Cohomology of Commutative Algebras

Michael Barr

A Dissertation
in

Mathematics

Presented to the Faculty of the Graduate School of Arts and Sciences of the University of Pennsylvania in partial fulfillment of the requirements of the degree of Doctor of Philosophy.

1962

D. K. Harrison
Supervisor of Dissertation

I. J. Schoenberg
Chairman of Department

Bibliography

1. Cartan H. and Eilenberg, S.: Homological Algebra, Princeton: Princeton University Press, 1956.

2. Harrison, D. K., Commutative algebras and cohomology, Trans. Am. Math. Soc. **104** (1962), 191–204.

3. Hochschild, G.: On the cohomology groups of an associative algebra, Ann. of Math., **46** (1945), 58–67.

4. Hochschild, G.: On the cohomology theory for associative algebras, Ann. of Math., **47** (1947), 565–579.

5. Tate, J.: Homology of Noetherian rings and local rings, Illinois J. Math. **1** (1957), 14–27.

6. Zelinsky, D.: Cohomology of function fields and other algebras, International symposium of algebraic number theory, Tokyo: Science Council of Japan, 1956.

Preface

I would like to thank Professor D. K. Harrison who proposd the original problem and without whose inspiration and generous help this would not have been completed.

2023 preface

I acquired an electronic version of the thesis and decided to retype it (doing it from a paper copy would have been much harder). In the process, I silently corrected a number of obvious mistakes (often failure to insert a non-typable character by hand) and added a few notes on alternative ways to do things. But this is the thesis that was presented in 1962.

The essence of the thesis was the splitting of the Hochschild cohomology of a commutative ring over a field of characteristic not 2 or 3 with coefficients in a symmetric module into a commutative part and a skew part, but only in degrees ≤ 4. The proof is based on an explicit construction of what has come to be called the **shuffle idempotents** in those degrees. These are idempotents in the group ring of the symmetric groups $k(S_n)$ for $n = 2, 3, 4$.

Eventually, I discovered a way to give an inductive construction of these idempotents in all degrees, but only for fields of characteristic 0.[1]

[1] See M. Barr, Harrison homology, Hochschild homology and triples. J. Algebra **8** (1968), 314–323.

Introduction

In [3] and [4] Hochschild defined cohomology groups for associative algebras and discussed some of their properties. In [2] Harrison defines special cohomology groups for commutative algebras. The question naturally arises as to the relation, for commutative algebras, between the two theories. We let $H^n(A,E)$ and $H_c^n(A,E)$ denote the Hochschild and Harrison cohomology groups, respectively, of the commutative algebra A with coefficients in the a-module E. Then we know that $H^1(A,E) = H_c^1(A,E)$ and that $H_c^2(A,E)$ is naturally isomorphic to a subgroups of $H^2(A,E)$. We show here that it is possible, provided the characteristic of the field over which A is defined is not two or three, to define groups $H_s^n(A,E)$ for $n = 2, 3, 4$ so that $H^n(A,E) \cong H_c^n(A,E) \oplus H_s^n(A,E)$. Consequently it would be desirable to extend this result to all n, even in the case of characteristic zero, although it has not been possible as yet. Unfortunately, the computations even for $n = 3, 4$ are almost prohibitive. They are valuable because a) in applications to the study of algebras the cases $n = 2, 3, 4$ are the most important ones, and b) the computations for small n can give considerable insight into more general situation.[2]

In the second chapter of this paper we derive some computational results about these cohomology groups and give a more natural interpretations of a theorem of Tate in the case of radical algebras with the maximal condition. Using this we can show that $H_c^2(A,E) = 0$ for all A-modules E implies that $H_c^3(A,E) = H_c^4(A,E) = 0$. There are two directions of improvements possible for this result: a) to extend it to all commutative algebras, and b) to extend it to higher dimensions. Just as above, however, the results are of considerable value in themselves, and extension to higher n would almost certainly come as a corollary to to construction of groups $H_s^n(A,E)$ for larger n.

In a short appendix, we give a more natural proof of a theorem of Harrison that if A is an algebra and S a multiplicatively closed subset of A with $1 \in S$, $0 \notin S$, and E is an A_S-module, then $H_c^2(A,E) = H_c^2(A_S,E)$. We also add a few supplementary results which should prove useful in future investigations along these lines.

1. Chapter I

Throughout this chapter, A will denote a commutative algebra over a field k with characteristic not 2 or 3. $\mathfrak{a}$ will denote an ideal of A and R denote the factor algebra $A/\mathfrak{a}$. We let $A^{(n)}$ denote the tensor product over k of n copies of A and $R^{(n)}$ the tensor product over k of n copies of R. We let $\mathfrak{a}^{(n)}$ denote the kernel of the canonical homomorphism of $A^{(n)}$ onto $R^{(n)}$. It will be shown that

$$\mathfrak{a}^{(n)} = \mathfrak{a} \otimes A \otimes \cdots \otimes A + A \otimes \mathfrak{a} \otimes A \otimes \cdots \otimes A + \cdots + A \otimes \cdots \otimes A \otimes \mathfrak{a}$$

the tensor product taken over k; i.e. that $\mathfrak{a}^{(n)}$ is generated by all all elements $a_1 \otimes \cdots a_n \in A^{(n)}$ such that $a_i \in \mathfrak{a}$ for at least one integer i. Let E be an A-module with $\mathfrak{a} \cdot E = 0$, so

[2] This turned out to be false. When the general problem was solved, the solution was simple and made no use of the detailed computations here. See M. Barr, *op. cit.*

that in a natural fashion E becomes an R-module. We let

$$C^n(A, E) = \mathrm{Hom}_k(A^{(n)}, E)$$

$$C^n(R, E) = \mathrm{Hom}_k(R^{(n)}, E)$$

$$C^n(A, \mathfrak{a}, E) = \mathrm{Hom}_k(\mathfrak{a}^{(n)}, E)$$

1.1. Proposition.

$$\mathfrak{a}^{(n)} = \mathfrak{a} \otimes A \otimes \cdots \otimes A + A \otimes \mathfrak{a} \otimes A \otimes \cdots \otimes A + \cdots + A \otimes \cdots \otimes A \otimes \mathfrak{a}$$

Proof. Clearly the right hand side is mapped to 0 under the canonical homomorphism of $A^{(n)}$ onto $R^{(n)}$. To go the other way, we need

1.2. Lemma. *If $\phi : U \longrightarrow V$ and $\phi' : U' \longrightarrow V'$ are linear transformations over a field k, and $W = \ker \phi$, $W' = \ker \phi'$, then kernel of $\phi \otimes \phi' : U \otimes U' \longrightarrow V \otimes V'$ is $W \otimes U' + U \otimes W'$.*

Proof. Clearly, by replacing, if necessary, V and V' by the images of ϕ and ϕ', respectively, we may assume that ϕ and ϕ' are epimorphisms. Moreover, since k is a field, all modules are projective so that tensor is an exact functor[3]. Then we get the following commutative exact diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W \otimes W' & \xrightarrow{i\otimes I} & U \otimes W' & \xrightarrow{\phi\otimes I} & V \otimes W' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I\otimes i'} & & \downarrow{\scriptstyle I\otimes i'} & & \downarrow{\scriptstyle I\otimes i'} & & \\
0 & \longrightarrow & W \otimes U' & \xrightarrow{i\otimes I} & U \otimes U' & \xrightarrow{\phi\otimes I} & V \otimes U' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I\otimes \phi'} & & \downarrow{\scriptstyle I\otimes \phi'} & & \downarrow{\scriptstyle I\otimes \phi'} & & \\
0 & \longrightarrow & W \otimes V' & \xrightarrow{i\otimes I} & U \otimes V' & \xrightarrow{\phi\otimes I} & V \otimes V' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where i and i' represent the injection maps of W and W', respectively, and I repreents the identity map of any space. Now $\phi \otimes \phi' = (\phi \otimes I) \circ (I \otimes \phi')$. Suppose $x \in U \otimes U'$ with $(\phi \otimes \phi')(x) = 0$. Then $(\phi \otimes I)((I \otimes \phi')(x)) = 0$ so $\exists y \in W \otimes V'$ with $(i \otimes I)(y) = (I \otimes \phi')(x)$.

[3] Although this is correct, in fact the argument needs only that the tensor product be right exact and therefore works for any commutative ring k

Since $I\otimes\phi'$ is an epimorphism, choose $z \in W\otimes U'$ with $(I\otimes\phi')(z) = y$. Let $x' = (i\otimes I)(z)$. Then by the commutativity of the diagram,

$$\begin{aligned}(I \otimes \phi')(x - x') &= (I \otimes \phi')(x) - (i \otimes I)(I \otimes \phi'(z)) \\ &= (I \otimes \phi')(x) - (i \otimes I)(y) = 0\end{aligned}$$

Hence $\exists z' \in U \otimes W'$ with $(I \otimes i')(z') = x - x'$. Let $x" = (I \otimes i')(z')$, then $x = x' + x"$ with $x' \in (I\otimes i')(W\otimes U')$ and $x" \in (i\otimes I(W\otimes U'))$. But $i\otimes I$ and $I\otimes i'$ are the natural injections of $W \otimes U'$ and $U \otimes W'$ into $U \otimes U'$ which completes the proof.

Now back to the proposition. We see from the lemma that $\mathfrak{a}^{(2)} = \mathfrak{a} \otimes A + A \otimes \mathfrak{a}$ and that $\mathfrak{a}^{(n)} = \mathfrak{a}^{(n-1)} \otimes A + A^{(n-1)} \otimes \mathfrak{a}$ since $\mathfrak{a}^{(n)}$ is the kernel of the homomorphism of $A^{(n-1)} \otimes A$ onto $R * (n-1) \otimes R$. Assuming, by induction, the result for $n-1$, the proposition follows.

Now we have an exact sequence

$$0 \longrightarrow \mathfrak{a}^{(n)} \longrightarrow A^{(n)} \longrightarrow R^{(n)} \longrightarrow 0$$

which induces an exact sequence

$$0 \longrightarrow C^n(R, E) \longrightarrow C^n(A, E) \longrightarrow C^n(A, \mathfrak{a}, E) \longrightarrow 0$$

since we are operating over the field k. If $f \in C^n(A, E)$, define $\delta f \in C^{n+1}(A, E)$ by

$$\begin{aligned}\delta f(a_1, \ldots, a_{n+1}) &= a_1 f(a_2, \ldots, a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} a_{n+1} f(a_1, \ldots, a_n)\end{aligned}$$

As usual $\delta\delta f = 0$. Now consider the diagram in which the rows are exact,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{n+1}(R, E) & \longrightarrow & C^{n+1}(A, E) & \longrightarrow & C^{n+1}(A, \mathfrak{a}, E) & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle\delta} & & & & \\
0 & \longrightarrow & C^{n}(R, E) & \longrightarrow & C^{n}(A, E) & \longrightarrow & C^{n}(A, \mathfrak{a}, E) & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle\delta} & & & & \\
0 & \longrightarrow & C^{n-1}(R, E) & \longrightarrow & C^{n-1}(A, E) & \longrightarrow & C^{n-1}(A, \mathfrak{a}, E) & \longrightarrow & 0
\end{array}$$

1.3. Proposition. *There are maps* $\delta : C^n(R, E) \longrightarrow C^{n+1}(R, E)$ *and* $\delta : C^n(A, \mathfrak{a}, E) \longrightarrow C^{n+1}(A, \mathfrak{a}, E)$ *which make the above commutative abd such that* $\delta\delta = 0$.

Proof. Let $f \in C^n(R,E)$ and define

$$\delta f(r_1,\dots,r_{n+1}) = r_1 f(r_2,\dots,r_{n+1})$$
$$+\sum_{i=1}^{n}(-1)^i f(r_1,\dots,r_i r_{i+1},\dots,r_{n+1}) + (-1)^{n+1} r_{n+1} f(r_1,\dots,r_n)$$

If we let $a \mapsto \overline{a}$ denote the projection of A onto R and $f \mapsto \overline{f}$ the induced map from $C^n(R,E)$ to $C^n(A,E)$, then $\overline{f}(a_1,\dots,a_n) = f(\overline{a}_1,\dots,\overline{a}_n)$ so that

$$\delta\overline{f}(a_1,\dots,a_{n+1}) = a_1\overline{f}(a_2,\dots,a_{n+1})$$
$$+\sum_{i=1}^{n}(-1)^i\overline{f}(a_1,\dots,a_i a_{i+1},\dots,a_{n+1}) + (-1)^{n+1}a_{n+1}\overline{f}(a_1,\dots,a_n)$$

so that

$$\delta f(\overline{a}_1,\dots,\overline{a}_{n+1}) = \overline{a}_1 f(\overline{a}_2,\dots,\overline{a}_{n+1})$$
$$+\sum_{i=1}^{n}(-1)^i f(\overline{a}_1,\dots,\overline{a}_i\overline{a}_{i+1},\dots,a_{n+1}) + (-1)^{n+1}\overline{a}_{n+1} f(\overline{a}_1,\dots,\overline{a}_n)$$

$= \delta f(\overline{a}_1,\dots,\overline{a}_{n+1})$ since $ae = \overline{a}e$ and $\overline{ab} = \overline{a}\overline{b}$ for all $a,b \in A$ and all $e \in E$. Clearly $\delta\delta = 0$ as before. Now let $f \in C^n(A,\mathfrak{a},E)$ and choose $f' \in C^n(A,E)$ with $f'|_{\mathfrak{a}^{(n)}} = f$. Define $\delta f = \delta f'|_{\mathfrak{a}}^{(n)}$. This clearly makes the diagram commutative and we continue to have $\delta\delta = 0$ since we can take $\delta f'$ as the map extending δf. It is only necessary to show that δf does not depend on f'. If $a_1 \otimes \cdots \otimes a_n \in \mathfrak{a}^{(n)}$, then at least one $a_i \in \mathfrak{a}$.

$$\delta f(\overline{a}_1,\dots,\overline{a}_{n+1}) = \overline{a}_1 f'(\overline{a}_2,\dots,\overline{a}_{n+1})$$
$$+\sum_{i=1}^{n}(-1)^i f'(\overline{a}_1,\dots,\overline{a}_i\overline{a}_{i+1},\dots,a_{n+1}) + (-1)^{n+1}\overline{a}_{n+1} f'(\overline{a}_1,\dots,\overline{a}_n)$$

If $a_i \in \mathfrak{a}$ for some $1 < i < n+1$, then in every term at least one variable is in $\mathfrak{a}$ while if $a_1 \in \mathfrak{a}$, the first term is zero since $\mathfrak{a}\cdot E = 0$ and all other terms contain a variable in $\mathfrak{a}$ and similarly if $a_{n+1} \in \mathfrak{a}$. Hence we have a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{n-1}(R,E) & \longrightarrow & C^{n-1}(A,E) & \longrightarrow & C^{n-1}(A,\mathfrak{a},E) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & C^{n}(R,E) & \longrightarrow & C^{n}(A,E) & \longrightarrow & C^{n}(A,\mathfrak{a},E) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & C^{n+1}(R,E) & \longrightarrow & C^{n+1}(A,E) & \longrightarrow & C^{n+1}(A,\mathfrak{a},E) & \longrightarrow & 0
\end{array}$$

and if we use $Z*n$, B^n and H^n to denote the cycles, boundaries, and homology classes[4], respectively, the fundamental lemma of homological algebra gives an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(R,E) \longrightarrow H^1(A,E) \longrightarrow H^1(A,\mathfrak{a},E) \\ &\longrightarrow H^2(R,E) \longrightarrow H^2(A,E) \longrightarrow H^2(A,\mathfrak{a},E) \longrightarrow \cdots \\ &\longrightarrow H^n(R,E) \longrightarrow H^n(A,E) \longrightarrow H^n(A,\mathfrak{a},E) \longrightarrow \cdots \qquad (*) \end{aligned}$$

The groups $H^n(R,E)$ and $H^n(A,E)$ are the Hochschild cohomology groups.

