# Nullifying Cobordism in Quantum Gravity


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# The swampland program

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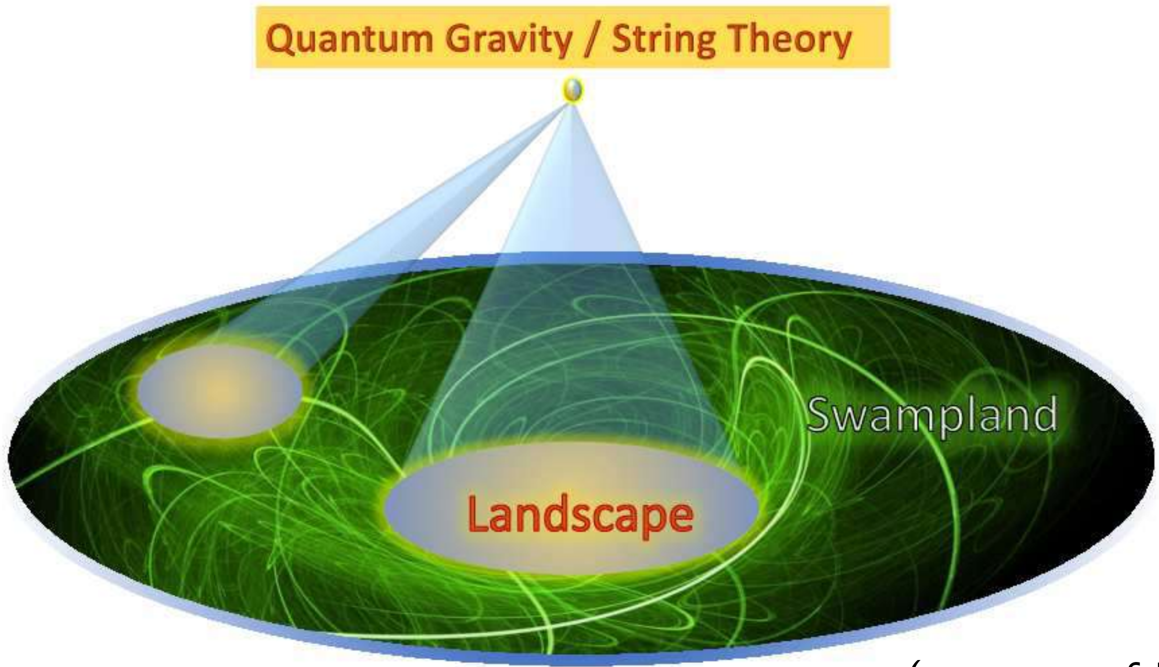
Aim: To characterize the features that distinguishes effective actions admitting a consistent UV completion (landscape) from those (swampland) that do not.




(courtesy of Eran Palti)

Reviews: (Palti, 1903.06239),(van Beest,Calderón-Infante,Mirfendereski,Valenzuela, 2102.01111), (Graña,Herráez, 2107.00087).

# Global symmetries

# Global symmetries

Swampland Conjecture: No global symmetries in QG!

- Quasi-classical arguments based on Black-Hole evaporation
- String Theory: global symmetry on the 2D word-sheet $\rightarrow$ gauge theory in 10D target space

If one seems to detect one, it actually needs to be gauged or broken

- Gauging a continuous symmetry in $d$ dimensions means:

$$d \star F_{d-n+1} = J_n$$

- Breaking a continuous symmetry means:

$$dJ_n = I_{n+1}$$

# Cobordism Conjecture

# Cobordism Conjecture

QG theory in $d$ dimensions: (McNamara,Vafa, 1909.10355)

- A non-vanishing cobordism group $\Omega_n^\xi \neq 0$ gives rise to a $(d-n-1)$-form global symmetry:

