BOUSFIELD LOCALIZATION AND THE HASSE SQUARE

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1. Bousfield localization

The general idea of localization at a spectrum E is to associate to any spectrum X the "part of X that E can see", denoted by $L_E X$. In particular, it is desirable that L_E is a functor with the following equivalent properties:

$$E \wedge X \simeq * \Longrightarrow L_E X \simeq *$$

$$\text{If } X \to Y \text{ induces an equivalence } E \wedge X \to E \wedge Y \text{ then } L_E X \xrightarrow{\simeq} L_E Y.$$

Definition 1.1. A spectrum X is called *E-acyclic* if $E \wedge X \simeq *$. It is called *E-local* if for each E-acyclic T, $[T, E] = 0$, where $[T, E]$ denotes the group of stable homotopy classes. This is equivalent to the statement that for each E-equivalence $S \to T$, the induced map $[T, X] \to [S, X]$ is an isomorphism.

A spectrum Y with a map $X \to Y$ is called an *E-localization* of X if Y is E-local and $X \to Y$ is an E-equivalence.

If a localization of X exists, then it is unique up to homotopy and will be denoted by $X \xrightarrow{\eta_E} L_E X$.

Localizations of this kind were first studied by Adams [Ada74], but set-theoretic difficulties prevented him from actually constructing them. Bousfield found a way of overcoming these problems in the unstable category [Bou75]; for spectra, he showed in [Bou79] that localization functors exist for arbitrary E.

We start by collecting a couple of easy facts about localizations.

Lemma 1.2. *Module spectra over a ring spectrum E are E-local.*

Proof. Since any map from a spectrum W into an E-module spectrum M can be factored through $E \wedge W$, it follows that there are no essential maps from an E-acyclic W into M. □

Lemma 1.3. *If $v\colon E \to E$ is a self-map of a spectrum E (possibly of nonzero degree), then $L_E \simeq L_{v^{-1}E \vee E/v}$, where E/v denotes the cofiber and $v^{-1}E$ the mapping telescope.*

Proof. It suffices to show that the class of E-acyclics agrees with the class of $(v^{-1}E \vee E/v)$-acyclics. Since the latter is a module spectrum over the former, E-acyclics are clearly $(v^{-1}E \vee E/v)$-acyclic; conversely, if $E/v \wedge W \simeq *$ then $v\colon E \wedge W \to E \wedge W$ is a homotopy equivalence, hence $E \wedge W \simeq v^{-1}E \wedge W$. Thus if also $v^{-1}E \wedge W \simeq *$, W is E-acyclic. □

Lemma 1.4. *Homotopy limits and retracts of E-local spectra are E-local.*

Proof. The statement about retracts is obvious. For the statement about limits, first observe that a spectrum X is E-local if and only if the mapping spectrum $\operatorname{Map}(T, X)$ is contractible for all E-acyclic T. This is obvious because $\pi_k \operatorname{Map}(T, X) = [\Sigma^k T, X]$, and if T is E-acyclic then so are all its suspensions.

Now if $F\colon I \to \{\text{spectra}\}$ is a diagram of E-local spectra, the claim follows from the equivalence

$$\operatorname{Map}(T, \operatorname{holim} F) \simeq \operatorname{holim} \operatorname{Map}(T, F)$$

□

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The following lemma characterizes E-localizations.

Lemma 1.5. *The following are equivalent for a map of spectra $X \to Y$:*

- $X \to Y$ *is an E-localization;*

① $X \to Y$ *is the initial map into an E-local target;*

② $X \to Y$ *is the terminal map which is an E-equivalence.*

Proof. Obvious from the axioms. □

This characterization suggests two ways of constructing $X \to L_E X$:

① $L_E X = \operatorname*{holim}_{\substack{X \to Y \\ Y\, E\text{-local}}} Y$ or

② $L_E X = \operatorname*{hocolim}_{\substack{X \to Y \\ E\text{-equivalence}}} Y$.

