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Vector Bundles on Projective 3-Space

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§ 1. Introduction

On a compact complex manifold X it is an interesting problem to compare the continuous and holomorphic vector bundles. The case of line-bundles is classical and is well understood in the framework of sheaf theory. On the other hand for bundles E with $\dim E \geqq \dim X$ we are in the stable topological range and one can use K-theory. Much is known in this direction, for example the topological and holomorphic K-groups of all complex projective spaces are isomorphic.

This paper deals with what is perhaps the simplest case not covered by the methods indicated above. We shall consider 2-dimensional complex vector bundles over the 3-dimensional complex projective space P_3. Our aim is to prove

(1.1) **Theorem.** *Every continuous 2-dimensional vector bundle over P_3 admits a holomorphic structure.*

The corresponding result for P_2 was proved by Schwarzenberger [13], but this falls into the class of stable problems. In particular 2-dimensional vector bundles over P_2 are determined by their Chern classes c_1, c_2. This is no longer true on P_3 and therein lies the main difficulty and also the interest of this paper. In fact Horrocks in [10] has already constructed holomorphic (actually algebraic) bundles with arbitrary c_1, c_2 subject only to the topologically necessary condition that $c_1 c_2$ be even [8; p. 166].

It is not hard to see that, topologically, there are at most two bundles on P_3 with given c_1, c_2. The two possibilities arise because the homotopy group

$$\pi_5(U(2)) \cong \pi_5(S^3)$$

which classifies 2-dimensional bundles over S^6, and acts on the bundles over P_3, has order 2. It turns out that there are two sharply different cases depending on the parity of c_1.

In §2 we study the case of even c_1. By tensoring with line-bundles one reduces to the case of $c_1 = 0$ in which case the structure group is $SU(2) \cong Sp(1)$. We view our 2-dimensional complex vector bundle as a quaternion line-bundle and this simplifies the classification because quaternion line-bundles over P_3 are already *stable.* The abelian group $\widetilde{KSP}(P_3)$ which classifies these is shown to be $Z_2 \oplus Z$ so that (Theorem (2.8)) in addition to the integer invariants $c_1(E), c_2(E)$ there is also in this case a mod 2 invariant $\alpha(E)$. In other words $\pi_5(U(2))$ acts freely when c_1 is even.

In §3 we examine the case of odd c_1. To apply similar methods to those of §2 it is necessary to consider twisted symplectic bundles and the corresponding twisted K-theories. The conclusion (Theorem (3.3)) is that $c_1(E), c_2(E)$ are now the only invariants, or that $\pi_5(U(2))$ acts trivially when c_1 is odd.

In view of Horrocks' result, Theorem (1.1) for odd c_1 follows from Theorem (3.3). The interesting and more subtle case is for c_1 even. In §4 we show that, if E is holomorphic, $\alpha(E)$ can be computed in terms of sheaf cohomology. Namely if $2n = -(c_1(E)+4)$, and if $E(n) = E \otimes H^n$, where H is the Hopf bundle, then

(1.2) $\alpha(E) = \dim H^0(P_3, E(n)) + \dim H^2(P_3, E(n)) \bmod 2.$

This identification of the topological invariant $\alpha(E)$ with a holomorphic semi-characteristic is a special case of the general index theorem of [5]. Another related special case was considered in [3] and we take this opportunity in §4 of giving a more general version (Theorem (4.2)), valid for any odd-dimensional compact complex manifold. Algebraic geometers may find Theorem (4.2) of some independent interest.

With the explicit formula (1.2) it is then quite easy to compute $\alpha(E)$ for the Horrocks examples. This is carried out in §5 and the final formulae are given in Theorem (5.9).

To prove Theorem (1.1) for c_1 even it is then only necessary to show that, by varying the parameters in Horrocks' construction, we get all possible triples c_1, c_2, α. This involves a certain amount of elementary computation which is the content of §6.

In §7 we show (Theorem (7.2)) that only *one* of the two bundles E on P_3 with given c_1, c_2 (c_1 even) extends topologically to P_4. Moreover we show that the α-invariant of the one which extends is given by

$$\alpha(E) = \frac{\Delta(\Delta-1)}{12} \bmod 2$$

where $4\Delta = c_1(E)^2 - 4c_2(E)$. Again this is best understood in a wider context and we establish a more general result, both holomorphically (Theorem (7.4)) and topologically (Theorem (7.9)).

On P_n we have standard generators for $H^{2q}(P_n, Z)$ ($q \leq n$) and it is convenient therefore to identify cohomology classes with integers. In particular we shall regard the Chern classes $c_i(E)$ as integers.

§2. Topology of Symplectic Bundles

In this section we shall give the complete topological classification of 2-dimensional complex vector bundles over P_3 with *even* first Chern class. Because of the formula

$$c_1(E \otimes L) = c_1(E) + 2c_1(L)$$

where E is 2-dimensional and L is a line-bundle, we can normalize the problem by fixing $c_1(E)$ to be any even multiple of the generator of $H^2(P_3, Z)$.

If we take $c_1(E)=0$ the structure group of E reduces canonically from $U(2)$ to $SU(2)$: this follows by using the fibration $BSU(2)\to BU(2)\to BS^1$ and the fact that $H^1(P_3, Z)=0$.

Now $SU(2)\cong Sp(1)$, so we can view E as a symplectic line-bundle. The advantage of this is that, over a base space of dimension $\leqq 6$, symplectic line-bundles are already *stable*. In other words the inclusion $Sp(1)\to Sp(n)$ given in quaternion matrix form by

$$Q \mapsto \begin{pmatrix} Q & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

induces a bijection between $Sp(1)$-bundles and $Sp(n)$-bundles. The reason is that $Sp(n)/Sp(1)$ is 6-connected (recall the fibrations $Sp(1)\mapsto Sp(2)\to S^7$ etc.) and so $BSp(1)\mapsto BSp(n)$ is a homotopy equivalence up to dimension 6.

Note that $U(2)$ bundles over P_3 are not stable as unitary bundles because $U(3)/U(2)=S^5$: we have to go to $U(3)$ before unitary stability sets in.

Stable symplectic bundles over a space X form an abelian group which in K-theory notation is denoted by $\widetilde{KSP}(X)$. In virtue of the Bott periodicity theorems this group is isomorphic to $\widetilde{KO}^{-4}(X)$. Complete information about all the groups $KO^q(P_n)$ is given by the following proposition which will be used also in the subsequent sections.

(2.1) **Proposition.**

(i) $\widetilde{KO}^q(P_{2n})=0$ *for q odd*

$=Z^n$ *for q even,*

(ii) $\widetilde{KO}^q(P_{2n-1})\cong KO^q(P_{2n-2})\oplus KO^{q-4n+2}$ (point),

(iii) $\widetilde{K}^q(P_{2n})\to KO^q(P_{2n})$ *is surjective for all q.*

Proof. We prove (i) by induction on n. For $n=1$ this is contained in a much more general theorem of R. Wood which asserts that, for any X, we have a canonical isomorphism

$$KO^q(X\times P_2, X\times P_0)\cong K^q(X).$$

The proof of Wood's theorem consists in comparing the exact KO-sequence of the triple $(X\times P_2, X\times P_1, X\times P_0)$ with the exact sequence [2; (3.4)]

(2.2) $$KO^{q-2}(X)\to K^q(X)\to KO^q(X)\xrightarrow{\eta} KO^{q-1}(X)$$

and showing that they are naturally isomorphic (where η denotes multiplication by the generator of $KO^{-1}(\text{point})=Z_2$).