For $n = 2$, 3, 4 we define maps $\pi_n : C^n \longrightarrow C^n$ where C^n stands for any of the groups $C^n(A,E)$, $C^n(R,E)$ or $C^n(A,\mathfrak{a},E)$ Recall that we have assumed that the characteristic of k is not 2 or 3.

$$\pi_2 f(a,b) = 1/2[f(a,b) - f(b,a)]$$

$$\pi_3 f(a,b,c) = 1/6[4f(a,b,c) + 2f(c,b,a) + f(c,a,b) + f(b.c.a) - f(b,a,d) - f(a,c,b)]$$

$$\begin{aligned} \pi_4 f(a,b,c,d) = 1/12[&9f(a,b,c,d) + f(c,d,b,a) + f(c,a,b,d) + f(d,b,c,a) \\ &+ f(a,d,c,b) + f(c,d,a,b) + f(c,b,a,d) + f(d,c,a,b) \\ &+ f(a,c,d,b) + f(b,c,a,d) + f(b,d,a,c) + f(a,d,b,c) \\ &+ 3f(d,c,b,a) - f(a,b,d,c) - f(d,b,a,c) - f(a,c,b,d) \\ &- f(b,c,d,a) - f(b,a,d,c) - f(d,a,b,c) - f(b,a,c,d) \\ &- f(b,d,c,a) - f(d,a,c,b) - f(c,a,d,b) - f(c,b,d,a)] \end{aligned}$$

1.4. Proposition. *For $n = 2$, 3, 4, π_n is idempotent.*

Proof. If $n = 2$,

$$\begin{aligned} \pi_2^2 f(a,b) &= 1/2[\pi_2 f(a,b) - \pi_2 f(b,a)] \\ &= 1/4[f(a,b) - f(b,a) - f(b,a) + f(a,b)] \\ &= 1/2[f(a,b) - f(b,a)] = \pi_2 f(a,b)] \end{aligned}$$

For $n = 3$, we define maps σ_3 and τ_3 which will prove useful later.

$$\sigma_3\, abc = 1/2[\, abc + \, cba]$$

$$\tau_3\, abc = 1/3[\, abc + \, cab + \, bca]$$

Then we claim that σ_3 and τ_3 are idempotents and commute with each other.

$$\begin{aligned} \sigma_3^2 f(a,b,c) &= 1/2[\sigma_3 f(a,b,c) + \sigma_3 f(c,b,a)] \\ &= 1/4[f(a,b,c) + f(c,b,a) + f(c,b,a) + f(a,b,c) \\ &= 1/2[f(a,b,c) + f(c,b,a)] = \sigma_3 f(a,b,c)] \end{aligned}$$

[4]Actually, they should be called cocycles, coboundaries, and cohomology classes

$$\begin{aligned}\tau_3^2 f(a,b,c) &= 1/3[\tau_3 f(a,b,c) + \tau_3 f(c,a,b) + \tau_3 f(b,c,a)] \\ &= 1/9[f(a,b,c) + f(c,a,b) + f(b,c,a) + f(c,a,b) + f(b,c,a) \\ &\quad + f(a,b,c) + f(b,c,a) + f(a,b,c) + f(c,a,b) \\ &= 1/3[f(a,b,c) + f(c,a,b) + f(b,c,a)] = \tau_3 f(a,b,c)]\end{aligned}$$

$$\begin{aligned}\sigma_3\tau_3 f(a,b,c) &= 1/2[\tau_3 f(a,b,c) + \tau_3 f(c,b,a)] \\ &= 1/6[f(a,b,c) + f(c,a,b) + f(b,c,a) + f(c,b,a) + f(a,c,b) + f(b,a,c)] \\ &= 1/3[\sigma_3 f(a,b,c) + \sigma_3 f(c,a,b) + \sigma_3 f(b,c,a)] = \tau_3\sigma_3 f(a,b,c)\end{aligned}$$

Moreover, $(\sigma_3 + \tau_3 - \sigma_3\tau_3)f(a,b,c)$

$$\begin{aligned}&= 1/2[f(a,b,c) + f(c,b,a)] + 1/3[f(a,b,c) + f(c,a,b) + f(b,c,a)] \\ &- 1/6[f(a,b,c) + f(c,a,b) + f(b,c,a) + f(c,b,a) + f(a,c,b) + f(b,a,c)] \\ &= 1/6[4f(a,b,c) + 2f(c,b,a) + f(c,a,b) + f(b,c,a) - f(b,a,c) - f(a,c,b)] \\ &= \pi_3 f(a,b,c)\end{aligned}$$

So that $\pi_3 = \sigma_3 + \tau_3 - \sigma_3\tau_3$ which gives that

$$\begin{aligned}\pi_3^2 &= \sigma_3^2 + \tau_3^2 + \sigma_3^2\tau_3^2 + 2\sigma_3\tau_3 - 2\sigma_3^2\tau_3 - 2\tau_3\sigma_3^2 \\ &= \sigma_3 + \tau_3 - \sigma_3\tau_3 = \pi_3\end{aligned}$$

For $n = 4$, we introduce maps σ_4 and τ_4 as follows:

$$\sigma_4 f(a,b,c,d) = 1/2[f(a,b,c,d) + f(d,c,b,a)]$$

$$\begin{aligned}\tau_4 f(a,b,c,d) = [&f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) \\ &- f(a,c,b,d) - f(b,c,d,a) + f(c,d,a,b) \\ &- f(d,a,b,c) - f(b,a,c,d) + f(a,c,d,b) \\ &+ f(b,c,a,d) - f(c,a,d,b) + f(a,d,b,c)]\end{aligned}$$

Then we claim that σ_4 commutes with τ_4 and that if $\sigma_4 f = 0$, then $\tau_4^2 f = \tau_4 f$, i.e. that $\tau_4^2(1 - \sigma_4) = \tau_4$, and that σ_4 is idempotent.

$$\begin{aligned}\sigma_4^2 f(a,b,c,d) &= 1/2[\sigma_4 f(a,b,c,d) + \sigma_4 f(d,c,b,a)] \\ &= 1/4[f(a,b,c,d) + f(d,c,b,a) + f(d,c,b,a) + f(a,b,c,d)] \\ &= 1/2[f(a,b,c,d) + f(d,c,b,a)] = \sigma_4 f(a,b,c,d)\end{aligned}$$

$$\begin{aligned}
\sigma_4\tau_4 f(a,b,c,d) &= 1/2[\tau_4 f(a,b,c,d) + \tau_4 f(d,c,b,a)] \\
&= 1/12[3f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) - f(a,c,b,d) \\
&- f(b,c,d,a) + f(c,d,a,b) - f(d,a,b,c) - f(b,a,c,d) \\
&+ f(a,c,d,b) + f(b,c,a,d) - f(c,a,d,b) + f(a,d,b,c) \\
&+ 3f(d,c,b,a) - f(d,c,a,b) + f(b,d,c,a) - f(d,b,c,a) \\
&- f(c,b,a,d) + f(b,a,d,c) - f(a,d,c,b) - f(c,d,b,a) \\
&+ f(d,b,a,c) + f(c,b,d,a) - f(b,d,a,c) + f(d,a,c,b)]
\end{aligned}$$

$$\begin{aligned}
&= 1/6[3\sigma_4 f(a,b,c,d) - \sigma_4 f(a,b,d,c) + \sigma_4 f(c,a,b,d) - \sigma_4 f(a,c,b,d) \\
&- \sigma_4 f(b,c,d,a) + \sigma_4 f(c,d,a,b) - \sigma_4 f(d,a,b,c) - \sigma_4 f(b,a,c,d) \\
&+ \sigma_4 f(a,c,d,b) + \sigma_4 f(b,c,a,d) - \sigma_4 f(c,a,d,b) + \sigma_4 f(a,d,b,c)] \\
&= \tau_4\sigma_4 f(a,b,c,d)
\end{aligned}$$

Now suppose $\sigma_4 f = 0$. Then $\tau_4^2 f(a,b,c,d)$

$$= 1/6[3\tau_4 f(a,b,c,d) - \tau_4 f(a,b,c,d) + \tau_4 f(c,a,b,d) - \tau_4 f(a,c,b,d)$$
$$- \tau_4 f(b,c,d,a) + \tau_4 f(c,d,a,b) - \tau_4 f(d,a,b,c) - \tau_4 f(b,a,c,d)$$
$$+ \tau_4 f(a,c,d,b) + \tau_4 f(b,c,a,d) - \tau_4 f(c,a,d,b) + \tau_4 f(a,d,b,c)$$
$$= 1/36[9f(a,b,c,d) - 3f(a,b,d,c) + 3f(c,a,b,d) - 3f(a,c,b,d)$$
$$- 3f(b,c,d,a) + 3f(c,d,a,b) - 3f(d,a,b,c) + 3f(b,a,c,d)$$
$$+ 3f(a,c,d,b) + 3f(b,c,a,d) - 3f(c,a,d,b) + 3f(a,d,b,c)$$
$$- 3f(a,b,d,c) + f(a,b,c,d) - f(d,a,b,c) + f(a,d,b,c)$$
$$+ f(b,d,c,a) - f(d,c,a,b) + f(c,a,b,d) - f(b,a,d,c)$$
$$- f(a,d,c,b) - f(b,d,a,c) + f(d,a,c,b) - f(a,c,b,d)$$
$$+ 3f(c,a,b,d) - f(c,a,d,b) + f(b,c,a,d) - f(c,b,a,d)$$
$$- f(a,b,d,c) + f(b,d,c,a) - f(d,c,a,b) - f(a,c,b,d)$$
$$+ f(c,b,d,a) + f(a,b,c,d) - f(b,c,d,a) + f(c,d,a,b)$$
$$- 3f(a,c,b,d) + f(a,c,d,b) - f(b,a,c,d) + f(a,b,c,d)$$
$$+ f(c,b,d,a) - f(b,d,a,c) + f(d,a,c,b) + f(c,a,b,d)$$
$$- f(a,b,d,c) - f(c,b,a,d) + f(b,a,d,c) - f(a,d,c,b)$$
$$- 3f(b,c,d,a) + f(b,c,a,d) - f(d,b,c,a) + f(b,d,c,a)$$
$$+ f(c,d,a,b) - f(d,a,b,c) + f(a,b,c,d) + f(c,b,d,a)$$
$$- f(b,d,a,c) - f(c,d,b,a) + f(d,b,a,c) - f(b,a,c,d)$$
$$+ 3f(c,d,a,b) - f(c,d,b,a) + f(a,c,d,b) - f(c,a,d,b)$$
$$- f(d,a,b,c) + f(a,b,c,d) - f(b,c,d,a) - f(d,c,a,b)$$
$$+ f(c,a,b,d) + f(d,a,c,b) - f(a,c,b,d) + f(c,b,d,a)$$

$$
\begin{aligned}
&- 3f(d,a,b,c) - f(b,c,d,a) + f(b,d,a,c) + f(d,b,a,c) \\
&+ f(a,b,c,d) - f(b,c,d,a) + f(c,d,a,b) + f(a,d,b,c) \\
&- f(d,b,c,a) - f(a,b,d,c) + f(b,d,c,a) - f(d,c,a,b) \\
&- 3f(b,a,c,d) + f(b,a,d,c) - f(c,b,a,d) + f(b,c,a,d) \\
&+ f(a,c,d,b) - f(c,d,b,a) + f(d,b,a,c) + f(a,b,c,d) \\
&- f(b,c,d,a) - f(a,c,b,d) + f(c,b,d,a) - f(b,d,a,c) \\
&+ 3f(a,v,d,b) - f(a,c,b,d) + f(d,a,c,b) - f(a,d,c,b) \\
&- f(c,d,b,a) + f(d,b,a,c) - f(b,a,c,d) - f(c,a,d,b) \\
&+ f(a,d,b,c) + f(c,d,a,b) - f(d,a,b,c) + f(a,b,c,d) \\
&+ 3f(b,c,a,d) - f(b,c,d,a) + f(a,b,c,d) - f(b,a,c,d) \\
&- f(c,a,d,b) + f(a,d,b,c) - f(d,b,c,a) - f(c,b,a,d) \\
&+ f(b,a,d,c) + f(c,a,b,d) - f(a,b,d,c) + f(b,d,c,a) \\
&- 3f(c,a,d,b) + f(c,a,b,d) - f(d,c,a,b) + f(c,d,a,b) \\
&+ f(a,d,b,c) - f(d,b,c,a) + f(b,c,a,d) + f(a,c,b,d) \\
&- f(c,d,b,a) - f(a,d,b,c) + f(d,c,b,a) - f(c,b,a,d) \\
&+ 3f(a,d,b,c) - f(a,d,c,b) + f(b,a,d,c) - f(a,b,d,c) \\
&- f(d,b,c,a) + f(b,c,a,d) - f(c,a,b,d) - f(d,a,b,c) \\
&+ f(a,b,c,d) + f(d,b,a,c) - f(b,a,c,d) + f(a,c,b,d)]
\end{aligned}
$$

Collecting terms and using that $\sigma_4 f = 0$, we get

$$
\begin{aligned}
1/6[&3f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) - f(a,c,b,d) \\
&- f(b,c,a,d) + f(c,d,a,b) - f(d,a,b,c) - f(b,a,c,d) \\
&+ f(a,c,d,b) + f(b,c,a,d) - f(c,a,b,d) + f(a,d,b,c)]
\end{aligned}
$$

