NONCOMPACT COMPLEX SYMPLECTIC AND HYPERKÄHLER MANIFOLDS

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Contents

1. Introduction

The aim of this course is to introduce some aspects of the geometry of hyperkähler manifolds (and more general complex symplectic manifolds) focusing on basic ideas and examples.

The principal motivation is to give some of the background material for further study of complex symplectic and hyperkähler manifolds and their applications. Some examples of such applications that we have in mind (but will not be covered here!) include:

• Nakajima's work on the representation theory of quantum algebras [Nak94, Nak98, Nak01],

• The approach of Witten and collaborators [KW07, GW06, Wit08] to the geometric Langlands program.

• The approach of Gaiotto–Moore–Neitzke [GMN08] to certain "wall-crossing formulae" of Kontsevich–Soibelman [KS08], as consistency conditions for the existence of a hyperkähler metric.

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Indeed for the most part we will try to be as down to earth as possible and focus on basic examples. (A surprisingly large number of features of the more complicated spaces appear even in the simplest cases.)

Berger's List.

From a mathematical perspective hyperkähler manifolds first appeared in M.Berger's work on Riemannian holonomy groups, which we will now briefly recall. Given a Riemannian manifold (M, g) and a point $p \in M$ we can associate an orthogonal transformation

$$\tau(\gamma) \in \mathrm{O}(T_pM)$$

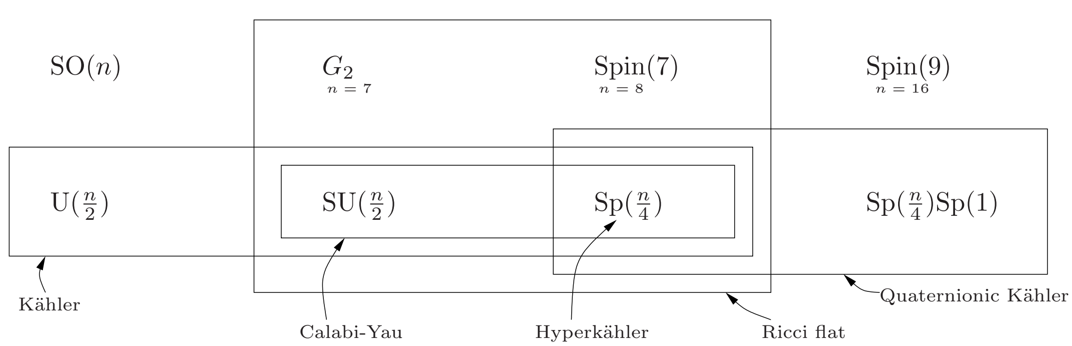
of the tangent space T_pM to any loop $\gamma : S^1 \to M$ in M based at p (i.e. so that $\gamma(1) = p$). This is defined by restricting (pulling back) the tangent bundle of M to the loop S^1. The Levi-Civita connection of g then becomes a flat connection over S^1 and we may use it to parallel translate any vector $v \in T_pM$ around the loop and back to T_pM. This defines an orthogonal transformation $\tau(\gamma)$ of T_pM. The holonomy group Hol_p is then defined to be the group generated by all these transformations as the loop γ varies:

$$\mathrm{Hol}_p = \langle \tau(\gamma) \bigm| \gamma : S^1 \to M, \gamma(1) = p \rangle \subset \mathrm{O}(T_pM).$$

If we change base points then the holonomy groups are identified, once we make a choice of path between the two points, and we may speak of "the" holonomy group Hol.

The holonomy groups were classified as follows:

Theorem 1.1 (M. Berger 1955). *Let (M, g) be an oriented simply connected Riemannian manifold of dimension n which is neither locally a product nor symmetric. Then* Hol *is one of:*



Thus hyperkähler manifolds are those lying in the intersection of all the boxes: they have dimension divisible by 4 and holonomy group the compact symplectic group (the quaternionic unitary group). The first example was constructed in dimension 4 in 1978 by Eguchi–Hanson, and then in 1979 Calabi found examples in every dimension (and coined the name "hyperkähler" for them). These examples will be described in some detail later on.

The emphasis on the noncompact case is mainly because that is what arises in the applications we are interested in (namely quiver varieties in [Nak94, Nak98, Nak01], moduli spaces of Higgs bundles in [KW07, GW06, Wit08], and a certain hyperkähler four manifold in [GMN08]). Also there are many more known examples and constructions. Indeed in the compact case one has the $K3$ surfaces, the abelian surfaces (with the flat metric), two infinite families constructed out of Hilbert schemes of points on these hyperkähler surfaces, and two other (deformation classes of) examples of complex dimension 6 and 10 due to O'Grady [O'G99, O'G03]. See Beauville [Bea84] for the construction of the families, which rests on Yau's solution of the Calabi conjecture for the existence of the metric. In the noncompact case there are other ways of obtaining hyperkähler manifolds (i.e. constructive methods), which often give a lot more information about the hyperkähler metric than just an existence theorem.

2. Basic examples: Calogero–Moser spaces and Hilbert schemes

One of the key features of hyperkähler geometry is that hyperkähler manifolds have families of complex structures. In other words one may have two non-isomorphic complex manifolds, which are naturally isomorphic as real manifolds, due to the fact that they are the *same* hyperkähler manifold simply viewed in two different complex structures. Thus we are able to see relations between certain complex manifolds which are simply "hidden" from a purely complex viewpoint. In this section we will describe two classes of complex manifolds of independent interest (one from integrable systems and the other from algebraic geometry). We will see they are not isomorphic as complex manifolds, but later in the course we will see they are the same hyperkähler manifold viewed in two different complex structures. This gives some concrete motivation for a lot of the course.

Calogero–Moser spaces.

Let $V = \mathbb{C}^n$ and consider the space

$$C_n = \{(X, Z) \in \mathrm{End}(V) \times \mathrm{End}(V) \bigm| [X, Z] + \mathrm{Id}_V \text{ has rank one}\}/\mathrm{GL}_n(\mathbb{C})$$

consisting of two square matrices X, Z whose commutator $[Z, X]$ differs from the identity by a rank one matrix. Note (by taking the trace) that it is impossible to find two square matrices whose commutator is the identity. Thus we are asking, in some sense, for the "best approximation". [Here $\mathrm{GL}_n(\mathbb{C})$ is acting by conjugation, and the quotient means simply taking the set of orbits.]

An alternative description will also be useful. Consider the space

$$\{(X, Z, v, \alpha) \in \mathrm{End}(V)^2 \times V \times V^* \bigm| [X, Z] + \mathrm{Id}_V = v \otimes \alpha\}/\mathrm{GL}_n(\mathbb{C})$$

where $g \in \mathrm{GL}_n(\mathbb{C})$ acts in the natural way, as

$$g(X, Z, v, \alpha) = (gXg^{-1}, gZg^{-1}, gv, \alpha \circ g^{-1}).$$

It is easy to show that this action is free and one may show (cf. [Wil98] p.5) the result is a smooth (connected) *affine* variety of dimension $2n$—it is the affine variety associated to the ring of invariant functions.

To go between the two descriptions consider the map from the second description to the first got by forgetting v, α. This is clearly surjective, and moreover (since the rank one matrix is always nonzero—having trace n) $v_1 \otimes \alpha_1 = v_2 \otimes \alpha_2$ iff there is a nonzero scalar relating the pairs (v, α). Now the scalar subgroup of $\mathrm{GL}_n(\mathbb{C})$ acts trivially on the pair X, Z so we see the map on orbits is bijective.

Let us describe a big open subset of C_n. (We will later relate it to the Calogero–Moser integrable system.) Consider the subset $C'_n \subset C_n$ where X is diagonalizable. Then, moving within the orbit we may assume X is diagonal, and write $X = \mathrm{diag}(x_1, \ldots, x_n)$. Thus

$$[X, Z]_{ij} = (x_i - x_j)z_{ij}$$

which should equal the ij entry of $-\mathrm{Id}_V + v\alpha$. Taking $i = j$ we see $v_i\alpha_i = 1$, so no components of v or α vanish. Thus if $i \neq j$ we see $(x_i - x_j)z_{ij} = v_i\alpha_j \neq 0$ so $x_i \neq x_j$, i.e. X is regular semisimple (its stabilizer is the diagonal maximal torus). Within the orbits of this torus action there is a unique point with $v_i = 1$ for all i (and then since $v_i\alpha_i = 1$ we will also have $\alpha_i = 1$). [Then $v\alpha$ is the rank one matrix with a 1 in every entry.] Now note that the off-diagonal parts of Z are determined uniquely: $z_{ij} = 1/(x_i - x_j)$. Thus if we let $p_i = Z_{ii}$ we have coordinates (x_i, p_j) on C'_n defined upto permutation of the indices, i.e. the map

$$\begin{aligned}(\mathbb{C}^n \setminus \text{diagonals}) \times \mathbb{C}^n \ &\to C'_n\\ (x_i, p_j) \mapsto &\big(\mathrm{diag}(x_i), Z, (1, \ldots, 1)^T, (1, \ldots, 1)\big)\end{aligned}$$

is a covering with group Sym_n, where Z is as above with diagonal part $\mathrm{diag}(p_j)$ and offdiagonal entries $1/(x_i - x_j)$.

Thus $C'_n \cong ((\mathbb{C}^n \setminus \text{diagonals}) \times \mathbb{C}^n)/\operatorname{Sym}_n = T^*((\mathbb{C}^n \setminus \text{diagonals})/\operatorname{Sym}_n)$ is the cotangent bundle of the configuration space of n-identical particles on the complex plane $\mathbb{C}$, i.e. the phase space for n distinct identical particles, and so C_n is a partial compactification of this. This is interesting since C_n contains the trajectories as the particle collide: As will be explained later (Exercise 3.28), if we consider n particles on $\mathbb{C}$ with an inverse square potential, then for real initial positions there are no collisions, but on the complex plane there can be collisions, where the flows enter $C_n \setminus C'_n$: the flows are incomplete on C'_n, but complete on C_n.

Hilbert Scheme of points on $\mathbb{C}^2$.

Now let us describe the algebraic geometer's way to partially compactify the set of n-tuples of distinct unordered point of $\mathbb{C}^2$ (this will contain the collisions, but not points tending to infinity in $\mathbb{C}^2$).

Set $X = \mathbb{C}^2$. Given n distinct points $x_i \in X$ we can consider the corresponding subscheme $Z \subset X$. This has structure sheaf

$$\mathcal{O}_Z = \bigoplus_{i=1}^{n} \mathbb{C}_i$$

where $\mathbb{C}_i$ is the skyscraper sheaf supported at x_i, and corresponds to the ideal

$$I = \{f \in \mathbb{C}[z_1, z_2] \bigm| f(x_1) = \cdots = f(x_n) = 0\} \subset \mathbb{C}[z_1, z_2]$$

of functions vanishing at these points. Such an ideal is an example of an ideal "of colength n" i.e. such that the $\mathbb{C}$-vector space

$$\mathbb{C}[z_1, z_2]/I$$

has dimension n. Thus the natural algebro-geometric partial compactification of $(X^n \setminus \text{diagonals})/\operatorname{Sym}_n$ is to consider the set of all such ideals:

$$X^{[n]} := \{ \text{ ideals } I \subset \mathbb{C}[z_1, z_2] \bigm| \dim_{\mathbb{C}}(\mathbb{C}[z_1, z_2]/I) = n \ \}.$$

Such an ideal corresponds to a subscheme of X with fixed (constant) Hilbert polynomial $P(t) = n$ (and so justifies the name "Hilbert scheme"). In general $P(m) = \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(m))$.

Consider a point $x \in X$ and a tangent vector $v \in T_xX$ to X at x. Then we may define an ideal

$$I = \{f \bigm| f(x) = 0, df_x(v) = 0\}$$

of functions vanishing at x and having zero derivative in the direction v. This is an ideal of colength 2, so represents a point of $X^{[2]}$. In particular we see that the Hilbert scheme retains more information than just the positions of the points as they collide–e.g. this ideal determines the direction of v as well.

Some basic properties (due to Fogarty in general) of the Hilbert scheme of points on X are as follows:

1) $X^{[n]}$ is smooth and has dimension $2n$,

2) There is a map (the Hilbert–Chow morphism) $\pi : X^{[n]} \to S^n X$ to the symmetric product given by

$$Z \mapsto \sum_{x \in X} \dim_{\mathbb{C}}(\mathcal{O}_{Z,x})[x]$$

where Z is the subscheme corresponding to an ideal I. We will give a direct definition of the multiplicities for this situation below, here we just define it as the dimension over $\mathbb{C}$ of the stalk at x of the structure sheaf of Z. The symmetric product[1] $S^n X = X^n / \operatorname{Sym}_n$ is singular (if $n > 1$) and we have that:

3) π is a resolution of singularities (in particular it is an isomorphism away from the singularities).