$$dJ_n = 0\,.$$

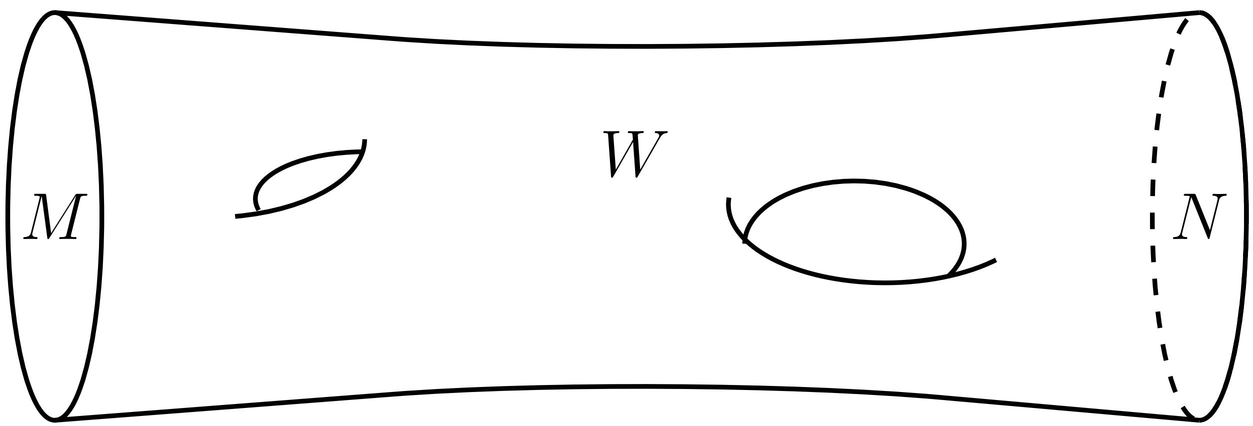
- Hence, the cobordism needs to be nullified, i.e extending the structure $\xi \to QG$ such that $\Omega_n^{QG} = 0$.
- Literature beyond this talk: (Dierigl,Heckmann, 2012.00013), (Montero,Vafa, 2008.11729), (Sati,Schreiber, 2103.01877), (Andriot,Carqueville,Cribiori, 2204.00021), (Velázquez,De Biasio,Lüst, 2209.10297), (Dierigl,Heckman,Montero,Torres, 2212.05077), (Debray,Dierigl,Heckman,Montero, 2302.00007),. . . .

# Cobordism

# Cobordism

$\xi$-cobordism $\Omega_n^\xi$ are equivalence classes of $n$-dim. manifolds with structure $\xi$, where $M$ and $N$ are equivalent if

