Journal of Pure and Applied Algebra 4 (1974) 1–8. © North-Holland Publishing Company

RIGHT EXACT FUNCTORS

Michael BARR
Department of Mathematics, McGill University, Montreal, Canada

Communicated by P.J. Hilton
Received 20 January 1973
Revised 30 May 1973

0. Introduction

Let X be a category with finite limits. We say that X is *exact* if:

(1) whenever

$$\begin{tikzcd} X_1 \arrow[r, "g"] \arrow[d] & X_2 \arrow[d] \\ X_3 \arrow[r, "f"] & X_4 \end{tikzcd}$$

is a pullback diagram and f is a regular epimorphism (the coequalizer of some pair of maps), so is g;

(2) whenever $E \subset X \times X$ is an equivalence relation (see Section 1 below), then the two projection maps $E \rightrightarrows X$ have a coequalizer; moreover, E is its kernel pair.

When X lacks all finite limits, a somewhat finer definition can be given. We refer to [3, I(1.3)] for details.

In this note, we explore some of the properties of right exact functors – those which preserve the coequalizers of equivalence relations. For functors which preserve kernel pairs, this is equivalent (provided that the domain is an exact category) to preserving regular epimorphisms. I had previously tried – in vain – to show that functors which preserve regular epimorphisms have some of the nice properties which right exact functors do have. For example, what do you need to assume about a triple on an exact category to insure that the category of algebras is exact? The example of the torsion-free-quotient triple on abelian groups shows that preserving regular epimorphisms is not enough. On the other hand, as I showed in [3, I.5.11], the algebras for a finitary theory in any exact category do form an exact category. And that will remain true for any theory if the n^{th} power functor preserves regular epimorphisms for all cardinals n. The missing link which connects all these disparate results is that the n^{th} power functor preserves kernel pairs (in fact all limits) and is (right) exact as soon as it preserves regular epimorphisms.

We say that a category is EX5 if it is exact, if it has all finite inverse limits and filtered direct limits, and if finite inverse limits commute with filtered direct limits. We say that a sequence of objects and maps

$$X' \rightrightarrows X \to X''$$

is *exact in some category* if it is simultaneously a coequalizer and a kernel pair. A functor is *right exact* if it takes such a sequence into a coequalizer. A functor is called *finitary* if it commutes with filtered direct limits.

In an exact category, every map has a canonical factorization as a regular epimorphism followed by a monomorphism (see [3, I.2.3]). When we speak of image, we mean in terms of this factorization. We will say that a pair of maps

$$Y \rightrightarrows X$$

in a category is a *reflexive pair* if the image of Y in $X \times X$ contains the diagonal $\Delta X \subset X \times X$. This is rather weaker than the usual definition, in which one supposes the two maps to have a common right inverse. However, it will be a useful notion in this paper.

The most surprising consequence of right exactness is that a finitary right exact functor whose domain is EX5 preserves the coequalizers of reflexive pairs. From this we derive the fact that under the same conditions, pushout diagrams consisting of regular epimorphisms (regular co-intersections) are also preserved. Finally we use that to give a very general solution to a problem arising in the theory of automata – the existence of minimal machines.

As usual, a map denoted $\twoheadrightarrow$ is assumed to be a regular epimorphism; a map denoted $\rightarrowtail$ is assumed to be a monomorphism.

1. Reflexive relations

In an exact (or just regular) category we define a (binary) relation on an object X to be a subobject of $X \times X$. More generally, one can define a relation between X and Y as a subobject of $X \times Y$. In Grillet [4, I.4] an efficient calculus of relations is developed. If R and S are relations, he defines $R \circ S$ as the image in $X \times X$ of a pullback $R \times_X S$ and shows that this rule of composition is associative (here is where regularity comes in) and unitary (the diagonal subobject ΔX is the unit) and satisfies

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

(inverse defined in the usual way via the switching map $X \times X \to X \times X$). The relation R is an equivalence relation if and only if

(i) R is reflexive, i.e. $\Delta X \subset R$;
(ii) R is symmetric, i.e. $R^{-1} \subset R$;
(iii) R is transitive, i.e. $R \circ R \subset R$.

