THE ADAMS SPECTRAL SEQUENCE FOR tmf

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The Adams-Novikov spectral sequence is the easiest way to compute tmf_*, and that is how most people do the computation. However, sometimes you want the classical Adams spectral sequence. That is what is contained in these notes.

1. $p = 3$

The homology of tmf as a comodule over the Steenrod algebra is given by

$$H_*(tmf) = \mathbb{F}_3[b_4, \xi_1^3 - b_4\xi_1, \xi_2, \xi_3, \dots] \otimes E[\tau_3, \tau_4, \dots]$$

where $|b_4| = 8$. Define a Hopf algebroid (B, Γ) by

$$B = E[b_4]$$
$$\Gamma = B[\xi_1] \otimes E[\tau_0, \tau_1, \tau_2]/(\xi_1^3 - b_4\xi_1)$$

This is actually a Hopf algebra, because $\eta_L = \eta_R$. The coproduct formulas are just as in the Steenrod algebra. A change of rings theorem implies

$$\mathrm{Ext}_A(\mathbb{F}_3, H_*(tmf)) = \mathrm{Ext}_\Gamma(B, B).$$

We will compute the polynomial part of the latter with the May spectral sequence. Then we will proceed with a sequence of Bockstein spectral sequences. Let $\Gamma(2)$ denote the subalgebra

$$B[\xi_1]/(\xi_1^3 - b_1\xi_1)$$

The E_1 term of the May spectral sequence for $H^*(\Gamma(2))$ is given by

$$B[b_0] \otimes E[h_0].$$

Here the bifiltrations (s, t) are given by

$$|h_0| = (1, 4)$$
$$|b_0| = (2, 12)$$

There are no differentials, and we have $H^*(\Gamma(2))$. Since we had the relation $\xi^3 = b_4\xi_1$ in Γ, we shall define

$$h_1 := h_0 b_4$$

in $H^*(\Gamma(2))$. Observe that it is the image of the element $h_1 = [\xi_1^3]$ in $H^*(A)$.

Letting $\Gamma(1) = \Gamma(2) \otimes E[\tau_2]$, since τ_2 is primitive modulo (τ_0, τ_1), we have

$$H^*(\Gamma(1)) = H^*(\Gamma(2)) \otimes P[v_2]$$

with $|v_2| = (1, 17)$. For $\Gamma(0)\Gamma(1) \otimes E[\tau_1]$, we have a v_1-BSS

$$H^*(\Gamma(1)) \otimes P[v_1] \Rightarrow H^*(\Gamma(0)).$$

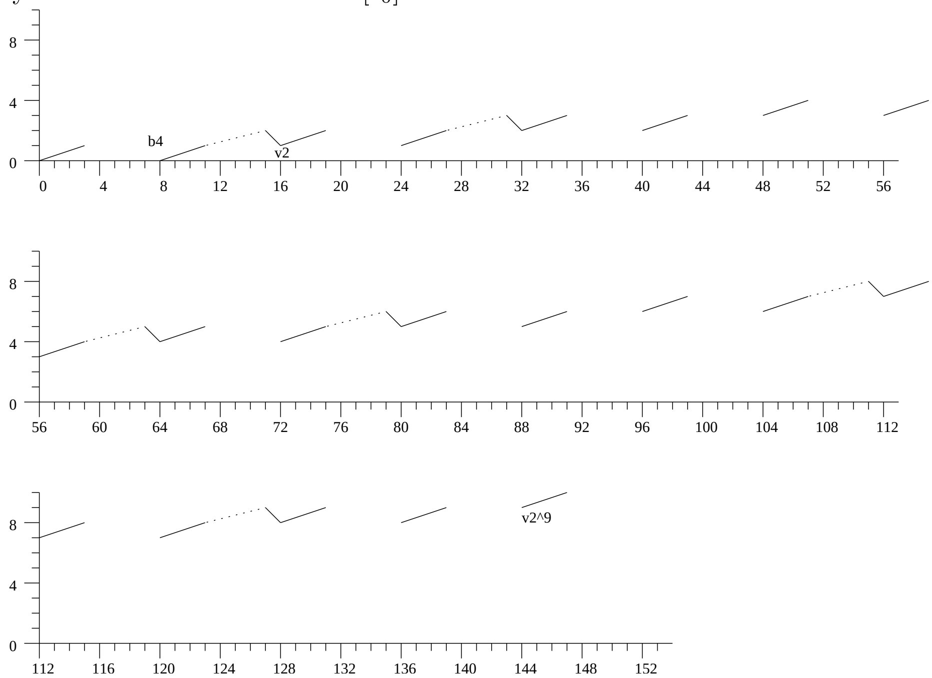
The main differential is

$$d_1(v_2) = v_1 h_1$$

which follows immediately from the coproduct formula for τ_2. It follows that

$$d_1(v_2^i) = \pm h_1 v_1 v_2^{i-1} = \pm h_0 b_4 v_1 v_2^{i-1} \quad \text{for } i \not\equiv 0 \pmod 3.$$

