GEOMETRIC FUNCTION THEORY

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Dear Cafe Patrons,

In this guest post I want to briefly discuss correspondences, integral transforms and their categorification as they apply to representation theory, a topic that might be called geometric function theory. This has been one of the fundamental paradigms of geometric representation theory (together with localization of representations and, on a much grander scale, the Langlands program) for at least the past twenty years (some key names to mention in this context prior to the last decade are Kazhdan, Lusztig, Beilinson, Bernstein, Drinfeld, Ginzburg, I. Frenkel, Nakajima and Grojnowski). These ideas are closely related to topics often discussed in this Cafe (in particular groupoidification and extended topological field theories), and my hope is to facilitate communication between the schools by focussing on some toy examples of geometric function theory and suppressing techincal details. I will conclude self-centeredly by describing my recent work "Integral transforms and Drinfeld centers in derived algebraic geometry" (arXiv:0805.0157) with John Francis and David Nadler, in which we prove some basics of categorified function theory using tools from higher category theory and derived algebraic geometry. (Of course this is out of all proportion to its role relative to the seminal works I glancingly mention or omit, but it provides my excuse to be writing here, so please indulge me.) We were motivated by a desire to understand aspects of one of the more exciting developments of the past five years, namely the convergence of categorified representation theory (in particular the geometric Langlands program), derived algebraic geometry and topological field theory, which was at the center of last year's special program at the IAS. I will not attempt to describe these developments here but refer readers to collected lecture notes on my webpage.

First some references: one of my favorite books is

• N. Chriss and V. Ginzburg, Representation Theory and Complex Geometry. Birkhäuser Boston, Inc., Boston, MA, 1997.

which is an excellent introduction to geometric representation theory. Ginzburg also wrote a shorter survey

• V. Ginzburg, Geometric Methods in Representation Theory of Hecke Algebras and Quantum Groups (arXiv:math/9802004)

of the same material plus some more applications. A wonderful survey (with much overlap with my post) is the following book review by one of the leading experts:

• Ivan Mirković, Book review of Chriss-Ginzburg. Bulletin of the AMS Volume 37, Number 3, available at http://www.ams.org/bull/2000-37-03/S0273-0979-00-00864-8/

Another excellent book covering some related ideas from a different perspective is

• H. Nakajima, Lectures on Hilbert schemes of points on surfaces. University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.

See also the ICM addresses of Ginzburg (1986), Lusztig (1990) and Nakajima (2002). On a more technical note, categorified Hecke algebras and their role in representation theory are the topic of the epic Chapter 7, Hecke Patterns, of the seminal unpublished book (circa 1995)

• A. Beilinson and V. Drinfeld, Quantization of Hitchin's Hamiltonians and Hecke Eigensheaves. Available at http://www.math.uchicago.edu/ mitya/langlands.html

An expansion of much of the material of that chapter (in particular, a discussion of group actions on categories) is available in the appendices of a more recent classic,

• E. Frenkel and D. Gaitsgory, Local geometric Langlands correspondence and affine Kac-Moody algebras. arXiv:math/0508382

This list of course is horribly and offensively incomplete, and I encourage other suggestions in the comments!

I would also like to apologize in advance for the highly informal and imprecise style, inaccuracies and mis- or un-attributions. Finally I want to thank Urs Schreiber and Bruce Bartlett for their encouragement towards the writing of this post.

1. Warmup: Functions on Finite Sets

I'd like to start with a toy model, namely function theory on finite sets. For a finite set we have an unambiguous notion of complex-valued functions $Fun(X)$. Some of the key properties of functions (i.e., of the assignment $X \mapsto Fun(X)$) are that we can multiply functions pointwise, we can pull them back along any map $\pi : X \to Y$, and crucially that we can push them forward along any π, by summing along the fibers of the map. This allows us to use functions $K \in Fun(Z)$ on any correspondence

$$X \leftarrow Z \rightarrow Y$$

as operators (or *integral transforms*) $K * - : Fun(X) \to Fun(Y)$, via the assignment $K * f = (\pi_Y)_*(K \cdot \pi_X^* f)$. K is known as the *integral kernel* representing the integral transform. (Cultural note: practitioners in algebraic geometry and representation theory generally refer to such Z as correspondences – these are of course also what regulars of the Cafe will recognize as spans.)