We let $R^{(n)} = R \circ R \circ \ldots \circ R$ denote the n-fold iterated circle composite of R.

1.1. Theorem. *Let X be an* EX5 *category with exact direct limits. Then for any reflexive relation R on X, the smallest equivalence relation containing R is*

$$E = S \cup S \circ S \cup S^{(3)} \cup \ldots \cup S^{(n)} \cup \ldots ,$$

where $S = R \circ R^{-1}$.

Proof (compare the proof of [4, I.6.8]). Since $\Delta X \subset R$, $\Delta X \subset R^{-1}$, and then

$$R = R \circ \Delta X \subset R \circ R^{-1} = S .$$

It is evident that S is reflexive and symmetric, and then so is any power of S, so that E is reflexive and symmetric. Since direct limits are exact, $E \times_X E$ is a direct limit of $S^{(i)} \times_X S^{(j)}$ whose image is contained in $S^{(i+j)}$. Thus E is also transitive. Conversely, if F is an equivalence relation containing R, it is clear that $R^{-1} \subset F^{-1} \subset F$,

$$S = R \circ R^{-1} \subset F \circ F \subset F$$

and inductively that $S^{(n)} \subset F$, whence $E \subset F$.

1.2. Proposition. *Let $U : X \to Y$ be an exact functor between exact categories. Then U preserves the calculus of relations.*

Proof. An exact functor preserves finite limits, hence products and subobjects and hence relations. Furthermore, it preserves pullbacks and regular epimorphisms, hence images, and so preserves composition of relations. It is trivial that it preserves inverses of relations.

1.3. Theorem. *Let X be an* EX5 *category, Y an exact category, and $U : X \to Y$ a finitary exact functor. Then U preserves the coequalizer of any reflexive pair of maps.*

Proof. Let $Y \rightrightarrows X$ be a reflexive pair and R be the image of Y in $X \times X$. Since $Y \twoheadrightarrow R$, $R \rightrightarrows X$ has the same coequalizer as $Y \rightrightarrows X$. Since also $UY \twoheadrightarrow UR$, $UY \rightrightarrows UX$ has the same coequalizer as $UR \rightrightarrows UX$. Hence it suffices to prove this for a reflexive relation. As above, let $S = R \circ R^{-1}$ and

$$E = S \cup S^{(2)} \cup S^{(3)} \cup \ldots \cup S^{(n)} \cup \ldots .$$

Then $US = UR \circ (UR)^{-1}$, and

$$UE = US \cup (US) \circ (US) \cup \ldots \cup (US)^{(n)} \cup \ldots .$$

Just as in the proof of Theorem 1.1, UE must be contained in any equivalence relation containing UR. But now if

$$R \rightrightarrows X \to Z$$

is the coequalizer,

$$E \rightrightarrows X \to Z$$

is exact and hence so is

$$UE \rightrightarrows UX \to UZ .$$

This shows that UE is an equivalence relation and is the smallest one containing UR Hence

$$UR \rightrightarrows UX \to UZ$$

must also be exact.

1.4. Remark. Both the hypotheses that X have exact filtered direct limits and that R be reflexive are necessary. The underlying functor from compact Hausdorff spaces to sets does not preserve the coequalizer of

$$d^0, d^1 : [0, 1] \cup [0, 1] \to [0, 1] ,$$

where each map is the identity on the first component, while on the second d^0 is the identity and d^1 is multiplication by an irrational $t < 1$. The underlying functor from groups to sets does not preserve the coequalizer of

$$d^0, d^1 : Z \to Z ,$$

where d^0 is the zero map and d^1 is multiplication by 2. This second example is especially interesting: of the three conditions defining an equivalence relation, that of being reflexive seems the most negligible.

