
f \rhd f \Rightarrow 1_B & \\ \hline\hline
1_B \Rightarrow B(f \rhd f, 1) & ((-)^* \text{ is locally fully faithful}) \\ \hline\hline
1_B \Rightarrow B(f,1) \blacktriangleleft B(f,1) & \text{(pointwise left extension)} \\ \hline\hline
B(f,1) \Rightarrow B(f,1) & \text{(right lift)}
\end{array}$$

to the identity on $B(f,1)$, is invertible.

The important property of density is the following.

Lemma 5.1.9. *Let $j, f, g \colon A \to E$ be 1-cells in $\mathcal{K}$. If j is dense, then 2-cells $f \Rightarrow g$ are in bijection with 2-cells $E(j,f) \Rightarrow E(j,g)$.*

Proof. 2-cells $E(j,f) \Rightarrow E(j,g)$ are, by Corollary 5.1.7, in bijection with 2-cells $f \,;(j \rhd j) \Rightarrow g$, hence 2-cells $f \Rightarrow g$ when j is dense so that $j \rhd j \Rightarrow 1_E$ is invertible. □

5.1.1 Lax idempotent pseudomonads

We have mentioned that the motivating example of a proarrow equipment is the inclusion **Cat** $\to$ **Prof**. Explicitly, it is not so difficult to see why profunctors capture the structure of hom-sets and operations thereon.

However, there is also an abstract perspective of the bicategory of profunctors that sheds light into its importance: namely, **Prof** is the Kleisli bicategory for the presheaf construction [Hyl14a; Fio+18], which is the free cocompletion of small categories [Ulm68]. As a free cocompletion, the presheaf construction forms a lax idempotent pseudomonad relative to the inclusion **Cat** $\hookrightarrow$ **CAT** of small categories into locally-small categories [Fio+18, §5], which extends to the small-presheaf construction, forming a lax idempotent pseudomonad on **CAT** for the small-cocompletion of locally-small categories. As mentioned at the beginning of this chapter, a 2-category with a lax idempotent pseudomonad is an alternative context for formal category theory (cf. [BF99]). Equipping a 2-category with a lax idempotent pseudomonad $\mathcal{P}$ is stronger than equipping it with a proarrow equipment, in the sense that every lax idempotent pseudomonad canonically induces a proarrow equipment (Definition 5.1.12). It will be useful in some cases to assume that our proarrow equipment has arisen in this way, as it permits us to consider certain 1-cells $f \colon A \to B$ in $\mathcal{N}$ as represented by 1-cells $f \colon A \to \mathcal{P}B$ in $\mathcal{K}$. In particular, this is true for 2-categories of enriched categories (assuming a nice enough base of enrichment), which form a motivating class of examples.

Lax idempotent pseudomonads were introduced independently (in a slightly stricter form than necessary) by Kock [Koc95] and Zöberlein [Zöb76] to characterise cocompletions of categories. Later, Marmolejo [Mar97] gave a more general definition eliminating unnecessary strictness. The terminology *lax idempotent* was introduced by Kelly and Lack [KL97].

Definition 5.1.10 ([Mar97, §10])**.** A pseudomonad $(\mathcal{P}, \mu, よ)$ is *lax idempotent* if either of the following two equivalent conditions hold.

- There is an adjunction $\mathcal{P}よ_A \dashv \mu_A$ with invertible unit for every object $A \in \mathcal{K}$.
- There is an adjunction $\mu_A \dashv よ_{\mathcal{P}A}$ with invertible counit for every object $A \in \mathcal{K}$.

In this case, we say that a $\mathcal{P}$-algebra is $\mathcal{P}$*-cocomplete* and a $\mathcal{P}$-homomorphism is $\mathcal{P}$*-cocontinuous*. A lax idempotent monad is *locally fully faithful* if its unit is componentwise representably fully faithful (cf. [BF99, Remark 1.12]).

There is much that can be said about the theory of lax idempotent pseudomonads, but we shall need very little of it here. It will suffice to define the notion of *admissibility* (cf. [SW78]), which, conceptually, identifies those 1-cells for which we can perform a nerve construction.

Definition 5.1.11 ([BF99, Definition 1.1])**.** Let $(\mathcal{P}, \mu, よ)$ be a lax idempotent pseudomonad on a 2-category $\mathcal{C}$. A 1-cell $f \colon A \to B$ in $\mathcal{C}$ is $\mathcal{P}$*-admissible* if $\mathcal{P}f \colon \mathcal{P}A \to \mathcal{P}B$ has a right-adjoint $\mathcal{P}^*f \colon \mathcal{P}B \to \mathcal{P}A$. In this case, we denote by $n_f = よ_B \mathbin{;} \mathcal{P}^*f \colon B \to \mathcal{P}A$ the *nerve* of f. Denote by $\mathbf{Adm}_{\mathcal{P}}(\mathcal{C})$ the locally full sub-2-category of $\mathcal{C}$ on the $\mathcal{P}$-admissible 1-cells.

Conceptually, the $\mathcal{P}$-admissible 1-cells are those that are "small" in a suitable sense. In particular, when $\mathcal{P}$ is the small presheaf construction, a functor $f \colon A \to B$ is $\mathcal{P}$-admissible just when the presheaf $B(f-, b)$ is a small presheaf for all $b \in B$ (i.e. admissible in the sense of Definition 3.2.1). Our notation f^* is intended to be indicative of that for proarrow equipments, as explicated by the following definition.

For any locally fully faithful lax idempotent pseudomonad $\mathcal{P}$ on $\mathcal{C}$, the pseudofunctor $\mathbf{Adm}_{\mathcal{P}}(\mathcal{C}) \to \mathcal{C} \to \mathbf{Kl}(\mathcal{P})$ forms a canonical proarrow equipment (cf. [DL19, Definition 4.1]), since for every admissible 1-cell $f \colon A \to B$, the right-adjoint $\mathcal{P}^*f \colon \mathcal{P}B \to \mathcal{P}A$ is $\mathcal{P}$-cocontinuous [BF99, Proposition 1.3].

Definition 5.1.12. A proarrow equipment $(-)_* \colon \mathcal{K} \to \mathcal{N}$ *corresponds* to a lax idempotent pseudomonad $\mathcal{P}$ on $\mathcal{C}$ if $\mathbf{Adm}_{\mathcal{P}}(\mathcal{C}) \to \mathcal{C} \to \mathbf{Kl}(\mathcal{P})$ is equivalent to $(-)_*$ as a proarrow equipment: namely, if there is a 2-equivalence $\mathbf{Adm}_{\mathcal{P}}(\mathcal{C}) \simeq \mathcal{K}$ and a biequivalence $\mathbf{Kl}(\mathcal{P}) \sim \mathcal{N}$ making the following diagram commute up to pseudonatural equivalence.

$$\begin{tikzcd}
\mathbf{Adm}_{\mathcal{P}}(\mathcal{C}) \arrow[r, hook] \arrow[d, "\simeq"'] & \mathcal{C} \arrow[r, "K_{\mathcal{P}}"] & \mathbf{Kl}(\mathcal{P}) \arrow[d, "\sim"] \\
\mathcal{K} \arrow[rr, "(-)_*"'] & & \mathcal{N}
\end{tikzcd}$$

In this case, for a $\mathcal{P}$-admissible 1-cell $f\colon A \to B$, we have $B(1,f) \overset{\text{def}}{=} f \mathbin{;} よ_B\colon A \to \mathcal{P}B$ and $B(f,1) \overset{\text{def}}{=} n_f\colon B \to \mathcal{P}A$.

Finally, we will occasionally need to make a smallness assumption on an object of $\mathcal{K}$ to ensure enough pointwise extensions exist.

Definition 5.1.13 ([DL18, Definition 2.33])**.** Let $(\mathcal{P}, よ)$ be a lax idempotent pseudomonad on $\mathcal{K}$. An object A in $\mathcal{K}$ is *$\mathcal{P}$-small* if every 1-cell in $\mathcal{K}$ with domain A is $\mathcal{P}$-admissible.

In particular, when $\mathcal{P}$ is the small-presheaf construction on **CAT**, this is precisely the classical notion of smallness for a category [DL18, Proposition 2.34].

5.2 Relative monads and relative adjunctions

We shall work henceforth in a proarrow equipment $(-)_*\colon \mathcal{K} \to \mathcal{N}$. Though relative monads are the primary object of *interest* in this chapter, it is relative adjunctions that will be the primary object of *study*. A relative adjunction acts as a kind of presentation for a relative monad: since a relative adjunction is defined in terms of a universal property, it is often more convenient to work with than its corresponding relative monad, whose axioms can be tiresome to check. With that in mind, we begin by defining relative adjunctions, and then show how every relative adjunction induces a relative monad. Thenceforth, it will often be unnecessary to deal directly with relative monads.

Definition 5.2.1 ([Woo82, §3; Ulm68, Definition 2.2])**.** A *relative adjunction* comprises

$$\begin{tikzcd} & B \arrow[dr, "r"] & \\ A \arrow[ur, "\ell"] \arrow[rr, "j"'] \arrow[rr, phantom, shift left=4, "\dashv"] & & E \end{tikzcd}$$

1. a 1-cell $j\colon A \to E$ in $\mathcal{K}$, the *root*;
2. a 1-cell $\ell\colon A \to B$ in $\mathcal{K}$, the *left relative adjoint*;
3. a 1-cell $r\colon B \to E$ in $\mathcal{K}$, the *right relative adjoint*;
4. an isomorphism in $\mathcal{N}$,
$$\sharp\colon B(\ell,1) \cong E(j,r) :\flat$$
the *transposition operators*.

We denote this situation by $\ell \mathbin{{}_j\dashv} r$, and call B the *apex*. A *j-relative adjunction* (or simply *j-adjunction*) is a relative adjunction with root j.

Note that, while having an isomorphism $B(\ell,1) \cong E(j,r)$ is a *property*, justifying the usual practice in category theory of eliding the specific nature of the isomorphism, it will be important for our purposes to treat it as a *structure*, making the transposition operators explicit.

Remark 5.2.2. When the root j is fully faithful, j-relative adjunctions are sometimes called *partial adjunctions* (cf. [Kel82, §1.11]), as the left relative adjoint is defined on a sub-object of E.

Recall that adjunctions may be presented in several forms (cf. [Mac98, Theorem IV.1.2]). A relative adjunction does not evidently have the same flexibility, because it is not possible to define a counit without forming a composite $r \mathbin{;} \ell$ (which does not type-check). (Conversely, a relative coadjunction, i.e. a relative adjunction in $\mathcal{K}^{\text{co}}$, has a counit rather than a unit.) However, it is possible to give a definition in terms of a unit and a 2-cell $E(j,r) \Rightarrow B(\ell,1)$, without the 2-cell $B(\ell,1) \Rightarrow E(j,r)$.

Lemma 5.2.3. *Let $j\colon A \to E$, $\ell\colon A \to B$, and $r\colon B \to E$ be 1-cells. The following are in bijection, natural in ℓ and r.*

1. *2-cells* $\sharp \colon B(\ell, 1) \Rightarrow E(j, r)$.
2. *2-cells* $\eta \colon j \Rightarrow \ell \,;\, r$.

Proof. (1) $\implies$ (2). Given a 2-cell $\sharp \colon B(\ell, 1) \Rightarrow E(j, r)$, we may form

$$A(1,1) \xRightarrow{\overline{\ell}} B(\ell, \ell) \xRightarrow{\sharp\ell} E(j, r\ell)$$

which is equivalently a 2-cell $j \Rightarrow \ell \,;\, r$.

(2) $\implies$ (1). Given a 2-cell $\eta \colon j \Rightarrow \ell \,;\, r$, we may form

$$B(\ell, 1) \xRightarrow{B(\ell,1)\overline{r}} E(r\ell, r) \xRightarrow{E(\eta,r)} E(j, r)$$

That these transformations are inverse follows from the triangle identities. □

A relative adjunction may therefore equivalently be specified by a 2-cell $\eta \colon j \Rightarrow \ell \,;\, r$ for which the induced $B(\ell, 1) \Rightarrow E(j, r)$ is invertible.

Just as is the case for (non-relative) adjunctions, left relative adjoints are unique up to isomorphism; the same is true for right relative adjoints in the presence of dense roots, though not in general (see [Ulm68, (2.5)] for a counterexample).

Lemma 5.2.4. *If $\ell \;{}_j\!\dashv r$ and $\ell' \;{}_j\!\dashv r$, then $\ell \cong \ell'$. If $\ell \;{}_j\!\dashv r$ and $\ell \;{}_j\!\dashv r'$ and j is dense, then $r \cong r'$.*

Proof. For essential uniqueness of the left relative adjoint, we have that $B(\ell, 1) \cong E(j, r) \cong B(\ell', 1)$ implies $B(1, \ell) \cong B(1, \ell')$ since an isomorphism of right adjoints implies an isomorphism of left adjoints, which in turn implies $\ell \cong \ell'$ by local full faithfulness of $(-)_*$. For essential uniqueness of the right relative adjoint, assuming j is dense, we have $E(j, r) \cong B(\ell, 1) \cong E(j, r')$ and hence $E(1, r) \cong E(1, r')$. □

As expected, relative adjunctions subsume non-relative adjunctions.

Proposition 5.2.5. *Let $\ell \colon E \to B$ and $r \colon B \to E$ be 1-cells in $\mathcal{K}$. Then $\ell \;{}_{1_E}\!\dashv r$ if and only if $\ell \dashv r$.*

Proof. If $\ell \dashv r$, then $B(\ell, 1) \cong E(1, r)$ since $(-)_*$, being a pseudofunctor, preserves adjunctions. Conversely, if $B(\ell, 1) \cong E(1, r)$, then $E(1, r)$ is right adjoint to $B(1, \ell)$ since $B(1, \ell) \dashv B(\ell, 1)$, so that $\ell \dashv r$ since $(-)_*$ is locally fully faithful and hence reflects adjunctions. □

We now introduce the notion of relative monad in a proarrow equipment. The definition here is new, but essentially follows that of [ACU10, Definition 1] in **CAT** (cf. [Lob20, Definition 2.1]).

Definition 5.2.6 (Relative monad)**.** A *relative monad* comprises

1. a 1-cell $j \colon A \to E$ in $\mathcal{K}$, the *root*;
2. a 1-cell $t \colon A \to E$ in $\mathcal{K}$, the *underlying 1-cell*;
3. a 2-cell $\tau \colon j \Rightarrow t$ in $\mathcal{K}$, the *unit*;
4. a 2-cell $\dagger \colon E(j, t) \Rightarrow E(t, t)$ in $\mathcal{N}$, the *extension operator*,

such that the following diagrams commute:

1.

$$\begin{tikzcd} E(j,t) \arrow[r, "\dagger"] \arrow[dr, equal] & E(t,t) \arrow[d, "E(\tau,t)"] \\ & E(j,t) \end{tikzcd}$$

2.

$$\begin{tikzcd} A(1,1) \arrow[r, "\overline{j}"] \arrow[drr, "\overline{t}"'] & E(j,j) \arrow[r, "E(j,\tau)"] & E(j,t) \arrow[d, "\dagger"] \\ & & E(t,t) \end{tikzcd}$$

3.

$$\begin{tikzcd} E(j,t) \odot E(j,t) \arrow[r, "E(j,t)\odot\dagger"] \arrow[d, "\dagger\odot\dagger"'] & E(j,t) \odot E(t,t) \arrow[r, "\mu_{j,t,t}"] & E(j,t) \arrow[d, "\dagger"] \\ E(t,t) \odot E(t,t) \arrow[rr, "\mu_{t,t,t}"'] & & E(t,t) \end{tikzcd}$$

A j*-relative monad* (or simply j*-monad*) is a relative monad with root j. A *morphism* of j-monads from $(t, \tau, (-)^\dagger)$ to $(t', \tau', (-)^{\dagger'})$ is a 2-cell $\varphi \colon t \Rightarrow t'$ such that the following diagrams commute:

1.

$$\begin{tikzcd} t \arrow[rr, "\varphi"] & & t' \\ & j \arrow[ul, "\tau"] \arrow[ur, "\tau'"'] & \end{tikzcd}$$

2.

$$\begin{tikzcd} E(j,t) \arrow[r, "E(j,\varphi)"] \arrow[d, "\dagger"'] & E(j,t') \arrow[r, "\dagger'"] & E(t',t') \arrow[d, "E(\varphi,t')"] \\ E(t,t) \arrow[rr, "E(t,\varphi)"'] & & E(t,t') \end{tikzcd}$$

j-monads and their morphisms form a 1-category $\mathbf{RMnd}(j)$.

We shall not define a 2-category of relative monads, as there are subtleties with the appropriate definition of morphism[5]. Instead, we focus on the 1-category of relative monads with a fixed root.

Example 5.2.7. For any 1-cell $j \colon A \to E$ in $\mathcal{K}$, there is a canonical j-monad $(j, j, 1_j, 1_{E(j,j)})$, which is the initial object in $\mathbf{RMnd}(j)$. In a certain sense, this relative monad acts as a surrogate for the identity monad[6].

Relative monads generalise ordinary monads in $\mathcal{K}$: observe that a 2-cell $\dagger \colon E(1,t) \Rightarrow E(t,t)$ is equivalently a 2-cell $E(1,t) \odot E(1,t) \Rightarrow E(1,t)$ by transposing, and hence a 2-cell $t\,;t \Rightarrow t$ by local full faithfulness of $(-)_*$. We note that essentially the same observation appears in [MW10, Lemma 9.1], where 1_E-relative monads are called *extension systems*[7].

Proposition 5.2.8. *For every object E of $\mathcal{K}$, there is an isomorphism of 1-categories*

$$\mathbf{RMnd}(1_E) \cong \mathbf{Mnd}(E)$$

Proof. Given an 1_E-relative monad $(t, \tau, \dagger)$, upon defining $E(1, \mu)$ to be the transpose of $\dagger$, axioms (1 – 3) of the relative monad exactly correspond to the left unit, right unit, and associativity laws for a monad (t, τ, μ); and axioms (1 & 2) for a morphism of 1_E-monads correspond to the unit and multiplication laws for a monad morphism. $\square$

Just as every adjunction induces[8] a monad, every relative adjunction induces a relative monad. We will later also introduce morphisms of relative adjunctions and relate them to relative monad morphisms.

Proposition 5.2.9. *Every relative adjunction induces a relative monad.*

[5] There is evidence to suggest that the most obvious notion of morphism (cf. [Lob20, Definition 3.1]) is too strict to recover theorems of interest.

[6] This intuition may be formalised by characterising relative monads as monoids in skew-monoidal hom-categories (cf. [ACU15, §3]), for which j is the unit.

[7] In their terminology, 1_E-relative comonads would be called *lifting systems*.

[8] The literature varies on the appropriate verb for the construction of a monad from an adjunction, *induce* (cf. [Hub61, §4]) and *generate* (cf. [Str72a, Theorem 2]) being two popular terms. We shall prefer the former, the latter having connotations of computation.

Proof. Let $(j, \ell, r, \sharp, \flat)$ be a relative adjunction. We have a unit $\eta \colon j \Rightarrow \ell \mathbin{;} r$ using Lemma 5.2.3, which is explicitly defined by

$$E(1, j) \cong E(1, j) \odot A(1, 1) \xrightarrow{E(1,j)\odot\overline{\ell}} E(1, j) \odot B(\ell, \ell) \xrightarrow{E(1,j)\odot\sharp} E(1, j) \odot E(j, r\ell) \xrightarrow{\mu_j} E(1, r\ell)$$

so that the transpose is

$$E(r\ell, 1) \cong B(\ell, 1) \odot E(r, 1) \xrightarrow{\sharp\odot E(r,1)} E(j, r) \odot E(r, 1) \xrightarrow{\mu_r} E(j, 1)$$

Define an extension operator $\dagger \colon E(j, r\ell) \Rightarrow E(r\ell, r\ell)$ by

$$E(j, r\ell) \xrightarrow{\flat} B(\ell, \ell) \xrightarrow{\overline{r}} E(r\ell, r\ell)$$

$(\ell \mathbin{;} r, \eta, \dagger)$ forms a j-monad: the axioms for a relative monad are validated through sizeable but elementary commutative diagrams as follows.

The first axiom:

$$\begin{tikzcd}
E(j, r\ell) \arrow[r, "\flat"] \arrow[dr, equal, bend right] & B(\ell, \ell) \arrow[r, "\overline{r}"] \arrow[d, equal] & E(r\ell, r\ell) \arrow[d, equal] \\
& B(\ell, 1) \odot B(1, \ell) \arrow[r, "B(\ell,1)\odot\overline{r}"] \arrow[d, dashed, "\sharp\odot B(1,\ell)"] & B(\ell, 1) \odot E(r, r\ell) \arrow[d, "\sharp\odot E(r,r\ell)"] \\
& E(j, r) \odot B(1, \ell) \arrow[r, "E(j,r)\odot\overline{r}"] \arrow[dr, equal] & E(j, r) \odot E(r, r\ell) \arrow[d, "\mu_r"] \\
& & E(j, r\ell)
\end{tikzcd}$$

The second axiom (using the transpose of the unit directly):

$$\begin{tikzcd}
A(1, 1) \arrow[r, "\overline{\ell}"] \arrow[ddrr, "\overline{r\ell}"'] & B(\ell, \ell) \arrow[r, "\sharp"] \arrow[dr, equal] & E(j, r\ell) \arrow[d, "\flat"] \\
& & B(\ell, \ell) \arrow[d, "\overline{r}"] \\
& & E(r\ell, r\ell)
\end{tikzcd}$$

The third axiom[9]:

$$
\begin{tikzcd}
E(j,r\ell) \odot E(j,r\ell) \arrow[r, "E(j,r\ell)\odot\flat"] \arrow[dd, "\flat\odot\flat"'] & E(j,r\ell) \odot E(\ell,\ell) \arrow[r, "E(j,r\ell)\odot\overline{r}"] \arrow[d, equal] \arrow[ddl, bend right, "\flat\odot B(\ell,\ell)"'] & E(j,r\ell) \odot E(r\ell,r\ell) \arrow[r, equal] \arrow[rrr, bend left, "\mu_{r\ell}"] & E(j,r) \odot B(1,\ell) \odot B(\ell,1) \odot E(r,r\ell) \arrow[r, "\mu_\ell"] & E(j,r) \odot E(r,r\ell) \arrow[r, "\mu_r"] & E(j,r\ell) \arrow[dd, "\flat"] \\
& E(j,r) \odot B(1,\ell) \odot B(\ell,1) \odot B(1,\ell) \arrow[rrr, "\mu_\ell"] \arrow[d, dashed, "\flat\odot B(1,\ell)\odot B(\ell,1)\odot B(1,\ell)"] & & & E(j,r) \odot B(1,\ell) \arrow[u, dashed, "E(j,r)\odot\overline{r}"] \arrow[ur, equal] & \\
B(\ell,\ell) \odot B(\ell,\ell) \arrow[r, equal] \arrow[d, "\overline{r}\odot\overline{r}"'] & E(\ell,1) \odot B(1,\ell) \odot B(\ell,1) \odot B(1,\ell) \arrow[r, "\mu_\ell"] \arrow[d, dashed, "\overline{r}\odot B(1,\ell)\odot b(\ell,1)\odot\overline{r}"] & B(\ell,1) \odot B(1,\ell) \arrow[rrr, equal] \arrow[d, dashed, "\overline{r}\odot\overline{r}"] & & & B(\ell,\ell) \arrow[d, "\overline{r}"] \\
E(r\ell,r\ell) \odot E(r\ell,r\ell) \arrow[r, equal] & E(r\ell,r) \odot B(1,\ell) \odot B(\ell,1) \odot E(r,r\ell) \arrow[r, "\mu_\ell"'] \arrow[rrrr, bend right, "\mu_{r\ell}"'] & E(r\ell,r) \odot E(r,r\ell) \arrow[rrr, "\mu_r"'] & & & E(r\ell,r\ell)
\end{tikzcd}
$$

□

9 A strong advocate for the use of string diagrams over commutative diagrams.

The previous proposition motivates the following definition.

Definition 5.2.10. Let $j \colon A \to E$ be a 1-cell in $\mathcal{K}$, and let T be a j-monad. A *resolution*[10] of T is a j-adjunction that induces T. A *morphism* of resolutions of T from $\ell \;{}_j\!\dashv\; r$ to $\ell' \;{}_j\!\dashv\; r'$ is a 1-cell $b \colon B \to B'$ between the apices making the following diagrams commute.

$$\begin{tikzcd} & B \arrow[dd, "b"] \arrow[dr, "r"] & \\ A \arrow[ur, "\ell"] \arrow[dr, "\ell'"'] & & E \\ & B' \arrow[ur, "r'"'] & \end{tikzcd} \qquad \begin{tikzcd} B(\ell, 1) \arrow[r, "\bar{b}"] \arrow[dr, "\sharp"'] & B'(\ell', b) \arrow[d, "\sharp'"] \\ & E(j, r) \end{tikzcd}$$

Resolutions of T and their morphisms form a 1-category $\mathbf{Res}(T)$.

Though it is not necessarily the case for an arbitrary proarrow equipment that a given relative monad has a resolution, we will be interested in the proarrow equipments for which this is the case. In particular, we are concerned with relative monads having canonical resolutions given by universal properties.

Definition 5.2.11. A j-adjunction $\ell \;{}_j\!\dashv\; r$ inducing a j-monad T is

1. *j-opmonadic* (or *Kleisli*) if it is initial in $\mathbf{Res}(T)$.
2. *j-monadic* (or *Eilenberg–Moore*) if it is terminal in $\mathbf{Res}(T)$.

