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Harrison Homology, Hochschild Homology and Triples*[1]

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1. INTRODUCTION

We consider the following situation: a field k, a commutative k-algebra R and a left R-module M. Since R is commutative, M may also be considered as an R-R bimodule with the same operation on each side (such modules are often termed *symmetric*). With these assumptions we have the Harrison (co-) homology groups $\mathrm{Harr}_*(R, M)$ ($\mathrm{Harr}^*(R, M)$), the Hochschild (co-) homology groups $\mathrm{Hoch}_*(R, M)$ ($\mathrm{Hoch}^*(R, M)$) and the symmetric algebra triple (co-) homology groups $\mathrm{Sym}_*(R, M)$ ($\mathrm{Sym}^*(R, M)$). (Harrison cohomology is introduced in [*8*], homology may be defined similarly; Hochschild cohomology is introduced in [*9*], a good account of the homology and cohomology is found in [*10*]; general triple homology and cohomology are described in [*4*] and [*5*], and the symmetric algebra cotriple is described at the beginning of Section 3 below.)

There are obvious natural transformations

$$\phi_* = \phi_*(R, M) : \mathrm{Hoch}_*(R, M) \to \mathrm{Harr}_*(R, M)$$

and $\phi^* = \phi^*(R, M) : \mathrm{Harr}^*(R, M) \to \mathrm{Hoch}^*(R, M)$ which will be described in detail below. It is the purpose of this paper to prove the following two theorems.

THEOREM 1.1. *Suppose k has characteristic zero. Then ϕ_* is a split epimorphism and ϕ^* is a split monomorphism. Moreover the splitting maps are also natural.*

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THEOREM 1.2. *Suppose k has characteristic zero. Then there are natural equivalences*

$$\mathrm{Sym}_n(R, M) \simeq \mathrm{Harr}_{n+1}(R, M),$$
$$\mathrm{Sym}^n(R, M) \simeq \mathrm{Harr}^{n+1}(R, M).$$

Moreover, in the coefficient variable they are natural equivalences of connected sequences of functors.

Theorem 1.1 is easily seen to be false for finite characteristic and a recent example of André shows that Theorem 1.2 is false there as well (see Section 4 below). The results of this paper are obtained, essentially, for degrees $\leqslant 4$ by direct computations in [2].

We use the following notation. $A \otimes B$, Tor (A, B), $\mathrm{Hom}(A, B)$ for $A \otimes_k B$, $\mathrm{Tor}^k(A, B)$, $\mathrm{Hom}_k(A, B)$ respectively. $A^{(n)}$ denotes $A \otimes \cdots \otimes A$ (n factors).

2. PROOF OF THEOREM 1.1.

The Hochschild homology of R may be defined by choosing a complex B_*R of R-R-bimodules and, for any R-R-bimodule M, defining $\mathrm{Hoch}_*(R, M)$ to be the homology $H(B_*R \otimes_{R\otimes R} M)$. Requiring that M is symmetric is equivalent to saying that $M \simeq R \otimes_R M$ as an $R \otimes R$ module, with $R \otimes R$ operating on R by multiplication. Thus

$$B_*R \otimes_{R\otimes R} M \simeq (B_*R \otimes_{R\otimes R} R) \otimes_R M,$$

and so we may work with the complex of left R-modules which we will denote by $C_*R = B_*R \otimes_{R\otimes R} R$. Then $\mathrm{Hoch}_*(R, M) \simeq H(C_*R \otimes_R M)$. Similarly, $\mathrm{Hoch}^*(R, M) \simeq H(\mathrm{Hom}_R(C_*R, M))$.

