MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Almost Ring Theory I - Bhargav Bhatt

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Summary: This talk is an overview of the subject of "almost ring theory", which is an important technical tool needed for working with perfectoid spaces. The speaker gives the basic definitions of the category of almost modules over a ring, and then builds up some commutative algebra that's needed in the "almost" world. The goal is to define the appropriate "almost" version of finite étale extensions.

Section I: Almost mathematics. The theory is due to Faltings, and a very general version is studied in book of Gabber-Ramero). We'll specialize to the case we care about.

Definition 1. A *perfectoid field* K is a complete nonarchimedean field (with valuation ring K°) such that:

1. The residue characteristic is p.
2. The associated rank-1 valuation is nondiscrete.
3. The Frobenius map $\Phi : K^\circ/p \to K^\circ/p$ is surjective.

The examples to keep in mind are the following. First of all, we can take is $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$, the completion of the algebraic closure of $\mathbb{Q}_p$. Another example is

$$K = \mathbb{Q}_p(1/p^{1/\infty})^\wedge,$$

where we adjoin all p-power roots of p to $\mathbb{Q}_p$ and complete. Can also take

$$K = \left(\varinjlim_{\text{Frob}} \mathbb{F}_p((t)) \right)^\wedge .$$

A non-example is $K = \mathbb{Q}_p$, because the valuation here is discrete!

If K is a perfectoid field, let $\mathfrak{m}$ be the maximal ideal of K°. We can observe that $\mathfrak{m}^2 = \mathfrak{m} = \mathfrak{m} \otimes \mathfrak{m}$. Then, if we let

$$\Sigma = \{\mathfrak{m}\text{-torsion modules}\} \subseteq K^\circ\text{-}\mathbf{mod},$$

this is a (thick) abelian Serre subcategory, which means that it's closed under subobjects, quotients, and (most importantly) extensions. This is the ideal situation for discussing localizations of categories!

Definition 2. A K°-module is called *almost zero* if it's in Σ, i.e. it's killed by $\mathfrak{m}$. We let $K^{\circ a}$-**mod** be K°-**mod**$/\Sigma$, the localization of the category K°-**mod** by the Serre subcategory Σ. (This means the objects of the two categories are the same, but we change the hom-sets so that everything in Σ is isomorphic to 0).

This category $K^{\circ a}$-**mod** where we do "almost mathematics"! For example, $K^\circ/\mathfrak{m}$ is an almost zero module, so it becomes zero in $K^{\circ a}$-**mod**. On the other hand, K°/p is not almost zero.

In general categorical localizations can be abstract and difficult to work with, but our situation here lets us actually compute things. The key fact is that the "almost" functor K°-**mod** $\to K^{\circ a}$-**mod** (which we denote by $M \mapsto M^a$) has a right adjoint $N \mapsto N_*$ and a left adjoint $N \mapsto N_!$. This means that we have

$$\mathrm{Hom}_{K^\circ}(M_!, N) = \mathrm{Hom}_{K^{\circ a}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(M, N_*).$$

What are these adjoint functors? They are actually very simple; if $M = T^a$ is an almost module then

$$(T^a)_* = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, T) \qquad\qquad (T^a)_! = m \otimes T.$$

We call M_* the module of *almost elements* of M.

Remarks:
(1) The notation comes topology. If $j : U \hookrightarrow X$ is the inclusion of an open subset, then $j^* : \mathbf{Sh}(X) \to \mathbf{Sh}(U)$ has left and right adjoints $j_!$ and j_*, respectively. So by analogy we think of the almost functor as being restriction of sheaves to an open subset. Recall that topologically, $j_!$ is exact. Exercise: check that $N_!$ is exact.
(2) If $N \in K^{\circ a}$-**mod**, we have $(N_!)^a = N = (N_*)^a$; so both of the adjoints are sections of the almost functor, and both maps $N \mapsto N_*, N_!$ are fully faithful.
(3) The converse of remark 2 is false: if M is a K°-module then neither $(M^a)_!$ nor $(M^a)_*$ is necessarily M. For instance $(K^{\circ a})_! = \mathfrak{m}$ and $(\mathfrak{m}^a)_* = K^\circ$. Together, (2) and (3) say that going from from the almost world to the concrete world and back doesn't change anything, but going from concrete to almost and back does.
(4) The Serre subcategory Σ is an "ideal" for tensor products, i.e. if you tensor any module by an almost zero module you get an almost zero module. This means that $K^{\circ a}$-**mod** inherits a tensor product from K°-**mod**. We can then talk about any construction we can do in a tensor category; for instance we can define $K^{\circ a}$-algebras, and then modules over algebras.
(5) The tensor category $K^{\circ a}$-**mod** has internal Hom: if M, N are $K^{\circ a}$-modules, the internal Homs are the "almost homomorphisms"

$$\mathrm{alHom}(M, N) = \mathrm{Hom}_{K^{\circ a}}(M, N)^a$$

with the expected functors. (The set $\mathrm{Hom}(M,N)$ is the homomorphisms in $K^{\circ a}$-**mod**, and it is given a K°-module structure via the structure of this category; applying the almost functor gives that $\mathrm{alHom}(M,N)$ is an element of $K^{\circ a}$-**mod**).

Section II: Almost commutative algebra. All of the above discussion was essentially category; we now want to translate commutative algebra over to this category. We start by defining flatness.

Definition 3. Let A be a $K^{\circ a}$-algebra and M an A-module.

