Notes from "On Braids and Cobordism Theories," given on February 6, 2022 at the University of Glasgow

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1 Preliminaries

This talk primarily concerns the category of *spectra*, denoted $\mathcal{S}p$, and the category of ∞*-groupoids*, denoted $\mathcal{S}$, which I'll often refer to as *spaces*. I will implicitly be using the ∞-categorical manifestations of these objects, in the sense of [Lur14, Lur09], but specific knowledge of that formalism is not necessary in the slightest. Indeed, most of the contents of this talk is essentially "classical." This talk represents joint work with Jack Morava [BM18]. It is also available on the arXiv at `https://arxiv.org/abs/1710.05992`.

Remark 1.1. I won't try to define the category of spectra but here are some important facts about it:

1. Spectra are essentially the same as *generalized homology theories*, i.e. functors from spaces to graded abelian groups $h_*\colon \mathcal{S} \to GrAb$ satisfying the *Eilenberg-Steenrod axioms.*

2. Probably the most well-known spectrum is the one which represents singular homology with coefficients in an abelian group $X \mapsto H_*(X; A)$, which we'll denote by HA. These are called Eilenberg-MacLane spectra. There is also the stable homotopy groups functor $X \mapsto \pi_*^{st}(X)$, denoted $\mathbb{S}$, which approximates the usual homotopy groups of spaces. Finally, there are the topological K-theory functors, denoted KU and KO, for complex and real topological K-theory. These last are more naturally *cohomology* theories, but every spectrum gives both a homology and cohomology theory anyway.

3. Spectra have a closed symmetric monoidal structure called the *smash product*, often denoted $\wedge$. The tensor unit for this structure is the sphere spectrum $\mathbb{S}$, the generalized homology theory for stable homotopy groups. For a spectrum E I'll denote the right adjoint to $E \wedge -$ by $Map(E, -)$.

4. There is symmetric monoidal functor from pointed spaces to spectra, $\Sigma^\infty \colon \mathcal{S}_* \to \mathcal{S}p$. I might want to precompose with the free pointing functor $\mathcal{S} \to \mathcal{S}_*$, in which case I'll write Σ^∞_+.

5. Spectra have *homotopy groups* which roughly generalize the stable homotopy groups of spaces. I'll just write $\pi_* \colon \mathcal{S}p \to GrAb$ for this functor if we need it, and whether I mean the homotopy groups of a spectrum or a space should be clear from context.

6. By including spaces into spectra we can literally "represent" (co)homology theories. Specifically, given a spectrum E, there are: the homology theory associated to E which is computed as $\pi_*(\Sigma^\infty_+ X \wedge E)$; and the cohomology theory associated to E, which is computed as $\pi_*(Map(\Sigma^\infty_+ X, E)$. In particular, $\pi_*(\Sigma^\infty_+ X \wedge \mathbb{S}) \cong \pi_*(\Sigma^\infty_+ X) \cong \pi_*^{st}(X)$, $\pi_*(Map(\Sigma^\infty_+ X, HA)) \cong H^*(X; A)$ and $\pi_*(\Sigma^\infty_+ X \wedge HA) \cong H_*(X; A)$.

2 Thom Spectra

The sphere spectrum $\mathbb{S}$ has an infinite loop space of homotopy automorphisms which I'll denote by $GL_1(\mathbb{S})$. Its delooping, or classifying space, is $BGL_1(\mathbb{S})$. One of the nice things about working with ∞-categories is that spaces, i.e. ∞-groupoids, and ∞-categories, are all on the same footing now. So $BGL_1(\mathbb{S})$ is just a special kind of ∞-category (one in which all of the morphisms are equivalences), so it makes sense to talk about functors out of it. And, indeed, there is a very special functor $BGL_1(\mathbb{S}) \to \mathcal{S}p$ which describes the action of the homotopy automorphisms of $\mathbb{S}$ on $\mathbb{S}$ itself. See [ABG+14] for more about this.

Definition 2.1. Given a space X and a map $f \colon X \to BGL_1(\mathbb{S})$, write Mf for the colimit of the composite functor $X \to BGL_1(\mathbb{S}) \to \mathcal{S}p$.