1.5. Corollary. *Suppose U, X, Y are as in the theorem. Then U preserves pushout diagrams in which every map is a regular epimorphism.*

Proof. Let

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_3 \end{array}$$

be a commutative square with every map a regular epimorphism. Let $E \rightrightarrows X$ be the kernel pair of $X \to X_1$. Then one may easily show, using universal mapping properties, that the square is a pushout if and only if the sequence

$$E \rightrightarrows X_2 \to X_3$$

is a coequalizer. Then supposing the square is a pushout, the sequence is a coequalizer. The maps $E \rightrightarrows X$ are a reflexive pair and since $X \twoheadrightarrow X_2$, the pair $E \rightrightarrows X_2$ is also a reflexive pair. By the theorem,

$$UE \rightrightarrows UX_2 \to UX_3$$

is also a coequalizer, whence

$$\begin{array}{ccc} UX & \longrightarrow & UX_1 \\ \downarrow & & \downarrow \\ UX_2 & \longrightarrow & UX_3 \end{array}$$

is a pushout.

2. Right exact functors

A functor $U : \boldsymbol{X} \to \boldsymbol{Y}$ is called *right exact* provided that whenever

$$X' \rightrightarrows X \to X''$$

is exact in $\boldsymbol{X}$,

$$UX' \rightrightarrows UX \to UX''$$

is right exact, i.e. a coequalizer, in $\boldsymbol{Y}$. A triple $\mathcal{T} = (T, \eta, \mu)$ is right exact provided that T is.

2.1. Theorem. *Let $\boldsymbol{X}$ be an exact category and $\mathcal{T}$ a right exact triple on $\boldsymbol{X}$. Then the category $\boldsymbol{X}^{\mathcal{T}}$ of $\mathcal{T}$-algebras is exact; if $\mathcal{T}$ is finitary and $\boldsymbol{X}$ is* EX5, *then $\boldsymbol{X}^{\mathcal{T}}$ is* EX5 *as well as the underlying functor $\boldsymbol{X}^{\mathcal{T}} \to \boldsymbol{X}$ is finitary. Hence it preserves the coequalizers of reflexive pairs.*

Proof. If T is right exact, then it certainly preserves regular epimorphisms. According to [5, (2.8)], $\boldsymbol{X}^{\mathcal{T}}$ is then regular. Now suppose that $E \to A \times A$ is an equivalence relation on A in $\boldsymbol{X}^{\mathcal{T}}$. That is, $A = (X, x)$ is an algebra, $E = (Y, y)$ is a subalgebra, and $Y \to X \times X$ is an equivalence relation. Let Z be the coequalizer, so that

$$Y \rightrightarrows X \to Z$$

is exact. Apply T to get

$$\begin{tikzcd} TY \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "y"'] & TX \arrow[r] \arrow[d, "x"'] & TZ \arrow[d] \\ Y \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & Z \end{tikzcd}$$

Since the upper line is a coequalizer, there is a unique $z : TZ \to Z$ making the diagram commute. The argument that $B = (Z, z)$ is a $\mathcal{T}$-algebra is standard. Clearly

$$E \rightrightarrows A \to B$$

is a coequalizer,,and since U creates limits, it is also a kernel pair, and hence exact. To finish the proof, we remark that one easily shows, in a way analogous to the above, that U creates any direct limits which are preserved by T, in particular filtered direct limits. It also creates inverse limits (always). Hence to show they commute in $\boldsymbol{X}^{\mathcal{T}}$ it suffices to assume they do in $\boldsymbol{X}$.