The Harrison homology is defined by forming a subcomplex $\mathrm{Sh}_*R \subset C_*R$, the subcomplex of *shuffles* (defined below), and letting Ch_*R denote C_*R/Sh_*R. Then $\mathrm{Harr}_*(R, M) \simeq H(Ch_*R \otimes_R M)$ and $\mathrm{Harr}^*(R, M) \simeq H(\mathrm{Hom}_R(Ch_*R, M))$. Let $f_* : C_*R \to Ch_*R$ be the projection. Then $\phi_*(R, M) = H(f_* \otimes_R M)$ and $\phi^*(R, M) = H(\mathrm{Hom}_R(f_*, M))$ are the natural transformations mentioned above.

Explicitly these complexes are defined as follows. Let $C_nR = R \otimes R^{(n)}$ where R operates on the first factor. If we denote the element

$$r_0 \otimes r_1 \otimes \cdots \otimes r_n \in C_nR \qquad \text{by} \quad r_0[r_1, \dots, r_n],$$

then the boundary $\partial : C_nR \to C_{n-1}R$ is the R-linear map such that

$$\partial[r_1, \dots, r_n] = r_1[r_2, \dots, r_n] + \sum (-1)^i [r_1, \dots, r_i r_{i+1}, \dots, r_n] + (-1)^n r_n[r_1, \dots, r_{n-1}].$$

In what follows we use the natural equivalence $C_nR \otimes_R C_mR \simeq C_{n+m}R$ to identify $[r_1, ..., r_n] \otimes [r_{n+1}, ..., r_{n+m}]$ with $[r_1, ..., r_n, r_{n+1}, ..., r_{n+m}]$. Then we define maps $s_{i,n-i} : C_nR \to C_nR$ by requiring that they be R-linear, $s_{0,n} = s_{n,0} = C_nR$ (that is, the identity map; we will consistently identify the identity map of an object in a category with the object itself), and

$$s_{i,n-i}[r_1, ..., r_n] = [r_1] \otimes s_{i-1,n-i}[r_2, ..., r_n] \\ + (-1)^i[r_{i+1}] \otimes s_{i,n-i-1}[r_1, ..., r_i, r_{i+2}, ..., r_n].$$

These are the so-called *shuffle-products* (see [7]). Intuitively, $s_{i,n-i}$ is the alternating sum of all permutations of n cards obtained by shuffling i of them with $n - i$ of them. We also observe that if C_nR is made into a left module over the symmetric group S_n by letting $\pi^{-1}[r_1, ..., r_n] = [r_{\pi 1}, ..., r_{\pi n}]$, then $s_{i,n-i}$ may be thought of as multiplication by a well defined element of the rational group algebra $\mathbf{Q}(S_n)$ (recall that R is an algebra over a field of characteristic zero).

Sh_nR is defined as the $\sum_{i=1}^{n-1} \mathrm{Im}\, s_{i,n-i}$. That the Sh_nR form a subcomplex will follow readily from Proposition 2.2. Theorem 1.1 will easily follow if we can find, for each $n \geqslant 2$, natural transformations $e_n : C_nR \to \mathrm{Sh}_nR$ which are the identity on Sh_nR and such that $\partial e_n = e_{n-1}\partial$, for then $f_n : C_nR \to Ch_nR$ has a natural splitting by chain maps. (Only $n \geqslant 2$ is necessary, for $\mathrm{Sh}_nR = 0$, $n = 0,1$). In fact the e_n will be multiplication by a suitable element of $\mathbf{Q}(S_n)$, from which naturality, at least in the technical sense, is automatic.

The alternating representation $\mathrm{sgn} : S_n \to \mathbf{Q}$, defined as usual by $\mathrm{sgn}\, \pi = +1$ or -1 according as π is or is not in the alternating subgroup, extends to an algebra homomorphism which we will also denote by $\mathrm{sgn} : \mathbf{Q}(S_n) \to \mathbf{Q}$. We let $\epsilon_n \in \mathbf{Q}(S_n)$ be $1/n! \sum (\mathrm{sgn}\, \pi)\pi$. Then for any $u \in \mathbf{Q}(S_n)$, $u\epsilon_n = \mathrm{sgn}\, u \cdot \epsilon_n$.

