On the Cubical Model of Homotopy Type Theory
— *work in progress* —

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Why Cubical HoTT?

- Basic MLTT has a **constructive character** that makes it well-suited for use in computational proof assistants: strong normalization of terms, decidability of type-checking, decidability of judgemental equality, canonicity, etc.
- But when we add new **axioms** like Univalence and HITs, this constructive character is spoiled. Instances of UA cannot always be eliminated, and new primitive terms of higher Id-type need not reduce to normal forms.
- A "**normalization up to homotopy**" algorithm could partially restore the constructive character of the system.
- But, as recently shown by **Coquand** et al., a system with additional cubical structure seems to allow for such extensions while still retaining a constructive character.
- This could lead to a proof of normalization up to homotopy for the **original** system via an interpretation. Moreover, it could also serve on its own as the basis of a new generation of proof assistants based on **cubical** HoTT.

Cubical HoTT: Recent work

Why cubical?

- Some success was had by Licata-Harper (2011) and Shulman (2013) in verifying the homotopy canonicity conjecture at **low dimensions**, using methods based on **groupoids**.
- In recent and current work, Coquand and collaborators have devised an approach based on a **constructive** interpretation of HoTT in (different versions of) **cubical sets**, which are a form of ∞-**groupoids**.
- Cubical sets are a **combinatorial** model of homotopy theory, introduced by Kan and still used in algebraic topology. Like the more familiar simplicial sets, they provide a more **algebraic** setting to study the homotopy theory of spaces.
- Voevodsky's original model of UA used classical **simplicial** sets and is **not constructive**. Known models of HITs are also based on **classical methods** from the theory of ∞-toposes.

Cubical HoTT: Recent success

Cubes rule!

- The cubical model suggests enriching the type theory itself with some additional **cubical operations** and **equations** which are present in the model, and which allow calculations that are otherwise available only “up-to-homotopy”. This makes the system **more computational**.
- Coquand et al. have programmed a **proof checker** for such a **cubical type theory**, in which all terms — including those involving UA and some HITs — compute to normal forms.
- Brunerie and Licata (LICS 2015) have a variant system of **cubical HoTT** in which e.g. the proof that $\mathbf{T}^2 \simeq \mathbf{S}^1 \times \mathbf{S}^1$ is short and sweet (in contrast to the original “heroic” proof in plain HoTT first given by Sojakova in 2013).
- The cubical setting seems to be **better suited to HoTT** than the simplicial one (or the globular one). It may also be of some use in **homotopy theory** (cf. recent work by Jardine, Grandis, Williamson, and others).

Variations on cubical sets

The category of **cubical sets** is the functor category

$$\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$$

of **presheaves** on the category $\mathbb{C}$ of cubes.

There are various different flavors of cubical sets in the literature, based on different categories $\mathbb{C}$ of cubes:

- $\mathbb{C}_m$ = the free **monoidal** category on an **interval** $1 \to \mathtt{I} \leftarrow 1$,
- $\mathbb{C}_{mc}$ = the free monoidal category on an **interval with connections** $\wedge$ and $\vee$.
- $\mathbb{C}_s$ = the free **symmetric monoidal** category on an interval.
- $\mathbb{C}_c$ = the free **cartesian** category on an interval.
- $\mathbb{C}_d$ = the free cartesian category on a **distributive lattice**.

The more structure one puts into the index category of cubes, the more "algebraic" the resulting model of type theory will be.

Cartesian cubes

Like the simplicial category Δ, each of these cube categories can be presented by generating face and degeneracy maps (plus others).

But the **cartesian** cube category also has a simple description in terms of its Lawvere dual:

Definition

The **cartesian cube category** $\mathbb{C} = \mathbb{C}_c$ is the opposite of the category $\mathbb{B}$ of finite, strictly **bipointed sets**,

$$\mathbb{C} \ =_{\mathsf{def}} \ \mathbb{B}^{\mathsf{op}} \,.$$

Write the bipointed sets:

$$[n] = \{0, x_1, ..., x_n, 1\}$$

So $\mathbb{C}$ has the objects: $[0], [1], ..., [n], ...$, which we regard dually as the basic **n-cubes**.

Cartesian cubes

$\mathbb{C}$ is the free finite-product category on the **bipointed object**:

$$[0] \to [1] \leftarrow [0]\,,$$

which is then the universal cartesian interval.

The basic cubes are then just the finite powers of [1],

$$[n] \;=\; [1] \times\; ... \times [1]\,.$$

The maps are those that can be composed from the $\times$-structure and the basic points $0, 1 : [0] \rightrightarrows [1]$.

They can also be represented syntactically as the terms of a very simple **algebraic theory**.