There is no room for any more differentials. The v_1 BSS with differentials is displayed below as a module over $P[b_0]$.



Next we compute the v_0-BSS. The v_0 BSS breaks up into patterns of two types. The first pattern (pattern 1) arises from the elements in $\mathrm{Ext}(tmf \wedge V(0))$ in

$$P[v_0, b_0] \otimes E[h_0]\{v_2^k\}$$

or those in

$$P[v_0, b_0] \otimes E[h_0]\{b_4 v_2^{k-1}\}$$

for a fixed $k \equiv 0 \pmod 3$ The second pattern (pattern 2) arises from the elements of $\mathrm{Ext}(tmf \wedge V(0))$ in

$$P[b_0, v_0] \otimes E[h_0]/(v_0 h_0)\{b_4 v_2^{k-1}\} \oplus P[b_0, v_0]\{v_2^k h_0\}$$

for $k \equiv 1, 2 \pmod 3$.

The basic differential is given by

$$d_1(v_1) = v_0 h_0. \tag{1.1}$$

The fundamental observation is that in the v_0-BSS, v_1^3 is a permanent cycle. This is because after the differential $d_1(b_4 v_1) = v_0 b_4 h_0$, there are no remaining targets for a Bockstein differential supported by v_1^3. Thus, on v_1 multiples, any pattern of differentials is periodic on v_1^3. We shall henceforth refer to v_1^3 as c_6, since that is what it detects in homotopy.

Let us first consider pattern 1, based at $v_2^k b_4^e$ for $e = 0, 1$ and $k \cong -e \pmod p$. There is no target for a Bockstein differential supported by $v_2^k b_4$, so it must be a permanent cycle in the BSS. There are differentials

$$d_1(v_1^i \cdot v_2^k b_4^e) = \pm v_1^{i-1} v_0 h_0 v_2^k b_4^e$$

for $i = 1, 2$ which follow from 1.1 and these are then propagated by multiplication by b_0, and c_4.

Turning our attention to those elements of the form $v_1^i h_0 v_2^k b_4^e$, the only such elements which can support Bockstein differentials are those for which $i \equiv 2 \pmod 3$. This is because for $i \not\equiv 2 \pmod 3$ these elements were targets of Bockstein differentials. There is no room for a d_1. In the E_2 term we can use the Massey product to obtain