Now $(\sigma_4 + \tau_4 - \sigma_4\tau_4)f(a,b,c,d)$

$$
\begin{aligned}
&= 1/2[f(a,b,c,d) + f(d,c,b,a) \\
&+ 1/6[3f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) - f(a,c,b,d) \\
&- f(b,c,d,a) + f(c,d,a,b) - f(d,a,b,c) - f(b,a,c,d) \\
&+ f(a,c,d,b) + f(b,c,a,d) - f(c,a,d,b) + f(a,d,b,c)]
\end{aligned}
$$

$$
\begin{aligned}
&- 1/12[3f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) - f(a,c,b,d)\\
&- f(b,c,a,d) + f(c,d,a,b) - f(d,a,b,c) + f(b,a,c,d)\\
&+ f(b,c,d,a) + f(b,c,a,d) - f(c,a,b,d) + f(a,d,b,c)\\
&+ 3f(d,c,b,a) - f(c,d,b,a) + f(d,b,a,c) - f(d,b,c,a)\\
&- f(a,d,c,b) + f(b,a,d,c) - f(c,b,a,d) - f(d,c,a,b)]\\
&= 1/12[9f(a,b,c,d) - f(a,b,d,c) - f(d,b,a,c) - f(a,c,b,d)\\
&- f(b,c,d,a) - f(b,a,d,c) - f(d,a,b,c) - f(b,a,c,d)\\
&- f(b,d,a,c) - f(d,a,c,b) - f(c,a,d,b) - f(c,b,d,a)\\
&+ 3f(d,c,b,a) + f(c,d,b,a) + f(c,a,b,d) + f(d,b,c,a)\\
&+ f(a,d,b,c) + f(c,d,a,b) + f(c,b,a,d) + f(d,c,a,b)\\
&+ f(a,c,d,b) + f(b,c,a,d) + f(b,d,a,c) + f(a,d,b,c)\\
&= \pi_4\tau_4 f(a,b,c,d)
\end{aligned}
$$

Now $\sigma_4(1-\sigma_4) = 0$ so that $\tau_4^2(1-\sigma_4) = \tau_4(1-\sigma_4)$. Hence

$$
\begin{aligned}
\pi_4^2 &= \sigma_4^2 + \tau_4^2 + \sigma_4^2\tau_4^2 + 2\sigma_4\tau_4 - 2\sigma_4^2\\
&= \sigma_4 + \tau_4^2 + \sigma_4^2\tau_4^2 + 2\sigma_4\tau_4 - 2\sigma_4^2\tau_4 - 2\sigma_4\tau_4^2\\
\sigma_4 + \tau_4^2 + \sigma_4\tau_4^2 - 2\sigma_4\tau_4^2 &= \sigma_4 + t^2 - s\tau_4^2 = \sigma_4 + \tau_4^2(1-\sigma_4)\\
&= \sigma_4 + \tau_4(1-\sigma_4) = \sigma_4 + \tau_4 - \sigma_4\tau_4 = \pi_4
\end{aligned}
$$

This completes the proof of the proposition.

We will let $C_c^n(A,E)$ denote the chain groups used by Harrison in [3] of A with coefficients in an A-module E.

1.5. Proposition. *For* $n = 2, 3, 4$,

$$C_c^n = \operatorname{Ker}(\pi_n : C^n(A,E) \longrightarrow C^n(A,E))$$

Proof. For $n = 2$, $f \in C_c^2(A,E)$ if and only if $f(a,b) = f(b,a)$ for all $a, b \in A$ if and only if $\pi_2 f = 0$. For $n = 3$, $f \in C_c^n(A,E)$ if and only if f satisfies,

$$(i) \qquad f(a,b,c) - f(a,c,b) + f(c,a,b) = 0$$

for all a, b, $c \in A$. Now if satisfies (i), we have $f(a,b,c) - f(a,c,b) + f(c,a,b) = 0$ and $f(a,c,b) - f(a,b,c) + f(b,a,c) = 0$, and adding we get $f(c,a,b) + f(b,a,c) = 0$ or that

$\sigma_3 f = 0$. Using this

$$\tau_3 f(a,b,c) = 1/3[f(a,b,c) + f(c,a,b) + f(b,c,a)]$$
$$= 1/3[f(a,b,c) - f(a,c,b) + f(c,a,b)] = 0$$

so that $\pi_3 f = 0$.

Conversely, suppose $\pi_3 f = 0$. Then

$$0 = \sigma_3 \pi_3 f = (\sigma_3^2 + \sigma_3 \tau_3 - \sigma_3^2 \tau_3) f = \sigma_3 f$$

and we see that $\sigma_3 f = 0$ also and putting these together, we get

$$0 = 3\tau_3 f(a,b,c) = f(a,b,c) + f(b,c,a) + f(c,a,b)$$
$$= f(a,b,c) - f(a,c,b) + f(c,a,b)$$

so that $f \in C_c^3(A,E)$, since f satisfies (i).

For $n = 4$, $C_c^4(A,E)$ consists of those $f \in C^4(A,E)$ with

$$(ii) \qquad f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) = 0$$

and

$$(iii) \quad f(a,b,c,d) - f(a,c,d,b) + f(c,a,b,d) - f(c,a,d,b) + f(a,c,d,b) + f(c,d,a,b) = 0$$

for all a, b, c, $d \in A$. Suppose $f \in C_c^4(A,E)$; we wish to show that $\pi_4 f = 0$. First we show that $\sigma_4 f = 0$ and use this show that $\tau_4 f = 0$ which implies that $\pi_4 f = 0$. From (ii) we infer that

$$f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) = 0$$

and

$$f(d,c,b,a) - f(c,d,b,a) + f(c,b,a,d) + f(c,b,d,a) + f(b,c,d,a) + f(c,d,b,a) = 0$$

Adding these three equations gives $f(a,b,c,d) + f(d,c,b,a) = 0$ or $\sigma_4 f = 0$. Also from (ii) we have that

$$f(a,b,c,d) - f(a,b,d,c) + f(a,d,b,c) - f(d,a,b,c) = 0$$
$$f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) = 0$$
$$f(a,c,d,b) - f(a,c,b,d) + f(a,b,c,d) - f(b,a,c,d) = 0$$
$$f(c,a,b,d) - f(c,a,d,b) + f(c,d,a,b) - f(d,c,a,b) = 0$$

We add these, using that $\sigma_4 f = 0$, to obtain

$$\begin{aligned} 0 &= 3f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) - f(a,c,b,d) \\ &- f(b,c,d,a) + f(c,d,a,b) - f(d,a,b,c) - f(b,a,c,d) \\ &+ f(a,c,d,b)f(b,c,a,d) - f(c,a,b,d) + f(a,d,b,c) \\ &= 6\tau_4 f(a,b,c,d) \end{aligned}$$

Conversely, suppose $\pi_4 f = 0$. Then,

$$0 = \sigma_4 \pi_4 f = (\sigma_4^2 + \sigma_4 \tau_4 - \sigma_4^2) f = \sigma_4 f$$

from which we can see that $\tau_4 f = 0$ also. Hence

$$\begin{aligned} 0 &= \tau_4 f(a,b,c,d) - \tau_4 f(b,a,c,d) + \tau_4 f(b,a,c,d) - \tau_4 f(b,c,d,a) \\ &= 1/6[3f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) - f(a,c,b,d) \\ &- f(b,c,d,a) + f(c,d,a,b) - f(d,a,b,c) - f(b,a,c,d) \\ &+ f(a,c,d,b) + f(b,c,a,d) - f(c,a,d,b) + f(a,d,b,c) \\ &- 3f(b,a,c,d) + f(b,a,d,c) - f(c,b,a,d) + f(b,c,a,d) \\ &+ f(a,c,d,b) - f(c,d,b,a) + f(d,b,a,c) + f(a,b,c,d) \\ &- f(b,c,d,a) - f(a,c,b,d) + f(c,b,d,a) - f(b,d,a,c) \\ &+ 3f(b,c,a,d) - f(b,c,d,a) + f(a,b,c,d) - f(b,a,c,d) \\ &- f(c,a,d,b) + f(a,d,b,c) - f(d,b,c,a) - f(c,b,d,a) \\ &+ f(b,a,d,c) + f(c,a,b,d) - f(a,b,d,c) + f(b,d,c,a) \\ &- 3f(b,c,d,a) + f(b,c,a,d) - f(d,b,c,a) + f(b,d,c,a) \\ &+ f(c,d,a,b) - f(d,a,b,c) + f(a,b,c,d) + f(c,b,d,a) \\ &- f(b,d,a,c) - f(c,d,b,a) + f(d,b,a,c) - f(b,a,c,d) \end{aligned}$$

Collecting terms and using that $\sigma_4 f = 0$, this reduces to

$$0 = f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)$$

which is (ii) above. Hence

$$0 = f(c,a,b,d) - f(a,c,b,d) + f(a,b,c,d) - f(a,b,d,v)$$

and

$$0 = f(a,c,d,b) - f(c,a,d,b) + f(c,d,a,b) - f(c,d,b,a)$$

Adding these and using that $\sigma_4 f = 0$, we get

$$0 = f(a,b,c,d) - f(a,c,b,d) + f(c,a,b,d) - f(c,a,d,b) + f(a,c,d,b) + f(c,d,a,b)$$

which is (iii) above. This completes the proof of the proposition.

It is true, of course, that

$$C_c^n = \mathrm{Ker}(\pi_n : C^n(R,E) \longrightarrow C^n(R,E))$$

Now define

$$C_c^n(A,\mathfrak{a},E) = \mathrm{Ker}(\pi_n C^n(A,\mathfrak{a},E) \longrightarrow C^n(A,\mathfrak{a},E))$$

Also we define $C_s^n = \mathrm{Im}(C^n \longrightarrow C^n)$, where C^n stands for any of the groups considered.

1.6. Proposition.

$$0 \longrightarrow C_c^n(R,E) \longrightarrow C_c^n(A,E) \longrightarrow C_c^n(A,\mathfrak{a},E) \longrightarrow 0$$

and

$$0 \longrightarrow C_s^n(R,E) \longrightarrow C_s^n(A,E) \longrightarrow C_s^n(A,\mathfrak{a},E) \longrightarrow 0$$

are exact for $n = 2, 3, 4$*, the maps being the restrictions of the corresponding maps on the full chain groups.*

Proof. [5] We can write

$$\pi_n f(a_1,\dots,a_n) = \sum_p r_p f(a_{p(1)},\dots,a_{p(n)})$$

where p runs over the permutations of $1,\dots n$ which depend only on the n (not on the particular group). If, as before, we let $a \mapsto \overline{a}$ denote the map from A onto R and $f \mapsto \overline{f}$ the induced map from $C^n(R,E) \longrightarrow C^n(A,E)$, then

$$\begin{aligned}\pi_n \overline{f}(a_1,\dots,a_n) &= \sum_p r_p \overline{f}(a_{p(1)},\dots,a_{p(1)}) \\ &= \sum_p r_p f(\overline{a}_{p(1)},\dots,\overline{a}_{p(1)}) \\ &= \pi_n f(\overline{a}_1,\dots,\overline{a}_n) \\ &= \overline{\pi_n f}(a_1,\dots,a_n)\end{aligned}$$

[5] Clearly $C^n = C_c^n \oplus C_s^n$. It is fairly easy to show that if the direct sum of two sequences is exact, each of the constituents is. However, we give the original argument here.

Similarly, if $f \mapsto f'$ is the map from $C^n(A,E) \longrightarrow C^n(A,\mathfrak{a},E)$,

$$\begin{aligned}\pi_n f'(a_1,\dots,a_n) &= \sum_p r_p f(a_{p(1)},\dots,a_{p(1)}) \\ &= \sum_p r_p f(a_{p(1)},\dots,a_{p(1)}) \\ &= \pi_n f(a_1,\dots,a_n) \\ &= (\pi_n f)'(a_1,\dots,a_n)\end{aligned}$$

Since $f \mapsto f'$ is just the restriction map.

1.7. PROPOSITION. *(i)* $\delta C_c^n \subseteq c_c^{n+1}$*, for* $n = 1,\, 2, 3$*, (ii)* $\delta C_s^n \subseteq C_s^{n+1}$*, for* $n = 1,\, 2$ *where* $C_c^1 = C^1$.

PROOF. (i) is shown in [2]. (ii) for $n = 2$, suppose $g \in C_s^2$ and $f = \delta g$, then

$$\begin{aligned}\pi_3 f(a,b,c) = 1/6[&4f(a,b,c) + 2f(c,a,b) + f(c,a,b) \\ &+ f(b,c,a) - f(b,a,c) - f(a,c,b)] \\ &= 1/6[4ag(b,c) + 2cg(b,a) + cg(a,b) + bg(c,a) - bg(a,c) - ag(c,b) \\ &- 4g(ab,c) - 2g(cb,a) - g(ca,b) - g(bc,a) + g(ba,c) + g(ca,b) \\ &+ 4g(a,bc) + 2g(c,ba) + g(c,ab) + g(b,ca) - g(b,ac) - g(c,ab) \\ &- 4cg(a,b) - 2ag(c,b) - bg(c,a) - ag(b,c) + cg(b,a) + bg(c,a)]\end{aligned}$$