$$\partial W = M \sqcup \overline{N} \,.$$

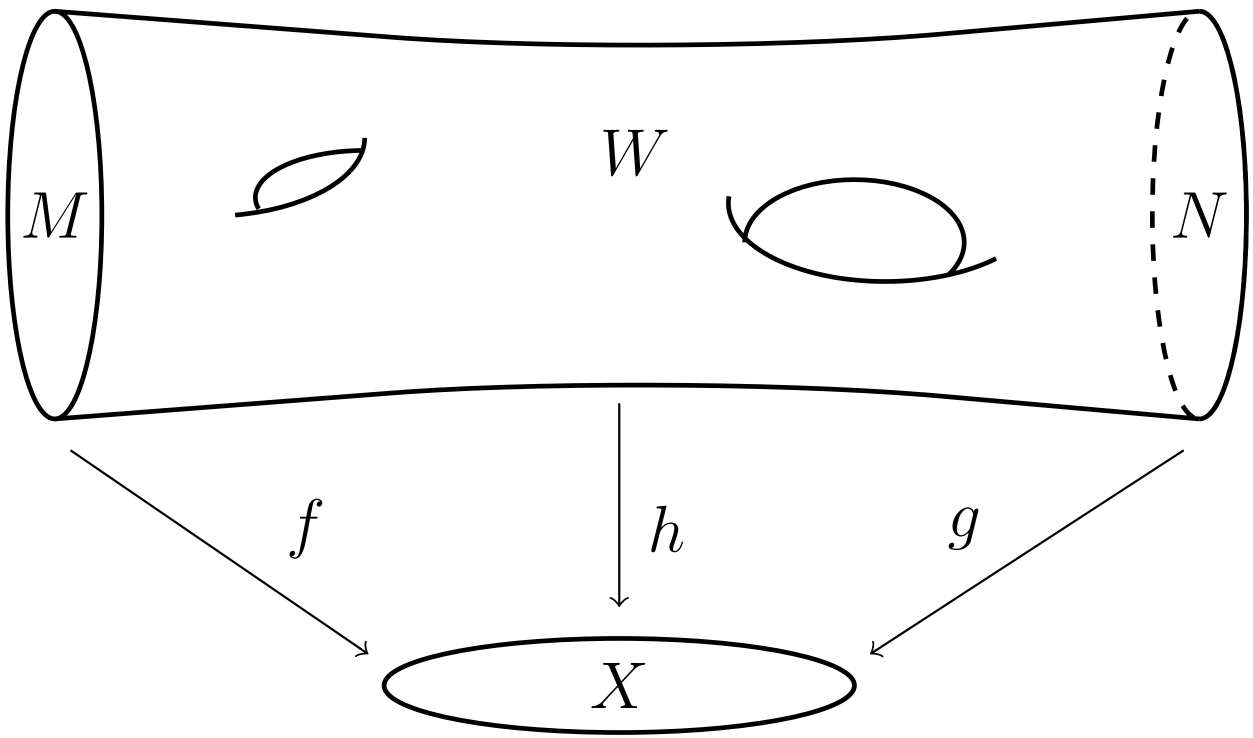



- In the $d-n$-dimensional theory: A cobordism $W$ is a domain wall separating $M$ and $N$.
- Cobordism conjecture: topologically all QG configurations are connected to nothing via finite energy domain walls.(McNamara,Vafa, 1909.10355)

# Generalized cobordism

# Generalized cobordism

Generalization to $\Omega_n^G(X)$: cobordism groups relative to $X$ consisting of pairs $(M, f)$ modulo equivalence:




$X = pt$ is the former case.

# Breaking of symmetry

# Breaking of symmetry

**Breaking of the symmetry**: Introducing defects so that

$$0 \neq dJ_n = I_{n+1} = \sum_{\mathrm{def},j} \delta^{(n+1)}(\Delta_{d-n-1,\,j})\,.$$

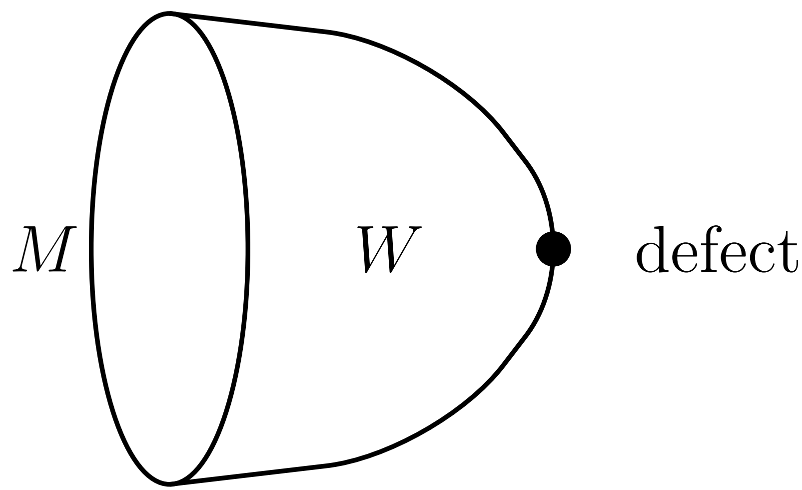
Defects: codim=1 in $d-n$ dimensions

Elements in the kernel of the map

$$f_B : \Omega_n^{\xi} \to \Omega_n^{\xi+\mathrm{defect}}$$

are killed in the full theory.

$(W+$ defect) trivializes $M$:



# Gauging of symmetry

# Gauging of symmetry

**Gauging of the symmetry**: Elements in the cokernel of

$f_G : \Omega_n^{\xi+\mathrm{def};U(1)} \to \Omega_n^{\xi} \oplus \Omega_n^{\mathrm{def}}$