In both cases, these limits are not guaranteed to exist because the indexing categories are not small. This is more than a set-theoretic nuisance and requires a deeper study of the structure of the background categories.

I will first briefly discuss what can be done with approach ①. The main construction will be closer to method ②.

① **Localizations as limits.** Instead of indexing the homotopy limit over all $X \to Y$ with Y E-local, we could use the spaces in the Adams tower for E:

$$X \to \operatorname{Tot}^n \left(E^{\wedge(\bullet+1)} \wedge X \right),$$

which is a subdiagram because $E \wedge X$ is E-local for any X by Lemma 1.2, and E-locality satisfies the 2-out-of-3 property for cofibration sequences of spectra.

If we are lucky, $X \to X\hat{_E} = \operatorname{Tot}(E^{\wedge(\bullet+1)} \wedge X)$ is an E-localization. This is not always the case – $X \to X\hat{_E}$ sometimes fails to be an E-equivalence. Whether or not $L_E X \simeq X\hat{_E}$, the latter is what the E-based Adams-Novikov spectral sequence converges to and thus is of independent interest. If $L_E X$ can be built from E-module spectra by a finite sequence of cofiber extensions and retracts, then $L_E X \simeq X\hat{_E}$ [Bou79, Thm 6.10] (such spectra are called E-prenilpotent). For some spectra E, every X is E-prenilpotent; these spectra have the characterizing property that their Adams spectral sequence has a common horizontal vanishing line at E_∞ and a horizontal stabilization line at every E_r for every finite CW-spectrum [Bou79, Thm 6.12]. A necessary condition for this is that E is *smashing*, i.e., that $L_E X = X \wedge L_E S^0$ for every spectrum X.

② **Localizations as colimits.** Bousfield's approach to constructing localizations uses colimits. The basic idea for cutting down the size of the diagram the colimit is formed over is the following observation:

To check if X is E-local, it is enough to show that for any E-equivalence $S \to T$ with $\#S$, $\#T < \kappa$ for some cardinal κ depending only on E, $[T, X] \xrightarrow{\cong} [S, X]$.

At this point, it is not crucial what exactly we mean by $\#S$. For a construction of $L_E X$ that is functorial up to homotopy, it is enough to define $\#S$ to be the number of stable cells.

Given this observation, $L_E X$ can be constructed in a small-object-argument-like fashion by forming homotopy pushouts

$$\begin{array}{ccc} \coprod_{\substack{S \to T \\ E\text{-eq.}}} S & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{\substack{S \to T \\ E\text{-eq.}}} T & \longrightarrow & X_{(1)} \end{array}$$

and iterating this transfinitely (using colimits at limit ordinals). When the cardinal κ is reached, $X_{(\kappa)}$ is E-local because it satisfies the lifting condition for "small" $S \to T$.

Theorem 1.6. *The category of spectra has a model structure with*

- *cofibrations the usual cofibrations of spectra, i.e. levelwise cofibrations $A_n \to B_n$ such that*
$$\mathbb{S}^1 \wedge B_n \cup_{\mathbb{S}^1 \wedge A_n} A_{n+1} \to B_{n+1}$$
are also cofibrations;
- *weak equivalences the (stable) E-equivalences;*
- *fibrations given by the lifting property*

The fibrant objects in this model structure are the E-local Ω-spectra.

Here are some explicit examples of localization functors.

Example 1.7.
(1) $E = \mathbb{S}^0$. In this case, L_E is the functor that replaces a spectrum by an equivalent Ω-spectrum.
(2) $E = M(\mathbb{Z}_{(p)}) =$ Moore spectrum. In this case $L_E X \simeq X_{(p)}$ is the classical p-localization. This is an example of a *smashing localization*, i.e. $L_E X \simeq X \wedge L_E \mathbb{S}^0$, which in this case is also the same as $X \wedge E$.
(3) $E = M(\mathbb{Z}/p)$. For connective X, $L_E X \simeq X\hat{_p}$ is the p-completion functor
$$X\hat{_p} = \operatorname{holim}\{\cdots \to X \wedge M(\mathbb{Z}/p^2) \to X \wedge M(\mathbb{Z}/p)\}.$$
(4) $E = M(\mathbb{Q}) = H\mathbb{Q}$. As in (2), $L_E X = L_\mathbb{Q} X = X \wedge L_\mathbb{Q}\mathbb{S}^0 = X \wedge H\mathbb{Q}$ is smashing; it is the classical rationalization of X.