For the inductive step we use the exact sequence

$$(2.3) \quad \to KO^q(P_{2n}, P_{2n-2}) \to KO^q(P_{2n}, P_0) \to KO^q(P_{2n-2}, P_0) \to 0.$$

Since P_{2n}/P_{2n-2} is the Thom space of $(2n-1)$-copies of the Hopf bundle over P_1, it is the $(2n-2)$-suspension of P_2, and so

$$KO^q(P_{2n}, P_{2n-2}) \cong KO^{q-2n+2}(P_2, P_0).$$

Substituting this in (2.3) we see that (i) follows and moreover (2.3) breaks up into short exact sequences. Comparing (2.3) with the exact sequence for $(P_{2n-1}, P_{2n-2}, P_0)$ it follows that this latter also breaks up into short exact sequences

$$(2.4) \quad 0 \to KO^q(P_{2n-1}, P_{2n-2}) \to KO^q(P_{2n-1}, P_0) \to KO^q(P_{2n-2}, P_0) \to 0.$$

Since the last group in this sequence is free abelian (2.4) splits. Since

$$KO^q(P_{2n-1}, P_{2n-2}) = \widetilde{KO}^q(S^{4n-2}) = KO^{q-4n+2}(\text{point}),$$

this proves (ii). Finally, since $KO^q(P_{2n}, P_0)$ is torsion-free, the homomorphism η in (2.2) applied with $X = (P_{2n}, P_0)$ is zero, and hence the preceding homomorphism is surjective, proving (iii).

There is a canonical choice for the splitting of (2.4), i.e. for the isomorphism (ii), arising from the fact that P_{2n-1} is a *Spin-manifold.* We recall that, if X, Y are Spin-manifolds of dimension m, n respectively, then a proper map $f\colon X \to Y$ induces a natural homomorphism (for KO with compact supports)

$$f_!\colon KO^q(X) \to KO^{q+n-m}(Y).$$

In particular taking f to be first the inclusion of a point in a compact manifold X and then the projection $X \to$ point, we see that $KO^{q-n}(\text{point})$ is a direct factor of $KO^q(X)$. Moreover, for the inclusion i: point $\to X$ $i_!$ is defined as the composition of the Thom isomorphism

$$KO^{q-n}(\text{point}) \to KO^q(U)$$

(U an open ball around the point) and the natural map

$$KO^q(U) \to KO^q(X).$$

This shows that $i_!\colon KO^{q-n}(\text{point}) \to KO^q(X)$ coincides with the homomorphism

$$f^*\colon \widetilde{KO}^q(S^n) \to KO^q(X)$$

induced by the collapsing map $X \to S^n$ of degree one obtained by compactifying U.

Since $c_1(P_{2n-1}) = 2n$ the second Stiefel-Whitney class ω_2, which is $c_1 \bmod 2$, vanishes. Hence P_{2n-1} has a Spin structure and, since P_{2n-1} is simply -connected, the Spin-structure is unique. Hence, applying the above remarks concerning direct image homomorphisms, we see that the projection

$$(2.5) \quad \alpha\colon KO^q(P_{2n-1}, P_0) \to KO^{q-4n+2}(\text{point})$$

in the decomposition (ii) can be taken to be the direct image associated to the map $P_{2n-1} \to$ point.

We now return to the special case of P_3 in which we are interested. Putting $q=-4$, and $n=2$ in (2.1)(ii), we get (using (i) also)

(2.6) $\widetilde{KSP}(P_3) \cong Z_2 \oplus Z$

the projection onto the Z_2-component being given by the direct image homomorphism α. The Z-component is easily identified with the second Chern class. Adding a component for the augmentation to (2.6) we clearly get a decomposition for the unreduced group

(2.7) $KSP(P_3) \cong Z_2 \oplus Z \oplus Z.$

Now as we shall see at the end of § 4 (Remark (3)) we have $\alpha(1)=0$, where 1 denotes the trivial quaternionic line bundle. This is also a consequence of the index theorem of [5; (3.3)], identifying $\alpha(1)$ with the dimension mod 2 of the space of harmonic spinors, together with Lichnerowicz's theorem [11] that this space is zero if the scalar curvature is positive. Thus the projection onto the Z_2-component in (2.7) is still given by α.

If E is a 2-dimensional complex vector bundle over P_3 with vanishing c_1, we shall also denote by $\alpha(E)$ the value of α on the symplectic class $[E] \in KSP(P_3)$. More generally if $c_1(E)$ is even we shall put $\alpha(E)=\alpha(E \otimes L)$ where L is the line-bundle with $c_1(L)=-\frac{1}{2}c_1(E)$, so that $c_1(E \otimes L)=0$. We can now summarize our results as follows:

(2.8) **Theorem.** *A 2-dimensional complex vector bundle E over P_3 with even first Chern class is uniquely determined by the three invariants $c_1(E)$, $c_2(E)$, $\alpha(E)$. Moreover all possible triples occur.*

§ 3. Twisted Symplectic Bundles

In § 2, for bundles with c_1 even, we converted an unstable problem into a stable one by passing from the unitary group to the symplectic group. If c_1 is odd we shall now show that an analogous device works provided we "twist" the symplectic theory in an appropriate sense.

Consider in general a complex line-bundle L over a space X. By an L-symplectic bundle over X we shall mean a complex vector bundle E over X together with a non-degenerate skew-symmetric L-valued bilinear form:

$$\phi: E \otimes E \to L.$$

For example a vector bundle E of dimension 2 has a canonical L-symplectic structure, where $L=\Lambda^2(E)$ is the second exterior power. In particular $E_0=L\oplus 1$, where 1 is the trivial line-bundle, is naturally L-symplectic.

If E, F are L-symplectic, $E\oplus F$ has a natural L-symplectic structure so we can form a Grothendieck group $KSP_L(X)$. As usual the corresponding reduced group $\widetilde{KSP}_L(X)$ can be identified with stable L-symplectic bundles, where we stabilize by adding copies of E_0. We just have to observe that any L-symplectic bundle E is locally isomorphic to nE_0, for some integer n, and hence using a partition of unity we can construct an epimorphism $NE_0 \to E$ (compatible with the bilinear forms). Moreover stability sets in, i.e. $E \mapsto E \oplus E_0$ gives a bijection on L-symplectic classes, as soon as $\dim_{\mathbb{C}} E \geqq \frac{1}{2}\dim X - 1$. In fact if $\dim_{\mathbb{C}} F \geqq \frac{1}{2}\dim X + 1$ there is a non-zero section s of F, unique up to homotopy, and the sub-bundle E_0 can be constructed from s: it is spanned by s and $\mathrm{Ann}(s)^{\perp}$, where $\mathrm{Ann}(s) \subset F$ is the codimension one bundle of elements annihilating s (for the bilinear form) and $\perp$ denotes an orthogonal complement (using any auxiliary metric).

Suppose now that $\dim X \leqq 6$, then 2-dimensional L-symplectic bundles are stable and so $\widetilde{KSP_L}(X)$ classifies complex 2-dimensional vector bundles E over X together with a choice (up to homotopy) of isomorphism $\Lambda^2 E \cong L$. Since the automorphisms (up to homotopy) of any line-bundle L are given by $H^1(X, Z)$ it follows that, if $H^1(X, Z) = 0$, $\widetilde{KSP_L}(X)$ classifies 2-dimensional bundles E with $c_1(E) = c_1(L)$. In particular this holds in our case when $X = P_3$.

If E is L-symplectic and N is another line-bundle, then $E \otimes N$ is $L \otimes N^2$-symplectic, and so

$$KSP_L(X) \cong KSP_{L \otimes N^2}(X)$$

so that the different twisted theories are really parametrized by $H^2(X, Z)$ reduced modulo 2. For $X = P_3$ there are therefore just two theories corresponding to the parity of c_1. The even case is just the ordinary symplectic theory used in § 2. The group $KSP_H(P_3)$ is the group we now want: it classifies 2-dimensional E with $c_1(E) = 1$.

The twisted K-groups can be computed from the usual K-groups in virtue of the following version of the Thom isomorphism theorem:

(3.1) **Proposition.** $KSP_L(X) \cong KO^{-2}(L)$, *where we use K-theory with compact supports.*

Postponing till later the proof of (3.1) we proceed to apply it in our case. Taking $L = H$, the Hops bundle on P_3, we can identify the Thom space of H with P_4 and so

(3.2) $$KSP_H(P_3) \cong KO^{-2}(P_4, P_0) \cong Z^2 \quad \text{by (2.1)}$$

or

$$\widetilde{KSP_H}(P_3) \cong Z.$$

Thus there is no torsion and the integer invariant, which can be computed rationally, is certainly determined by c_2: in fact it is $\frac{1}{2}c_2$. Hence we have proved:

(3.3) **Theorem.** *A 2-dimensional complex vector bundle over P_3 with c_1 odd is uniquely determined by c_1, c_2.*

We return now to (3.1) which we shall prove by reinterpreting our twisted K-groups in terms of equivariant K-theory and then using the Thom isomorphism theorem of [1].