In this cases, we call the apex of the j-adjunction *Kleisli* or *Eilenberg–Moore* respectively.

$$\begin{tikzcd} & \mathbf{Kl}(T) \arrow[dr, "v_T"] & \\ A \arrow[ur, "k_T"] \arrow[rr, "j"'] \arrow[rr, phantom, shift left=3ex, "\dashv"] & & E \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Kl}(T) \arrow[r, "{[]_T}"] & B \arrow[r, "\langle\rangle_T"] \arrow[dr, "r"] & \mathbf{EM}(T) \arrow[d, "u_T"] \\ A \arrow[u, "k_T"] \arrow[ur, "\ell"] \arrow[rr, "j"'] \arrow[rr, phantom, shift left=3ex, "\dashv"] & & E \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{EM}(T) \arrow[dr, "u_T"] & \\ A \arrow[ur, "f_T"] \arrow[rr, "j"'] \arrow[rr, phantom, shift left=3ex, "\dashv"] & & E \end{tikzcd}$$

Supposing their existence, we denote by $k_T \;{}_j\!\dashv\; v_T$ the chosen Kleisli j-adjunction, and by $f_T \;{}_j\!\dashv\; u_T$ the chosen Eilenberg–Moore j-adjunction; and denote by $[]_T \colon \mathbf{Kl}(T) \to B$ the unique mediating morphism of resolutions from a Kleisli object, and by $\langle\rangle_T \colon B \to \mathbf{EM}(T)$ the unique mediating morphism of resolutions to an Eilenberg–Moore object.

Kleisli and Eilenberg–Moore resolutions have universal properties with respect to a single relative monad (namely, that which they induce). However, in many proarrow equipments of interest, they have a further universal property with respect to relative monad morphisms. To express these, we must introduce a notion of morphism of relative adjunction. In fact, there are two choices: one of which is necessary to express the universal property of Kleisli morphisms (which we dub *right-morphisms*), and one of which is necessary to express the universal property of Eilenberg–Moore morphisms (which we dub *left-morphisms*). There is some symmetry in the definitions, but, unlike in the setting of non-relative monads, they are not formally dual.

Definition 5.2.12. Let $j \colon A \to E$ be a 1-cell in $\mathcal{K}$. A *right-morphism* of j-adjunctions from $\ell \;{}_j\!\dashv\; r$ to $\ell' \;{}_j\!\dashv\; r'$ comprises

$$\begin{tikzcd} & B' \arrow[dr, "r'"] & \\ A \arrow[ur, "\ell'"] \arrow[r, "\ell"'] & B \arrow[u, "b"] \arrow[r, "r"'] \arrow[ur, phantom, "\Nearrow\rho"] & E \end{tikzcd}$$

1. a 1-cell $b \colon B \to B'$ such that $\ell \,;\, b = \ell'$;
2. a 2-cell $\rho \colon r \Rightarrow b \,;\, r'$,

[10] Resolutions of monads were introduced by [Cop70], called there *factorisations* of a monad. Resolutions of relative monads were called *splittings* in [ACU10]. We follow the terminology of [LS88, Definition 0.6.4].

rendering commutative the following diagram:

$$\begin{tikzcd}
B(\ell, 1) \arrow[r, "\sharp"] \arrow[d, "\overline{b}"'] & E(j, r) \arrow[d, "{E(j,\rho)}"] \\
B'(\ell', b) \arrow[r, "\sharp'"'] & E(j, r'b)
\end{tikzcd}$$

It is *strict* if ρ is the identity. j-adjunctions and right-morphisms form a 1-category $\mathbf{RAdj}_r(j)$.

Just as strict morphisms of adjunctions may be specified by several equivalent conditions (cf. [Mac98, §IV.7]), in terms of (either of) the transposition operations, the unit, or the counit, so right-morphisms of relative adjunctions may be expressed in terms of (either of) the transposition operations or the unit.

Lemma 5.2.13. *Let $b \colon B \to B'$ be a 1-cell between apices of j-adjunctions $\ell \; {}_j\!\dashv r$ and $\ell' \; {}_j\!\dashv r'$ and let $\rho \colon r \Rightarrow b ; r'$ be a 2-cell. Then (b, ρ) is a right-morphism if and only if the following equality of 2-cells holds, where η is defined as in Lemma 5.2.3.*

$$\begin{tikzcd}
 & B' \arrow[dr, bend left, "r'"] & \\
 & B \arrow[u, dashed, "b"] \arrow[dr, "r"] \arrow[r, phantom, "\overset{\rho}{\Rrightarrow}"] \arrow[d, phantom, "\eta\Uparrow"] & \\
A \arrow[uur, bend left, "\ell'"] \arrow[ur, "\ell"] \arrow[rr, "j"'] & {} & E
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & B' \arrow[dr, "r'"] \arrow[d, phantom, "\eta'\Uparrow"] & \\
A \arrow[ur, "\ell'"] \arrow[rr, "j"'] & {} & E
\end{tikzcd}$$

Proof. Suppose that (b, ρ) is a right-morphism, and observe that, by the axiom for a right-morphism, the following diagram commutes:

$$\begin{tikzcd}
A(1,1) \arrow[r, "\overline{\ell}"] \arrow[d, equal] & B(\ell, \ell) \arrow[r, "\sharp"] \arrow[d, dashed, "\overline{b}"] & E(j, r\ell) \arrow[d, "{E(j,\rho\ell)}"] \\
A(1,1) \arrow[r, "\overline{\ell'}"'] & B'(\ell', \ell') \arrow[r, "\sharp'"'] & E(j, r'\ell')
\end{tikzcd}$$

By tensoring on the left by $E(1, j)$ and then composing along j, we recover the coherence law for the units. The converse holds by Lemma 5.2.3. □

We are now in a position to show that the induction of a relative monad from a relative adjunction is functorial with respect to right-morphisms (and hence trivially strict morphisms, when ρ is invertible).

Proposition 5.2.14. *The induction of a j-monad from a j-adjunction as defined in Proposition 5.2.9 extends to a 1-functor $\oslash \colon \mathbf{RAdj}_r(j) \to \mathbf{RMnd}(j)$.*

Proof. Let (b, ρ) be a right-morphism between relative adjunctions $\ell \; {}_j\!\dashv r$ and $\ell' \; {}_j\!\dashv r'$. Left-whiskering ℓ on to ρ produces a 2-cell $(\ell ; \rho) \colon \ell ; r \Rightarrow \ell' ; r'$. This preserves the unit by Lemma 5.2.13.

Observe that, for all $s \colon B \to E$ and $\phi \colon r \Rightarrow s$, the following diagram commutes.

$$\begin{tikzcd}
E(1, r\ell) \odot B(\ell, 1) \arrow[rr, "{E(1,\phi\ell) \odot \overline{s}}"] \arrow[d, "{E(1,r\ell) \odot \overline{r}}"'] \arrow[dr, "\mu_\ell"] & & E(1, s\ell) \odot E(s\ell, s) \arrow[dd, "\mu_s"] \\
E(1, r\ell) \odot E(r\ell, r) \arrow[r, "\mu_r"'] \arrow[d, "{E(1,r\ell) \odot E(r\ell,\phi)}"'] & E(\ell, r) \arrow[dr, "{E(\ell,\phi)}"] & \\
E(1, r\ell) \odot E(r\ell, s) \arrow[rr, "\mu_{r\ell}"'] & & E(1, s)
\end{tikzcd}$$

Taking $s = b\,;r'$, $\phi = \rho$, tensoring on the left by $E(1, r)$ and the right by $B(1, \ell)$, so does the following:

$$\begin{tikzcd}
E(r\ell, r\ell) \odot B(\ell, \ell) \arrow[rr, "E(r\ell,\rho\ell)\odot\overline{r'b}"] \arrow[d, "E(r\ell,r\ell)\odot\overline{r}"'] \arrow[dr, "\mu_\ell"] & & E(r\ell, r'\ell') \odot E(r'\ell', r'\ell') \arrow[dd, "\mu_{r'\ell'}"] \\
E(r\ell, r\ell) \odot E(r\ell, r\ell) \arrow[r, "\mu_{r\ell}"'] \arrow[d, "E(r\ell,r\ell)\odot E(r\ell,\rho\ell)"'] & E(r\ell, r\ell) \arrow[dr, "E(r\ell,\rho\ell)"] & \\
E(r\ell, r\ell) \odot E(r\ell, r'\ell') \arrow[rr, "\mu_{r\ell}"'] & & E(r\ell, r'\ell')
\end{tikzcd}$$

And therefore also the following:

$$\begin{tikzcd}
B(\ell, \ell) \arrow[r, "\overline{r'b}"] \arrow[d, equal] & E(r'\ell', r'\ell') \arrow[r, equal] & A(1, 1) \odot E(r'\ell', r'\ell') \arrow[d, "\overline{r\ell}\odot E(r'\ell',r'\ell')"] \\
A(1, 1) \odot B(\ell, \ell) \arrow[d, "\overline{r\ell}\odot B(\ell,\ell)"'] & & B(r\ell, r\ell) \odot E(r'\ell', r'\ell') \arrow[d, "E(r\ell,\rho\ell)\odot E(r'\ell',r'\ell')"] \\
E(r\ell, r\ell) \odot B(\ell, \ell) \arrow[rr, "E(r\ell,\rho\ell)\odot\overline{r'b}"] \arrow[d, "E(r\ell,r\ell)\odot\overline{r}"'] & & E(r\ell, r'\ell') \odot E(r'\ell', r'\ell') \arrow[dd, "\mu_{r'\ell'}"] \\
E(r\ell, r\ell) \odot E(r\ell, r\ell) \arrow[d, "E(r\ell,r\ell)\odot E(r\ell,\rho\ell)"'] & & \\
E(r\ell, r\ell) \odot E(r\ell, r'\ell') \arrow[rr, "\mu_{r\ell}"'] & & E(r\ell, r'\ell')
\end{tikzcd}$$

Which, by the cancellation law for composition, is equivalent to commutativity of the following diagram.

$$\begin{tikzcd}
B(\ell, \ell) \arrow[r, "\overline{r'b}"] \arrow[dd, "\overline{r}"'] & E(r'\ell', r'\ell') \arrow[r, equal] & A(1, 1) \odot E(r'\ell', r'\ell') \arrow[d, dashed, "\overline{r\ell}\odot E(r'\ell',r'\ell')"] \\
& & E(r\ell, r\ell) \odot E(r'\ell', r'\ell') \arrow[d, dashed, "E(r\ell,\rho\ell)\odot E(r'\ell',r'\ell')"] \\
E(r\ell, r\ell) \arrow[r, "E(r\ell,\rho\ell)"'] & E(r\ell, r'\ell') & E(r\ell, r'\ell') \odot E(r'\ell', r'\ell') \arrow[l, "\mu_{r'\ell'}"]
\end{tikzcd}$$

Furthermore, since $\sharp$ and $\sharp'$ are invertible, the following diagram commutes.

$$\begin{tikzcd}
B(\ell, 1) \arrow[d, "\bar{b}"'] & E(j, r) \arrow[l, "\flat"'] \arrow[d, "E(j,\rho)"] \\
B'(\ell', b) & E(j, r'b) \arrow[l, "\flat'"]
\end{tikzcd}$$

Therefore, the following diagram commutes, demonstrating that the operator for the induced j-monad is preserved.

$$\begin{tikzcd}
E(j, r\ell) \arrow[r, "E(j,\rho\ell)"] \arrow[d, "\flat"'] & E(j, r'\ell') \arrow[r, "\flat'"] & B'(\ell', \ell') \arrow[r, "\overline{r'}"] & E(r'\ell', r'\ell') \arrow[dd, "E(\rho\ell,r'\ell')"] \\
B(\ell, \ell) \arrow[urr, "\bar{b}"] \arrow[urrr, "\overline{r'b}"'] \arrow[d, "\overline{r}"'] & & & \\
E(r\ell, r\ell) \arrow[rrr, "E(r\ell,\rho\ell)"'] & & & E(r\ell, r'\ell')
\end{tikzcd}$$

$\square$

When j is dense, Lemma 5.2.4 states that right relative adjoints are uniquely determined by their left relative adjoints. A similar statement holds for right-morphisms of relative adjunctions: when j is dense, the 2-cell between the right relative adjoints is uniquely determined by the 1-cell between the apices.

Lemma 5.2.15. *Let $j \colon A \to E$ be a dense 1-cell in $\mathcal{K}$. A right-morphism of j-adjunctions (b, ρ) from $\ell \;{}_j\!\dashv r$ to $\ell' \;{}_j\!\dashv r'$ is uniquely determined by b. Conversely, every 1-cell b under A underlies a right-morphism of j-adjunctions.*

Proof. The right-morphism axiom requires the following diagram to commute:

$$\begin{tikzcd} B(\ell, 1) \arrow[r, "\sharp"] \arrow[d, "\bar{b}"'] & E(j, r) \arrow[d, "E(j,\rho)"] \\ B'(\ell', b) \arrow[r, "\sharp'"'] & E(j, r'b) \end{tikzcd}$$

Since $\sharp$ is invertible, this is equivalent to commutativity of the following diagram:

$$\begin{tikzcd} B(\ell, 1) \arrow[d, "\bar{b}"'] & E(j, r) \arrow[l, "\flat"'] \arrow[d, "E(j,\rho)"] \\ B'(\ell', b) \arrow[r, "\sharp'"'] & E(j, r'b) \end{tikzcd}$$

Hence $E(j, \rho)$ is uniquely determined by b. Since j is dense, the 2-cell ρ is consequently also uniquely determined by b, and conversely. $\square$

Consequently, we have the following.

Corollary 5.2.16. *Let $j \colon A \to E$ be a dense 1-cell. The forgetful 1-functor $\mathbf{RAdj}_r(j) \to A/\mathcal{K}$, given by sending a j-adjunction to its left adjoint and a right-morphism (b, ρ) to b is fully faithful.*

Proof. By Lemma 5.2.15, right-morphisms are determined by their 1-cells when j is dense, and so the function $(b, \rho) \mapsto b$ is bijective. $\square$

As mentioned earlier, we will be interested in 2-categories admitting initial resolutions of relative monads, which are appropriately functorial with respect to relative monad morphisms. We axiomatise this property through adjointness as follows.

Definition 5.2.17. Let $\mathcal{K}$ be a 2-category and let $j \colon A \to E$ be a 1-cell. $\mathcal{K}$ *admits Kleisli constructions for j-monads* if $\oslash \colon \mathbf{RAdj}_r(j) \to \mathbf{RMnd}(j)$ has a fully faithful left adjoint, which we denote $\mathbf{Kl}_j \colon \mathbf{RMnd}(j) \to \mathbf{RAdj}_r(j)$, and if the transpose of an identity j-monad morphism is a strict right-morphism.

Conceptually, a fully faithful left adjoint to $\oslash$ gives an assignment of a resolution to each monad, as well as an assignment of a right-morphism between Kleisli resolutions given a relative monad morphism between the corresponding relative monads; asking that the transpose of an invertible morphism be strict means that right-morphisms between resolutions of the same relative monad are strict: this is necessary for initiality of the Kleisli resolution.

The following lemma is useful in practice to establish that particular 2-categories admit Kleisli constructions.

Lemma 5.2.18. *The following are equivalent.*

1. *$\mathcal{K}$ admits Kleisli constructions for j-monads.*
2. (a) *Every j-monad admits an initial resolution;*

(b) (invertible) j-monad morphisms are in natural bijection with (strict) right-morphisms between their initial resolutions;

(c) for every j-monad T and j-adjunction $\ell' \ {}_j\!\dashv r'$ with apex B', a right-morphism $\mathbf{Kl}(T) \to B'$ has a unique lift along $[]_{T'}$:

$$\begin{tikzcd}
& \mathbf{Kl}(T') \arrow[dr, "{[]_{T'}}"] & \\
\mathbf{Kl}(T) \arrow[ur, dashed] \arrow[rr, "b"] & & B' \\
& A \arrow[ul, "k_T"] \arrow[ur, "\ell'"'] &
\end{tikzcd}$$

Proof. (1) $\implies$ (2).

(a) Let T be a j-monad, let $\ell \ {}_j\!\dashv r$ be a resolution of T, and denote by $k_T \ {}_j\!\dashv v_T$ the j-adjunction $\mathbf{Kl}_j(T)$ with apex $\mathbf{Kl}(T)$. Since $\mathbf{Kl}_j \dashv \oslash$, we have a natural isomorphism $\mathbf{RAdj}_r(j)(\mathbf{Kl}_j(T), \ell \ {}_j\!\dashv r) \cong \mathbf{RMnd}(j)(T, T)$, and so the identity 1_T exhibits a right-morphism $\mathbf{Kl}(T) \to B$, which is strict by assumption. Conversely, let $b \colon \mathbf{Kl}(T) \to B$ be a strict morphism of j-adjunctions, hence a right-morphism of j-adjunctions. We have an induced j-monad morphism $T \Rightarrow T$, which is the identity because the morphism was strict. Thus $\mathbf{Kl}_j(T)$ is initial amongst resolutions of T.

(b) For the bijection between j-monad morphisms and right-morphisms between Kleisli resolutions, observe that $\mathbf{RAdj}_r(j)(\mathbf{Kl}_j(T), \mathbf{Kl}_j(T')) \cong \mathbf{RMnd}(j)(T, T')$. Furthermore, this bijection preserves and reflects isomorphisms, since $\mathbf{Kl}_j$ is a fully faithful functor.

(c) Finally, given a right-morphism $b \colon \mathbf{Kl}(T) \to B'$, we have

$$\mathbf{RAdj}_r(j)(\mathbf{Kl}_j(T), \ell' \ {}_j\!\dashv r') \cong \mathbf{RMnd}(j)(T, T') \cong \mathbf{RAdj}_r(j)(\mathbf{Kl}_j(T), \mathbf{Kl}_j(T'))$$

and so there is a right-morphism $\mathbf{Kl}_j(T) \to \mathbf{Kl}_j(T') \to B'$ necessarily equal to b since it induces the same j-monad morphism. Therefore b factors through the lift uniquely.

(2) $\implies$ (1). We define a functor $\mathbf{Kl}_j \colon \mathbf{RMnd}(j) \to \mathbf{RAdj}_r(j)$ as follows. On objects, $\mathbf{Kl}_j(T)$ is given by the initial resolution $k_T \ {}_j\!\dashv v_T$. By assumption, we have that j-monad morphisms are in bijection with right-morphisms between Kleisli resolutions, which defines the action on morphisms. This functor is a section of $\oslash$, since each j-monad is sent to a resolution, so that $\mathbf{Kl}_j$ is fully faithful. Finally, to each right-morphism $\mathbf{Kl}(T) \to B'$, we have a right-morphism $\mathbf{Kl}(T) \to \mathbf{Kl}(T')$, hence a j-monad morphism $T \Rightarrow T'$; conversely, given an (invertible) j-monad morphism $T \Rightarrow T'$, hence an (invertible) right-morphism $\mathbf{Kl}(T) \to \mathbf{Kl}(T')$, we obtain a (strict) right-morphism $\mathbf{Kl}(T) \to B'$ by postcomposing $[]_{T'}$. This correspondence is naturally bijective, and exhibits $\mathbf{Kl}_j$ as left-adjoint to $\oslash$. □

The following consequence will be crucial to the formal relative monad–theory correspondence: it states that relative monads and their morphisms may be represented by their relative left adjoints and right-morphisms between them, at least when j is dense.

Corollary 5.2.19. *Let $j \colon A \to E$ be a dense 1-cell and suppose that $\mathcal{K}$ admits Kleisli constructions for j-monads. There is a fully faithful functor $\mathbf{RMnd}(j) \to A/\mathcal{K}$ sending each j-monad to the Kleisli inclusion $k_T \colon A \to \mathbf{Kl}(T)$.*

Proof. There is a fully faithful functor $\mathbf{RMnd}(j) \to \mathbf{RAdj}_r(j)$ by Lemma 5.2.18(a & b), which we compose with the fully faithful functor $\mathbf{RAdj}_r(j) \to A/\mathcal{K}$ of Corollary 5.2.16. □

We shall use left-morphisms of relative adjunctions only sparingly, so we omit a detailed treatment; for now, we shall content ourselves simply by defining them. In addition, we take the unit preservation condition to be primary here, rather than a condition involving preservation of the transposition operators: this is purely for subsequent convenience.

Definition 5.2.20. Let $j \colon A \to E$ be a 1-cell in $\mathcal{K}$. A *left-morphism* of j-adjunctions from $\ell \mathbin{{}_j\dashv} r$ to $\ell' \mathbin{{}_j\dashv} r'$ comprises

$$\begin{tikzcd} & B' \arrow[dr, "r'"] & \\ A \arrow[ur, "\ell'"] \arrow[r, "\ell"'] \arrow[r, phantom, shift left=3, "\Searrow"] & B \arrow[u, "b"] \arrow[r, "r"'] & E \end{tikzcd}$$

1. a 1-cell $b \colon B \to B'$ such that $b\, ;\, r' = r$;
2. a 2-cell $\lambda \colon \ell' \Rightarrow \ell\, ;\, b$,

such that

$$\begin{tikzcd} & B \arrow[d, "b"] \arrow[ddr, bend left, "r"] & \\ & B' \arrow[dr, "r'"] \arrow[d, phantom, "\Uparrow \eta'"] & \\ A \arrow[uur, bend left, "\ell"] \arrow[ur, "\ell'"] \arrow[ur, phantom, shift left=4, "\Uparrow \lambda"] \arrow[rr, "j"'] & {} & E \end{tikzcd} \quad = \quad \begin{tikzcd} & B' \arrow[dr, "r"] \arrow[d, phantom, "\eta \Uparrow"] & \\ A \arrow[ur, "\ell"] \arrow[rr, "j"'] & {} & E \end{tikzcd}$$

It is *strict* if λ is the identity. j-adjunctions and left-morphisms form a 1-category $\mathbf{RAdj}_l(j)$.

5.3 Algebras

Just as a 2-category may admit Kleisli constructions for relative monads, so too may it admit Eilenberg–Moore constructions. However, in some instances, these universal properties are too weak. Recall that Kleisli objects and Eilenberg–Moore objects for (non-relative) monads may be characterised in two ways: either as the apices of initial and terminal resolutions respectively, or as universal right- and left-modules (i.e. (op)algebras). The former is more commonly encountered in classical category theory, while the latter is the property axiomatised by Street in the formal theory of monads [Str72a, §1]. From this property, it is possible to prove the resolution-based universal property [Str72a, Theorem 3], and vice versa [Aud74, Théoreme 4.4]. However, this correspondence makes crucial use of the fact that Eilenberg–Moore objects may be characterised representably as Eilenberg–Moore categories for monads on hom-categories [Str72a, Theorem 8]. This fact does not evidently carry across to relative monads[11]. Therefore, the two definitions of Kleisli and Eilenberg–Moore objects for monads generalise to relative monads seemingly in two different directions. We use *Kleisli object* and *Eilenberg–Moore object* to refer to the definitions in terms of universal resolutions, as this aligns with the original motivation for Kleisli and Eilenberg–Moore categories; and use *opalgebra-object* and *algebra-object* for the definitions in terms of universal right- and left-modules, as this aligns with Street's terminology. For our purposes, it suffices to consider for Kleisli objects only the resolution universal property, but we shall require Eilenberg–Moore objects with both universal properties.

To define algebra-objects, we must introduce the notion of left-module for a relative monad.

Definition 5.3.1 (cf. [Lob20, Definitions 4.1 & 4.2])**.** Let T be a j-monad and let M be an object of $\mathcal{K}$. An *M-indexed left-module* for T comprises

1. a 1-cell $m \colon M \to E$;
2. a 2-cell $\ddagger \colon E(j, m) \Rightarrow E(t, m)$,

such that

[11] We shall spell out where the problem arises for the interested reader. Given a relative monad T in $\mathcal{K}$ and an object $X \in \mathcal{K}$, there is a relative monad $\mathcal{K}[X, T]$ in $\mathcal{K}$ induced by postcomposition. Furthermore, the left-modules for T (Definition 5.3.1) induce algebras for $\mathcal{K}[X, T]$. However, the converse is not true in general, essentially because operators in the sense of [Lob20, Definition 1.2] are not in bijection with 2-cells (in other words, the analogue of [MV17, Lemma 2.2] for arbitrary operators does not hold). Therefore, the desired isomorphism corresponding to [Str72a, Theorem 8] does not hold in general for relative monads.

1.

$$\begin{tikzcd} E(j,m) \arrow[r, "\ddagger"] \arrow[dr, equal] & E(t,m) \arrow[d, "E(\tau,m)"] \\ & E(j,m) \end{tikzcd}$$

2.

$$\begin{tikzcd} E(j,t) \odot E(j,m) \arrow[r, "E(j,t)\odot\ddagger"] \arrow[d, "\dagger\odot\ddagger"'] & E(j,t) \odot E(t,m) \arrow[r, "\mu_t"] & E(j,m) \arrow[d, "\ddagger"] \\ E(t,t) \odot E(t,m) \arrow[rr, "\mu_t"'] & & E(t,m) \end{tikzcd}$$

A *morphism* of M-indexed left-modules for T from $(m, \ddagger)$ to $(m', \ddagger')$ is a 2-cell $\psi \colon m \Rightarrow m'$ such that the following diagram commutes:

$$\begin{tikzcd} E(j,m) \arrow[r, "E(j,\psi)"] \arrow[d, "\ddagger"'] & E(j,m') \arrow[r, "\ddagger'"] & E(t',m') \arrow[d, "E(\psi,m')"] \\ E(t,m) \arrow[rr, "E(t,\psi)"'] & & E(t,m') \end{tikzcd}$$

M-indexed left-modules for T and their morphisms form a 1-category $\mathbf{Mod}_l(T, M)$, functorial contravariantly in T and M.