The universal case of a correspondence is $Z = X \times Y$ with its two projections (any other Z will have a unique map to $X \times Y$), and in this case the construction gives an isomorphism

$$Fun(X \times Y) \simeq Hom_{\mathbb{C}}(Fun(X), Fun(Y))$$

between integral kernels and linear transformations.This is simply the realization that (ordering the points of X and Y) functions on $X \times Y$ are matrices, which we identify with linear transformations.

Likewise, we have a relative version of this construction: if X, X' both map to Y then functions on X and X' become modules over functions on Y, and we can identify $Fun(Y)$-linear maps with functions on the *fiber product* $X \times_Y X'$ (the set of pairs (x, x') whose images in Y are identified):

$$Fun(X \times_Y X') \simeq Hom_{Fun(Y)}(Fun(X), Fun(X')).$$

In matrix notation, these are matrices that are block-diagonal, where the blocks are labelled by points of Y.

Once we have established these basics of function theory, we can use them to study algebras. For example, functions on the square

$$Fun(X \times X) = End_{\mathbb{C}}(Fun(X))$$

form the algebra of square matrices, while functions on the fiber square

$$Fun(X \times_Y X) = End_{Fun(Y)}(Fun(X))$$

form the algebra of block-diagonal square matrices. One useful feature of these matrix algebras is *Morita equivalence*: the category of representations of the algebra of square matrices $Fun(X \times X)$ is equivalent to the category of vector spaces (i.e., every representation is isomorphic to a direct sum of copies of the standard representation $Fun(X)$). Likewise the algebra $Fun(X \times_Y X)$ of block diagonal matrices is Morita equivalent to the commutative algebra $Fun(Y)$ - which we can realize as block-scalar matrices - independently of X (as long as $X \to Y$ is surjective). The functor realizing this equivalence is

$$(-) \otimes_{Fun(Y)} Fun(X) : Fun(Y) - mod \longrightarrow Fun(X \times_Y X) - mod.$$

More geometrically, this equivalence is an example of *descent*. In brief, the category of modules over $Fun(Y)$ can be identified with that of vector bundles on the finite set Y, modules for $Fun(X \times_Y X)$ are identified with vector bundles on X that are fiberwise trivial, and the Morita equivalence is given by pullback of vector bundles from Y to X. This gives a complete picture of the representation theory of block matrix algebras (concretely representations are identified with vector bundles on Y).

2. Functions on finite orbifolds

Continuing with the toy model of finite sets, we can vastly expand our repertoire of examples by taking into account symmetry. Namely, we'll look at finite sets with group actions, and replace all functions by invariant functions. A more compact way to say this is to pass to the quotients, but in order, for example, to push forward correctly (accounting for multiplicities) we must keep track also of the stabilizers. Thus we will consider functions on finite orbifolds, i.e., finite unions of points with finite isotropy groups $\coprod_i pt/G_i$. We may also consider these as finite groupoids, i.e., groupoids with finite isotropy group and finitely many isomorphism classes, or as toy models of *stacks*. The function theory we discussed before extends nicely to this setting, but we need to remember two things. First, a point with stabilizer G is counted with multiplicity inverse to the order of G, so that the pushforward of the function 1 from pt/G to pt is $\frac{1}{|G|}$. Second, fibers (and more general fiber products) of groupoids need to be calculated in the correct categorical fashion: for $X \to Y \leftarrow X'$ the fiber product $X \times_Y X'$ is the groupoid whose objects are triples $(x, x', \gamma : \pi(x) \simeq \pi'(x'))$ consisting of pairs of points and an identification of their images. In particular, to evaluate the pushforward of the function 1 from pt/H to pt/G for a subgroup $H \subset G$, we need to counts the number of points on the (unique) fiber, which is the set of cosets G/H.