We let $S(L)$ denote the unit circle bundle of L. This is a free G-space where $G=U(1)$ so that the K-theory of X can be reinterpreted as the G-equivariant theory of $S(L)$. For example

(3.4) $KSP_G(S(L)) \cong KSP(X)$.

To obtain the twisted theories we now introduce the connected double covering $\tilde{G} \to G$. Let N denote the trivial bundle $S(L) \times \mathbb{C}$, with $\tilde{G}$ acting on $\mathbb{C}$ via its standard faithful representation. Note that N^2, which may be viewed as a G-vector bundle, corresponds to the line-bundle L on X. Now it is easy to see that any $\tilde{G}$-vector bundle E over $S(L)$ has a canonical decomposition of the form

$$E \cong E^+ \oplus (N^* \otimes E^-)$$

where E^+ and E^- are (lifts of) G-vector bundles. A symplectic structure on E, compatible with the action of $\tilde{G}$, corresponds to symplectic structures on E^+ and on $N^* \otimes E^-$. Thus E^+ gives a symplectic bundle on X, but the pairing

$$(N^* \otimes E^-) \otimes (N^* \otimes E^-) \to \mathbb{C}$$

gives a pairing

$$E^- \otimes E^- \to N^2$$

and so E^- gives an L-symplectic bundle on X. Hence, when G is replaced by $\tilde{G}$, (3.4) becomes

(3.5) $KSP_{\tilde{G}}(S(L)) \cong KSP(X) \oplus KSP_L(X)$.

The decomposition of (3.5) is canonical and it may be regarded as a grading into even and odd parts.

We now consider the $\tilde{G}$-vector bundle N^2 on $S(L)$. Since it has a square-root, namely N, it is a Spin-bundle and so the equivariant Thom isomorphism theorem [1; §6] holds:

(3.6) $KO^q_{\tilde{G}}(S(L)) \cong KO^{q+2}_{\tilde{G}}(N^2)$.

In particular, letting $q=-4$, we get

(3.7) $KSP_{\tilde{G}}(S(L)) \cong KO^{-2}_{\tilde{G}}(N^2)$.

Since the fundamental class for the Thom isomorphism is constructed from the Spin representation it is an *odd* element for our grading and so (3.6) interchanges the even and odd parts. Hence using (3.5) and (3.7) we get

$$KSP_L(X) \cong (KO^{-2}_{\tilde{G}}(N^2))_{\text{even}} = KO^{-2}(L)$$

proving (3.1).

We conclude this section with a few further remarks about these twisted K-theories. Although not necessary for our particular problems on P_3 they are relevant for other manifolds and help to clarify the general situation.

First it is clear that there is an analogous twisted orthogonal theory in which we consider symmetric bilinear forms $E \otimes E \to L$. The corresponding Grothendieck group is denoted by $KO_L(X)$ and (3.1) becomes

(3.8) $KO_L(X) \cong KO^2(L)$.

A small point of difference from the symplectic case arises because E need not have even dimension. However since $E \cong E^* \otimes L$ we have $2c_1(E) = mc_1(L)$ where $m = \dim E$. Thus, if m is odd, $c_1(L)$ must be even so that L has a square root and the twisted theory is isomorphic to the usual theory.

If we put as usual

$$KO_L^{-q}(X) = KO_L(R^q \times X),$$

then the Thom isomorphisms (3.1) and (3.8) imply that

(3.9) $KO_L^{-4}(X) \cong KSP_L(X)$.

If X is a complex manifold it is particularly interesting to take $L = K$, the canonical line-bundle. As a routine consequence of the twisted Thom isomorphisms it follows that a proper map $f: X \to Y$ of complex manifolds induces a natural direct image homomorphism

(3.10) $f_!: KO_K^q(X) \to KO_K^{q-2n+2m}(Y)$.

Here, n, m denote the complex dimensions of X, Y and the suffix K denotes the appropriate canonical line-bundle in each case. It is not necessary that f be a *holomorphic* map, nor in fact that X, Y be complex analytic. All that is necessary is that X, Y should be Spin^c-manifolds, i.e. that the structure group of their tangent bundles is lifted from $SO(k)$ to $\mathrm{Spin}^c(2k) = \mathrm{Spin}(2k) \times_{Z_2} U(1)$. Also we can twist one stage further and define a direct image

(3.11) $f_!: KO_{K \otimes f^*(L)}^q(X) \to KO_{K \otimes L}^{q-2n+2m}(Y)$

for any line-bundle L on Y. In fact (3.11) can be deduced from (3.10) by putting $f^*(L)$ and L for X, Y. It is also not hard to check that (3.10) is compatible with the K-theory direct image by the functor $KO_K \to K$, and similarly for (3.11).

Taking Y to be a point and X to be compact in (3.10) we get the special cases which concern us most, namely homomorphisms

(3.12) $\alpha_K: KO_K(X) \to KO^{-2}(\text{point}) = Z_2 \qquad n \equiv 1 \bmod 4$

$\alpha_K: KSP_K(X) \to KO^{-2}(\text{point}) = Z_2 \qquad n \equiv -1 \bmod 4$.

Finally we should remark that twisted K-theories were introduced by Donovan and Karoubi in [7], where they are treated by more categorical methods. The approach followed above, reducing to the equivariant case, is due to G.B. Segal.

§ 4. Analysis of Symplectic Bundles

Our aim now is to identify the α-invariant of § 2 in the case of holomorphic bundles. We shall show that α can be computed in terms of sheaf cohomology as a mod 2 semi-characteristic. The particular features of P_3 play no special role in this section and our results will be valid much more generally.

Let X be a compact complex n-dimensional manifold and let E be a holomorphic vector bundle over X. Then we have the sheaf cohomology groups $H^q(X, E)$ and the corresponding Euler characteristic

(4.1) $\chi(X, E) = \sum(-1)^q \dim H^q(X, E)$

which figures in the Riemann-Roch theorem. If K denotes the canonical line-bundle (so that holomorphic sections of K give holomorphic differential n-forms) then the Serre duality theorem asserts that $H^q(X, E)$ and $H^{n-q}(X, E^* \otimes K)$ are dual vector spaces, where E^* is the dual bundle of E. In particular

$$\dim H^q(X, E) = \dim H^{n-q}(X, E^* \otimes K).$$

Hence, if n is odd and if $E \cong E^* \otimes K$, the terms in (4.1) cancel in pairs and $\chi(X, E) = 0$. The isomorphism $E \cong E^* \otimes K$ is equivalent to giving a non-degenerate holomorphic bilinear form ϕ on E with values in K:

$$\phi: E \otimes E \to K.$$

If ϕ is symmetric or skew-symmetric we shall show, for appropriate values of n, that we can introduce an interesting "semi-characteristic". This is in formal analogy with the Kervaire semi-characteristic which is defined for odd-dimensional manifolds – where the usual Euler-Poincaré characteristic vanishes.

We shall prove the following theorem which constitutes the main result of this section:

(4.2) **Theorem.** *Let X be a compact complex manifold of odd dimension n, and let E be a holomorphic vector bundle over X with a non-degenerate holomorphic bilinear form ϕ with values in the canonical line-bundle K. Assume further that ϕ is symmetric if $n \equiv 1 \bmod 4$ (so that E is K-orthogonal) and skew-symmetric if $n \equiv -1 \bmod 4$ (so that E is K-symplectic). Then the holomorphic semi-characteristic*

$$\beta(E) = \sum_q \dim H^{2q}(X, E) \mod 2$$

is a deformation invariant. It can be computed topologically by applying the direct image homomorphisms of (3.12)

$$\alpha_K: KO_K(X) \to KO^{-2}(\text{point}) = Z_2 \qquad n \equiv 1 \bmod 4$$

$$\alpha_K: KSP_K(X) \to KO^{-2}(\text{point}) = Z_2 \qquad n \equiv -1 \bmod 4$$

to the twisted K-class of E: $\beta(E) = \alpha_K[E]$.