There are two canonical examples of left-modules for relative monads. First, every relative monad forms a left-module for itself, akin to how every monoid forms a left-action for itself.

Lemma 5.3.2. *Let $T = (t, \tau, \dagger)$ be a j-monad. Then $(t, \dagger)$ is an A-indexed left-module for T.*

Proof. The unit and compatibility axioms follow directly from the left-unit and associativity axioms for a relative monad. □

Second, every resolution of a relative monad induces a left-module for that relative monad.

Lemma 5.3.3. *Let $\ell \;{}_j\!\dashv\; r$ be a relative adjunction inducing a j-monad T. Then r may be equipped with the structure of a left-module for T.*

Proof. The extension operator $\ddagger \colon E(j, r) \Rightarrow E(r\ell, r)$ is given by

$$E(j,r) \overset{\flat}{\Rightarrow} B(\ell, 1) \xRightarrow{B(\ell,1)\odot\overline{r}} E(r\ell, r)$$

The left-module axioms are then given as in the proofs of the first and third axioms for a relative monad in Proposition 5.2.9. □

When j is the identity, left-modules (morphisms) reduce to the usual notion of left-module (morphism) for a monad in essentially the same way as Proposition 5.2.8: a 2-cell $E(1, m) \Rightarrow E(t, m)$ is equivalent to a 2-cell $E(1, t\,; m) \Rightarrow E(1, m)$ by transposition, and hence a 2-cell $t\,; m \Rightarrow m$ in $\mathcal{K}$, and the axioms correspond by the usual manipulation of units and counits. We define an algebra-object to be a universal left-module (cf. [KS74, §3.3]).

Definition 5.3.4 (cf. [Lob20, Definition 4.3])**.** Let T be a j-monad. An *algebra-object* for T is a 2-representation for $\mathbf{Mod}_l(T, -) \colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$.

Suppose that $\mathbf{Alg}(T)$ is an algebra-object for a j-monad T. The identity on $\mathbf{Alg}(T)$ induces an $\mathbf{Alg}(T)$-indexed left-module for T, i.e. a 1-cell $u_T \colon \mathbf{Alg}(T) \to E$ and a 2-cell $E(j, u_T) \Rightarrow E(t, u_T)$. In addition, the A-indexed left-module $(t, \dagger)$ induces a 1-cell $f_T \colon A \to \mathbf{Alg}(T)$ such that $f_T \,;\, u_T = t$. Therefore, we have a 2-cell $\tau \colon j \Rightarrow f_T \,;\, u_T$ given by the unit of T, hence a 2-cell $\sharp \colon \mathbf{Alg}(T)(f_T, 1) \Rightarrow E(j, u_T)$ by Lemma 5.2.3. It does not appear to be the case that $f_T \;{}_j\!\dashv\; u_T$ in general, i.e. that $\sharp$ is invertible[12]; however, we shall require this stronger condition to hold.

Definition 5.3.5. An algebra-object $\mathbf{Alg}(T)$ is *Eilenberg–Moore* if $\sharp \colon \mathbf{Alg}(T)(f_T, 1) \Rightarrow E(j, u_T)$ has an inverse, denoted $\flat \colon E(j, u_T) \Rightarrow \mathbf{Alg}(T)(f_T, 1)$, and if

$$E(j, t) \xRightarrow{\flat f_T} \mathbf{Alg}(T)(f_T, f_T) \xRightarrow{\overline{u}_T} E(t, t)$$

is equal to $\dagger$.

Note that, since $f_T \,;\, u_T = t$, the relative adjunction associated to an Eilenberg–Moore algebra-object for T is necessarily a resolution of T (since the units and extension coincide). Furthermore, it is necessarily a terminal resolution, justifying our nomenclature.

Lemma 5.3.6. *Let $\mathbf{Alg}(T)$ be an Eilenberg–Moore algebra-object for T. The resolution $f_T \;{}_j\!\dashv\; u_T$ is terminal in $\mathbf{Res}(T)$.*

Proof. Let $\ell \;{}_j\!\dashv\; r$ be a resolution of T. Then r forms a left-module for T by Lemma 5.3.3, and so there is a unique 1-cell $\langle\rangle_T \colon B \to \mathbf{Alg}(T)$ such that $\langle\rangle_T ; u_T = r$ and $\langle\rangle_T ; \ddagger = B(\ell, 1) \odot \overline{r}$. We have $\ell ; \langle\rangle_T ; u_T = \ell ; r = t$ and likewise for the extension operator $\dagger$, so that $\ell ; \langle\rangle_T$ induces the same left-module as f_T, so that $\ell ; \langle\rangle_T = f_T$. Therefore, $\langle\rangle_T$ is a morphism of resolutions from $\ell \;{}_j\!\dashv\; r$ to $f_T \;{}_j\!\dashv\; u_T$. Conversely, for any such morphism $b \colon B \to \mathbf{EM}(T)$ of resolutions, we have $b ; \ddagger = B(\ell, 1) \odot \overline{r}$ since $\ddagger$ is induced by $f_T \;{}_j\!\dashv\; u_T$, so that $b = \langle\rangle_T$. □

Eilenberg–Moore algebra-objects satisfy a useful universal property with respect to left-morphisms of relative adjunctions. The reader should note the strong resemblance of the following to Lemma 5.2.18(2)(c) (we note that methods similar to those used in the proof of that lemma could be used to give an alternative proof of the following).

Lemma 5.3.7. *Let $j \colon A \to E$ be a 1-cell and let T and T' be j-monads having Eilenberg–Moore algebra-objects. Consider a resolution $\ell \;{}_j\!\dashv\; r$ of T. For every left-morphism $b \colon B \to \mathbf{EM}(T')$, there is a unique extension along $\langle\rangle_{T'}$:*

$$\begin{tikzcd}
 & \mathbf{EM}(T) \arrow[dr, dashed] & \\
B \arrow[ur, "\langle\rangle_T"] \arrow[rr, "b"] \arrow[dr, "r"'] & & \mathbf{EM}(T') \arrow[dl, "u_{T'}"] \\
 & E &
\end{tikzcd}$$

Proof. Observe that from the left-morphism (b, λ), we can construct a j-monad morphism

$$t' = f_{T'} \,;\, u_{T'} \xRightarrow{\lambda} \ell \,;\, b \,;\, u_{T'} = \ell \,;\, r = t$$

The Eilenberg–Moore object $\mathbf{EM}(T)$ is equipped with a 2-cell $E(j, u_T) \Rightarrow E(t, u_T)$, from which we construct a 2-cell

$$E(j, u_T) \Rightarrow E(t, u_T) \Rightarrow E(t', u_T)$$

by precomposition, equipping $u_T \colon \mathbf{EM}(T) \to E$ with the structure of a left-module for T'. Thus there is an induced 1-cell $\mathbf{EM}(T) \to \mathbf{EM}(T')$ from the universal property of $\mathbf{EM}(T')$, which is a unique extension of b along $\langle\rangle_T$, again by the universal property. □

[12] This may suggest that our definition of algebra-object is not quite correct, and that we should be looking for a definition involving the structure of the proarrow equipment. At present, it is not clear for what structure we should be looking, and so we leave this as an open question.

While it is possible to describe the functoriality of the association of (Eilenberg–Moore) algebra-objects to relative monads (in which case, we obtain a 2-adjoint characterisation of the construction of algebras similar to that in [Str72a, §1]), we shall not need it for the purposes of the formal monad–theory correspondence, and therefore omit a treatment here. However, we shall record one property that we derive here directly, but which may be derived by more general methods, along the lines of Corollary 5.2.19, given a 2-functorial understanding of algebra-objects.

Proposition 5.3.8. *Let $j\colon A \to E$ be a 1-cell and assume $\mathcal{K}$ admits algebra-objects for every j-monad. The assignment of $u_T\colon \mathbf{Alg}(T) \to E$ to a j-monad T extends to a fully faithful functor $\mathbf{Alg}_j\colon \mathbf{RMnd}(j)^{\mathrm{op}} \to \mathcal{K}/E$.*

Proof. Let T and T' be j-monads and let $\varphi\colon T \to T'$ be a j-monad morphism. We may construct a 1-cell $\mathbf{Alg}(T') \to \mathbf{Alg}(T)$ as follows. We have a 2-cell

$$E(j, u_{T'}) \overset{\ddagger}{\Rightarrow} E(t', u_{T'}) \xRightarrow{E(\varphi, u_{T'})} E(t, u_{T'})$$

which exhibits $u_{T'}$ as an $\mathbf{Alg}(T')$-indexed left-module for T, the left-module laws following from those for $u_{T'}$ as a left-module for T'. By the universal property of $\mathbf{Alg}(T)$, there is therefore an induced 1-cell $!\colon \mathbf{Alg}(T') \to \mathbf{Alg}(T)$ satisfying $!\,;u_T = u_{T'}$. Functoriality follows by the uniqueness of induced 1-cells by the universal property of $\mathbf{Alg}(T)$.

Conversely, suppose that $m\colon \mathbf{Alg}(T') \to \mathbf{Alg}(T)$ is a 1-cell over E. By the universal property of $\mathbf{Alg}(T)$, this corresponds to an $\mathbf{Alg}(T')$-indexed left-module for T, i.e. a 1-cell $u_{T'} = m\,;u_T$ and 2-cell $m\,;\ddagger\colon E(j, m\,; u_T) \Rightarrow E(t, m\,;u_T)$ satisfying the left-module laws. Precomposing by $f_{T'}$, since $f_{T'}\,;m\,;u_T = f_{T'}\,;u_{T'} = t'$, we obtain a 2-cell $f_{T'}\,;m\,;\ddagger\colon E(j, t') \Rightarrow E(t, t')$, from which applying $\tau'\colon j \Rightarrow t'$ produces a 2-cell $t \Rightarrow t'$ satisfying the laws for a j-monad morphism. That these two constructions are mutually inverse follows again from the uniqueness property of the universal property of $\mathbf{Alg}(T)$. □

Remark 5.3.9. While it is possible to define opalgebra-objects, we shall not need them for the purposes of the formal monad–theory correspondence, and therefore omit a treatment here. The interested reader is referred to Lobbia [Lob20, Definitions 6.1, 6.2 & 6.3] from which the appropriate definitions may be derived.

5.4 Restriction and extension

To prove a formal monad–theory correspondence, it will be necessary to relate relative monads to non-relative monads. The crucial invariant in any such relationship is the preservation of algebras: this is crucial, because the monad–theory correspondence should commute with the construction of algebras. To this end, we introduce the concepts of *j-ary monads* and *realisable j-monads*. Intuitively, for $j\colon A \to E$ a 1-cell in $\mathcal{K}$, a monad on E is j-ary when its algebras are determined by its action on A.

Definition 5.4.1. Let $j\colon A \to E$ be a 1-cell. A monad T' on E admitting an Eilenberg–Moore algebra-object is *j-ary* when there exists a j-monad T (necessarily unique up to isomorphism) admitting an algebra-object, and an isomorphism $\mathbf{EM}(T) \cong \mathbf{EM}(T')$ in $\mathcal{K}/E$. Denote by $\mathbf{Mnd}_j(E)$ the full subcategory of $\mathbf{Mnd}(E)$ spanned by the j-ary monads.

Conversely, a relative monad is realisable when it can be extended to a monad with the same algebras.

Definition 5.4.2. Let $j\colon A \to E$ be a 1-cell. A j-monad T admitting an Eilenberg–Moore algebra-object is *realisable* when there exists a monad T' on E (necessarily unique up to isomorphism) admitting an algebra-object, and an isomorphism $\mathbf{EM}(T) \cong \mathbf{EM}(T')$ in $\mathcal{K}/E$. Denote by $\mathbf{RMnd}^E(j)$ the full subcategory of $\mathbf{RMnd}(j)$ spanned by the realisable j-monads.

The categories of realisable j-monads and the j-ary monads are equivalent, essentially by definition. This equivalence will provide the core of the formal monad–theory correspondence, once we identify theories with relative monads.

Proposition 5.4.3. *The categories of realisable j-monads and j-ary monads are equivalent:*

$$\mathbf{RMnd}^E(j) \simeq \mathbf{Mnd}_j(E)$$

and this equivalence commutes with taking algebras.

Proof. By definition, there is a fully faithful assignment $\mathbf{RMnd}^E(j) \to \mathbf{Mnd}(E)$ commuting with taking algebras, the action on j-monad morphisms induced by full faithfulness of $\mathbf{Alg}_j : \mathbf{RMnd}^E(j)^{\mathrm{op}} \to \mathcal{K}/E$ and $\mathbf{Alg}_{1_E} : \mathbf{Mnd}(E)^{\mathrm{op}} \to \mathcal{K}/E$ (Proposition 5.3.8). Necessarily, each monad in the image of this assignment is j-ary, defining a retraction. Conversely, each j-ary monad T is assigned to a realisable j-monad that is realised by a monad T' such that $\mathbf{Alg}(T) \cong \mathbf{Alg}(T')$. Therefore $T \cong T'$ by Proposition 5.3.8, exhibiting the assignment as an equivalence. □

There is an elementary characterisation of the realisable monads, assuming algebra-objects are Eilenberg–Moore, as demonstrated by the following lemma.

Lemma 5.4.4. *Let $j : A \to E$ be a 1-cell, and let T be a j-monad admitting an algebra-object. Assume that algebra-objects for relative monads are Eilenberg–Moore. Then T is realisable if and only if u_T has a left adjoint. In this case, u_T is strictly monadic, and T is realised by the induced monad.*

$$\begin{tikzcd}
& \mathbf{EM}(T) \arrow[d, bend left, "u_T"] \arrow[d, phantom, "\dashv"] \\
A \arrow[ur, "f_T"] \arrow[r, "j"'] & E \arrow[u, dashed, bend left]
\end{tikzcd}$$

Proof. Assume there exists a 1-cell $f : E \to \mathbf{EM}(T)$ such that $f \dashv u_T$, and denote by T' the induced monad. Since $f_T \;{}_j\!\dashv u_T$ and $j \,;\, f \;{}_j\!\dashv u_T$, we have $f_T \cong j \,;\, f$ by uniqueness of relative left adjoints. Hence, by terminality of $\mathbf{EM}(T')$, there is a unique 1-cell $! : \mathbf{EM}(T) \to \mathbf{EM}(T')$ making the following diagram commute.

$$\begin{tikzcd}
& E \arrow[dl, "f"'] \arrow[dr, "f_{T'}"] & \\
\mathbf{EM}(T) \arrow[rr, "!"] \arrow[dr, "u_T"'] & & \mathbf{EM}(T') \arrow[dl, "u_{T'}"] \\
& E &
\end{tikzcd}$$

Conversely, by terminality of $\mathbf{EM}(T)$, there is a unique 1-cell $¡ : \mathbf{EM}(T') \to \mathbf{EM}(T)$ making the following diagram commute.

$$\begin{tikzcd}
& A \arrow[dl, "j;f_{T'}"'] \arrow[dr, "j;f"] & \\
\mathbf{EM}(T') \arrow[rr, "¡"] \arrow[dr, "u_{T'}"'] & & \mathbf{EM}(T) \arrow[dl, "u_T"] \\
& E &
\end{tikzcd}$$

However, clearly $! \,;\, ¡ : \mathbf{EM}(T) \to \mathbf{EM}(T)$ is a morphism of j-adjunctions from $j \,;\, f \;{}_j\!\dashv u_T$ to $j \,;\, f \;{}_j\!\dashv u_T$, and hence the identity by terminality of $\mathbf{EM}(T)$. Conversely, $¡ \,;\, ! : \mathbf{EM}(T') \to \mathbf{EM}(T')$ is a morphism of adjunctions from $f_{T'} \dashv u_{T'}$ to $f \,;\, ! \dashv u_{T'}$, since the following diagram commutes, and hence the identity by terminality of $\mathbf{EM}(T')$.

$$\begin{tikzcd}
& E \arrow[dl, "f"'] \arrow[dr, "f_{T'}"] & & \\
\mathbf{EM}(T) \arrow[rr, "!"] \arrow[rrr, equal, bend right] & & \mathbf{EM}(T') \arrow[r, "¡"] & \mathbf{EM}(T) \arrow[r, "!"] & \mathbf{EM}(T')
\end{tikzcd}$$

Therefore $!$ is an isomorphism with $!^{-1} = ¡$, exhibiting u_T as strictly monadic, and so $\mathbf{EM}(T) \cong \mathbf{EM}(T')$ in $\mathcal{K}/E$, exhibiting T as realisable.

Conversely, if T is realisable, then there exists a monad T' such that $\mathbf{EM}(T) \cong \mathbf{EM}(T')$ in $\mathcal{K}/E$. Since $u_{T'}$ has a left adjoint, so does u_T, inherited from $u_{T'}$ via the isomorphism. □

We may also give several alternative characterisations of j-ary monads. First, we show that every j-ary monad is exhibited by a canonical j-monad, and then give several sufficient conditions for demonstrating that a monad is j-ary.

Let T be a monad on E admitting an algebra-object and suppose that the j-monad $j\,;T$ also admits an algebra-object. Then the 2-cell $E(1, u_T) \Rightarrow E(t, u_T)$ induces a 2-cell $E(j, u_T) \Rightarrow E(j\,; t, u_T)$ by postcomposition with $E(j, 1)$, which is a left-module for $j\,;T$. Hence there is a canonical 1-cell $\langle\rangle_{j;T} \colon \mathbf{Alg}(T) \to \mathbf{Alg}(j\,;T)$ by the universal property of $\mathbf{Alg}(j\,;T)$. If the algebra-objects are Eilenberg–Moore, then this is equivalently the canonical 1-cell $\langle\rangle_{j;T} \colon \mathbf{EM}(T) \to \mathbf{EM}(j\,;T)$ induced by precomposing the j-adjunction $f_T\ {}_j\!\dashv u_T$ by j.

Lemma 5.4.5. *Let T be a monad on E admitting an algebra-object and suppose that the j-monad $j\,;T$ also admits an algebra-object. Assume that algebra-objects for relative monads are Eilenberg–Moore. Then T is j-ary if and only if the canonical 1-cell $\langle\rangle_{j;T} \colon \mathbf{EM}(T) \to \mathbf{EM}(j\,;T)$ is invertible.*

Proof. If $\langle\rangle_{j;T} \colon \mathbf{EM}(T) \to \mathbf{EM}(j\,;T)$ is invertible, then T is j-ary by definition.

Conversely, supposing that T is j-ary, there is a j-monad T' and invertible 1-cell $! \colon \mathbf{EM}(T) \rightleftarrows \mathbf{EM}(T') : ¡$ over E. By the universal property of $\mathbf{EM}(T')$ in Lemma 5.3.7, we have a commutative triangle as follows.

$$\begin{tikzcd}
 & \mathbf{EM}(j\,;T) \arrow[dr, "\overline{!}"] & \\
\mathbf{EM}(T) \arrow[ur, "\langle\rangle_{j;T}"] \arrow[rr, "!"'] & & \mathbf{EM}(T')
\end{tikzcd}$$

We have

$$\langle\rangle_{j;T}\,;\overline{!}\,;¡ = !\,;¡ = 1_{\mathbf{EM}(T)}$$

Conversely, by the universal property of $\mathbf{EM}(j\,;T)$, there is a unique 1-cell $\overline{\langle\rangle_{j;T}}$ making the following triangle commute, which is hence necessarily the identity.

$$\begin{tikzcd}
 & \mathbf{EM}(j\,;T) \arrow[dr, "\overline{\langle\rangle_{j;T}}"] & \\
\mathbf{EM}(T) \arrow[ur, "\langle\rangle_{j;T}"] \arrow[rr, "\langle\rangle_{j;T}"'] & & \mathbf{EM}(j\,;T)
\end{tikzcd}$$

Therefore, since

$$\langle\rangle_{j;T}\,;\overline{!}\,;¡\,;\langle\rangle_{j;T} = \langle\rangle_{j;T}$$

we have

$$\overline{!}\,;¡\,;\langle\rangle_{j;T} = \overline{\langle\rangle_{j;T}} = 1_{\mathbf{EM}(j;T)}$$

exhibiting $\langle\rangle_{j;T}$, and hence $\overline{!}$, as invertible. □

Theorem 5.4.6. *Let T' be a j-monad admitting an algebra-object. Assume that algebra-objects for relative monads are Eilenberg–Moore. The following are equivalent for each monad T on E admitting an algebra-object, and uniquely identify T up to isomorphism in $\mathbf{Mnd}(E)$, exhibiting T' as realisable.*

1. *T is j-ary, and $j\,;T \cong T'$ in $\mathbf{RMnd}(j)$.*
2. *There exists an isomorphism $s \colon \mathbf{EM}(T') \cong \mathbf{EM}(T)$ such that $s\,;u_T = u_{T'}$.*
3. *There exists an equivalence $s \colon \mathbf{EM}(T') \simeq \mathbf{EM}(T)$ and isomorphism $s\,;u_T \cong u_{T'}$.*
4. *The forgetful 1-cell $u_{T'} \colon \mathbf{EM}(T') \to E'$ has a left adjoint, and the induced monad is isomorphic to T in $\mathbf{Mnd}(E)$.*

Proof. (1) $\iff$ (2). In both cases, T is j-ary, the former by assumption and the latter by definition, so that $\mathbf{EM}(T) \cong \mathbf{EM}(j\,;T)$ over E' using Lemma 5.4.5. The result then follows in both directions by applying Proposition 5.3.8 and composing with the isomorphism that was assumed. Uniqueness of T up to isomorphism is then evident.

(2) $\implies$ (3) $\implies$ (4) is clear.

(4) $\implies$ (2). By Lemma 5.4.4, $u_{T'}$ is strictly monadic, exhibiting $\mathbf{EM}(T') \cong \mathbf{EM}(T)$, since the induced monad is T. □

It will be helpful to have a method to extend relative adjunctions to adjunctions. In particular, relative adjunctions with dense roots may be extended to (non-relative) adjunctions assuming the existence of certain pointwise extensions (Definition 5.1.3).

Proposition 5.4.7. *Let $j \colon A \to E$ be a dense 1-cell, and let $\ell \; {}_j\!\dashv r$ be a relative adjunction. Then r has a left adjoint if and only if the pointwise left extension $j \rhd \ell$ exists, in which case $j \rhd \ell \dashv r$.*

$$\begin{tikzcd} & B \arrow[d, bend left, "r"] \arrow[d, phantom, "\dashv"] \\ A \arrow[ur, "\ell"] \arrow[r, "j"'] & E \arrow[u, dashed, bend left] \end{tikzcd}$$

Proof. We have

$$\begin{aligned} E(1, r) &\cong E(j \rhd j, 1) \odot E(1, r) && (j \text{ is dense}) \\ &\cong (E(j,1) \blacktriangleleft E(j,1)) \odot E(1,r) && (\text{pointwise extension}) \\ &\cong E(j, r) \blacktriangleleft E(j,1) && (\text{Lemma 5.1.6}) \\ &\cong B(\ell, 1) \blacktriangleleft E(j,1) && (\ell \; {}_j\!\dashv r) \\ &\overset{(*)}{\cong} B(j \rhd \ell, 1) && (\text{pointwise extension}) \end{aligned}$$

where the $(*)$ holds if and only if the pointwise extension $j \rhd \ell$ exists; if there exists some ℓ' such that $\ell' \dashv r$, then this has the universal property of the pointwise extension $j \rhd \ell$. □

Consequently, for particularly nice j, along which pointwise extensions are plentiful, *all* j-monads are realisable by monads. In particular, this will be shown in Section 7.3 to be the case for the monad–theory correspondences appearing in the literature for enriched categories.

Corollary 5.4.8. *Let $j \colon A \to E$ be a dense 1-cell and suppose that j-monads admit Eilenberg–Moore algebra-objects. If $\mathcal{K}$ admits all pointwise left extensions along j, then every j-monad is realisable, and there is an equivalence*

$$\mathbf{RMnd}(j) \simeq \mathbf{RMnd}^E(j) \simeq \mathbf{Mnd}_j(E)$$

Proof. Let T be a j-monad. Since the left extension $j \rhd f_T$ exists by assumption, $j \rhd f_T \dashv u_T$ by Proposition 5.4.7 and hence T is realisable by Lemma 5.4.4. The equivalence then follows from Proposition 5.4.3. □

5.5 Embedding

There remains one aspect of the formal theory of relative monads that we require for the formal monad–theory correspondence, which will facilitate a characterisation of the algebras for a theory. With this motivation in mind, we investigate the process of embedding the category of j-monads into the category of $(j \mathbin{;} j')$-monads for a fully faithful 1-cell j'.