Thus we can follow the same rules as above: integral kernels $K \in Fun(Z)$ on a correspondence

$$X \leftarrow Z \to Y$$

of orbifolds give maps $K * - : Fun(X) \to Fun(Y)$. Likewise for a map $X \to Y$ of orbifolds we have a natural algebra structure (convolution relative to Y) on $Vect(X \times_Y X)$.

Let's consider some examples. First let's recall the notion of a group algebra. For a finite group G, functions $Fun(G)$ form an associative algebra under convolution. Namely, given functions $f, g \in Fun(G)$, we first consider the function $f \times g \in Fun(G \times G)$ (i.e., pull f and g back to $G \times G$ along the two projections to G and then multiply). Next we push forward (or sum up) $f \times g$ along the multiplication map $G \times G \to G$, obtaining a new function $f * g \in Fun(G)$. Concretely, in the basis of $Fun(G)$ given by delta functions δ_g ($g \in G$) this product is given by $\delta_g * \delta_h = \delta_{gh}$. The importance of the group algebra comes from its relation to representations: a $\mathbb{C}$-linear representation of G is the same thing as a module for the $\mathbb{C}$-algebra $Fun(G)$.

To fit this into our framework let $X = pt$ and $Y = pt/G$. The fiber product $X \times_Y X = pt \times_{pt/G} pt$ is simply the group G itself, and the convolution structure on $Fun(X \times_Y X)$ is precisely the group algebra structure. Of course this is just a fancy way to say the group algebra is a subalgebra of G by G matrices $Fun(G \times G) = End(Fun(G))$, consisting of functions invariant under the diagonal action of G.

We can expand this example to the case of a finite group G and a subgroup H, giving rise to a map of orbifolds $X = pt/H \to Y = pt/G$. The fiber product $X \times_Y X$ in this case is the double coset orbifold $H\backslash G/H$. Thus we find that functions on double cosets (i.e. H-bi-invariant functions on G) have a natural convolution structure (they form a subalgebra of the group algebra). This algebra

$$\mathcal{H}(G, H) = Fun(H\backslash G/H)$$

is called the *Hecke algebra* for G and H, and can be described as the endomorphism algebra

$$\mathcal{H}(G, H) = End_{G-mod}(Fun(G/H))$$

of the induced representation $Fun(G/H)$. Since the latter represents the functor

$$(-)^H : G - mod \to Vect$$

of H-invariants, the Hecke algebras is the endomorphism algebra of the functor of invariants. More prosaically, the Hecke algebra is precisely what acts on the vector space of H-invariants in any representation of G. This is the key to their importance: we try to understand big representations of a big group G by looking at the smaller vector spaces of invariants for suitably chosen $H \subset G$, with the residual action of the algebra $\mathcal{H}(G, H)$. This is for example the principle behind highest weight theory, and provides a kind of universal pattern for understanding representations. Of course this can't always work - e.g. if $H = G$ the Hecke algebra is trivial, while if H is trivial we gain nothing. We'll see below that the situation is very different after categorification.

3. Topological version

If we replace finite sets by a more involved class of spaces, we have to think harder about what kind of "functions" to consider - there are many different function spaces, and only with some restrictions on the functions or the maps do we have products, pull backs and integration/pushforward. One nice story is a kind of topological function theory, and involves considering compact, oriented manifolds X, and taking the cohomology $H^*(X)$ (say with complex coefficients for consistency) as $Fun(X)$ (another alternative is K-theory, or some other nice cohomology theory with orientations and Künneth formulas). This choice satisfied all the basic properties we wanted for a function theory: we can multiply and pullback cohomology classes, and thanks to Poincaré duality we can also push forward (or integrate). In particular given a correspondence $X \leftarrow Z \rightarrow Y$ and an integral kernel $K \in H^*(Z)$ we get a map $K * - : H^*(X) \rightarrow H^*(Y)$. The Künneth formula and Poincaré duality imply that

$$H^*(X \times Y) \simeq Hom(H^*(X), H^*(Y))$$

, so again all linear maps are given by integral transforms! As for the class of spaces, much of the literature works in the context of algebraic geometry, replacing manifolds with varieties, but this is mostly an issue of language and I will suppress the distinctions.