Collwcting terms and using that $\pi_2 g = g$, we get

$$ag(b,c) - g(ab,c) + g(a,bc) - cg(a,b) = f(a,b,c)$$

Now in the case $n = 4$, we shall prove something stronger, in fact that $\pi_4\delta = \delta\pi_3$. Let $f \in C^3(A,E)$. Then

$$\begin{aligned}\pi_4\delta f f(a,b,c,d) = 1/12\delta[&9f(a,b,c,d) - f(a,b,d,c) + f(c,a,b,d) - f(a,c,b,d) \\ &- f(b,c,d,a) + f(c,d,a,b) - f(d,a,b,c) - f(b,a,c,d) \\ &+ f(a,c,d,b) + f(b,c,a,d) - f(c,a,d,b) + f(a,d,b,c) \\ &+ 3f(d,c,b,a) + f(c,d,b,a) - f(d,b,a,c) + f(d,b,c,a) \\ &+ f(a,d,c,b) - f(d,a,b,c) + f(b,d,a,c) - f(c,b,d,a)]\end{aligned}$$

$$
\begin{aligned}
&= 1/12[9af(b,c,d) - 9f(ab,c,d) + 9f(a,bc,d) - 9f(a,b,cd) + 9df(a,b,c) \\
&\quad - af(b,d,c) + f(ab,d,c) - f(a,bd,c) + f(a,b,dc) - cf(a,b,d) \\
&\quad + cf(a,b,d) - f(ca,b,d) + f(c,ab,d) - f(c,a,bd) + df(c,a,b) \\
&\quad - af(c,b,d) + f(ac,b,d) - f(a,cb,d) + f(a,c,bd) - df(a,c,b) \\
&\quad - bf(c,d,a) + f(bc,d,a) - f(b,cd,a) + f(b,c,da) - af(b,c,d) \\
&\quad + cf(d,a,b) - f(cd,a,b) + f(c,da,b) - f(c,d,ab) + bf(c,d,a) \\
&\quad - df(a,b,c) + f(da,b,c) - f(d,ab,c) + f(d,a,bc) - cf(d,a,b) \\
&\quad - bf(a,c,d) + f(ba,c,d) - f(b,ac,d) + f(b,a,cd) - df(b,a,c) \\
&\quad + af(c,d,b) - f(ac,d,b) + f(a,cd,b) - f(a,c,db) + bf(a,c,d) \\
&\quad + bf(c,a,d) - f(bc,a,d) + f(b,ca,d) - f(b,c,ad) + df(b,c,a) \\
&\quad - af(d,b,c) + f(ad,b,c) - f(a,db,c) + f(a,d,bc) - cf(a,d,b) \\
&\quad + 3df(c,b,a) - 3f(dc,b,a) + 3f(d,cb,a) - 3f(d,c,ba) + 3af(d,c,b) \\
&\quad + cf(d,b,a) - f(cd,b,a) + f(c,db,a) - f(c,d,ba) + af(c,d,b) \\
&\quad - df(b,a,c) + f(db,a,c) - f(d,ba,c) + f(d,b,ac) - cf(d,b,a) \\
&\quad + af(d,b,c) - f(ad,b,c) + f(a,db,c) - f(a,d,bc) + cf(a,d,b) \\
&\quad + af(d,c,b) - f(ad,c,b) + f(a,dc,b) - f(a,d,cb) + bf(a,d,c) \\
&\quad - bf(a,d,c) + f(ba,d,c) - f(b,ad,c) + f(b,a,dc) - cf(b,a,d) \\
&\quad + cf(b,a,d) - f(cb,a,d) + f(c,ba,d) - f(c,b,ad) + df(c,b,a) \\
&\quad + df(c,a,b) - f(dc,a,b) + f(d,ca,b) - f(d,c,ab) + bf(d,c,a) \\
&\quad - bf(d,c,a) + f(bd,c,a) - f(b,dc,a) + f(b,d,ca) - af(b,d,c) \\
&\quad - df(a,c,b) + f(da,c,b) - f(d,ac,b) + f(d,a,cb) - bf(d,a,c) \\
&\quad + bf(d,a,c) - f(bd,a,c) + f(b,da,c) - f(b,d,ac) + cf(b,d,a) \\
&\quad - cf(b,d,a) + f(cb,d,a) - f(c,bd,a) + f(c,b,da) - af(c,b,d)
\end{aligned}
$$

$$= 1/6[4af(b,c,d) - 4f(ab,c,d) + 4f(a,bc,d) - 4f(a,b,cd) + 4df(a,b,c)$$

$$+ 2af(d,c,b) - 2f(d,c,ab) + 2f(d,bc,a) - 2f(cd,b,a) + 2df(c,b,a)$$

$$+ af(c,d,b) - f(c,d,ab) + f(bc,d,a) - f(b,cd,a) + df(b,c,a)$$

$$+ af(d,b,c) - f(d,ab,c) + f(a,d,bc) - f(cd,a,b) + df(c,a,b)$$

$$- af(b,d,c) + f(ab,c,d) - f(a,d,bc) + f(a,cd,b) - df(a,c,b)$$

$$- af(c,b,d) + f(c,ab,d) - f(bc,a,d) + f(b,a,cd) - df(b,a,c)]$$

$$a\pi_3 f(b,c,d) - \pi_3 f(ab,c,d) + \pi_3 f(a,bc,d) - \pi_3 f(a,b,cd) + d\pi_3 f(a,b,c)$$

$$= \delta\pi_3 f(a,b,c,d)$$

This completes the proof.

1.8. PROPOSITION. *$\pi_n(Z^n) \subseteq Z^n$, for $n = 2$, 3, 4.*

PROOF. We know that $\delta\sigma_2 = \pi_3\delta$ and that $\delta\pi_3 = \pi_4\delta$. So suppose that $f \in Z^4$. We must show that $\pi_4 f \in Z^4$. First we note that $\delta f = 0$ means that

$$\delta\sigma_4 f(a,b,c,d,e)$$

$$= a\sigma_4 f(b,c,d)e - \sigma_4 f(ab,c,d)e + \sigma_4 f(a,bc,d)e - \sigma_4 f(a,b,cd)e + \sigma_4 f(a,b,c)de - e\sigma_4 f(a,b,c)d$$

$$= 1/2[af(b,c,d,e) - f(ab,c,d,e) + f(a,bc,d,e) - f(a,b,cd,e) + f(a,b,c,de) - ef(a,b,c,d)$$

$$- ef(d,c,b,a) + f(ed,c,b,a) - f(e,dc,b,a) + f(e,d,cb,a) - f(e,d,c,ba) + af(e,d,c,b)]$$

$$= 1/2[\delta f(a,b,c,d,e) - \delta f(e,d,c,b,a)] = 0$$

Hence it suffices to show that $f \in Z^4$ implies $\tau_4(1-\sigma_4)f = 0$. Now $\sigma_4(1-\sigma_4) = 0$ and $(1-\sigma_4)f \in Z^4$ if f is, so that it is even sufficient to assume that $f \in Z^4$ with $\sigma_4 f = 0$ and show that $\pi_4 f \in Z^4$. We now compute $6\delta\tau_4 f(a,b,c,d,e)$

$$= \tau_4 af(b,c,d,e) - \tau_4 f(ab,c,d,e) + \tau_4 f(a,bc,d,e) - \tau_4 f(a,b,cd,e) + \tau_4 f(a,b,c,de) - \tau_4 ef(a,b,c,d)$$

to which we can freely add

$$3\delta f(a,b,c,d,e) + \delta f(d,e,c,b,a) + \delta f(e,c,b,a,d) - \delta f(e,c,b,d,a) + \delta f(e,c,d,b,a) - \delta f(e,b,c,d,a)$$

$$+ \delta f(b,a,c,d,e) - \delta f(d,e,b,c,a) + \delta f(e,b,c,a,d) - \delta f(e,b,c,d,a) + \delta f(a,c,b,d,e) - \delta f(a,b,d,e,c)$$

$$- \delta f(c,a,b,d,e) - \delta f(a,c,d,b,e) + \delta f(a,d,b,e,c) + \delta f(c,a,d,b,e) + \delta f(b,e,c,d,a) + \delta f(e,c,d,a,b)$$

Since $\delta f = 0$.

$$
\begin{aligned}
&3af(b,c,d,e) - af(b,c.e,d) + af(d,b,c,e) - af(b,d,c.e) - af(c,d,e,b) + af(d,e,b,c) \\
&- af(e,b,c,d) - af(c,b,d,e) + af(b,d,e,c) + af(c,d,b,e) - af(d,b,e,c) + af(b,e,c,d) \\
&- 3f(ab,c,d,e) + f(ab,c.e,d) - f(d,ab,c,e) + f(ab,d,c.e) + f(c,d,e,ab) - f(d,e,ab,c) \\
&+ f(e,ab,c,d) + f(c,ab,d,e) - f(ab,d,e,c) - f(c,d,ab,e) + f(d,ab,e,c) - f(ab,e,c,d) \\
&+ 3f(a,bc,d,e) - f(a,bc.e,d) + f(d,a,bc,e) - f(a,d,bc.e) - f(bc,d,e,a) + f(d,e,a,bc) \\
&- f(e,a,bc,d) - f(bc,a,d,e) + f(a,d,e,bc) + f(bc,d,a,e) - f(d,a,e,bc) + f(a,e,bc,d) \\
&- 3f(a,b,cd,e) + f(a,b.e,cd) - f(cd,a,b,e) + f(a,cd,b.e) + f(b,cd,e,a) - f(cd,e,a,b) \\
&+ f(e,a,b,cd) + f(b,a,cd,e) - f(a,cd,e,b) - f(b,cd,a,e) + f(cd,a,e,b) - f(a,e,b,cd) \\
&+ 3f(a,b,c,de) - f(a,b.de,c) + f(c,a,b,de) - f(a,c,b.de) - f(b,c,de,a) + f(c,de,a,b) \\
&- f(de,a,b,c) - f(b,a,c,de) + f(a,c,de,b) + f(b,c,a,de) - f(c,a,de,b) + f(a,de,b,c) \\
&- 3ef(a,b,c,d) + ef(a,b.d,c) - ef(c,a,b,d) + ef(a,c,b.d) + ef(b,c,d,a) - ef(c,d,a,b) \\
&+ ef(d,a,b,c) + ef(b,a,c,d) - ef(a,c,d,b) - ef(b,c,a,d) + ef(c,a,d,b) - ef(a,d,b,c) \\
&- 3af(b,c,d,e) + 3f(ab,c,d,e) - 3f(a,bc,d,e) + 3f(a,b,cd,e) - 3f(a,b,c,de) + 3dff(a,b,c,d) \\
&+ df(e,c,b,a) - f(de,c,b,a) + f(d,ec,b,a) - f(d,e,cb,a) + f(d,e,c,ba) - af(d,e,c,b) \\
&+ ef(c,b,a,d) - f(ec,b,a,d) + f(e,cb,a,d) - f(e,c,ba,d) + f(e,c,b,ad) - df(e,c,b,a) \\
&- ef(c,d,b,a) + f(ec,d,b,a) - f(e,cd,b,a) + f(e,c,db,a) - f(e,c,d,ba) + af(e,c,d,b) \\
&+ ef(c,d,b,a) - f(ec,d,b,a) + f(e,cd,b,a) - f(e,c,db,a) + f(e,c,d,ba) - af(e,c,d,b) \\
&+ af(c,d,e,b) - f(ac,d,e,b) + f(a,cd,e,b) - f(a,c,de,b) + f(a,c,d,eb) - bf(a,c,d,e) \\
&+ bf(a,c,d,e) - f(ba,c,d,e) + f(b,ac,d,e) - f(b,a,cd,e) + f(b,a,c,de) - ef(b,a,c,d) \\
&+ df(e,b,c,a) - f(de,b,c,a) + f(d,eb,c,a) - f(d,e,bc,a) + f(d,e,b,ca) - af(d,e,b,c) \\
&+ ef(b,c,a,d) - f(eb,c,a,d) + f(e,bc,a,d) - f(e,b,ca,d) + f(e,b,c,ad) - df(e,b,c,a) \\
&+ af(d,b,e,c) - f(ad,b,e,c) + f(a,db,e,c) - f(a,d,be,c) + f(a,d,b,ec) - cf(a,d,b,e) \\
&- ef(b,c,d,a) + f(eb,c,d,a) - f(e,bc,d,a) + f(e,b,cd,a) - f(e,b,c,da) + af(e,b,c,d) \\
&+ af(c,b,d,e) - f(ac,b,d,e) + f(a,cb,d,e) - f(a,c,bd,e) + f(a,c,b,de) - ef(a,c,b,d) \\
&- af(b,d,e,c) + f(ab,d,e,c) - f(a,bd,e,c) + f(a,b,de,c) - f(a,b,d,ec) + cf(a,b,d,e) \\
&- cf(a,b,d,e) + f(ca,b,d,e) - f(c,ab,d,e) + f(c,a,bd,e) - f(c,a,b,de) + ef(c,a,b,d)
\end{aligned}
$$

$$+cf(a,d,b,e) - f(ca,d,b,e) + f(c,ad,b,e) - f(c,a,db,e) + f(c,a,d,be) - ef(c,a,d,b)$$
$$+bf(e,c,d,a) - f(be,c,d,a) + f(b,ec,d,a) - f(b,e,cd,a) + f(b,e,c,da) - af(b,e,c,d)$$
$$+ef(c,d,a,b) - f(ec,d,a,b) + f(e,cd,a,b) - f(e,c,da,b) + f(e,c,d,ab) - bf(e,c,d,a)$$

Now, collecting terms and using that $\sigma_4 f = 0$, this reduces to

$$-f(ab,c,d,e) + f(e,ab,c,d) - f(d,e,ab,c) + f(c,d,e,ab)$$
$$-f(ab,e,c,d) + f(d,ab,e,c) - f(c,d,ab,e) + f(e,c,d,ab)$$
$$-f(bc,a,d,e) + f(e,bc,a,d) - f(d,e,bc,a) + f(a,d,e,bc)$$
$$+f(bc,d,a,e) - f(e,bc,d,a) + f(a,e,bc,d) - f(d,a,e,bc)$$
$$-f(bc,d,e,a) + f(a,bc,d,e) - f(e,a,bc,d) + f(d,e,a,bc)$$
$$-f(cd,e,b,a) + f(a,cd,e,b) - f(b,a,cd,e) + f(e,b,a,cd)$$
$$-f(cd,e,a,b) + f(b,cd,e,a) - f(a,b,cd,e) + f(e,a,b,cd)$$
$$+f(cd,e,a,b) - f(b,cd,e,a) + f(a,b,cd,e) - f(e,a,b,cd)$$
$$-f(cd,e,a,b) + f(b,cd,e,a) - f(a,b,cd,e) + f(e,a,b,cd)$$
$$-f(de,a,b,c) + f(c,de,a,b) - f(b,c,de,a) + f(a,b,c,de)$$
$$-f(de,b,c,a) + f(a,de,b,c) - f(c,a,de,b) + f(b,c,a,de)$$
$$-f(ec,b,a,d) + f(d,ec,b,a) - f(a,d,ec,b) + f(b,a,d,ec)$$
$$-f(ad,b,e,c) + f(c,ad,b,e) - f(e,c,ad,b) + f(b,e,c,ad)$$
$$-f(ac,d,e,b) + f(b,ac,d,e) - f(e,b,ac,d) + f(d,e,b,ac)$$
$$-f(be,c,a,d) + f(d,be,c,a) - f(a,d,be,c) + f(c,a,d,be)$$

Now define $g \in C^4$ by

$$g(a,b,c,d) = f(a,b,c,d) - f(d,a,b,c) + f(c,d,a,b) - f(b,c,d,a)$$

Then it is clear that $g(a,b,c,d) = -g(d,c,b,a)$ and that

$$g(a,b,c,d) = -g(d,a,b,c) = g(c,d,a,b) = -g(b,c,d,a)$$

from which it follows that $g(a,b,c,d) = -g(b,c,d,a)$. Then have that

$$\begin{aligned}\delta\tau_4 f(a,b,c,d) = &-g(ab,c,d,e) - g(ab,e,c,d) - g(bc,a,d,e) + g(bc,d,a,e)\\ &- g(bc,d,e,a) - g(cd,e,b,a) + g(cd,a,e,b) - g(cd,e,a,b)\\ &- g(de,a,b,c) - g(de,b,c,a) - g(ec,b,d,a) - g(ad,b,e,c)\\ &- g(ac,d,e,b) - g(be,c,a,d)\end{aligned}$$

At this time we need

1.9. Lemma.

$$g(ab,c,d,e) + g(ea,b,c,d) + g(de,a,b,c) + g(cd,e,a,b) + g(bc,d,e,a) = 0$$

Proof.

$$\begin{aligned}0 &= -\delta f(a,b,c,d,e) - \delta f(e,a,b,c,d) - \delta f(d,e,a,b,c) - \delta f(c,d,e,a,b) - \delta f(b,c,d,e,a)\\ &= -af(b,c,d,e) + f(ab,c,d,e) - f(a,bc,d,e) + f(a,b,cd,e) - f(a,b,c,de) + ef(a,b,c,d)\\ &- ef(a,b,c,d) + f(ea,b,c,d) - f(e,ab,c,d) + f(e,a,bc,d) - f(e,a,b,cd) + df(e,a,b,c)\\ &- df(e,a,b,c) + f(de,a,b,c) - f(d,ea,b,c) + f(d,e,ab,c) - f(d,e,a,bc) + cf(d,e,a,b)\\ &- cf(d,e,a,b) + f(cd,e,a,b) - f(c,de,a,b) + f(c,d,ea,b) - f(c,d,e,ab) + bf(c,d,e,a)\\ &- bf(c,d,e,a) + f(bc,d,e,a) - f(b,cd,e,a) + f(b,c,de,a) - f(b,c,d,ea) + af(b,c,d,e)\\ &= g(ab,c,d,e) + g(ea,b,c,d) + g(de,a,b,c) + g(cd,e,a,b) + g(bc,d,e,a) = 0\end{aligned}$$

from which the lemma follows.