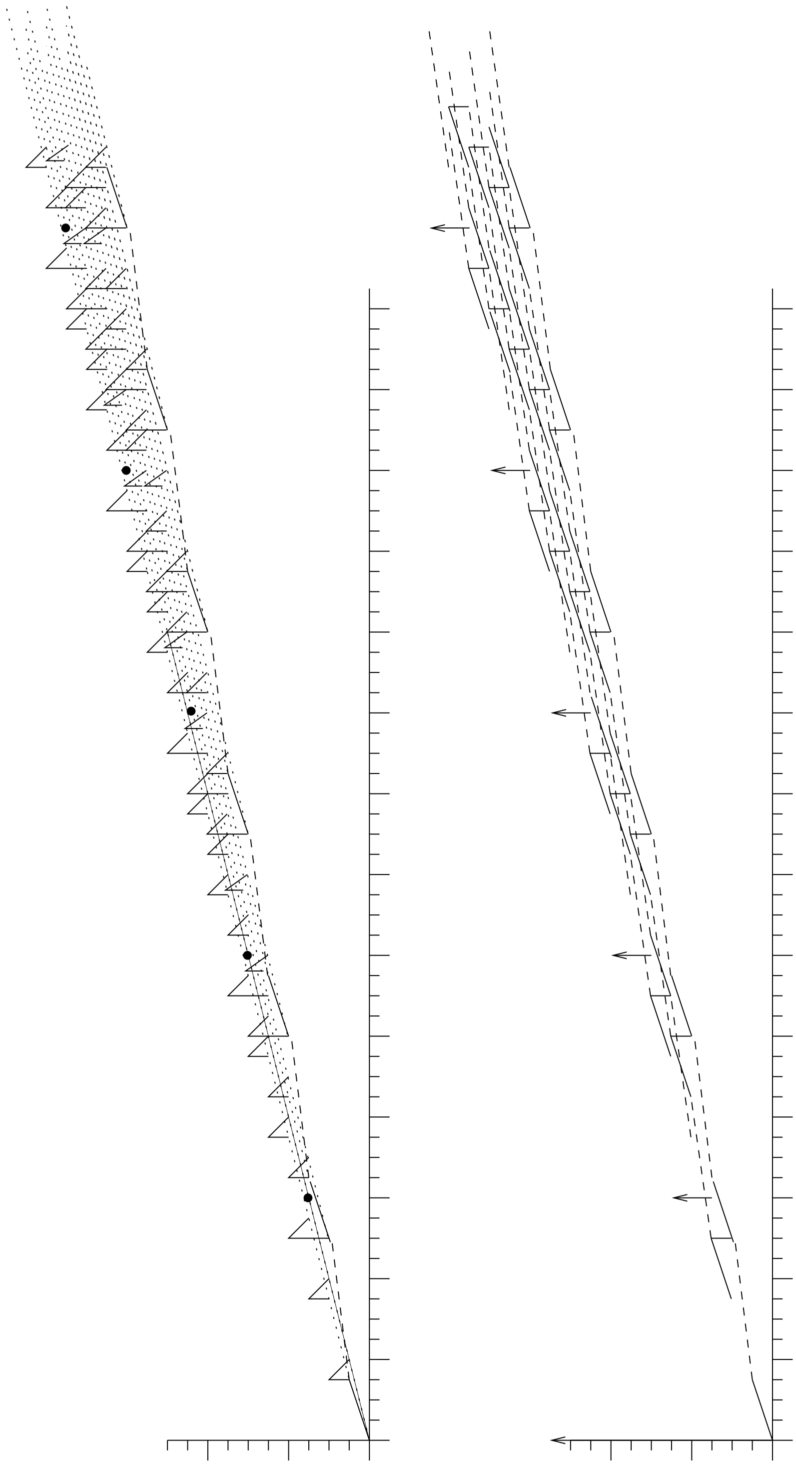
$$\begin{aligned} d_2 v_1^2 h_0 &= \pm d\langle v_1, v_0 h_0, h_0\rangle \\ &= \pm\langle v_0 h_0, v_0 h_0, h_0\rangle \\ &= \pm v_0^2 \langle h_0, h_0, h_0\rangle \\ &= \pm v_0^2 b_0. \end{aligned}$$