Things start getting interesting when we consider the equivariant version of this construction. If our space X carries an action of a topological group, we can consider the "invariant functions on X" in the sense of cohomology – this is the equivariant cohomology group $H^*_G(X)$. Equivalently, we may pass to the quotient X/G, but as before consider this quotient in a rich enough context to remember the structure of stabilizers. This means we consider the quotient X/G not as a naked space but as a *stack* (which just means consistently replacing *sets* by *groupoids*). I won't embark on an exposition of stacks, but their formal properties are just as in the finite toy model. The role of the quotient stack X/G is to carry precisely the equivariant geometry of X, no more and no less. In particular the equivariant cohomology (or K-theory) of X is precisely the cohomology (or K-theory) of this new geometric object $H^*_G(X) = H^*(X/G)$, equivariant vector bundles on X are vector bundles on X/G, and so on.

One of the great advances in representation theory (due in large part to some of the names listed at the top) was the realization that many of the algebras of greatest interest can be realized as convolution algebras of the form $Fun(X \times_Y X)$, where X, Y are topological spaces or groupoids (in fact algebraic varieties or stacks) and Fun is a function theory, such as cohomology and K-theory. (Equivalently, we are realizing algebras as the *equivariant* cohomologies or K-theories of spaces with group actions.) Moreover once one realizes an algebra of interest in such fashion there is a relatively straightforward procedure to construct and classify its representations geometrically! (Roughly speaking, we expect some version of the Morita equivalence from the first section to hold, so that representations come from the function theory of the base Y.) Moreover, in many cases one can use the geometry to go much further, in particular finding canonical bases for the representations as the classes of explicit geometric cycles. Among the algebras that have been constructed and studied in such a fashion are Heisenberg and Kac-Moody algebras (more precisely their enveloping algebras), quantum groups, affine Hecke algebras, Cherednik's double affine Hecke algebras, Hall algebras, and many others. The relevant geometries include flag varieties and their cotangent bundles, moduli spaces of instantons, quiver varieties, and various other moduli spaces.

4. Categorification

We would like to consider now the categorification of this story. This again is an old and well established theme in representation theory, with its origins in Grothendieck's function-sheaf dictionary and the work on Kazhdan-Lusztig theory in the early 1980s. Rougly speaking, we will replace numbers by vector spaces, and functions by vector bundles or more generally sheaves.

Let's start again with the toy model of finite sets and orbifolds X, and consider the category $Vect(X)$ of vector bundles on X (of arbitrary, not necessarily constant, rank). We know what this means when X is a finite set, but what about orbifolds? For $X = pt/G$, $Vect(X)$ are G-equivariant vector bundles on a point, i.e. representations of G: $Vect(X) = Rep(G)$. The category of vector bundles on a disjoint union is the product of categories of vector bundles, so we now know what to assign to any X.

Vector bundles always admit a multiplication (tensor product) and pullback. But in this finite situation we can also push vector bundles forward along maps $\pi : X \to Y$. For maps of finite sets this means sum the vector spaces along the fibers, while for maps $pt/G \to pt$ this means take invariants of G-representations. (In general it is a combination of summation and coinduction of representations.) Thus the assignment $X \mapsto Vect(X)$ behaves like a function theory! In particular we can follow the same rules as above: integral kernels $K \in Vect(Z)$ on a correspondence

$$X \leftarrow Z \rightarrow Y$$

of orbifolds give functors $K * - : Vect(X) \to Vect(Y)$. Likewise if $X \to Y \to X'$ are orbifolds and we consider the fiber product $X \times_Y X'$, then we have an equivalence of categories

$$Vect(X \times_Y X') \simeq Funct_{Vect(Y)}(Vect(X), Vect(X'))$$

between integral kernels relative to Y and $Vect(Y)$-linear functors. In particular for $X = X'$ we have a natural monoidal structure (convolution relative to Y) on $Vect(X \times_Y X)$, as a categorified matrix algebra over Y.

