On productively Lindelöf spaces

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Abstract. We study conditions on a topological space that guarantee that its product with every Lindelöf space is Lindelöf. The main tool is a condition discovered by K. Alster and we call spaces satisfying his condition Alster spaces. We also study some variations on scattered spaces that are relevant for this question.

1 Introduction It is well known that a product of two Lindelöf spaces need not be Lindelöf. On the other hand, many spaces are known whose product with every Lindelöf space is Lindelöf. Let us call such a space **productively Lindelöf**.

K. Alster, [Alster (1988)], discovered (and we rediscovered, [Barr, Kennison, & Raphael (2006), Section 4] and called *amply Lindelöf*) a property which we will here call **Alster's condition** that is sufficient—and possibly necessary—for a space to be productively Lindelöf . Our formulation of Alster's condition follows. It looks rather different from Alster's but the two are readily shown to be equivalent.

Definition 1. *A space satisfies Alster's condition if every cover by G_δ sets that covers each compact set finitely contains a countable subcover. A space that satisfies this condition will be called an* **Alster space***.*

The point about covering each compact set finitely is crucial. In the space of real numbers, every point is a G_δ but the cover by points has no proper subcover. But the reals are σ-compact and it is obvious that every σ-compact space is an Alster space. It is a trivial observation that if a space has the property that every compact set is a G_δ (this is the case in any metric space), then it is Alster if and only if it is σ-compact.

Alster proves, assuming CH, that a space of weight of at most $\aleph_1$ is productively Lindelöf if and only if it is Alster. However, it is quite evident in his paper that the "if" direction uses neither CH nor the weight condition; it is fair to say that he showed that his condition implies productively Lindelöf . Neither he nor any of us is aware of any productively Lindelöf space that is *not* Alster, no matter the weight or set theory.

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A somewhat different proof that Alster implies productively Lindelöf is found in [Barr, Kennison, & Raphael (2006), Theorem 4.5] where it is also shown that the product of two Alster spaces is Alster.

Oddly, despite interest in the question of productively Lindelöf, Alster's paper does not seem to be widely known. We have found only two citations, one by Alster himself and one in a paper that is widely unavailable (and we have not seen it).

The paper [Telgarsky, 1971] studies C-scattered spaces in some detail (see Section 5 for the definition). One result we will be showing is that Lindelöf C-scattered spaces (and some more general ones) satisfy Alster's condition and therefore are productively Lindelöf.

2 Definitions and basic properties All spaces considered here are completely regular and Hausdorff. We denote by $C(X)$ the ring of continuous real-valued functions on the space X.

We will be dealing with covers by G_δ sets. Since a finite union of G_δ sets is again a G_δ set, we can, and often will, suppose that the covers are closed under finite unions.

We recall that a continuous map $\theta : X \to Y$ is called **perfect** if it closed and if $\theta^{-1}(y)$ is compact for all $y \in Y$. It can be shown that whenever $B \subseteq Y$ is compact, $\theta^{-1}(B)$ is also compact. It is not always assumed that a perfect map is continuous, but we will suppose that it is. The inclusion map of a subspace is perfect if and only if the subspace is closed.

We also recall from [Barr, Kennison, & Raphael (2006), 2.2] that any $\theta : X \to Y$ induces three maps on subsets, the direct image also denoted $\theta : \mathcal{P}(X) \to \mathcal{P}(Y)$, the inverse image $\theta^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ and the universal image $\theta_\# : \mathcal{P}(X) \to \mathcal{P}(Y)$. These are characterized by the fact that if $A \subseteq X$ and $B \subseteq Y$, then $\theta(A) \subseteq B$ if and only if $A \subseteq \theta^{-1}(B)$ and $\theta^{-1}(B) \subseteq A$ if and only if $B \subseteq \theta_\#(A)$. Since $\theta_\#(A) = Y - \theta(X - A)$, we see that when θ is closed, $\theta_\#$ takes open sets to open sets.

We also recall that if X is a space, a point $p \in X$ is called a **P-point** if for any $f \in C(X)$, the set $\{q \in X \mid f(q) = f(p)\}$ is a neighbourhood of p. A **P-space** is a space in which every point is a P-point. It is immediate that P-spaces are characterized by the fact that G_δ sets are open.

A not-necessarily-open cover of a space is called **ample** if it covers every compact set finitely. One of the main results of this paper is that Lindelöf ORC-scattered spaces (defined in Section 5) are Alster (see 26).

We will say that a point $p \in X$ satisfies the **open refinement condition (ORC)** if every ample G_δ cover that is closed under finite union contains a neighbourhood of p. If p is a P-point, it satisfies the ORC because one element of the cover contains p and a G_δ that contains p is a neighbourhood of p. If p has a compact neighbourhood A then some member of the cover contains A and hence is a neighbourhood. Thus this condition is a common generalization of being a P-point and having a compact neighbourhood. We will say that a space **satisfies the ORC** or that it is **an ORC space** if every point satisfies the ORC. This is a common generalization of P-space and locally compact space. We will see that the class of ORC spaces is closed under finite products and closed subspaces and hence gives a much broader class than simply the union of the P-spaces and the locally compact spaces.

Theorem 2. *Of the following properties on a space:*

1. *discrete*
2. *P-space;*
3. *locally compact;*
4. *ORC;*

5. *Alster;*

6. *productively Lindelöf.*

1 implies 2 and 3, each of which implies 4. For Lindelöf spaces, 4 implies 5 implies 6.