Therefore

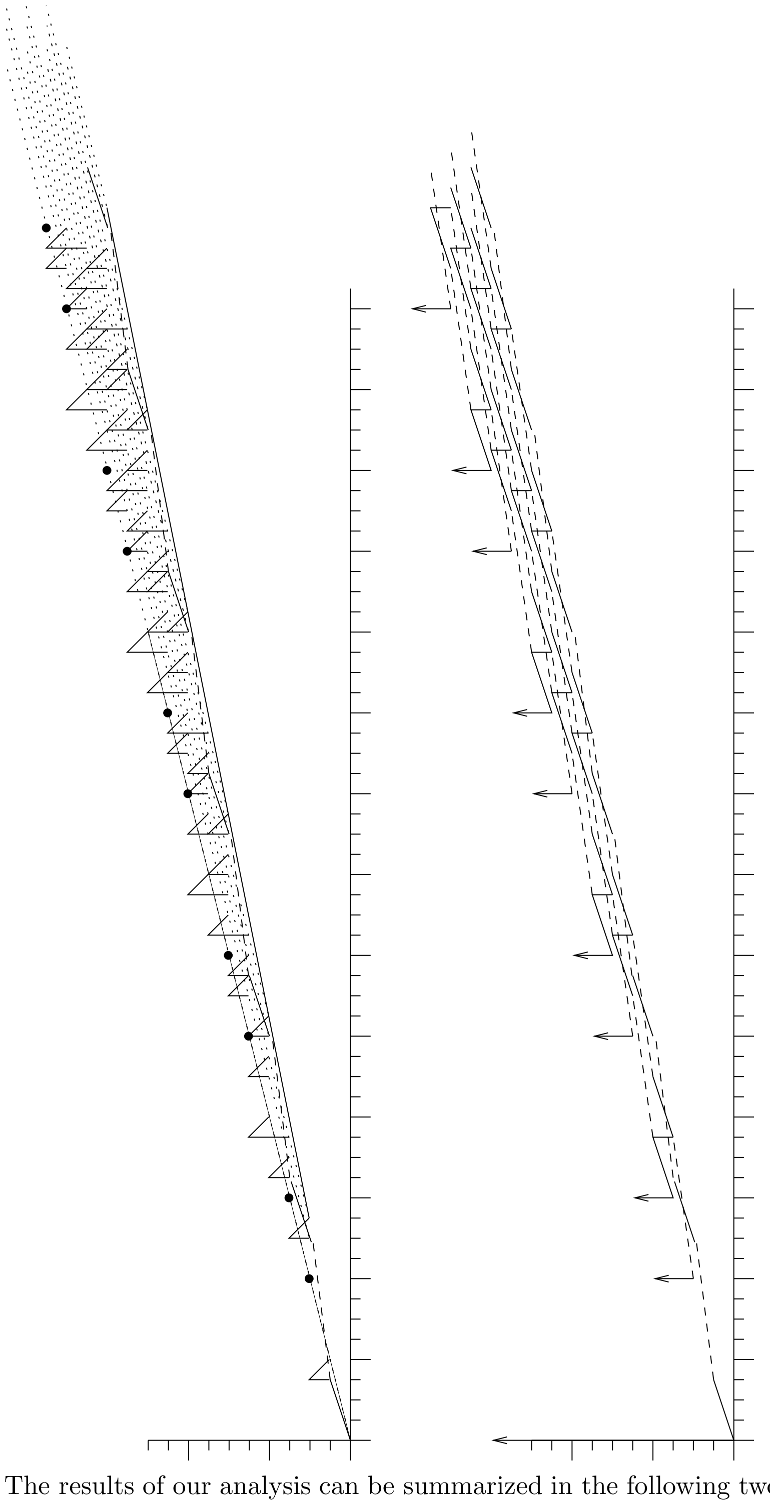
$$d_2 v_1^2 h_0 \cdot v_2^k b_4^e = \pm v_0^2 b_0 \cdot v_2^k b_4^e.$$

This differential is then propagated by c_6 and b_0 multiplication.

The result of these differentials on pattern 1 is displayed below. The top chart displays the Bockstein differentials, and the bottom chart displays the result after the differentials are taken. In the top chart, multiplication by h_0 is given by solid lines, as are multiplications by v_0 and Bockstein differentials. The solid circles indicate that the element supports an infinite v_0 tower, i.e. it is not the target of a Bockstein differential. The Toda bracket $\langle h_0, h_0, -\rangle$ is indicated with a dashed line. Multiplication by b_0 may be implicitly read off from the composition of h_0 multiplication and $\langle h_0, h_0, -\rangle$. In the first chart, the v_1-towers are indicated with dotted lines.

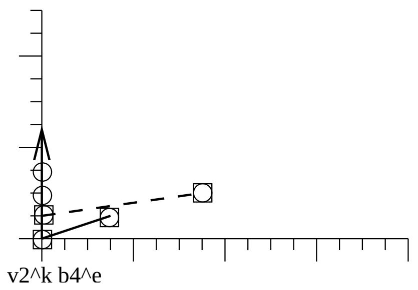


A similar analysis yields the Bockstein differentials on pattern 2. The result is displayed below (here b_0 multiplication is indicated with solid lines).

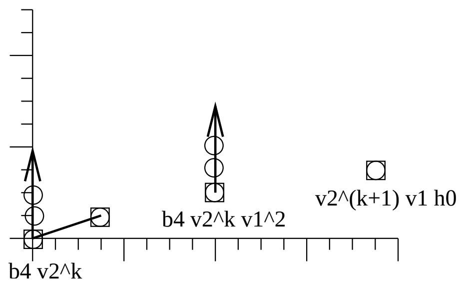


The results of our analysis can be summarized in the following two charts. This is effectively a summary of the ASS E_2-term. In these charts, circles represent a polynomial algebra $P[c_6]$, squares represent a polynomial algebra $P[b_0]$ and circles and squares represent a $P[c_6, b_0]$.