PROPOSITION 2.1. *For any chain* $[r_1, ..., r_n]$, $\partial\epsilon_n[r_1, ..., r_n] = 0$. *If* $u \in \mathbf{Q}(S_n)$ *is such that* $\partial u[r_1, ..., r_n] = 0$ *for all* R, *then* $u = \mathrm{sgn}\, u \cdot \epsilon_n$.

Proof. In the computation of $n!\partial\epsilon_n[r_1, ..., r_n]$ the term $r_{\pi 1}[r_{\pi 2}, ..., r_{\pi n}]$ appears with coefficient $\mathrm{sgn}\, \pi$ in the first term of the boundary of $\pi^{-1}[r_1, ..., r_n]$ and with coefficient $\mathrm{sgn}\, \pi \cdot \mathrm{sgn}\, \tau(-1)^n$ in the last term of the boundary of $(\pi\tau)^{-1}[r_1, ..., r_n]$ where τ is the cycle $(12...n)$. But $\mathrm{sgn}\, \tau = (-1)^{n-1}$, so they cancel. Similarly the term $[r_{\pi 1}, ..., r_{\pi i}r_{\pi(i+1)}, ..., r_{\pi n}]$ appears with coefficient $(-1)^i \mathrm{sgn}\, \pi$ in the boundary of $\pi^{-1}[r_1, ..., r_n]$ and with coefficient $(-1)^{i+1}\mathrm{sgn}\, \pi$ in the boundary of $(i\, i + 1)\pi^{-1}[r_1, ..., r_n]$. Moreover, it is clear that for "general" R, say if R is the polynomial algebra $k[r_1, ..., r_n]$, this is the only possible cancellation. Thus if $\partial u = 0$, then for each $\pi \in S_n$ and each $1 \leqslant i \leqslant n$ the coefficient of π and $\pi(i\, i + 1)$ in u must have the same magnitude but opposite sign. This suffices to guarantee that u is a multiple of ϵ_n and clearly that multiple is sgn u.

PROPOSITION 2.2.

$$\partial s_{i,j}[r_1, ..., r_{i+j}] = s_{i-1,j}(\partial[r_1, ..., r_i] \otimes [r_{i+1}, ..., r_{i+j}]) + (-1)^i s_{i,j-1}([r_1, ..., r_i] \otimes \partial[r_{i+1}, ..., r_{i+j}]).$$

Proof. We may consider C_*R as a simplicial module by defining $\partial^0[r_1, ..., r_n] = r_1[r_2, ..., r_n]$, $\partial^n[r_1, ..., r_n] = r_n[r_1, ..., r_{n-1}]$ and $\partial^i[r_1, ..., r_n] = [r_1, ..., r_i r_{i+1}, ..., r_n]$ for $0 < i < n$. Also define $s^i[r_1, ..., r_n] = [r_1, ..., r_i, 1, r_{i+1}, ..., r_n]$. Define $[r_1, ..., r_n] \times [r_1', ..., r_n'] = [r_1 r_1', ..., r_n r_n']$. Then with these definitions, all the considerations of [7], p. 64 *ff*. apply and the result follows from Theorem 5.2 of that paper.

PROPOSITION 2.3. *For any* $1 \leqslant i \leqslant n$,

$$\partial[r_1, ..., r_{n+1}] = \partial[r_1, ..., r_{i+1}] \otimes [r_{i+2}, ..., r_{n+1}] + (-1)^i [r_1, ..., r_i] \otimes \partial[r_{i+1}, ..., r_{n+1}].$$

Proof. Trivial.

Now we define $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$. On the basis of the last two propositions, together with the fact that $\partial : C_1R \to C_0R$ is zero, it is now easily proved that

PROPOSITION 2.4. $\partial s_n = s_{n-1}\partial$ *for* $n \geqslant 1$, $(s_1 = s_0 = 0)$.