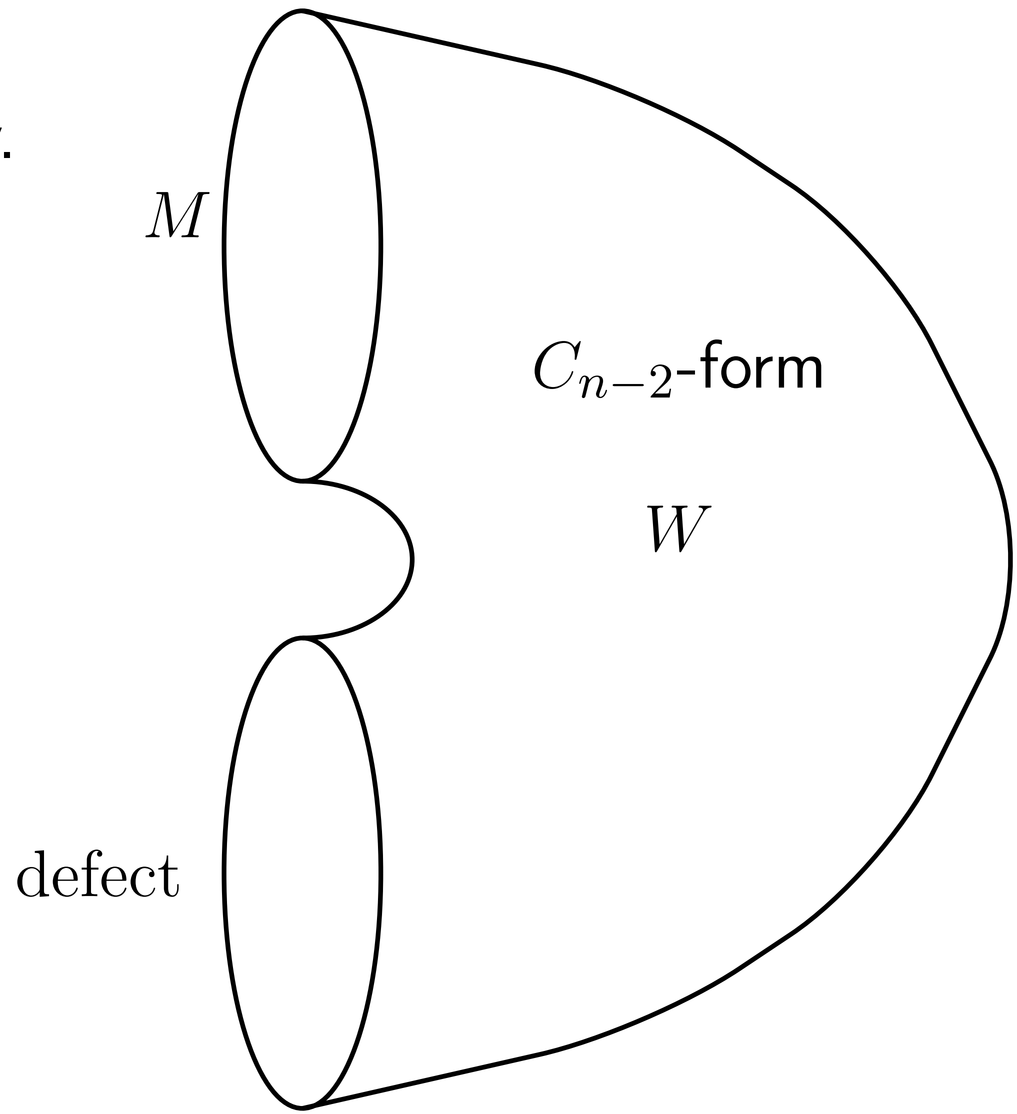
are co-killed in the full theory.

$$dF_{n-1} = \underbrace{J_n^{\xi} + J_n^{\mathrm{def}}}_{\text{trivialized}}$$

$J_n^{\xi}$: cobordism invariant

Defects: codim=0 in $(d-n)$ dimensions

Example: String cobordism
$dH_3 = p_1(M)/2$



# Contents

# Contents

Here, we are seeking answers to the general questions:

- How are these ideas realized in string theory?
- What are the codim=0 defects appearing for gauging?
- The breaking gives rise to codim=1 defects. Are these (all) known in string theory?

More involved questions:

- How far can one develop this picture without relying on string theory? Is there a bottom-up approach?
- One expects that the physically motivated maps $f_B$ and $f_G$ are part of spectral sequence. How does this work mathematically?

# Gauging in string theory

# Gauging in string theory

In string theory we have $p$-form gauge fields satisfying Bianchi identities of the generic form

$$d\tilde{F}_{n-1} = \sum_i N_i \underbrace{\delta^{(n)}(\Sigma_i)}_{\text{localized brane}} + \sum_a \underbrace{J_a^{(n)}}_{\text{top. invariants}}$$

- For RR-forms we have D-branes carrying a K-theory charge.
- Type IIB: $K_{2n}(pt) = \mathbb{Z}$ gives the charge of a BPS D-brane with dim$=(10-2n)$ world-volume.
- Type I: $KO_n(pt) \in \{\mathbb{Z}, \mathbb{Z}_2\}$ gives the charge of a stable (non-)BPS D-brane with dim$=(10-n)$ world-volume

What about the geometric contributions?