Using the lemma and the fact that $g(a,b,c,d) = g(a,d,c,b)$, we get

$$\begin{aligned}0 = &- g(ab,c,e,d) - g(bc,e,d,a) - g(ce,d,ab) - g(ed,a,b,c) - g(da,b,c.e)\\ &- g(bd,a,c,e) - g(da,c,e,b) - g(ac,e,bd) - g(ce,b,d,a) - g(eb,d,a.c)\\ &+ g(ba,c,e,d) + g(ac,e,d,b) + g(ce,d,ba) + g(ed,b,a,c) + g(db,a,c.e)\\ &- g(de,b,a,c) - g(eb,a,c,d) - g(ba,c,de) - g(ac,d,e,b) - g(cd,e,b.a)\\ &+ g(be,a,c,d) + g(ea,c,d,b) + g(ac,d,be) + g(cd,b,e,a) + g(db,e,a.c)\\ &- g(ae,c,d,b) - g(ec,d,b,a) - g(cd,b,ae) - g(db,a,e,c) - g(ba,e,c.d)\\ &+ g(bd,e,a,c) + g(de,a,c,b) + g(ea,c,bd) + g(ac,b,d,e) + g(cb,d,e.a)\\ &- g(ea,c,b,d) - g(ac,b,d,e) - g(cb,d,ea) - g(bd,e,a,c) - g(de,a,c.b)\end{aligned}$$

$$
\begin{aligned}
&= -g(ab,c,d,e) - g(ab,e,c,d) - g(bc,a,d,e) + g(bc,d,e,a)\\
&\quad - g(bc,d,e,a) - g(cd,e,b,a) + g(cd,a,e,b) - g(cd,e,a,b)\\
&\quad - g(de,a,b,c) - g(de,b,c,a) - g(ec,b,a,d) - g(ad,b,e,c)\\
&\quad - g(ac,d,e,b) - g(be,c,a,d)\\
&\quad = \delta\tau_4 f(a,b,c,d,e)
\end{aligned}
$$

which completes the proof.

1.10. Proposition. *$\pi_n(B^n) \subseteq B^n$ for $n = 2, 3, 4$.*

Proof. If $n = 2$, suppose $f \in B^2$. Then

$$f(a,b) = ag(b) - g(ab) + bg(a) = f(b,a)$$

so that $0 = \pi_2 f \in B^2$. If $n = 3$ and $f \in B^3$, then $\pi_3 f(a,b,c)$

$$= 1/6[4f(a,b,c) + 2f(c,b,a) + f(c,a,b) + f(b,c,a) - f(b,a,c) - f(a,c,b)$$

$$
\begin{aligned}
&= 1/6[4ag(b,c) - 4g(ab,c) + 4g(a,bc) - 4cg(a,b)\\
&\quad + 2cg(b,a) - 2g(cb,a) + 2g(c,ba) - 2ag(c,b)\\
&\quad + cg(a,b) - g(ca,b) + g(c,ab) - bg(c,a)\\
&\quad + bg(c,a) - g(bc,a) + g(b,ca) - ag(b,c)\\
&\quad - bg(a,c,) + g(ba,c) - g(b,ac) + cg(b,a) - ag(c,b,) + g(ac,b) - g(a,cb) + bg(a,c)]\\
&= 1/2[ag(b,c) - g(ab,c) + g(a,bc) - cg(a,b)\\
&\quad - ag(c,b,) + g(ac,b) - g(a,cb) + bg(a,c)]\\
&= \pi_2 ag(b,c) - \pi_2 g(ab,c) + \pi_2 g(a,bc) - \pi_2 cg(a,b)
\end{aligned}
$$

For $n = 4$, we already know that $\delta\pi_3 = \pi_4\delta$

1.11. Definition. *$Z^n_c = Z^n \cap C^n_c$, $Z^n_s = Z^n \cap C^n_s$, $B^n_c = B^n \cap C^n_c$, $B^n_s = B^n \cap C^n_s$, $H^n_c = Z^n_c/B^n_c$, $H^n_s = Z^n_s/B^n_s$, for $n = 1, 2, 3, 4$, where we let $C^1_c = C^1$ and $C^1_s = 0$.*

1.12. Theorem. *Let $0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow R \longrightarrow 0$ be exact. Then the following sequences are exact and the sequence $(*)$ on page 7 is the direct sum of them:*

$$
\begin{aligned}
0 &\longrightarrow H^1_c(R,E) \longrightarrow H^1_c(A,E) \longrightarrow H^1_c(A,\mathfrak{a},E)\\
&\longrightarrow H^2_c(R,E) \longrightarrow H^2_c(A,E) \longrightarrow H^2_c(A,\mathfrak{a},E)\\
&\longrightarrow H^3_c(R,E) \longrightarrow H^3_c(A,E) \longrightarrow H^3_c(A,\mathfrak{a},E)\\
&\longrightarrow H^4_c(R,E) \longrightarrow H^4_c(A,E) \longrightarrow H^4_c(A,\mathfrak{a},E)
\end{aligned}
$$

$$\begin{aligned} 0 \longrightarrow H^1_s(R,E) \longrightarrow H^1_s(A,E) \longrightarrow H^1_s(A,\mathfrak{a},E) \\ \longrightarrow H^2_s(R,E) \longrightarrow H^2_s(A,E) \longrightarrow H^2_s(A,\mathfrak{a},E) \\ \longrightarrow H^3_s(R,E) \longrightarrow H^3_s(A,E) \longrightarrow H^3_s(A,\mathfrak{a},E) \\ \longrightarrow H^4_s(R,E) \longrightarrow H^4_s(A,E) \longrightarrow H^4_s(A,\mathfrak{a},E) \end{aligned}$$

Proof. First we show that $H^n \cong H^n_c \oplus h^n_s$. $\pi_n(Z^n) \subseteq Z^n \cap Z^n_s$ and $(1-\pi_n)(Z^n) \subseteq Z^n \cap C^n_c$ give $Z^n = Z^n_c \oplus Z^n_s$ and similarly we see that $B^n = B^n_c \oplus B^n_s$, so that

$$H^n = \frac{Z_n}{B^n} = \frac{Z^n_c \oplus Z^n_s}{B^n_c \oplus B^n_s} \cong \frac{C^n_c}{B^n_c} \oplus \frac{Z^n_s}{B^n_s} = H^n_c \oplus H^n_s$$

To finish the proof we must show that the maps used in the exact sequence $(*)$ on 7 map commutative (skew commutative) to commutative (skew commutative) cocycles. In the case of the maps induced by the injection of $\mathfrak{a}$ into A and the projection of A onto R, it is clear. The dimension raising map $H^{n-1}(A,\mathfrak{a},E) \longrightarrow H^n(R,E)$ is given as follows: Let $f : \mathfrak{a}^{(n-1)} \longrightarrow E$ with $\delta f = 0$. Choose $\overline{f} \in C^n(A,E)$ which extends f. If f is commutative (skew commutative), this may be expressed by saying that f vanishes on a certain subspace $B \subseteq \mathfrak{a}^{(n)}$. We can write $B = C \cap \mathfrak{a}(n)$ where C is the subspace of $A^{(n)}$ on which all commutative (skew commutative) maps vanish. Then we can assume $\overline{f}$ vanishes on C also since are merely linear maps; i.e. $\overline{f} \in C^n_c(A,E)$ $(C^n_s(A,E))$. Then $\delta f(a_1, \dots a_{n+1}) = 0$ if any $a_i \in \mathfrak{a}$ since $\overline{f}|_{\mathfrak{a}^{(n)}} \in Z^n(A,\mathfrak{a},E)$ so that $\overline{f}$ is a well defined map in $C^{n+1}(R,E)$. Also $\delta\delta\overline{f} = 0$ so $\delta\overline{f} \in Z^{n+1(R,E)}$. Since δ maps commutative (skew commutative) maps to commutative (skew commutative) ones the theorem is proved.

2. Chapter II

This chapter is devoted to an examination of some of these groups. It is known that $H^1(A,\mathfrak{a},E)$ is isomorphic to all the groups $\mathrm{Hom}_A(\mathfrak{a},E)$, $\mathrm{Hom}(\mathfrak{a}/\mathfrak{a}^2,E)$, $\mathrm{Hom}_R(\mathfrak{a}/\mathfrak{a}^2,E)$ where since $\mathfrak{a}\cdot\mathfrak{a}/\mathfrak{a}^2 = 0$, $\mathfrak{a}/\mathfrak{a}^2$ becomes an r-module.

2.1. Theorem. *There are exact sequences*

(*i*) $$0 \longrightarrow H^2_c(A,\mathfrak{a},E) \longrightarrow \mathrm{Ext}^1_A(\mathfrak{a},E) \longrightarrow \mathrm{Hom}_A(\mathfrak{a}\otimes_A\mathfrak{a},E)$$

(*ii*) $$0 \longrightarrow H^2_s(A,\mathfrak{a},E) \longrightarrow \mathrm{Hom}_A(\mathfrak{a},H^1(A,E)) \longrightarrow \mathrm{Hom}_A(\mathfrak{a}\otimes_A\mathfrak{a},E)$$

(*iii*) $$0 \longrightarrow H^2(A,\mathfrak{a},E) \longrightarrow \mathrm{Ext}^1_{A\otimes A}(\mathfrak{a},E) \longrightarrow \mathrm{Hom}_A(\mathfrak{a}\otimes_A\mathfrak{a},E)$$

Proof. Let $f \in Z_c^2(A, \mathfrak{a}, E)$. We define and extension

$$0 \longrightarrow E \xrightarrow{\theta} B \xrightarrow{\phi} \mathfrak{a} \longrightarrow 0$$

as follows: Let B be the additive group $\mathfrak{a} \oplus E$ and define $a(\alpha, e) = (a\alpha, ae + f(a, \alpha))$, for all $a \in A$, $\alpha \in \mathfrak{a}$, $e \in E$. To see that this defines an associative operation of A on B, we compute $(ab)(\alpha, e)$ and $a(b(\alpha, e))$.

$$(ab)(\alpha, e) = (ab\alpha, abe + f(ab, \alpha))$$

$$a(b(\alpha, e)) = a(b\alpha, be + f(b, \alpha)) = (ab\alpha, abe + af(b, \alpha) + f(a, b\alpha))$$

so that we must show that $af(b, \alpha) - f(ab, \alpha) + f(a, b\alpha) = 0$, but this just says that $\delta f(a, b, \alpha) = 0$[6]. We let $\theta : E \longrightarrow B$ by $\theta(e) = (0, e)$ and $\phi : B \longrightarrow A$ by $\phi(\alpha, e) = \alpha$. $\theta(ae) = (0, ae) = a(0, e) = a\theta(e)$ [and] $\phi(a, (\alpha, e)) = \phi(a\alpha, ae + f(a, \alpha)) = a\alpha = a\phi(\alpha, e)$ so that $(\theta, B, \phi) \in \mathrm{Ext}^1(\mathfrak{a}, E)$. Suppose that (θ, B, ϕ) is equivalent to the split extension; i.e. that have a map $t : \longrightarrow \mathfrak{a} \oplus E$ such that the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \xrightarrow{\theta} & B & \xrightarrow{\phi} & \mathfrak{a} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle t} & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & \mathfrak{a} \oplus E & \longrightarrow & \mathfrak{a} & \longrightarrow & 0
\end{array}$$

where unlabeled maps are the obvious ones. If $t(\alpha, e) = (\alpha', e')$, then by the commutativity of the diagram $\alpha = \alpha'$. Also, $t(0, e) = (0, e)$ by the commutativity of the diagram which gives $t(\alpha, e) = t(\alpha, 0) + t(0, e) = (\alpha, e' - e)$ so that $e' - e$ depends only on α. Let $g(\alpha) = e' - e$ and we get that $t(\alpha, e) = (\alpha, e + g(\alpha))$. Than

$$t(a(\alpha, e)) = t(a\alpha, ae + f(a, \alpha)) = (a\alpha, ae + f(a, \alpha) + g(\alpha))$$

while

$$at(\alpha, e) = a(\alpha, e + g(\alpha)) = (a\alpha, ae + ag(\alpha))$$

and setting these equal,

$$f(a, \alpha) = ag(\alpha) - g(a\alpha) = \delta g(a, \alpha)$$

Now suppose there is a map $g : \mathfrak{a} \longrightarrow E$ with $f = \delta g$. Map $t : B \longrightarrow \mathfrak{a} \oplus E$ by $t(\alpha, e) = e + g(\alpha)$. Then

$$\begin{aligned} t(a(\alpha, e)) &= t(a\alpha, ae + f(a, \alpha)) = t(a\alpha, ae + ag(\alpha) + g(a\alpha)) \\ &= (a\alpha, ag(\alpha) - g(a\alpha) + g(a\alpha)) = a(\alpha, e + g(\alpha)) = at(\alpha, e) \end{aligned}$$