PROPOSITION 2.5. *For each* $n \geqslant 2$ *there is an* $e_n \in \mathbf{Q}(S_n)$ *with the following properties*:

(i) e_n is a polynomial in s_n without constant term;

(ii) sgn $e_n = 1$;

(iii) $\partial e_n = e_{n-1}\partial$;

(iv) $e_n{}^2 = e_n$;

(v) $e_n s_{i,n-i} = s_{i,n-i}$ for $1 \leqslant i \leqslant n - 1$.

Remark. (i) and (v) together imply that the principal right ideal $e_n\mathbf{Q}(S_n) = \sum s_{i,n-i}\mathbf{Q}(S_n)$. But then $\mathrm{Sh}_n R = e_n C_n R$, and since $e_n{}^2 = e_n$, we have $C_n R = e_n C_n R + (1 - e_n) C_n R$, from which $Ch_n R \simeq (1 - e_n) C_n R$. Then (iii) implies that these form subcomplexes and Theorem 1.1 is proved.

Proof of Proposition 2.5. Let us denote the binomial coefficient corresponding to n things taken i at a time by $c_{i,n-i}$. Then it follows easily from the inductive definitions of $s_{i,n-i}$ and $c_{i,n-i}$ that sgn $s_{i,n-i} = c_{i,n-i}$. Thus

$\operatorname{sgn} s_n = 2^n - 2 \neq 0$. Let $e_2 = \epsilon_2 = 1/2\, s_2$. Having found $e_2, \ldots, e_{n-1}$ satisfying the above conditions, suppose $e_{n-1} = p(s_{n-1})$ (where p is the polynomial assumed by (i) to exist). Then define

$$e_n = p(s_n) + (1 - p(s_n))\, s_n/\operatorname{sgn} s_n .$$

It is easily seen that e_n satisfies (i) and (ii). Also we have

$$\begin{aligned} \partial e_n &= p(s_{n-1})\, \partial + (1/\operatorname{sgn} s_n)(1 - p(s_{n-1}))\, s_{n-1}\partial \\ &= e_{n-1}\partial + (1/\operatorname{sgn} s_n)(1 - e_{n-1})\, s_{n-1}\partial \\ &= e_{n-1}\partial + (1/\operatorname{sgn} s_n)(s_{n-1} - e_{n-1}s_{n-1})\, \partial \\ &= e_{n-1}\partial. \end{aligned}$$

Then $\partial e_n{}^2 = e_{n-1}^2\partial = e_{n-1}\partial = \partial e_n$, so that, by Proposition 2.1, $e_n{}^2 - e_n = (\operatorname{sgn} e_n{}^2 - \operatorname{sgn} e_n)\epsilon_n = 0$. Finally, for any chain $[r_1, \ldots, r_n]$, we have

$$\begin{aligned} \partial e_n s_{i,n-i}[r_1, \ldots, r_n] &= e_{n-1}\partial s_{i,n-i}[r_1, \ldots, r_n] \\ &= e_{n-1}s_{i-1,n-i}(\partial[r_1, \ldots, r_i] \otimes [r_{i+1}, \ldots, r_n]) \\ &\quad + (-1)^i\, e_{n-1}s_{i,n-i-1}([r_1, \ldots, r_i] \otimes \partial[r_{i+1}, \ldots, r_n]) \\ &= s_{i-1,n-1}(\partial[r_1, \ldots, r_i] \otimes [r_{i+1}, \ldots, r_n]) \\ &\quad + (-1)^i\, s_{i,n-i-1}([r_1, \ldots, r_i] \otimes \partial[r_{i+1}, \ldots, r_n]) \\ &= \partial s_{i,n-i}[r_1, \ldots, r_n]. \end{aligned}$$

Thus by Proposition 2.1, $e_n s_{i,n-i} - s_{i,n-i} = (\operatorname{sgn} e_n \cdot \operatorname{sgn} s_{i,n-i} - \operatorname{sgn} s_{i,n-i})\epsilon_n = 0$.