Proposition 5.5.1. *Let $j \colon A \to E$ and $j' \colon E \to E'$ be 1-cells. Suppose that j' is fully faithful. Then there is a fully faithful functor $(-) \mathbin{;} j' \colon \mathbf{RMnd}(j) \hookrightarrow \mathbf{RMnd}(j \mathbin{;} j')$.*

Proof. Let $(t, \tau, \dagger)$ be a j-monad. We define an extension operator $\dagger \mathbin{;} j' \colon E(j \mathbin{;} j', t \mathbin{;} j') \Rightarrow E(t \mathbin{;} j', t \mathbin{;} j')$ as follows:

$$E(j \mathbin{;} j', t \mathbin{;} j') \xrightarrow{(\overline{j'})^{-1}} E(j, t) \xrightarrow{\dagger} E(t, t) \xrightarrow{\overline{j'}} E(t \mathbin{;} j', t \mathbin{;} j')$$

Then it is easy to check that $(t\, ;\, j', \tau\, ;\, j', \dagger\, ;\, j')$ forms a $(j\, ;\, j')$-monad. Similarly, given a j-monad morphism $\tau\colon t \Rightarrow t'$, the 2-cell $\tau\, ;\, j'\, ;\, t\, ;\, j' \Rightarrow t'\, ;\, j'$ is easily seen to be a $(j\, ;\, j')$-monad morphism. This assignment is trivially functorial, and furthermore fully faithful since every 2-cell $t\, ;\, j' \Rightarrow t'\, ;\, j'$ is equivalently a 2-cell $t \Rightarrow t'$ by full faithfulness of j', the former being a $(j\, ;\, j')$-monad morphism if and only if the latter is a j-monad morphism. □

This embedding does not necessarily preserve algebras. However, there is a strong relationship between algebras for j-monads and the algebras for their induced $(j\, ;\, j')$-monads, which we shall shortly use to great effect.

Theorem 5.5.2. *Let $j\colon A \to E$ be a 1-cell, let $j'\colon E \to E'$ be a fully faithful 1-cell, and let T be a j-monad. Suppose that $T\, ;\, j'$ admits an algebra-object. Then a slice $\cdot \to E$ may be equipped with an algebra-object structure for T if and only if there is a strict 2-pullback square in $\mathcal{K}$ as follows.*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}(T\, ;\, j') \arrow[d, "u_{T;j'}"] \\ E \arrow[r, "j'"'] & E' \end{tikzcd}$$

In particular, when an algebra-object for T exists, the following square is a strict 2-pullback in $\mathcal{K}$.

$$\begin{tikzcd} \mathbf{Alg}(T) \arrow[r, "\langle\rangle_{T;j'}"] \arrow[d, "u_T"'] & \mathbf{Alg}(T\, ;\, j') \arrow[d, "u_{T;j'}"] \\ E \arrow[r, "j'"'] & E' \end{tikzcd}$$

Proof. Observe first that, assuming the algebra-object exists, the bottom square commutes by the universal property of $\mathbf{Alg}(T\, ;\, j')$. We will show that the algebra-object satisfies the same universal property as the 2-pullback.

Recall that 1-cells into an algebra-object for T each comprise a 1-cell $u\colon X \to E$ with a 2-cell $E(j, u) \Rightarrow E(t, u)$. Given that j' is fully faithful, this is equivalently a 2-cell $E'(j'j, j'u) \Rightarrow E'(j't, j'u)$. Similarly, 1-cells into an algebra-object for $T\, ;\, j'$ each comprise a 1-cell $u'\colon X' \to E'$ with a 2-cell $E'(j'j, u') \Rightarrow E'(j't, u')$ satisfying the two axioms.

Now, 1-cells into the 2-pullback each comprise a 1-cell $u\colon X \to E$ and a 1-cell into $\mathbf{Alg}(T\, ;\, j')$ for which the forgetful 1-cell u' is equal to $u\, ;\, j'$: in other words, they are given by a 2-cell $E'(j'j, j'u) \Rightarrow E'(j't, j'u)$. But this is exactly the data for a 1-cell into an algebra-object for T, the axioms being satisfied by virtue of the axioms for the algebra-object for $T\, ;\, j'$. Therefore, 1-cells into the 2-pullback are in bijection with 1-cells into $\mathbf{Alg}(T)$, which is easily seen to be natural, so that the two objects satisfy the same 1-categorical universal property. Furthermore, according to the same reasoning, a 2-cell between 1-cells into the 2-pullback are in bijection with left-module morphisms for T, establishing the mutual satisfaction of the 2-categorical university property, and hence an isomorphism between the algebra-object $\mathbf{Alg}(T)$ and the apex of the specified 2-pullback. □

It is often the case that one is interested in 2-categories admitting certain bicategorical structure (for instance, pseudolimits rather than 2-limits), and in this light the previous result may seem too strict. Happily, this may be rectified. First, we shall need the following result (cf. [BG19, Example 3]).

Lemma 5.5.3. *Let $j\colon A \to E$ be a 1-cell and let T be a j-monad admitting an algebra-object. The forgetful 1-cell $u_T\colon \mathbf{Alg}(T) \to E$ is a discrete isofibration (Definition 2.3.4).*

Proof. Consider a triangle as follows, commutative up to isomorphism.

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[dr, "e"'] \arrow[r, phantom, shift right=3ex, "\cong"] & \mathbf{Alg}(T) \arrow[d, "u_T"] \\ & E \end{tikzcd}$$

By the universal property of $\mathbf{Alg}(T)$, the 1-cell $x \colon \cdot \to \mathbf{Alg}(T)$ specifies a left-module for T by postcomposition with u_T, with a 2-cell $E(j, u_T x) \Rightarrow E(t, u_t x)$. This induces a 2-cell

$$E(j, e) \cong E(j, u_T x) \Rightarrow E(t, u_t x) \cong E(t, e)$$

equipping $e \colon \cdot \to E$ with a left-module structure for T, and hence a 1-cell $\cdot \to \mathbf{Alg}(T)$ by the universal property of the latter, making the lower triangle in the following commute.

$$\begin{tikzcd} \cdot \arrow[r, bend left, "x"] \arrow[r, bend right] \arrow[r, phantom, "\cong"] \arrow[dr, "e"'] & \mathbf{Alg}(T) \arrow[d, "u_T"] \\ & E \end{tikzcd}$$

Uniqueness of this lifting follows again from the universal property. □

Corollary 5.5.4. *The strict 2-pullback of Theorem 5.5.2 is a pseudopullback.*

Proof. Since $u_{T;n_j}$ is relatively monadic, it is a discrete isofibration by Lemma 5.5.3, hence has invertible-path lifting. The result then follows by [JS93, Corollary 1]. □

Furthermore, when $\mathbf{Alg}(T \mathbin{;} j')$ is Eilenberg–Moore, so is $\mathbf{Alg}(T)$.

Proposition 5.5.5. *Let $j \colon A \to E$ be a 1-cell, let $j' \colon E \to E'$ be a dense fully faithful 1-cell, and let T be a j-monad. Suppose that T admits an algebra-object, that $T \mathbin{;} j'$ admit an Eilenberg–Moore algebra-object, and that the mediating morphism $\langle\rangle_{T;j'}$ is fully faithful. Then $f_T \;{}_j\!\dashv u_T$ if and only if $f_{T;j'} \;{}_{j;j'}\!\dashv u_{T;j'}$.*

Proof. We have:

$$\begin{aligned} \mathbf{Alg}(T)(f_T, 1) &\cong \mathbf{EM}(T \mathbin{;} j')(\langle\rangle_{T;j'} f_T, \langle\rangle_{T;j'}) \\ &\cong \mathbf{EM}(T \mathbin{;} j')(f_{T;j'}, \langle\rangle_{T;j'}) \\ &\cong E'(j'j, u_{T;j'} \langle\rangle_{T;j'}) \\ &\cong E'(j'j, j'u_T) \\ &\cong E(j, u_T) \end{aligned}$$

□

To demonstrate the value of Theorem 5.5.2, it will be useful henceforth to impose a stronger condition on our proarrow equipment, as described in Section 5.1.1, asking for it to correspond to a lax idempotent pseudomonad $(\mathcal{P}, \mu, よ)$: this is in particular the case for 2-categories of enriched categories. In this case, given a dense 1-cell $j \colon A \to E$ in $\mathcal{K}$, there is a canonical fully faithful 1-cell from E given by the nerve $n_j \colon E \to \mathcal{P}A$. We can then apply Theorem 5.5.2 taking $j' = n_j$.

The following generalises [Str74a, Theorem 35], [SW78, Proposition 22], and [Woo85, Proposition 7] from monads to relative monads.

Corollary 5.5.6. *Let $(\mathcal{P}, よ)$ be a locally fully faithful lax idempotent pseudomonad on $\mathcal{K}$ and take $(-)_* \colon \mathbf{Adm}_{\mathcal{P}}(\mathcal{K}) \to \mathcal{N}$ to be the proarrow equipment induced by the Kleisli inclusion of $\mathcal{P}$. Let $j \colon A \to E$ be a dense 1-cell in $\mathcal{K}$ with $\mathcal{P}$-small domain, and let T be a j-monad. Suppose that $T \mathbin{;} n_j$ admits an algebra-object. Then a slice $\cdot \to E$ is the algebra-object for T if and only if there is a strict 2-pullback square as follows.*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \mathbf{Alg}(T \mathbin{;} n_j) \arrow[d, "u_{T;n_j}"] \\ E \arrow[r, "n_j"'] & \mathcal{P}A \end{tikzcd}$$

Proof. Since j is dense, n_j is fully faithful, from which the result follows by Theorem 5.5.2. □

Crucially, our proof of this theorem illuminates the nature of this characterisation, originally observed in the case of non-relative monads on categories by Linton [Lin69a, Observation 1.1]. Namely, this is a special case of a more general theorem, which is only visible when viewed from the perspective of relative monads (even when we are only concerned with the case $j = 1_E$).

Remark 5.5.7. Though our characterisation in Corollary 5.5.6 of the category of algebras for a monad reduces to Linton's characterisation, it is not entirely direct. In particular, we must observe that in many 2-categories, given a monad T, we have a commutative triangle:

$$\begin{tikzcd} \mathbf{EM}(T \,;\, \text{よ}_A) \arrow[rr, "\simeq"] \arrow[dr, "u_{T;n_j}"'] & & \mathcal{P}(\mathbf{Kl}(T)) \arrow[dl, "{k_T}^*"] \\ & \mathcal{P}A & \end{tikzcd}$$

This holds in particular for $\mathcal{K} = \mathcal{W}$-$\mathbf{Cat}$, for $\mathcal{W}$ a locally cocomplete bicategory, because Kleisli objects coincide with Eilenberg–Moore objects in $\mathcal{W}$-$\mathbf{Prof}$ [CKW87, Proposition 3.3], and $\mathbf{EM}(T \,;\, \text{よ}_A) \simeq \mathbf{EM}(\mathcal{P}(T))$ and $\mathcal{P}(\mathbf{Kl}(T)) \simeq \mathbf{Kl}(\mathcal{P}(T))$, which are Eilenberg–Moore and Kleisli objects for the monad $\mathcal{P}(T)$ in $\mathcal{W}$-$\mathbf{Prof}$. We leave the study of when this property holds more generally for future work, though note it is sufficient to assume that the proarrow equipment satisfies Wood's Axiom 5 [Woo85].

A practicable consequence of this characterisation is a sufficient condition for the existence of algebra-objects for relative monads, which is frequently easily verifiable. However, we must first make note that the equivalence between realisable relative monads and j-ary monads (Proposition 5.4.3) takes on a particularly elegant form when j is the unit of a lax idempotent pseudomonad.

Theorem 5.5.8. *Let $(\mathcal{P}, \text{よ})$ be a locally fully faithful lax idempotent pseudomonad on $\mathcal{K}$ and let A be a $\mathcal{P}$-small object of $\mathcal{K}$. Take $(-)_* \colon \mathbf{Adm}_{\mathcal{P}}(\mathcal{K}) \to \mathcal{N}$ to be the proarrow equipment induced by the Kleisli inclusion of $\mathcal{P}$. Then a monad on $\mathcal{P}A$ is よ_A-ary if and only if it is $\mathcal{P}$-cocontinuous.*

Proof. By the universal property of $\mathcal{P}$, $\mathcal{P}$-cocontinuous 1-cells $\mathcal{P}A \to \mathcal{P}A$ are equivalently given by 1-cells $A \to \mathcal{P}A$, mediated by the adjoint equivalence induced by left extension.

$$\begin{tikzcd} \mathcal{K}_{\mathcal{P}}[A, A] \arrow[r, shift left=1.5, "\text{よ}_A \rhd (-)"] \arrow[r, phantom, "\simeq"] & \mathcal{K}^{\mathcal{P}}[\mathcal{P}A, \mathcal{P}A] \arrow[l, shift left=1.5, "\text{よ}_A ; (-)"] \end{tikzcd}$$

This adjoint equivalence lifts to an adjoint equivalence between $\mathcal{P}$-cocontinuous monads on $\mathcal{P}A$ and よ_A-monads.

$$\begin{tikzcd} \mathbf{RMnd}(\text{よ}_A) \arrow[r, shift left=1.5, "\text{よ}_A \rhd (-)"] \arrow[r, phantom, "\simeq"] \arrow[d] & \mathbf{Mnd}_{\mathcal{P}}(\mathcal{P}A) \arrow[l, shift left=1.5, "\text{よ}_A ; (-)"] \arrow[d] \\ \mathcal{K}_{\mathcal{P}}[A, A] \arrow[r, shift left=1.5, "\text{よ}_A \rhd (-)"] \arrow[r, phantom, "\simeq"] & \mathcal{K}^{\mathcal{P}}[\mathcal{P}A, \mathcal{P}A] \arrow[l, shift left=1.5, "\text{よ}_A ; (-)"] \end{tikzcd}$$

The proof is the same as that of [ACU15, Theorems 4.6 – 4.8], observing that よ_A is *well-behaved* in the sense of [ACU15, Definition 4.1]. Finally, this equivalence respects the process of taking algebras by the discussion in [ACU15, §4.4]. □

In particular, when j is fully faithful and $j' = n_j$, we have $j \,;\, j' \cong \text{よ}_A$, and hence $(j \,;\, j')$-monads are equivalent to $\mathcal{P}$-cocontinuous monads on $\mathcal{P}A$.

Corollary 5.5.9. *Every finitely 2-complete 2-category with a locally fully faithful lax idempotent pseudomonad $\mathcal{P}$ admits Eilenberg–Moore algebra-objects for relative monads with dense fully faithful roots and $\mathcal{P}$-small domain.*

Proof. Every finitely 2-complete 2-category admits Eilenberg–Moore algebra-objects for monads [Gra74, §1, 7.12.4], from which the result follows by Corollary 5.5.4 and Proposition 5.5.5, since $u_{T;n_j} : \mathbf{Alg}(T ; n_j) \to \mathcal{P}A$ is monadic when j is fully faithful by the preceding remark. □

We conclude this chapter by giving a representation theorem for relative monads, which will shed light on several definitions arising previously in the literature. First, we have a converse to Proposition 5.5.1, stating that every $(j \mathbin{;} j')$-monad of the form $t \mathbin{;} j'$ is necessarily induced by postcomposition by j'.

Proposition 5.5.10. *Let $j : A \to E$ and $j' : E \to E'$ be 1-cells in $\mathcal{K}$, and suppose that j' is fully faithful. The image of* $\mathbf{RMnd}(j) \hookrightarrow \mathbf{RMnd}(j \mathbin{;} j')$ *is the full subcategory spanned by $(j \mathbin{;} j')$-monads whose underlying 1-cell is of the form $t \mathbin{;} j'$ for a 1-cell $t : A \to E$.*

Proof. Trivially, if T is a j-monad, then its image under $(-) \mathbin{;} j'$ is given by $t \mathbin{;} j'$, and is hence of the desired form. Conversely, let $T' = (t \mathbin{;} j', \eta, (-)^\dagger)$ be a $(j \mathbin{;} j')$-monad. Since j' is fully faithful, the unit $\eta : j \mathbin{;} j' \Rightarrow t \mathbin{;} j'$ determines a 2-cell $j \Rightarrow t$. Similarly, the extension 2-cell $\dagger : E'(j'j, k't) \Rightarrow E'(j't, j't)$ determines an extension 2-cell

$$E(j,t) \xRightarrow{\overline{j'}} E'(j'j, j't) \xrightarrow{\dagger'} E'(j't, j't) \xRightarrow{\overline{j'}^{-1}} E(t,t)$$

Hence T' induces a j-monad, to which applying $(-) \mathbin{;} j'$ produces T', exhibiting T' as being in the image of $(-) \mathbin{;} j'$. □

As before, in the presence of a lax idempotent pseudomonad, we can take j' to be the canonical fully faithful 1-cell given by the nerve n_j, assuming that j is dense. This permits us to represent j-monads fully faithfully as certain monads in the Kleisli bicategory of $\mathcal{P}$. We will relate this to various results in the literature in Chapter 7.

Corollary 5.5.11. *Let $j : A \to E$ be a dense fully faithful 1-cell in $\mathcal{K}$ with $\mathcal{P}$-small domain.* $\mathbf{RMnd}(j)$ *is isomorphic to the full subcategory of* $\mathbf{Mon}(\mathcal{K}_{\mathcal{P}}[A, A])$ *spanned by monoids whose underlying 1-cell is of the form $t \mathbin{;} n_j$ for $t : A \to E$.*

Proof. Since j is dense, n_j is fully faithful, and we may consider the embedding $\mathbf{RMnd}(j) \hookrightarrow \mathbf{RMnd}(j \mathbin{;} n_j)$. Since j is fully faithful, $j \mathbin{;} n_j \cong よ_A$, and so the embedding is given by

$$\mathbf{RMnd}(j) \hookrightarrow \mathbf{RMnd}(よ_A) \simeq \mathbf{Mnd}_{\mathcal{P}}(\mathcal{P}A) \simeq \mathbf{Mon}(\mathcal{K}^{\mathcal{P}}[\mathcal{P}A, \mathcal{P}A]) \simeq \mathbf{Mon}(\mathcal{K}_{\mathcal{P}}[A, A])$$

by Theorem 5.5.8. The result follows by Proposition 5.5.10. □

Chapter 6

The formal monad–theory correspondence

In this chapter, we will build upon the understanding of the monad–theory correspondence presented in Chapter 3 to develop a monad–theory correspondence in an arbitrary 2-category with some minimal structure. The primary objective is to provide a conceptual explanation for the existence of the correspondence in **CAT**: to do so, we must necessarily abstract away from the concrete setting of categories, functors, and natural transformations, as we wish to avoid taking advantage of any of the phenomena that occur, so to speak, incidentally. A consequence, and secondary objective, for such an understanding will be a framework for monad–theory correspondences applicable in many 2-categories besides **CAT**: in the following chapter, we will show how correspondences for enriched categories follow very simply from the results of this chapter. It is worth noting that, while the motivation for the classical monad–theory correspondence is highly logical, serving to relate two frameworks for universal algebra, the formal monad–theory correspondence may essentially be seen as a contribution to the formal theory of monads: in essence, the presence of a well-behaved factorisation system on a 2-category will be shown to be greatly beneficial to the study of Kleisli objects, and recovers many of the properties with which we are familiar in the study of monads in **CAT**. We expect these observations will consequently be valuable from a purely categorical perspective in addition to the classical logical perspective.

Let us begin by outlining the conceptual explanation for the existence of monad–theory correspondences, which will act as a guide for the rest of the chapter. In the classical setting (namely, in the context of the 2-category **CAT**), every relative monad is induced canonically by two relative adjunctions: the Kleisli resolution, and the Eilenberg–Moore resolution [ACU15, Theorem 2.12]. This generalises the situation, known since the early years of category theory, to hold for monads [EM65; Kle65; Mar66]. Consequently, there are bijections between relative monads, Kleisli resolutions (namely, adjunctions that are initial amongst the adjunctions realising a given relative monad), and Eilenberg–Moore resolutions (namely, adjunctions that are terminal amongst the adjunctions realising a given relative monad). Furthermore, it is possible to characterise those resolutions that are initial solely in terms of their left relative adjoint: they are precisely the identity-on-objects left relative adjoints. Thus the bijection between relative monads and Kleisli resolutions may be rephrased as a bijection between relative monads and identity-on-objects functors equipped with right relative adjoints. In some cases, such as when j is the inclusion of a category into its cocompletion under a class of weights, the condition to be a left relative adjoint is equivalent to a colimit-preservation property, and j-relative monads furthermore extend to certain colimit-preserving monads: in this way the classical monad–theory correspondence is recovered.

As a category theorist, one ought seldom to be satisfied by bijections; happily, the bijection above may be extended to an equivalence of categories. For a dense functor $j \colon A \to E$, functors between Kleisli categories for j-relative monads that commute with the Kleisli inclusions correspond to relative monad morphisms, and vice versa. These are precisely the classical morphisms of algebraic theories. For every dense functor j, therefore, there is an equivalence of categories $\mathbf{Th}(j) \simeq \mathbf{RMnd}(j)$, and for those j for which j-monads extend to monads, j-monad morphisms also extend to monad morphisms.

In the setting of a 2-category, the situation is more subtle, but also more illuminating. On one hand, several of the properties of **CAT** that we rely on for the classical monad–theory correspondence, such as the fact that every monad is induced by an adjunction, do not hold in arbitrary 2-categories. On the other hand, these same obstructions force us to understand precisely which assumptions are necessary for the existence of the theory, and why they are the appropriate assumptions to make.

The first obstruction we encounter when approaching a formal correspondence, discussed in more detail in the previous chapter, is fundamental: unlike monads, it is not clear how to define the concept of relative monad in an arbitrary 2-category. The difficulty lies in the extension operator for a relative monad, which in **CAT** maps morphisms $jx \to ty$ to morphisms $tx \to ty$. To define an extension operator, we must assume some setting for formal category theory: in our case, a proarrow equipment.

Having assumed a proarrow equipment structure, and defined relative monads in our 2-category $\mathcal{K}$, we must then assume, for a given 1-cell $j \colon A \to E$, that initial resolutions for j-monads exist. In this case, we have a correspondence between initial resolutions and relative monads, essentially by definition. We wish to characterise initial resolutions in terms of their left relative adjoints. Recall that, in **CAT**, this property was given by being an bijective-on-objects functor. It is well-known that bijective-on-objects functors form the left-class of an orthogonal factorisation system on **CAT**, and one might suspect that an orthogonal factorisation system may be needed. This is certainly not a novel assumption: for instance, the *proto-theories* of Avery [Ave17, Definition 6.1.2] are precisely 1-cells in the left-class of an orthogonal factorisation system on (the underlying 1-category of) a 2-category. However, it is clear that not any orthogonal factorisation system will do: for instance, **CAT** has many orthogonal factorisation systems that do not classify Kleisli inclusions. Our contribution in this regard is an additional assumption on the orthogonal factorisation system that we call *resoluteness* (after its relation to resolutions of relative monads). We shall show that 1-cells in the left-class of a resolute factorisation system characterise initial resolutions.

The next step is to extend the correspondence between initial resolutions and relative monads to an equivalence of categories. For this, we need a stronger assumption than having initial resolutions for j-monads, which does not suffice to ensure that relative monad morphisms are given by morphisms between their Kleisli objects: we need $\mathcal{K}$ to admit Kleisli constructions (Definition 5.2.17) to ensure that this relationship is functorial.

Finally, we must have a way to extend relative monads to monads. This has already been discussed in Section 5.4, and we shall not need to dwell on it further here: the results of the previous chapter will be directly instantiated in our concrete examples.

Remark 6.0.1. Upon reading the previous paragraphs, the reader may have wondered, since relative monads are in bijection with Eilenberg–Moore resolutions as well as Kleisli resolutions, whether there might be an analogous monad correspondence from that, dual, point of view. This is indeed the case. Just as we can characterise initial resolutions as left relative adjoint identity-on-objects functors, so too can we characterise terminal resolutions as right relative adjoint functors creating certain colimits. In the setting of monads, this is a consequence of Beck's monadicity theorem [Bec]. In the setting of relative monads, an adaption of Paré's monadicity theorem [Par71] is more appropriate [Ark21]. While such a development is possible in a formal context, it is tangential to the subject of this thesis, and we shall leave the details for exposition elsewhere.

6.1 Resolute factorisation systems

In this subsection, we introduce a kind of orthogonal factorisation system that permits the characterisation (Section 6.1.1) and construction (Section 6.1.2) of those relative adjunctions that are opmonadic. There are thus two aspects of such factorisation systems: essentially, the former ensures that the 1-cells in the left class are well-behaved, and the latter ensures that the 1-cells in the right-class are well-behaved.

6.1.1 Hom-action factorisations

We shall require the 1-cells in the left class of our factorisation system to be sufficiently like identity-on-objects functors. The following definitions facilitate this intuition.

Definition 6.1.1. Let $k\colon A \to K$ and $\ell\colon A \to B$ be 1-cells in $\mathcal{K}$. A *hom-action* from k to ℓ is a 2-cell $\delta\colon K(k,k) \Rightarrow B(\ell,\ell)$ such that the following diagrams commute:

$$\begin{tikzcd} A(1,1) \arrow[r, "\overline{k}"] \arrow[dr, "\overline{\ell}"'] & K(k,k) \arrow[d, "\delta"] \\ & B(\ell,\ell) \end{tikzcd} \qquad \begin{tikzcd} K(k,k) \odot K(k,k) \arrow[r, "\mu_k"] \arrow[d, "\delta\odot\delta"'] & K(k,k) \arrow[d, "\delta"] \\ B(\ell,\ell) \odot B(\ell,\ell) \arrow[r, "\mu_\ell"'] & B(\ell,\ell) \end{tikzcd}$$

We make note of the resemblance of our definition of hom-action to the *extraordinary natural transformations* of [SW78, p. 369], but shall not attempt to make a formal comparison here.

Definition 6.1.2. A 1-cell $k\colon A \to K$ *factors hom-actions* if, for every 1-cell $\ell\colon A \to B$ and hom-action δ from k to ℓ, there is a unique 1-cell $d\colon K \to B$ such that $k \mathbin{;} d = \ell$ and $\delta = \overline{d}$.

1-cells that factor hom-actions capture a formal notion of identity-on-objects functor: intuitively, such a 1-cell ensures that every hom-action therefrom is induced by postcomposition.