2. The Sullivan arithmetic square

The arithmetic square is a homotopy cartesian square that allows one to reconstruct a space if, roughly, all of its mod-p-localizations and its rationalization are known. For the case of nilpotent spaces, which is similar to spectra, this was first observed by Sullivan [Sul05].

Lemma 2.1. *For any spectrum X, the following diagram is a homotopy pullback square:*

$$\begin{array}{ccc} X & \xrightarrow{\prod \eta_p} & \prod_p L_p X \\ {\scriptstyle \eta_\mathbb{Q}}\downarrow & & \downarrow{\scriptstyle \prod L_\mathbb{Q}} \\ L_\mathbb{Q} X & \xrightarrow{L_\mathbb{Q}(\prod \eta_p)} & L_\mathbb{Q}\left(\prod_p L_p X\right) \end{array}$$

This is a special case of

Proposition 2.2. *Let E, F, X be spectra with $E_*(L_F X) = 0$. Then there is a homotopy pullback square*

$$\begin{array}{ccc} L_{E\vee F} X & \xrightarrow{\eta_E} & L_E X \\ {\scriptstyle \eta_F}\downarrow & & \downarrow{\scriptstyle \eta_F} \\ L_F X & \xrightarrow{L_F(\eta_E)} & L_F L_E X \end{array}$$

In the case of Prop. 2.1, $E = \bigvee_p M(\mathbb{Z}/p)$, $F = H\mathbb{Q}$. To see that $L_E = \prod_p L_p$, we have to show that there are no nontrivial homotopy classes from an E-acyclic space, which is immediate, and that
$$M(\mathbb{Z}/p)_*(X) \xrightarrow{\cong} M(\mathbb{Z}/p)_*\left(\prod_l L_l X\right)$$
is an isomorphism for all p. The latter holds because smashing with $M(\mathbb{Z}/p)$ commutes with products since $M(\mathbb{Z}/p)$ is a finite (two-cell) spectrum (use Spanier-Whitehead duality).

Furthermore, the condition $E_*(L_F X) = E_*(H\mathbb{Q} \wedge X) = 0$ is satisfied because $H_*(M(\mathbb{Z}/p);\mathbb{Q}) = 0$.

Proof of the proposition. Note that the map denoted η_E in the diagram is the unique factorization of $\eta_E\colon X \to L_E X$ through $L_{E\vee F}X$, which exists because $X \to L_{E\vee F}X$ is an E-equivalence. The same holds for η_F, and furthermore, these maps are E- and F-equivalences, respectively. Now let P be the pullback. We need to see that (1) P is $(E \vee F)$-local and (2) the induced map $X \to P$ is an E- and an F-equivalence. For (1), take a spectrum T with $E_*T = F_*T = 0$. Then in the Mayer-Vietoris sequence for the pullback,

$$\dots \to [T, P] \to [T, L_E X] \oplus [T, L_F X] \to [T, L_F L_E X] \to \dots,$$

the two terms on the right are zero, hence so is $[T, P]$.