Let's consider the categorified analogue of the Hecke algebras we considered above. If $X = pt/H \to Y = pt/G$, we find the *Hecke category* $Vect(X \times_Y X) = Vect(H\backslash G/H)$. The two extreme examples are $H = G$ ($X = Y$), in which case we recover $Vect(pt/G) = Rep(G)$, and H trivial, in which case $X \times_Y X = G$ and we recover the Vect-valued (or categorified) group algebra of G, i.e., $Vect(G)$ with its convolution product induced from the multiplication in G. In fact, results of Müger and Ostrik tell us that the Hecke categories $Vect(H\backslash G/H)$ are all Morita equivalent – i.e., they have equivalent 2-categories of module categories. Note that this result is analogous to the Morita equivalence we discussed before for the block matrix algebra $Fun(X \times_Y X)$ and the block-scalar matrices $Fun(Y)$. However on the level of functions the Morita equivalence fails if we replace finite sets by finite orbifolds: it's certainly not true that the Hecke algebras of functions on double cosets $H\backslash G/H$ are Morita equivalent for all H. Thus we find that by categorifying we've increased the applicability of the Morita equivalence statement from maps of finite sets to maps of finite orbifolds.

One role of this result is to clarify the notion of a G-category. One natural way to define a G-category is as a module category for the categorified group algebra $Vect(G)$. The Morita equivalence tells us that such module categories are identified with modules over the symmetric monoidal category $Rep(G) = Vect(pt/G)$, i.e. a "category over the stack pt/G" (cf. Gaitsgory's paper arXiv:0507192). We will see below in particular how to generalize this statement from finite groups to arbitrary linear algebraic groups over $\mathbb{C}$.

5. Geometric Categorification

We would now like to leave the toy setting of finite orbifolds for the more fertile geometric setting of manifolds or varieties, orbifolds or stacks. Of course we still have a nice notion of vector bundle and these still multiply and pull back, but they no longer push forward. This is precisely analogous to the distinction between functions and distributions or measures: the former pull back, the latter push forward.

Let's consider two illustrative examples. First, a vector bundle on a closed submanifold $Z \subset X$ does not extend functorially to a vector bundle on the full manifold. So we should allow into our function theory more singular objects which are vector bundles on Z and vanish outside of Z. Second, if we consider the map $X \to pt$, what should we assign as the "integral" of the trivial vector bundle on X? In the decategorified situation, a good way to measure spaces (that behaves well under cutting and pasting for example) is to assign to X its Euler characteristic. In the categorified setting, we can assign instead the actual cohomology of X, as a graded vector space, or better, as a complex of vector spaces (with zero differentials). Thus the Euler characteristic map allows us to categorify integers (not just positive integers) by complexes of vector spaces. In fact since we have a graded vector space, or complex, we can get more than the Euler characteristic - we can form the Poincaré polynomial or graded character, which is a Laurent polynomial in q with positive integer coefficients, whose value at -1 is the Euler characteristic. (A note to purists: in typical geometric situations, we will find bigraded vector spaces with a cohomological gradation, with indexing variable t, and an additional "weight" decomposition, for which we ought to reserve q. However in the most common situations in representation theory the two gradings coincide.)

The result is that one can define nicely behaved categorified function theories for arbitrary spaces, varieties or stacks by generalizing vector bundles to *complexes of sheaves*. We can think of them as functions whose values are complexes of vector spaces (categorified analogues of integers or q-series). So we will replace vector spaces of functions by *derived categories* of sheaves, whose objects are complexes of sheaves of various kinds. As in classical harmonic analysis, in this categorified function theory there are many kinds of smoothness, analyticity, boundedness, integrability and other analytic conditions to impose on sheaves or complexes of sheaves. These give rise to different kinds of derived categories of (quasi)coherent sheaves, perverse sheaves, $\mathcal{D}$-modules, etc.