Lemma 6.1.3. *Let $k \mathbin{{}_j{\dashv}} v$ be a j-adjunction for which k factors hom-objects. Denote by T the j-monad induced thereby.*

$$\begin{tikzcd} & K \arrow[dr, "v"] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "k"] \arrow[rr, "j"'] & {} & E \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, "r"] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "\ell"] \arrow[rr, "j"'] & {} & E \end{tikzcd}$$

For any resolution $\ell \mathbin{{}_j{\dashv}} r$ of T, there is a unique morphism of resolutions from $k \mathbin{{}_j{\dashv}} v$ to $\ell \mathbin{{}_j{\dashv}} r$. Thus $k \mathbin{{}_j{\dashv}} v$ is initial in $\mathbf{Res}(T)$.

Proof. Let $\ell \mathbin{{}_j{\dashv}} r$ be a resolution of T. We may form a 2-cell

$$\delta\colon K(k,k) \xRightarrow{\sharp k} E(j, k \mathbin{;} v) = E(j, \ell \mathbin{;} r) \xRightarrow{\flat \ell} B(\ell,\ell) \tag{6.1}$$

which satisfies the conditions to be a hom-action, since both j-adjunctions have the same unit and extension. Therefore, there is a unique $d\colon K \to B$ such that $k \mathbin{;} d = \ell$ and $\overline{d} = \delta$. To see that d commutes with the relative right adjoints, observe that $k \mathbin{;} d \mathbin{;} r = \ell \mathbin{;} r = k \mathbin{;} v$ and

$$\begin{aligned} \overline{d \mathbin{;} r} &= \sharp k \mathbin{;} \flat \ell \mathbin{;} \overline{r} && (\overline{d} = \delta) \\ &= \sharp k \mathbin{;} \flat k \mathbin{;} \overline{v} && \text{(resolutions of same } j\text{-monad)} \\ &= \overline{v} && (\sharp = \flat^{-1}) \end{aligned}$$

so that, since k factors hom-actions, we have $d \mathbin{;} r = v$. Thus $d\colon K \to B$ forms a morphism of resolutions from $k \mathbin{{}_j{\dashv}} v$ to $\ell \mathbin{{}_j{\dashv}} r$.

Conversely, assume there is a morphism of resolutions $d\colon K \to B$ from $k \mathbin{{}_j{\dashv}} v$ to $\ell \mathbin{{}_j{\dashv}} r$. We have

$$\begin{aligned} \overline{d} \mathbin{;} \sharp \ell &= \overline{d \mathbin{;} r} \mathbin{;} E(\tau, \ell \mathbin{;} r) && \text{(definition of } E(\tau, \ell \mathbin{;} r)) \\ &= \overline{v} \mathbin{;} E(\tau, \ell \mathbin{;} r) && (\overline{d \mathbin{;} r} = \overline{v}) \\ &= \overline{v} \mathbin{;} E(\tau, k \mathbin{;} v) && \text{(resolutions of same } j\text{-monad)} \\ &= \sharp k && \text{(definition of } E(\tau, k \mathbin{;} v)) \end{aligned}$$

so that $\overline{d} = \sharp k \mathbin{;} \flat \ell$ and hence, since k factors hom-actions, d must be the unique such morphism of resolutions. □

6.1.2 Resolute factorisations

We shall require the 1-cells in the right class of our factorisation system to be sufficiently like fully faithful functors, though in practice we only need a slightly weaker concept than fully faithful 1-cells. We shall first need the concept of a resolute composable pair of 1-cells.

Definition 6.1.4. Let $\ell_1 \colon A \to L$ and $\ell_2 \colon L \to B$ be 1-cells in $\mathcal{K}$. The pair (ℓ_1, ℓ_2) is *resolute* if

$$L(\ell_1, 1) \xrightarrow{\overline{\ell_2}L(\ell_1,1)} B(\ell_2\ell_1, \ell_2)$$

is invertible, so that there is a canonical relative adjunction $\ell_1 \;{}_{\ell_1;\ell_2}\!\dashv \ell_2$.

Resoluteness of a pair (ℓ_1, ℓ_2) may be considered a weakening of full faithfulness, where we only require full faithfulness for 2-cells whose domain is of the form $\ell_1(-)$. In particular, it holds whenever ℓ_2 is fully faithful (and, in practice, this is how our examples arise).

Lemma 6.1.5. *Let $\ell = \ell_1 \mathbin{;} \ell_2$ in $\mathcal{K}$, and suppose that ℓ_2 is fully faithful. Then (ℓ_1, ℓ_2) is resolute.*

Proof. Invertibility of $\overline{\ell_2}L(\ell_1, 1)$ follows immediately from invertibility of $\overline{\ell_2}$, which is full faithfulness of ℓ_2. □

The motivating property of resolute factorisations is the following, demonstrating that resolute factorisation *preserves resolutions* in a suitable sense, allowing us to transfer the right cell of a factorisation of a left relative adjoint from the left relative adjoint to the right relative adjoint, without altering the induced relative monad.

Proposition 6.1.6. *Let $\ell_1 \mathbin{;} \ell_2 \;{}_j\!\dashv r$ be a j-adjunction. The following are equivalent.*

1. *$\ell_1 \;{}_j\!\dashv \ell_2 \mathbin{;} r$; this j-adjunction induces the same j-monad as $\ell_1 \mathbin{;} \ell_2 \;{}_j\!\dashv r$; and ℓ_2 forms a morphism of resolutions therebetween.*
2. *(ℓ_1, ℓ_2) is resolute.*

$$\begin{tikzcd} L \arrow[r, "\ell_2"] & B \arrow[dr, "r"] & \\ A \arrow[u, "\ell_1"] \arrow[ur, "\ell"] \arrow[rr, "j"'] & \dashv & E \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} & L \arrow[dr, "\ell_2;r"] & \\ A \arrow[ur, "\ell_1"] \arrow[rr, "j"'] & \dashv & E \end{tikzcd}$$

Proof. ($1 \implies 2$) Since ℓ_2 is a morphism of resolutions, we have the following,

$$\begin{tikzcd} L(\ell_1, 1) \arrow[r, "\overline{\ell_2}L(\ell_1,1)"] \arrow[dr, "\cong"'] & B(\ell_2\ell_1, \ell_2) \arrow[d, "\cong"] \\ & E(j, r) \end{tikzcd}$$

exhibiting (ℓ_1, ℓ_2) as resolute.

($2 \implies 1$) Since $B(\ell_2\ell_1, 1) \cong E(j, r)$, it follows that $B(\ell_2\ell_1, \ell_2) \cong E(j, r\ell_2)$. Hence $L(\ell_1, 1) \cong E(j, r\ell_2)$ if and only if $L(\ell_1, 1) \cong B(\ell_2\ell_1, \ell_2)$, which holds in particular if (ℓ_1, ℓ_2) is resolute. The j-monad induced by $\ell_1 \;{}_j\!\dashv \ell_2 \mathbin{;} r$ has underlying 1-cell given by $\ell_1 \mathbin{;} (\ell_2 \mathbin{;} r) = (\ell_1 \mathbin{;} \ell_2) \mathbin{;} r = \ell \mathbin{;} r$; the units coincide directly by resoluteness; and the operators coincide by commutativity of the following diagram.

$$\begin{tikzcd} E(j, r\ell_2\ell_1) \arrow[r, "\flat'"] \arrow[dr, "\flat"'] & L(\ell_1, \ell_1) \arrow[r, "\overline{\ell_2;r}"] \arrow[d, "\overline{\ell_2}"] & E(r\ell_2\ell_1, r\ell_2\ell_1) \\ & B(\ell_2\ell_1, \ell_2\ell_1) \arrow[ur, "\overline{r}"'] & \end{tikzcd}$$

Finally, that ℓ_2 is a morphism of resolutions follows directly from resoluteness. □

Definition 6.1.7. A *resolute factorisation system* on a proarrow equipment $(-)_* \colon \mathcal{K} \to \mathcal{N}$ is an orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ on the underlying 1-category of $\mathcal{K}$ for which every $\mathcal{E}$-cell factors hom-actions and for which, for every factorisation $f = e \mathbin{;} m$, the pair (e, m) is resolute.

The informal slogan is that equipping a proarrow equipment $\mathcal{K}$ with a resolute factorisation system means that the formal theory of (relative) monads in $\mathcal{K}$ will behave like that of **CAT**: this idea will be justified in what follows.

In any proarrow equipment with a resolute factorisation system, we can characterise the relative opmonadic 1-cells in terms of the left-class of the factorisation system; this is the theorem that motivates the definition of resoluteness.

Theorem 6.1.8 (Relative opmonadicity)**.** *Let $\ell \;{}_j\!\dashv r$ be a j-adjunction in a proarrow equipment with a resolute factorisation system $(\mathcal{E}, \mathcal{M})$. Then $\ell \;{}_j\!\dashv r$ is j-opmonadic (Definition 5.2.11) if and only if ℓ is an $\mathcal{E}$-cell.*

Proof. Denote by T the j-monad induced by $\ell \;{}_j\!\dashv r$ and by $k \mathbin{;} i = \ell$ the resolute factorisation of ℓ. By Proposition 6.1.6, $k \;{}_j\!\dashv i \mathbin{;} r$, and this j-adjunction is opmonadic by Lemma 6.1.3. Therefore if $\ell \;{}_j\!\dashv r$ is j-opmonadic, i is an isomorphism, so that ℓ is an $\mathcal{E}$-cell since $\mathcal{E}$-cells are closed under isomorphism. Conversely, if ℓ is an $\mathcal{E}$-cell, then so is i via the cancellation property for $\mathcal{E}$-cells. Therefore, i is both an $\mathcal{E}$-cell and an $\mathcal{M}$-cell, and hence an isomorphism. □

In this way, in any proarrow equipment with a resolute factorisation system, relative opmonadicity is a property that depends only on the left relative adjoint ℓ, and not the right relative adjoint r, or even the root j. This means that if ℓ is the left relative adjoint of several relative adjunctions (with different right adjoints if j is not dense, or different roots), the relative adjunctions will share a Kleisli object. Furthermore, in a certain sense, the $(\mathcal{E}, \mathcal{M})$-factorisation of a 1-cell f that is not left relative adjoint may be thought of as constructing what the Kleisli object *would* be, if f were left relative adjoint. This is the direct analogue of the full image in **CAT**, which is obtained (up to isomorphism) by taking the (bijective-on-objects, fully faithful)-factorisation of a functor.

It follows that, given a relative monad T in $\mathcal{K}$, if T admits any resolution, it admits an initial resolution, since we may always take the $(\mathcal{E}, \mathcal{M})$-factorisation of the left relative adjoint to obtain the initial resolution.

Corollary 6.1.9. *Let $\ell \;{}_j\!\dashv r$ be a resolution of a j-monad T, and let $k \mathbin{;} i \cong \ell$ be the $(\mathcal{E}, \mathcal{M})$-factorisation of ℓ. Then $k \;{}_j\!\dashv i \mathbin{;} r$ is the initial resolution of T.*

Proof. By Proposition 6.1.6, taking the $(\mathcal{E}, \mathcal{M})$-factorisation of ℓ gives a relative adjunction $k \;{}_j\!\dashv i \mathbin{;} r$ with the same resolution, and this is furthermore j-opmonadic by Theorem 6.1.8 since k is an $\mathcal{E}$-cell. □

This process is functorial in a suitable sense, as exhibited by the following proposition.

Proposition 6.1.10. *Let $\mathcal{K}$ be a 2-category with a resolute factorisation system $(\mathcal{E}, \mathcal{M})$. Suppose that $\oslash \colon \mathbf{RAdj}_r(j) \to \mathbf{RMnd}(j)$ has a section, and that the image of an invertible relative monad morphism under the section is a strict right-morphism. Then $\mathcal{K}$ admits Kleisli constructions.*

Proof. We shall show that the conditions of Lemma 5.2.18(ii) are satisfied. First, the existence of a section implies that every j-monad has a resolution $\ell \;{}_j\!\dashv r$, and hence an initial resolution by Corollary 6.1.9. It furthermore implies that every j-monad morphism $\varphi \colon T \Rightarrow T'$ induces a right-morphism of j-adjunctions $(b, \rho) \colon (\ell \;{}_j\!\dashv r) \to (\ell' \;{}_j\!\dashv r')$, for which $\oslash(\ell \;{}_j\!\dashv r) = T$, $\oslash(\ell' \;{}_j\!\dashv r') = T'$, and $\oslash((b, \rho)) = \varphi$.

$$
\begin{tikzcd}
 & B' \arrow[dr, "r'"] & \\
A \arrow[ur, "\ell'"] \arrow[r, "\ell"'] & B \arrow[u, dashed, "b"] \arrow[r, "r"'] \arrow[ur, phantom, "\Nearrow\rho"] & E
\end{tikzcd}
$$

By orthogonality, we have a unique lift $\mathbf{Kl}(T) \to \mathbf{Kl}(T')$, since k_T is an $\mathcal{E}$-cell and $[]_{T'}$ is an $\mathcal{M}$-cell by Corollary 6.1.9.

$$\begin{tikzcd}
A \arrow[r, "k_T"] \arrow[d, "k_{T'}"'] & \mathbf{Kl}(T) \arrow[d, "{[]_T ; b}"] \arrow[dl, dashed] \\
\mathbf{Kl}(T') \arrow[r, "{[]_{T'}}"'] & B'
\end{tikzcd}$$

This lift is an $\mathcal{E}$-cell, since $k_{T'}$ is, and induces a right-morphism of j-adjunctions from the right-morphism (b, ρ).

$$\begin{tikzcd}
& \mathbf{Kl}(T') \arrow[r, "{[]_{T'}}"'] \arrow[rrd, bend left, "v_{T'}"] & B' \arrow[rd, "r'"] & \\
A \arrow[ru, "k_{T'}"] \arrow[r, "k_T"'] & \mathbf{Kl}(T) \arrow[u, dashed] \arrow[r, "{[]_T}"] \arrow[rr, bend right, "v_T"'] & B \arrow[u, "b"] \arrow[r, "r"] \arrow[ru, phantom, "\Rightarrow"] & E
\end{tikzcd}$$

It remains to show that we have unique lifts. Let $b \colon \mathbf{Kl}(T) \to B$ be a 1-cell under A, and consider its $(\mathcal{E}, \mathcal{M})$-factorisation $b = e \, ; m$. We have $k_T \, ; e \, ; m = \ell$. The $(\mathcal{E}, \mathcal{M})$-factorisation of ℓ is therefore given by $(k_T \, ; e, m)$, since k_T is an $\mathcal{E}$-cell. However, by Proposition 6.1.6 and Theorem 6.1.8 as before, this exhibits $k_T \, ; e$ as $k_{T'}$ and m as $[]_{T'}$. Therefore, the $(\mathcal{E}, \mathcal{M})$-factorisation of b provides a unique factorisation through $\mathbf{Kl}(T')$.

$$\begin{tikzcd}
& \mathbf{Kl}(T') \arrow[rd, "m"] & \\
\mathbf{Kl}(T) \arrow[ru, "e"] \arrow[rr, "b"] & & B \\
& A \arrow[lu, "k_T"] \arrow[ru, "\ell'"'] &
\end{tikzcd}$$

The result then follows by Lemma 5.2.18. □

In particular, assuming the existence of Eilenberg–Moore objects, there is a canonical resolution from which we may obtain the Kleisli resolution.

Corollary 6.1.11 (cf. [SW78, Proposition 24; Woo85, Proposition 6]). *Let T be a j-monad for which there exists an Eilenberg–Moore algebra-object. The resolute factorisation of $f_T \colon A \to \mathbf{EM}(T)$ is the Kleisli inclusion of T. Therefore, the unique morphism of resolutions $\mathbf{Kl}(T) \to \mathbf{EM}(T)$ is an $\mathcal{M}$-cell.*

Proof. Follows directly from Corollary 6.1.9. □

Often $\mathcal{M}$ will comprise fully faithful 1-cells, in which case this establishes that Kleisli objects fully faithfully embed into Eilenberg–Moore objects, as is known to be the case for $\mathcal{K} = \mathbf{CAT}$.

Remark 6.1.12. In [Str74a, p. 171], Street discusses the relationship between Kleisli and Eilenberg–Moore categories. Observe that, in **CAT**, we may construct the Eilenberg–Moore category for a (relative) monad from the Kleisli category via pullback (cf. Corollary 5.5.6 and Remark 5.5.7). Conversely, we may obtain the Kleisli category from the Eilenberg–Moore category in two ways: either by taking the (bijective-on-objects, fully faithful)-factorisation of f_T, or (up to Cauchy completion) by identifying $\mathbf{EM}(T \, ; \text{よ}_A) \simeq \mathcal{P}(\mathbf{Kl}(T))$. Street does not have the formalisms in [Str74a] to recover either construction (though some attempts are made on the former in [SW78]). We contend that the concept of resolute factorisation system finally gives a fully satisfactory formalisation of the first of these constructions; see Remark 5.5.7 for a discussion of the second.

Note, however, that admitting algebra-objects is not sufficient to admit (functorial) Kleisli constructions in the presence of a resolute factorisation system: though Kleisli objects will exist, it will not necessarily be the case that every j-monad morphism is induced by a right-morphism of j-adjunctions[1].

6.2 j-theories

We may define theories in any proarrow equipment with a resolute factorisation system: the definition mirrors that of a classical algebraic theory. We henceforth assume a proarrow equipment $(-)_* \colon \mathcal{K} \to \mathcal{N}$ with a resolute factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{K}$.

Definition 6.2.1. Let $j \colon A \to E$ be a dense 1-cell in $\mathcal{K}$. A *j-theory* is a left j-adjoint $\mathcal{E}$-cell. Denote by $\mathbf{Th}(j)$ the full subcategory of the coslice category $A/\mathcal{K}$ spanned by j-theories. Explicitly, a morphism of j-theories from $k \colon A \to B$ to $k' \colon A \to B'$ is a 1-cell $b \colon B \to B'$ such that $k \mathbin{;} b = k'$.

We require j to be dense so that being left j-adjoint is a property, rather than structure. Without this assumption, a theory would be given by a pair of 1-cells comprising the left and right relative adjoints; in this sense, density is a necessity for a formal theory of theories. Since density is a necessary requirement also for much of the theory of relative monads, this is not a particularly restrictive assumption.

The equivalence between j-theories and j-monads follows directly from the characterisation of relative adjunctions in terms of their left adjoints for dense j, together with the characterisation of relative opmonadicity.

Theorem 6.2.2. *Let $\mathcal{K}$ be a 2-category with a resolute factorisation system $(\mathcal{E}, \mathcal{M})$. Let $j \colon A \to E$ be a dense 1-cell for which $\mathcal{K}$ admits the construction of Kleisli objects. The categories of j-theories and of j-monads are equivalent.*

$$\mathbf{Th}(j) \simeq \mathbf{RMnd}(j)$$

Proof. By Corollary 5.2.19, the functor $\mathbf{RMnd}(j) \to A/\mathcal{K}$ is fully faithful. Hence, $\mathbf{RMnd}(j)$ is equivalent to the full subcategory of $A/\mathcal{K}$ spanned by j-opmonadic 1-cells: by Theorem 6.1.8, these are precisely the left j-adjoint $\mathcal{E}$-cells, i.e. the j-theories. □

In absolute generality, this is the most precise "monad–theory correspondence" for which one could hope: which is to say, in general, theories correspond to *relative monads* rather than monads. In other words, one ought really to think of theories as *being* relative monads. We believe the traditional desire for *monad*–theory correspondences is caused by the misunderstanding that theories correspond to monads, rather than a true desire to recover monads rather than relative monads. Of course, this is an entirely natural misunderstanding, given that the modern formulation of relative monad is a recent development [ACU10], and that Diers's formulation [Die75] has been overlooked until now.

That said, in some cases, we really are interested in monads: for instance, it is true that monads are more convenient objects of study than relative monads. In addition, we – meaning the author and readers of this thesis – are necessarily interested in monads, because we wish to demonstrate that *monad*–theory correspondences arise from *relative monad*–theory correspondences. To that end, we note that when the root j is dense, and $\mathcal{K}$ admits pointwise left extensions therealong, Corollary 5.4.8 applies, and we have an equivalence

$$\mathbf{Th}(j) \simeq \mathbf{RMnd}(j) \simeq \mathbf{Mnd}_j(E)$$

However, we wish to emphasise that, in general, we cannot make any comparison between relative monads and monads, because it is not the case that every relative monad is realisable.

[1] It seems unlikely that even having a functorial construction of Eilenberg–Moore objects would be sufficient, since morphisms of Eilenberg–Moore objects are classified by left-morphisms of j-adjunctions rather than right-morphisms.

Chapter 7

The enriched monad–theory correspondence

In this chapter, we justify the formal development of the monad–theory correspondence of the previous chapter by demonstrating that we recover all known monad–theory correspondences for enriched and internal categories through simple instantiations of the general theory. The one setting for monad–theory correspondences in the literature that we do not recover is that of monads on $(\infty, 1)$-categories (cf. [Kos21; HM21]): to express correspondences at this generality would likely require a theory of relative monads in $(\infty, 2)$-categories[1]. We begin by giving a short history of the monad–theory correspondence. We then discuss the application of our general theory to enriched categories, for concreteness taking a locally bicomplete closed bicategory as the base of enrichment, showing that, in many cases, we may easily prove that the necessary assumptions for the monad–theory correspondence are satisfied via general existence theorems. In the final sections, we show that the enriched monad–theory correspondences of Lucyshyn-Wright [Luc16] and of Bourke and Garner [BG19], which are the two most general correspondences appearing in the literature, are thereby subsumed, and briefly outline how the internal monad–theory correspondence of Johnstone and Wraith [JW78] may be seen as a special case of the enriched setting.

7.1 A short history of the monad–theory correspondence

The monad–theory correspondence has a rich history spanning over half a century. We shall give a brief overview of the developments in this area, both to put our work in context, and to establish the key insights that have led to the modern understanding of the correspondence. In particular, significant work in the early years of category theory has often been overlooked due to obscurity, and it is useful to have a survey explicating their contributions. We shall focus primarily on progress directly relevant to the monad–theory correspondence, though this line of research may be seen to fit into a wider context of developments in categorical logic and algebra. For ease of comprehension, we use notation consistent with our usage throughout the thesis, rather than the notations used in the original sources.

The story begins with the introduction of *algebraic theories*: coproduct-preserving bijective-on-objects functors from $\mathbb{F}(1)$, the free category with finite strict coproducts on a single object, with (locally-)small codomain. First appearing in Lawvere's 1963 thesis [Law63], algebraic theories were intended to serve as a presentation-invariant axiomatisation of finitary, monosorted universal algebras, as introduced by Birkhoff [Bir35].

Contemporaneously, Hall in 1965 introduced *abstract clones* [Coh65, Exercise III.3.3], intended to capture the same universal algebraic structures as algebraic theories, modulo a caveat as to whether nullary operations

[1] We note that it may be possible to use 2-categorical methods for the special case $(\infty, 1)$-**Cat**, such as in the framework of Riehl and Verity [RV18], but this too requires a development of relative $(\infty, 1)$-monads that is orthogonal to our interests here.

are permitted. Assuming one permits nullary operations (which is the natural choice), the categories of algebraic theories and of abstract clones are concretely isomorphic, and their tight connection was appreciated essentially immediately [Lin66b]. Abstract clones play an important role in the monad–theory correspondence and we shall return to them shortly.

In 1966, Linton was motivated to extend the algebraic theories of Lawvere from finitary operations to infinitary operations, following an analogous development of Słomiński [Sło59] in the classical, set-theoretic setting. Linton's *varietal equational theories* are product-preserving bijective-on-objects functors from $\mathbf{Set}^{\mathrm{op}}$, the free category with small products on a single object, with locally-small codomain. Such functors are equivalently the bijective-on-objects right-adjoint functors from $\mathbf{Set}^{\mathrm{op}}$ with locally-small codomain, and in turn are in bijection with monads[2] on $\mathbf{Set}$. Furthermore, in a preliminary report made in the same year, Linton observed that this bijection could be extended to an equivalence of categories [Lin66c]. Having made this observation, albeit without proof, Linton suggested that one could define a *theory* over any category E as a bijective-on-objects right-adjoint functor from E^{op} with locally-small codomain: such theories are in bijection with monads on E [Lin66b, §6]. It is this observation that is the origin of the monad–theory correspondence. Note that Linton's correspondence takes place in the setting of infinitary algebraic theories, rather than the subtler setting of finitary algebraic theories; the relation to finitary algebraic theories would not be explicated for another decade.