Cartesian cubical sets

Definition

The category **cSet** of (cartesian) **cubical sets** is the **presheaves** on $\mathbb{C}$. It is thus equal to the **covariant** functors on $\mathbb{B}$,

$$\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} = \mathbf{Set}^{\mathbb{B}}\,.$$

The **cubes** in **cSet** are the *representable functors*:

$$\mathtt{I}^n = \mathrm{hom}_{\mathbb{C}}(-,[n])\,.$$

The **interval** object $\mathtt{I} = \mathrm{hom}_{\mathbb{C}}(-,[1])$ generates all the other cubes, which are closed under finite products and satisfy:

$$\mathtt{I}^n \times \mathtt{I}^m \cong \mathtt{I}^{n+m}\,.$$

Cartesian cubical sets

The interval $1 + 1 \to \mathtt{I}$ in **cSet** is **universal**, in the following sense.

Theorem (A. 2015)

The category **cSet** *of cubical sets is the* **classifying topos** *for strictly bipointed objects* $(X, a, b, a \neq b)$.

- This allows us to relate **cSet** to other logical and homotopical models in toposes.
- Other models of type theory, such as **Top** and **sSet**, have a **canonical comparision** with **cSet**.
- Since $\mathbb{C}$ is a **test category** in the sense of Grothendieck, **cSet** has "the same" homotopy theory as classical spaces.
- Moreover, the geometric realization **cSet** $\longrightarrow$ **Top** preserves finite products.

Path spaces in cubical sets

The interval $1 + 1 \to \mathtt{I}$ endows each cubical set A with a **canonical path object**,

$$A^{\mathtt{I}} \to A^{1+1} \cong A \times A\,.$$

The object $A^{\mathtt{I}}$ has the special property,

$$\begin{aligned} A^{\mathtt{I}}_n \;&\cong\; \hom(\mathtt{I}^n, A^{\mathtt{I}}) \;\cong\; \hom(\mathtt{I}^n \times \mathtt{I}, A) \\ &\cong\; \hom(\mathtt{I}^{n+1}, A) \;\cong\; A_{n+1}. \end{aligned}$$

So an n-cube of **paths** in A is an $n+1$-cube in A.

This **combinatorial specification** makes this path object very well-behaved. For example, it has not only a **left** adjoint ("cylinder") but also a **right** adjoint,

$$X \times \mathtt{I} \;\dashv\; Y^{\mathtt{I}} \;\dashv\; Z_{\mathtt{I}}\,.$$

Path spaces in cubical sets

Lemma

The interval $\mathtt{I}$ *in* **cSet** *satisfies the "domain equation"*

$$\mathtt{I}^{\mathtt{I}} \;\cong\; \mathtt{I} + 1\,.$$

Something similar happens in the object classifier and in the Schanuel topos. We can use this to calculate the right adjoint $Z_{\mathtt{I}}$.

Corollary

For the "amazing right adjoint" $Z_{\mathtt{I}}$*, we have:*

$$\begin{aligned}
Z_{\mathtt{I}}(n) &\cong \mathrm{Hom}(\,\mathtt{I}^n,\, Z_{\mathtt{I}}) \;\cong\; \mathrm{Hom}(\,(\mathtt{I}^n)^{\mathtt{I}},\, Z) \\
&\cong \mathrm{Hom}(\,(\mathtt{I}^{\mathtt{I}})^n,\, Z) \;\cong\; \mathrm{Hom}(\,(\mathtt{I}+1)^n,\, Z) \\
&\cong \mathrm{Hom}(\,\mathtt{I}^n + C^n_{n-1}\mathtt{I}^{n-1} + \cdots + C^n_1\mathtt{I} + 1,\, Z) \\
&\cong Z_n \times Z_{n-1}^{C^n_{n-1}} \times \cdots \times Z_1^{C^n_1} \times Z_0,
\end{aligned}$$

where $C^n_k = \binom{n}{k}$ *is the usual binomial coefficient.*

Path spaces as identity types

We will use the canonical pathobject $A^{\mathtt{I}}$ to interpret the $\mathtt{Id}$-**type**,

$$\mathtt{Id}_A = A^{\mathtt{I}}\,.$$

This implies some new type-theoretic **equations** and **conditions**, such as:

$$\mathtt{Id}_{\mathtt{Id}_A} = (A^{\mathtt{I}})^{\mathtt{I}} \cong A^{\mathtt{I}\times\mathtt{I}}\,,$$
$$\mathtt{Id}_{A+B} = (A+B)^{\mathtt{I}} \;\cong\; A^{\mathtt{I}} + B^{\mathtt{I}} = \mathtt{Id}_A + \mathtt{Id}_B\,,$$

and generally, the $\mathtt{Id}$-type of a colimit is a colimit of $\mathtt{Id}$-types.

The interpretation is thus **not** expected to be **conservative** — indeed, one hopes to determine some new **cubical laws** that may be soundly added to the original theory

Path spaces as identity types

In order to use $A^{\mathtt{I}}$ as the `Id`-type, we are led to ask:

When does $A^{\mathtt{I}} \to A \times A$ satisfy the rules for `Id`-types?