A point to note is that $X^{[n]}$ is *not* affine. For example in the case $n = 2$ we have $X^{[n]} \cong \mathbb{C}^2 \times T^*\mathbb{P}^1$ (which has a compact complex submanifold, $\mathbb{P}^1$).

Let us give a more explicit description of $X^{[n]}$. Given a point I of $X^{[n]}$ we can associate an n-dimensional complex vector space $V = \mathbb{C}[z_1, z_2]/I$. This has the following properties:

1) The action of z_i (by multiplication) on $\mathbb{C}[z_1, z_2]$ yields elements

$$B_i \in \operatorname{End}(V)$$

for $i = 1, 2$.

2) The elements B_i commute: $[B_1, B_2] = 0$.

3) The element $1 \in \mathbb{C}[z_1, z_2]$ maps to a vector $v \in V$

4) v is a cyclic vector: any element of V is a linear combination of elements of the form wv where w is a word in the B_i.

In other words V is a "cyclic $\mathbb{C}[z_1, z_2]$-module". Now the fact is that this data determines I: Given data (V, B_1, B_2, v) satisfying these conditions we may define a map $\varphi : \mathbb{C}[z_1, z_2] \to V$ by setting

$$\varphi(f) = f(B_1, B_2) \cdot v \in V.$$

Since v is cyclic this is surjective and we take I to be the kernel of φ. It is an ideal of colength n. This establishes the following [cf. [Nak99], chapter 1]:

Proposition 2.1. *$X^{[n]}$ is isomorphic to the set of $\mathrm{GL}_n(\mathbb{C})$ orbits in the space of matrices*

$$\left\{ (B_1, B_2, v) \in \operatorname{End}(V)^2 \times V \;\middle|\; \begin{array}{c} [B_1, B_2] = 0, \text{ and if} \\ U \subset V \text{ with } v \in U \text{ and} \\ B_i(U) \subset U, i = 1, 2, \text{ then } U = V \end{array} \right\}$$

[1] which can be viewed as the set of degree n formal linear combination of points of X i.e. finite formal sums of the form $\sum_{x \in X} n_x[x]$ with $n_x \in \mathbb{Z}_{\geq 0}$.

where $V = \mathbb{C}^n$*.*

Thus C_n and $X^{[n]}$ are both smooth (algebraic) complex manifolds of dimension $2n$ and both involve "adding extra material to account for collisions" in one sense or another. But they are not isomorphic as complex manifolds, since one is affine and the other is not. Later in this course we will see however that they are in fact the same hyperkähler manifold viewed in two different complex structures. One may say there is a "hidden" symmetry group changing the complex structure moving from one space to the other.

To end this section we will make some aspects more explicit in terms of this "matricial" description of the Hilbert scheme. In terms of this description the Hilbert–Chow map (to the symmetric product) is as follows. Given commuting matrices B_1, B_2 we may simultaneously put them in upper triangular form, with diagonal entries $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ respectively say. The corresponding point of $S^n X$ is then

$$\sum_{i=1}^{n} [(\lambda_i, \mu_i)].$$

Note that if all these n points of X are distinct then (in the basis in which they are upper triangular) both B_1, B_2 are necessarily diagonal matrices (stabilized by the maximal diagonal torus of $\mathrm{GL}_n(\mathbb{C})$). In this basis, each component of the cyclic vector v must be nonzero, and so there is a unique element of the torus mapping v to the vector $(1, 1, \ldots, 1)^T$. This shows the open part of $X^{[n]}$ where both B_1, B_2 are diagonal maps isomorphically onto the smooth locus of the symmetric product (where the points are distinct).

Let us look at the simplest case where two points coincide, in $X^{[2]}$. Let us consider the locus lying over the point $2[(0, 0)] \in S^2 X$. These will be represented by matrices of the form

$$B_1 = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{C}$. Clearly if $\alpha = \beta = 0$ then v is not cyclic (whatever v is). Otherwise one easily sees the set of cyclic vectors is $\{\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)\}$ with $q \neq 0$. The subgroup stabilizing B_1, B_2 is $\{g = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)\}$ with x nonzero, and this acts simply transitively on the set of cyclic vectors. The larger group $\{h = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)\}$ with both x, z nonzero preserves the form of the matrices B_1, B_2 (i.e. maps them to a pair of matrices of the same form). This larger group acts on the pair α, β as follows:

$$h(\alpha, \beta) = (t\alpha, t\beta)$$

where $t = x/z \in \mathbb{C}^*$ for $h = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$. This shows that the set of orbits of such matrices (the fibre of $\pi : X^{[2]} \to S^2 X$) is a projective line $\mathbb{P}^1(\mathbb{C})$, with homogeneous coordinates $[\alpha : \beta]$. By definition (after choosing $v = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$) the corresponding ideal is the kernel of the map

$$f \in \mathbb{C}[z_1, z_2] \mapsto f(B_1, B_2)\begin{pmatrix}0\\1\end{pmatrix}$$

Expanding such f in a Taylor expansion at 0 we see (easily) that the kernel I consists of functions f with $f(0) = 0$ and first derivatives at zero constrained so that:

$$\alpha\frac{\partial f}{\partial z_1}(0) + \beta\frac{\partial f}{\partial z_2}(0) = 0.$$

The left hand side of this is $\langle \alpha\frac{\partial}{\partial z_1} + \beta\frac{\partial}{\partial z_2}, df\rangle$ and so $[\alpha : \beta]$ should be viewed as a (complex) tangent direction; i.e. a point of the projectivised tangent space $\mathbb{P}T_0X$ to $X = \mathbb{C}^2$ at the origin; it corresponds to the ray in $T_0X \cong \mathbb{C}^2$ through the vector

$$\alpha\frac{\partial}{\partial z_1} + \beta\frac{\partial}{\partial z_2}.$$

A little bit more work will enable us to identify $X^{[2]} \cong \mathbb{C}^2 \times T^*\mathbb{P}^1$. First note there is a free action of the additive group $\mathbb{C}^2$ on $X^{[2]}$ by translating z_1, z_2. Quotienting by this action amounts to restricting to the subset $X_0^{[2]} \subset X^{[2]}$ (the "punctual" Hilbert scheme) where the points have centre of mass zero. One has $X^{[2]} \cong \mathbb{C}^2 \times X_0^{[2]}$ (and similarly on the symmetric products). Then one observes that S_0^2X is just $\mathbb{C}^2/\{\pm 1\}$. This is the A_1 singularity and the resolution of this (got by simply blowing up the singular point once) is the total space of the bundle $\mathcal{O}(-2) \to \mathbb{P}^1$, which is isomorphic to the cotangent bundle of $\mathbb{P}^1$.

Exercise 2.2 (Nakajima [Nak99]). Suppose we have $B_1, B_2 \in \mathrm{End}(V)$, a cyclic vector $v \in V$ and an element $\phi \in V^*$ such that

$$[B_1, B_2] + v \otimes \phi = 0.$$

Show, as follows, that $\phi = 0$, so in fact $[B_1, B_2] = 0$.

1) Show that $\phi(v) = 0$,

Suppose now that $\phi(wv) = 0$ for all products (words) w of the B_i of length $< k$.

2) Deduce that $\phi \circ w_1 B_2 B_1 w_2 = \phi \circ w_1 B_1 B_2 w_2$ for any words w_1, w_2 such that w_1 has length $< k$.

3) Deduce that $\phi \circ w = \phi \circ B_1^{k_1} B_2^{k_2}$ for any word w of length k, where k_i is the number of B_i's occuring in w (for $i = 1, 2$).

4) Verify that $[X^k, Y] = \sum_{l=0}^{k-1} X^l [X, Y] X^{k-l-1}$ for any square matrices X, Y.

5) Use 4) to show that

$$\phi w v = \mathrm{Tr}(wv \otimes \phi) = -\mathrm{Tr}(w[B_1, B_2])$$

$$= -\mathrm{Tr}(B_1^{k_1}[B_2^{k_2}, B_1]B_2) = \cdots = -\sum_{l=0}^{k_2-1} \phi B_2^{k_2-l} B_1^{k_1} B_2^l v$$

where $w = B_1^{k_1} B_2^{k_2}$ as in 3),

6) Deduce from 3) and 5) that $\phi w v = -k_2 \phi w v$, so $\phi w v = 0$

7) Deduce that $\phi = 0$.

This immediately yields the following alternative description of the Hilbert scheme, looking a little more like the definition of the Calogero–Moser spaces.

Corollary 2.3. *$X^{[n]}$ is isomorphic to the set of $\mathrm{GL}_n(\mathbb{C})$ orbits in the space*

$$\left\{ (B_1, B_2, v, \phi) \in \mathrm{End}(V)^2 \times V \times V^* \;\middle|\; \begin{array}{c} [B_1, B_2] + v \otimes \phi = 0, \textit{ and if} \\ U \subset V \textit{ with } v \in U \textit{ and} \\ B_i(U) \subset U, i = 1, 2, \textit{ then } U = V \end{array} \right\}$$

where $V = \mathbb{C}^n$.

3. Real and complex symplectic geometry

We will quickly cover the basics of symplectic (and holomorphic symplectic) geometry. The aim is to get to the definition of the moment map, consider some examples and define the symplectic quotient construction.

Symplectic vector spaces.

First it is useful to recall (from basic linear algebra) the canonical form of a skew-symmetric bilinear form on a vector space.

Lemma 3.1. *Let V be a finite dimensional vector space (over a field of characteristic not equal to 2) and let ω be a skew-symmetric bilinear form on V. Then there is a basis*

$$u_1, \dots, u_k, v_1, \dots, v_k, e_1, \dots, e_l$$

of V such that

$$\omega(u_i, v_i) = 1 = -\omega(v_i, u_i)$$

for $i = 1, \dots, k$ and ω is zero on all other pairs of basis vectors.

Proof. If $\omega \neq 0$ then there are $u, v \in V$ such that $\omega(u, v) \neq 0$ and we may scale u such that $\omega(u, v) = 1$. Clearly u, v are linearly independent (since $\omega(u, u) = 0$ if char$\neq 2$), so we may set $u_1 = u, v_1 = v$. Let $V_1 \subset V$ be the span of u_1, v_1, and set

$$U = V_1^{\perp} = \{x \in V \mid \omega(x, u) = \omega(x, v) = 0\}.$$

If $x \in V$ then $x' := x - u_1\omega(x, v_1) + v_1\omega(x, u_1)$ is in U, and so $V = V_1 \oplus U$. Now if $\omega|_U \neq 0$ we may iterate until we find $V = V_1 \oplus \cdots V_k \oplus U$ and $\omega|_U = 0$. □

Here we are only interested in the real and complex cases.

A (real) symplectic vector space (V, ω) is a real vector space V together with a skew-symmetric bilinear form

$$\omega : V \otimes V \to \mathbb{R}$$

which is nondegenerate in the sense that the associated linear map

$$\omega^\flat : V \to V^*; \quad v \mapsto \iota_v\omega = \omega(v, \cdot) \tag{3.1}$$

from V to its dual space, is an isomorphism. By Lemma 3.1 we may then find a basis $p_1, \dots, p_n, q_1, \dots, q_n$ of V^* such that

$$\omega = \sum_1^n dp_i \wedge dq_i$$

for some integer n. In particular V has even real dimension $2n$.

Similarly a complex symplectic vector space $(V, \omega_{\mathbb{C}})$ is a complex vector space V together with a skew-symmetric $\mathbb{C}$-bilinear form

$$\omega_{\mathbb{C}} : V \otimes V \to \mathbb{C}$$

which is nondegenerate in the sense that the associated linear map

$$\omega_{\mathbb{C}}^{\flat} : V \to V^*; \quad v \mapsto \iota_v \omega_{\mathbb{C}} = \omega_{\mathbb{C}}(v, \cdot) \tag{3.2}$$

from V to its dual space, is an isomorphism. (Beware now V^* denotes the complex dual space, of $\mathbb{C}$-linear maps $V \to \mathbb{C}$.) Again by Lemma 3.1 we can find a basis $p_1, \ldots, p_n, q_1, \ldots, q_n$ of V^* such that

$$\omega_{\mathbb{C}} = \sum_1^n dp_i \wedge dq_i$$

for some integer n. In particular V has even complex dimension $2n$.

A simple example should clarify the distinction between real and complex symplectic vector spaces (the simplest examples of real and complex symplectic manifolds). Take $V = \mathbb{C}^2 \cong \mathbb{R}^4$ with complex (linear) coordinates $z, w \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Then $\omega_{\mathbb{C}} = dz \wedge dw$ is a complex symplectic form. If we write $z = x + iy, w = u + iv$ for real coordinates $x, y, u, v \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ then

$$\omega_{\mathbb{C}} = dx \wedge du + dv \wedge dy + i(dy \wedge du + dx \wedge dv),$$

and we see that both the real and imaginary parts of $\omega_{\mathbb{C}}$ are real symplectic forms on $\mathbb{R}^4$.

Basic definitions for symplectic manifolds.