Linton continued this study of algebraic theories in a 1969 *tour de force* dedicated to a general investigation of the structure–semantics adjunction [Lin69a], a process devised by Lawvere to pass between algebraic theories and their categories of algebras [Law63, Theorem III.1.2]. Fixing a functor $j\colon A \to E$ between locally-small categories, Linton developed an adjunction between the opposite of the category of bijective-on-objects coslices over A^{op} (there called A-*clones*), and the category of slices over E [Lin69a, Theorem 4.1]. For $j = (\mathbb{F}(1) \simeq \mathbf{FinSet} \hookrightarrow \mathbf{Set})$, this essentially recovers Lawvere's original structure–semantics adjunction; for a set S, and $j = (\mathbb{F}(S) \simeq {\mathbf{FinSet}^S}_f \hookrightarrow \mathbf{Set}^S)$, the structure–semantics adjunction of S-sorted algebraic theories developed by Bénabou [Bén68] (where ${\mathbf{FinSet}^S}_f$ is the subcategory of $\mathbf{FinSet}^S$ spanned by S-indexed sets with finite support; equivalently, the free category with finite coproducts on S); and for $j = 1_{\mathbf{Set}}$, a structure–semantics adjunction for infinitary algebraic theories. Initially, no adjointness or limit-preservation conditions were assumed of the coslices. However, Linton went on to consider bijective-on-objects left-adjoint functors from E (i.e. for which $j = 1_E$) and proved a monad correspondence in this setting [Lin69a, Lemma 10.2], which was shown to fit naturally into the general structure–semantics adjunction [Lin69a, Theorem 10.1]. There is one further contribution of Linton's work that is crucial for the later discussion: the observation that the Eilenberg–Moore category for a monad can be characterised in terms of a subcategory of presheaves on the Kleisli category. If T is a monad on a small category E, then the following square forms a pullback [Lin69a, Observation 1.1], where we denote by $k_T\colon E \to \mathbf{Kl}(T)$ the Kleisli inclusion, by $u_T\colon \mathbf{EM}(T) \to E$ the Eilenberg–Moore forgetful functor, and by よ$_E\colon E \to [E^{\mathrm{op}}, \mathbf{Set}]$ the Yoneda embedding.

$$\begin{tikzcd} \mathbf{EM}(T) \arrow[r] \arrow[d, "u_T"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbf{Kl}(T)^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "{[{k_T}^{\mathrm{op}}, \mathbf{Set}]}"] \\ E \arrow[r, "よ_E"'] & {[E^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd} \tag{7.1}$$

The importance of this characterisation to later monad–theory correspondences will be discussed shortly.

Linton worked at the level of generality of [Lin69a] only in the unenriched setting; a definition of *finitary $\mathcal{V}$-theory* for $\mathcal{V}$ a cartesian-closed category appeared briefly in a contemporaneous paper of Day [Day70, Example 4.3], but the theory was not developed.

The history of the monad–theory correspondence is replete with developments that have since been forgotten or overlooked. One of the earliest examples is the *device-theoretic* approach to universal algebra, due to Walters [Wal69; Wal70]. In a 1969 paper, Walters defines the notion of *device* and its algebras [Wal69, §1], intended essentially as an alternative to monads and their algebras. Through a modern lens, a device captures exactly the notion of j-*relative monad* (cf. [ACU10]), for $j\colon A \to E$ an injective-on-objects functor. The main

[2] The reader desiring to study these references should be aware that in Linton's papers, and those of contemporaries, the term *triple* is used for what is now known by *monad*.

theorem of [Wal69] is a relationship between (the categories of algebras for) devices over **Set** and varietal categories, i.e. the category of algebras for a finitary monosorted universal algebra, which is reminiscent of Linton's earlier characterisation of the infinitary varietal categories as the categories monadic over **Set**. The definition of device was generalised the following year in Walters's thesis to permit $j \colon A \to E$ to be an arbitrary functor [Wal70, Definitions 1.1.1, 1.1.3], and devices were further studied, Walters providing universal properties for their Kleisli and Eilenberg–Moore categories (though the relationship with relative adjunctions was not not observed), and establishing various properties of the categories of algebras for a device [Wal70]. Though devices provide an early account of relative monads, this has not previously been appreciated, with modern references to Walters's work acknowledging only the non-relative aspects (cf. [MW12, §1]).

In 1970, in the setting of enrichment in $\mathcal{V}$ a complete well-powered closed symmetric monoidal category, Dubuc generalised the enriched structure–semantics adjunction of [Lin69b] to slices $u \colon X \to E$ that are not necessarily right-adjoint, but admit codensity monads[3] [Dub70] (see also [Dub06, Chapter II]). In particular, every right-adjoint functor admits a codensity monad, and so Linton's setting is recovered. Dubuc also gave a definition of $\mathcal{V}$*-theory*, which is an identity-on-objects power-preserving $\mathcal{V}$-functor from $\mathcal{V}$. A monad–theory correspondence follows as a consequence of the structure–semantics adjunction [Dub70, Theorem III]. For a modern reader, Dubuc's theorem is the first that is unambiguously a monad–theory correspondence; in contrast, Linton's presentation requires some work to interpret this way, since the relationship between right-adjointness and limit-preservation is left implicit. It should be noted that Dubuc considers primarily $\mathcal{V}$-monads on $\mathcal{V}$, rather than on arbitrary $\mathcal{V}$-categories, though the appropriate generalisation of $\mathcal{V}$-theories to identity-on-objects right-adjoint functors from arbitrary $\mathcal{V}$-categories is outlined [Dub70, §6].

A profunctorial perspective on the structure–semantics adjunction was developed in 1971 by Thiébaud [Thi71], who showed that the A-clones of Linton are the same as monads on A in **Prof**, the bicategory of profunctors [Thi71, Proposition II.1.5] (cf. [Jus68, p. 6.22]). For the most part, Thiébaud's treatment is orthogonal to the consideration of the monad–theory correspondence; however, the connection with profunctors will provide helpful insight.

Up to this point, monad–theory correspondences had been considered only for arbitrary monads; there was consequently no such correspondence for classical, finitary algebraic theories. This may seem surprising in light of Linton's development of the structure–semantics adjunction relative to a functor $j \colon A \to E$, which in particular captured finitary algebraic theories when $j = (\mathbf{FinSet} \hookrightarrow \mathbf{Set})$, but less surprising in light of the general unawareness at that time of an appropriate generalisation of monad to j-relative monad. However, in 1968, Ulmer [Ulm68] had introduced the notion of j*-relative adjunction*: a pair of functors $\ell \colon A \to B$ and $r \colon B \to E$ satisfying $B(\ell x, y) \cong E(jx, ry)$ natural in $x \in A$ and $y \in B$. (Notably, Ulmer mentions in a footnote that Linton appreciated there is a connection with algebraic theories, though this is never expounded.) In 1974, Diers made use of relative adjunctions to define the notion of *algebraic* j*-theory* for a dense fully faithful functor $j \colon A \to E$ [Die74]. An algebraic j-theory is an identity-on-objects left- j-relative adjoint functor $k \colon A \to B$ [Die74, Définition 4.1.0]. Following Linton [Lin69a, §5], Diers defined the algebras for a theory as a pullback, which was directly inspired by the pullback characterisation of the Eilenberg–Moore category for a monad.

$$
\begin{tikzcd}
\mathbf{Alg}(k) \arrow[r] \arrow[d, "u_k"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[B^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "{[k^{\mathrm{op}}, \mathbf{Set}]}"] \\
E \arrow[r, "N_j"'] & {[A^{\mathrm{op}}, \mathbf{Set}]}
\end{tikzcd}
\tag{7.2}
$$

In particular, for $j = (\mathbf{FinSet} \hookrightarrow \mathbf{Set})$, one recovers finitary algebraic theories as the algebraic j-theories; for $j = ({\mathbf{FinSet}^S}_f \hookrightarrow \mathbf{Set}^S)$, one recovers S-sorted finitary algebraic theories; and for $j = 1_{\mathbf{Set}}$, one recovers infinitary algebraic theories [Die74, Exemple 4.1.1].

The following year, Diers defined a notion of j-relative monad, necessary to establish a refinement of the monad–theory correspondence to algebraic j-theories [Die75]. For $j \colon A \to E$ a dense fully faithful functor, Diers defined a j*-monad* to be a triple $T = (t, \eta, \mu)$, where $t \colon A \to E$ is a functor, $\eta \colon j \Rightarrow t$ is a natural transformation, and $\mu \colon E(j-, t-) \circ E(j-, t-) \to E(j-, t-)$ is a 2-cell in **Prof**, forming a monad on

[3] Codensity monads are sometimes referred to as *monadic completions* [Man03, Definition 3.18], as they are a universal solution to constructing a monad from a functor.

A in **Prof** [Die75, Définitions 1.0]. To justify the definition, Diers established an equivalence between algebraic j-theories and j-monads, in particular establishing that finitary algebraic theories were equivalent to (**FinSet** $\hookrightarrow$ **Set**)-monads. This may be seen as a relative monad–theory correspondence. Diers also considered conditions for the category of algebras for a j-monad to be equivalent to the category of algebras for a non-relative monad, from which one can deduce the monad–theory correspondence for finitary algebraic theories [Die75, Théorème 3.4]. Diers's work on algebraic j-theories and j-monads already contains the essential ingredients for a general monad–theory correspondence subsuming later correspondences, at least in the unenriched setting. However, the work has been sadly overlooked, though algebraic j-theories appear in a handful of contemporary papers [Die76; Wis76; Day77; BD77].

In 1976, Manes introduced an alternative presentation of monad inspired by the definition of Kleisli category [Man76, Exercise 1.3.12]. Though it was not observed until later (cf. [MW12, §1]), Manes's *algebraic theories in extension form* (called *Kleisli triples* in [Mog91, Definition 1.2] and *monads in extension form* in [Man03, Definition 2.13]) are essentially the same as the *full devices* of Walters [Wal70], or monads relative to the identity functor (cf. [Wal70, Theorem 1.4.1]).

Linton and Diers were not the only ones to appreciate the importance of relative adjunctions to theories. Recall that every adjunction gives rise to a monad, and to every monad T we may associate two canonical adjunctions inducing T: the Kleisli and Eilenberg–Moore adjunctions, which are respectively initial and terminal [Kle65; EM65; Mar66]. In this way, we may reason about a monad in terms of its Kleisli or Eilenberg–Moore adjunction, and furthermore could do so even without having a definition of monad. Analogously, it would be plausible to imagine that one might reason about a notion of relative monad purely through universal relative adjunctions. This was the starting point for the 1977 thesis of Lee [Lee77], whose motivation was to capture finitary algebraic theories using monadic techniques. Lee's work is independent of Diers's, and there is a substantial overlap in their results. For instance, Lee defines Kleisli and Eilenberg–Moore relative adjunctions, and proves a relative monadicity theorem, but does not attempt to define a notion of relative monad. Instead, Lee observes that every j-relative adjunction, for $j \colon A \to E$, induces a cocontinuous monad on $[E^{\mathrm{op}}, \mathbf{Set}]$, and this acts as a surrogate for the hypothetical notion of relative monad; this observation is strongly related to the work of Thiébaud and Diers as we will explain later. Unfortunately, the work of Lee has also been overlooked.

Finitary enriched algebraic theories were first studied by Gray [Gra75] in 1975, who was motivated primarily by **Cat**-enrichment (cf. [Gra73]). Gray's setting was that of enrichment in a bicomplete closed symmetric monoidal category $\mathcal{V}$ (frequently further assumed cartesian-closed). While the precise definition of the $\mathcal{V}$*-algebraic theories* ibid. is somewhat involved, they are in particular bijective-on-objects $\mathcal{V}$-functors preserving finite coproducts, as with Lawvere's original definition. Gray considered several aspects of the theory of algebraic theories, including a structure–semantics adjunction, but did not consider the relationship with $\mathcal{V}$-enriched monads.

The early 1970s saw the introduction of (elementary) topos theory and it became popular to understand categorical logic from this foundational perspective [Joh77]. Algebraic theories were no exception, and in 1978 Johnstone and Wraith produced an extensive study of finitary and infinitary algebraic theories internal to a topos $\mathcal{E}$ with a natural numbers object [JW78]. In particular, the authors showed that finitary algebraic theories internal to $\mathcal{E}$ corresponded to monads locally-internal to $\mathcal{E}$ [JW78, Corollary 7.6]. This is the first example of a monad–theory correspondence in a setting other than enriched categories (though we will later see how (locally-)internal categories may be seen as enriched categories via a suitable base of enrichment).

Finitary algebraic theories were considered from the monadic perspective, independently from those aforementioned, by Borceux and Day [BD80] in 1980. Their motivation was to develop aspects of universal algebra in the setting of $\mathcal{V}$-categories, for $\mathcal{V}$ a π*-category*: a complete and cocomplete closed symmetric monoidal category such that taking products with a fixed object of $\mathcal{V}$ preserves sifted colimits, and satisfying a commutation condition between products and coends. In this context, they defined a $\mathcal{V}$*-theory* to be an essentially-surjective-on-objects $\mathcal{V}$-functor from $\mathcal{V}_f$, the opposite of the subcategory of finite copowers of I, preserving finite powers of I, and established a structure–semantics adjunction for $\mathcal{V}$-theories. Of particular relevance to the monad–theory correspondence is an equivalence, due to Kelly, between the category of $\mathcal{V}$-theories, and the category of monoids in the $\mathcal{V}$-functor category $[\mathcal{V}_f, \mathcal{V}]$ [BD80, Proposition 2.6.1]. This bears resemblance to the characterisation of monads as monoids in endofunctor categories; in fact, this equivalence may be seen in a modern light as a correspondence between $\mathcal{V}$-theories and j-relative monads [ACU15, Theorem 3.5]. Modulo the differences in the bases of enrichment, this correspondence may therefore be understood as a restriction of Dubuc's to the finitary setting.

In 1999, Altenkirch and Reus defined the notion of *Kleisli structure* to capture type theoretic structure, which is a special case of the definition of j-relative monad for j having discrete domain [AR99, Definition 10].

In the same year, Power proved a monad–theory correspondence in a similar setting to that of Borceux and Day, but with different conditions on the enriching category $\mathcal{V}$, requiring a locally finitely presentable closed monoidal category, and considering preservation of all finite cotensors rather than finite cotensors with I, showing finite-cotensor-preserving identity-on-objects $\mathcal{V}_t$-functors[4] from $\mathcal{V}_f$ to be equivalent to finitary $\mathcal{V}$-monads on $\mathcal{V}$ [Pow99, Theorem 4.3]. Later, in 2009, assuming additionally that $\mathcal{V}$ is symmetric, Nishizawa and Power generalised the correspondence to finitary $\mathcal{V}$-monads on arbitrary locally finitely presentable $\mathcal{V}$-categories [NP09] (cf. the paper of Lack and Power [LP09] of the same year, which explores the same correspondence from a different perspective). In this setting, $\mathcal{V}$-theories are required to preserve all finite $\mathcal{V}$-limits.

The next enhancement to the monad–theory correspondence was made by Melliès in 2010. Three years prior, Weber had introduced *monads with arities* [Web07, Definition 4.1] to generalise the Segal condition for simplicial sets to the setting of well-behaved monads (where the classical Segal condition is recovered for the free-category monad on **Graph**). Melliès observed that the definition of monads with arities was related to Linton's characterisation of the Eilenberg–Moore category for a monad in terms of its Kleisli category. In particular, the Segal condition for a simplicial set arises exactly as the condition for a presheaf to be in the image of the nerve $\mathbf{Cat} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ in the pullback (7.2) [Mel10, §I & §III]. Furthermore, the limit-preservation condition of an algebraic theory can be expressed in terms of a representability-preservation condition. This suggests a general definition of *Lawvere theory with arities* j corresponding to that of monad with arities j, for $j \colon A \to E$ a dense fully faithful functor with small domain, and in this setting Melliès proved a monad–theory correspondence [Mel10, §V]. This work was later developed further by Berger, Melliès and Weber in 2012, for instance by generalising the characterisation of the Segal condition in terms of a sheaf condition to monads with arities [BMW12, Lemma 3.6], but they did not prove a more general monad–theory correspondence (cf. [BMW12, Theorem 3.4]).

In a separate development the following year, Lack and Rosický generalised the work of [NP09], taking the base of enrichment $\mathcal{V}$ to be a complete and cocomplete closed symmetric monoidal category, and relaxing finite limits to Φ-limits (for Φ a class of weights for which Φ-continuous weights are Φ-flat). In this setting, they proved an equivalence between $\mathcal{V}$-theories and Φ-accessible $\mathcal{V}$-monads on a locally Φ-presentable $\mathcal{V}$-category [LR11, Theorem 7.7].

Relative monads were defined and studied by Altenkirch, Chapman and Uustalu [ACU10] in 2010, who also rediscovered the notion of relative adjunction. Though their definition is equivalent to that of Walters's devices, their presentation, in the style of Manes, is more convenient in practice. In their 2010 paper, as well as in an extended version of the paper five years later [ACU15], they established various constructions and results regarding relative monads, such as the construction of the Kleisli and Eilenberg–Moore categories, and the equivalence between relative monads and monoids in skew-monoidal functor categories. Furthermore, they observed that the *monads with arities* of [BMW12] were induced by relative monads [ACU15, §4.3]. Notably, specialising the definition of j-relative monad to $j = (\mathbb{F} \hookrightarrow \mathbf{Set})$ essentially recovers the definition of abstract clone as defined by Hall. Thus, the equivalence between algebraic theories and abstract clones may alternatively be seen to be an equivalence between algebraic theories and $(\mathbb{F} \hookrightarrow \mathbf{Set})$-relative monads. Relative monads enriched in a symmetric monoidal category $\mathcal{V}$ were later defined by Staton in 2013 under the name *enriched clone*, where j is a fully faithful $\mathcal{V}$-functor [Sta13a, Definition 4] (cf. the *j-abstract $\mathcal{V}$-clones* of [Fio17b, Definition 1.1], which provides another definition of enriched relative monad, but with different restrictions on j).

Recall that Justesen [Jus68] and Thiébaud [Thi71] characterised identity-on-objects functors as monads in **Prof**. This characterisation may be extended to capture finitary algebraic theories, by incorporating product-preservation. Hyland [Hyl14a] illustrated this in a 2014 paper, giving a bicategorical understanding of finitary

[4] Here, $\mathcal{V}_t$ is the monoidal category whose delooping is the one-object bicategory $(\Sigma\mathcal{V})^{\mathrm{op}}$.

algebraic theories. The 2-monad on **Cat** for small categories with finite products extends to **Prof**, the Kleisli bicategory for the presheaf construction [Fio+18]. An S-sorted finitary algebraic theory is then precisely a monad on (the discrete category) S in the Kleisli bicategory for the extended 2-monad on **Prof** [Hyl14a, §4.3]. One thereby recovers the classical monad–theory correspondence.

In the same year, Garner [Gar14] presented a different perspective on the classical monad–theory correspondence. A finitary monad on **Set** is equivalently a monoid in $[\mathbb{F}, \mathbf{Set}]$, hence a one-object $[\mathbb{F}, \mathbf{Set}]$-enriched category. From this perspective, one may analyse various results regarding algebraic theories, including the monad–theory correspondence, using enriched category theory. In particular, Garner showed how one can view the process of assigning to a finitary algebraic theory its corresponding finitary monad as a $[\mathbb{F}, \mathbf{Set}]$-enriched Cauchy completion. This perspective was extended three years later in joint work with Power [GP17], to the generality of [NP09; LP09]. To do so, it is necessary to move from categories enriched in functor categories to categories enriched in the bicategory of locally finitely presentable categories.

Most of the monad–theory correspondences described thus far are orthogonal, in that few had been subsumed by more general correspondences. In fact, in 2015, only the work of Nishizawa and Power had been generalised by later work. Notably, no frameworks after Linton and Dubuc captured the classic examples of infinitary algebraic theories and monads on **Set**, because they all made essential use of smallness assumptions.

In 2016, Lucyshyn-Wright proved a significant generalisation of the previous monad–theory correspondences, subsuming many of the previous frameworks. Considering enrichment in $\mathcal{V}$ a closed symmetric monoidal category, and fixing a dense fully faithful strong symmetric $\mathcal{V}$-functor $j\colon A \to \mathcal{V}$ satisfying certain well-behavedness conditions, Lucyshyn-Wright defined a notion of *j-theory* and proved a correspondence between j-theories and monads on $\mathcal{V}$ preserving certain weighted colimits [Luc16, Theorem 11.8]. In particular, since A is not assumed small, this captures the correspondences with large algebraic theories of Linton [Lin69a] and Dubuc [Dub70]. However, monad–theory correspondences for $\mathcal{V}$-monads on categories other than $\mathcal{V}$ are not captured: for instance, that of Nishizawa and Power [NP09].

The most recent contribution to the monad–theory correspondence at the time of writing is the 2019 paper of Bourke and Garner [BG19]. Working in the setting of enrichment in a locally presentable closed symmetric monoidal category $\mathcal{V}$, Bourke and Garner fixed a dense fully faithful $\mathcal{V}$-functor $j\colon A \to E$ with small domain and locally presentable codomain, and considered an adjunction between *A-pretheories*, which are coslices over A, and monads on E. They proved that the fixed points of the adjunction are respectively the *A-theories* and the *A-nervous monads*. This framework was shown to subsume many monad–theory correspondences, including that of Berger, Melliès and Weber, though it does not include the classic correspondence of monads on **Set**, due to the requirement that A be small.

Monad–theory correspondences for higher-categories

In this thesis, we are concerned with monad–theory correspondences that may be carried out within a 2-category. However, monad–theory correspondences have also been developed in the setting of (unenriched) $(\infty, 1)$-categories, both following the approach of Berger–Melliès–Weber's monads with arities [Kos21] and of Bourke–Garner's nervous monads [HM21]. We expect the perspective and techniques presented in this thesis will be amenable to establishing general monad–theory correspondences subsuming those aforementioned, in the same manner they do their 1-dimensional counterparts, but do not pursue such a development here. In particular, our theory relies fundamentally on relative monads, a concept that, at the time of writing, has not been developed internally to $(\infty, 2)$-categories.

7.2 Monads and theories in $\mathcal{W}$-CAT

Let $\mathcal{W}$ be a base of enrichment for which we have a notion of *$\mathcal{W}$-enriched category* (or simply *$\mathcal{W}$-category*): for instance, a monoidal category [Bén65; Mar65], bicategory [Wal81, §1; Str83, §2], multicategory [Lam69, p. 106], double category [Lei02], or suchlike. In each case, $\mathcal{W}$-categories form a 2-category $\mathcal{W}$-**CAT**, and it is reasonable to ask under what assumptions we may establish monad–theory correspondences therein. While it is possible to give explicit constructions validating the necessary assumptions, for instance to define Kleisli and Eilenberg–Moore $\mathcal{W}$-categories and prove their universal properties, we shall prefer to avoid explicit constructions where possible, as they are typically protracted and unenlightening. Fortunately, the formal theory of relative monads provides us with a relatively minimal set of assumptions from which the others may be derived. More precisely, it will only be necessary to show that:

1. $\mathcal{W}$-**CAT** has a (identity-on-objects, fully faithful)-factorisation system.
2. $\mathcal{W}$-**CAT** admits finite 2-limits.
3. $\mathcal{W}$-**CAT** is equipped with a lax idempotent pseudomonad (in particular, the free small cocompletion).

(1) provides $\mathcal{W}$-**CAT** with a resolute factorisation system, while (2) ensures that $\mathcal{W}$-**CAT** has Eilenberg–Moore objects for (non-relative) monads. Using Corollary 5.5.9, having a lax idempotent pseudomonad will then ensure $\mathcal{W}$-**CAT** also has Eilenberg–Moore objects for relative monads. Kleisli objects may be obtained by factorising the terminal resolution of each relative monad. For any reasonable notion of enrichment, (1) will always hold, while (2) and (3) will typically hold in the presence of sufficient limits in $\mathcal{W}$.

We will prove that (1 – 3) hold when $\mathcal{W}$ is a locally bicomplete closed bicategory. This permits us to efficiently use several known results about the 2-category $\mathcal{W}$-**CAT**, rather than having to derive them ourselves. First, we review the notion of category enriched in a bicategory [Wal81; Str83], the theory of which is analogous to that of categories enriched in monoidal categories (cf. [Kel82]), but where we associate to each pair of objects of the category not a hom-*object*, but a hom-*morphism*.

Notation 7.2.1. We shall use notation reminiscent of a monoidal category for the base of enrichment, with the intent to aid the reader familiar only with enrichment in a monoidal category. To that end, we denote by $\otimes$ diagrammatic composition in the bicategory $\mathcal{W}$, and by I the identity for composition (technically both are parameterised by objects of $\mathcal{W}$, but we will often leave this implicit for readability).