Before proceeding to the proof we shall make a few explanatory remarks. The words "deformation invariant" mean that if X, E, ϕ depend continuously on a real parameter t then β is independent of t. This statement has a clear analogue in abstract algebraic geometry (for algebraic dependence on t) and it would be interesting to produce an algebraic proof. The proof which we shall give is analytical and has no obvious translation to the algebraic context.

If the first Chern class of X is even, X is a Spin-manifold and the different Spin-structures correspond precisely to holomorphic square roots L of K [3; (3.2)]. Choosing such an L we can replace E by $E \otimes L^{-1}$ which will now be an orthogonal or symplectic bundle in the usual sense. Instead of the twisted direct images of (4.2) we now have the usual direct images for a Spin-manifold:

$\alpha: KO(X) \to Z_2 \qquad n \equiv 1 \bmod 4$

$\alpha: KSP(X) \to Z_2 \qquad n \equiv -1 \bmod 4$

and $\alpha_K(E) = \alpha(E \otimes L^{-1})$, so that (4.2) becomes

(4.3) $\beta(E) = \alpha[E \otimes L^{-1}]$.

The two simplest cases of Theorem (4.2) arise when $n=1$ and $n=3$. In the first case X is a Riemann surface and $\beta(E) = \dim H^0(X, E)$. In particular if $E=L$ is a line-bundle then ϕ is just an isomorphism $L^2 \cong K$ and this case was investigated in detail in [3]. For $n=3$ the simplest case is when E is 2-dimensional. Then ϕ is just an isomorphism $\Lambda^2(E) \cong K$, where Λ^2 is the second exterior power; in particular ϕ (if it exists) is unique up to a constant (provided X is connected). This is the only case needed for § 2, but it seemed better to state and prove Theorem (4.2) in full generality so that its real features would stand out more clearly.

The proof of (4.2) is a straightforward extension of the methods explained in [3]. The idea is that $\oplus H^{2q}(X, E)$ can be identified with the null-space of a certain elliptic skew-adjoint anti-linear operator P. The dimension of this null-space mod 2 is then a homotopy invariant as proved in [6]. It is moreover a mod 2 "index" for which the index theorem of [5] yields a K-theory formula in terms of the symbol of P. This leads to the direct image formula.

We proceed to spell out the details. As far as possible we shall use the same notation as in [3]. The bilinear form $\phi: E \otimes E \to K$ induces a homomorphism

$$\Omega^{0,q}(E) \otimes \Omega^{0,r}(E) \to \Omega^{0,q+r}(K) \cong \Omega^{n,q+r}$$

denoted by $u \wedge v$: here $\Omega^{p,q}$ denotes forms of type (p, q) and $\Omega^{p,q}(E)$ the E-valued forms of type (p, q). Since ϕ is holomorphic it follows that

(4.4) $\bar\partial(u \wedge v) = \bar\partial u \wedge v + (-1)^q u \wedge \bar\partial v$.

We now choose hermitian metrics on X and E and define (for all q) an anti-linear isomorphism $h: \Omega^{0,q}(E) \to \Omega^{0,n-q}(E)$ by

(4.5) $\langle u, h(\omega)\rangle = \int_X u \wedge \omega$.

Here $u \in \Omega^{0,q}(E)$, $\omega \in \Omega^{0,n-q}(E)$ and $\langle\ ,\ \rangle$ denotes the inner product induced by the metrics on X and E. Assuming that ϕ is ε-symmetric ($\varepsilon = \pm 1$), i.e. that $\phi(x, y) = \varepsilon\phi(y, x)$, we have

$$\omega \wedge u = \varepsilon(-1)^{q(n-q)} u \wedge \omega = \varepsilon u \wedge \omega \qquad \text{since } n \text{ is odd.}$$

Hence

$$\langle u, h(\omega)\rangle = \int_X u \wedge \omega = \varepsilon \int_X \omega \wedge u = \varepsilon\langle \omega, h(u)\rangle = \varepsilon\overline{\langle h(u), \omega\rangle}$$

which implies that, as a real linear operator $h^* = \varepsilon h$ (because the real inner product is given by $\operatorname{Re}\langle\ ,\ \rangle$). Together with (4.4) this shows that the anti-linear operator

$$h\bar\partial: \Omega^{0,q}(E) \to \Omega^{0,n-q-1}(E)$$

satisfies

(4.6) $(h\bar\partial)^* = (-1)^{q+1}\varepsilon(h\bar\partial)$

or equivalently $\bar\partial^* = (-1)^{q+1} h\bar\partial h^{-1}$.

We now consider the anti-linear operator

$$P: \sum_p \Omega^{0,2p}(E) \to \sum \Omega^{0,2p}(E)$$

defined on $\Omega^{0,2p}(E)$ by

$$(4.7) \quad P = (-1)^{p+1} h\bar{\partial} + (-1)^p \varepsilon \bar{\partial} h^{-1}.$$

Using (4.6) a little computation shows that

$$(4.8) \quad P^* = -\varepsilon\eta P$$

where $\eta = \pm 1$ according as $n \equiv \pm 1 \bmod 4$. But the hypothesis of our theorem is precisely that $\eta = \varepsilon$ and so $P^* = -P$. If we define an operator j by

$$\begin{aligned} j(\phi) &= (-1)^{p+1} h(\phi) \qquad \text{for } \phi \in \Omega^{0,2p}(E) \\ &= (-1)^p \varepsilon h^{-1}(\phi) \qquad \text{for } \phi \in \Omega^{0,2p+1}(E) \end{aligned}$$

then $j^2 = -1$ and P can be written in the form

$$(4.9) \quad P = j(\bar{\partial} + \bar{\partial}^*) = (\bar{\partial} + \bar{\partial}^*) j.$$

In particular this shows that P is elliptic (since $\bar{\partial} + \bar{\partial}^*$ has square equal to the complex Laplacian $\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$) and hence $\dim \operatorname{Ker} P \bmod 2$ is a well-defined deformation invariant. But by (4.9)

$$\begin{aligned} \operatorname{Ker} P &= \operatorname{Ker}(\bar{\partial} + \bar{\partial}^*) \qquad \text{on } \sum \Omega^{0,2p}(E) \\ &= \sum_p \operatorname{Ker} \Delta_{2p} \\ &= \sum_p H^{2p}(X, E) \qquad \text{by the Dolbeault isomorphism.} \end{aligned}$$

This proves the first part of the theorem. To compute this mod 2 index we can now appeal to the general formula of [5]. The skew-adjoint anti-linear operator P has a symbol class $[\sigma(P)] \in KR^{-2}(TX)$ and the main theorem of [5] asserts that the mod 2 index of P is obtained by applying the topological index

$$KR^{-2}(TX) \to KR^{-2}(\text{point}) = Z_2$$

to $[\sigma(P)]$.

It remains therefore to identify $\operatorname{ind}[(P)]$ with $\alpha_K[E]$. Now we have Thom isomorphisms

$$KO_K(X) \cong KO^{-2}(K) \cong KR^{-2}(TK) \cong KR^{-2}(TX) \qquad n \equiv 1 \bmod 4,$$

$$KSP_K(X) \cong KO^{-2}(K) \cong KR^{-2}(TK) \cong KR^{-2}(TX) \qquad n \equiv -1 \bmod 4$$

and it is a routine matter (cf. [9; §4.2]) to check that these are compatible with α_k and ind_t. Thus we have commutative diagrams

$$\begin{array}{ccc} KO_K(X) & \to & KR^{-2}(TX) \\ & \searrow^{\alpha_K} \quad \swarrow_{\operatorname{ind}_t} & \\ & Z_2 & \end{array} \qquad \begin{array}{ccc} KSP_K(X) & \to & KR^{-2}(TX) \\ & \searrow^{\alpha_K} \quad \swarrow_{\operatorname{ind}_t} & \\ & Z_2 & \end{array}$$

$n \equiv 1(4)$ $\qquad\qquad$ $n \equiv -1(4)$

To complete the proof of Theorem (4.2) we must finally check that the element $[E]$ in the twisted K-theory corresponds to $[\sigma(P)]$ under the above isomorphisms. For simplicity we shall only carry out this verification in the case that K has a square root L so that X is a Spin-manifold. This is more than adequate for our purposes, when $X=P_3$, and the general case follows by passing to equivariant K-theory as in the proof of (3.1).