Definition 3.2. *A (real)* symplectic manifold *is a pair (M, ω) consisting of a differentiable manifold M and a real two-form ω, such that:*

- *ω is closed: i.e. $d\omega = 0$, and*
- *ω is nondegenerate: the associated linear map $T_m M \to T_m^* M; v \mapsto \omega_m(v, \cdot)$ from the tangent space to the cotangent space, is an isomorphism at each point $m \in M$.*

Thus firstly the tangent space $T_m M$ to a symplectic manifold is a symplectic vector space at each point $m \in M$, but there is also the nonalgebraic condition that ω is closed.

In the holomorphic (or complex algebraic) category one uses a different notion (and one should not confuse the two), as follows:

Definition 3.3. *A* complex symplectic manifold *is a pair $(M, \omega_{\mathbb{C}})$ consisting of a complex manifold M and a* holomorphic *two-form $\omega_{\mathbb{C}}$ (of type $(2, 0)$) such that:*

- *$\omega_{\mathbb{C}}$ is closed: i.e. $d\omega_{\mathbb{C}} = 0$, and*

• $\omega_{\mathbb{C}}$ *is nondegenerate: the associated linear map* $T_mM \to T^*_mM; v \mapsto (\omega_{\mathbb{C}})_m(v,\cdot)$ *from the* holomorphic *tangent space to the holomorphic cotangent space, is an isomorphism at each point* $m \in M$.

These are sometimes also referred to as *holomorphic* symplectic manifolds. (Note that if one just stipulates that $\omega_{\mathbb{C}}$ is a C^∞ global $(2,0)$ form, then requiring it to be closed implies it is in fact holomorphic.)

Basic examples of symplectic manifolds.

Example 3.4 (Cotangent bundles)**.** Let N be a manifold and let $M = T^*N$ be the total space of its cotangent bundle. This has a natural symplectic structure, which may be defined locally as follows. Choose local coordinates $x_1, \ldots, x_n$ on N. Then the one-forms $dx_1, \ldots, dx_n$ provide a local trivialisation of T^*N, so we obtain local coordinate functions $p_1, \ldots p_n$ on the fibres of T^*N (fixing the values of these determine the point $\sum p_i dx_i$ of the fibre). Thus M has local coordinates x_i, p_i. We may define a one form

$$\theta = \sum_1^n p_i dx_i$$

locally on M and it turns out that this local definition in fact defines a global one-form (the "Liouville form") on M. The exterior derivative

$$\omega = d\theta$$

is a natural symplectic form on M. Clearly it is closed (since it is exact), and it is nondegenerate because in local coordinates it is just

$$\sum_1^n dp_i \wedge dx_i.$$

This is the basic class of symplectic manifolds crucial to classical mechanics and much else, and indeed any symplectic manifold is locally of this form (the Darboux theorem). However there are many other symplectic manifolds which are not cotangent bundles globally.

The intrinsic definition of the Liouville form θ is as follows. Given a point $m = (p, x) \in M = T^*N$ (with $x \in N, p \in T^*_xN$) and a tangent vector $v \in T_mM$ the one form θ should produce a number $\langle\theta, v\rangle_m$. This number is obtained from v as follows: the derivative at m of the projection $\pi : M \to N$ is a map $d\pi_m : T_mM \to T_xN$, and we simply pair the image of v with p:

$$\langle\theta, v\rangle_m := \langle p, d\pi_m(v)\rangle.$$

This example may be read in both a real fashion (to obtain a real symplectic form) or in a complex fashion, if M is a complex manifold, using the holomorphic (co)tangent bundles and coordinates, to get a complex symplectic form on the total space of the holomorphic cotangent bundle.

Example 3.5 (Coadjoint orbits). Let G be a Lie group and let $\mathfrak{g} = T_eG$ be its Lie algebra. Let $\mathfrak{g}^* = T_e^*G$ be the dual vector space to $\mathfrak{g}$. If G acts on itself by conjugation, this fixes the identity $e \in G$ and so we have an induced action on $\mathfrak{g}$ and $\mathfrak{g}^*$, the adjoint action Ad and the coadjoint action Ad^*. These are related as follows, where $X \in \mathfrak{g}, \alpha \in \mathfrak{g}^*, g \in G$:

$$\langle \mathrm{Ad}_g^*(\alpha), X\rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}}(X)\rangle,$$

where the brackets denote the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Infinitesimally (writing $g = \exp(Xt)$ for $X \in \mathfrak{g}$ and taking the derivaive at $t = 0$) yields the corresponding Lie algebra actions:

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}; \qquad Y \mapsto \mathrm{ad}_X(Y) = [X, Y]$$
$$\mathrm{ad}_X^* : \mathfrak{g}^* \to \mathfrak{g}^*; \qquad \alpha \mapsto \mathrm{ad}_X^*(\alpha)$$

which in turn are related by:

$$\langle \mathrm{ad}_X^*(\alpha), Y\rangle = -\langle \alpha, \mathrm{ad}_X(Y)\rangle.$$

It follows immediately from these definitions that if $\mathcal{O} \subset \mathfrak{g}$ is an arbitrary orbit for the adjoint action, and $Y \in \mathcal{O}$ then the tangent space to $\mathcal{O}$ at Y is $\{\mathrm{ad}_X(Y) \mid X \in \mathfrak{g}\}$. E.g. for $G = \mathrm{GL}_n(\mathbb{C})$ the orbit through Y is just the set of conjugate $n \times n$ matrices:

$$\mathcal{O} = \{gYg^{-1} \mid g \in G\} \subset \mathfrak{gl}_n(\mathbb{C}).$$

Then writing $g = \exp(Xt)$ for $X \in \mathfrak{g}$ and taking the derivaive at $t = 0$ yields

$$T_Y\mathcal{O} = \{XY - YX \mid X \in \mathfrak{g}\} \subset \mathfrak{gl}_n(\mathbb{C})$$

as stated, since here $\mathrm{ad}_X(Y) = [X, Y]$ agrees with the commutator $XY - YX$ of matrices. Indeed in general one has

$$\frac{d}{dt}\left(\mathrm{Ad}_{\exp(Xt)}(Y)\right)\Big|_{t=0} = [X, Y] \in \mathfrak{g}.$$

Similarly (basically by definition) if $\mathcal{O} \subset \mathfrak{g}^*$ is an arbitrary orbit for the coadjoint action, and $\alpha \in \mathcal{O}$ then the tangent space to $\mathcal{O}$ at α is

$$T_\alpha\mathcal{O} = \{\mathrm{ad}_X^*(\alpha) \mid X \in \mathfrak{g}\}.$$

An important fact about coadjoint orbits is the following:

Theorem 3.6 (Kostant–Kirillov–Souriau). *Coadjoint orbits are symplectic manifolds: Any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ has a natural G-invariant symplectic structure, given by the formula:*

$$\omega_\alpha(\mathrm{ad}_X^*(\alpha), \mathrm{ad}_Y^*(\alpha)) = \langle \alpha, [X, Y]\rangle = \alpha([X, Y])$$

for all $\alpha \in \mathcal{O} \subset \mathfrak{g}^*, X, Y \in \mathfrak{g}$.

Proof. To simplify notation, for $X \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$, write $[X, \alpha] := \mathrm{ad}^*_X(\alpha) \in \mathfrak{g}^*$. First we check ω is well defined and nondegenerate. Note that

$$\begin{aligned}[X_1, \alpha] = [X_2, \alpha] &\Leftrightarrow \langle [X_1, \alpha], Y\rangle = \langle [X_2, \alpha], Y\rangle \quad \text{for all } Y \in \mathfrak{g}\\ &\Leftrightarrow \alpha([Y, X_1]) = \alpha([Y, X_2]) \quad \text{for all } Y \in \mathfrak{g}.\end{aligned}$$

Thus ω is independent of the choice of X, Y representing the tanget vectors. Moreover (putting $X_2 = 0$) we see ω is nondegenerate. It is straightforward to see ω is G invariant, so we need just check it is closed. Given $X \in \mathfrak{g}$ write v_X for the vector field on $\mathcal{O}$ taking the value $v_{X,\alpha} := -[X, \alpha] \in T_\alpha\mathcal{O}$ at α for each $\alpha \in \mathcal{O}$. Also let H_X be the function on $\mathcal{O}$ defined by $H_X(\alpha) = \alpha(X)$. We then claim that

$$dH_X = \omega(\cdot, v_X)$$

as one-forms on $\mathcal{O}$, for all $X \in \mathfrak{g}$. Indeed any tangent vector is of the form v_Y for some $Y \in \mathfrak{g}$ and

$$\omega(v_Y, v_X) = \alpha([Y, X]) = -\langle [Y, \alpha], X\rangle = H_X(-[Y, \alpha])$$

which equals the derivaive of H_X at α along v_Y (since it is linear). Now since ω is G-invariant and the infinitesimal G-action maps $X \in \mathfrak{g}$ to (minus) the vector field v_X (it is the "fundamental vector field" of the action, to be defined below) we have

$$0 = \mathcal{L}_{v_X}\omega = (d\iota_{v_X} + \iota_{v_X}d)\omega = -d(dH_X) + \iota_{v_X}d\omega = \iota_{v_X}d\omega.$$

But the vector fields of the form v_X span the tangent space to $\mathcal{O}$ at each point so we deduce $d\omega = 0$. $\square$

This becomes more explicit in the case where $\mathfrak{g}$ admits a nondegenerate invariant symmetric bilinear form $\mathcal{B}$. Then the adjoint orbits and coadjoint orbits may be identified, with $\alpha \in \mathfrak{g}^*$ corresponding to $A \in \mathfrak{g}$ such that $\alpha = \mathcal{B}(A, \cdot)$. The adjoint and coadjoint actions then correspond to each other. It follows then that the adjoint orbits obtain a symplectic structure, and this may be written as

$$\omega_A([X, A], [Y, A]) = \mathcal{B}(A, [X, Y])$$

for $A \in \mathcal{O} \subset \mathfrak{g}$.

For example taking $G = \mathrm{SO}_3(\mathbb{R})$ we have $\mathfrak{g} \cong \mathbb{R}^3$ (with Lie bracket given by the cross-product) and the adjoint action corresponds to the standard action of G on $\mathbb{R}^3$. The orbits are two-spheres of fixed radius (and the origin). These have a symplectic structure since $\mathcal{B}(A, B) = \mathrm{Tr}(AB)$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ the 3×3 real skew-symmetric matrices.

These are compact so clearly not isomorphic to cotangent bundles.

Exercise 3.7. Check that $\mathfrak{so}_3(\mathbb{R}) \cong \mathbb{R}^3$ (with Lie bracket given by the cross-product).

Another example is to choose n real numbers $\lambda_1, \ldots, \lambda_n$ and consider the set $\mathcal{O}$ of $n \times n$ Hermitian matrices with these eigenvalues. Multiplying by i identifies the Hermitian matrices with the skew-Hermitian matrices, the Lie algebra of the unitary group $\mathrm{U}(n)$. This has nondegenerate invariant symmetric bilinear form $\mathcal{B}(A, B) = \mathrm{Tr}(AB)$. Moreover the adjoint action is simply given by matrix conjugation, so $\mathcal{O}$ is indeed an adjoint orbit and thus a symplectic manifold.

Exercise 3.8 (*)**.** Show that one obtains the projective spaces $\mathbb{P}^{n-1}$ (the space of lines in $\mathbb{C}^n$) and the Grassmannians $\mathrm{Gr}_k(\mathbb{C}^n)$ (the spaces of k-dimensional subspaces of $\mathbb{C}^n$) as examples of adjoint orbits of $\mathrm{U}(n)$. Identify the other adjoint orbits of $\mathrm{U}(n)$ with flag manifolds.

If G is a complex Lie group then we obtain a complex symplectic structure on its coadjoint orbits. For example take $G = \mathrm{GL}_n(\mathbb{C})$, so $\mathfrak{g}$ is just the set of $n \times n$ complex matrices. The pairing $\mathcal{B}(A, B) = \mathrm{Tr}(AB)$ is a nondegenerate invariant symmetric *complex* bilinear form, and the adjoint action is given by matrix conjugation: $\mathrm{Ad}_g(A) = gAg^{-1}$ (and $[X, A] = XA - AX$). (Beware that this is not the Killing form, which is degenerate for $\mathrm{GL}_n(\mathbb{C})$.) Thus the orbit $\mathcal{O} \subset \mathfrak{g}$ of a matrix $A \in \mathfrak{g}$ is simply the set of matrices with the same Jordan normal form as A. This has a complex symplectic structure, given by the same formula as above:

$$\omega_A([X, A], [Y, A]) = \mathrm{Tr}(A[X, Y]).$$

These are basic examples of complex symplectic manifolds.

Exercise 3.9. Take $G = \mathrm{SL}_2(\mathbb{C})$, so $\mathfrak{g} \cong \mathbb{C}^3$ is the space of of 2×2 tracefree complex matrices. Set $\mathcal{B}(A, B) = \mathrm{Tr}(AB)$. (Co)adjoint orbits of G are the simplest nontrivial examples of complex symplectic manifolds.