We note that (ii) implies

$$e_n\epsilon_n = \epsilon_n, \qquad (1 - e_n)\epsilon_n = 0.$$

3. Proof of Theorem 1.2.

We begin with a brief exposition of the triple cohomology. Let U denote the underlying functor from k-algebras to k-modules. It has a left adjoint F which is most easily described by saying that if V is a k-module, FV is the tensor algebra $k + V + V^{(2)} + \cdots + V^{(n)} + \cdots$ modulo its commutator ideal. Then if $\alpha : 1 \to UF$ and $\epsilon : FU \to 1$ are the adjointness morphisms, $\mathbf{G} = (G, \epsilon, F\alpha U)$ is a cotriple on the category of commutative k-algebras known as the symmetric algebra cotriple.

The (co-) homology groups are described as follows. Given $\sigma : S \to R$ where S is any commutative k-algebra, we define an R-module Diff S as $R \otimes S$ (made into an R-module by multiplication on the first factor) modulo the submodule generated by $\{r \otimes ss' - r\sigma(s) \otimes s' - r\sigma(s') \otimes s \mid r \in R, s, s' \in S\}$. Diff S is characterized by the formula $\mathrm{Hom}_R(\mathrm{Diff}\, S, M) \simeq \mathrm{Der}(S, M)$ where the latter stands for the group of k-linear derivations of S to M, where M is made into an S-module via σ. Then the homology groups $\mathrm{Sym}_n(R, M)$ are the homology of $\cdots \to \mathrm{Diff}\, G^{n+1}R \otimes_R M \to \cdots \to \mathrm{Diff}\, G^2R \otimes_R M \to \mathrm{Diff}\, GR \otimes_R M \to 0$ where $G^{n+1}R \to R$ is any appropriate composite of ϵ's and $d : \mathrm{Diff}\, G^{n+1}R \otimes_R M \to \mathrm{Diff}\, G^nR \otimes_R M$ is $\sum (-1)^i \,\mathrm{Diff}\, G^i\epsilon G^{n-i}R \otimes_R M$. Similarly the cohomology groups are defined as the homology of

$$0 \to \mathrm{Hom}_R(\mathrm{Diff}\, GR, M) \to \mathrm{Hom}_R(\mathrm{Diff}\, G^2R, M) \to \cdots$$
$$\to \mathrm{Hom}_R(\mathrm{Diff}\, G^{n+1}R, M) \to \cdots$$

where the coboundary is induced by the analogous alternating sum. We note that the second complex could also be written as

$$0 \to \mathrm{Der}(GR, M) \to \mathrm{Der}(G^2R, M) \to \cdots \to \mathrm{Der}(G^{n+1}R, M) \to \cdots .$$

It is shown in [3] that this (co-) homology is the same as that obtained for the free (polynomial) algebra cotriple when k is a field.

The proof of Theorem 1.2 is based on a method first employed by M. André (see [1], pp. 6–8), which is reminiscent of acyclic models. The latter method will not work in this case unless the acyclicity asserted in the next proposition can be given by a natural contracting homotopy.

PROPOSITION 3.1. *Let $R = k[X]$ be the algebra of polynomials over a set X. Then for any R-module M,* $\mathrm{Harr}^n(R, M) = 0 = \mathrm{Harr}_n(R, M)$ *for* $n > 1$, $\mathrm{Harr}^1(R, M) \simeq \mathrm{Der}(R, M) \simeq M^X$, *and* $\mathrm{Harr}_1(R, M) \simeq \mathrm{Diff}\, X \otimes_R M \simeq X \cdot M$, *the latter denoting a direct sum of X copies of the module M.*

Proof. The idea of the proof is to show that $H_1(Ch_*R) \simeq \mathrm{Diff}\, R \simeq X \cdot R$ and $H_n(Ch_*R) = 0$ for $n > 1$. The first is a simple exercise and is left to the reader. Each Ch_nR is R-projective, being a retract of an R-free module, and $\partial : Ch_1R \to Ch_0R$ is zero, so that ignoring Ch_0R the rest of this complex will merely be a projective resolution of a projective module and thus be contractible.