(Bhg,Cribiori, 2112.07678)

# Gauging global symmetries

# Gauging global symmetries

Examples:

- F-theory/type IIB compactified on $M$ to 8D:

$$d\tilde{F}_1 = \sum_i N_i \, \delta^{(2)}(\Delta_{8,i}) + a_1^{(2)} \frac{c_1(M)}{2} \,.$$

For $M = \mathbb{P}^1$ the base of $T^2 \to K3 \to \mathbb{P}^1$: $a_1^{(2)} = -24$.

- F-theory/type IIB compactified on $M$ to 4D: D3-brane tadpole

$$d\tilde{F}_3 = \sum_i N_i \, \delta^{(6)}(\Delta_{4,i}) + a_1^{(6)} \frac{c_2(M) c_1(M)}{24} + a_2^{(6)} \frac{c_1^3(M)}{2} \,.$$

For $M = B_3$ the base of a smooth elliptically fibered CY fourfold $Y$: $a_1^{(6)} = -12$, $a_2^{(6)} = -30$.

# Spin$^c$ cobordism

# Spin$^c$ cobordism

For Spin$^c$ structure, the non-vanishing cobordisms are

| n | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $\Omega_n^{\mathrm{Spin}^c}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ |
| $\Sigma_{n,i}$ | pt$^+$ | $\mathbb{P}^1$ | $\mathbb{P}^2 \oplus (\mathbb{P}^1)^2$ | $\begin{matrix}\mathbb{P}^2 \\ \mathbb{P}^2\end{matrix}\begin{bmatrix}2 \\ 2\end{bmatrix} \oplus \begin{matrix}\mathbb{P}^3 \\ \mathbb{P}^1\end{bmatrix}\begin{bmatrix}3 \\ 1\end{bmatrix}$ |
| Inv. | 1 | $\mathrm{td} = c_1/2$ | $\mathrm{td},\ c_1^2$ | $\mathrm{td} = (c_2 c_1)/24,\ c_1^3/2$ |
| $K_n(pt)$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |

- Geometric pieces in the Bianchi identities are described by the cobordism invariants of $\Omega_n^{\mathrm{Spin}^c}(pt)$
- Here: Ungauging makes the various global charges in $K_n(pt)$ and $\Omega_n^{\mathrm{Spin}^c}(pt)$ visible

# Gauging global symmetries

# Gauging global symmetries

- For Type I, one can find a similar relation between $KO_n(pt)$ and Spin-cobordism $\Omega_n^{\mathrm{Spin}}(pt)$
- Can one fix the relative coefficients $a_i^{(n)}$ from first principles?

Indeed, there exists a mathematical relation between these K-theory and cobordism classes.

Atiyah–Bott–Shapiro (ABS): There exist ring homomorphisms

$$\alpha^c : \Omega_*^{\mathrm{Spin}^c}(pt) \to K_*(pt)\,, \qquad \alpha : \Omega_*^{\mathrm{Spin}} \to KO_*(pt)\,.$$

with $K_n(pt) = \tilde{K}(S^n)$. When restricted to a fixed grade $n$

$$\alpha_n^c([M]) = \mathrm{Td}(M) \in \mathbb{Z}\,, \quad \alpha_n([M]) = {}''\hat{\mathrm{A}}(M)''\,.$$

# Hopkins–Hovey isomorphism

# Hopkins–Hovey isomorphism

The maps $\alpha^c$ and $\alpha$ can be promoted to isomorphisms leading to the Hopkins–Hovey theorem (generalisation of a classic theorem by Conner-Floyd)

$$\Omega_*^{\mathrm{Spin}}(X) \otimes_{\Omega_*^{\mathrm{Spin}}} KO_* \to KO_*(X),$$

$$\Omega_*^{\mathrm{Spin}^c}(X) \otimes_{\Omega_*^{\mathrm{Spin}^c}} K_* \to K_*(X)$$

are isomorphisms for any topological space $X$.

It involves the cobordism $\Omega_n^{\mathrm{Spin}^c}(X)$ and K-theory $K_n(X)$ classes depending on a background $X$.

- How to compute them?
- What is their string theoretic relevance?