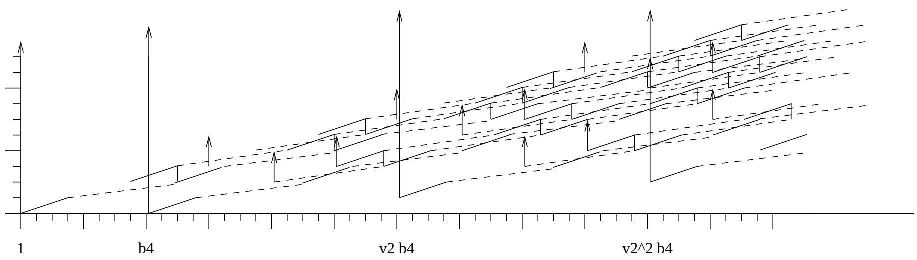
Pattern 1



Pattern 2



For instance, the E_2-term begins in the following manner.



We now determine Adams differentials. We will first determine Adams d_2's. This will leave a pattern that is essentially the Adams-Novikov E_2 term. The remaining differentials are then well known.

The primary differential that we need to consider is

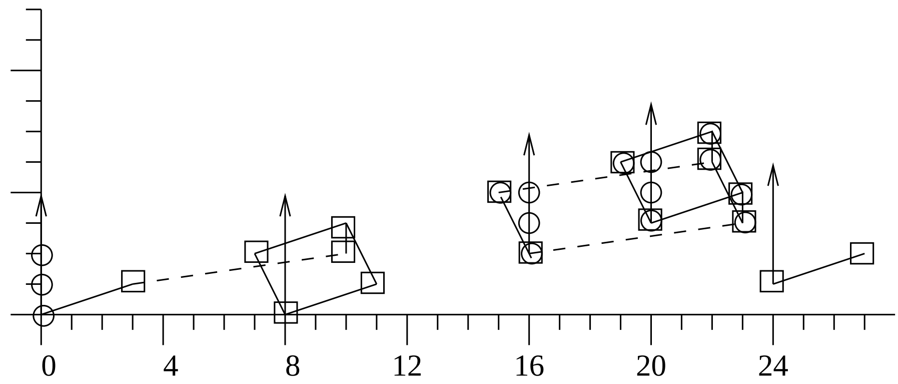
$$d_2 b_4 = v_1 h_0. \tag{1.2}$$

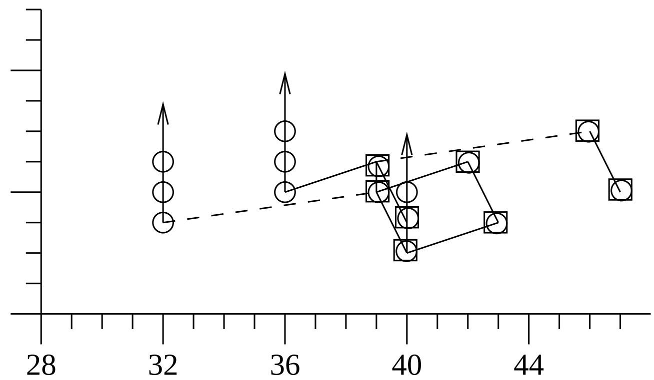
This differential can be deduced from the fact that in the ASS for the sphere