1. We say M is *flat* if $M \otimes_A -$ is exact.
2. We say M is *almost projective* if $\mathrm{alHom}_A(M,-)$ is exact.
3. If $M = N^a$ and $A = R^a$, we say M is *almost finitely generated* if, for every $\varepsilon \in \mathfrak{m}$, there exists a finitely-generated R-module N_ε and a morphism $f_\varepsilon : N_\varepsilon \to N$ such that $\ker f$ and $\operatorname{coker} f$ are killed by ε. (This allows different N_ε and f_ε's as we vary ε)!
4. If M is almost finitely generated and the number of generators we need for N_ε is bounded independently of ε, we say it's *uniformly almost finitely generated*.
5. There is a similar notion of *almost finitely presented*.

We emphasize that all of the statements about exactness are computed in the abelian category A-**mod** of almost modules over the almost algebra A!

Remarks:
(1) If $A = R^a$ and $M = N^a$, then (because the almost functor has two-sided adjoints), M is flat iff $\mathrm{Tor}_i^R(N,-)$ is almost zero for all $i > 0$. Similarly M is almost projective iff $\mathrm{Ext}^i_R(N,-)$ is almost zero for all i.
(2) The reason we use the term "almost projective" is because it is *not* the same as projectivity in $K^{\circ a}$-**mod** in the categorical sense! Exercise: If $M \in K^{\circ a}$-**mod** is projective, then $M = 0$. (Related to previous exercise about $N_!$).

Examples:
(1) Any finitely-generated ideal $I \subseteq K^\circ$ is uniformly almost finitely generated.
(2) Fix a real number $r > 0$ that doesn't arise as a value of K^*. Set

$$I_r = \{f \in K^\circ : \mathrm{val}(f) > r\}.$$

Then this is not finitely-generated (and not almost-isomorphic to a finitely-generated module) but is almost finitely-generated.

Section III: Diagonal idempotents. Our goal in this talk to define "almost étale extensions"; we've been building up the almost commutative algebra we need to. We now take a digression to the usual (non-almost) case of commutative algebra to review étale maps.

If $A \to B$ is a finite étale map of commutative rings, then there's a closed and open embedding $\mathrm{Spec}(B) \hookrightarrow \mathrm{Spec}(B \otimes B)$. Then there exists a unique *diagonal idempotent* $e \in B \otimes_A B$, which satisfies:

1. $e^2 = e$.
2. $\mu(e) = 1$ for $\mu : B \otimes_A B \to B$ the multiplication map.
3. $\ker(\mu) \cdot e = 0$.

Actually this works more generally for unramified maps, but we just need the étale case.

Example: If $A \to B$ is a Galois extension of fields with group G, then $B \otimes_A B \cong \prod_g B$ with $b_1 \otimes b_2 \mapsto (b_i \cdot g(b_2))_{g \in G}$. Then, e corresponds to the tuple $(1, 0, \ldots, 0)$.

Key fact: If $e = \sum_{i=1}^N x_i \otimes y_i \in B \otimes B$, then:

1. $\mathrm{tr}(e) = \sum \mathrm{tr}(x_i y_i) = \deg(B/A)$.
2. If we take the map $B \mapsto A^{\oplus N} \to B$ with the first one given by $b \mapsto (\mathrm{tr}(b \cdot x_i))$ and the second by $(a_i) \mapsto \sum a_i y_i$, the composite of these is the identity.

We will use this key fact as an alternate formulation of "finite étale" that can be generalized to the almost setting.

Section IV: Almost étale extensions. Let K be a perfectoid field, $f : A \to B$ a map of $K^{\circ a}$-algebras.

Definition 4. We say f is *unramified* if there exists an almost element $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\mu(e) = 1$, and $\ker(\mu) \cdot e = 0$. We say f is *étale* if it's flat and unramified. We say f is *finite étale* if it's étale and B is an almost finitely presented projective A-module.

We let $A_{\text{fét}}$ denote the category of all finite étale A-algebras. There is a good deformation theory here: if there is a nilpotent ideal $I \subseteq A$ then $A_{\text{fét}}$ is equivalent to $(A/I)_{\text{fét}}$.

Finally, want to give one real example of an almost étale map (related to the almost purity theorem). Fix a perfectoid field K of characteristic p. Fix a nonzero element $t \in \mathfrak{m}$ (thinking about the completion of the perfection of $\mathbb{F}_p((t))$, say). Let A be a flat K°-algebra integrally closed in $A[1/t]$, and B' a finite étale $A[1/t]$-algebra. Take B to be the integral closure of A in B'. This map will not be étale. If we were in characteristic zero, the purity theorem in algebraic geometry would control how badly this fails, and say we could get it to be étale by base change. In characteristic p, our "almost purity theorem" says that if A is perfect, then $A^a \to B^a$ is finite étale.

Proof: Let $e \in B' \otimes_A B'$ be a diagonal idempotent. Then there exists $N > 0$ such that $t^N e \in B \otimes_A B$. Because A is perfect, B' is perfect and thus B is perfect. Can apply the inverse of Frobenius, which does nothing to e, and then get $(t^N)^{1/p^n} e \in B \otimes_A B$. Iterating this we can conclude $e \in (B \otimes_A B)_*$. This gives that the extension is unramified. For the rest, fix $\varepsilon \in \mathfrak{m}$ and write $\varepsilon e = \sum_{i=1}^N x_i \otimes y_i$. Have the maps $B \to A^{\oplus N} \to B$ as before. Compute that the composite is given by multiplication by ε. Conclude that B is uniformly almost finitely generated projective.