For (2), observe that $P \to L_F X$ is an F-equivalence because it is the pullback of η_F on $L_E X$, and since $X \to L_F X$ is also an F-equivalence, so is $X \to P$. The same argument works for $P \to L_E X$ except that here, the bottom map is an E-equivalence for the trivial reason that both terms are E-acyclic by the assumption. □

3. Morava K-theories and related ring spectra

Given a complex oriented even ring spectrum E and an element $v \in \pi_*E$, we would like to construct a new complex oriented ring spectrum E/v such that $\pi_*(E/v) = (\pi_*E)/(v)$. This is clearly not always possible. The machinery of commutative $\mathbb{S}$-algebras of [EKMM97] (or any other construction of a symmetric monoidal category of spectra, such as symmetric spectra) allows us to make this work in many cases where more classical homotopy theory has to rely on ad-hoc constructions (such as the Baas-Sullivan theory of bordism of manifolds with singularities).

In this section, let E be a complex oriented even commutative $\mathbb{S}$-algebra and A an E-module spectrum with a commutative ring structure in the homotopy category of E-modules, and which is also a complex oriented even ring spectrum. Let us call this an E-even ring spectrum. A commutative E-algebra would of course be fine, but we need the greater generality.

Theorem 3.1 ([EKMM97, Chapter V]). *For any $v \in \pi_*E$, $v^{-1}A$ carries the structure of an E-even ring spectrum. Furthermore, if v is a non-zero divisor then A/v is also an E-even ring spectrum.*

Even if A is a commutative E-algebra (for example, $A = E$), the resulting spectrum is usually not a commutative $\mathbb{S}$-algebra.

Of course, this construction can be iterated to give

Corollary 3.2. *Given a graded ideal $I \lhd \pi_*E$ generated by a regular sequence and a graded multiplicative set $S \subset \pi_*E$, one can construct an E-even ring spectrum $S^{-1}A/I$ with $\pi_*S^{-1}A/I = S^{-1}(\pi_*A)/I$.*

In particular, this can be done for $E = MU$. For example, BP can be constructed in this way by taking $I = \ker(MU_* \to BP_*)$, which is generated by a regular sequence. It is currently not known whether BP is a commutative $\mathbb{S}$-algebra. However, the methods above allow as to construct all the customary BP-ring spectra by pulling regular sequence back to $E = MU_*$ and letting $A = BP$. For example,

$$\begin{aligned} E(n) &= v_n^{-1}BP/(v_{n+1}, v_{n+2}, \dots) \\ K(n) &= v_n^{-1}BP/(p, v_1, \dots, v_{n-1}, v_{n+1}, \dots) \\ P(n) &= BP/(p, v_1, \dots, v_{n-1}) \\ B(n) &= v_n^{-1}BP/(p, v_1, \dots, v_{n-1}). \end{aligned}$$

Any MU-even ring spectrum A give rise to a Hopf algebroid (A_*, A_*A) and an Adams-Novikov spectral sequence

$$E^2_{**} = \mathrm{Cotor}_{A_*A}(A_*, A_*X) \Longrightarrow \pi_*X\hat{_A}.$$

If $\mathcal{M}_A$ denotes the stack associated to the Hopf algebroid (A_*, A_*A) and F_X the graded sheaf associated with the comodule A_*X, this E^2-term can be expressed as

$$E^2_{**} = H^{**}(\mathcal{M}_a, F_X).$$

In particular, if $f\colon A \to B$ is a morphism of MU-even ring spectra, we get a morphism of spectral sequences, and if f induces an equivalence of the associated stacks, then f induces an isomorphism of spectral sequences from the E_2-term on. In particular, in this case, $X\hat{_A} \simeq X\hat{_B}$ if we can assure that the spectral sequences converge strongly. Note that we do not need an inverse map $B \to A$.

Theorem 3.3. *If $f\colon A \to B$ is a morphism of MU-even ring spectra inducing an equivalence of associated stacks, then $L_A \simeq L_B$.*

Proof. The argument outline above gives an almost-proof of this fact, but it puts us at the mercy of the convergence of the Adams-Novikov spectral sequences to the localizations $L_A X$ and $L_B X$. We give an argument that doesn't require such additional assumptions. Note that it is sufficient to show that $A_* X = 0$ if and only if $B_* X = 0$. Assume $A_* X = 0$. Then the A-based Adams-Novikov spectral sequence is 0 from E^1 on, thus the B-based Adams-Novikov spectral sequence is also trivial from E^2 on. This time, the spectral sequence converges strongly because it is conditionally convergent in the sense of Boardman [Boa99], which implies strong convergence if the derived E_∞-term is 0 – but this is automatic since the E_r-terms are all trivial for $r \geq 0$. Thus $X\hat{_B}$ is contractible.