1) Choose a diagonal matrix $A \in \mathfrak{g}$ and consider its orbit $\mathcal{O}$. Write down an algebraic equation for $\mathcal{O} \subset \mathbb{C}^3$, showing it is a (smooth) affine surface (i.e. a real four-manifold). Show, by considering the holomorphic map $\mathcal{O} \to \mathbb{P}^1$ taking the first eigenspace (or otherwise), that $\mathcal{O}$ is not isomorphic to $\mathbb{C}^2$.

2) Choose a nonzero nilpotent matrix $A \in \mathfrak{g}$ and consider its orbit $\mathcal{O}$. Write down an equation satisfied by the points of $\mathcal{O} \subset \mathbb{C}^3$, and observe that zero is also a point of the zero locus of this equation. Deduce that $\mathcal{O}$ is the smooth locus of a singular affine surface. (This is the A_1 surface singularity $\cong \{xy = z^2\} \subset \mathbb{C}^3$.)

One motivation for coadjoint orbits is the (heuristic) "orbit method" of representation theory: one would like to quantise coadjoint orbits of G to obtain representations (see e.g. [Kir99, Vog00]). For example for a compact group Borel–Weil theory may be viewed as saying that any irreducible representation arises by geometric quantisation of the coadjoint orbit through the highest weight. Our direct motivation is however more geometric: complex coadjoint orbits will provide basic examples of hyperkähler manifolds.

Hamiltonian vector fields and Poisson brackets.

Given a function f on a symplectic manifold (M, ω), we obtain a vector field v_f, by taking the derivative of f and then using the isomorphism between the tangent and cotangent bundle furnished by the symplectic form. This is the *Hamiltonian vector field* of the function f. Explicitly v_f is defined by the formula:

$$\boxed{df = \omega(\cdot, v_f).}$$

In other words $df = -\iota_{v_f}\omega = -\omega^\flat(v_f)$. (The minus sign is put so that Lemma 3.11 below holds—i.e. we get a Lie algebra morphism.) Thus we obtain a map from the functions on M to the Lie algebra of vector fields on M. Not all vector fields arise in this way, since for example we have:

Lemma 3.10. *The flow of a Hamiltonian vector field preserves the symplectic form.*

Proof. This follows from Cartan's formula:

$$\begin{aligned}\mathcal{L}_{v_f}\omega &= (d\iota_{v_f} + \iota_{v_f} d)\omega \\ &= d\iota_{v_f}\omega \\ &= -d(df) \\ &= 0\end{aligned}$$

□

However in many instances the vector fields we are interested in are Hamiltonian, and so their study is essentially reduced to studying functions. Indeed the notion of Poisson bracket lifts the Lie algebra structure from the vector fields to the functions on M. More precisely, the symplectic form determines a bilinear bracket operation on the functions

$$\{\cdot, \cdot\} : Fun(M) \otimes Fun(M) \to Fun(M)$$

defined by

$$\{f, g\} = \omega(v_f, v_g).$$

This is the Poisson bracket associated to ω. In the real symplectic case we may take $Fun(M) = C^\infty(M)$. In the complex symplectic case we should take the holomorphic functions, or better (since there may not be many global holomorphic functions), the sheaf of holomorphic functions. In this setting the Poisson bracket provides, for any open subset $U \subset M$, an operation

$$\{\cdot, \cdot\} : Fun(U) \otimes Fun(U) \to Fun(U)$$

where $Fun(U)$ denotes the holomorphic functions on U. (Since the proofs are the same we will generally omit this extra level of complexity from the notation/statements.) The main properties of the Poisson bracket are as follows.

Lemma 3.11. *0)* $\{f, g\} = -\{g, f\}$,

1) $v_f = \{f, \cdot\}$ *as derivations acting on functions,*

2) The map $f \to v_f$ *from functions to vector fields satisfies* $v_{\{f,g\}} = [v_f, v_g]$,

3) The Poisson bracket makes the functions $Fun(M)$ *on* M *into a Lie algebra (and* $Fun(\cdot)$ *in to a sheaf of Lie algebras).*

Proof. 0) is clear, and 1) is immediate since:

$$v_f(g) = \langle v_f, dg\rangle = \langle v_f, \omega(\cdot, v_g)\rangle = \omega(v_f, v_g) = \{f, g\}.$$

For 2) we need to show $\omega(\cdot, [v_f, v_g]) = d\{f, g\}$, i.e. that $\iota_{[v_f,v_g]}\omega = -d\{f, g\}$. However the standard formula $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$ combined with the Cartan formula $\mathcal{L}_X = d\iota_X + \iota_X d$ for the Lie derivative acting on forms implies:

$$\iota_{[X,Y]} = d\iota_X\iota_Y - \iota_X d\iota_Y - \iota_Y d\iota_X + \iota_X\iota_Y d$$

as operators acting on forms for any vector fields X, Y. Thus in our situation we obtain

$$\iota_{[v_f,v_g]}\omega = d(\omega(v_g, v_f)) = -d\{f, g\}$$

as required (since the other terms vanish).

For 3) we need just to check the Jacobi identity. This now follows directly from 1) and 2):

$$\{\{f, g\}, h\} = v_{\{f,g\}}(h) = [v_f, v_g](h) = v_f(v_g(h)) - v_g(v_f(h)) = \{f, \{g, h\}\} - \{g, \{f, h\}\}.$$

□

Thus the map from function to vector fields $f \mapsto v_f$ is a Lie algebra homomorphism. Note that the Poisson bracket is only bilinear over the base field ($\mathbb{R}$ or $\mathbb{C}$); indeed 1) here implies that $\{f, gh\} = \{f, h\}g + \{f, g\}h$.

Lie group actions on symplectic manifolds.

Suppose a Lie group G acts on a manifold M. Let $\mathfrak{g}$ denote the Lie algebra of G. For any $X \in \mathfrak{g}$ we denote by v_X the *fundamental vector field* of X. By definition this is obtained by taking (minus) the tangent vector to the flow on M generated by X. Explicitly X determines a one-parameter subgroup e^{Xt} of G, and we act with this on M:

$$m \mapsto e^{Xt} \cdot m.$$

The derivative of this at $t = 0$ is a vector field on M, and (by convention) we define v_X to be minus this vector field:

$$v_X = -\frac{d}{dt}\left(e^{Xt}\cdot m\right)\big|_{t=0}.$$

The reason for the sign added here is as follows:

Lemma 3.12. *The map $\mathfrak{g} \to \mathcal{X}(M); X \mapsto v_X$ from the Lie algebra of $\mathfrak{g}$ to the set of global vector fields on M is a Lie algebra homomorphism. In other words $[v_X, v_Y] = v_{[X,Y]}$ for all $X, Y \in \mathfrak{g}$.*

Sketch. Suppose $X \in \mathfrak{g}$ and $g \in G$. Then the action of G on M enables us to view g as an automorphism of M, and the induced action on vector fields will be denoted $g \cdot v$ (for v a vector field on M). First one may show directly that

$$v_{\mathrm{Ad}_g(X)} = g \cdot v_X. \tag{3.3}$$

(this is basically the chain rule). Then, from the definition of the Lie bracket of vector fields, we have:

$$\begin{aligned}
[v_X, v_Y] &= \frac{d}{dt}\Big|_{t=0} g \cdot v_Y && \text{where } g = e^{Xt}\\
&= \frac{d}{dt}\Big|_{t=0} v_{\mathrm{Ad}_g(Y)} && \text{by (3.3)}\\
&= v_{[X,Y]}
\end{aligned}$$

as required. The point is that there is a minus sign in the definition of the Lie bracket of two vector fields v, w:

$$[v, w]_m = \mathcal{L}_v(w)_m = \frac{d}{dt}\Big|_{t=0} d\Phi_{-t}(w_{\Phi_t(m)})$$

(where Φ_t is the (local) flow of v and $m \in M$). We put a minus sign in the definition of v_X so that Φ_{-t} is the action of e^{Xt} when $v = v_X$. $\square$

On the other hand recall that given a function f on a symplectic manifold (M, ω), we obtain a Hamiltonian vector field v_f which preserves the symplectic form.

Now suppose a Lie group G acts on a symplectic manifold (M, ω) preserving the symplectic form. The role of a moment map is to combine the above two situations, i.e. we would like a collection of (Hamiltonian) functions, one for each element of the Lie algebra of G, whose Hamiltonian vector fields are the corresponding fundamental vector fields. This is encompassed by the following definition.

Definition 3.13. *A* moment map *for the G action on (M, ω) is a G-equivariant map*

$$\mu : M \to \mathfrak{g}^*$$

from M to the dual of the Lie algebra of G, such that:

$$\boxed{d\langle \mu, X\rangle = \omega(\cdot, v_X) \qquad \text{for all } X \in \mathfrak{g}.}$$

Here G acts on $\mathfrak{g}^*$ by the coadjoint action.

Thus for any $X \in \mathfrak{g}$ we obtain a function $\mu^X := \langle \mu, X\rangle$ on M, the X-component of μ, and we demand that this has Hamiltonian vector field equal to the fundamental vector field v_X of X.

Said differently the Poisson bracket makes the functions $Fun(M)$ on M into a Lie algebra. Taking Hamiltonian vector fields yields a Lie algebra map $Fun(M) \to \mathcal{X}(M)$ to the vector fields on M. The moment map provides a lift of the map $\mathfrak{g} \to \mathcal{X}(M)$ to a Lie algebra morphism $\mathfrak{g} \to Fun(M)$.

We will see the notion of moment map is crucial for many reasons (in particular for constructing hyperkähler manifolds).

Remark 3.14 (Shifting moment maps)**.** Note that if $\mu : M \to \mathfrak{g}^*$ is a moment map and we choose any element $\lambda \in \mathfrak{g}^*$ which is preserved by the coadjoint action (i.e. $\mathrm{Ad}^*_g(\lambda) = \lambda$ for all $g \in G$), then the map $\mu - \lambda$:

$$m \mapsto \mu(m) - \lambda$$

is also a moment map, since it is still equivariant and has the same derivative.

Exercise 3.15. Suppose G acts on (M, ω) with moment map μ, and that $H \subset G$ is a Lie subgroup. Thus the derivative at the identity of the inclusion is a map $\mathfrak{h} \to \mathfrak{g}$, and the dual linear map is a map $\pi : \mathfrak{g}^* \to \mathfrak{h}^*$. Show that $\pi \circ \mu$ is a moment map for the action of the subgroup H on M.

Exercise 3.16. Suppose G acts on two symplectic manifold (M_i, ω_i) with moment maps μ_i, for $i = 1, 2$. Show that

$$\mu_1 + \mu_2 : M_1 \times M_2 \to \mathfrak{g}^*; (m_1, m_2) \mapsto \mu_1(m_1) + \mu_2(m_2) \tag{3.4}$$

is a moment map for the diagonal action of G on the product $M_1 \times M_2$, defined by $g(m_1, m_2) = (g \cdot m_1, g \cdot m_2)$.

Examples of moment maps.

Lemma 3.17. *Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit of a Lie group G. Let μ be inclusion map*

$$\mu : \mathcal{O} \to \mathfrak{g}^*.$$

Then μ is a moment map for the coadjoint action of G on $\mathcal{O}$.

Proof. This was established in the proof of Theorem 3.6. In any case it is a straightforward (if confusing) unwinding of the definitions: given $X \in \mathfrak{g}$ we must show $d\langle \mu, X\rangle = \omega_\alpha(\cdot, v_X)$ at each point $\alpha \in \mathcal{O}$. In other words

$$\langle \beta, d\langle \mu, X\rangle\rangle = \omega_\alpha(\beta, v_X)$$

for any $\beta \in T_\alpha\mathcal{O} \subset \mathfrak{g}^*$. Since μ is the inclusion, the left-hand side is just $\beta(X)$. On the other hand $v_X = -\mathrm{ad}^*_X(\alpha)$ and so by definition the right-hand side is

$$\omega_\alpha(\mathrm{ad}^*_X(\alpha), \beta) = \langle \alpha, [X, Y]\rangle$$

for any $Y \in \mathfrak{g}$ such that $\beta = \mathrm{ad}^*_Y(\alpha)$. Now $\langle \alpha, [X, Y]\rangle = -\langle \alpha, \mathrm{ad}_Y(X)\rangle = \langle \mathrm{ad}^*_Y(\alpha), X\rangle = \beta(X)$ as required. □

Lemma 3.18. *Let (V, ω) be a symplectic vector space and let $G = \mathrm{Sp}(V)$ be the group of linear automorphisms of V preserving the symplectic form. Then the map*

$$\mu : V \to \mathfrak{g}^*; \qquad v \mapsto \text{“}A \mapsto \frac{1}{2}\omega(Av, v)\text{”}$$

is a moment map for the action of $G = \mathrm{Sp}(V)$ on V.