Proof. We have already discussed the facts that 2 and 3 imply 4. It is clear that, for Lindelöf spaces, 4 implies 5 and Alster showed that 5 implies 6 (see also [Barr, Kennison, & Raphael (2006), Theorem 4.5]).

The space of rational numbers gives an example that is Alster but not ORC, since every compact set is a G_δ and no compact set has a non-empty interior.

3 Permanence properties In studying the properties of G_δ covers, we can usually suppose, without loss of generality, that the cover is closed under finite union and the formation of G_δ subsets. If it is, we call it a G_δ *ideal cover.* We make the same definition, substituting open subsets for G_δ subsets, for *open ideal covers.*

Theorem 3. *The product of two ORC spaces is a ORC space.*

We use several lemmas.

Lemma 4. *Let X and Y be spaces and $\mathcal{W}$ be an ample G_δ ideal cover of $X \times Y$. Then for any compact sets $A \subseteq X$ and $B \subseteq Y$ and any $W \in \mathcal{W}$ such that $A \times B \subseteq W$ there are G_δ sets $U \subseteq X$ and $V \subseteq Y$ with $A \times B \subseteq U \times V \subseteq W$.*

Proof. Let $W = \bigcap_{n\in\mathbf{N}} W_n$ with each W_n open. According to [Kelley (1955), Theorem 5.12] there are, for each $n \in \mathbf{N}$ open sets $U_n \subseteq X$ and $V_n \in Y$ with $A \times B \subseteq U_n \times V_n \subseteq W_n$. If we let $U = \bigcap U_n$ and $V = \bigcap V_n$, then U and V are G_δ sets and $A \times B \subseteq U \times V \subseteq W$. □

Lemma 5. *Let X and Y be spaces with X being ORC. Let $\mathcal{W}$ be an ample G_δ ideal cover of $X \times Y$. Then for any compact set $B \subseteq Y$, there is an open ideal cover $\mathcal{U}(B)$ of X with the property that whenever $U \in \mathcal{U}(B)$ there is a G_δ set $V \subseteq Y$ with $B \subseteq V$ and $U \times V \in \mathcal{W}$.*

Proof. Let $\mathcal{U}'(B)$ denote the set of all G_δ sets $U \in X$ for which there is a $V \supseteq B$ with $U \times V \in \mathcal{W}$. If U_1, $U_2 \in \mathcal{U}'(B)$ there are G_δ sets V_1, V_2 containing B such that $U_i \times V_i \in \mathcal{W}$ for $i = 1, 2$. But then $B \subseteq V_1 \cap V_2$ and

$$(U_1 \cup U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cup (U_2 \times V_2) \in \mathcal{W}$$

It is trivial to see that $\mathcal{U}'(B)$ is closed under G_δ subsets and hence is a G_δ ideal. Finally, if $A \subseteq X$ is compact, the ampleness of $\mathcal{W}$ implies that there is some $W \in \mathcal{W}$ that contains $A \times B$ and the preceding lemma gives G_δ sets U and V with $A \times B \subseteq U \times V \subseteq W$. Thus $U \in \mathcal{U}'(B)$. This shows that $\mathcal{U}'(B)$ is an ample G_δ ideal cover of X. The set $\mathcal{U}(B)$ of open sets in $\mathcal{U}'(B)$ is the required open ideal cover. □

Lemma 6. *If in addition to the hypotheses of the preceding lemma, Y is also a ORC space, then for each compact $A \in X$, there is an open cover $\mathcal{V}(A)$ of Y such that for each $V \in Y$ there is an open set $U \subseteq X$ such that $A \subseteq U$ and $U \times V \in \mathcal{W}$.*

Proof. Let $\mathcal{V}'(A)$ denote the set of all G_δ sets $V \subseteq Y$ for which there is an open set $U \subseteq X$ that contains A and for which $U \times V \in \mathcal{W}$. To show that $\mathcal{V}'(A)$ is ample, let $B \in Y$ be compact. According to the preceding lemma, there is an open ideal cover $\mathcal{U}(B)$ of X with the property that for all $U \in \mathcal{U}(B)$ there is a G_δ set $V \in Y$ such that $B \subseteq V$ and $U \times V \in \mathcal{W}$. Since $\mathcal{U}(B)$ is an open ideal cover, there is a $U \in \mathcal{U}(B)$ with $A \subseteq U$ and this shows that $V \in \mathcal{V}'(A)$. The fact that $\mathcal{V}'(A)$ is an ideal cover follows exactly as in the preceding lemma. The set $\mathcal{V}(A)$ of open sets in $\mathcal{V}'(A)$ is the required cover. □

Proof of 3. Now given any point $(x, y) \in X \times Y$, let $V \in \mathcal{V}(\{x\})$ that contains y. By definition, there is an open set $U \subseteq X$ with $x \in U$ such that $(x, y) \in U \times V \in \mathcal{W}$. Thus the open sets in $\mathcal{W}$ cover $X \times Y$. □

Theorem 7. *The following table expresses the permanence of these properties. In this table, D means discrete, P means P-space, LC means locally compact, A means Alster, and PL means productively Lindelöf. The + signs indicate the property is preserved, while a blank may express our ignorance or else that the property is not preserved.*

	D	*P*	*LC*	*ORC*	*A*	*PL*	
Finite products	+	+	+	+	+	+	
Perfect preimage	*	*	+	+	+	+	
Closed subspaces	+	+	+	+	+	+	
Continuous image					+	+	
Quotient	+	+			+	+	
Open image	+	+	+	+	+	+	
Perfect image	+	+	+	+	+	+	

* *Preserved, provided the inverse image of each point is finite.*

Proof. Certain properties follow from others and will not be mentioned explicitly: a closed subspace is a perfect preimage and both open images and closed images are quotients while quotients are continuous images. We will take each class of spaces in turn.