Now the Hurewicz map $X \to B \wedge X$ factors as $X \to L_B X \to X\hat{_B} \to B \wedge X$ by the universal property ① of the localization, since $X\hat{_B}$ is B-local. Thus $X \to B \wedge X$ is trivial. Using the ring spectrum structure on B, we see that $B \wedge X \to B \wedge B \wedge X \xrightarrow{\mu} B \wedge X$, which is the identity, is also trivial, so $B \wedge X \simeq *$. ☐

In particular, this applies to the following cases:

Theorem 3.4. *We have*

$$L_{B(n)} \simeq L_{K(n)}$$

Let $I_n = (p, v_1, \dots, v_{n-1}) \lhd BP_*$ and $E(k,n) = E(n)/I_k$ for $0 \leq k \leq n \leq \infty$. Then

$$L_{v_k^{-1}E(k,n)} \simeq L_{K(k)}.$$

Proof. The first part is due to Ravenel [Rav84] and Johnson-Wilson [JW75], but they give a different proof without the Adams-Novikov spectral sequence.

To apply Theorem 3.3, it is useful to extend the ground ring of the homology theories in question from $\mathbb{F}_p$ to $\mathbb{F}_{p^n}$, which does not change their localization functors. The Hopf algebroids for $B(n) \otimes \mathbb{F}_{p^n}$ and $K(n) \otimes \mathbb{F}_{p^n}$ both classify formal groups of height n. By Lazard's theorem, there is only one such group over $\mathbb{F}_{p^n}$ up to isomorphism, which shows that the quotient map $B(n) \otimes \mathbb{F}_{p^n} \to K(n) \otimes \mathbb{F}_{p^n}$ induces an isomorphism of Hopf algebroids.

The second part works similarly by considering the maps of Hopf algebroids induced from

$$v_k^{-1}E(k,n) \leftarrow B(k)/(v_{n+1}, v_{n+2}, \dots) \to K(k)$$

which again all represent the stack of formal groups of height k.

☐

Theorem 3.5. *We have that*

$$L_{E(n)} \simeq L_{K(0) \vee K(1) \vee \cdots \vee K(n)} \simeq L_{v_n^{-1}BP}.$$

Proof. With the notation of Theorem 3.4, since $E(n,n) = K(n)$ and $E(0,n) = E(n)$, it suffices to show that

$$L_{E(k,n)} \simeq L_{K(k) \vee E(k+1,n)}.$$

By Lemma 1.3, $L_{E(k,n)} \simeq L_{v_k^{-1}E(k,n) \vee E(k+1,n)}$. By Theorem 3.4, $L_{v_k^{-1}E(k,n)} \simeq L_{K(k)}$, and the result follows by induction.

The second equivalence can be proved by a similar argument, not needed in this paper, and left to the reader. ☐

Theorem 3.6. *There is a homotopy pullback square*

$$\begin{array}{ccc} L_{K(1)\vee K(2)}X & \xrightarrow{\eta_{K(2)}} & L_{K(2)}X \\ {\scriptstyle \eta_{K(1)}}\downarrow & & \downarrow{\scriptstyle \eta_{K(1)}} \\ L_{K(1)}X & \xrightarrow{L_{K(1)}(\eta_{K(2)})} & L_{K(1)}L_{K(2)}X \end{array}$$

Proof. This is an application of Prop. 2.2. We need to see that $K(2)_*(L_{K(1)}X) = 0$ for any X. To see this, let $\alpha\colon \Sigma^k M(\mathbb{Z}/p) \to M(\mathbb{Z}/p)$ be the Adams map, which induces multiplication with a power of v_1 in $K(1)$ and is trivial in $K(2)$. Here $k = 2p-2$ for odd p and $k = 8$ for $p = 2$).