Remark 2.2. In the case that X is connected we have $X \simeq B\Omega X$ so the map $X \to BGL_1(\mathbb{S})$ describes an action of the ∞-group ΩX on $\mathbb{S}$. In that case, I might write $\mathbb{S}/\!\!/\Omega X$ for Mf since it is quite literally the (homotopy) orbits spectrum associated to the action coming from $\Omega X \to GL_1(\mathbb{S})$.

Theorem 2.3 ([ABG+14]). *The above notion of Thom spectrum recovers the classical notion of the Thom spectrum associated to a stable spherical bundle. In particular:*

1. *If $X \simeq BO$ and $BO \to BGL_1(\mathbb{S})$ is the J-homomorphism then $MJ = \mathbb{S}/\!\!/O \simeq MO$, the spectrum of unoriented cobordism.*

2. *If $X \simeq BU$ and $BU \to BO \to BGL_1(\mathbb{S})$ is the (delooping of the colimit of) the usual inclusion $U(n) \to O(2n)$, the resulting Thom spectrum is the complex cobordism spectrum $MU \simeq \mathbb{S}/\!\!/U$.*

3 Eilenberg-MacLane spectra as Thom spectra

Recall that $\pi_1(BO) \cong \mathbb{Z}/2$, whcih allows us to make the following definition:

Definition 3.1. Writing $\eta\colon S^1 \to BO$ for the generator, let $\overline{\eta}\colon \Omega^2\Sigma^2 S^1 \to \Omega^2\Sigma^2 BO \to BO$ be the map of $\mathbb{E}_2$-spaces generated by the adjunction between $\mathbb{E}_2$-spaces and pointed connected spaces, i.e. the image of η under the isomorphism $Hom_{\mathcal{S}_*}(S^1, BO) \cong Hom_{\mathbb{E}_2}(\Omega^2 S^3, BO)$.

Theorem 3.2 (Mahowald [Mah79]). *There is an equivalence of $\mathbb{E}_2$-ring spectra $M\eta \simeq H\mathbb{Z}/2$. Moreover, if $p\colon \Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^3$ is the universal cover then there is an equivalence of ring spectra $M(\eta p) \simeq H\mathbb{Z}$.*

Remark 3.3. Note that the above theorem can be interpreted as saying that $H\mathbb{Z}/2$ and $H\mathbb{Z}$ are the cobordism spectra for manifolds whose stable normal bundles admits lifts:

$$\begin{tikzcd} & \Omega^2 S^3 \arrow[d, "\eta"] \\ M \arrow[r] \arrow[ur, dashed] & BO \end{tikzcd} \qquad \begin{tikzcd} & \Omega^2(S^3\langle 3\rangle) \arrow[d, "p\eta"] \\ M \arrow[r] \arrow[ur, dashed] & BO \end{tikzcd}$$

Now let $B\mathcal{B}_\infty = colim_n(BB_n)$, where B_n is the n-stranded braid group (which includes into the $(n+1)$-stranded braid group by inserting an unbraided strand on the right side). Then:

Theorem 3.4 (Cohen, [Coh78]). *There is a commutative diagram*

$$\begin{tikzcd} B\mathcal{B}_\infty \arrow[r, "\rho"] \arrow[d, "\theta"] & BO \\ \Omega^2 S^3 \arrow[ur, "\overline{\eta}"'] & \end{tikzcd}$$

in which ρ is the "underlying permutation map" $B\mathcal{B}_\infty \to B\Sigma_\infty$ followed by the regular representation $B\Sigma_\infty \to BO$, and θ is a homology equivalence.

Remark 3.5. Some consequences of Cohen's theorem are the following:

1. We can think of $H\mathbb{Z}/2$ as the quotient of the sphere spectrum by the infinite stranded braid group $\mathbb{S}/\!\!/\mathcal{B}_\infty$.

2. We can describe homology classes $x \in H_*(X;\mathbb{Z}/2)$ as bordism classes $(M, M \xrightarrow{f} X)$ in which M has a lift of its stable normal bundle:

$$\begin{tikzcd} & B\mathcal{B}_\infty \arrow[d] \\ M \arrow[r] \arrow[ur, dashed] & BO \end{tikzcd}$$

   Cohen calls such lifts "braid orientations" and gives some examples of manifolds that are braid oriented (in particular, the so-called solvmanifolds).