Discrete: Obvious.

P-space: See [Gillman & Jerison (1960), 4K] for products. Closure under subspaces and quotient mappings is obvious. It is known that a perfect preimage of a P-space need not be a P-space. We will prove it here under the additional hypothesis that the inverse image of each point is a singleton or doubleton. Any finite-to-one map will work, but the notation gets ugly. So let $\theta : Y \to X$ be such a map with X a P-space. Let $y \in Y$ and $f : Y \to [0, 1]$ be continuous with $f(y) = 0$. Suppose that $\theta(y') = \theta(y)$ and $f(y') = 1$. The case that there is no such y' or that $f(y') = 0$ is easier and we omit it. For each pair of positive integers m and n, the set

$$U_{mn} = \{p \mid f(p) < 1/m\} \cup \{q \mid 1 - f(q) < 1/n\}$$

is an open neighbourhood of $\{y, y'\}$ and hence $\theta_{\#}(U_{mn})$ is an open neighbourhood of $\theta_{\#}(y, y') = \theta(y)$. Since X is a P-space, $\bigcap \theta_{\#}(U_{mn}) = \theta_{\#}(\bigcap U_{mn})$ is a neighbourhood of $\theta(y)$ and hence $\theta^{-1}(\theta_{\#}(\bigcap U_{mn})) \subseteq \bigcap U_{mn}$ is a neighbourhood of $\{y, y'\}$. But $\bigcap U_{mn} = f^{-1}(0) \cup f^{-1}(1)$ and the only way it can be a neighbourhood of $\{y, y'\}$ is for the first component to be a neighbourhood of y and the second a neighbourhood of y'.

Locally compact: It is well-known that a finite product of locally compact spaces is locally compact. It is shown in [Engelking, (1989), p. 189] that local compactness is closed under perfect image and preimage. It is obvious that the open image of a locally compact space is locally compact.

ORC: The closure under products is Theorem 3.

If $\theta : Y \to X$ is perfect and X is ORC, let $\mathcal{V}$ be an ample G_δ cover of Y. Assume it is closed under finite unions. For any $y \in Y$, $\theta^{-1}(\theta(y))$ is compact and hence contained in some $V \in \mathcal{V}$. It follows that $\theta(y) \in \theta_{\#}(\theta^{-1}(\theta(y))) \subseteq \theta_{\#}(V)$ so that $\theta_{\#}(\mathcal{V})$ is a G_δ

cover of X. A similar argument shows it is ample. The finite sum closure has an open cover refinement and the inverse image of that refinement refines $\mathcal{V}$.

Let $\theta : X \to Y$ be perfect and assume that X is ORC. If $\mathcal{V}$ is an ample G_δ cover of Y, closed under finite unions, then $\theta^{-1}(\mathcal{V})$ is an ample G_δ cover of X closed under finite unions. It therefore has an open cover refinement $\mathcal{U}$, which may be assumed to be closed under finite unions. then each compact set in X, in particular every set of the form $\theta^{-1}(y)$, is contained in a single set of $U \in \mathcal{U}$. This implies that $y \in \theta_\#(U)$. Thus $\theta_\#(\mathcal{U})$ is an open cover refinement of $\mathcal{V}$.

If $\theta : X \to Y$ is open and X is ORC, let $\mathcal{V}$ be an ample G_δ cover of Y, which we will suppose closed under finite union. Then $\theta^{-1}(\mathcal{V}) = \{\theta^{-1}(V) \mid V \in \mathcal{V}\}$ is an ample G_δ cover of X and also closed under finite union. Thus there is an open cover $\mathcal{U}$ such that for all $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ with $U \subseteq \theta^{-1}(V)$, which implies that $\theta(U) \subseteq V$. Moreover $\theta(U)$ is open by hypothesis and hence $\theta(\mathcal{U}) = \{\theta(U) \mid U \in \mathcal{U}\}$ is an open refinement of $\mathcal{V}$.

If $\theta : X \to Y$ is perfect and X is ORC, let $\mathcal{V}$ be an ample G_δ cover of Y, which we will suppose closed under finite union. As above, there is an open refinement $\mathcal{U}$ of $\theta^{-1}(\mathcal{V})$, which we may suppose is closed under finite union. If $y \in Y$, the set $\theta^{-1}(y)$ is compact and hence there is a $U \in \mathcal{U}$ and $V \in \mathcal{V}$ with $\theta^{-1}(y) \subseteq U \subseteq \theta^{-1}(V)$. But then $\{y\} = \theta_\#(\theta^{-1}(y)) \subseteq \theta_\#(U) \subseteq \theta_\#(\theta^{-1}(V)) = V$ so that $\theta_\#(\mathcal{U}) = \{\theta_\#(U) \mid U \subseteq \mathcal{U}\}$ is an open refinement of $\mathcal{V}$.