Let X be $K(1)$-local. Then so is $X \wedge M(\mathbb{Z}/p)$, and since $\Sigma^k X \wedge M(\mathbb{Z}/p) \xrightarrow{\alpha} X \wedge M(\mathbb{Z}/p)$ is a $K(1)$-isomorphism, it is a homotopy equivalence. On the other hand, $\alpha_*\colon K(2)_*(\Sigma^k X \wedge M(\mathbb{Z}/p)) \to K(2)_*(X \wedge M(\mathbb{Z}/p))$ is trivial, thus $K(2)_*(X \wedge M(\mathbb{Z}/p)) = 0$. By the Künneth isomorphism, $K(2)_*(X) = 0$. □

The same result holds true for any $K(m)$ and $K(n)$ with $m < n$; $M(\mathbb{Z}/p)$ and α then have to be replaced by a type-m complex and its v_m-self map in the argument. We briefly recall some basic facts around the periodicity theorem.

Definition 3.7. A finite p-local CW-spectrum X has *type n* if $K(n)_*(X) \neq 0$ but $K(k)_*(X) = 0$ for $k < n$. For example, the sphere has type 0, the Moore spectrum $M(\mathbb{Z}/p)$ has type 1, and the the cofiber of the Adams map has type 2.

Theorem 3.8 ([DHS88, HS98]). *Every type-n spectrum X admits a v_n-*self map*, i. e. a map $f\colon \Sigma^? X \to X$ which induces multiplication by a power of v_n in $K(n)_*(X)$.*

The periodicity theorem implies that there exist type-n complexes for every $n \in \mathbb{N}$. They can be constructed iteratively, starting with the sphere, by taking cofibers of v_k-self maps. Thus, there exist multi-indices $I = (i_0, \dots, i_{n-1})$ and spectra $\mathbb{S}^0/(v^I)$ such that $BP_*(\mathbb{S}^0/(v^I)) = BP_*/(v^I)$, where $(v^I) = (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$. These are sometimes called *generalized Moore spectra*. It is an open question what the minimal values of I are (they certainly depend on the prime.)

4. The Hasse square

In this section, we will study algebraic interpretations of $K(n)$-localization in terms of formal groups and elliptic curves.

Proposition 4.1. *Let E be a complex oriented ring spectrum and define*

$$E' = \operatorname*{holim}_{(i_0,\dots,i_{n-1})\in\mathbb{N}^n} v_n^{-1}E/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}).$$

Then $L_{K(n)}E \simeq E'$.

Proof. As $v_n^{-1}E/I_n$ is a $B(n)$-module spectrum, it is $B(n)$-local by Lemma 1.2, thus by Theorem 3.4 also $K(n)$-local. Each spectrum $v_n^{-1}E/(v^I)$ (using multi-index notation) is constructed from this by a finite number of cofibration sequences, thus it is also $K(n)$-local. Since homotopy limits of local spectra are again local (Lemma 1.4), E' is $K(n)$-local, and it remains to show that $K(n)_*(E) \cong K(n)_*(E')$. The coefficient rings of the Morava K-theories $K(n)$ are graded fields, hence they have Künneth isomorphisms. Thus it suffices to show that $E \wedge X \to E' \wedge X$ is a $K(n)$-equivalence for some X with nontrivial $K(n)_*(X)$. Choose $X = \mathbb{S}^0/(v^J)$ to be a generalized Moore spectrum of type n, for some multi-index J. Then

$$E' \wedge X \simeq \operatorname*{holim}_{I\in\mathbb{N}^n} \left(v_n^{-1}E/(v^I) \wedge \mathbb{S}^0/(v^J)\right) \simeq v_n^{-1}E/(v^J).$$