[6] Note that $\mathfrak{a}E = 0$ so that the fourth term of $\delta f(a, b, \alpha)$ is 0

Hence we get a monomorphism of $H_c^2(A, \mathfrak{a}, E)$ into $\mathrm{Ext}_A^1(\mathfrak{a}, E)$. Now let

$$0 \longrightarrow E \xrightarrow{\phi} B \xrightarrow{\theta} \mathfrak{a} \longrightarrow 0$$

be an extension of E by $\mathfrak{a}$. Choose a linear map $g : \mathfrak{a} \longrightarrow B$ with θg the identity map. Then for $a \in A$, $\alpha \in \mathfrak{a}$, $f(a, \alpha) = ag(\alpha) - g(a\alpha) \in E$. We define $\overline{f} : \mathfrak{a} \oplus \mathfrak{a} \longrightarrow E$ by $\overline{f}(\alpha, \beta) = f(\alpha, \beta) - f(\beta, \alpha) = \alpha g(\beta) - \beta g(\alpha)$.

$$\begin{aligned}\overline{f}(a\alpha, \beta) - a\overline{f}(\alpha, \beta) &= a\alpha g(\beta) - \beta g(a\alpha) - a\alpha g(\beta) + \beta a g(\alpha)\\ &= \underline{(}ag(\alpha) - g(a\alpha)) = 0\end{aligned}$$

since $\mathfrak{a} \cdot E = 0$, and

$$\begin{aligned}\overline{f}(a\alpha, \beta) - \overline{f}(\alpha, a\beta) &= a\alpha g(\beta) - g(a\alpha\beta) - \alpha g(a\beta) + g(a\alpha\beta)\\ &= \alpha(ag(\beta)) - g(a\beta) = 0\end{aligned}$$

so that $\overline{f} \in \mathrm{Hom}(\mathfrak{a} \otimes \mathfrak{a}, E)$. If g' is another choice for g, then $g'(\alpha) - g(\alpha) \in E$ for all $\alpha \in \mathfrak{a}$, so that if $f' = \delta g'$ and $\overline{f}'(\alpha, \beta) - f'(\beta, \alpha)$, then

$$\begin{aligned}\overline{f}(\alpha, \beta) - \overline{f}'(\alpha, \beta) &= f(\alpha, \beta) - f(\beta, \alpha) - f'(\alpha, \beta) + f'(\beta, \alpha)\\ &= \alpha g(\beta) - \beta g(\alpha) - \alpha g'(\beta) + \beta g'(\alpha)\\ &= \alpha(g(\beta) - g'(\beta)) - \beta(g(\alpha) - g'(\alpha)) = 0\end{aligned}$$

since $\mathfrak{a} \cdot E = 0$. $f \mapsto 0$ under this map if and only if $f(\alpha, \beta) = f(\beta, \alpha)$ for all α, $\beta \in \mathfrak{a}$. Clearly if f is in the image of $H_c^2(A, \mathfrak{a}, E)$ it satisfies this condition. Conversely, suppose f satisfies this condition. Then by choosing a basis for $\mathfrak{a}$ and extending it to a basis for A, we can find $f^* : \mathfrak{a}^{(2)} \longrightarrow E$ which extends f and continues to satisfy $f^*(a, b) = f^*(b, a)$ for all a, $b \in \mathfrak{a}^{(2)}$. Now if $\alpha \in \mathfrak{a}$ and a, $b \in A$, we have

$$\begin{aligned}f^*(\alpha a, b) - f^*(\alpha, ab) = bf^*(\mathfrak{a}, a) &= f^*(b, \alpha a) - f^*(ba, \alpha) + bf^*(a, \alpha)\\ &= f(b, \alpha a) - f(ba, \alpha) + bf(a, \alpha) = 0\end{aligned}$$

and

$$\begin{aligned}af^*(\alpha, b) - f^*(a\alpha, b) + f^*(a, \alpha b) - bf^*(a, \alpha) &= af(b, \alpha) - f(b, a\alpha) + f(a, \alpha b) - bf(a, \alpha)\\ = abg(\alpha) - ag(b\alpha) - bg(a\alpha) + g(ba\alpha) &+ ag(\alpha b) - g(a\alpha b) - bag(\alpha) + bg(a\alpha) = 0\end{aligned}$$

and we see from this that $f^* \in Z_c^2(A, \mathfrak{a}, E)$. Clearly f^* induces that given extension, which proves that (i) is exact.

Now let $f \in Z_s^2(A, \mathfrak{a}, E)$, $\alpha \in \mathfrak{a}$, and a, $b \in A$. Then

$$f(\alpha a, b) - f(\alpha, ab) + bf(\alpha, a) = 0$$

$$-af(\alpha,b)+f(b\alpha,a)-f(b,\alpha a)+af(b,\alpha)=0$$
$$-f(b,\alpha)+f(ab,\alpha)-f(a,b\alpha)=0$$

Adding as using that $f(a,b)=-f(b,a)$, we get after divison by 2, $f(\alpha a,b)-f(\alpha,ab)+f(a,b\alpha)=0$. Subtracting this from the first equation above, we get $f(\alpha b,a)=bf(\alpha,a)$. From this we see that $f(\alpha,ab)=af(\alpha,b)+bf(\alpha,a)$, so that each $\alpha\in\mathfrak{a}$, $f(\alpha,\cdot)$ is a derivation of A to E. Hence if we make $H^1(A,E)$ into an A-module by $(ag)(b)=ag(b)$ for all a, $b\in A$ and $g\in H^1(A,E)$, we see that $f\in \mathrm{Hom}_A(\mathfrak{a},H^1(A,E))$, and that $f(\alpha,\beta)=-f(\beta,\alpha)$, for all α, $\beta\in\mathfrak{a}$. Conversely, let $f\in\mathrm{Hom}_A(\mathfrak{a},H^1(A,E))$ satisfy $f(\alpha,\beta)=-f(\beta,al)$ for all α, $\beta\in\mathfrak{a}$. Now we think of f as a map from $\mathfrak{a}\otimes A\longrightarrow E$. Extend to a map $f^*:A^{(2)}\longrightarrow E$ which continues to satisfy $f^*(a,b)=-f^*(b,a)$.[7] Then

$$\begin{aligned}f^*(\alpha a,b)-f^*(\alpha,ab)+bf^*(\alpha,a)&=f(\alpha a,b)-f(\alpha,ab)+bf(\alpha,a)\\&=af(\alpha,b)-af(\alpha,b)-bf(\alpha,a)+bf(\alpha,a)=0\end{aligned}$$

and

$$af^*(\alpha,b)-f^*(a\alpha,b)+f^*(a,\alpha b)-bf^*(a,\alpha)$$
$$=-af(b,\alpha)+af(b,\alpha)+bf(a,\alpha)-bf(a,\alpha)=0.$$

Hence the image of $H^2_s(A,\mathfrak{a},E)$ is

$$\{f\in\mathrm{Hom}_A(\mathfrak{a},H^1(A,E))\mid f(\alpha,\beta)=-f(\beta,\alpha)\text{ for all }\alpha,\ \beta\in\mathfrak{a}\}$$

Now map $\mathrm{Hom}_A(\mathfrak{a},H^1(A,E))$ into $\mathrm{Hom}(\mathfrak{a}\otimes\mathfrak{a},E)$ as follows: Let $f\in\mathrm{Hom}(\mathfrak{a},H^1(A,E))$, then $f:\mathfrak{a}\otimes A\longrightarrow E$. If α, $\beta\in\mathfrak{a}$, let $f'(\alpha,\beta)=f(\alpha,\beta)+f(\beta,\alpha)$. Then if a, $b\in A$,

$$f'(a\alpha,\beta)=f(a\alpha,\beta)+f(\beta,a\alpha)=af(\alpha,\beta)+af(\beta,\alpha)+\alpha f(\beta,a)=a(f(\alpha,\beta)+f(\beta,\alpha))=af'(\alpha,\beta)$$

Since $f'(\alpha,\beta)=f'(\beta,\alpha)$ we see that $f'(\alpha,a\beta)=f'(a\beta,\alpha)=af'(\beta,\alpha)$, so that f can be thought of as a map in $\mathrm{Hom}(\mathfrak{a}\otimes_A\mathfrak{a},E)$. The kernel of this map is clearly the image of $H^2_s(A,\mathfrak{a},E)$ so that the sequence (ii) is exact.

$\mathrm{Ext}^1_{A\otimes A}(A,E)$ is the group of extensions

$$0\longrightarrow E\xrightarrow{\theta}B\xrightarrow{\phi}\mathfrak{a}\longrightarrow 0$$

where B is a two-sided A-module with possibly different operations on each side, and θ and ϕ are two-sided A-homomorphisms. $\mathrm{Ext}^1_A(\mathfrak{a},E)$ can be thought of as that subgroup consisting in which the operations are the same on each side. Suppose $f\in Z^2(A,\mathfrak{a},E)$. Let B be the additive group $\mathfrak{a}\oplus E$ and define $a(\alpha,e)=(a\alpha,ae+fa,\alpha)$ and $(\alpha,e)a=(\alpha a,ea+f(\alpha,a))$. The verifications that $(ab)(\alpha,e)=a(b(\alpha,e))$ and that $(\alpha,e)(ab)=((\alpha,e)a)b$ are the same as in the proof of the exactness of (i).

$$\begin{aligned}(a(\alpha,e))b&=(a\alpha,ae+f(a\alpha))b=(ab\alpha,abe+bf(a,\alpha)+f(a\alpha,b))\\&=(ab\alpha,abe+f(a,\alpha b)+af(\alpha,b))=a(b\mathfrak{a},be+f(\alpha,b))=a((\alpha,e)b)\end{aligned}$$

[7] The original here is $f^*:\mathfrak{a}^{(2)}\longrightarrow E$, but this makes no sense.

since $\delta f = 0$. Suppose that f induces the split extension so that we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & B & \longrightarrow & \mathfrak{a} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle t} & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & \mathfrak{a} \oplus E & \longrightarrow & \mathfrak{a} & \longrightarrow & 0
\end{array}$$

As before we see that $t(\alpha, e) = (\alpha, e + g(\alpha))$ and that $f = \delta g$. Conversely, if $f = \delta g$. then f is commutative and makes A operate the same on sides of B and then just as before f induces the split extension.

Suppose

$$0 \longrightarrow E \xrightarrow{\theta} B \xrightarrow{\phi} \mathfrak{a} \longrightarrow 0$$

is a two-sided extension of E by $\mathfrak{a}$. Let $g : \mathfrak{a} \longrightarrow B$ a linear map such that ϕg is the identity. Let $f_1(a, \alpha) = ag(\alpha) - g(a\alpha)$ and $f_2(\alpha, a) = g(\alpha)a - g(a\alpha)$ for all $a \in A$ and $\alpha \in \mathfrak{a}$. Then

$$af_1(b, \alpha) - f_1(ab, \alpha) + f_1(a, b\alpha) = abg(\alpha) - ag(b\alpha) - abg(\alpha) + g(ab\alpha) + ag(b\alpha) - g(ab\alpha) = 0$$

while

$$f_2(\alpha a, b) - f_2(\alpha, ab) + f_2(\alpha, a)b = g(\alpha a)b - g(\alpha ab) - g(\alpha)ab + g(\alpha ab) + g(\alpha)ab - g(\alpha a)b = 0$$

and

$$\begin{aligned}
&af_2(\alpha, b) - f_2(a\alpha, b) + f_1(a, \alpha b) - f_1(a, \alpha)b \\
&= ag(\alpha)b - ag(\alpha b) - g(a\alpha)b + g(a\alpha b) + ag(\alpha b) - g(a\alpha b) - ag(\alpha)b + g(a\alpha)b = 0
\end{aligned}$$

We associate the extension with the pair (f_1, f_2). Suppose g' is another choice for g and f_1', f_2' are the corresponding maps. Let $h = g - g'$. Then $(f_1 - f_1')(a, \alpha) = ah(\alpha) - h(a\alpha)$ and $(f_2 - f_2')(\alpha, a) = h(\alpha)a - h(a\alpha) = ah(\alpha) - h(a\alpha)$ since $h(\alpha) \in E$, and A operates the same on both sides of E. In accordance with this we will say $(f_1, f_2) \sim (f_1', f_2')$ if and only if there is an $h \in C^1(A, \mathfrak{a}, E)$ with $(f_1 - f_1')(a, \alpha) = \delta h(a, \alpha)$ and $(f_2 - f_2')(\alpha, a) = \delta h(\alpha, a)$. The map from $Z^2(A, \mathfrak{a}, E) \longrightarrow \mathrm{Ext}^1_{A \otimes A}(\mathfrak{a}, E)$ associates with f the pair $(f_1.f_2)$ where $f_1(a, \alpha) = f(a, \alpha)$ and $f_2(\alpha, a) = f(\alpha, a)$ for all $a \in A$ and $\alpha \in \mathfrak{a}$. From the previous discussion we see that the kernel of this map is exactly $B^2(A, \mathfrak{a}, E)$. Hence we have a monomorphism from $H^2(A, \mathfrak{a}, E)$ to $\mathrm{Ext}^1_{A \otimes A}(\mathfrak{a}, E)$. If (f_1, f_2) is in the image, then for all α, $\beta \in \mathfrak{a}$, $f_1(\alpha, \beta) = f_2(\alpha, \beta)$. Conversely, suppose this is satisfied. Then by the usual basis argument we can find a map $f \in C^2(A, \mathfrak{a}, E)$ with $f|_{A \otimes \mathfrak{a}} = f_1$ and $f|_{\mathfrak{a} \otimes A} = f_2$. From the relations satisfied by f_1 and f_2 we infer that $\delta f = 0$ and hence f is in the image of $H^2(A, \mathfrak{a}, E)$. Now given a pair (f_1, f_2) corresponding to an extension, let $f'(\alpha, \beta) = f_1(\alpha, \beta) - f_2(\alpha, \beta)$ for all α, $\beta \in \mathfrak{a}$. We then see that $f' = 0$ if and only if

(f_1, f_2) is in the image of $H^2(A, \mathfrak{a}, E)$, so that $f' = 0$.

$$\begin{aligned} f'(a\alpha, \beta) - af'(\mathfrak{a}, \beta) &= f_1(a\alpha, \beta) - f_2(a\alpha, \beta) - af_1(\alpha, \beta) + af_2(\alpha, \beta) \\ &= a\alpha g(\beta) - g(a\alpha\beta) - g(a\alpha)\beta + g(a\alpha\beta) - a\alpha g(\beta) + ag(\alpha\beta) - g(a\alpha)\beta \\ &= (ag(\alpha)\beta - g(a\alpha))\beta = 0 \end{aligned}$$

since $E \cdot \mathfrak{a} = 0$. Similarly, $f'(\alpha, a\beta) = af'(\alpha, \beta) = f'(\alpha a, \beta)$ so that $f' \in \mathrm{Hom}_A(\mathfrak{a} \otimes \mathfrak{a}, E)$. This completes the proof.