This work is inspired by trying to give a similar interpretation for the case of $\Omega^2(S^3\langle 3\rangle)$.

Construction 3.6. *Consider the short exact sequence of groups*

$$\mathcal{B}_n^0 := [\mathcal{B}_n, \mathcal{B}_n] \overset{i}{\hookrightarrow} \mathcal{B}_n \xrightarrow{w} \mathbb{Z}$$

where w is the "writhe" map taking a braid to the difference of its "left-over-right" crossings and its "right-over-left" crossings. Note that w is of course also the abelianization map of $\mathcal{B}_n$. This induces a "fiber sequence" of monoidal categories

$$\mathbb{B}^0 := \coprod_n B\mathcal{B}_n^0 \xrightarrow{\mathbb{I}} \mathbb{B} := \coprod_n B\mathcal{B}_n \xrightarrow{\mathbb{W}} \coprod_n B\mathbb{Z} \simeq \mathbb{N} \times \mathbb{Z}$$

Note that $\mathbb{I}$ is a monoidal functor and $\mathbb{W}$ is braided monoidal.

The monoidal structures of $\mathbb{B}$ and $\mathbb{B}^0$ are given by addition of natural numbers and juxtaposition of braids. However, the braiding in $\mathbb{B}$, in which $n+m \to m+n$ crosses n strands over m strands, has writhe nm, so $\mathbb{B}^0$ is not braided monoidal. As a result, when we take geometric realization and then group complete we get a commutative diagram

$$\begin{tikzcd} \Omega B|\mathbb{B}^0| \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \Omega B|\mathbb{B}| \arrow[r] & \mathbb{Z} \times S^1 \end{tikzcd}$$

which is a (homotopy) pullback square but *not* of $\mathbb{E}_2$-spaces, only of $\mathbb{E}_1$ spaces.

To deal with this issue, we restrict to the base point components of each space and use:

Theorem 3.7 (Segal). *There are equivalences*

$$\Omega B|\mathbb{B}| \simeq \Omega^2 S^2 \simeq \mathbb{Z} \times \Omega^2 S^3$$

Additionally, we have that the bottom horizontal map in the above pullback diagram, $\Omega B|\mathbb{B}| \to \mathbb{Z} \times S^1$, is the 1-truncation $\Omega^2 S^3 \to \Omega^2 S^3\langle 1\rangle \simeq S^1$ on each connected component.

Now we have a pullback square of $\mathbb{E}_2$-spaces

$$\begin{tikzcd}
\Omega B|\mathbb{B}^0|_0 \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega^2 S^3 \arrow[r] & S^1
\end{tikzcd}$$

This implies the following theorem:

Theorem 3.8 (B.-Morava). *There is an equivalence of $\mathbb{E}_2$-spaces*

$$\Omega B|\mathbb{B}^0|_0 \simeq \Omega^2(S^3\langle 3\rangle)$$

This is analogous Segal's theorem: $\Omega B|\mathbb{B}|_0 \simeq \Omega^2 S^3$. So then there is a natural question about whether or not there is a version of Cohen's theorem in this setting. Specifically:

Question. *Is there a homology equivalence between $\Omega^2(S^3\langle 3\rangle)$ and $B\mathcal{B}^0_\infty$, where $\mathcal{B}^0_\infty$ is the colimit of the writhe-free braid groups $\mathcal{B}^0_n$?*

This would, for instance, give a description of integral homology classes as cobordism classes of manifolds with "writhe-free braid orientations."

References

[ABG+14] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk, *An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology*, J. Topol. **7** (2014), no. 3, 869–893.

[BM18] Jonathan Beardsley and Jack Morava, *Toward a Galois theory of the integers over the sphere spectrum*, J. Geom. Phys. **131** (2018), 41–51. MR3815226

[Coh78] F. R. Cohen, *Braid orientations and bundles with flat connections*, Invent. Math. **46** (1978), no. 2, 99–110. MR493954

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur14] ———, *Higher algebra*, 2014. `math.harvard.edu/~lurie/papers/higheralgebra.pdf`.

[Mah79] Mark Mahowald, *Ring spectra which are Thom complexes*, Duke Math. J. **46** (1979), no. 3, 549–559. MR544245