Proof. Given $A \in \mathfrak{g}$ we should verify that $d\langle \mu, A\rangle = \omega(\cdot, v_A)$. Since ω is closed the left hand side is

$$\frac{1}{2}\left(\omega(Adv, v) + \omega(Av, dv)\right) = \omega(Av, dv)$$

since A is in the Lie algebra of $\mathrm{Sp}(V)$. Thus we must show $\omega(Av, w) = \omega(w, v_A)$ for any $w \in V$. Now at $v \in V$ the vector field v_A takes the value $-Av$ (which is minus tangent to the flow), so the result follows by skew-symmetry. □

Example 3.19. The name “moment map” (or momentum map), comes from the following example. Suppose G acts on a manifold N. Then there is an induced action on the symplectic manifold $M = T^*N$, the *cotangent lift* of the action on N. The action of G on M is then Hamiltonian and the moment map is given by pairing the momentum (the fibre coordinates in $M \to N$) with the fundamental vector field for the G action on N. More precisely:

Lemma 3.20. *The map*

$$\mu : M = T^*N \to \mathfrak{g}^*; \qquad (p, x) \mapsto \text{“}X \mapsto \langle p, v_X^N\rangle\text{”}$$

is a moment map for the action of G on M, where $x \in N, p \in T^*_xN$ and $v_X^N \in \Gamma(TN)$ is the fundamental vector field for the action of G on N.

Proof. More generally suppose we are in the situation where the symplectic form ω is exact: $\omega = d\theta$, and the G action preserves θ (so $\mathcal{L}_{v_X}\theta = 0$ for all $X \in \mathfrak{g}$). Then consider the map $\mu : M \to \mathfrak{g}^*$ defined by

$$m \mapsto \text{“}X \mapsto \langle \theta, v_X \rangle_m\text{”}$$

where v_X is the fundamental vector field on M. This is a moment map since, by Cartan's formula

$$\iota_{v_X}\omega = \iota_{v_X} d\theta = \mathcal{L}_{v_X}\theta - d\iota_{v_X}\theta = -d\langle \mu, X\rangle.$$

(See e.g. [LM87] p.192 for proof of equivariance, omitted here.) Now in our situation, where $\theta = \sum p_i dx_i$ is the Liouville form, we must check this definition of μ coincides with that above, i.e. that

$$\langle \theta, v_X \rangle_{(p,x)} = \langle p, v_X^N \rangle_x$$

where $x \in N, p \in T_x^*N$. This however is immediate by definition of θ, and the fact that v_X is a lift of v_X^N. (θ the projection of v_X to N, and pairs it with p.) □

For example let V be a complex vector space, and take $G = \mathrm{GL}(V)$ acting on $N = V$ in the natural way. Then $M = T^*V \cong V^* \times V$ and the moment map is $\mu(\alpha, v)(X) = -\alpha(Xv)$ where $\alpha \in V^*, X \in \mathfrak{g} = \mathrm{End}(V)$. If we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the pairing $\mathrm{Tr}(AB)$ then we have:

$$\mu(\alpha, v) = -v \otimes \alpha \in \mathrm{End}(V). \tag{3.5}$$

Similarly if $G = \mathrm{GL}(V)$ acts on $N = \mathrm{End}(V)$ by conjugation. Then $M = T^*\,\mathrm{End}(V) \cong \mathrm{End}(V) \times \mathrm{End}(V)$ and the moment map is

$$\mu(B_1, B_2)(X) = -\mathrm{Tr}(B_1[X, B_2]) = \mathrm{Tr}(X[B_1, B_2]) \tag{3.6}$$

where $B_i \in \mathrm{End}(V)$ $(B_2 \in N, B_1 \in T^*_{B_1}\mathrm{End}(V))$, $X \in \mathfrak{g} = \mathrm{End}(V)$. If we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the pairing $\mathrm{Tr}(AB)$ then we have:

$$\mu(B_1, B_2) = [B_1, B_2] \in \mathrm{End}(V),$$

i.e. the moment map is given by the commutator.

Symplectic Quotient Construction.

The aim of this subsection is to define a way construct new symplectic manifolds from old ones. Note that in general the quotient of a symplectic manifold by a Lie group may not be even dimensional (e.g. S^2/S^1 is an interval, where the circle acts by rotation), so cannot be symplectic in general. The following construction shows how the use of the moment map guides the way to obtain symplectic manifolds.

First some linear algebra. Let (V, ω) be a symplectic vector space and $W \subset V$ a subspace. Define the symplectic orthogonal $W^\perp \subset V$ as

$$W^\perp = \{v \in V \mid \omega(w, v) = 0 \text{ for all } w \in W\} \subset V$$

and the annihilator $W^\circ \subset V^*$ as

$$W^\circ = \{\alpha \in V^* \;\big|\; \alpha(w) = 0 \text{ for all } w \in W\} \subset V^*.$$

Let $\omega^\flat : V \to V^*$ be the linear map determined by ω:

$$\omega^\flat(v) = \iota_v\omega = \text{“}u \mapsto \omega(v,u)\text{”}$$

where $u, v \in V$. The following easy observation will be useful:

Lemma 3.21. *Let V be a symplectic vector space and $W \subset V$ a subspace. Then*

$$\omega^\flat(W) = (W^\perp)^\circ$$

as subspaces of V^*.

Proof. Clearly $\omega^\flat$ is bijective, and if $\alpha = \omega^\flat(w)$ for some $w \in W$, then $\alpha(u) = \omega(w,u) = 0$ for any $u \in W^\perp$. Thus $\omega^\flat(W) \subset (W^\perp)^\circ$, so they must be equal as they have the same dimension. □

The basic idea of symplectic quotients is as follows. Suppose a Lie group G acts on a symplectic manifold (M,ω) and this action admits a moment map μ. Let

$$Z = \mu^{-1}(0) = \{m \in M \;\big|\; \mu(m) = 0\}$$

be the inverse image of zero under the moment map. Since the moment map is equivariant and zero is preserved by the coadjoint action, then Z is G invariant and we can consider the quotient

$$M/\!\!/G := Z/G = \mu^{-1}(0)/G.$$

The point is that under some mild conditions this is again a symplectic manifold, the *symplectic quotient* (or Marsden–Weinstein quotient) of M by G.

First we will explain how the quotient inherits a symplectic form. Let us denote by ι the inclusion $\iota : Z \hookrightarrow M$ and by π the projection $\pi : Z \to M/\!\!/G$.

Choose a point $m \in Z$. If Z is a submanifold of M then the tangent space to Z is

$$\mathrm{Ker}(d\mu_m) = T_mZ \subset T_mM.$$

Moreover the G-orbit through m is within Z, so we can consider the tangent space to the orbit. Since the action is Hamiltonian this tangent space is simply the span of the Hamiltonian vector fields at m. We will denote it by $\mathfrak{g}_m$:

$$\mathfrak{g}_m = \{v_X \;\big|\; X \in G\} \subset T_mZ \subset T_mM.$$

Now the quotient Z/G parameterises the set of G orbits in Z, so if it is a manifold (and we have a local slice to the action) then its tangent space at $\pi(m)$ will be isomorphic to the quotient $T_mZ/\mathfrak{g}_m$. Thus the main step is to show that this vector space inherits a symplectic form from that on T_mM. Clearly we can restrict ω to T_mZ, where it will become degenerate. We need to show that the subspace we quotient by $(\mathfrak{g}_m)$ coincides with the degenerate directions, i.e. that

Proposition 3.22. $\mathfrak{g}_m = (\operatorname{Ker} d\mu_m)^\perp$.

Proof. This is equivalent to showing $\mathfrak{g}_m^\perp = \operatorname{Ker} d\mu_m$, or in turn that $(\mathfrak{g}_m^\perp)^\circ = (\operatorname{Ker} d\mu_m)^\circ$. Now by definition the dual linear map to the map $d\mu_m : T_m M \to T_0\mathfrak{g}^* = \mathfrak{g}^*$ is the map $\mathfrak{g} \to T_m^* M; X \mapsto \omega(\cdot, v_X)$. In general the kernel of a linear map is the annihilator of the image of the dual map, so (taking annihilators) we have

$$\begin{aligned}(\operatorname{Ker} d\mu_m)^\circ &= \{\omega(\cdot, v_X) \mid X \in \mathfrak{g}\} \\ &= \{\omega(\cdot, v) \mid v \in \mathfrak{g}_m\} \\ &= \omega^\flat(\mathfrak{g}_m) \subset T_m^* M\end{aligned}$$

Now we use Lemma 3.21 (with $W = \mathfrak{g}_m$) to see $\omega^\flat(\mathfrak{g}_m) = (\mathfrak{g}_m^\perp)^\circ$ and so the result follows. □

Thus $T_m Z/\mathfrak{g}_m$ is a symplectic vector space. Moreover since ω is G invariant, for any $g \in G$ the action of G yields a symplectic isomorphism $T_m Z/\mathfrak{g}_m \cong T_{g(m)} Z/\mathfrak{g}_{g(m)}$ and so we get a well defined nondegenerate two-form $\overline{\omega}$ on Z/G. To see it is closed we argue as follows. It is defined such that

$$\pi^*\overline{\omega} = \iota^*\omega.$$

Moreover $\pi^* d\overline{\omega} = d\pi^*\overline{\omega} = d\iota^*\omega = \iota^* d\omega = 0$. But this forces $d\overline{\omega} = 0$ since π is surjective on tangent vectors.

This establishes the following:

Theorem 3.23. *Suppose a Lie group G acts on a symplectic manifold (M, ω) with moment map μ such that $Z = \mu^{-1}(0)$ is a smooth submanifold of M, and the quotient Z/G is a smooth manifold. Then $M/\!\!/G := Z/G = \mu^{-1}(0)/G$ is a symplectic manifold.*

There are various conditions that may be added to ensure the various criteria are met, for example if G is a compact group acting freely and 0 is a regular value of the moment map. Note in this case one has:

$$\dim(M/\!\!/G) = \dim(M) - 2\dim(G).$$

The same argument works in the complex symplectic category (with the words holomorphic/complex added throughout):

Theorem 3.24. *Suppose a complex Lie group G acts holomorphically on a complex symplectic manifold $(M, \omega_{\mathbb{C}})$ with moment map $\mu_{\mathbb{C}}$ such that $Z = \mu_{\mathbb{C}}^{-1}(0)$ is a smooth complex submanifold of M, and the quotient Z/G is a smooth complex manifold. Then $M/\!\!/G := Z/G = \mu_{\mathbb{C}}^{-1}(0)/G$ is a complex symplectic manifold.*

Remark 3.25. Recall from Remark 3.14 that the moment map is not uniquely determined: if $\mu : M \to \mathfrak{g}^*$ is a moment map then so is $\nu := \mu - \lambda$ for any $\lambda \in \mathfrak{g}^*$ which is fixed by the coadjoint action. Now

$$\nu^{-1}(0)/G = \mu^{-1}(\lambda)/G$$

so we see that we may perform symplectic reduction at any invariant value λ of the moment map and not necessarily at zero (provided the various conditions hold to ensure the quotient is smooth).

Remark 3.26. More generally we may perform "symplectic reduction at any coadjoint orbit", as follows. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. Let $\mathcal{O}^-$ denote the orbit $\mathcal{O}$ with the symplectic form negated. Then

$$\begin{aligned}\mu^{-1}(\mathcal{O})/G &= \{(m,x) \in M \times \mathcal{O} \bigm| \mu(m) = x\}/G \\ &= \{(m,x) \in M \times \mathcal{O} \bigm| \mu(m) + (-x) = 0\}/G \\ &= (M \times \mathcal{O}^-)/\!\!/G\end{aligned}$$

is the symplectic quotient by G of the product of M and $\mathcal{O}^-$, so is again symplectic in general. (An invariant element $\lambda \in \mathfrak{g}^*$ is the simplest example of a coadjoint orbit—just a single point.)

Example 3.27. We can now obtain the Calogero–Moser spaces as complex symplectic quotients. Let $V = \mathbb{C}^n$ and consider $N = \mathrm{End}(V) \times V$, with the natural action of $G = \mathrm{GL}(V)$. Let $M = T^*N \cong \mathrm{End}(V) \times \mathrm{End}(V) \times V \times V^*$ be the total space of the holomorphic cotangent bundle of N with its standard complex symplectic form. By (3.4), (3.5) and (3.6) this has moment map

$$(X, Z, v, \alpha) \mapsto [X, Z] - v \otimes \alpha \in \mathrm{End}(V).$$

Clearly the complex symplectic quotient $\mu_{\mathbb{C}}^{-1}(-\mathrm{Id})/G$ is the Calogero–Moser space C_n, which we thus see has a natural complex symplectic structure.

Exercise 3.28 (Calogero–Moser flows)**.**

1) Compute the restriction of this symplectic form to the open part C_n' of C_n where X is diagonalisable (explicitly in terms of the eigenvalues of X). Show it agrees with the standard symplectic structure on $T^*((\mathbb{C}^n \setminus \text{diagonals})/\,\mathrm{Sym}_n)$.