Alster: For finite products, see [Barr, Kennison, & Raphael (2006), Theorem 4.5]. Suppose $\theta : Y \to X$ is perfect and X is Alster. Let $\mathcal{V}$ be an ample G_δ cover of Y. If $p \in Y$, $\theta^{-1}(t(p))$ is compact and hence contained in some $V \in \mathcal{V}$ so that $\theta(p) = \theta_\#(\theta^{-1}(t(p))) \in \theta_\#(V)$ and thus $\theta_\#(\mathcal{V})$ is a cover of X. Since $\theta_\#$ preserves open sets and meets, $\theta_\#(\mathcal{V})$ is a G_δ cover. If $A \in X$ is compact, $\theta^{-1}(A)$ is compact and therefore contained in some $V \in \mathcal{V}$ whence $A = \theta_\#(\theta^{-1}(A)) \subseteq \theta_\#(V)$. Thus $\theta_\#(\mathcal{V})$ is an ample G_δ cover of X and has a countable subcover $\mathcal{U}$. If $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $U \subseteq \theta_\#(V)$ from which we conclude that $\theta^{-1}(U) \subseteq \theta^{-1}(\theta_\#(V)) \subseteq V$. Suppose $\theta : X \to Y$ is a continuous surjection and X is Alster. Let $\mathcal{V}$ be an ample G_δ cover of Y. Then $\theta^{-1}(\mathcal{V})$ is a G_δ cover of X. Since the image of a compact space is compact, it is clear that $\theta^{-1}(\mathcal{V})$ is ample. There is a countable subcover $\{\theta^{-1}(V_1), \theta^{-1}(V_2), \ldots,\}$ and the corresponding $\{V_1, V_2, \ldots\}$ is a countable subcover of $\mathcal{V}$.

Productively Lindelöf: Closure under finite products follows from the definition. The remaining properties all follow from the corresponding properties of Lindelöf spaces. □

Proposition 8. *If every point of a space has a neighbourhood that satisfies the ORC, so does the space.*

Proof. Suppose X is such a space. If U is a neighbourhood of $p \in X$ that satisfies the ORC, then $\mathrm{int}(U)$ is the union of the closed neighbourhoods of p contained in $\mathrm{int}(U)$. A closed neighbourhood satsifes the ORC and their union is an open image of their sum. Clearly the sum of spaces that satisfy the ORC does and hence $\mathrm{int}(U)$ does. But the union of the interiors of all those ORC neighbourhoods is again an open image of a sum and hence satisfies the ORC. □

4 Derived spaces If S is a property of topological spaces, let us call a space that has that property an S**-space**. An S-subspace is a subspace having property S; if it is a neighbourhood of some point, we will call it an S-neighbourhood of that point. We have in mind mainly the following four properties:

D: being discrete;

P: being a P-space;

C: being compact;

R: being ORC.

Hypothesis 9. *We will suppose that S satisfies the following conditions:*

1. *A closed subspace of an S-space is an S-space.*
2. *The union of two closed S-subspace is an S-subspace.*
3. *The product of two S-spaces is an S-space.*

Proposition 10. *The four examples D, P, C, and R satisfy these conditions.*

Proof. The first and third of these is either evident or follows from Theorem 7. As for the second, the union of two closed subspaces is a perfect image of their sum and it is obvious that these conditions are all preserved by sums. □

Define

$$L_S(X) = \{p \in X \mid p \text{ has an } S\text{-neighbourhood}\}$$

and $D_S(X) = X - L_S(X)$. We define $D_S^\alpha(X)$ for any ordinal α inductively by $D_S^\alpha = D_S(D_S^\beta)$ when $\alpha = \beta + 1$ and, if α is a limit ordinal, then $D_S^\alpha = \bigcap_{\beta<\alpha} D_S^\beta$. Evidently, $D_S^\alpha(X)$ is closed in X for all α.

From now on we will usually suppress the S and write $L(X)$ and $D(X)$.

Proposition 11. *If A is an open or closed subset of X, then $L(A) \supseteq A \cap L(X)$ and $D(A) \subseteq A \cap D(X)$. When A is open these inclusions are equalities.*

Proof. Suppose first that A is closed. If $p \in A$ and $U \subseteq X$ is an S-neighbourhood of p, then $A \cap U$ is closed in U and is therefore an S-neighbourhood of p in A and so $p \in L(A)$. We have

$$D(A) = A - L(A) \subseteq A - (A \cap L(X)) \subseteq A \cap (X - L(X)) = A \cap D(X)$$

When A is open, suppose $p \in A \cap L(X)$. Let U be an S-neighbourhood of p in X and let V be a closed neighbourhood of p inside A. Then $U \cap V$ is an S-neighbourhood of p so that $p \in L(A)$. Conversely, if $p \in L(A)$ then p has an S-neighbourhood inside A, but, A being open in X, this is also an S-neighbourhood in X. □

Proposition 12. *If A is a closed or open subset of X, then $D^\alpha(A) \subseteq A \cap D^\alpha(X)$ for all α; when A is open, the inclusion is an equality.*

Proof. First suppose that A is closed. If we suppose that $D^\beta(A) \subseteq D^\beta(X)$ then $D^\beta(A)$ is closed in A, which is closed in X and therefore $D^\beta(A)$ is closed in $D^\beta(X)$ so that

$$D^{\beta+1}(A) = D(D^\beta(A)) \subseteq D(D^\beta(X)) = D^{\beta+1}(X)$$

from which the conclusion is obvious. The same conclusion holds at limit ordinals by taking intersections.