Thus $K(n)_*(E \wedge X) = K(n)_*(E/v^J) = K(n)_*(v_n^{-1}E/v^J) = K(n)_*(E' \wedge X)$. □

Now we will specialize to an elliptic spectrum E defined over the ring E_0 with associated elliptic curve C_E over $\operatorname{Spec} E_0$. Proposition 4.1 in particular tells us that

$$\pi_0 L_{K(1)} E \cong \lim_i v_1^{-1} E_0/(p^i),$$

which is the ring of functions on $\operatorname{Spf}((E_0)^{\hat{}}_p)^{\mathrm{ord}}$, the ordinary locus of the formal competion of $\operatorname{Spec} E_0$ at p, i.e. the sub-formal scheme over which C_E is ordinary. In particular, if E_0 is an $\mathbb{F}_p$-algebra, $\pi_0 L_{K(1)} E \cong v_1^{-1} E_0$ is just the (non-formal) ordinary locus of E_0. Similarly,

$$\pi_0 L_{K(2)} E \cong \lim_{i_0, i_1} v_2^{-1} E_0/(p^{i_0}, v_1^{i_1}) = \lim_{i_0, i_1} E_0/(p^{i_0}, v_1^{i_1})$$

is the ring of functions on the formal completion of $\operatorname{Spec} E_0$ at the supersingular locus at p. The last equality holds because any elliptic curve has height either 1 or 2 over $\mathbb{F}_p$, thus v_2 is a unit in $E_0/(p, v_1)$ and hence in $E_0/(p^{i_0}, v_1^{i_1})$.

Lemma 4.2. *Any p-local elliptic spectrum E is $E(2)$-local.*

Proof. We need to show that for any W with $E(2)_* W = 0$, we have that $E_* W = 0$. By Theorems 3.4 and 3.5, this is equivalent to $B(i) = 0$ for $0 \leq i \leq 2$. That is,

$$\begin{aligned} p^{-1} BP \wedge W &\simeq * \\ v_1^{-1} BP/p \wedge W &\simeq * \\ v_2^{-1} BP/(p, v_1) \wedge W &\simeq *. \end{aligned}$$

Now since E is a BP-ring spectrum, the same equalities hold with BP replaced by E. It follows from Lemma 1.3 that

$$\begin{aligned} E/(p, v_1) \wedge W \simeq v_2^{-1} E/(p, v_1) \wedge W \simeq * \text{ and } v_1^{-1} E/p \wedge W \simeq * \quad &\Rightarrow \quad E/p \wedge W \simeq * \\ E/p \wedge W \simeq * \text{ and } p^{-1} E \wedge W \simeq * \quad &\Rightarrow \quad E \wedge W \simeq *. \end{aligned}$$

□

Corollary 4.3 (the "Hasse square"). *For any elliptic spectrum E, there is a pullback square*

$$\begin{array}{ccc} E^{\hat{}}_p & \longrightarrow & L_{K(2)} E \\ \downarrow & & \downarrow \\ L_{K(1)} E & \longrightarrow & L_{K(1)} L_{K(2)} E. \end{array}$$

Proof. It follows from Lemma 3.6 that the pullback is $L_{K(1) \vee K(2)} E$. Now consider the arithmetic square

$$\begin{array}{ccc} L_{K(0) \vee K(1) \vee K(2)} E & \longrightarrow & L_{K(1) \vee K(2)} E \\ \downarrow & & \downarrow \\ L_{K(0)} E & \longrightarrow & L_{K(0)} L_{K(1) \vee K(2)} E. \end{array}$$

Since $L_p L_{K(0)} X = L_p L_{\mathbb{Q}} X = *$, applying the p-completion functor L_p, we see that top horizontal map

$$L_p E \simeq L_p L_{K(0) \vee K(1) \vee K(2)} E \to L_p L_{K(1) \vee K(2)} E \simeq L_{K(1) \vee K(2)} E$$

is an equivalence, hence the result. □

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