I would like to black-box the details of these categorified function theories, and just briefly hint at their crucial role in modern representation theory. One of the central points of Kazhdan-Lusztig theory, as developped in the late '70s and early '80s, is the realization of Hecke algebras as a decategorification of a Hecke category, built out of the convolution of sheaves on $B\backslash G/B = pt/B \times_{pt/G} pt/B$ where G is a semisimple (or reductive) group and B its Borel subgroup. Here the Hecke algebras are q-analogues of the group algebra of the Weyl group of G, and the q as comes naturally by decategorification as above. Beilinson and Bernstein realized that different categorified function theories on the flag variety G/B were equivalent to the representation theory of the Lie algebra of G and of all its real forms, with tremendous consequences. These categorifications are one of the main inspirations for the definition of Khovanov homology, one of the most popular forms of categorification today. In another (related) direction, Lusztig categorified the notion of character of a representation, creating the theory of *character sheaves* with remarkable applications to the representation theory of finite groups of Lie type. More recently, and inspired by these ideas, the geometric Langlands program aims at a categorified version of harmonic analysis on moduli spaces of bundles on Riemann surfaces. The main algebraic objects in this harmonic analysis are Hecke categories, made up of sheaves on double coset spaces for loop groups. And as it turns out all of these examples fit very naturally into the framework of extended topological field theory!

6. Postlude

I wanted to conclude with a brief synopsis of my work "Integral transforms and Drinfeld centers in derived algebraic geometry" (arXiv:0805.0157) with Francis and Nadler. Our goal was to provide some basic foundations for categorified harmonic analysis, motivated by applications in geometric representation theory and topological field theory. Because of their importance in applications, we would like to be able to work with derived categories of sheaves with the same facility and comfort level that one is used to working with ordinary function spaces. In particular we'd like to understand categorical representations (group actions on categories) and their characters. While there are nice operations of tensor product, pullback and pushforward on complexes of sheaves, it is widely recognized that it is essentially impossible to really do algebra with derived categories.

It has long been recognized that triangulated categories (of which derived categories are examples) are terribly behaved under very basic algebraic operations - even passing to internal hom (functor categories) takes you out of the triangulated world! Luckily, there is a beautiful answer to this conundrum – rather than considering complexes of sheaves as forming a triangulated category, one ought to consider them as forming an object known as a *stable* $(\infty, 1)$-*category*. (I will risk offending Cafe regulars by dropping the $(-, 1)$ notation henceforth..) Jacob Lurie has developed very complete foundations of noncommutative and commutative algebra in this homotopical context. In fact this algebra provides the foundations for *derived algebraic geometry*, a synthesis of homotopy theory and algebraic geometry developed by Toën-Vezzosi and Lurie (our work relies very heavily on these foundations and the guidance of their developers, for which I'm very grateful!). I'd like to refer interested readers to the long introduction and preliminaries section of our paper, where these ideas are explained much more expansively.

While our main applications involve sheaves on simple spaces like flag varieties and classifying spaces of groups, it is useful to work in much greater generality. One way to formulate the basic idea of derived algebraic geometry is to extend the world of varieties in two directions simultaneously: by allowing good quotients and more general simplicial constructions (leading to higher stacks), as well as good intersections, fiber products and more general cosimplicial constructions (leading to derived schemes and stacks). We work throughout in the context of homotopical algebra, which means roughly that all operations are derived: before performing any operation to an object you first replace it by a nice resolution. We consider a class of spaces (perfect stacks) which admit such natural operations as quotients, products, and fiber products and contain as examples all algebraic varieties, all common stacks over $\mathbb{C}$ (such as finite orbifolds, moduli of bundles on Riemann surfaces, classifying spaces of affine algebraic groups etc), and many objects of a more mysterious nature appearing from homotopy theory (for example, there is a perfect stack associated to any E_∞-ring spectrum, such as the sphere spectrum or complex cobordism theory). On this class of spaces we consider a particularly nice and natural geometric function theory – we assign to each X the collection of complexes of quasicoherent sheaves on X, which forms a stable ∞-category $QC(X)$. In the case of a finite orbifold, quasicoherent sheaves are simply vector bundles, so we are indeed generalizing the previous discussion! (A paper in preparation with Nadler proves some analogous results for a more difficult function theory, that of $\mathcal{D}$-modules.)