Now let A be open. If we suppose that $D^\beta(A) = A \cap D^\beta(X)$, then since A is open, $A \cap D^\beta(X)$ is open in $D^\beta(X)$ so that

$$D^{\beta+1}(A) = D(D^\beta(A)) = D(A \cap D^\beta(X)) = A \cap D^\beta(X) \cap D^{\beta+1}(X) = A \cap D^{\beta+1}(X)$$

Again, the same conclusion holds at limit ordinals by taking intersections. □

Proposition 13. *If A and B are both open or both closed subsets of X, then $D(A \cup B) = D(A) \cup D(B)$.*

Proof. For open sets, we have from Proposition 12 that $D(A) = A \cap D(X)$ and $D(B) = B \cap D(X)$ so that $D(A) \cup D(B) = (A \cup B) \cap D(X) = D(A \cup B)$. For closed sets, we have from Proposition 12 that $D(A) \subseteq A \cap D(A \cup B)$ and $D(B) \subseteq B \cap D(A \cup B)$ so that $D(A) \cup D(B) \subseteq (A \cup B) \cap D(A \cup B) = D(A \cup B)$. For the reverse inequality we must show that

$$A \cup B - L(A \cup B) \subseteq (A - L(A)) \cup (B - L(B))$$

In other words, that if $p \in A \cup B$ and $p \notin L(A \cup B)$, then either $p \in A$ and $p \notin L(A)$ or $p \in B$ and $p \notin L(B)$.

If $p \in A - B$ and $p \in L(A)$, then p has an S-neighbourhood and, since B is closed, p has a closed neighbourhood disjoint from B. Their intersection is an S-neighbourhood disjoint from B, which is then an S-neighbourhood of p in $A \cup B$ so that $p \in L(A \cup B)$. If $p \in B - A$, we have the same argument. Finally we consider the case that $p \in A \cap B$ and $p \in L(A) \cap L(B)$. Then p has a closed S-neighbourhood $U \subseteq A$ and a closed S-neighbourhood $V \subseteq B$. Let U' and V' be $A \cup B$-neighbourhoods of p such that $U' \cap A = U$ and $V' \cap B = V$. Then $U' \cap V'$ is an $(A \cup B)$-neighbourhood of p and

$$U' \cap V' = (U' \cap V') \cap (A \cup B) = (U' \cap V' \cap A) \cup (U' \cap V' \cap B) = (U \cap V') \cup (U' \cap V) \subseteq U \cup V$$

so that $U \cup V$ is an S-neighbourhood of p in $(A \cup B)$. □

Corollary 14. *If A and B are both open or both closed subsets of X, then for any ordinal α, $D^\alpha(A \cup B) = D^\alpha(A) \cup D^\alpha(B)$.* □

Proposition 15 (Leibniz formula)**.** *For any spaces X and Y, $D(X \times Y) \subseteq (X \times D(Y)) \cup (D(X) \times Y)$.*

Proof. Since $L(X)$ and $L(Y)$ satisfy S, so does $L(X) \times L(Y) \subseteq X \times Y$ so that $L(X \times Y) \supseteq L(X) \times L(Y)$ from which the conclusion is clear. □

Corollary 16. *For all $n \in \mathbf{N}$, $D^n(X \times Y) \subseteq \bigcup_{i+j=n}(D^i(X) \times D^j(Y))$.* □

Proposition 17. *For all $n > 0$ in $\mathbf{N}$, $D^{2n-1}(X \times Y) \subseteq (X \times D^n(Y)) \cup (D^n(X) \times Y)$.*

Proof.

$$\begin{aligned} D^{2n-1}(X \times Y) &\subseteq \bigcup_{i=0}^{n-1}(D^i(X) \times D^{2n-1-i}Y) \cup \bigcup_{i=n}^{2n-1}(D^i(X) \times D^{2n-1-i}Y) \\ &\subseteq (X \times D^n(Y)) \cup (D^n(X) \times Y) \end{aligned}$$

□

Corollary 18. $D^\omega(X \times Y) \subseteq (X \times D^\omega(Y)) \cup (D^\omega(X) \times Y)$.

Proof.

$$D^{\omega}(X \times Y) = \bigcap((X \times D^n(Y)) \cup (D^n(X) \times Y)) = \bigcap(X \times D^n(Y)) \cup (\bigcap D^n(X) \times Y)$$
$$= (X \times D^{\omega}(Y)) \cup (D^{\omega}(X) \times Y)$$

where the commutation of the meet and join is justified by the fact that the sequences of $D^n(X)$ and $D^n(Y)$ are descending. □

Theorem 19. *Assume the Hypothesis 9. Then for any limit ordinal α, we have $D^{\alpha}(X \times Y) \subseteq (X \times D^{\alpha}(Y)) \cup (D^{\alpha}(X) \times Y)$.*

Proof. Either $\alpha = \beta + \omega$ with β a limit ordinal, or $\alpha = \bigcup \beta$, the latter union over all the limit ordinals below α. In the first case, we can suppose by induction that the result is valid for β and then we see that $D^{\beta+2n-1}(X \times Y) \subseteq D^{\beta}(X) \times D^{\beta+n}(Y) \cup D^{\beta+n}X \times (Y)$. Forming the meet over all n, we conclude the result for α. In the second case, we assume inductively that the conclusion is true for all $\beta < \alpha$ and form the meet over all such β. □

5 Scattered spaces If S is a property of topological spaces, we will say that a space X is **S-scattered** if for some ordinal α, $D^{\alpha}(X) = \emptyset$. The following is an immediate consequence of the preceding section.