2) Consider the function $H = \mathrm{Tr}(Z^2)/2$ on M. Show (upto signs) that the flow of the corresponding Hamiltonian vector field on M is given by

$$(X, Z, v, \alpha) \mapsto (X + tZ, Z, v, \alpha) \tag{3.7}$$

for $t \in \mathbb{C}$. Observe that this flow commutes with the action of G and preserves the level set $\mu_{\mathbb{C}}^{-1}(-\mathrm{Id})$ of the moment map.

3) Show that the restriction of H to $\mu_{\mathbb{C}}^{-1}(-\mathrm{Id})$ descends to a function on C_n, and on the open part C_n' equals

$$H = \mathrm{Tr}(Z^2)/2 = \sum_i \frac{1}{2}p_i^2 - \sum_{i<j}(x_i - x_j)^2$$

(where p_i, x_i are the coordinates on C_n'). This is the Calogero–Moser Hamiltonian, modelling the flows of n identical particles on the complex plane with an inverse square potential. Whilst on C_n' the flows are complicated and incomplete, we see by "unwinding" the symplectic quotient the flows become as in (3.7) very simple (and thus complete on the partial compactification C_n).[2]

[2]This "unwinding" is due to Kazhdan–Kostant–Sternberg [KKS78] and this complex case has been studied by G. Wilson [Wil98].

4. Quick review of Kähler geometry

We will quickly review (mostly without proof) some of the basics of Kähler geometry. This will be useful to highlight the analogies with the hyperkähler case later.

Linear algebra.

A *complex structure* on a real vector space V is an real linear endomorphism whose square is minus the identity: $\mathbf{I} : V \to V, \mathbf{I}^2 = -1$. This enables to view V as a complex vector space, with $i \in \mathbb{C}$ acting on V as $\mathbf{I}$ (this entails that V has even real dimension).

A *Kähler vector space* is a real vector space V together with a real symplectic form $\omega \in \bigwedge^2 V^*$ and a complex structure $\mathbf{I}$ such that:

a) the complex structure preserves the symplectic form

$$\omega(\mathbf{I}v, \mathbf{I}w) = \omega(v, w)$$

and

b) the associated real bilinear form g defined by

$$g(v, w) := \omega(v, \mathbf{I}w)$$

is positive definite (i.e. a metric—it is necessarily symmetric due to the invariance and antisymmetry of ω).

Now let V be a finite dimensional complex vector space. Recall that a (positive definite) *Hermitian form* on V associates a complex number $h(v, w)$ to a pair of $v, w \in V$. It is $\mathbb{C}$-linear in the first slot and is such that

$$h(v, w) = \overline{h(w, v)}$$

for all $v, w \in V$, and such that $h(v, v) > 0$ for all nonzero $v \in V$. Note that this implies $h(\mathbf{I}v, \mathbf{I}w) = h(v, w)$, i.e. the complex structure preserves the Hermitian form.

Note that the real part of a positive definite Hermitian form is a real metric (i.e. a positive definite real symmetric bilinear form):

$$g(v, w) := \operatorname{Re} h(v, w) = \operatorname{Re} \overline{h(v, w)} = \operatorname{Re} h(w, v) = g(w, v).$$

Also the imaginary part (and any nonzero real multiple of it) is a (real) symplectic form:

$$\operatorname{Im} h(v, w) = -\operatorname{Im} \overline{h(v, w)} = -\operatorname{Im} h(w, v)\,.$$

To check this skew-form is nondegenerate note that

$$\operatorname{Im} h(v, w) = -\operatorname{Re}\big(ih(v, w)\big) = -\operatorname{Re} h(\mathbf{I}v, w) = -g\,(\mathbf{I}v, w) \tag{4.1}$$

so $\operatorname{Im} h(\mathbf{I}v, v) = g(v, v)$ is zero only if $v = 0$.

Note that (4.1) also shows that specifying a positive definite Hermitian form is the same as specifying a real metric g such that $g(\mathbf{I}v, \mathbf{I}w) = g(v, w)$. Such a real metric is called a Hermitian metric; this may cause some confusion since it is not a Hermitian form, however little confusion is possible since the corresponding Hermitian form h is determined by g (from (4.1)): if $g = \mathrm{Re}(h)$ and $\omega(v, w) = g(\mathbf{I}v, w)$ then $\omega = -\operatorname{Im} h$):

$$\boxed{\begin{array}{c} h = g - \sqrt{-1}\omega \\ \omega(v, w) = g(\mathbf{I}v, w), \qquad g(v, w) = \omega(v, \mathbf{I}w) \end{array}}$$

Said differently a Kähler vector space is the same thing as a (positive definite) Hermitian vector space (i.e. a finite dimensional Hilbert space).

Standard/Example Formulae: $V = \mathbb{C}^n$, Real coordinates x_i, y_i (in real dual), with derivatives dx_i, dy_i. Complex coordinates $z_i = x_i + \sqrt{-1}y_i$.

Standard symplectic form:

$$\omega = \sum dx_i \wedge dy_i = \frac{\sqrt{-1}}{2} \sum dz_i \wedge d\overline{z}_i$$

since $dz_i \wedge d\overline{z}_i = 2\sqrt{-1}dx_i \wedge dy_i$. Hermitian form

$$h = \sum dz_i \otimes d\overline{z}_i$$

so $h(v, w) = \sum v_i \overline{w}_i$ where $v_i = z_i(v), \overline{w}_i = \overline{z}_i(w)$. Now

$$dz \otimes d\overline{z} = dxdx + dydy + idydx - idxdy$$

so the real part of the Hermitian form is

$$g = \sum dx_i^2 + dy_i^2$$

and the imaginary part is

$$-\omega = \sum dy_i \wedge dx_i$$

i.e. ω is minus the imaginary part of the Hermitian form, as expected.

Some group theory I.

The unitary group $\mathrm{U}(n)$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ preserving the standard positive definite Hermitian form on $\mathbb{C}^n$. This is equivalent to preserving both the real and imaginary parts, so we have

$$\boxed{\mathrm{U}(n) = \mathrm{O}_{2n}(\mathbb{R}) \cap \mathrm{Sp}_{2n}(\mathbb{R})}$$

(since the real part is the standard positive definite bilinear form on $\mathbb{R}^{2n}$ and the imaginary part is the standard real symplectic form, and moreover these parts determine the complex structure). Note that the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ is not compact, e.g for $n = 1$ it is just $\mathrm{SL}_2(\mathbb{R})$ (since a symplectic form on a real two dimensional vector space is just a volume form). Do not confuse it with the compact group $\mathrm{Sp}(n)$, the "quaternionic unitary group" to be defined below—they are two different real forms of the complex symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$.

Since the real symplectic form is preserved, a real volume form is preserved (an orientation—here the top exterior power of the symplectic form), i.e. $\mathrm{Sp}_{2n}(\mathbb{R}) \subset \mathrm{SL}_{2n}(\mathbb{R})$, so in fact

$$\mathrm{U}(n) \subset \mathrm{SO}_{2n}(\mathbb{R}).$$

The complexification of this fact will be important below.

Kähler manifolds.

Recall that an *almost-complex structure* on a real manifold M is an endomorphism $\mathbf{I} \in \mathrm{End}(TM)$ of the tangent bundle such that $\mathbf{I}^2 = -1$. (Thus each tangent space is a complex vector space via the action of $\mathbf{I}$.) A complex structure on M is an *integrable* almost-complex structure, i.e. M is a complex manifold (we can find local $\mathbf{I}$-holomorphic coordinates etc.)

A *Kähler manifold* is a complex manifold M together with a (real) symplectic form ω on M which is compatible with the complex structure and induces a metric: if $\mathbf{I} \in \mathrm{End}(TM)$ is the complex structure, then

$$\omega(\mathbf{I}v, \mathbf{I}w) = \omega(v, w)$$

and such that the associated real (symmetric) bilinear form

$$g(v, w) = \omega(v, \mathbf{I}w)$$

is positive definite, at each point of M. (One may easily check that the skew-symmetry of ω and the invariance of ω under $\mathbf{I}$ implies the symmetry of g.) Thus g is a Riemannian metric and is invariant under $\mathbf{I}$.

Note that Theorem 4.3 on p.148 of [KN69] (combined with the Newlander–Nirenberg theorem) says that:

Theorem 4.1. *If M is a Riemannian manifold with metric g and a global section $\mathbf{I} \in \Gamma(\mathrm{End}(TM))$ such that $\mathbf{I}^2 = -1, g(\mathbf{I}u, \mathbf{I}v) = g(u, v)$ then the following are equivalent:*

a) $\nabla \mathbf{I} = 0$,

b) $\mathbf{I}$ is an integrable complex structure (making M into a complex manifold) and the associated two form ω (defined by $\omega(\cdot, \cdot) = g(\mathbf{I}\cdot, \cdot)$) is closed.

If either of these hold then M is a Kähler manifold.

Corollary 4.2. *Let M be a complex manifold with complex structure $\mathbf{I} \in \mathrm{End}(TM)$ and with a Hermitian metric g. Then g is Kähler (i.e. the associated $(1,1)$-form ω is closed) iff the complex structure $\mathbf{I}$ is parallel for the Levi-Civita connection.*

Kähler manifolds arise in abundance since complex submanifolds of Kähler manifolds are Kähler:

Proposition 4.3. *Let M be a Kähler manifold with Kähler two-form ω and let N be a complex manifold. Suppose that*

$$f : N \to M$$

is a complex immersion (holomorphic map injective on tangent vectors). Then N is a Kähler manifold with two-form $f^*\omega$.

Proof. Clearly $f^*\omega$ is closed. The holomorphicity of f means that df_* intertwines the complex structures. The corresponding symmetric bilinear form is $g_N(v,w) = (f^*\omega)(v, \mathbf{I}w) = \omega(df_*v, df_*(\mathbf{I}w))$ and this equals $\omega(df_*v, \mathbf{I}df_*(w)) = g_M(df_*v, df_*w) = f^*(g_M)(v,w)$ since f is holomorphic. Thus $g_N = f^*g_M$ is positive definite since f is an immersion. This also shows $f^*\omega$ is nondegenerate (by setting $w = v$ for example). Again since f is holomorphic g_N is preserved by the complex structure on N. □

Kähler quotients.

The main result is the following

Theorem 4.4 (see [HKLR87])**.** *If $M/\!\!/G$ is a symplectic manifold obtained as the (real) symplectic quotient of a Kähler manifold M, by the action of a group G that preserves the Kähler structure on M, then $M/\!\!/G$ is also naturally Kähler.*

The proof is similar to the hyperkähler case we will give in detail below. Let us discuss some aspects of this situation which will be helpful later. Suppose G is a compact group acting freely and 0 is a regular value of μ so that $Z = \mu^{-1}(0)$ is a smooth submanifold of M. Choose a point $m \in Z$, and write $\pi : Z \to N := Z/G = M/\!\!/G$ for the quotient. Thus the tangents to the action (the span of the fundamental vector fields at m) $\mathfrak{g}_m$ is a subspace of T_mZ isomorphic to $\mathfrak{g}$, via the map $X \mapsto (v_X)_m$. Since we now have a metric, we can define the orthogonal subspace $H_m = \mathfrak{g}_m^\perp \subset T_mZ$ to $\mathfrak{g}_m$ (using the metric, not the symplectic form). Thus

$$T_mZ = H_m \oplus \mathfrak{g}_m$$

and the projection π identifies H_m with the tangent space $T_{\pi(m)}N$ of the quotient. On the other hand using the metric we can consider the normal directions to Z in M:

these arise as the gradient vector fields $grad\mu^X$ of components μ^X of μ (for $X \in \mathfrak{g}$). If Y is any vector field, since μ is a moment map, we have

$$g(grad\mu^X, Y) = d\mu^X(Y) = -\omega(v_X, Y) = -g(\mathbf{I}v_X, Y)$$

so $grad\mu^X = -\mathbf{I}v_X$; the gradients are obtained by rotating the fundamental vector fields by $-\mathbf{I}$. Thus $\mathbf{I}\mathfrak{g}_m$ is the set of normal directions to T_mZ in T_mM, and we have a decomposition:

$$T_mM = H_m \oplus \mathfrak{g}_m \oplus \mathbf{I}\mathfrak{g}_m.$$

Clearly the last two factors make up a complex vector space $\cong \mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ and in particular H_m is identified with the quotient of two complex vector spaces $T_mM/\mathfrak{g}_\mathbb{C}$ so is complex itself. [It follows that the metric on N (defined by restricting from T_mM to H_m) is Kähler since the Levi-Civita connection is obtained by orthogonal projection to H_m, and this commutes with the complex structures.] The point we wish to emphasize here is that H_m arises by quotienting T_mM by $\mathfrak{g}_m \otimes \mathbb{C}$. Thus N appears as a quotient by a local (Lie algebra) action of $G_\mathbb{C}$ (the complexification of G). If this comes from an action of $G_\mathbb{C}$ then we expect that

$$N = M^{ss}/G_\mathbb{C}$$

where $M^{ss} \subset M$ is the union $G_\mathbb{C} \cdot Z$ of the $G_\mathbb{C}$ orbits in M which meet Z. In other words we obtain an alternate viewpoint on the Kähler quotient as the quotient of an (open) subset $M^{ss} \subset M$ by the complexified group. This will be made precise when we discuss GIT quotients later. For now we will give a simple example.