$$d_2 h_1 = v_0 b_0.$$

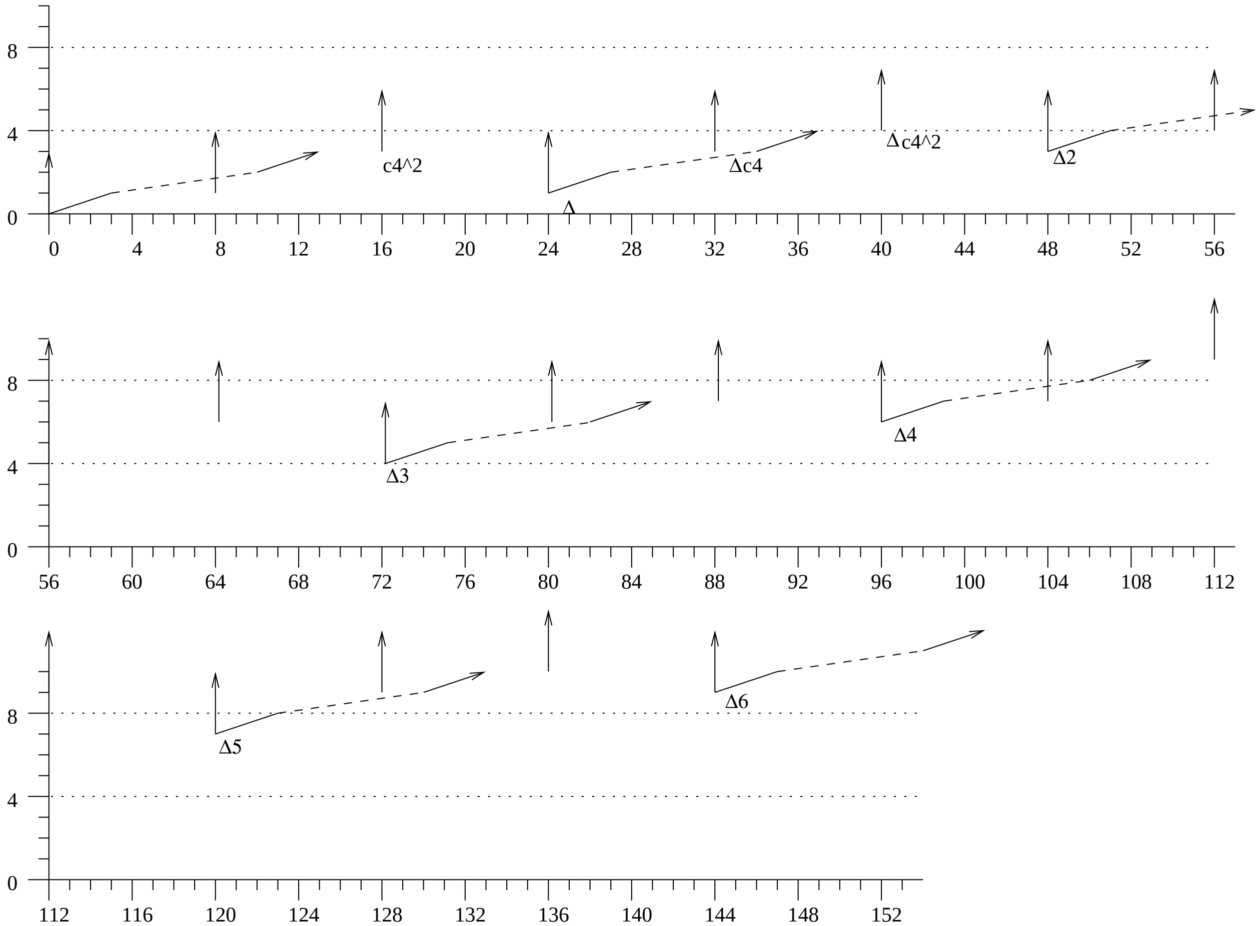
The Hurewicz image of h_1 is $b_4 h_0$, so the differential 1.2 must occur. Alternatively, one can use the formula for $\eta_R(b_4)$ in the elliptic curve Hopf algebroid.

The differential 1.2 propagates to give all of the rest of the d_2's in the ASS. This pattern below is periodic on v_2^3. The notation is the same as before.





What is left is the E_3-term, which is isomorphic to the ANSS. It is displayed below. All of the v_0 towers are periodic on c_6. The labels c_4^i and Δ^j are ANSS names. Most of the multiplicative extensions are not seen in the ASS.


c4^2
Δ
Δc4
Δ c4^2
Δ2
Δ3
Δ4
Δ5
Δ6
0
4
8
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56
56 60 64 68 72 76 80 84 88 92 96 100 104 108 112
112 116 120 124 128 132 136 140 144 148 152

All of the differentials, modulo the different filtrations, are exactly as those in the ANSS. Namely, they are those differentials generated by

$$d_4(\Delta) = h_0 b_0^2$$
$$d_3(\Delta^2) = h_0 b_0^2 \Delta$$
$$d_6(\Delta^2 h_0) = b_0^5$$
$$d_3(\Delta^4) = \Delta^3 h_0 b_0^2$$
$$d_4(\Delta^5) = \Delta^4 h_0 b_0^2$$
$$d_6(\Delta^5 h_0) = \Delta^3 b_0^5$$

These differentials are propagated by b_0 multiplication. The whole spectral sequence is periodic on the permanent cycle $\Delta^3 = v_2^9$.