As before put $\varepsilon=\pm 1$ according as $n\equiv\pm 1 \bmod 4$. Then $F=E\otimes L^{-1}$ has an ε-symmetric bilinear form. The metric on X gives a metric on K, hence one on L and together with the metric on E this gives a metric on F. Using this metric the bilinear form converts as usual into an anti-linear operator $\gamma: F\to F$ with $\gamma^*=\varepsilon\gamma$. Now let S^+, S^- denote the two complex $\frac{1}{2}$-Spin bundles of X. Since $\dim_R X=2n\equiv 2 \bmod 4$ there is a natural anti-isomorphism[1] $\delta: S^+\to S^-$ with $\delta^2=-\varepsilon$ (recall that $\varepsilon=\pm 1$ as $n\equiv\pm 1 \bmod 4$). This follows from properties of the Clifford algebras (see [4; § 4]). Hence

$$\gamma\otimes\delta: F\otimes S^+\to F\otimes S^-$$

is anti-linear and satisfies $(\gamma\otimes\delta)^2=-1$.

Now it is well-known (see [4; (5.11)]) that there are natural bundle isomorphisms

$$\Lambda^{0,\,\mathrm{even}}\otimes L\cong S^+, \qquad \Lambda^{0,\,\mathrm{odd}}\otimes L\cong S^-$$

where $\Lambda^{0,p}=\Lambda^{0,p}(T^*X)$ is the $(0,p)$ part of the exterior power. In terms of spaces of sections

$$\Omega^{0,\,\mathrm{even}}(L)\cong\Gamma(S^+) \qquad \Omega^{0,\,\mathrm{odd}}(L)\cong\Gamma(S^-)$$

where Γ denotes C^∞ sections. Tensoring with F gives

$$\Omega^{0,\,\mathrm{even}}(E)\cong\Gamma(F\otimes S^+) \qquad \Omega^{0,\,\mathrm{odd}}(E)\cong\Gamma(F\otimes S^-).$$

Comparing definitions one can check that, via these isomorphisms.

(4.10) $j=\gamma\otimes\delta.$

Moreover [9; § 2.1] the operator $\bar\partial+\bar\partial^*$ has (up to a factor $\sqrt{2}$) the same symbol (Clifford multiplication) as the Dirac operator (with coefficients in F)

$$D_F: \Gamma(F\otimes S^\pm)\to\Gamma(F\otimes S^\mp).$$

In view of (4.9) and (4.10) it follows that P has the same symbol $(\gamma\otimes\delta)_{D_F}$ acting on $\Gamma(F\otimes S^+)$. But

(4.11) $[\sigma((\gamma\otimes\delta)_{D_F})]=[F,\gamma]\,[\sigma(\delta D)]\in KR^{-2}(TX).$

Here F with the antilinear map γ, satisfying $\gamma^2=\varepsilon$, gives the element in $KR^{2-2\varepsilon}(X)$ which is just the orthogonal or sympletic class of F (with its bilinear form). The anti-linear operator δD satisfies

$$(\delta D)^*=-\varepsilon(\delta D)$$

[1] There is an ambiguity of sign in the choice of δ which can be made explicit, but since our final index is in Z_2 the sign is unimportant and will be left in decent obscurity.

and hence has a symbol in $KR^{-2\varepsilon}(TX)$ (cf. [9; § 4.2]), so that the product in (4.11) ends up in $KR^{-2}(TX)$ in both cases ($\varepsilon = \pm 1$). Finally we need to know that $[\sigma(\delta D)] \in KR^{-2\varepsilon}(TX)$ is precisely the fundamental class for the Thom isomorphism

$$KR(X) \cong KR^{-2n}(TX) = KR^{-2\varepsilon}(TX)$$

(recall that $2n \equiv 2\varepsilon \bmod 8$). For the verification of this see [9; § 4.2].

Remarks. 1) If X is a Kähler manifold, and if we take the unique connection on E which preserves both the metric and the bilinear form, one can check that the operators $\bar{\partial} + \bar{\partial}^*$ and D_F not only have the same symbol, but they actually coincide (cf. [9; § 2.1] for similar results). Thus $\sum H^{2p}(X, E)$ can be identified with the space of harmonic spinors with coefficients in E.

2) The rather tedious verification above that $\alpha_K(E)$ coincides with the topological index is not vital for our application to P_3. We could simply replace α by ind_t in § 2: it does just as well.

3) We can use Theorem (4.2) to show that the splitting (2.7) of $KSP(P_3)$, given by α, coincides with the splitting obtained by lifting the geometric generators of $KSP(P_2) \cong Z \oplus Z$. In fact the two generators are the trivial bundle $H^0 \oplus H^0$ and $H \oplus H^{-1}$, and

$$\alpha(H^0 \oplus H^0) = 2\beta(H^{-2}) = 0,$$

$$\alpha(H \oplus H^{-1}) = \beta(H^{-1}) + \beta(H^{-3}) = 0.$$

§ 5. The Horrocks Examples

We shall now combine the results of § 2 and § 4 and apply them to the explicit examples of algebraic vector bundles over P_3 constructed by Horrocks in [10].

Let E be a holomorphic 2-dimensional vector bundle over P_3 with c_1 even. We then put

(5.1) $\quad n = -\frac{1}{2}(c_1(E) + 4)$

so that $E(n) = E \otimes H^n$ (where H is the Hopf bundle) satisfies

$$c_1(E(n)) = -4 = c_1(K),$$

K being the canonical line bundle of P_3. Since holomorphic line-bundles on P_3 are determined by c_1 it follows that $\Lambda^2(E(n)) \cong K$, and hence that there is a skew-symmetric non-degenerate holomorphic linear map

$$\phi\colon E(n) \otimes E(n) \to K$$

unique up to a constant. By Theorem (4.2)

$$\beta(E(n)) = \dim H^0(P_3, E(n)) + \dim H^2(P_3, E(n)) \mod 2$$

is then equal to the invariant $\alpha(E)$ of § 2. According to Theorem (2.6) the underlying topological bundle of E is determined by $c_1(E)$, $c_2(E)$ and $\alpha(E)$. Since $c_1(E)$ and $c_2(E)$ are easy to compute, and since $\alpha(E)$ can be replaced by $\beta(E(n))$, we are now in a good position to compute all the topological invariants of any explicitly given holomorphic vector bundle E.

In [10] Horrocks gave a construction for algebraic (and therefore holomorphic) vector bundles over P_3 and he computed their Chern classes. He showed that, by varying the parameters in his construction, all possible pairs c_1, c_2 with $c_1 c_2$ even could be realized. We shall compute the α-invariant of Horrocks' bundles and show again that varying the parameter gives all possible triples c_1, c_2, α with c_1 even.

We begin by reviewing briefly the main points of Horrocks' construction, which is done in the framework of coherent sheaves: we recall that locally free sheaves correspond to vector bundles. Horrocks considers locally free sheaves $\mathscr{F}_i$ of rank 2 which fit into exact sequences

(5.2) $0 \to \mathcal{O}(-p) \to \mathscr{F}_i \to \mathcal{O} \to \mathscr{R}_i \to 0$

where $\mathcal{O}$ is the structure sheaf on P_3, $\mathcal{O}(n)$ is the sheaf of sections of H^n, $\mathscr{R}_i$ is a sheaf supported on a disjoint set of lines in P_3 and $p \geqq 2$. Horrocks' main result is that two sequences of the form (4.2), taking $i=1,2$, can be combined to give a third ($i=3$) provided $\mathscr{R}_1, \mathscr{R}_2$ have disjoint supports. Moreover we then have $\mathscr{R}_3 = \mathscr{R}_1 \oplus \mathscr{R}_2$. The first Chern class of the sheaves $\mathscr{F}_i$ is always equal to $-p$.