In the next part of this chapter we apply some of these results to the theory of local noetherian algebras. Let A now denote a commutative radical algebra with ascending chain condition and let A^* denote with an identity adjoined; i.e. A^* is that additive group $A \oplus k$ made into an algebra by defining $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$ for all a, $b \in A$ and $\lambda, \mu \in k$. Saying that A is a radical algebra is the same thing as saying that A^* is a local algebra with maximal ideal A.

2.2. Proposition. *$H_c^n(A, E) \cong H^n(A^*, E)$ and $H_s^n(A, E) \cong H_s^n(A^*, E)$ for $n = 2$, 3, 4.*

Proof. In [4], it is shown that $H^n(A, E) \cong H^n(A^*, E)$, for all $n \geq 1$, with the isomorphism by restriction. If we consider the sequence

$$\begin{aligned} 0 \longrightarrow H^1(A, E) &\xrightarrow{j_1} H^1(A^*, E) \xrightarrow{i_1} H^1(A^*, A, E) \\ &\xrightarrow{d_2} H^2(A, E) \xrightarrow{j_2} H^2(A^*, E) \xrightarrow{i_2} H^2(A^*, A, E) \\ &\xrightarrow{d_3} H^3(A, E) \xrightarrow{j_3} H^3(A^*, E) \xrightarrow{i_3} H^3(A^*, A, E) \\ &\xrightarrow{d_4} H^4(A, E) \xrightarrow{j_4} H^4(A^*, E) \xrightarrow{i_4} H^4(A^*, A, E) \end{aligned}$$

we see that $i_1 = d_2 = i_2 = d_3 = i_3 = d_4 = i_4 = 0$ since j_2, j_3 and j_4 are isomorphisms. Hence the commutative and skew commutative parts of these maps are 0 and we have $0 \longrightarrow H_c^n(A, E) \longrightarrow H_c^n(A^*, E) \longrightarrow 0$ and $0 \longrightarrow H_c^n(A, E) \longrightarrow H_s^n(A^*, E) \longrightarrow 0$ are exact for $n = 2$, 3, 4. k becomes an A-module if we let $a\lambda = 0$ for all $a \in A$, $\lambda \in k$.

2.3. Proposition. *$H^n(A, k) \cong \mathrm{Tor}_n^{A^*}(k, k)$.*

Proof. $H^n(A, k) \cong H^n(A^*, k)$ from above. From [1], page 170, we see that $H^n(A^*, \mathrm{Hom}_k(k, k)) \cong \mathrm{Ext}^n_{A^*}(k, k)$. But of course $\mathrm{Hom}_k(k, k) \cong k$, so we have $H^n(A, k) \cong \mathrm{Ext}^n_{A^*}(k, k)$. Now we need

2.4. Proposition. *$\dim_k \mathrm{Ext}^n_{A^*}(k, k)$ is finite.*

PROOF. We have $0 \longrightarrow A \longrightarrow A^* \longrightarrow k \longrightarrow 0$ the beginning of a free resolution for k. Suppose we have

$$0 \longrightarrow K_n \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow A^* \longrightarrow k \longrightarrow 0$$

exact with every F_i free and finitely generated. Then K_n is a submodule of F_n a finitely generated A^*-module and A^* is noetherian, so K_n is finitely generated. Hence we can find a finitely generated free module F_{n+1} mapping onto K_n. Let K_{n+1} be the kernel of this map, so we have

$$0 \longrightarrow K_{n+1} \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow \cdots \longrightarrow A^* \longrightarrow k \longrightarrow 0$$

exact with every F_i free and finietly generated. Hence, inductively, we have a free resolution for k in which every F_i is finitely generated, $F_i = \sum A^*$, the sum being finitely so

$$\mathrm{Hom}_{A^*}(F_i, k) \cong \mathrm{Hom}_{A^*}(\sum A^*, k) \cong \sum \mathrm{Hom}_{A^*}(A^*, k) \cong \sum k$$

and hence is a finite dimensional space. $\mathrm{Ext}^i_{A^*}(k, k)$ is a quotient of a subspace of this and hence is a finite dimensional space. This completes the lemma.

Now by [1], page 120, $\mathrm{Tor}^{A^*}(\mathrm{Hom}_k(k,k), k) \cong \mathrm{Hom}_k(\mathrm{Ext}^n_{A^*}(,k,), k)$ but again $\mathrm{Hom}_k(k,k) \cong k$ and since $\mathrm{Ext}^n_{A^*}(k,k)$ is finite dimensional, we also have $\mathrm{Hom}_k(\mathrm{Ext}^n_{A^*}(k,k), k) \cong \mathrm{Ext}^n_{A^*}(k,k)$, so that $\mathrm{Tor}^A_n(k,k) \cong \mathrm{Ext}^n_{A^*}(k,k) \cong H^n(A,k)$.

Now, in [5], it is shown that if A^* is a local algebra with maximal ideal A and $\dim_k(A/A^2) = n$, then $\dim_k(\mathrm{Tor}^{A^*}_i) \geq \left(\begin{array}{c} n \\ i \end{array}\right)$ with equality for all $i > 1$ if and only if equality for any $i > 1$ if and only if A^* is regular.

Let $P^m(A,k)$ be the set of all $f \in Z^m(A,k)$ with $f(a_{\sigma(1)}, \ldots, a_{\sigma(m)}) = \mathrm{sgn}\sigma f(a_1, \ldots, a_m)$ for all permutations σ of 1, ..., m where $\mathrm{sgn}\sigma$ is $+1$ or -1 according as σ is even or odd (i.e. the alternating maps).

2.5. LEMMA. *If the characteristic of k does not divide m and $f \in P^m(A,k)$ then $f(a_1, \ldots, a_m) = 0$ if any $a_i \in A^2$.*

PROOF. Since $f \in P^m(A,k)$ and $A \cdot k = 0$, we get that

$$\begin{aligned} 0 = \delta f(a_2, a_3, \ldots, a_{m+1}, a_1) &= -f(a_2 a_3, a_4, \ldots, a_{m+1}, a_1) + \cdots \\ &+ (-1)^{m-1} f(a_2, \ldots, a_m a_{m+1}, a_1) + (-1)^m f(a_2, \ldots, a_{m+1}, a_1) \end{aligned} \qquad (i)$$

and

$$\begin{aligned} 0 &= \delta f(a_3, a_4, \ldots, a_{m+1}, a_1, a_2) = -f(a_3, a_4, \ldots, a_{m+1}, a_1, a_2) + \cdots \\ &+ (-1)^m f(a_3, \ldots, a_m a_{m+1} a_1, a_2) + (-1)^m f(a_3, \ldots, a_{m+1}, a_1 a_2) \\ &= (-1)^m f(a_2, a_3 a_4, \ldots, a_1) + \cdots + f(a_2, a_3, \ldots, a_{m+1} a_1) + (-1)^m f(a_1 a_2, a_3, \ldots, a_{m+1}) \end{aligned}$$

and multiplying by $(-1)^m$ we get

$$0 = -f(a_2, a_3a_4, \ldots, a_{m+1}, a_1) + \cdots + (-1)^{m-1}f(a_2, a_3, \ldots, a_{m+1}a_1)$$
$$+ (-1)^m f(a_1a_2.a_3, \ldots, a_{m+1}) \qquad (ii)$$

Adding (i) and (ii) gives

$$-f(a_2a_3, \ldots, a_{m+1,a_1}) + (-1)^m f(a_1a_2, \ldots, a_{m+1}) = 0$$

or

$$(-1)^m f(a_1, a_2a_3, \ldots, a_{m+1}) + (-1)^m f(a_1a_2, a_3, \ldots, a_{m+1}) = 0$$

and finally

$$f(a_1, a_2a_3, \ldots, a_{m+1}) = -f(a_1a_2, a_3, \ldots, a_{m+1})$$

Then using again that f is an alternating map, we get

$$f(a_1, \ldots, a_ja_{j+1}, \ldots) = -f(a_1 \ldots, a_{j-1}a_j, \ldots a_{m+1})$$
$$= \cdots = (-1)^{j-1}f(a_1a_2, a_2, \ldots, a_{m+1})$$

Then

$$0 = \delta f(a_1, a_2, \ldots, a_{m+1}) = \sum(-1)^j f(a_1, \ldots, a_ja_{j+1}, \ldots, a_{m+1})$$

$$\sum -f(a_1, a_2, a_3, \ldots, a_{m+1}) = -mf(a_1a_2, a_3, \ldots, a_{m+1})$$

and if the characteristic of k does not divide m, $f(a_1a_2, a_3, \ldots .a_{m+1}) = 0$. Since f this implies that $f(a_1, \ldots, a_m) = 0$ if any $a_i \in A^2$.

Hence f induces an m-linear map from A/A^2 to k. Conversely, any m-linear alternating map from A/A^2 to k comes from such a cocycle. For if $g : A/A^2 \longrightarrow k$ is such a map, let

$$f(a_1, a_2, \ldots, a_m) = g(a_1 + A^2, a_2 + A^2, \ldots a_m + A^2)$$

Then since g alternates, so does f. Moreover $\delta g = 0$ since every term has a variable in A^2. It is well known that the dimension of the space of m-linear alternating maps of an n-dimensional vector space into its coefficient field is $\begin{pmatrix} n \\ m \end{pmatrix}$. Hence we have

2.6. Proposition. $\dim_k P^m(A, k) = \begin{pmatrix} n \\ m \end{pmatrix}$.

2.7. Proposition. *If $f \in B^m(A, k)$, then $\sum \operatorname{sgn}\sigma f(a_{\sigma(1)}, \ldots, a_{\sigma(m)}) = 0$, where the sum is taken over all permutations of $1, \ldots, m$.*

Proof. see [6].

2.8. Theorem. *If the characteristic of k does not divide $m!$,*[8] *then $B^m(A, k) \cap P^m(A, k) = 0$.*

[8]Obviously this means that the characteristic is $> m$

Proof. For if $f \in P^m(A,k)$, then

$$0 = \sum \mathrm{sgn}\sigma f(a_{\sigma(1)}, \ldots, a_{\sigma(m)}) = \sum f(a_1, \ldots, a_m) = m! f(a_1, \ldots, a_m)$$

which gives that $f(a_1, \ldots, a_m) = 0$.

2.9. Proposition. *$P^m(A,k) \subseteq Z_s^m(A,k)$ for $m = 2, 3, 4$.*

Proof. It is sufficient to show that if $f \in P^m(A,k)$, then $\pi_m f = f$.

$$\pi_2 f(a,b) = 1/2[f(a,b) - f(b,a)] = f(a,b)$$

$$\pi_3 f(a,b,c) = 1/6[4f(a,b,c) + 2f(c,b,a) - f(b,a,c) + f(b,c,a) - f(a,c,b) + f(c,a,c)]$$

$$= 1/6[(4 - 2 + 1 + 1 + 1 + 1)f(a,b,c)] = f(a,b,c)$$

$$\pi_4 f(a,b,c,d) = 3/4 f(a,b,c,d) + 1/4 f(d,c,b,a) + D$$

where D is a sum of terms of the form $1/12[f(a,y,z,t) - f(t,z,y,x)$ and is 0 for an alternating map, so $\pi_4 f = f$.

2.10. Theorem. *If $H^2(A,k) = 0$, then A is regular. If A is regular, then $H^m(A,k) = 0$, for $m = 2, 3, 4$.*

Proof. Using Tate's theorem and Proposition 2.3, we get that $\dim_k H^m(A,k) \geq \begin{pmatrix} n \\ m \end{pmatrix}$ with equality if and only A is regular. Consequently, since $\dim_k P^2(A,k) = \begin{pmatrix} n \\ m \end{pmatrix}$, A is reguar if and only if $H^m(A,k) = P^m(A,k)$. Now for $m = 2$, $P^(A,k) = H_s^2(A,k)$ so that A is regular if and only if $H_c^2(A,k) = 0$. Moreover, by Proposition 2.9, $P_m(A,k) \subseteq Z_s^m(A,k)$ so that if A is regular, $H_c^m(A,k) = 0$, for $m = 3, 4$.

3. Appendix

Here we give a more natural proof of Theorem 16 of [2] and add a few miscellaneous results.

3.1. THEOREM. *Let A be a commutative algebra with identity, and S a multiplicatively closed subset of A with $1 \in A$, $0 \notin A$ and A_S the algebra of quotients. Suppose E is an A_S-module, then $H_c^2(A_S, E) \cong H_c^2(A, E)$.*

PROOF. We first consider the case in which there are no zero divisors of S in A. In this case A is a subalgebra of A_S. We know from [2] that $H_c^2(A, E)$ is the group of extensions

$$0 \longrightarrow E \xrightarrow{\theta} B \xrightarrow{\phi} A \longrightarrow 0$$

in which B is a commutative algebra, E is an ideal of B with $E^2 = 0$ and $be = \phi(b)$ for all $b \in B$ and $e \in E$.

3.2. LEMMA. *If B is such an extension, then B has an identity.*

PROOF. Choose $b \in B$ with $\phi(b) = 1$. If $e \in E$, $be = e$. $\phi(b^2 - b) = 0$, so if we let $e = b^2 - b$, then since $e^2 = 0$,

$$(b - e)^2 = b^2 - 2be = b^2 - 2eb = b^2 - e + b - b^2 = b - e$$

so that replacing, if necessary, b by $b - e$ we may assume that $b^2 = b$. Then if $b' \in B$, $bb' - b' \in E$, so that $bb' - b = b(bb' - b') = b^2b' - bb' = 0$ and we get that $b = 1$.[9]

Now let $T = \phi^{-1}(S)$.

3.3. LEMMA. *There are no zero divisors of T in B.*

PROOF. For if $b \in B$. $t \in T$ with $tb = 0$, then $0 = \phi(tb) = \phi(t)\phi(b)$ and $\phi(t) \in S$, so that $\phi(b) = 0$. Then $b \in E$ so that $0 = tb = \phi(t)b$. But E is an A_S-module, so that $b = 0$. Hence we can form the ring of quotients B_T. Map $\phi_T : B_T \longrightarrow A_S$ by $\phi_T(b/t) = \phi(b)/\phi(t)$ and $\theta_T : E \longrightarrow B_T$ as θ followed by the inclusion of B into B_T.

3.4. LEMMA. *The sequence*

$$0 \longrightarrow E \xrightarrow{\theta_T} B_T \xrightarrow{\phi_T} A_S \longrightarrow 0$$

is exact.