Since $\otimes$ commutes with colimits, e_n, being idempotent, commutes with all limits and colimits, and homology commutes with directed colimits, then the functor $H(Ch_*R)$ commutes with directed colimits (i.e., direct limits). Each $k[X]$ is the colimit of $k[X_\alpha]$ where X_α varies over all finits subsets $X_\alpha \subset X$, which forms a directed system. Thus it suffices to prove this proposition for X finite.

Let $X^\#$ be a set isomorphic to and disjoint from X by an isomorphism which associates to each $x \in X$ an element denoted by $x^\# \in X^\#$, and let $R^\# = k[X^\#]$. It is clear that $R \otimes R \simeq R \otimes R^\#$, and we will identify them.

Now let $X' = \{x' = 1 \otimes x^{\#} - x \otimes 1 \in R \otimes R^{\#} \mid x \in X\}$ and $R' = k[X']$. It is clear that $R \otimes R^{\#} = R \otimes R'$. Here we are thinking of an "internal tensor product" analogous to internal direct sum. But R' operates trivially on R, that is via the augmentation $R' \to k$. Hence $R \simeq R \otimes k$ as an $R \otimes R'$-module, R operating on the first factor, R' on the second. Then according to Theorem X.3.1 of [*6*],

$$\begin{aligned}\mathrm{Tor}^{R\otimes R'}(R, R) &\simeq \mathrm{Tor}^{R\otimes R'}(R \otimes k, R \otimes k)\\ &\simeq \mathrm{Tor}^{R}(R, R) \otimes \mathrm{Tor}^{R'}(k, k) \simeq R \otimes \mathrm{Tor}^{R}(k, k).\end{aligned}$$

But

$$\mathrm{Tor}^{R\otimes R'}(k, k) \simeq \mathrm{Tor}^{R\otimes R}(k, k) \simeq \mathrm{Hoch}_*(R, R) = HC_*R.$$

Thus HC_*R and hence HCh_*R consist of R-projective modules.

At this point we need

PROPOSITION 3.2. *Let P be an R-projective (or, in fact, any submodule of a free) R-module such that $P \otimes_R k = 0$. Then $P = 0$.*

Proof. Let M be the ideal of R generated by X. Then $0 \to M \to R \to k \to 0$ is exact and so is $P \otimes_R M \to P \otimes_R R \to P \otimes_R k \to 0$. If the last term is zero, then $P \otimes_R M \to P$ by right multiplication is onto. But then $PM^i = P$ for all i. Now M^i consists of all polynomials of which every terms has total degree at least i. Clearly $\bigcap M^i = 0$, from which it is clear that if F is free, $\bigcap FM^i = 0$. Then if $P \subset F$, $P = \bigcap PM^i \subset \bigcap FM^i = 0$.

From this it follows that it suffices to show $H_n(Ch_*R) \otimes_R k = 0$ for $n > 1$. Since Ch_*R and $H(Ch_*R)$ consist of R-projectives, Theorem V.10.1 of [*10*] implies that $H_n(Ch_*R) \otimes_R k \simeq H_n(Ch_*R \otimes_R k)$. We call a cycle $\gamma \in C_nR \otimes_R k$ ***alternating*** if $\gamma = \epsilon_n \gamma$.