We shall only be concerned here with the case of even first Chern class so we put $p=2q$ ($q \geqq 1$). The integer n of (5.1) is therefore equal to $q-2$ and

$$\alpha(\mathscr{F}) = \beta(\mathscr{F}(q-2)) = \dim H^0(P_3, \mathscr{F}(q-2)) + \dim H^2(P_3, \mathscr{F}(q-2)) \mod 2.$$

In fact it is convenient to apply Serre duality and replace H^2 here by H^1 so that (since $\mathscr{F}(q-2)^* \otimes K \cong \mathscr{F}(q-2)$)

(5.3) $\alpha(\mathscr{F}) = \dim H^0(P_3, \mathscr{F}(q-2)) + \dim H^1(P_3, \mathscr{F}(q-2)) \mod 2.$

To compute this we use the following lemma:

(5.4) **Lemma.** $\alpha(\mathscr{F}) = \dim H^0(P_3, \mathcal{O}(q-2)) + \dim H^1(P_3, \mathscr{R}(q-2)) \mod 2.$

Proof. We tensor (4.2) with $\mathcal{O}(q-2)$ and then break it up into two short exact sequences

(5.5) $0 \to \mathcal{O}(-q-2) \to \mathscr{F}(q-2) \to \mathscr{A} \to 0,$

(5.6) $0 \to \mathscr{A} \to \mathcal{O}(q-2) \to \mathscr{R}(q-2) \to 0.$

Since $q \geqq 1$ we have

$$H^i(P_3, \mathcal{O}(-q-2)) = 0 \quad \text{for } i<3,$$
$$H^i(P_3, \mathcal{O}(q-2)) = 0 \quad \text{for } i>0.$$

(5.5) then shows that $\mathscr{F}(q-2)$ and $\mathscr{A}$ have isomorphic cohomology groups in dimensions 0, 1. The Lemma now follows from the 4-term exact cohomology sequence arising from (5.5).

(5.7) **Corollary.** $\alpha(\mathscr{F}) + \dim H^0(P_3, \mathcal{O}(q-2)) \mod 2$ *is additive for the Horrocks construction.*

Proof. This follows at one from (5.4) and the additivity of the sheaf $\mathscr{R}$ in the Horrocks construction.

To obtain interesting bundles Horrocks starts from the case $\mathscr{F}=\mathcal{O}(-m)\oplus\mathcal{O}(-k)$ where $m+k=p=2q$ and $m, k>0$. The roles of the two factors of $\mathscr{F}$ are symmetric so we may assume $m\leqq q$. Then

$$\mathscr{F}(q-2)=\mathcal{O}(q-m-2)\oplus\mathcal{O}(m-q-2)$$

and so $\alpha(\mathscr{F})=\dim H^0(P_3, \mathcal{O}(q-m-2)) \bmod 2$. Taking a sequence $m_1, \ldots, m_r$ of such integers m, Horrocks combines the corresponding sheaves $\mathscr{F}_1, \ldots, \mathscr{F}_r$ to obtain finally a new locally free sheaf $\mathscr{F}$. Applying (5.7), and the above formula for each $\alpha(\mathscr{F}_i)$, we get

$$\alpha(\mathscr{F})+d(q-2)=\sum_{i=1}^{r}(d(q-2)+\alpha(\mathscr{F}_i))$$

or

$$(5.8)\quad \alpha(\mathscr{F})=(r-1)\,d(q-2)+\sum_{i=1}^{r} d(q-m_i-2)$$

where we have put for brevity $d(s)=\dim H^0(P_3, \mathcal{O}(s)) \bmod 2$.

Now $H^0(P_3, \mathcal{O}(s))$ is the space of homogeneous polynomials of degree s in 4 variables and so

$$\begin{aligned} d(s)&=0 \quad \text{for } s<0\\ &=\tfrac{1}{6}(s+1)(s+2)(s+3) \bmod 2 \quad \text{for } s\geqq 0\\ &=1 \quad \text{only if } s \text{ is divisible by } 4. \end{aligned}$$

Together with Horrocks' computation of the Chern class c_2 we have therefore proved:

(5.9) **Theorem.** *Let $q\geqq 1$ be an integer and let $m_1, \ldots, m_r$ be integers with $0<m_i\leqq q$. Then there exists a locally free sheaf (or holomorphic vector bundle) $\mathscr{F}$ of rank 2 on P_3 such that*

$$\begin{aligned} c_1(\mathscr{F})&=-2q,\\ c_2(\mathscr{F})&=\sum_{i=1}^{r} m_i(2q-m_i),\\ \alpha(\mathscr{F})&=(r-1)\,d(q-2)+\sum_{i=1}^{r} d(q-m_i-2) \end{aligned}$$

where $d(s)=1$ *if* $s\equiv 0 \bmod 4$ *and* $s\geqq 0$

$=0$ *otherwise.*

It is convenient to define Δ by

$$4\Delta=c_1^2-4c_2$$

Since c_1 is even Δ is an integer. It has the merit of being unchanged under tensoring with line-bundles. For the bundle in Theorem (5.9) we have

$$(5.10)\quad \Delta(\mathscr{F})=q^2-\sum_{i=1}^{r} m_i(2q-m_i).$$

Replacing $\mathscr{F}$ by $\mathscr{F}(r)$ we can alter c_1 to any even integer without changing $\Delta(\mathscr{F})$ or, because of its very definition, $\alpha(\mathscr{F})$. Thus to show that every triple (c_1, c_2, α)

with c_1 even arises from some $\mathscr{F}(r)$ we must show that, for any $k \in Z$, $\alpha \in Z_2$, the equations

(5.11) $$k = q^2 - \sum_{i=1}^{r} m_i(2q - m_i)$$

$$\alpha = (r-1)d(q-2) + \sum_{i=1}^{r} d(q - m_i - 2)$$

can always be solved for $r, q, m_1, \ldots, m_r$. This involves some elementary but lengthy computation and will be treated in the next section.

§ 6. Arithmetical Computations

In this section we shall prove the following result:

(6.1) **Proposition.** *Let $k \in Z$, $\alpha \in Z_2$, be given. Then there exist integers $q, r \geqq 1$ and integers $m_1, \ldots, m_r$ with $0 < m_i \leqq q$ such that*

$$k = q^2 - \sum_{i=1}^{r} m_i(2q - m_i),$$

$$\alpha = (r-1)d(q-2) + \sum_{i=1}^{r} d(q - m_i - 2)$$

where $d(s)$ is the function defined in (5.9).

For the proof it will suffice to consider $q \equiv 3 \bmod 4$ and $q \geqq 15$. Moreover we shall take $m_i = 1, 2, 3, 10$ for t, x, y, z values of i respectively. Some of the integers t, x, y, z may be zero, but at least one must be non-zero (so that $r \geqq 1$). Because of our restrictions on q, α equals $t \bmod 2$. The equation for k becomes

$$k = q^2 - t(2q-1) - 2x(2q-2) - 3y(2q-3) - 10z(2q-10).$$

If $t = 2s$, we get $\alpha = 0$ and

(6.2) $$q^2 - k = f_q(s, x, y, z)$$

where

$$f_q(s, x, y, z) = s(4q-2) + x(4q-4) + y(6q-9) + z(20q-100).$$

If $t = 2s+1$, then $\alpha = 1$ and

(6.3) $$(q-1)^2 - k = f_q(s, x, y, z).$$

Now a well-known result of elementary number theory says that, if $l, m, n, \ldots$ are positive integers with no common factor, then the semi-group $P(l, m, n, \ldots)$ *of* Z which they generate contains all sufficiently large integers N, say $N \geqq \sigma(l, m, n, \ldots)$ – with σ the minimum choice.

Since the integers

$$4q-2, \quad 4q-4, \quad 6q-9, \quad 20q-100$$

have no common factor the equations (6.2) and (6.3) will be soluble (with $s, x, y, z \geqq 0$) provided

$$(q-1)^2 - k \geqq \sigma(4q-2, 4q-4, 6q-9, 20q-100).$$

Since k is given and we are allowed to choose q it will be sufficient to show that

(6.4) $\sigma(4q-2, 4q-4, 6q-9, 20q-100) \leqq cq^2$ for large q

for some constant $c<1$. Now

$$\sigma(4q-2, 4q-4, 6q-9, 20q-100) \leqq 6q-9+2\sigma(2q-1, 2q-2, 10q-50)$$

and so the following lemma with $n=2q-2$, will establish (6.4) with $c=16/17$.