Theorem 20. *An open or closed subspace of an S-scattered space, a union of two open or two closed S-scattered subspaces and a product of two S-scattered spaces, is S-scattered.* □

The following is an immediate consequence of Proposition 12:

Proposition 21. *A space X is S-scattered if and only if every non-empty closed subset $A \subseteq X$ contains an S-space whose A-interior is non-empty.*

Corollary 22. *A space that is S-scattered for $S =$ D, P, or C is also R-scattered.*

Proposition 23. *Suppose $X = Y \cup Z$ and both Y and Z are S-scattered. If one of Y or Z is either open or closed in X, then X is also S-scattered.*

Proof. Suppose Y is open. From Proposition 12, we have that for all ordinals α, $D^{\alpha}(Y) = Y \cap D^{\alpha}(X)$. If α is chosen so that $D^{\alpha}(Y) = \emptyset$, we conclude that $D^{\alpha}(X) \subseteq Z$ and then the result follows since Z is scattered.

Now suppose that Y is closed. Then $X - Y$ is an open subset of X and therefore scattered, so the result follows from the first part applied to $(X - Y) \cup Y$. □

Corollary 24. *The union of finitely many S-scattered subspaces, each of which is either open or closed, is S-scattered.* □

One can show that if a Lindelöf space X contains an open subspace U for which U and $X - U$ are P-spaces, then X is Alster. This is a special case of the following (a space is δ-Lindelöf if the δ-topology, in which every G_{δ} set is open, is Lindelöf.)

Theorem 25 ([Henriksen *et. al.*, (to appear)])**.** *A Lindelöf P-scattered space is δ-Lindelöf.*

Using a similar transfinite induction argument, we will prove:

Theorem 26. *A Lindelöf R-scattered space is Alster.*

It follows from Corollary 22 that this theorem will show that any Lindelöf D, P, C, or R-scattered space is Alster.

Proof. We will make the inductive hypothesis that for any Lindelöf space Y and for any $\beta < \alpha$, if $D^\beta(Y) = \emptyset$, then Y is Alster. We first consider the case that α is a limit ordinal. In that case, $\bigcap_{\beta<\alpha} D^\beta(X) = \emptyset$ which implies that $\{X - D^\beta(X)\}$ is an open cover of X. Since

$$D^\beta(X - D^\beta(X)) = (X - D^\beta(X)) \cap D^\beta(X) = \emptyset$$

and $\beta < \alpha$, the inductive hypothesis implies that each $X - D^\beta(X)$ is Alster. Since X is Lindelöf, countably many of them cover X and so X is a union of countably many Alster spaces, hence is Alster.

Now suppose that $\alpha = \beta + 1$ is a successor. In that case, every element of $Y = D^\beta(X)$ has an open ORC neighbourhood. It follows from Proposition 8 that Y is ORC. Let $\mathcal{U}$ be an ample G_δ cover of X. From [Barr, Kennison, & Raphael (2006), 4.8] we may suppose, without loss of generality, that $\mathcal{U}$ consists of zerosets. Since a finite union of zerosets is a zeroset, we can suppose that $\mathcal{U}$ is closed under finite unions. Then $\{Y \cap U \mid U \in \mathcal{U}\}$ has an open refinement, which has a countable refinement by cozerosets, say $\{Y \cap V_n\}$. For each n, there is a $U_n \in \mathcal{U}$ such that $Y \cap V_n \subseteq U_n$. Now $X - \bigcup V_n$ is closed in X and thus

$$D^\beta\left(X - \bigcup V_n\right) \subseteq \left(X - \bigcup V_n\right) \cap Y \subseteq (X - Y) \cap Y = \emptyset$$

and the inductive hypothesis implies that $X - \bigcup V_n$ is countably covered by $\mathcal{U}$. Each set $V_n - U_n$ is the difference of a cozeroset and a zeroset, which is a cozeroset and hence an F_σ. If $V_n - U_n = \bigcup_m A_{nm}$ with each A_{nm} closed, we have

$$D^\beta(A_{nm}) \subseteq A_{nm} \cap Y \subseteq (V_n - U_n) \cap Y \subseteq (V_n \cap Y) - U_n = \emptyset$$

so that the inductive hypothesis implies that A_{nm} is countably covered by $\mathcal{U}$ and then so is $\bigcup_n (V_n - U_n) = \bigcup_{n,m} A_{nm}$. Finally, $\bigcup_n U_n$ is countably covered by the U_n and so

$$X = \left(X - \bigcup V_n\right) \cup \left(\bigcup (V_n - U_n)\right) \cup \bigcup U_n$$

is countably covered. Thus X is Alster. □

Theorem 27. *When S is one of the classes D, P, C, or R, being S-scattered is invariant under perfect surjections and perfect preimages, provided in the latter case that when $S = D$ or P, the the preimage of each point is finite.*

The proof will proceed by a series of lemmas. Note that all four of the classes are invariant under perfect image (which implies closure under finite unions of closed subobjects) and, subject to the proviso in the statement, perfect preimage (see Theorem 7).