PROPOSITION 3.3. *If $R = k[X]$ as above, every cycle in $C_nR \otimes_R k$ is homologous to an alternating cycle.*

Proof. Let $X = \{x_1, \ldots, x_d\}$. To compute $H_n(R, k)$ we first observe, by the same argument used in the beginning of the proof of proposition 3.1, that $\mathrm{Hoch}_*(R, k) \simeq \mathrm{Tor}^{R\otimes R}(R, k) \simeq \mathrm{Tor}^{R}(R, k) \otimes \mathrm{Tor}^{R}(k, k) \simeq \mathrm{Tor}^{R}(k, k)$. It is well known (see [*10*] p. 205, for example) that $\mathrm{Tor}_n{}^R(k, k)$ is a k-space of dimension $c_{n,d-n}$. We shall show that there is a subspace of $C_nR \otimes_R k$ consisting entirely of alternating cycles, having dimension $c_{n,d-n}$ and independent modulo boundaries. It will follow that it contributes a subspace of $H_n(R, k)$ of exactly that dimension which, since k is a field, must be the whole thing.

For each set of integers $1 \leqslant i_1 < \cdots < i_n \leqslant d$, of which there are exactly

$c_{n,d-n}$, consider the chain $\epsilon_n[x_{i_1}, ..., x_{i_n}]$. These are all cycles, since $\partial\epsilon_n = 0$, and they are clearly linearly independent. To show that they are independent modulo boundaries, we first observe that $C_{n+1}R \otimes_R k$ has a k basis consisting of chains $[p_1, ..., p_{n+1}]$ where p_i is a monomial. If any $p_i = 1$, then it is directly verified that every term of its boundary has the same property except two which cancel. If no $p_i = 1$, then every term of the boundary contains an entry of degree at least two (except the first and last, which, however, are zero because the coefficients are in k). Thus no boundary can have only entrics of degree one.

Now suppose γ is a cycle in $Ch_nR \otimes_R k$ and $n > 1$. Then we can write $\gamma = \partial\gamma' + \epsilon_n\gamma''$ for $\gamma' \in C_{n+1}R \otimes_R k$ and $\gamma'' \in \mathrm{Sh}_nR \otimes_R k$. Then $\gamma = (1 - e_n)\gamma = (1 - e_n)\,\partial\gamma' + (1 - e_n)\,\epsilon_n\gamma'' = \partial(1 - e_{n+1})\gamma'$, and so γ is a boundary. This completes the proof of Proposition 3.1.

We are now ready to prove Theorem 1.2. We give the proof for cohomology. The proof for homology is similar. For any R, the Hochschild complex is $\mathrm{Hom}_R(C_nR, M) \simeq \mathrm{Hom}(R^{(n)}, M)$ with suitable coboundary. Let us write $\mathrm{Hom}(R^{(n)}, M)_c$ for the subgroup of commutative cochains (those which vanish on all shuffles; equivalently those f for which $fe_n = 0$). Let $E = \{E^{i,j}\}$ denote the double complex such that $E^{i,j} = \mathrm{Hom}((G^{j+1}R)^{(i+1)}, M)_c$, $i \geqslant 0$, $j \geqslant 0$ with coboundaries $\delta_{\mathrm{I}} : E^{i,j} \to E^{i+1,j}$ given by the Harrison coboundary formula (the restriction of the Hochschild coboundary; recall that each $G^{j+1}R$ acts on M via the ϵ's) and $\delta_{\mathrm{II}} : E^{i,j} \to E^{i,j+1}$ given by the triple coboundary formula. Proposition 3.1 implies that $H_{\mathrm{I}}E$ reduces to $\mathrm{Der}(G^{j+1}R, M)$ concentrated in bidegree $(0, j)$, and so $H_{\mathrm{II}}H_{\mathrm{I}}E \simeq \mathrm{Sym}^*(R, M)$. To compute in the other order, we first observe that the k-linear map $R \to GR$ (which is actually the front adjunction) induces natural transformations $\theta_iR : \mathrm{Hom}((GR)^{(i+1)}, M)_c \to \mathrm{Hom}(R^{(i+1)}, M)_c$ whose composite with $\mathrm{Hom}((\epsilon R)^{(i+1)}, M)_c$ is the identity. Now with i fixed, this is enough to imply the map whose jth component is θ_iG^jR is a contraction of the augmented cochain complex $0 \to \mathrm{Hom}(R^{(i+1)}, M)_c \to E^{i,0} \to \cdots \to E^{i,j} \to \cdots$. Thus $H_{\mathrm{II}}E \simeq \mathrm{Hom}(R^{(i+1)}, M)_c$ concentrated in bidegree $(i, 0)$ and $H_{\mathrm{I}}{}^iH_{\mathrm{II}}E \simeq \mathrm{Harr}^{i+1}(R, M)$, which proves theorem 1.2.