PROOF. Clearly θ_T is a monomorphism, ϕ_T is an epimorphism, and $\phi_T\theta_T = 0$. If $\phi_T(b/t) = 0$, then $\phi(b) - 0$ and $b \in E$. Then $b = \phi(t)(b/\phi(t)) = t(b/\phi(t))$ so that $b/t = b/\phi(t) \in E$.

Also as a corollary to the proof, we have

[9] A conceptually simpler argument is to use the multiplication $(a, e)(a', e') = (ae', a'e + ae' + f(a, a'))$ on $A \oplus E$. Let $g(a) = -f(1, a)$. Then $(f - \delta g)(1, a) = -1f(1, a) - af(1, 1)$. Using that $\delta f(1, 1, a) = 0$ yields $1f(1, a) - af(1, 1) = 0$, so that, replacing f by $f - \delta g$, we can suppose that $f(1, a) = 0$ and then $(1, 0)(a, e) = (a, e + f(1, a)) = (a, e)$.

3.5. Corollary. *$e/t = e/\phi(t)$ for all $e \in E$ and $t \in T$.*

We denote this extension (θ_S, B_T, ϕ_T) by $(\theta, B, \phi)_T$. It will be shown that the map $T : H^2_c(A, E) \longrightarrow H^2_c(A_S, E)$ given by $T(\theta, B, \phi) = (\theta, B, \phi)_T$ is well defined and an isomorphism of the groups.

Suppose (θ, B, ϕ) amd (θ', B', ϕ') are two extensions. Their sum, which we will denote by $(\theta, B, \phi) * (\theta', B', \phi')$ is given as follows. Let $M = \{(b, b') \in B \oplus B' \mid \phi(b) = \phi'(b')\}$, $N = \{(\theta(e), -\theta'(e')) \mid e \in E\}$, and $B*B' = M/N$. We denote the class containing (b, b') by $b*b'$. Define $\phi*\phi'$ by $(\phi*\phi')(b*b') = \phi(b) = \phi'(b')$ and $\theta*\theta'$ by $(\theta*\theta')(e) = \theta(e)*0 = 0*\theta'(e)$. Let $T' = \phi'^{-1}(S)$ and $T * T' = (\phi * \phi')^{-1}(S)$.

3.6. Lemma. *$(\theta, B, \phi)_T * (\theta', B', \phi')_{T'} \sim (\theta * \theta', B * B', \phi * \phi')_{T*T'}$.*

Proof. We define $f : B_t * B'_{T'} \longrightarrow (B * B')_{T*T'}$ as follows: Let $b/t * b'/t' \in B_T * B_{T'}$. Choose $u \in B'$ with $\phi'(u) = \phi(t)/\phi'(t')$. $\phi'(u)$ is invertible in A_S so there exists $v \in B'_{T'}$ with $uv - 1 = e \in E$. Then

$$u(v - e/\phi'(u)) = uv - ue/\phi'(u) = 1 + e - \phi'(u)e/\phi'(u) = 1$$

which gives that u is invertible in $B'_{T'}$. Then $b'/t' = ub'/ut'$,

$$\phi'(ut') = \phi'(t')\phi'(u) = \phi'(t')\phi(t)/\phi'(t') = \phi(t)$$

and

$$\phi'(ub') = \phi'(ut'b') = \phi'(ut')\phi'(b'/t') = \phi(t)\phi(b/t) = \phi(b)$$

so that $(b * ub')/(t * ut') \in (B * B'_{T*T'}$ and we let this be $f(b/t * b'/t')$. (Using the same argument as before it is see that $t * ut'$ is an invertible element of $(B * B')_{T*T'}$.) It is a straightforward calculation to show that does depend on u and defines a homomorphism of $B_T*B'_{T'}$ to $(B*B')_{T*T'}$. If $f(b/t*b'/t') = 0$, then there exists $e \in E$ with $b = \theta(e)$ and $ub' = -\theta'(e)$. Then from Corollary 3.5, $b/t = \theta(e/\phi(t))$, and $b'/t' = ub'/ut' = -\theta'(e/\phi'(ut')) = -\theta'(e/\phi(t))$, so that $b/t * b'/t' = 0$. Moreover, if $(b * b')/(t * t') \in (B * B')_{T*T'}$, then $f(b/t * b'/t') = (b * b')/(t * t')$ so that f is an isomorphism. Also we have

$$\begin{aligned}(\phi * \phi')_{T*T'} f(b/t * b'/t') &= (\phi * \phi')_{T*T'}((b * ub')/(t * ut')) \\ &= (\phi * \phi')(b * ub')/\phi * \phi'(t * ut') \\ &= \phi(b)/\phi(t) = \phi_T(b/t) = (\phi_T * \phi'_{T'})(b/t * b'/t')\end{aligned}$$

and

$$f(\theta_T * \theta'_{T'})(e) = f(\theta(e)/1 * 0/1) = (\theta(e) * 0)/(1 * 1) = (\theta * \theta')_{T*T'}(e)$$

This proves the lemma.

3.7. Lemma. *(θ, B, ϕ) is the split extension if and only if $(\theta, B\phi)_T$ is.*

PROOF. Suppose (θ, B, ϕ) splits, with $g : E \oplus A \longrightarrow B$ the isomorphism. Define $g_T : E \oplus A_S \longrightarrow B_T$ by $g_T(e, a/s) = g(e, 0) + g(0, a)/g(0, s)$ for all $e \in E$, $a \in A$ and $s \in S$. It is easily checked that g_T is an algebra homomorphism. If $g_T(e, a/s) = 0$, then $g(0, a) = -g(e, 0)g(0, s) = -g(es, 0)$, or since g is an isomorphism, $(0, a) = -(es, 0)$. Hence $(e, a/s) = 0$. Further, if $g(e, a)/g(e', s) \in B_T$, then

$$\begin{aligned} g(se - ae') + g(ae', as) &= g(se, as) + g(se', s^2)g(e/s - ae'/s^2, 0) + g(0, a)g(0, a)g(e', s) \\ &= g(e, a)g(0, s) + g(e/s - ae'/s^2, 0) + g(0, a)g(e', s) \\ &+ g_T(e/s - ae'/s^2, a/s))(g(e, a)/g(e', s) \end{aligned}$$

so that g_T is an isomorphism. It clearly induces an equivalence between $(\theta, B, \phi)_T$ and the split extension.

Conversely, suppose $(\theta, B, \phi)_T$ is the split extension and $h : E \oplus A_S \longrightarrow B_T$ is the isomorphism. If $b \in B$, choose $(e, a/s) \in E \oplus A_S$ with $h(e, a/s) = b$, then $a/s = \phi_T(e, a/s) = \phi(b) \in A$, so that s is an invertible element of A and $(e, a/s) \in E \oplus A$. If $(e, a) \in E \oplus A$, choose $b \in B$ with $\phi(b) = a$, and $(e', a'/s') \in E \oplus A_S$ with $h(e', a'/s') = b$. Then $a'/s' = \phi_T h(e', a'/s') = \phi(b) = a$, so that $h(e', a) \in B$. $h(e, a) = h(e', a) + h(e - e', 0) = h(e', a) + \theta(e - e') \in B$. Hence if we let $g = h|_{E\oplus A}$, then g defines an equivalence between (θ, B, ϕ) and the split extension. Hence the proof of the theorem in the special case in which S has no zero divisors in A reduces to

3.8. LEMMA. *The map T is an epimorphism.*

PROOF. Suppose (Γ, C, Δ) is an extension of E by A_S. Let $B = \Delta^{-1}(A) \supseteq \Delta^{-1}(0) = \Gamma(E)$. Let $\theta : E \longrightarrow B$ just be Γ and $\phi : B \longrightarrow A$ be the restriction of Δ. Clearly (θ, B, ϕ) is an extension of E by A. We claim that (
$th, B\phi)_T \sim (\Gamma, C, \Delta)$. Now if $t \in T$, then $\phi(t) = \Delta(t)$ so that by the same argument as in Lemma 3.6, t is invertible in C. Map $f : B_T \longrightarrow C$ by $f(b/t) = b/t$. Clearly f is an algebra monomorphism since the operations in B_T and C coincide. Let $c \in C$ and $\Delta(c) - a/s$. Choose $t \in T$ with $\phi(t) = s$, then $\Delta(ct) = \Delta(c)\Delta(t) = (a/s)s = a$ so that $ct \in B$ and $c = ct/t$ gives that f is an isomorphism. Moreover, it clearly gives an equivalence of the extensions.

The theorem will now follow from

3.9. LEMMA. *Let $f : A \longrightarrow A_S$. If $\mathfrak{a} = \ker(f)$ and $R = A/\mathfrak{a}$, then $H^2_c(A, E) \cong H^2_c(A_S, E)$.*

PROOF.

$$\mathrm{Hom}_A(\mathfrak{a}, E) \longrightarrow H^2_c(R, E) \longrightarrow H^2_c(A, E) \longrightarrow H^2_c(A, \mathfrak{a}, E)$$

is exact so it is sufficient to show that $\mathrm{Hom}_A(A, E) = H^2_c(A, \mathfrak{a}, E) = 0$. [10] Now we know that for all $\alpha \in \mathfrak{a}$, there exists $s \in S$ with $\alpha s = 0$ But E is an A_S-module so there are

[10]This is a consequence of Theorem 1.12 that actually asserts the exactness of

$$H^1(A, \mathfrak{a}, E) \longrightarrow H^2_c(R, E) \longrightarrow H^2(A, E) \longrightarrow H^2(A, \mathfrak{a}, E)$$

But since $H^1_c(A, \mathfrak{a}, E)$ is a subquotient of $\mathrm{Hom}_A(A, E)$ it will be 0 when the homset is.

no zero divisors of S in E. Hence if $f \in \mathrm{Hom}_A(\mathfrak{a}, E)$, $0 = f(s\alpha) = sf(\alpha)$ which implies that $f(\alpha) = 0$. Since α was arbitrary, $f = 0$. Now let $f \in Z^2(A, \mathfrak{a}, E)$, $\alpha \in \mathfrak{a}$, and $s, t \in S$ with $s\alpha = t\alpha = 0$. Then $0 = sf(t, \alpha) - f(st, \alpha) + f(s, t\alpha) - \alpha f(s, t)$ together with $t\alpha = 0$ and $\mathfrak{a} \cdot E = 0$ give $f(st, \alpha) = sf(t, \alpha)$ and similarly $f(st, \alpha) = tf(s, al)$ so that $(1/s)f(s\alpha) = (1/t)f(t, \alpha)$. If we let $g(\alpha) = (1/s)f(s, \alpha)$, it is easily checked that g is a linear map. Then if $a \in A$, $0 = sf(a, \alpha) + f(s, \alpha a) - af(s, \alpha) = g(\alpha a) - ag(\alpha)$. Hence $H^2(A, \mathfrak{a}, E) = 0$ and the theorem is proved.

3.10. Theorem. *If R is an affine algebra*[11] *and $H^2_c(R, E) = 0$ for all finitely generated R-modules E, then $H^2_c(R, E) = 0$ for all R-modules E.*

Remark. This improves Theorem 22 of [2].

Proof. If R is an affine algebra, we can find a polynomial algebra A in finitely many variables over k and an epimorphism $\phi : A \longrightarrow R$. Let $\mathfrak{a}$ be the kernel of this map. We have the exact sequencw

$$H^1(A, E) \longrightarrow \mathrm{Hom}(\mathfrak{a}, E) \longrightarrow H^2(R, E) \longrightarrow 0$$

since $H^2(A, E) = 0$ (see [2], Theorem 11). Hence $H^2_c(R, E) = 0$ if and only if the map from $H^1(A, E)$ to $Hom_A(\mathfrak{a}, E)$ is an epimorphism. If this is true of all finitely generated modules E, then we can take $E = \mathfrak{a}/\mathfrak{a}^2$ which is a quotient module of a submodule of a finitely generated module over a noetherian ring. Then there is a derivation $d : A \longrightarrow \mathfrak{a}/\mathfrak{a}^2$ which when restricted to $\mathfrak{a}$ bives the canonical projection of $\mathfrak{a}$ onto $\mathfrak{a}/\mathfrak{a}^2$, i.e. $d(\alpha) = \alpha + \alpha^2$ for all $\alpha \in \mathfrak{a}$. Now let E be any R-module, and $f : \mathfrak{a} \longrightarrow E$ an A-homomorphism. Since $\mathfrak{a} \cdot E = 0$, $f(\mathfrak{a}^2) = 0$ and f induces an R-homomorphism $\overline{f} : \mathfrak{a}/\mathfrak{a}^2 \longrightarrow E$. Consider $g = \overline{f}d : A \longrightarrow E$.

$$g(ab) = \overline{f}d(ab) = \overline{f}(ad(b) + bd(a)) = a\overline{f}d(b) + b\overline{f}d(a) = ag(b) + bg(a)$$

for all $a, b \in A$ so that $g \in H^1(A, E)$ and if $a \in \mathfrak{a}$, $\overline{f}d(\alpha) = \overline{f}(\alpha + \mathfrak{a}^2) = f(\alpha)$. This proves the theorem.

3.11. Theorem. *If A is an algebra with identity 1 and E is an A-module with $1 \cdot E = 0$, then $H^n(A, E) = 0$, for all $n > 0$.*

For $n = 1$, the result follows from $f(a) = f(1 \cdot a) = 1 \cdot f(a) + a \cdot f(1)$ for all $a \in A$, and $f \in Z^1(A, E)$. If $n > 1$, and $f \in Z^n(A, E)$, let $g(a_1, \ldots, a_{n-1}) = f(1, a_1, \ldots, a_{n-1})$. Then

$$0 = \delta f(1, \ldots, a_n) = 1 \cdot f(a_1) - \sum (-1)^i f(1, a_1, \ldots a_i a_(i+1), \ldots, a_n)$$
$$= -\sum (-1)^i g(a_1, \ldots, a_i a_{i+1}, \ldots, a_n)$$

from which the theorem can be seen.[12]

[11] a quotient of a polynomial ring in finitely many variables

[12] Clearly, if $1 \cdot E = 0$, then $A \cdot E = 0$.

3.12. Corollary. *If A has an identity, and E is an A-module, then* $H^n(A, E) \cong H^n(A, 1 \cdot E)$.

Proof. The result can be seen from the sequence

$$H^{n-1} \longrightarrow H^n(A, 1 \cdot E) \longrightarrow H^n(A, E) \longrightarrow H^n(A, E/1\cdot)$$

3.13. Corollary. $H_c^n(A, E) \cong H_c^n(A, 1 \cdot E)$ *and* $H_s^n(A, E) \cong H_s^n(A, 1 \cdot E)$, *for* $n =$ 1, 2, 3, 4.

Remark. This improves Theorems 17 and 22 of [2].