The simplest examples come from Kähler vector spaces. Let V be a complex vector space with a positive definite Hermitian form $(\cdot, \cdot)$. Let ω be minus the imaginary part of this Hermitian form. Let $G = \mathrm{U}(V)$ be the group of linear automorphisms of V preserving the Hermitian form. Clearly this is a subgroup of the group of symplectic automorphisms of V. The Lie algebra $\mathfrak{g}$ of G consists of the skew-Hermitian endomorphisms, i.e. those satisfying

$$(Av, w) + (v, Aw) = 0$$

for all $v, w \in V$.

Lemma 4.5. *The map*

$$\mu : V \to \mathfrak{g}^*; \qquad v \mapsto \text{“}A \mapsto \frac{i}{2}(Av, v)\text{”}$$

is a moment map for the action of $G = \mathrm{U}(V)$ on V.

Proof. This is immediate from Lemma 3.18 and Exercise 3.15. Here is the direct verification anyway: First note that $i(Av, v)$ is indeed real, since A satisfies $(Av, v) + (v, Av) = 0$ and $(v, Av) = \overline{(Av, v)}$ so (Av, v) is pure imaginary. Thus μ indeed maps

V to the real dual of $\mathfrak{g}$. Given $A \in \mathfrak{g}$ we should verify that $d\langle\mu, A\rangle = \omega(\cdot, v_A)$. The left hand side is $\frac{i}{2}\left((Adv, v) + (Av, dv)\right)$, so we must show

$$\frac{i}{2}\left((Aw, v) + (Av, w)\right) = \omega(w, v_A)$$

for any $w \in V$. Now at $v \in V$ the vector field v_A takes the value $-Av$ (which is minus tangent to the flow), so the right hand side is: $\mathrm{Im}(w, Av) = \frac{1}{2i}\left((w, Av) - (Av, w)\right)$ which indeed equals the left-hand side, since A is skew Hermitian. □

Note that if we use the standard Hermitian structure $h(v, w) = v^T\overline{w}$ on $V = \mathbb{C}^n$ and use the pairing $(A, B) \mapsto \mathrm{Tr}(AB)$ to identify $\mathfrak{g}$ with its dual, then the moment map is given by

$$\mu : V \to \mathfrak{g}; \qquad \mu(v) = \frac{i}{2} v \otimes v^\dagger$$

(where $\dagger$ dentoes the conjugate transpose) since $\frac{i}{2}\mathrm{Tr}(Av \otimes v^\dagger) = \frac{i}{2}v^\dagger Av = \frac{i}{2}v^T A^T \overline{v} = \frac{i}{2}(Av, v)$.

Example 4.6. Take $M = \mathbb{C}^n$ with its standard Kähler structure. Let $S^1 \subset U(n)$ be the circle subgroup of diagonal scalar unitary matrices, acting on M. Since $U(n)$ has moment map $v \mapsto \frac{i}{2}v \otimes v^\dagger$, and the dual of the derivative of the inclusion $S^1 \subset U(n)$ is given by the trace, this action has moment map $v \in M \mapsto \frac{i}{2}\|v\|^2$. Now since the coadjoint action of S^1 is trivial we may perform reduction at any value of the moment map (all coadjoint orbits are points). Taking the value $i/2$ we find

$$\mu^{-1}(i/2) = \{v \in M \mid \|v\| = 1\}$$

is the unit sphere in $\mathbb{C}^n$, and the symplectic quotient is

$$M /\!\!/ S^1 = \mu^{-1}(i/2)/S^1 \cong \mathbb{P}^{n-1}$$

the complex projective space of dimension $n-1$, since each complex one dimensional subspace of $\mathbb{C}^n$ intersects the sphere in a circle, and this circle is an S^1 orbit. Thus the projective space inherits a Kähler structure. This is the (well-known) Fubini–Study Kähler structure; the key point here is that we have obtained a nontrivial Kähler structure from the standard Kähler structure on a vector space. In this example the complexified group $G_{\mathbb{C}} = \mathbb{C}^*$ does indeed act holomorphically (by scalar multiplication), and M^{ss}, the union of the $\mathbb{C}^*$ orbits which meet $\mu^{-1}(i/2)$, is just $V \setminus \{0\}$. Thus in this case it is easy to see

$$M /\!\!/ S^1 = M^{ss}/\mathbb{C}^* = (V \setminus \{0\})/\mathbb{C}^*$$

which is closer to the usual description of the projective space as the space of one dimensional subspaces of V.

Let us record another example of moment map for later use:

Example 4.7. Take $V = \mathrm{End}(\mathbb{C}^n)$ with the Hermitian form $(X,Y) = \mathrm{Tr}(XY^\dagger)$, with $G = \mathrm{U}(n)$ acting by conjugation (we are thus embedding $\mathrm{U}(n)$ in $\mathrm{U}(n^2)$). Then a moment map $\mu : V \to \mathfrak{g}$ is given by

$$\mu(X) = \frac{i}{2}[X, X^\dagger]$$

(where again we use $(A,B) \mapsto \mathrm{Tr}(AB)$ to identify $\mathfrak{g}$ with its dual) since for $X \in V, A \in \mathfrak{g}$, the adjoint action of A on X is $[A,X]$, so we just observe that

$$\mu(X)(A) = \frac{i}{2}\mathrm{Tr}(A[X,X^\dagger]) = \frac{i}{2}\mathrm{Tr}([A,X]X^\dagger) = \frac{i}{2}([A,X],X)$$

as in Lemma 4.5.

5. Quaternions and hyperkähler vector spaces

Quaternions.

The quaternions are the real (noncommutative) algebra

$$\mathbb{H} = \{q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \bigm| x_i \in \mathbb{R}\} \cong \mathbb{R}^4$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the quaternion identities:

$$\boxed{\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.}$$

Thus each generator $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is a square root of -1, and we have $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}$ etc. Given a quaternion $q \in \mathbb{H}$ we define its real part to be x_0 and its imaginary part to be $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, so that

$$\mathrm{Im}\,\mathbb{H} = \{x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \bigm| x_i \in \mathbb{R}\} \cong \mathbb{R}^3.$$

An important property is the existence of a conjugation on $\mathbb{H}$:

$$\overline{q} = q - 2\,\mathrm{Im}\,q, \qquad \overline{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$$

negating the imaginary part. Thus q is real iff $q = \overline{q}$. Moreover the conjugation satisfies $\overline{(pq)} = (\overline{q})(\overline{p})$ so that $q\overline{q}$ is real. This enables us to define a norm:

$$|q|^2 = q\overline{q} = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

The nonzero quaternions are precisely those with nonzero norm which in turn are precisely those with a multiplicative inverse $q^{-1} = \overline{q}/|q|^2$. They constitute a four dimensional Lie group

$$\mathrm{GL}_1(\mathbb{H}) = \{q \in \mathbb{H} \bigm| q \neq 0\} = \mathbb{H}^*.$$

The unit quaternions (those of norm one) clearly constitute a three sphere, and moreover they form a subgroup:

$$\mathrm{Sp}(1) = \{q \in \mathbb{H} \mid |q| = 1\} \cong \mathrm{SU}(2) \cong S^3.$$

Note that the generators $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{H}$ are not really distinguished: there are lots of ordered triples of mutually orthogonal elements of norm one with the same algebraic properties: Their choice amounts to choosing an orthonormal basis of $\mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$; the set of such choices is a torsor for $\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{Sp}(1)/\{\pm 1\}$. Indeed any such triple is of the form

$$q(\mathbf{i}, \mathbf{j}, \mathbf{k})q^{-1} = (q\mathbf{i}q^{-1}, q\mathbf{j}q^{-1}, q\mathbf{k}q^{-1})$$

for some unit quaternion q and the stabiliser of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is $\{\pm 1\} \subset \mathrm{Sp}(1)$. In particular any element of the form $q\mathbf{i}q^{-1}$ is a square root of -1; the set of such constitutes the orbit of $\mathrm{SO}_3(\mathbb{R})$ through (the nonzero vector) $\mathbf{i} \in \mathbb{R}^3$, so is a two-sphere:

$$\{x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathrm{Im}\,\mathbb{H}.$$

In the sequel this will be the two-sphere of complex structures on a hyperkähler manifold (and in general they will not all be equivalent).

If we use the norm to define the constant flat metric $g(p, q) = \mathrm{Re}(p\overline{q})$ on $\mathbb{H}$ we then get a triple of symplectic form $\omega_{\mathbf{i}}, \omega_{\mathbf{j}}, \omega_{\mathbf{k}}$ defined as usual: $\omega_{\mathbf{i}}(v, w) = g(\mathbf{i}v, w)$ etc. The formulae are as follows:

$$g = \sum_0^3 dx_i^2$$

$$\begin{aligned} \omega_{\mathbf{i}} &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3, \\ \omega_{\mathbf{j}} &= dx_0 \wedge dx_2 + dx_3 \wedge dx_1, \\ \omega_{\mathbf{k}} &= dx_0 \wedge dx_3 + dx_1 \wedge dx_2. \end{aligned}$$

Given that $\mathbb{C}$ with coordinate $z = x + iy$ has symplectic structure $dx \wedge dy$, one can "see" these formulae directly by writing:

$$\begin{aligned} q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} &= (x_0 + x_1\mathbf{i}) + (x_2 + x_3\mathbf{i})\mathbf{j} \\ &= (x_0 + x_2\mathbf{j}) + (x_3 + x_1\mathbf{j})\mathbf{k} \\ &= (x_0 + x_3\mathbf{k}) + (x_1 + x_2\mathbf{k})\mathbf{i} \end{aligned}$$

respectively, since then the action of the complex structures (acting by left multiplication on $\mathbb{H}$) is clear, decomposing $\mathbb{H} \cong \mathbb{C} \times \mathbb{C}$ in three different ways.

Note that this may be encoded succinctly as:

$$\boxed{g - \mathbf{i}\omega_{\mathbf{i}} - \mathbf{j}\omega_{\mathbf{j}} - \mathbf{k}\omega_{\mathbf{k}} = dq \otimes d\overline{q}}$$

and using the quaternion multiplication to expand the right-hand side. This should be compared to $h = g - \sqrt{-1}\omega$.

Exercise: Check this, and show $g + \mathbf{i}\omega_{\mathbf{i}} + \mathbf{j}\omega_{\mathbf{j}} + \mathbf{k}\omega_{\mathbf{k}} \neq d\overline{q} \otimes dq$.

Remark 5.1. Recall the Hodge star operator, acting on forms, defined by the equation $a \wedge *b = (a, b)\,\mathrm{vol}$. This squares to one when restricted to two-forms on a four dimensional space, so the two-forms break up in to the ± 1 eigenspaces. The three real two forms appearing here are a basis of the *self-dual* two forms on $\mathbb{R}^4$, i.e. they are preserved by the Hodge star. This enables us to identify the imaginary quaternions with the self-dual two forms.

Suppose we work in the complex structure $\mathbf{i}$, and write $q = z + w\mathbf{j}$ with $z = x_0 + \mathbf{i}x_1, w = x_2 + \mathbf{i}x_3$ (so $z, w : \mathbb{H} \to \mathbb{C}$ are now holomorphic coordinates on $\mathbb{H}$). Then we claim that

$$\omega_{\mathbb{C}} := \omega_{\mathbf{j}} + \sqrt{-1}\omega_{\mathbf{k}} \quad \in \bigwedge^2 (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})^*$$

is a complex symplectic form (i.e. of type $(2, 0)$). Indeed it is just

$$\begin{aligned} dz \wedge dw &= (dx_0 + \sqrt{-1}dx_1) \wedge (dx_2 + \sqrt{-1}dx_3) \\ &= (dx_0 \wedge dx_2 - dx_1 \wedge dx_3) + \sqrt{-1}(dx_0 \wedge dx_3 + dx_1 \wedge dx_2). \end{aligned}$$

This will be seen much more generally below. Note that this corresponds to viewing $\mathbb{H} \cong \mathbb{C}^2$ as the cotangent bundle of the w-line: $\mathbb{H} = T^*\mathbb{C}_w$ (since this has symplectic form $d(zdw)$).