Lemma 28. *Suppose $\theta : X \to Y$ is a perfect surjection. Then $\theta(D(X)) \supseteq D(Y)$.*

Proof. We must show that $y \notin L(Y)$ implies that there is some $x \in \theta^{-1}(y)$ such that $x \notin L(X)$. Equivalently, we must show that $x \in L(X)$ for all $x \in \theta^{-1}(y)$ implies $y \in L(Y)$. Suppose that for each $x \in \theta^{-1}(y)$ there is an S-neighbourhood $U(x)$ of x. We may suppose that each $U(x)$ is closed. Since $\theta^{-1}(y)$ is compact, there is finite set $x_1, \ldots, x_n \in \theta^{-1}(y)$ such that the interiors of $U(x_1), \ldots, U(x_n)$ cover $\theta^{-1}(y)$. Thus $\theta^{-1}(y) \subseteq U = \bigcup_{i=1}^n U(x_i)$ and so $\theta_\#(U)$ is a neighbourhood of y. By Theorem 7, $\theta(U)$ is an S-subspace of Y and also a neighbourhood of y since $\theta(U) \supseteq \theta_\#(U)$.

Lemma 29. *Suppose $\theta : X \to Y$ is a perfect surjection. Then for all ordinals α, $\theta(D^\alpha(X)) \supseteq D^\alpha(Y)$.*

Proof. If we make the inductive hypothesis that $\theta(D^\alpha(X)) \supseteq D^\alpha(Y)$, it follows that there is a perfect surjection $X_\alpha = \theta^{-1}(D^\alpha(Y)) \cap D^\alpha(X) \to D^\alpha(Y)$. Since $X_\alpha \subseteq D^\alpha(X)$, we have that $D(X_\alpha) \subseteq D^{\alpha+1}(X)$ so that $\theta(D^{\alpha+1}(X)) \supseteq \theta(D(X_\alpha)) \supseteq D^{\alpha+1}(Y)$. Now suppose that α is a limit ordinal and $\theta(D^\beta(X)) \supseteq D^\beta(Y)$ for all $\beta < \alpha$. We want to show that $\theta\left(\bigcap_{\beta<\alpha} D^\beta(X)\right) \supseteq D^\alpha(Y)$. For each $y \in D^\alpha(Y)$ and each $\beta < \alpha$ the set $\{x \in D^\beta(X) \mid \theta^{-1}(y)\}$ is a non-empty closed subset of the compact set $\theta^{-1}(y)$ and hence their meet over all $\beta < \alpha$ is non-empty. ☐

Corollary 30. *If $\theta : X \to Y$ is a perfect surjection and X is S-scattered, then so is Y.* ☐

In order to simplify the statements of the following results, we will say that a map is *S-perfect* if it is perfect and, in case $S =$ D or P, that the inverse image of each point is finite.

Lemma 31. *Suppose $\theta : X \to Y$ is S-perfect. Then $\theta(D(X)) \subseteq D(Y)$.*

Proof. We have

$$\begin{aligned} \theta(D(X)) &\subseteq D(Y) && \text{if and only if} \\ \theta(X - L(X)) &\subseteq Y - L(Y) && \text{if and only if} \\ Y - \theta_\#(L(X)) &\subseteq Y - L(Y) && \text{if and only if} \\ L(Y) &\subseteq \theta_\#(L(X)) && \text{if and only if} \\ \theta^{-1}(L(Y)) &\subseteq \ L(X) \end{aligned}$$

If $y \in L(Y)$, then y has an S-neighbourhood U. Then $\theta^{-1}(U)$ is a neighbourhood of each point of $\theta^{-1}(y)$ and, from Theorem 7, is an S-subset and hence each point of $\theta^{-1}(y)$ is in $L(U)$. ☐

Lemma 32. *Suppose $\theta : X \to Y$ is S-perfect. Then for all ordinals α, $\theta(D^\alpha(X)) \subseteq D^\alpha(Y)$.*

Proof. Assume by induction that $\theta(D^\alpha(X)) \subseteq D^\alpha(Y)$. Then

$$\theta(D^{\alpha+1}(X)) = \theta(D(D^\alpha(X))) \subseteq \theta(D(D^\alpha)(X)) \subseteq D(D^\alpha(Y)) = D^{\alpha+1}(Y)$$

If α is a limit ordinal and $\theta(D^\beta(X)) \subseteq D^\beta(Y)$ for all $\beta < \alpha$, then

$$\theta(D^\alpha(X)) = \theta\left(\bigcap_{\beta<\alpha} D^\beta(X)\right) \subseteq \bigcap \theta(D^\beta(X)) \subseteq \bigcap D^\beta(Y) = D^\alpha(X)$$ ☐

Corollary 33. *If $\theta : X \to Y$ is S-perfect and Y is S-scattered, so is X.* ☐

This finishes the proof of Theorem 27. As an application, we have:

Corollary 34. *Suppose $X = \bigcup_{i\in I} X_i$ is a locally finite union of closed S-scattered spaces. Then X is S-scattered.*

Proof. The canonical map from the categorical sum to the union is easily seen to be closed with the inverse images of points being finite. ☐

6 An example In this section, we assume CH and give an example of a space that is Lindelöf and not productively Lindelöf but has an uncountable discrete subspace whose complement is countable. This example contradicts [Abu Osma and Henriksen, 2004, Theorem 3.8] in which the eleventh line of the claimed proof interchanges a join and a meet.

Let $\mathbf{R}$ denote the space of reals with the usual topology and $\mathbf{R}_\nu$ the same pointset with a new topology that we describe below. We denote by $\mathbf{Q}$ the space of rationals with the usual topology.