Theorem (A. 2015)

The path space $A^{\mathtt{I}} \to A \times A$ satisfies the rules for `Id`-types if

1. *The obect A is a Kan complex.*
2. *The dependent types $B \to A$ are Kan fibrations.*

The notions of **Kan complex** and **Kan fibration** are determined by the usual **box-filling conditions**.

Proof.

1. Reduce `Id`-elim to **transport** and **contraction**.
2. Transport follows from path-lifting, i.e. 1-box filling.
3. Contraction follows from 1-box filling for $A^{\mathtt{I}} \to A \times A$.
4. 1-box filling in $A^{\mathtt{I}} \to A \times A$ is 2-box filling in A. □

Path spaces and identity types

The last step of the foregoing is a special case of the following:

Lemma

The following are equivalent for a cubical set A.

1. $(n+1)$*-box filling in* A,
2. n*-box filling in* $A^{\mathtt{I}} \to A \times A$,
3. 1*-box filling in* $A^{\mathtt{I}^n} \to A^{\partial \mathtt{I}^n}$.

This can be used to prove a **converse** of the foregoing theorem: the box-filling conditions for cubical sets follow from the `Id`-rules together with Σ-types.

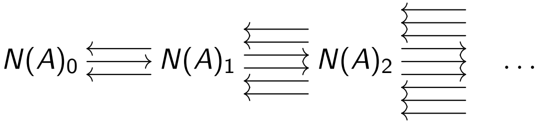
Cubical Lumsdaine

We can use the foregoing lemma to derive a **cubical version** of "Lumsdaine's Theorem" (aka "Lumsdaine-van den Berg-Garner"):

Theorem (A. 2015)

Every type A in MLTT gives rise to a **cubical** *∞-groupoid (a cubical set satisfying the box-filling conditions).*

We first need to determine the **cubical nerve of a type** A, i.e. a cubical set $N(A)$:



with:

$$N(A)_n \;\cong\; \text{“}n\text{-cubes in } A\text{”}$$

Cubical nerve of a type

A **pre-cubical** structure on a type A arises as follows:
Consider the **type-theoretic path object**:

$$P(X) \;=\; \sum_{x,y:X} \mathtt{Id}_X(x,y)\,.$$

We have the usual (reflexive) **globular** maps:

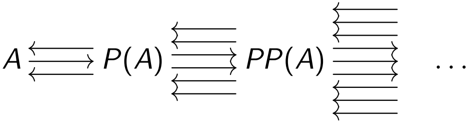
$$A \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; P(A) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; PP(A) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \quad \ldots$$

Since P also acts on maps by the “map on paths” operation, there are also the successive **images** of these maps under P:

$$\begin{array}{ccccccc}
A & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & P(A) & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & PP(A) & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \quad \ldots \\
 & & & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \\
 & & & & & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} &
\end{array}$$

Cubical nerve of a type

Rearranging, we find the usual cubical **structure**:



But we would need P to be **strictly functorial** for the cubical identities to hold!

Instead, we need a more elaborate dependent indexing of the successive steps to make the cubical identities hold. This is still **not** a **cartesian** cubical set (it lacks diagonals!), but only a **monoidal** one.

In **cubical** type theory we expect to have a **cartesian** cubical nerve.

Cubical nerve of a category

A similar example is the **cubical nerve** $N(\mathbb{A})$ of a category $\mathbb{A}$.
As a "pathobject" we can take the arrow category:

$$P(\mathbb{A}) \;=\; \mathbb{A}^{\to}$$

which **is strictly functorial**.

$N(\mathbb{A})_n$ is then the set of **commutative** n**-cubes** in $\mathbb{A}$, i.e.

$$\mathbf{Cat}(\mathbb{2}^n, \mathbb{A})\,,$$

where $\mathbb{2} = (\cdot \to \cdot)$ is the single-arrow category.

We also have the usual "realization $\dashv$ nerve" adjunction,

$$\mathbf{cSet} \overset{\longleftarrow}{\longrightarrow} \mathbf{Cat}\,,$$

given by Kan extension along $\mathbb{C} \longrightarrow \mathbf{Cat}$, the cartesian classifying map of the interval $\mathbb{1} \to \mathbb{2} \leftarrow \mathbb{1}$ in **Cat**.

Cubical nerve of a category

Theorem (A. 2015)

The **cartesian** *nerve functor* $N : \mathbf{Cat} \longrightarrow \mathbf{cSet}$ *is full and faithful.*

- This uses the diagonals in an essential way and **fails** for the **monoidal** version of cubical sets.
- As in **sSets**, the categories $\mathbb{A}$ with a Kan nerve $N(\mathbb{A})$ are exactly the **groupoids**.
- **Cubical analogues** of the **orientals**, the **homotopy coherent nerve**, and the notions of **quasicategory** and ∞-**topos** have not yet been studied.
- We expect the (cubical nerve of) the category of types in cubical homotopy type theory to be a cubical ∞-topos.