Definition 7.2.2 ([Bén67, Definition 5.5.1; Str83, §2]). Let $\mathcal{W}$ be a bicategory. A $\mathcal{W}$-*category* A consists of

1. a class $|A|$ of *objects*;
2. for each $x \in A$, an *extent* $\epsilon_x \in \mathcal{W}$;
3. for all $x, y \in A$, a *hom-morphism* $A(x, y) \colon \epsilon_x \to \epsilon_y$ in $\mathcal{W}$;
4. for all $x, y, z \in A$, a *composition* $\mu_{x,y,z} \colon A(x, y) \otimes A(y, z) \Rightarrow A(x, z)$ in $\mathcal{W}$;
5. for each $x \in A$, an *identity* $\iota_x \colon I \Rightarrow A(x, x)$ in $\mathcal{W}$,

such that the following diagrams commute

6.
$$\begin{tikzcd}
(A(w,x) \otimes A(x,y)) \otimes A(y,z) \arrow[rr, "\alpha_{A(w,x),A(x,y),A(y,z)}"] \arrow[dd, "\mu_{w,x,y} \otimes A(y,z)"'] & & A(w,x) \otimes (A(x,y) \otimes A(y,z)) \arrow[d, "A(w,x) \otimes \mu_{x,y,z}"] \\
& & A(w,x) \otimes A(x,z) \arrow[d, "\mu_{w,x,z}"] \\
A(w,y) \otimes A(y,z) \arrow[rr, "\mu_{w,y,z}"'] & & A(w,z)
\end{tikzcd}$$

7.
$$\begin{tikzcd}
& \epsilon_x \otimes A(x,y) \arrow[dl, "\lambda_{\epsilon_x}"'] \arrow[d, "\iota_x \otimes A(x,y)"] \\
A(x,y) & A(x,x) \otimes A(x,y) \arrow[l, "\mu_{x,x,y}"]
\end{tikzcd}$$

8.
$$\begin{tikzcd}
A(x,y) \otimes \epsilon_y \arrow[d, "A(x,y) \otimes \iota_y"'] \arrow[dr, "\rho_{\epsilon_y}"] & \\
A(x,y) \otimes A(y,y) \arrow[r, "\mu_{x,y,y}"'] & A(x,y)
\end{tikzcd}$$

A $\mathcal{W}$-*functor* $f\colon A \to B$ between $\mathcal{W}$-categories consists of

1. a function $|f|\colon |A| \to |B|$ such that f *preserves extent* in that $|f| \mathbin{;} \epsilon_B = \epsilon_A$;
2. for all $x, y \in A$, a 2-cell $f_{x,y}\colon A(x,y) \Rightarrow B(fx, fy)$,

such that the following diagrams commute

3.
$$\begin{tikzcd} A(x,y) \otimes A(y,z) \arrow[r, "f_{x,y} \otimes f_{y,z}"] \arrow[d, "\mu_{x,y,z}"'] & B(fx,fy) \otimes B(fy,fz) \arrow[d, "\mu_{fx,fy,fz}"] \\ A(x,z) \arrow[r, "f_{x,z}"'] & B(fx,fz) \end{tikzcd}$$

4.
$$\begin{tikzcd} I_{\epsilon_x} \arrow[r, "\iota_x"] \arrow[d, equal] & A(x,x) \arrow[d, "f_{x,x}"] \\ I_{\epsilon_{fx}} \arrow[r, "\iota_{fx}"'] & B(fx,fx) \end{tikzcd}$$

A $\mathcal{W}$-*transformation* $\alpha\colon f \Rightarrow g\colon A \to B$ between $\mathcal{W}$-functors consists of a family of 2-cells $\varphi_x\colon I \Rightarrow B(fx, gx)$ for each $x \in |A|$ making the following diagram commute for all $x, y \in |A|$:

$$\begin{tikzcd} & I_{A(x,y)} \otimes A(x,y) \arrow[r, "\varphi_x \otimes g_{x,y}"] & B(fx,gx) \otimes B(gx,gy) \arrow[dr, "\mu_{fx,gx,gy}"] & \\ A(x,y) \arrow[ur, "\lambda^{-1}_{A(x,y)}"] \arrow[dr, "\rho_{A(x,y)}"'] & & & B(fx,gy) \\ & A(x,y) \otimes I_{A(x,y)} \arrow[r, "f_{x,y} \otimes \varphi_y"'] & B(fx,fy) \otimes B(fy,gy) \arrow[ur, "\mu_{fx,fy,gy}"'] & \end{tikzcd}$$

We denote by $\mathcal{W}$-**CAT** the 2-category of $\mathcal{W}$-categories, $\mathcal{W}$-functors, and $\mathcal{W}$-transformations; and by $\mathcal{W}$-**Cat** the full sub-2-category spanned by small $\mathcal{W}$-categories (i.e. those for which the class of objects is a set).

We will assume $\mathcal{W}$ is locally bicomplete (that is, its hom-categories are small-complete and small-cocomplete) and closed (that is, has all right extensions and right lifts). The convenience of this assumption is demonstrated by the following theorems, stating that $\mathcal{W}$-**CAT** is closed under small 2-limits and admits cocompletions of small $\mathcal{W}$-categories (for the latter, we refer to [Fio+18] for the definition of lax idempotent relative pseudomonad).

Theorem 7.2.3 ([Bet+83, Theorem 10]). *Suppose that $\mathcal{W}$ is a locally bicomplete closed bicategory. Then the 2-category $\mathcal{W}$-**CAT** has small 2-limits.*

Theorem 7.2.4 ([Str83, §4]). *Suppose that $\mathcal{W}$ is a locally bicomplete closed bicategory. Then $\mathcal{W}$-**CAT** is equipped with a lax idempotent relative pseudomonad $\mathcal{P}$ exhibiting a biequivalence*

$$\mathcal{W}\text{-}\mathbf{Cocts}(\mathcal{P}A, B) \simeq \mathcal{W}\text{-}\mathbf{CAT}(A, B)$$

for all $\mathcal{W}$-categories A and B for which A is small.

By the bicategorical analogue of Corollary 5.4.8, $\mathcal{P}$ extends to a pseudomonad on $\mathcal{W}$-**CAT** expressing free small-cocompletion, which permits us to close large $\mathcal{W}$-categories under small colimits. It is possible to define a bicategory $\mathcal{W}$-**Prof** of small $\mathcal{W}$-categories, $\mathcal{W}$-profunctors, and $\mathcal{W}$-transformations [Str83], and a corresponding proarrow equipment $\mathcal{W}$-**Cat** $\to$ $\mathcal{W}$-**Prof**. However, we shall be interested in some $\mathcal{W}$-categories that are not small. To facilitate such a consideration, we observe that $\mathcal{W}$-**Prof** is biequivalent to the Kleisli bicategory $\mathbf{Kl}(\mathcal{P})$ (cf. [Fio+18]), with the Kleisli inclusion giving the proarrow equipment structure. We shall instead consider the proarrow equipment $\mathbf{Adm}_{\mathcal{P}}(\mathcal{W}\text{-}\mathbf{CAT}) \to \mathbf{Kl}(\mathcal{P})$, for which the 1-cells in the domain are those $\mathcal{W}$-functors $f\colon A \to B$ for which $\mathcal{P}f\colon \mathcal{P}A \to \mathcal{P}B$ has a right adjoint, as described in Section 5.1.1. In particular, this includes $\mathcal{W}$-functors from small $\mathcal{W}$-categories, as well as identity $\mathcal{W}$-functors, which allows us to consider relative monads with roots $j\colon A \to E$ for which E is a large $\mathcal{W}$-category. By $\mathcal{W}$-enriched relative monad, then, we mean a relative monad in the proarrow equipment $\mathbf{Adm}_{\mathcal{P}}(\mathcal{W}\text{-}\mathbf{CAT}) \to \mathbf{Kl}(\mathcal{P})$.

Just as with categories and functors, the 2-category $\mathcal{W}$-**CAT** has a (identity-on-objects, fully faithful)-factorisation system.

Definition 7.2.5. A $\mathcal{W}$-functor $f\colon A \to B$ is

1. *identity-on-objects* if $|f|\colon |A| \to |B|$ is the identity function;
2. *fully faithful* if, for all $x, y \in A$, the 2-cell $f_{x,y}\colon A(x, y) \Rightarrow B(fx, fy)$ is invertible.

It is evident that full faithfulness in the $\mathcal{W}$-enriched sense coincides with full faithfulness in the corresponding proarrow equipment.

Lemma 7.2.6. *Let $\mathcal{W}$ be a bicategory. The 2-category $\mathcal{W}$-**CAT** has a resolute factorisation system whose left class comprises the identity-on-objects $\mathcal{W}$-functors and whose right class comprises the fully faithful $\mathcal{W}$-functors.*

Proof. Let $f\colon A \to B$ be a $\mathcal{W}$-functor between $\mathcal{W}$-categories. Define the *full image* of f, denoted $\mathrm{im} f$, to be the $\mathcal{W}$-category whose objects are those of A and for which the extent of $x \in A$ is given by that of fx, and for which $\mathrm{im} f(x, y) = B(fx, fy)$. There are evident $\mathcal{W}$-functors $A \to \mathrm{im} f \to B$ composing to f, for which the former is identity-on-objects and for which the latter is fully faithful. Clearly these classes are closed under composition and identities, and the factorisation is unique up to unique isomorphisms of objects and hom-morphisms. The (identity-on-objects, fully faithful) $\mathcal{W}$-functors therefore form an orthogonal factorisation system. It remains to show that identity-on-objects $\mathcal{W}$-functors factor hom-actions (Definition 6.1.2). Let $k\colon A \to K$ be an identity-on-objects $\mathcal{W}$-functor and let $\ell\colon A \to B$ be a $\mathcal{W}$-functor. A hom-action $\delta\colon K(k, k) \Rightarrow B(\ell, \ell)$ is exactly the data of a $\mathcal{W}$-functor $d\colon K \to B$ on hom-morphisms. The data of the object-function $|d|\colon |K| \to |B|$ is given by the object-function of ℓ, since $|K| = |A|$. It is clear that $d\colon K \to B$ is the unique $\mathcal{W}$-functor factoring δ. □

This factorisation system is the (*surjection*, *inclusion*)-factorisation system of the proarrow equipment $\mathbf{Adm}_{\mathcal{P}}(\mathcal{W}\text{-}\mathbf{CAT}) \to \mathbf{Kl}(\mathcal{P})$ (cf. [Woo85]), and is consequently resolute, since each $\mathcal{M}$-cell is fully faithful. To establish a monad–theory correspondence, it remains to show that $\mathcal{W}$-**CAT** admits (enough) Kleisli and Eilenberg–Moore objects for relative monads. We shall restrict our attention to those relative monads having dense roots $j\colon A \to E$, which is in any case necessary for the monad–theory correspondence and allows us to avoid giving an explicit construction the of $\mathcal{W}$-categories involved.

Theorem 7.2.7. *Let $\mathcal{W}$ be a locally bicomplete closed bicategory. Then $\mathcal{W}$-**CAT** has Eilenberg–Moore algebra-objects and Kleisli resolutions for relative monads with dense fully faithful roots with small domain.*

Proof. $\mathcal{W}$-**CAT** has Eilenberg–Moore algebra-objects for relative monads with dense fully faithful roots with small domain by Corollary 5.5.9, since it has finite limits by Theorem 7.2.3 and a lax idempotent pseudomonad $\mathcal{P}$ by Theorem 7.2.4. It hence has Kleisli resolutions for the same by Corollary 6.1.11, which are given in the usual manner: for a j-monad T, the $\mathcal{W}$-category $\mathbf{Kl}(T)$ has as objects those of A, and the hom-morphism $\mathbf{Kl}(T)(x, y)$ is given by $E(jx, ty)$. We shall use Lemma 5.2.18 to show more generally that it admits Kleisli constructions (making use of Lemma 5.2.15 to characterise right-morphisms just in terms of their left relative adjoints). Let $j\colon A \to E$ be a dense $\mathcal{P}$-admissible $\mathcal{W}$-functor. Given a j-monad morphism $\phi\colon t \Rightarrow t'$, we construct an identity-on-objects $\mathcal{W}$-functor whose action on objects $x, y \in A$ is given by

$$\mathbf{Kl}(T)(x, y) = E(jx, ty) \xrightarrow{E(jx, \phi_y)} E(jx, t'y) = \mathbf{Kl}(T')(x, y)$$

In the other direction, since j is dense, an identity-on-objects $\mathcal{W}$-functor $\mathbf{Kl}(T) \to \mathbf{Kl}(T')$, which induces a $\mathcal{W}$-transformation $\mathcal{W}$-natural in $y \in A$,

$$E(j-, ty) = \mathbf{Kl}(T)(k_T-, y) \to \mathbf{Kl}(T')(k_T-, y) = E(j-, t'y)$$

is equivalently specified by a $\mathcal{W}$-transformation $t \Rightarrow t'$; $\mathcal{W}$-functoriality of $\mathbf{Kl}(T) \to \mathbf{Kl}(T')$ further implies that this is a morphism of $\mathcal{W}$-enriched relative monads. Clearly the relative monad morphism is the identity if and only if the $\mathcal{W}$-functor is the identity. Now consider a diagram such as the following.

$$\begin{tikzcd} & \mathbf{Kl}(T') \arrow[dr, "{[]_{T'}}"] & \\ \mathbf{Kl}(T) \arrow[ur, dashed] \arrow[rr, "b"] & & B' \\ & A \arrow[ul, "k_T"] \arrow[ur, "\ell'"'] & \end{tikzcd}$$

Observe that a $\mathcal{W}$-functor $\mathbf{Kl}(T) \to B'$ over A assigns a 2-cell for each pair $x, y \in A$

$$\mathbf{Kl}(T)(x, y) \xrightarrow{b_{x,y}} B(\ell' x, \ell' y) \cong E(jx, r'\ell' y) \cong E(jx, t'y) = \mathbf{Kl}(T')(x, y)$$

which canonically defines the requisite identity-on-objects $\mathcal{W}$-functor $\mathbf{Kl}(T) \to \mathbf{Kl}(T')$ giving a (necessarily unique) lift of $[]_{T'}$ along b. □

In particular, we recover Pumplün's characterisation of Kleisli categories and morphisms [Pum70, Satz 6] when $\mathcal{W}$ is the delooping of **Set** and j is the identity.

Remark 7.2.8. In fact, $\mathcal{W}$-**Cat** has Kleisli constructions for arbitrary relative monads and Eilenberg–Moore algebra-objects assuming the existence of certain limits, but at present we have no general tools for establishing such results, nor do we make use of relative monads without dense roots in what follows, so we shall not give an explicit construction.

We thus obtain a correspondence between $\mathcal{W}$-enriched j-theories and $\mathcal{W}$-enriched j-monads.

Corollary 7.2.9. *Let $j \colon A \to E$ be a $\mathcal{P}$-admissible dense $\mathcal{W}$-functor. Suppose either that A is small or that j is the identity*[5]*. The categories of j-theories and of j-monads are equivalent.*

$$\mathbf{Th}(j) \simeq \mathbf{RMnd}(j)$$

Proof. Follows from Theorem 6.2.2 in light of Lemma 7.2.6 and Theorem 7.2.7. □

Just as in the formal setting, not every $\mathcal{W}$-enriched j-monad is realisable; we provide sufficient conditions below.

7.3 Comparison with other frameworks

We conclude by justifying our claim that our general correspondence strictly subsumes those that have come before. It would be pleonastic to give an explicit comparison with every enriched monad–theory correspondence in the literature, so we instead start by observing that the correspondences described in Section 7.1 have generalised previous correspondences in one or more of the following ways.

(A) Permitting monads on categories other than the base of enrichment.

(B) Relaxing $j \colon A \to E$ from exhibiting cocompletions under weighted colimits to arbitrary dense fully faithful functors.

(C) Weakening assumptions on the base of enrichment $\mathcal{V}$ (e.g. permitting $\mathcal{V}$ to be nonsymmetric, nonbicomplete, etc.).

[5] These size assumptions are entirely for convenience, since we have only proven existence of Kleisli and Eilenberg–Moore $\mathcal{W}$-categories under these assumptions. Giving direct constructions would permit us to relax these assumptions, but we do not need to do so for our motivating results.

(D) Weakening size assumptions (e.g. permitting $\mathcal{V}$-monads on $\mathcal{V}$).

With respect to these considerations:

(A & B) We take as our j essentially any $\mathcal{P}$-admissible dense $\mathcal{W}$-functor, in particular imposing no restriction on the codomain.

(C) The weakest assumptions on the bases of enrichment that appear in the literature are the nonsymmetric $\mathcal{V}$ of Power [Pow99] and the non-bicomplete $\mathcal{V}$ of Lucyshyn-Wright [Luc16]. Since we consider as our base of enrichment an arbitrary locally bicomplete closed bicategory, we work in a strictly more general setting than Power; we will explain shortly how our assumption of local bicompleteness may be relaxed to subsume also Lucyshyn-Wright's setting.

(D) Though strictly speaking we do require j to be $\mathcal{P}$-admissible, this is only necessary to ensure the requisite presheaf and nerve constructions exist. In the presence of large cocompletions, this restriction is unnecessary. In any case, our motivating examples satisfy $\mathcal{P}$-admissibility.

From the perspective of these axes of generalisation, the correspondences of Power [Pow99] (A), Lucyshyn-Wright [Luc16] (AB), Melliès [Mel10] (CD), and Bourke and Garner [BG19] (CD) form a convex hull for enriched monad–theory correspondences. None strictly subsumes any of the others: however, save for nonsymmetric enrichment, [Pow99] is subsumed by [Luc16]; and save for the presentability assumption on E, [Mel10] is subsumed by [BG19, §7]. We will therefore focus only on the correspondences proven in [Luc16; BG19]; the others then trivially follow given our assumptions.

We should point out that while monad–theory correspondences are the main results of the papers we cite in this section, there are typically not the only results of interest to the authors. It is certainly the case that many, if not all, of these results can also be better understood from the perspective of relative monads. However, we shall focus on the monad–theory correspondence, leaving the investigation of related phenomena to future work.

7.3.1 Lucyshyn-Wright [Luc16]

Fix a closed symmetric monoidal category $\mathcal{V}$. The main result of Lucyshyn-Wright [Luc16] is an equivalence of categories between a full subcategory of $A/\mathcal{V}\text{-}\mathbf{CAT}$ spanned by the j-*theories* and the category of monads on $\mathcal{V}$ preserving a class of weighted colimits. In this sense, Lucyshyn-Wright's correspondence is very much in the spirit of early monad–theory correspondences, where theories and monads are characterised in terms of (co)limit-preservation properties. Conceptually, then, the correspondence follows in much the same way as that of Chapter 3. We recall terminology from [Luc16], though, in keeping with our convention, we dualise appropriately to facilitate a clearer comparison with our framework.

First, we define a *eleutheric system of arities*, which will play the role of the root of a relative monad.

Definition 7.3.1 ([Luc16, Definition 3.1, Proposition 3.10, Theorem 7.8])**.** A *system of arities* is a dense fully faithful strong symmetric monoidal $\mathcal{V}$-functor $j : A \to \mathcal{V}$. A system of arities is *eleutheric* if j exhibits $\mathcal{V}$ as the free cocompletion of A under N_j-weighted colimits.

Theories are then defined analogously to the classical setting (cf. [Lin66b; Dub70]).

Definition 7.3.2 ([Luc16, §4, Definition 4.1]). Let $j : A \to \mathcal{V}$ be a system of arities. A $\mathcal{V}$-functor *preserves* j*-copowers* if it preserves copowers with ja for each $a \in A$. A j*-theory* is an identity-on-objects $\mathcal{V}$-functor $f : A \to B$ that preserves j-copowers.

That j-copower preservation corresponds to relative adjointness could be proven in much the same way as Proposition 3.2.3 and Proposition 3.2.4 in the unenriched setting, but for convenience we prove it directly.

Lemma 7.3.3. *Let $j : A \to \mathcal{V}$ be a eleutheric system of arities. A j-theory in the sense of [Luc16] is precisely a j-theory in the sense of Definition 3.1.13.*

Proof. By the argument of [Luc16, Proposition 9.4], a $\mathcal{V}$-profunctor $P : B \to \mathcal{P}A$ sends j-copowers in A to j-powers if and only if P is *copresheaf-representable* in that there exists a $\mathcal{V}$-functor $u : B \to \mathcal{V}$ such that

$P \cong \mathcal{V}(j-, u-)$ [Luc16, Definition 9.2]. Let $f \colon A \to B$ be a $\mathcal{V}$-functor. Since hom-$\mathcal{V}$-functors preserve weighted limits in their first arguments, the $\mathcal{V}$-profunctor $B(f-, -)$ sends j-copowers in A to j-powers if and only if f is a j-theory in the sense of [Luc16], if and only if $B(f-, -) \cong \mathcal{V}(j-, u-)$, i.e. $f \; {}_j\!\dashv u$. □

In particular, in conjunction with Theorem 3.1.14, this proves an informal conjecture of [Luc16, §1], expressing j-theories as enriched relative monads. We immediately recover Lucyshyn-Wright's monad–theory correspondence.

Corollary 7.3.4 ([Luc16, Theorems 11.8 & 11.14])**.** *Let $\mathcal{V}$ be a closed symmetric monoidal category and let $j \colon A \to \mathcal{V}$ be an eleutheric system of arities for which A is small or j is the identity*[6]*. There is an equivalence of categories,*

$$\mathbf{Th}(j) \simeq \mathbf{Mnd}_{\{N_j\}}(\mathcal{V})$$

where $\mathbf{Mnd}_{\{N_j\}}(\mathcal{V})$ is the category of monads on $\mathcal{V}$ preserving N_j-weighted colimits[7]*, and this commutes with taking algebras.*

Proof. For an eleutheric system of arities, we are in the setting of Theorem 5.5.8, from which the result follows by taking the lax idempotent pseudomonad to be cocompletion under N_j-weighted colimits (cf. [KL00]). □

Universe enlargement

Lucyshyn-Wright does not assume that $\mathcal{V}$ is bicomplete. We shall briefly sketch how it ought to be possible to drop this assumption also in our setting; without a motivating example, we omit a precise treatment. Every 2-category $\mathcal{K}$ embeds into the bicomplete 2-presheaf category $\widehat{\mathcal{K}}$ via the 2-Yoneda embedding. A lax idempotent pseudomonad on $\mathcal{K}$ induces a lax idempotent pseudomonad on $\widehat{\mathcal{K}}$ via left 2-Kan extension and, since the 2-Yoneda embedding is 2-cocontinuous it preserves Eilenberg–Moore objects for admissible relative monads with dense roots. If $\mathcal{K}$ has a resolute factorisation system, $\widehat{\mathcal{K}}$ inherits it (cf. [Hyl14a, §3.2]), in which case $\widehat{\mathcal{K}}$ admits Eilenberg–Moore and Kleisli objects for admissible relative monads with dense roots. We may therefore carry out monad–theory correspondences in $\widehat{\mathcal{K}}$ without concern about existence of Eilenberg–Moore or Kleisli objects. Lastly, one may observe that the monad–theory correspondence preserves representability, in that a theory in $\mathcal{K}$ corresponds to a monad in $\widehat{\mathcal{K}}$ only if the monad is represented by a monad in $\mathcal{K}$, and vice versa. In this way, we may deduce monad–theory correspondences in $\mathcal{K}$ from those in $\widehat{\mathcal{K}}$ by restricting our attention to the representable theories and monads.

In the absence of a 2-categorical framework, a similar approach ought also to be possible, by modifying the base of enrichment rather than the 2-category itself. Every locally small bicategory $\mathcal{W}$ embeds into its local cocompletion $\mathcal{W}'$ [Kel+02, §5], which is locally bicomplete and closed, so that $\mathcal{W}'$-$\mathbf{CAT}$ admits Eilenberg–Moore and Kleisli objects. One may then carry out monad–theory correspondences for $\mathcal{W}'$-categories. These correspondences preserve representability of extent, so that a theory enriched in $\mathcal{W}$ corresponds to a $\mathcal{W}'$-monad only if the $\mathcal{W}'$-monad is represented by a $\mathcal{W}$-monad, and vice versa. In this way, we may deduce monad–theory correspondences in $\mathcal{W}$-$\mathbf{CAT}$ from those in $\mathcal{W}'$-$\mathbf{CAT}$ by restricting our attention to the theories and monads with representable extent. This is essentially the approach taken by Lucyshyn-Wright.

7.3.2 Bourke and Garner [BG19]

Fix a symmetric monoidal category $\mathcal{V}$ and a dense fully faithful $\mathcal{V}$-functor $j \colon A \to E$ with small domain and locally presentable codomain. The main result of Bourke and Garner [BG19] is an equivalence of categories between a full subcategory of $A/\mathcal{V}$-$\mathbf{CAT}$ spanned by the A-*theories* and the category of A-*nervous monads* on E. While the authors demonstrate the implications of this equivalence is broad-ranging, the conceptual meaning of the conditions for an identity-on-objects $\mathcal{V}$-functor $A \to B$ to be an A-theory, and for a monad on E to be nervous, may not appear evident, having been extracted from the fixed point conditions for an adjunction restricting to the monad–theory equivalence, rather than being motivated by first principles. However,

[6] [Luc16] does not have these size conditions, but as mentioned previously we impose them for simplicity. In any case, the examples ibid. satisfy these assumptions, and the extra generality provides no additional clarity.

[7] Here, we employ the notion of colimit weighted by a $\mathcal{V}$-profunctor, rather than simply a $\mathcal{V}$-functor [SW78, §4; Woo82, §2].

we will show that, interpreting their conditions from the perspective of relative monads, both A-theories and A-nervous monads turn out to have entirely natural reformulations that explain their appearance in the monad–theory correspondence of Bourke and Garner.

First, we recall a definition of [BG19] necessary to define the concept of A-theory. Categorical concepts defined using the Yoneda embedding may often be generalised by replacing the Yoneda embedding with the nerve functor $N_j = E(j-,-)$ for some (usually dense) $j\colon A \to E$, and this is typically the correct generalisation for relative monadic concepts: the following is the corresponding generalisation of representable functor.

Definition 7.3.5 (cf. [BG19, §2.1])**.** Let $j\colon A \to E$ be a $\mathcal{P}$-admissible $\mathcal{V}$-functor. A $\mathcal{V}$-presheaf $p\colon A^{\mathrm{op}} \to \mathcal{V}$ is *j-representable*[8] if there exists an object $e \in E$ such that $p \cong N_j(e)$.

Naturally, just as 1_E-adjoint functors are simply adjoint functors, 1_E-representable presheaves are simply representable presheaves on E. The condition for a $\mathcal{V}$-functor to be an A-theory is illuminated by the following lemma.