This assignment $X \mapsto QC(X)$ satisfies all the basic properties of the toy function theories we considered before: we have natural operations of multiplication (tensor product) making $QC(X)$ a categorified commutative ring (a symmetric monoidal ∞-category), and pullback and pushforward functors associated to maps $X \to Y$, satisfying standard properties (such as various adjunctions). Thus in particular for a correspondence

$$X \leftarrow Z \rightarrow Y$$

we obtain a functor $K \mapsto K*-$ from $QC(Z)$ to $Hom(QC(X), QC(Y))$, the ∞-category of continuous (i.e., colimit preserving) functors.

Our results extend all the basic properties of function theories that we saw before for finite orbifolds to this general setting. The first result shows that *all* continuous functors are given by integral transforms, sometimes known in this context as Fourier-Mukai transforms:

• The ∞ category of continuous functors $QC(X) \to QC(X')$ is equivalent to the ∞ category of integral transforms $QC(X \times X')$.

(This generalizes a theorem of Toën from schemes to perfect stacks.)

• The relative version also holds: for X, X' over Y, Y-linear maps from $QC(X)$ to $QC(X')$ are given by relative integral transforms, i.e., sheaves on the fiber product:

$$QC(X \times_Y X') \simeq Hom_{QC(Y)}(QC(X), QC(X')).$$

For $X \to Y$ surjective, $QC(X \times_Y X)$ (i.e., categorified Y-block diagonal matrices on X) forms an associative algebra (really monoidal ∞-category) under convolution, or via the identification with

$End_{QC(Y)}(QC(X))$. We then prove a generalization of the Morita equivalence for finite orbifolds above:

• The algebras (monoidal ∞-categories) $QC(X \times_Y X)$ for varying $X \twoheadrightarrow Y$ are all Morita equivalent: their $(\infty, 2)$-categories of module categories are all equivalent to $QC(Y) - mod$, the $(\infty, 2)$-category of "Y-linear ∞-categories".

This last assertion is in fact proven in a sequel paper in preparation, also with Francis and Nadler - the paper online only proves that these algebras have the same *Drinfeld center* (Hochschild cohomology category) independently of X. The Morita equivalence result means in particular that if we can represent a monoidal category as a convolution algebra for some map $X \to Y$ then we have a complete description of its representation theory (which is also further developed in the sequel, as part of a "spectral theory" for symmetric monoidal ∞-categories).

Let's conclude by returning to the example of Hecke categories. Now G is an arbiratry linear algebraic group (i.e. a group that can be embeeded in some GL_n) in characteristic zero, and H is a subgroup. Then the Hecke categories $QC(H\backslash G/H)$ are all Morita equivalent. In other words, the categorical representation theory of all the Hecke categories is equivalent. This is a statement about categorical representation theory of G. Namely, as with any notion of representation of a continuous group, we need to specify what kind of regularity we impose on the representations. This is nicely encoded by our choice of function theory on G - we can define different classes of representations as modules over different forms of the group algebra $Fun(G)$. In the case of algebraic groups G and actions on derived categories, there are two natural choices: we can look at module categories for the quasicoherent group algebra $QC(G)$ of G, as we do in this paper, or at its more topological counterpart, given by $\mathcal{D}$-modules on G, as we do in a forthcoming paper with Nadler. The Morita equivalence statement above identifies quasicoherent G-actions on stable ∞-categories with the notion of ∞-categories over the classifying space BG. Moreover it tells us that we lose no information about a quasicoherent G-action on an ∞-category by passing to H-equivariant objects (which is the corresponding module for the Hecke category $QC(H\backslash G/H)$). Our description of Hochschild (co)homology for these monoidal ∞-categories allows us to define *characters* (categorical traces) for representations, which are categorified class functions on G (objects of $QC(G)$ equivariant for the conjugation action). In the $\mathcal{D}$-module case this recovers Lusztig's character sheaves.

Thank you for reading!

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