(Bhg,Cribiori,Kneissl,Makridou, 2208.01656)

# Generalized cobordism

# Generalized cobordism

Clearly, this introduces some background dependence

$$\Omega_n^{\mathrm{Spin}^c}(X) \to \Omega_n^{\mathrm{Spin}^c}(pt) \to \Omega_n^{\mathrm{QG}} = 0$$

Physical expectation for $K^{-n}(X)$ with $k = \dim(X)$

- Classifies all $D$-brane charges in $D = 10 - k$ dimensions
- Contributions from wrapped 10D branes subject to Freed-Witten anomalies
- New tachyon decay channels can exist

Mathematically: compute $K^{-n}(X)$ via the Atiyah-Hirzebruch spectral sequence (AHSS)

- FW anomalies: non-trivial maps $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$
- New decay channels: extension problem at the end of AHSS, $e(\mathbb{Z}_2, \mathbb{Z}) = \{\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}\}$

# Generalized cobordism

# Generalized cobordism

For the simple examples $X \in \{S^k, T^k, K3, CY_3^{(\pi^1=0)}\}$ the spectral sequence stabilizes at the 2. page

$$K^{-n}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X) \cdot \underbrace{K^{-n-m}(pt)}_{\text{10D branes}} .$$

$\Omega_n^{\mathrm{Spin}^c}(X)$ are also computed via an AHSS

$$\Omega_{n+k}^{\mathrm{Spin}^c}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X) \cdot \Omega_{n+m}^{\mathrm{Spin}^c}(pt)$$

- Classifies all $(D-n-1)$-form global symmetries in the non-compact $D = d - k$ dimension.
- Consistent with dimensional reduction of gauging, tadpoles and breaking

# Dynamical Cobordism

# Dynamical Cobordism

Breaking of $\Omega^\xi \neq 0$ by codim=1 defects reminiscent of running solutions in string theory. (Buratti,Delgado,Uranga, 2104.02091) (Buratti,Calderón-Infante,Delgado,Uranga, 2107.09098) (Angius,Calderón-Infante,Delgado,Huertas,Uranga,2203.11240).

Now, follow recent paper (Bhg,Kneißl,Wang, 2303.03423) : Consider $D$-dimensional action of Dudas-Mourad type (hep-th/0004165)

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G} \left( R - \frac{1}{2} (\partial\phi)^2 \right) - \frac{\lambda}{\kappa_D^2} \int d^D x \sqrt{-G}\, e^{b\phi} \,,$$

- Sugimoto model=Non-supersymmetric orientifold of type IIB: $b = 3/2$
- Massive type IIA (RR 0-form flux): $b = 5/2$

No solution preserving $D$-dimensional Poincaré symmetry!

# Dynamical Cobordism

# Dynamical Cobordism

The general ansatz with $(D-1)$-dimensional Poincaré symmetry

$$ds^2 = e^{2A(y)} ds^2_{D-1} + e^{2B(y)} dy^2 ,$$

and a dilaton $\phi(y)$.

Construct explicit solutions for arbitrary parameter $b$:

- Solutions are of finite size $\Delta$ in the $y$-direction
- Singularities for dilaton and curvature with scaling

$$\Delta \sim e^{\mp \frac{\delta}{2} \mathcal{D}(y)} , \qquad |R| \sim e^{\pm \delta \mathcal{D}(y)}$$

- Type of solution depends on $b > b_{\rm cr}$ or $b < b_{\rm cr}$

$$b_{\rm cr} = \sqrt{\frac{2(D-1)}{(D-2)}} \, .$$

(Basile,Raucci,Thomée, 2209.10553)

# Dynamical Cobordism

# Dynamical Cobordism

Eventually, we find the following list of end-of-the-worlds(ETW) branes :

| region | $\epsilon_1$ | $\epsilon_2\epsilon_3$ | ETW-branes | |
|---|---|---|---|---|
| $b \geq b_{cr}$ | 1 | 1 | $\mathrm{ETW}^{(L,-)}_{(\delta=\sqrt{2}b_{cr})}$ | $\mathrm{ETW}^{(R,+)}_{(\delta=\sqrt{2}b)}$ |
| | -1 | -1 | $\mathrm{ETW}^{(L,+)}_{(\delta=\sqrt{2}b)}$ | $\mathrm{ETW}^{(R,-)}_{(\delta=\sqrt{2}b_{cr})}$ |
| $\lvert b\rvert \leq b_{cr}$ | 1 | 1 | $\mathrm{ETW}^{(L,-)}_{(\delta=\sqrt{2}b_{cr})}$ | $\mathrm{ETW}^{(R,+)}_{(\delta=\sqrt{2}b_{cr})}$ |
| | -1 | 1 | $\mathrm{ETW}^{(L,+)}_{(\delta=\sqrt{2}b_{cr})}$ | $\mathrm{ETW}^{(R,-)}_{(\delta=\sqrt{2}b_{cr})}$ |
| $b \leq -b_{cr}$ | 1 | -1 | $\mathrm{ETW}^{(L,-)}_{(\delta=-\sqrt{2}b)}$ | $\mathrm{ETW}^{(R,+)}_{(\delta=\sqrt{2}b_{cr})}$ |
| | -1 | 1 | $\mathrm{ETW}^{(L,+)}_{(\delta=\sqrt{2}b_{cr})}$ | $\mathrm{ETW}^{(R,-)}_{(\delta=-\sqrt{2}b)}$ |