Lemma 7.3.6. *Let $j\colon A \to E$ and $f\colon A \to B$ be $\mathcal{P}$-admissible $\mathcal{V}$-functors. Of the following, (1) implies (2); if j is dense, then (2) implies (1).*

1. *f is left-j-adjoint.*
2. *For every $y \in B$, the presheaf $B(f-,y)\colon A^{\mathrm{op}} \to \mathcal{V}$ is j-representable.*

Proof. (2) holds when, for every $y \in B$, there exists an $x \in E$ such that $B(f-,y) \cong E(j-,x)$. Define a function $uy := x$, so that $B(f-,y) \cong E(j-,uy)$. u is easily seen to extend uniquely to a $\mathcal{V}$-functor for which the bijections $B(f-,y) \cong E(j-,uy)$ are $\mathcal{V}$-natural in y: for any $v\colon y \to y'$, we have a morphism

$$E(j-,uy) \xrightarrow{\cong} B(f-,y) \xrightarrow{B(f-,v)} B(f-,y') \xrightarrow{\cong} E(j-,uy')$$

which, assuming density of j, hence full faithfulness of N_j, induces a morphism $uy \to uy'$. The isomorphism $B(f-,y) \cong E(j-,uy)$ is thus $\mathcal{V}$-natural in y, essentially by definition, so that $f \;{}_j\!\dashv\; u$. The converse is trivial. $\square$

It is direct that Bourke and Garner's notion of theory coincides with our own[9].

Corollary 7.3.7. *Let $j\colon A \to E$ be a dense $\mathcal{V}$-functor. An* A-theory *in the sense of [BG19, Definition 15] is precisely a j-theory in the sense of Definition 3.1.13.*

Proof. Using our notation, a $\mathcal{V}$-functor $f\colon A \to B$ is an A-theory when it is identity-on-objects and for every $y \in B$, the presheaf $B(f-,y)$ is j-representable. The claim thus follows from Lemma 7.3.6. $\square$

To explain the definition of A-nervous monad, the following lemma will be useful (cf. Corollary 5.5.6 and Remark 5.5.7).

Lemma 7.3.8. *Let $j\colon A \to E$ be a $\mathcal{P}$-admissible $\mathcal{V}$-functor, and let T be a j-monad. The following square commutes up to natural isomorphism if and only if $k_T \mathbin{;} \ell \;{}_j\!\dashv\; r$.*

$$\begin{tikzcd} X \arrow[r, "N_\ell"] \arrow[d, "r"'] & \mathcal{P}(\mathbf{Kl}(T)) \arrow[d, "{k_T}^*"] \\ E \arrow[r, "N_j"'] & \mathcal{P}A \end{tikzcd}$$

[8] j-representable $\mathcal{V}$-presheaves were called *j-nerves* in [BG19]. We rename them to be consistent with terminology such as *j-adjoint*, *j-absolute*, and so on.

[9] Marcelo Fiore has notified the author that he had, by different methods, independently observed in 2018 the equivalence between Bourke–Garner's A-theories, and $\mathcal{V}$-enriched j-monads.

Proof. First, observe that $N_\ell \mathbin{;} {k_T}^* = N_{k_T;\ell}$. Hence the square commutes up to natural isomorphism if and only if $N_{k_T;\ell} \cong r \mathbin{;} N_j$, i.e. if and only if $X(\ell k_T -, -) \cong E(j-, r-)$, which holds if and only if $k_T \mathbin{;} \ell \mathbin{{}_j{\dashv}} r$. □

We may now show that the nervousness condition of Bourke and Garner is equivalent to asking for the canonical functor $\mathbf{EM}(T) \to \mathbf{EM}(j \mathbin{;} T)$ to be essentially-surjective-on-objects and fully faithful, hence an equivalence[10]: which, by Theorem 5.4.6, is precisely the condition that T is j-ary.

Lemma 7.3.9. *Let $j\colon A \to E$ be a $\mathcal{P}$-admissible dense $\mathcal{V}$-functor. A monad is* A-nervous *in the sense of [BG19, Definition 17] precisely when it is j-ary in the sense of Definition 5.4.1.*

Proof. Using our notation, a monad T on E is A-nervous when

1. $q_T \mathbin{;} i_T \colon \mathbf{Kl}(j \mathbin{;} T) \to \mathbf{Kl}(T) \to \mathbf{EM}(T)$ is dense.
2. A presheaf $p\colon \mathbf{Kl}(j \mathbin{;} T)^{\mathrm{op}} \to \mathcal{V}$ is $(q_T \mathbin{;} i_T)$-representable if and only if ${k_{j;T}}^{\mathrm{op}} \mathbin{;} p\colon A^{\mathrm{op}} \to \mathbf{Kl}(j \mathbin{;} T)^{\mathrm{op}} \to \mathcal{V}$ is j-representable.

First, we show that $N_{q_T;i_T}$ is naturally isomorphic to $r_T \mathbin{;} N_{i_{j;T}} \colon \mathbf{EM}(T) \to \mathbf{EM}(j \mathbin{;} T) \to \mathcal{P}(\mathbf{Kl}(j \mathbin{;} T))$. To see this, observe that the following diagram commutes up to natural isomorphism,

$$\begin{tikzcd}
\mathbf{EM}(T) \arrow[r, "N_{q_T;i_T}"] \arrow[d, "u_T"'] & \mathcal{P}(\mathbf{Kl}(j \mathbin{;} T)) \arrow[d, "{k_{j;T}}^*"] \\
E \arrow[r, "N_j"'] & \mathcal{P}A
\end{tikzcd}$$

by Lemma 7.3.8, since $k_{j;T} \mathbin{;} q_T \mathbin{;} i_T = k_T \mathbin{;} i_T = f_T$ and $f_T \mathbin{{}_j{\dashv}} u_T$, and hence there is a mediating morphism $\mathbf{EM}(T) \to \mathbf{EM}(j \mathbin{;} T)$ by Corollary 5.5.6 and Remark 5.5.7 such that postcomposing $u_{j;T}$ gives u_T. Since r_T is the unique $\mathcal{V}$-functor with this property, the claim follows.

Consequently, (1) asks for $r_T \mathbin{;} N_{i_{j;T}}$ to be fully faithful, which, since $N_{i_{j;T}}$ is fully faithful by Corollary 5.5.6, implies that r_T is also fully faithful; and conversely. (2) asks that every presheaf P on $\mathbf{Kl}(j \mathbin{;} T)$ is in the essential image of $r_T \mathbin{;} N_{i_{j;T}}$ if and only if ${k_{j;T}}^*(P) \cong N_j(e)$ for some object $e \in E$. In particular, every presheaf in the image of $N_{i_{j;T}}$ satisfies this latter condition by Corollary 5.5.6 and Remark 5.5.7, and is hence in the image of $r_T \mathbin{;} N_{i_{j;T}}$, exhibiting r_T as being essentially-surjective-on-objects; and conversely.

Therefore, (1 & 2) ask exactly that r_T to be fully faithful and essentially-surjective-on-objects, i.e. an equivalence of categories $\mathbf{EM}(T) \simeq \mathbf{EM}(j \mathbin{;} T)$, which necessarily commutes up to isomorphism with the forgetful functors. Hence, by Theorem 5.4.6(3), T is A-nervous if and only if it is j-ary. □

The definitions of A-theories and A-nervous monads are thus seen to have natural reformulations in the language of relative monads. To demonstrate that the monad–theory correspondence of [BG19] follows as a consequence of our general theory, we observe finally that local presentability of the codomain of $j\colon A \to E$ is a particularly useful assumption, as it renders every j-monad realisable.

Lemma 7.3.10. *Let E be a locally-presentable $\mathcal{V}$-category, and let $j\colon A \to E$ be a dense functor with small domain. Every j-monad T is realisable.*

Proof. First observe that $\mathbf{EM}(T)$ is cocomplete, since the following square forms a (pseudo)pullback by Theorem 5.5.2,

$$\begin{tikzcd}
\mathbf{EM}(T) \arrow[r] \arrow[d, "u_T"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{EM}(T \mathbin{;} N_j) \arrow[d, "u_{T;N_j}"] \\
E \arrow[r, "N_j"'] & \mathcal{P}A
\end{tikzcd}$$

[10] We owe this observation to Dylan McDermott.

and the 2-category of locally presentable categories and right adjoint functors is closed under bilimits [Bir84, Theorem 6.11]. Since A is small and $\mathbf{EM}(T)$ is cocomplete, the pointwise left extension $j \rhd f_T$ exists.

$$
\begin{tikzcd}
 & \mathbf{EM}(T) \\
A \arrow[ur, "f_T"] \arrow[r, "j"'] & E \arrow[u, dashed, "j \rhd f_T"']
\end{tikzcd}
$$

Hence, Proposition 5.4.7 together with Lemma 5.4.4 implies that T is realisable. □

We then recover the main theorem of [BG19].

Corollary 7.3.11 ([BG19, Theorem 19]). *Let $\mathcal{V}$ be a locally presentable closed symmetric monoidal category, A be a small $\mathcal{V}$-category, E be a locally presentable $\mathcal{V}$-category, and $j\colon A \to E$ be a dense fully faithful $\mathcal{V}$-functor. There is an equivalence of categories*

$$\mathbf{Th}(j) \simeq \mathbf{Mnd}_j(E)$$

Proof. Since every j-monad is realisable, by Lemma 7.3.10, the result follows from Corollary 5.4.8 together with Corollary 7.2.9. □

Bourke and Garner obtain their main theorem as a restriction of an adjunction between *pretheories* (i.e. identity-on-objects functors from A) and monads on E. Unfortunately, this is not possible in much greater generality: constructing a monad from a pretheory makes crucial use of local presentability of E to guarantee the existence of an adjoint. In this sense, we do not view the monad–pretheory adjunction as a crucial aspect of the monad–theory correspondence. However, the monad–pretheory adjunction can be seen to arise as a facet of a more general phenomenon: the general structure–semantics adjunction of Linton [Lin69a]. In future work, we will explain how the (relative) monad–theory correspondence can be seen to arise from the structure–semantics adjunction, from which the monad–pretheory adjunction follows immediately assuming local presentability of E (cf. [Ark21]).

7.3.3 Johnstone and Wraith [JW78]

The one monad–theory correspondence that is not presented in the setting of enriched category theory is the internal monad–theory correspondence of Johnstone and Wraith [JW78]. At first glance, this appears to be a good candidate for an application of the formal monad–theory correspondence, working within the 2-category of categories internal to a topos $\mathcal{E}$. While this would be one reasonable approach, we may actually take advantage of an observation of Betti and Walters [BW85] to rephrase the correspondence as an enriched monad–theory correspondence. In particular, this demonstrates the strength of bicategory-enrichment over enrichment in a monoidal category. We shall briefly sketch how the internal monad–theory correspondence may be obtained from our general framework.

Let $\mathcal{E}$ be a topos with a natural numbers object. In [JW78, Definition 7.1], the authors define an *algebraic theory internal to a* $\mathcal{E}$ to be a monad internal to the 2-category of categories locally internal to $\mathcal{E}$. By [BW85, Proposition 4.3], this 2-category is biequivalent to the 2-category of $\mathbf{Span}(\mathcal{E})$-enriched categories with *restrictions* [BW85, Definition 1.8], which are a class of absolute weighted colimits. Though Johnstone and Wraith equate theories with monads, we may more precisely define $\mathcal{E}$-internal algebraic theories to be identity-on-objects $\mathbf{Span}(\mathcal{E})$-functors from $\mathcal{E}$ and $\mathcal{E}$-internal monads to be $\mathbf{Span}(\mathcal{E})$-monads on $\mathcal{E}$, from which we get an $\mathcal{E}$-internal monad–theory correspondence by Theorem 6.2.2.

For the finitary case, we may consider $j = (\mathcal{E}_f \hookrightarrow \mathcal{E})$, the inclusion of the free cocompletion of 1 under finite $\mathcal{E}$-indexed coproducts into the free cocompletion of 1 under $\mathcal{E}$-indexed colimits [Joh02, Example B3.2.9]. The category $[\mathcal{E}_f, \mathcal{E}]$ is the classifying topos of objects over $\mathcal{E}$ [Joh02, Example 4.2.4(a)], and so j-monads are precisely the *finitary algebraic theories internal to* $\mathcal{E}$ of [JW78, Definition 5.3]. We thus recover both the correspondence between *internal finitary algebraic theories* and *internal Lawvere theories* (i.e. j-theories) [Joh77, Theorem 5.14], and the correspondence between internal finitary algebraic theories and finitary $\mathbf{Span}(\mathcal{E})$-monads on $\mathcal{E}$ [JW78, Corollary 7.6].

7.3.4 Representations of relative monads

We shall conclude by rephrasing several definitions appearing in Section 7.1 in terms of relative monads, thus explaining their appearance in the monad–theory correspondence literature. First, we observe that monads relative to the Yoneda embedding have various equivalent formulations.

Proposition 7.3.12. *Let A be a small $\mathcal{W}$-category. The following categories are equivalent:*

1. *The full subcategory of $A/\mathcal{W}$-**Cat** spanned by identity-on-objects $\mathcal{W}$-functors.*
2. $\mathbf{Th}(よ_A)$*;*
3. $\mathbf{Mnd}_{\mathcal{P}}(\mathcal{P}A)$*.*
4. *The category of monads on A in $\mathcal{W}$-**Prof**.*

Proof. (1) $\iff$ (2) follows from the enriched analogue of Proposition 3.2.2, since every $\mathcal{W}$-functor with domain A is $\mathcal{P}$-admissible. (2) $\iff$ (3) follows from Theorem 5.5.8 together with Theorem 6.2.2. (3) $\iff$ (4) follows by the biequivalence between $\mathcal{W}$-**Prof** and the full sub-bicategory of $\mathcal{W}$-**Cocts** on the presheaf categories, given by the inclusion of $\mathbf{Kl}(\mathcal{P})$ in $\mathbf{EM}(\mathcal{P})$. □

These observations are not new (at least in the unenriched setting), though our proof, which relies on our conceptual understanding of the monad–theory correspondence, is new. (1) $\iff$ (4) is originally due to Justesen [Jus68, p. 6.22] (cf. [Thi71, Proposition II.1.5]); an enriched version appears in [Luc16, Corollary 10.4]. (3) $\iff$ (4) is originally due to Thiébaud [Thi71, Remark III.1.4] (in dual form). (2) $\iff$ (3) is originally due to Diers [Die75, Exemple 5.5].

Let $j \colon A \to E$ be a $\mathcal{W}$-functor. Since j-theories are in particular identity-on-objects $\mathcal{W}$-functors, we may ask how the different characterisations of Proposition 7.3.12 restrict correspondingly. The answer is given by Corollary 5.5.11: namely, the cocontinuous monads on $\mathcal{P}A$, and hence also the monads on A in $\mathcal{W}$-**Prof**, corresponding to j-theories are precisely those of the form $t;N_j$, for some $\mathcal{W}$-functor $t \colon A \to E$. Taking j to be fully faithful, we thereby recover Diers's definition of j-monad [Die75, Définitions 1.0], since, when j is dense and fully faithful, a 2-cell $\eta \colon j \Rightarrow t$ is equivalently given by a 2-cell $よ_A \cong j\,; n_j \Rightarrow t\,; n_j$, which is the unit of a monad on A in $\mathbf{Kl}(\mathcal{P})$. Furthermore, this characterisation justifies the approach of Lee, who represents relative monads by cocontinuous monads on presheaf categories [Lee77, Chapter 2]. Finally, we recover Lucyshyn-Wright's characterisation of j-monads (there given as a characterisation of j-theories) [Luc16, Theorem 10.5], who calls monads in $\mathcal{V}$-**Prof** of the form $t\,;N_j$ *copresheaf-representable* [Luc16, Definition 9.2].

7.3.5 Examples

We give a few examples of monad–theory correspondences arising as special cases of the general monad–theory correspondences for $\mathcal{W}$-**CAT**. Further examples may be found in the papers referenced in the previous section.

Example 7.3.13. *Arrows* on a small category A, in the sense of Hughes [Hug00], are $よ_A$-relative monads by [ACU15, Theorem 5.2], thus arbitrary identity-on-objects functors from A by Corollary 7.2.9. They are hence equivalent to cocontinuous monads on $\mathcal{P}A$.

Example 7.3.14. Let $\mathcal{V} = \mathbf{Cat}$ and $j = (\mathbb{F}(1) \hookrightarrow \mathbf{Sind}(\mathbb{F}(1)) \simeq \mathbf{Cat})$. Then j-theories are the *2-algebraic theories* of Gray [Gra73] and the *discrete finitary Lawvere $\mathcal{V}$-theories* of Power [Pow05] and Hyland and Power [HP06], which are equivalent to the *strongly finitary* monads on **Cat** of [KL93]. Taking instead $\aleph_1$-ary coproducts similarly recovers the *discrete countable Lawvere $\mathcal{V}$-theories* of [Pow05; HP06].

Example 7.3.15. Let $\mathbf{CGTop}_*$ denote the closed symmetric monoidal category of pointed compactly generated spaces. Considering enrichment in $\mathcal{V} = \mathbf{CGTop}_*$ and taking j to be the inclusion of finite copowers of I into $\mathcal{V}$, we recover the *finitary pointed topological theories* of Beck [Bec69, (6)].

Example 7.3.16. The *parameterised algebraic theories* of Staton [Sta13a; Sta13b] were exhibited in [Sta13a, Theorem 2] as enriched abstract clones, hence enriched relative monads. By Corollary 7.2.9, they are equivalently given by $\mathcal{P}\mathscr{L}$-enriched $(\mathbb{F}\mathscr{L} \to \mathcal{P}\mathscr{L})$-theories for $L \colon \mathbb{L} \to \mathscr{L}$ a parameterising algebraic theory, hence are equivalent to sifted-cocontinuous $\mathcal{P}\mathscr{L}$-monads on $\mathcal{P}\mathscr{L}$.

Example 7.3.17. As observed by Staton [Sta14], *Freyd categories* [PT97; PT99] are equivalently $\mathcal{V}$-enriched $(\mathcal{V}_f \to \mathcal{V})$-theories, for $\mathcal{V}$ a locally $\emptyset$-presentable cartesian-closed category in the sense of Adámek, Borceux, Lack and Rosický [Adá+02] ([Sta14, Theorem 3.3]); while *distributive Freyd categories* [Pow04; Pow06] are $\mathcal{V}$-enriched $(\mathcal{V}_f \to \mathcal{V})$-theories, for $\mathcal{V}$ a locally **FinProd**-presentable cartesian-closed category in the sense of Lack and Rosický [LR11] ([Sta14, Theorem 3.5]). As such, both correspond to classes of enriched monads preserving certain colimits.

Example 7.3.18. In [GP98], Gordon and Power study algebraic structure for categories enriched in bicategories. In particular, they are interested in (conically) finitary $\mathcal{W}$-monads on a closed locally (conically) finitely presentable bicategory A. Though they do not provide a monad–theory correspondence, our theory provides one. Their motivating example of $\mathcal{W}$ is (the nonsymmetric monoidal category) $\mathbf{LocOrd}_f$, the category of small locally-ordered categories equipped with the lax Gray tensor product [KP96].

Chapter 8

Concluding remarks

In these pages, we have studied two strands of structure, which at first appear entirely unrelated to one another, save for their common origin in categorical logic. Indeed, even setting aside for a moment the presence of higher-order structure, the relationship between algebraic structure and monadic structure is quite striking: while we have thoroughly examined the correspondence between algebraic theories and monads, to the point at which we feel justified in claiming to have dispelled any mystery behind the phenomenon, that is not to say that we find the correspondence unremarkable. Quite to the contrary, that algebraic theories and monads arose separately, motivated by entirely different questions, but happened to be perfectly in correspondence, still appears to us a paradigmatic example of the beauty of mathematics. That this relationship furthermore extends from the first-order setting to the higher-order setting is deeply telling of the expressivity of (relative) monadic structure.

We view the contributions of the two parts of this thesis quite differently. The formalism of higher-order algebraic theories is a valuable tool for reasoning about higher-order and variable-binding structure, and we have pursued a thorough treatment in order to facilitate the application of higher-order algebraic theories to problems in theoretical computer science and elsewhere. By establishing that many of the fundamental results in the theory of algebraic theories extend to higher-order algebraic theories (the monad–theory correspondence being just one example of such a result), it ought to be straightforward for anyone familiar with algebraic theories to use the same techniques to study higher-order algebraic theories. On the other hand, while we have developed a general framework for monad–theory correspondences, which may very well be used to recover new correspondences of interest, we view the main contribution of the second half of this thesis as conceptual rather than practical. In particular, with the understanding that theories are relative monads, the rest of the development flows naturally and is, in some sense, inevitable. The story we have told here is far from complete: some parts we have merely alluded to, whilst others are, as of yet, waiting to be told. We shall go into a little more detail below about the subsequent directions in which we expect our work to be developed.

8.1 Future work

There are a number of lacunae in the development we have presented, an inevitable consequence of temporal constraints. There are also a number of future directions we hope to pursue.

8.1.1 Higher-order algebraic theories

In Chapter 4, we established a (relative) monad–theory correspondence for higher-order algebraic theories. However, the (relative) monads involved are subject to a technical condition (namely, $+$-linearity) that may leave the reader feeling somewhat dissatisfied. With Dylan McDermott, we plan to rephrase the $+$-linearity condition in terms of an enrichment in a particular nonsymmetric monoidal category: the general monad–

theory correspondence of Chapter 7 will then exhibit the relationship between higher-order algebraic theories and $+$-linear (relative) monads as an enriched monad–theory correspondence. In fact, this has been a secondary motivation for taking our base of enrichment in Chapter 7 to be a bicategory, as there was previously no enriched monad–theory correspondence with general enough base to capture this setting of interest[1].

More generally, the inductive construction of the category of $(n+1)^{\text{th}}$-order algebraic theories in terms of the category of n^{th}-order algebraic theories suggests a fruitful pursuit of similar understandings for other structures in categorical logic: for instance, operads [May97] and substructural type theories [TP05; TP06], essentially algebraic theories [Fre72], and generalised algebraic theories [Car78]. That the latter in particular might be understood this way is suggested by the work of Uemura [Uem20]: indeed, gaining a better understanding of higher-order structure for dependent type theories by first understanding that of simple type theories was one of the original motivations for studying higher-order algebraic theories.

Finally, as we mentioned in the introduction, we should like to represent general simple type theories in a manner similar to algebraic theories. Here, we have described one aspect: that of higher-order structure. To describe simple type theories, we must additionally consider algebraic structure on the sorts of a (higher-order) algebraic theory. In the future, we intend to show that this structure is naturally understood from the perspective of enriched category theory. Furthermore, an understanding of algebraic type structure is not far removed from higher-order algebraic type structure, or from polymorphic type structure, facilitating an algebraic-theoretic perspective on the work of Fiore and Hamana [FH13]. We expect that such theories may also be understood monadically.

8.1.2 The formal theory of relative monads

As we stated in Chapter 5, we developed the formal theory of relative monads only to the extent necessary to establish our main results (namely, those directly pertinent to the monad–theory correspondence). Notable omissions include the consideration of relative monads as monoids in skew-monoidal hom-categories; the study of left-morphisms of relative adjunctions, and of right-modules (or *opalgebras*); a detailed comparison of relative adjunctions and (left- and right-)modules; and definitions[2] of 2-categories of relative monads in a proarrow equipment and its relation to the construction of (op)algebras.

8.1.3 The monad–theory correspondence

We focused in Chapter 7 only on monad–theory correspondences for enriched categories (consequently also recovering those for internal categories). While this setting is particularly useful, it is also, to some extent, the least interesting: though our correspondence is more general than appears previously in the literature, and to that extent may be used to obtain new monad–theory correspondences in specific cases of interest, in practice prior frameworks usually suffice. The power of our approach is that we may obtain correspondences in other 2-categories for which there do not currently exist general monad–theory correspondences, or for varieties of algebraic theories and monads that do not currently fit within a general framework. We have several open questions in this vein.

1. Can the correspondence between commutative algebraic theories [Lin66a, §6] and commutative monads [Koc70] be recovered as a monad–theory correspondence in a suitable 2-category? Similar questions may be posed for other varieties of algebraic theories (e.g. affine algebraic theories [Law68] and affine monads [Lin79]).
2. Can Diers's correspondence between multialgebraic theories and multimonads [Die80a; Die80b; Die83] be recovered using our 2-categorical framework?
3. Do essentially algebraic theories, viewed as partial algebraic theories in the sense of Di Liberti, Loregian, Nester and Sobociński [Di +21], form a monad–theory correspondence in a 2-category of restriction

[1] Power's correspondence for enrichment in a nonsymmetric monoidal category [Pow99] is close, but takes place in the setting of local presentability, rather than local strong presentability.

[2] There is evidence to suggest that the most natural definition of morphism of relative monads in a 2-category, for instance as in [Lob20, Definition 3.1], is too strong to recover certain phenomena of interest, and that it is worth instead studying (suitably defined) *left-* and *right-morphisms* of relative monads, in analogy to left- and right-morphisms of relative adjunctions.

categories [CL02] (perhaps using enrichment in a double category [CG14])?

More philosophically, with regards to (1), our perspective of theories as relative monads clarifies the role of theories in the study of monads, suggesting that, given some property φ of a relative monad in a proarrow equipment $(-)_* \colon \mathcal{K} \to \mathcal{N}$, it is fruitful to find some property φ' of its Kleisli embedding such that a relative monad T has property φ if and only if $k_T \colon A \to \mathbf{Kl}(T)$ has property φ'. It is likely in these situations that there is a proarrow equipment in which the relative monads are precisely the relative monads in $(-)_*$ satisfying φ, and that the Kleisli objects are those in $(-)_*$ satisfying φ'; and it may be possible in these cases to establish the correspondence abstractly using the general 2-categorical techniques we have outlined.

The monad–theory correspondence is part of a much larger story. Aspects absent from our treatment here involve the structure–semantics adjunction, relative monadicity, the construction of free relative monads, relative adjoint functor theorems, commutativity, duality, among others. In this thesis, we have given a small taste of the insight provided by the perspective of theories as relative monads; in future work, we will show that there remains much more to say.

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