(6.5) **Lemma.** $\sigma(n+1, n, 5n-40) \leqq \frac{2n^2}{17}$ *for large n.*

Proof. Note first that every interval $[rn, r(n+1)]$ is clearly contained in $P(n, n+1, 5n-40)$. Next we shall prove by induction on $l \geqq 0$ that

$$I_l = [(5n-40)l+40n, (5l+40)(n+1)] \subset P.$$

For $l=0$, $I_0=[40n, 40(n+1)] \subset P$ as just noted. Assume now that $I_l \subset P$, then adding $5n-40$

$$(5n-40)+I_l = [(5n-40)(l+1)+40n, 45n+5l(n+1)] \subset P.$$

Since this interval overlaps with $[rn, r(n+1)]$ for $r=5(l+1)+40$, their union, which is just I_{l+1}, is contained in P; this establishes the induction. Now the number of integers in I_l is

$$(5l+40)(n+1)-(5n-40)l-40n+1=45l+41.$$

If we choose l so that $45l+41 \geqq n$, the intervals $rn+I_l$ for $r=0, 1, 2, \ldots$ are contiguous and so

$$\sigma(n+1, n, 5n-40) \leqq (5n-40)l+40n.$$

If we take the smallest possible value for l, then $45(l-1)+41<n$, hence

$$\begin{aligned}\sigma(n+1, n, 5n-40) &\leqq \frac{(5n-40)(n+4)}{45}+40n\\ &\leqq \frac{n^2}{9}+40n\\ &< \frac{2n^2}{17} \quad \text{for large } n.\end{aligned}$$

This completes the proof of Proposition (6.1). In view of Theorem (5.9) this completes the proof of Theorem (1.1) for the case of c_1 even.

§ 7. Bundles on P_4

If a 2-dimensional vector bundle E on P_3 extends topologically to P_4 then its Chern classes must satisfy certain integrality relations arising from the Riemann-Roch formula [8; p. 155]. Tensoring by line-bundles does not affect the question of extension to P_4 so the conditions on c_1, c_2 can best be expressed in terms of

$\Delta = \frac{c_1^2 - 4c_2}{4}$ and the parity of c_1. If c_1 is even we may normalize it to be -4. Note that Δ is now an integer. Then applying the Riemann-Roch formula, i.e. the formula for the direct image $f_!$: $K(P_4) \to K(\text{point}) = Z$, a little computation gives:

(7.1) $f_![E] = \frac{\Delta(\Delta-1)}{12}$.

Alternatively, instead of carrying out the cohomological computations, it is sufficient to verify (7.1) (which is a polynomial in Δ) for the special cases $E = H^\gamma \oplus H^{-\gamma-4}$ $(\gamma \geqq 1)$, where H is the Hopf bundle. We can now use the analytical interpretation of $f_![E]$ as the holomorphic Euler characteristic $\chi(P_4, E)$. Substituting the known formulae for $\dim H^q(P_4, H^\gamma)$ we find

$$\begin{aligned}\chi(P_4, H^\gamma \oplus H^{-\gamma-4}) &= d_4(\gamma) + d_4(\gamma-1)\\ &= \frac{(\gamma+2)^2(\gamma+1)(\gamma+3)}{12}\\ &= \frac{\Delta(\Delta-1)}{12},\end{aligned}$$

where

$$d_4(\gamma) = \frac{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}{12}.$$

Thus, when c_1 is even, a necessary condition for E to extend to P_4 is that $\Delta(\Delta-1)$ should be divisible by 12. The sufficiency was shown in [12] and will be reproved more simply at the end of this section. Now we know from § 2 that there are *two* bundles on P_3 with given c_1, c_2 (if c_1 is even) and that these are distinguished by their α-invariant. It turns out that, when $\Delta(\Delta-1)/12$ is integral, only *one* of these two bundles on P_3 extends to P_4. Moreover there is a formula for α in terms of c_1, c_2 telling us which bundle extends, namely:

(7.2) **Theorem.** *Let E be a 2-dimensional complex vector bundle over P_3 with $c_1(E)$ even. If E extends to P_4 then*

$$\alpha(E) = \frac{\Delta(\Delta-1)}{12} \mod 2.$$

In view of (7.1) we see that (7.2) is equivalent to

(7.3) $\alpha(E) = f_![E] \mod 2$.

Note that $\alpha(E)$ is an invariant for bundles on P_3, while $f_![E]$ is an invariant for bundles on P_4.

Theorem (7.2), or equivalently (7.3), is a special case of a much more general result in the context of § 4. In the first instance we shall formulate and prove this in the holomorphic case:

(7.4) **Theorem.** *Let Y be a compact complex manifold of dimension $2n$, $X \subset Y$ a non-singular divisor, H the associated line-bundle on Y and $L = K_Y \otimes H$ where K_Y is the canonical line-bundle of Y. Assume now that E is a holomorphic vector* *bundle on Y which is L-orthogonal, for n odd, and L-symplectic for n even. Then we have*

$$\beta(X,E)=\chi(Y,E) \quad \mod 2$$

where

$$\beta(X,E)=\sum \dim H^{2q}(X,E) \quad \mod 2$$
$$\chi(Y,E)=\sum(-1)^q \dim H^q(Y,E)$$

are the holomorphic semi-characteristic (on X) and Euler characteristic (on Y) respectively.

Proof. Since $H|_X$ is the normal bundle of X in Y we have $L|_X \cong K_X$, the canonical bundle of X. Thus, restricting E to X we are in the situation of § 4. Now consider the standard cohomology sequence relating E on Y to E on X:

(7.5) $$\to H^{q-1}(X,E)\to H^q(Y,E\otimes H^{-1}) \xrightarrow{\phi_q} H^q(Y,E)\to H^q(X,E)\to \dots.$$

On Y we have

$$E\cong E^*\otimes L\cong E^*\otimes K_Y\otimes H$$

or

$$E\otimes H^{-1}\cong E^*\otimes K_Y$$

and so by Serre duality,

(7.6) $$H^q(Y,E)\cong[H^{2n-q}(Y,E\otimes H^{-1})]^*.$$

On the other hand, as in § 4, on X we have

$$E\cong E^*\otimes K_X$$

and so by Serre duality

(7.7) $$H^q(X,E)\cong[H^{2n-1-q}(X,E)]^*.$$

Using (7.6) and (7.7) we see that the terms in (7.5) occur in dual pairs. Taking alternating sums of dimension up to the middle in (7.5) we get:

$$\sum_{q=0}^{n}(-1)^q \dim H^q(Y,E\otimes H^{-1})-\sum_{q=0}^{n-1}(-1)^q \dim H^q(Y,E)$$
$$+\sum_{q=0}^{n-1}(-1)^q \dim H^q(X,E)=(-1)^n \operatorname{rank} \phi_n.$$

Reducing modulo 2 and using (7.6) and (7.7) this gives

(7.8) $$\beta(E)=\chi(Y,E)+\operatorname{rank}\phi_n \quad \mod 2.$$

To complete the proof of the Theorem it remains to show that rank ϕ_n is *even*.

Now we can define a bilinear form Φ_n on $H^n(Y,E\otimes H^{-1})$, associated to ϕ_n, by putting

$$\Phi_n(u,v)=u\cdot\phi_n(v)$$

where $u\cdot\omega$ is the Serre duality pairing. Thus Φ_n is induced by the cup-product on H^n, together with the bundle pairing

$$\psi'\colon (E\otimes H^{-1})\otimes(E\otimes H^{-1})\xrightarrow{1\otimes s}(E\otimes H^{-1})\otimes E\xrightarrow{\psi}L\otimes H^{-1}\cong K_Y$$

where s is the section of H vanishing on X and ψ is the given pairing $E\otimes E\to L$ tensored by the identity on H^{-1}. Since ψ is ε-symmetric, with $\varepsilon=(-1)^{n+1}$, the same is true of ψ'. Since the cup-product on H^n is $(-1)^n$-symmetric it follows that, in all cases, *Φ_n is skew-symmetric*. Hence

$$\operatorname{rank}\phi_n=\operatorname{rank}\Phi_n\equiv 0 \mod 2$$

completing the proof.