2.2. Corollary. *Let $\boldsymbol{X}$ and $\mathcal{T}$ be as in the theorem. Suppose also that objects of $\boldsymbol{X}$ have only a set of regular quotients. Let $f : UA \to X$ be a morphism in $\boldsymbol{X}$. Then there is a smallest quotient $g : A \to A'$ in $\boldsymbol{X}^{\mathcal{T}}$ such that there is a factorization*

$$\begin{tikzcd} UA \arrow[rr, "Ug"] \arrow[dr, "f"'] & & UA' \arrow[dl] \\ & X & \end{tikzcd} .$$

Proof. The functor U preserves pushouts of the form

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, two heads] & A_1 \arrow[d, two heads] \\ A_2 \arrow[r, two heads] & A_3 \end{tikzcd}$$

so that the family of all such quotients of A which satisfy the conclusion is filtered. Their direct limit $A \to A'$ exists and is easily seen to be a regular quotient of A. Since the direct limit is preserved, it has the requisite mapping property to have a map $UA' \to X$ with the desired property.

2.3. Remark. In the category of sets – indeed in any category in which every epimorphism splits – every exact sequence is a split coequalizer diagram. Hence every functor with such a category as domain is right exact. As noted, this is a much stronger condition than preservation of regular epimorphisms. This seems to explain much – if not all – of the nice structure possessed by categories tripleable over sets.

3. Application to categorical machines: the minimal realizable problem

In [1], the notion of a categorical machine is defined. The set-up is a category $\boldsymbol{X}$ and an endofunctor T which generates a free triple $\mathcal{T} = (T^*, \eta, \mu)$. A machine is (roughly) a $\mathcal{T}$-algebra Q equipped with a map – the output function – $UQ \to Y$ in $\boldsymbol{X}$. The minimal realization problem boils down to this: given a map $T^*I \to Y$, find a minimal (smallest, really) machine Q which "realizes" the given map; that is, a minimal quotient of T^*I which is still a $\mathcal{T}$-algebra and which factors the given map. Evidently, this will always exist if $\boldsymbol{X}$ and $\mathcal{T}$ satisfy the hypotheses of Theorem 2.1.

3.1. Theorem. *Let $\boldsymbol{X}$ be an* EX5 *category. Let T be a finitary right exact endofunctor. Then T generates a free triple $\mathcal{T} = (T^*, \eta, \mu)$ which is right exact and finitary. Moreover the minimal realization problem for $\mathcal{T}$-machines is always solvable.*

Proofs. The results of [2] apply here to get the free triple $\mathcal{T}$. The construction there is based on the identification of the category of $\mathcal{T}$-algebras (whether or not $\mathcal{T}$ exists!) as the category of $(T : \boldsymbol{X})$ whose objects are pairs (X, x), $x : TX \to X$, satisfying no conditions, and maps commuting with the structure in the obvious way. The underlying functor $U : (T : \boldsymbol{X}) \to \boldsymbol{X}$ forgets the structure, and if it has a left adjoint F, then $\mathcal{T}$ is the triple associated to the adjoint pair. (If U does not have an adjoint, a free triple does not exist.) At any rate, I showed there that a free triple exists if T has a rank, which it certainly does when T is finitary. Moreover, one easily shows, exactly as for $\boldsymbol{X}^{\mathcal{T}}$, that any direct limit preserved by T is preserved by Y. But F preserves all direct limits. Hence any direct limit preserved by T is preserved by $UF = T^*$. In particular, when T is finitary, so is T^*; when T is right exact, so is T^*. The last assertion then follows from the results of Section 2.

3.2. Remark. It is possible to generalize the notion of a machine so that the input process is a triple $\mathcal{T}$ rather than an endofunctor. Then a machine is a $\mathcal{T}$-algebra, except that there is no need for a free triple to exist.

References

[1] M.A. Arbib and E. Manes, Machines in a category, an expository introduction, Mimeographed Notes, Univ. of Massachusetts, Amherst, Mass.
[2] M. Barr, Coequalizers and free triples, Math. Z. 116 (1970) 307–322.
[3] M. Barr, Exact categories, Lecture Notes in Math. 236 (Springer, Berlin, 1971).
[4] P.A. Grillet, Regular categories, Lecture Notes in Math. 236 (Springer, Berlin, 1971).
[5] E. Manes, A triple theoretic construction of compact algebras, Lecture Notes in Math. 80 (Springer, Berlin, 1969).