In $\mathbf{R}_\nu$, every irrational point is open. A basic neighbourhood of a rational point q has the form $(a, b) - D$ where $a < q < b$ and D is a countable subset of irrational numbers. Since such a set is determined by the endpoints and a choice of D, the cardinality of such basic opens is ω_1. It is clear that since D consists of irrational numbers, $\mathbf{Q}$ appears as a subspace of $\mathbf{R}_\nu$ with its usual topology.

Proposition 35. *$\mathbf{R}_\nu$ is regular.*

Proof. We will show that whenever U is open and $p \in U$, then there is an open set V such that $p \in V \subseteq \mathrm{cl}(V) \subseteq U$. Since each irrational is clopen this is clear when $p \notin \mathbf{Q}$. Now suppose $p \in \mathbf{Q}$. It is sufficient to consider the case that U is basic, so suppose $U = (a, b) - D$ as above. If c and d are chosen so that $a < c < p < d < b$ and c and d are irrational, then (c, d) and $(c, d) - D$ are closed since each irrational is clopen. Then $p \in (c, d) - D \subseteq (a, b) - D$ is the required sequence. □

Proposition 36. *Every dense G_δ in $\mathbf{R}$ is uncountable.*

Proof. A dense G_δ is a countable meet of dense open sets. If it were countable, the meet could be extended by the complements of the points and then we would have an empty countable meet of dense open sets, which contradicts the Baire category theorem. □

Let us say that a countable set $\mathcal{B}$ of basic open sets of $\mathbf{R}_\nu$ (as defined above) is a **countable basic open cover of $\mathbf{Q}$** if it is a countable cover of $\mathbf{Q}$ in $\mathbf{R}_\nu$. Since there are ω_1-many basic open sets and such a cover is determined by a sequence of such covers, it is clear that there are ω_1-many such countable basic open covers. Let us enumerate them as $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\alpha, \ldots, \alpha < \omega_1$.

Proposition 37. *If $\mathcal{B}$ is a countable basic open cover of $\mathbf{R}_\nu$, then $\bigcup \mathcal{B}$ is a dense G_δ in $\mathbf{R}$.*

Proof. Suppose $\mathcal{B} = \{(a_n, b_n) - D_n) \mid n \in \mathbf{N}\}$ is a countable basic open cover. Then

$$\bigcup \mathcal{B} = \bigcup((a_n, b_n) - D_n) = \bigcup(a_n, b_n) \cap \bigcap_{x \in \bigcap D_n} (\mathbf{R}_\nu - \{x\})$$

which is the meet of an open set and a G_δ and hence a G_δ in $\mathbf{R}$. It is dense in $\mathbf{R}$ because it contains $\mathbf{Q}$. □

We will now choose an ω_1-indexed sequence $t_1, t_2, \ldots, t_\alpha, \ldots$ of irrational numbers. We let t_1 be any irrational. Suppose we have chosen t_β for all $\beta < \alpha$. Since $\bigcup \mathcal{B}_\beta$ is a dense G_δ of $\mathbf{R}$ for all $\beta < \alpha$ and there are only countably many $\beta < \alpha$, it follows that $\bigcap_{\beta<\alpha} \bigcup \mathcal{B}_\beta$ is a dense G_δ and therefore uncountable. The set $\{t_\beta \mid \beta < \alpha\}$ is countable and hence we can choose some

$$t_\alpha \in \left(\bigcap_{\beta<\alpha} \bigcup \mathcal{B}_\beta \right) - \{t_\beta \mid \beta < \alpha\} - \mathbf{Q}$$

Now we let $X = \mathbf{Q} \cup \{t_\alpha \mid \alpha < \omega_1\}$ with the topology inherited from $\mathbf{R}_\nu$.

Proposition 38. *X is Lindelöf and completely regular.*

Proof. Any open cover has a refinement by basic opens. Let $\mathcal{O}$ be such a cover of X. Since $\mathcal{O}$ contains a cover of $\mathbf{Q}$, some $\mathcal{B}_\alpha \subseteq \mathcal{O}$. But by construction, $t_\gamma \in \bigcup \mathcal{B}_\alpha$ for every $\gamma > \alpha$. Thus $\mathcal{B}_\alpha$, together with sets in $\mathcal{O}$ that cover the countably many t_β for $\beta < \alpha$ is a countable refinement of $\mathcal{O}$. Complete regularity follows from [Kelley (1955), 113]. ☐

Proposition 39. *X is not Alster and therefore, in the presence of CH not productively Lindelöf.*

Proof. We begin by observing that a since the irrationals are open a compact set can contain only finitely many of them. A compact set in $\mathbf{Q}$ must be compact and therefore closed and a G_δ in the usual topology, which makes it a G_δ in R_ν. Thus the cover consisting of all the compact sets of $\mathbf{Q}$ and all the singletons of $X - \mathbf{Q}$ is an ample G_δ cover without a countable refinement. Since X has weight ω_1, it follows from [Alster (1988), 1.1] that X cannot be productively Lindelöf. ☐

7 Some open questions

1. Is productively Lindelof weaker than Alster?
2. If a space is S-scattered, must each D_S^α be nowhere dense in $D_S^{\alpha+1}$? (This is known to be true in the cases D and P.)
3. Is Theorem 26 false if one replaces ORC with Alster, or productively Lindelöf?
4. Is there an example of Section 6 that does not use CH?

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