Remark. Topologists will note a close formal analogy between Theorem (7.4) and the situation of a real manifold Y with boundary X. The exact sequence (7.5) is the counterpart of the ordinary cohomology sequence of the pair (X,Y), and Poincaré-Lefschetz duality gives the analogues of (7.6) and (7.7). One concludes that the Kervaire semi-characteristic of X coincides with the Euler characteristic of Y, reduced modulo 2, provided $\dim Y\equiv 2 \bmod 4$. If one introduced an orthogonal or symplectic local coefficient system E the analogy would be even closer.

The topological version of (7.4) is

(7.9) **Theorem.** *Y, X and L being as in* (7.4) *we have commutative diagrams*

$$n \text{ even}\quad \begin{array}{ccc} KSP_L(Y) & \xrightarrow{\gamma_L} & Z \\ \downarrow & & \downarrow \rho \\ KSP_K(X) & \xrightarrow{\alpha_K} & Z_2 \end{array} \qquad n \text{ odd}\quad \begin{array}{ccc} KO_L(Y) & \xrightarrow{\gamma_L} & Z \\ \downarrow & & \downarrow \rho \\ KO_K(X) & \xrightarrow{\alpha_K} & Z_2 \end{array}$$

where α_K is the map of (4.2), *γ_L is the composition of $KSP_L(Y)\to K(Y)$ (or $KO_L(Y)\to K(Y)$) and the direct image $K(Y)\to K(\text{point})=Z$, and ρ is reduction* mod 2.

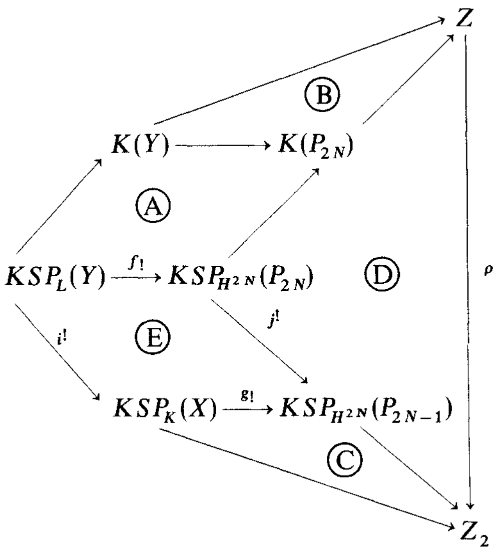
Clearly, if $KSP_L(Y)$ (or $KO_L(Y)$) is additively generated by *holomorphic* L-symplectic (or L-orthogonal) bundles (7.9) follows from (7.4), together with (4.2) and the Riemann-Roch theorem (which identify α_K and γ_L with β and χ). We apply this observation to the case $Y=P_{2n}$, $X=P_{2n-1}$. Then $c_1(K_X)=-2n$ is even, hence $L\cong H^{-2n}$ has a square root and so $KSP_L(P_{2n})\cong KSP(P_{2n})$, $KO_L(P_{2n})\cong KO(P_{2n})$. Proposition (2.1) asserts that $\tilde{K}(P_{2n})\to\widetilde{KSP}(P_{2n})$ and $\tilde{K}(P_{2n})\to\widetilde{KO}(P_{2n})$ are surjective. Since $K(P_{2n})$ is well-known to be generated by the holomorphic line-bundles H^k it follows that $KSP(P_{2n})$ and $KO(P_{2n})$ are generated by holomorphic bundles. Tensoring by H^{-n} the same is then true for $KSP_L(P_{2n})$ and $KO_L(P_{2n})$. This proves (7.9) for the case of projective spaces and in particular, taking $n=2$, this includes the special case of (7.2).

The general case of (7.9) can now be reduced to the case of projective spaces. For brevity we shall treat only the case of n odd: the case of n even is entirely similar. For N sufficiently large we can construct a commutative diagram of C^∞

embeddings

$$\begin{array}{ccc} Y & \xrightarrow{f} & P_{2N} \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ X & \xrightarrow{g} & P_{2N-1} \end{array}$$

where X is the transversal intersection of Y and P_{2N-1}. Choosing $N \equiv n \bmod 4$ and using the direct images explained in § 3 for Spin^c-manifolds we get a commutative diagram



Ⓐ commutes because direct images are compatible with the functor $KSP_L \to K$ as explained in § 3. Ⓑ and Ⓒ commute by the functoriality of direct images in K-theory and twisted KSP-theory. Ⓔ commutes because X is a transversal intersection and finally Ⓓ commutes by (7.4) for projective spaces which we have already proved. Following the perimeter of the diagram we get the assertion of Theorem (7.9), which completes the proof.

Finally, as promised, we shall reprove the result of [12] that, if Δ is any integer with $\Delta(\Delta-1)$ divisible by 12, then there is a 2-dimensional complex vector bundle E on P_4 with $c_1(E)=0$ and $c_2(E)=-\Delta$. The proof in [12] is rather complicated and, for the case of $c_1(E)$ odd treated below, requires further investigation, whereas our symplectic approach leads to a quick and simpler proof.

It will be enough to show that there is a stable symplectic bundle over P_4 with $c_4=0$ and $c_2=-\Delta$. In fact 2-dimensional symplectic bundles (i.e. $\dim_{\mathbb{C}}=4$) over P_4 are stable and $c_4=0$ is the condition for a non-zero section, hence reduces us to a symplectic line-bundle (i.e. $\dim_{\mathbb{C}}=2$). For a symplectic bundle the odd Chern classes c_1 and c_3 necessarily vanish so $c_4=0$, $c_2=-\Delta$ gives the following values for the components of the Chern character:

(7.10) $\mathrm{ch}_1=0, \quad \mathrm{ch}_2=\Delta, \quad \mathrm{ch}_3=0, \quad \mathrm{ch}_4=\frac{\Delta^2}{12}.$

Now the bundles $H\oplus H^*$ and $H^2\oplus(H^2)^*$ are clearly symplectic and so

$$\xi=A(H\oplus H^*)+B(H^2\oplus(H^2)^*),$$

for integer A, B gives a stable symplectic bundle. Computing its Chern character we find

$$\mathrm{ch}\,\xi=2A\cosh x+2B\cosh 2x$$

so

(7.11) $\mathrm{ch}_2\,\xi=A+4B, \quad \mathrm{ch}_4\,\xi=\frac{A+16B}{12}.$

Solving for A, B from (7.9) and (7.10) we get

$$A=\frac{4\Delta-\Delta^2}{3}, \quad B=\frac{\Delta^2-\Delta}{12}$$

which are integral if $\Delta^2-\Delta$ is divisible by 12. This completes the proof.

For completeness we can treat the case of c_1 odd in a similar way using the twisted symplectic theory of § 3. If $c_1=1$, $c_2=-a$, then Schwarzenberger's condition [9, p. 155] is that $a(a+4)$ must be divisible by 12. To prove the converse we shall exhibit a stable H-symplectic bundle with the right Chern classes. Note however that, since we now must stabilize by adding copies of $H\oplus 1$, the Chern classes depend on the dimension of our stable bundle. It is convenient to work with a stable bundle of virtual dimension zero and this alters the required Chern classes to $c_1=0$, $c_2=-a$, $c_3=a$, $c_4=-a$. The bundles $H^k\oplus H^{-k+1}$ are now H-symplectic and we put

$$\xi=A(H\oplus 1)+B(H^2\oplus H^{-1})+C(H^3\oplus H^{-2})$$

with $\dim_{\mathbb{C}}\xi=2(A+B+C)=0$. Taking

$$A=\frac{a^2-2a}{12}, \quad B=\frac{2a-a^2}{8}, \quad C=\frac{a^2-2a}{4}$$

gives $\mathrm{ch}\,\xi=ax^2+\frac{a}{2}x^3+\left(\frac{a^2}{12}+\frac{a}{6}\right)x^4$ and this has the required Chern classes. Note that $a(a+4)$ divisible by 12 implies that A, B, C are integers so that ξ is indeed a stable H-symplectic vector bundle.

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