Note: $\delta \geq \sqrt{2}b_{cr}$.

# Dynamical Cobordism

# Dynamical Cobordism

What is the nature of these ETW 8-brane?

Constraints:

- preserve 9D Poincare symmetry
- should feature the same singularities close to their core
- should be non-isotropic in the transverse direction.

Action of a neutral ETW-brane

$$S = \frac{1}{2}\int d^D x\sqrt{-G}\left(R - \frac{1}{2}(\partial\phi)^2\right) - \lambda_0 \int d^D x \sqrt{-g}\, e^{a_0\phi}\,\delta(y)$$

Boundary condition for ETW$^L$-defect:

- $y < 0$: there is nothing: $g_{\mu\nu} = g_s = 0$
- $y > 0$: non-trivial solution to bulk eom
- $y = 0$: jump conditions

# Dynamical Cobordism

# Dynamical Cobordism

- Consistency of the bulk eom implies (Raucci, 2209.06537)

$$a_0 = \mp\sqrt{\frac{(D-1)}{2(D-2)}} = \mp b_{\rm cr}/2\,.$$

- The tension is negative
- Precisely the scaling behavior of the $\mathrm{ETW}^{L}_{\sqrt{2}b_{\rm cr}}$ brane!

In 10D, the string frame brane action becomes

$$S\left(\mathrm{ETW}^{(L(R),\mp)}\right) = -T\int d^{10}x\sqrt{-g}\,\delta(y)\begin{cases} e^{-3\phi} \\ e^{-\frac{3}{2}\phi}\,. \end{cases} \qquad \text{(-19)}$$

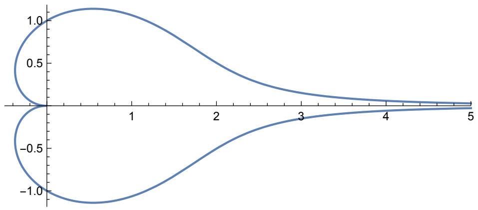
No known brane tension.

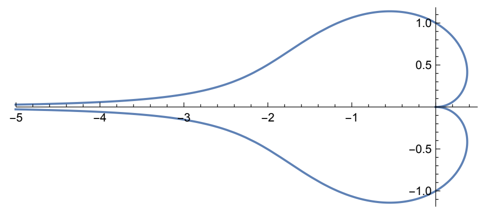
# Dynamical Cobordism Conjecture

# Dynamical Cobordism Conjecture

Generalization:

- The ETW brane with $\delta > \sqrt{2} b_{\rm cr}$ is given by a charged brane. ($O8$-plane for $b = 5/2$).
- The story can be generalized to the T-dual BF model
  (Bhg,Font, hep-th/0011269) (Bhg,Cribiori,Kneißl,Makridou, 2205.09782)




Dynamical Cobordism Conjecture: For a solution of an effective (super-)gravity theory featuring a singularity at finite space-time distance, consistency with QG requires the existence of an explicit solution for the ETW brane that closes off the space-time.

# Conclusions

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- Gauging leads to tadpole cancellation conditions known from orientifolds and F-theory.
- K-theory provides the brane charges, cobordism the geometric contributions to tadpoles
- Dynamical cobordism provide a (super)gravity description of ETW branes

There are still open questions

- Generalization to type IIA?
- Explicit computation of $\Omega_n^{\xi+\mathrm{def};U(1)_p}$ classes?
- Determination of relative coefficients in tadpoles.
- CFT description of ETW-branes?
- Unique bottom-up result for the final $\Omega_n^{QG} = 0$?