4. An Example in Characteristic p

We give here a slight modification of the example of André mentioned in the introduction. We show that if k has characteristic $p > 0$ then for any integer $m > 0$ the Harrison cohomology groups of the polynomial algebra $k[x]$ in dimension $n = 2p^m$ are nonzero.

Let $d_{i,j}$ denote the number of even permutations in $s_{i,j}$ less the number of odd ones. In fact $d_{i,j}$ is the value of $s_{i,j}$ under the trivial representation (the one which takes each $\pi \in S_n$ to 1). Upon examination of the inductive definition of $s_{i,j}$, $d_{i,j}$ is easily seen to satisfy the following functional equations:

$$d_{i,j} = \begin{cases} d_{i-1,j} + d_{i,j-1}, & \text{if } i \text{ is even} \\ d_{i-1,j} - d_{i,j-1}, & \text{if } i \text{ is odd;} \end{cases}$$

$$d_{i,j} = d_{j,i};$$

$$d_{i,1} = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

PROPOSITION 4.1. *$d_{2i,2j} = d_{2i+1,2j} = d_{2i,2j+1} = c_{i,j}$, and $d_{2i+1,2j+1} = 0$. (Recall that $c_{i,j}$ is the binomial coefficient corresponding to $i + j$ things taken i at a time.)*

Proof. The above functional equations clearly characterize d completely. Hence it is simply a matter of showing that these formulas satisfy these equations. This is an easy exercise.

PROPOSITION 4.2. *If $n = p^m$ for any integer $m > 0$, then $c_{i,n-i}$ is divisible by p for any $0 < i < n$.*

Proof. This is easily shown by counting directly the number of times p divides $n!$, $i!$ and $(n - i)!$.

COROLLARY 4.3. *If $i + j = 2p^m$ for $m > 0$, then p divides every $d_{i,j}$, $0 < i < 2p^m$.*

THEOREM 4.4. *If k is a field of characteristic $p > 0$ and $n = 2p^m$, $m > 0$, then* $\mathrm{Harr}^n(k[x], k) \neq 0$.

Proof. Here k is made into a $k[x]$ module by the map $k[x] \to k$, which sends x to 0. Define a cochain $f : k[x]^{(n)} \to k$ by letting

$$f[x^{i_1}, \ldots, x^{i_n}] = \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = 1 \\ 0, & \text{otherwise,} \end{cases}$$

and extending linearly. Then $f\, s_{i,n-i}[x, \ldots, x] = d_{i,n-i} f[x, \ldots, x] = 0$, since any permutation does not affect f, or, in other words, f represents S_n trivially. Then $f \in \mathrm{Hom}(k[x]^{(n)}, k)_c$. By using the fact that the coefficients are in k it is also readily checked that $\delta f = 0$. On the other hand, for any

$$g \in \mathrm{Hom}(k[x]^{(n-1)}, k)_c,$$

$$\begin{aligned} \delta g[x, \ldots, x] &= -g[x^2, x, \ldots, x] + g[x, x^2, x, \ldots, x] + \cdots - g[x, \ldots, x^2] \\ &= -g\, s_{1,n-2}[x^2, x, \ldots, x] = 0. \end{aligned}$$

This completes the proof.

Note that this is also a counter-example for Theorem 1.1, as $k[x]$ is also a free associative algebra and its higher Hochschild cohomology groups are all 0.

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