Moreover the quaternionic Hermitian form

$$dq \otimes d\overline{q} = g - \mathbf{i}\omega_{\mathbf{i}} - \mathbf{j}\omega_{\mathbf{j}} - \mathbf{k}\omega_{\mathbf{k}} = (g - \mathbf{i}\omega_{\mathbf{i}}) - (\omega_{\mathbf{j}} + \mathbf{i}\omega_{\mathbf{k}})\mathbf{j} = h - \omega_{\mathbb{C}}\mathbf{j}$$

is the sum of a (usual) Hermitian form minus $\mathbf{j}$ times a complex symplectic form. (This is the quaternionic analogue/complexification of the relation $h = g - \mathbf{i}\omega$.) In particular preserving $dq \otimes d\overline{q}$ is equivalent to preserving both h and $\omega_{\mathbb{C}}$. In particular we see

$$\mathrm{Sp}(1) = \mathrm{U}(2) \cap \mathrm{Sp}_2(\mathbb{C})$$

is the intersection of the unitary group (preserving h) and the complex symplectic group $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ preserving $\omega_{\mathbb{C}}$ (which here is a holomorphic volume form). This will be extended below.

Note that when working in complex structure $\mathbf{i}$ the extra structure of the quaternions on the complex vector space $\mathbb{C}^2$ is all encoded in the action of $\mathbf{j}$ by left multiplication (since $\mathbf{k} = \mathbf{ij}$). Explicitly this extra complex structure on $\mathbb{C}^2$ is given as follows

$$\mathbf{j}(z, w) = (-\overline{w}, \overline{z})$$

(as is easily seen by computing $\mathbf{j} \cdot (z + w\mathbf{j})$, given that $\mathbf{j}z = \overline{z}\mathbf{j}$ etc.)

2×2 **Matrices.** Under the isomorphism $\mathbb{C}^2 \cong \mathbb{H}; (z, w) \mapsto q = z + w\mathbf{j}$ the right action of $\mathbb{H}$ on itself ($x(q) = qx$), translates into a (right) action of $\mathbb{H}$ on $\mathbb{C}^2$ by $\mathbb{C}$-linear maps. In this way we get an embedding $\mathbb{H} \hookrightarrow \mathrm{End}(\mathbb{C}^2)$ of algebras. Viewing

$(z, w) \in \mathbb{C}^2$ as row vectors, one may readily verify that the elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ become the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2)$$

respectively (e.g. the computation $(z + w\mathbf{j})\mathbf{k} = iw + (iz)\mathbf{j}$, yields the formula for the image of $\mathbf{k}$). In particular $\mathrm{Im}\,\mathbb{H}$ corresponds to the set of skew-Hermitian matrices with trace zero—the Lie algebra of $\mathfrak{su}(2) \cong \mathfrak{sp}(1)$.

Quaternionic vector spaces, some group theory 2.

More generally let $V = \mathbb{H}^n \cong \mathbb{R}^{4n}$ be a "quaternionic vector space". We view it as a left $\mathbb{H}$-module by left multiplication. Thus V has three complex structures $\mathbf{I}, \mathbf{J}, \mathbf{K}$ given by left (component-wise) multiplication by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. Then we obtain a group

$$\mathrm{GL}_n(\mathbb{H}) \subset \mathrm{GL}_{2n}(\mathbb{C}) \subset \mathrm{GL}_{4n}(\mathbb{R})$$

of invertible quaternionic matrices (having real dimension $4n^2 < 8n^2 < 16n^2$). If we let $g \in \mathrm{GL}_{4n}(\mathbb{R})$ act on $\mathbf{q} \in V$ as $\mathbf{q}g^{-1}$ (where we view $\mathbf{q}$ as a row vector), then the action of $\mathrm{GL}_n(\mathbb{H})$ commutes with the action of the complex structures, i.e. it is "quaternionic-linear".

We may define a standard "quaternionic Hermitian product"

$$(\!(\mathbf{p}, \mathbf{q})\!) = \mathbf{p}\mathbf{q}^\dagger = \sum p_i \overline{q}_i \in \mathbb{H}$$

where $\mathbf{p} \in V$ has components $p_1, \dots p_n, \in \mathbb{H}$ etc. Note that $\overline{(\!(\mathbf{p}, \mathbf{q})\!)} = (\!(\mathbf{q}, \mathbf{p})\!)$. Equivalently $(\!(\cdot, \cdot)\!) = d\mathbf{q} \otimes d\mathbf{q}^\dagger = \sum dq_i \otimes d\overline{q}_i$. This decomposes as above in the $n = 1$ case

$$d\mathbf{q} \otimes d\mathbf{q}^\dagger = g - \mathbf{i}\omega_\mathbf{i} - \mathbf{j}\omega_\mathbf{j} - \mathbf{k}\omega_\mathbf{k} = (g - \mathbf{i}\omega_\mathbf{i}) - (\omega_\mathbf{j} + \mathbf{i}\omega_\mathbf{k})\mathbf{j} = h - \omega_\mathbb{C}\mathbf{j}$$

yielding the flat metric g, Kähler forms, Hermitian metric and complex symplectic form. (Compare with the relation $h = g - \mathbf{i}\omega$.) In local holomorphic coordinates (for $\mathbf{I}$) we have $\omega_\mathbb{C} = \sum dz_i \wedge dw_i$.

More abstractly one may define a *hyperkähler vector space* V to be a free left $\mathbb{H}$-module together with a (positive definite) quaternionic Hermitian form (i.e. an $\mathbb{R}$-bilinear map $(\!(\cdot, \cdot)\!) : V \otimes_\mathbb{R} V \to \mathbb{H}$, which is $\mathbb{H}$ linear in the first slot, satisfies $\overline{(\!(\mathbf{p}, \mathbf{q})\!)} = (\!(\mathbf{q}, \mathbf{p})\!)$, and is such that the real number $(\!(\mathbf{q}, \mathbf{q})\!)$ is positive unless $\mathbf{q} = 0$).

We define the quaternionic unitary group $\mathrm{Sp}(n)$ to be

$$\mathrm{Sp}(n) = \{g \in M_n(\mathbb{H}) \bigm| gg^\dagger = 1\}$$

where $g^\dagger$ is the transpose of the quaternionic conjugate matrix. This is the group preserving the quaternionic inner product, if we make $g \in \mathrm{GL}_n(\mathbb{H})$ act on $\mathbf{q} \in V$ as

$\mathbf{q}g^{-1}$, where we view $\mathbf{q}$, as a *row* vector of quaternions. Thus $g \in \mathrm{Sp}(n)$ iff

$$\big(\!\big(\mathbf{p}g^{-1}, \mathbf{q}g^{-1}\big)\!\big) = (\!(\mathbf{p}, \mathbf{q})\!)$$

for all $\mathbf{p}, \mathbf{q} \in V$. The Lie algebra of $\mathrm{Sp}(n)$ is thus the set of quaternionic skew-adjoint matrices:

$$\mathfrak{sp}(n) = \{X \in M_n(\mathbb{H}) \bigm| X + X^\dagger = 0\}.$$

In particular we deduce that $\dim_{\mathbb{R}}(\mathrm{Sp}(n)) = 3n + 4n(n-1)/2 = 2n^2 + n$.

Since $d\mathbf{q} \otimes d\mathbf{q}^\dagger = h - \omega_{\mathbb{C}}\mathbf{j}$ is the sum of a Hermitian form minus the complex symplectic form (times $\mathbf{j}$), preserving $d\mathbf{q} \otimes d\mathbf{q}^\dagger$ is equivalent to preserving both h and $\omega_{\mathbb{C}}$. It follows that the quaternionic unitary group of $\mathbb{H}^n \cong \mathbb{C}^{2n} = T^*\mathbb{C}^n$ is the intersection

$$\boxed{\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}_{2n}(\mathbb{C})}$$

of the unitary group (stabilising h) and the complex symplectic group (stabilising $\omega_{\mathbb{C}}$). Moreover clearly $\mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{SL}_{2n}(\mathbb{C})$ (since the top exterior power of the complex symplectic form is a holomorphic volume form) so that in fact

$$\mathrm{Sp}(n) \subset \mathrm{SU}(2n).$$

This means that hyperkähler manifolds are Ricci flat—they are Calabi–Yau manifolds of even complex dimension

Note also that $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, but (e.g. by comparing dimensions $2n^2 + n$ and $4n^2 - 1$ resp.) this inclusion is strict if $n > 1$.

6. Hyperkähler manifolds

Let M be a manifold of dimension $4n$.

Definition 6.1. *M is a hyperkähler manifold if it is equipped with*

1) A triple of global sections $\mathbf{I}, \mathbf{J}, \mathbf{K} \in \Gamma(\mathrm{End}(TM))$ of the tangent bundle, satisfying the quaternion identities:

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{IJK} = -1,$$

2) A Riemannian metric g such that

$$g(\mathbf{I}u, \mathbf{I}v) = g(\mathbf{J}u, \mathbf{J}v) = g(\mathbf{K}u, \mathbf{K}v) = g(u, v)$$

for all tangent vectors $u, v \in T_m M$ for all $m \in M$, and moreover $\mathbf{I}, \mathbf{J}, \mathbf{K}$ are covariant constant, i.e.

$$\nabla \mathbf{I} = \nabla \mathbf{J} = \nabla \mathbf{K} = 0,$$

where ∇ is the covariant derivative of the Levi-Civita connection of g.

Thus on a hyperkähler manifold, given three real numbers a_i whose squares sum one, we may define

$$\mathbf{I}_a = a_1 \mathbf{I} + a_2 \mathbf{J} + a_3 \mathbf{K} \in \Gamma\left(\mathrm{End}(TM)\right)$$

and verify that $\nabla \mathbf{I}_a = 0, \mathbf{I}_a^2 = -1, g(\mathbf{I}_a u, \mathbf{I}_a v) = g(u, v)$ so that, by Theorem 4.1

$$\boxed{(M, g, \mathbf{I}_a) \text{ is a Kähler manifold for any } a = (a_1, a_2, a_3) \in S^2}$$

i.e. it has a whole S^2 family of Kähler structure, hence the name.

Let us denote the triple of Kähler forms as follows:

$$\omega_{\mathbf{I}}(u, v) = g(\mathbf{I}u, v), \quad \omega_{\mathbf{J}}(u, v) = g(\mathbf{J}u, v), \quad \omega_{\mathbf{K}}(u, v) = g(\mathbf{K}u, v),$$

and we will also sometime write $\omega_1 = \omega_{\mathbf{I}}, \omega_2 = \omega_{\mathbf{J}}, \omega_3 = \omega_{\mathbf{K}}$.

A trivial example of hyperkähler manifold is a hyperkähler vector space, i.e. the flat metric and triple of complex structures on the $\mathbb{H}^n$ described earlier.

If we view M as a complex manifold using the complex structure $\mathbf{I}$ then we claim that the the complex two-form

$$\omega_{\mathbb{C}} = \omega_{\mathbb{C}}^{(\mathbf{IJK})} := \omega_{\mathbf{J}} + \sqrt{-1}\omega_{\mathbf{K}}$$

is a complex symplectic form. (Note that this depends the choice of the triple $\mathbf{I}, \mathbf{J}, \mathbf{K}$, an orthonormal basis of $\mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$, and not just $\mathbf{I}$.) We will first check it is of type

$(2,0)$. Indeed

$$\begin{aligned}\omega_{\mathbb{C}}(\mathbf{I}p,q) &= g(\mathbf{J}\mathbf{I}p,q)+\sqrt{-1}g(\mathbf{K}\mathbf{I}p,q)\\ &= -g(\mathbf{K}p,q)+\sqrt{-1}g(\mathbf{J}p,q)\\ &= \sqrt{-1}\left(g(\mathbf{J}p,q)+\sqrt{-1}g(\mathbf{K}p,q)\right)\\ &= \sqrt{-1}\omega_{\mathbb{C}}(p,q)\end{aligned}$$

which means that $\omega_{\mathbb{C}}$ is of type $(2,0)$. [Immediately this shows that the complex one-form $p\mapsto\omega_{\mathbb{C}}(p,q)$ is $(1,0)$ for any fixed q. Then by skew-symmetry $\omega_{\mathbb{C}}$ is $(2,0)$.] Clearly $\omega_{\mathbb{C}}$ is closed and it is nondegenerate since e.g. the above shows $\omega_{\mathbb{C}}(\mathbf{I}p,\mathbf{K}p)$ has nonzero real part whenever p is nonzero. Thus $(M,\mathbf{I})$ is a complex symplectic manifold. (Similarly, for example, $(M,\mathbf{J})$ and $(M,\mathbf{K})$ are complex symplectic with complex symplectic forms $\omega_{\mathbb{C}}^{\mathbf{JKI}},\omega_{\mathbb{C}}^{\mathbf{KIJ}}$.) Thus the notion of hyperkähler manifold is an *enrichment* of the notion of complex symplectic manifold. This is a useful viepoint since most examples arise in the first instance as complex symplectic manifolds.

It turns out that one obtains the same structure (of hyperkähler manifold on M) by asking for apparently weaker conditions: First note that the metric and the complex structures are determined by the three Kähler two-forms alone: For example since we have

$$\omega_{\mathbf{J}}(u,v)=g(\mathbf{J}u,v)=g(\mathbf{K}\mathbf{I}u,v)=\omega_{\mathbf{K}}(\mathbf{I}u,v). \tag{6.1}$$

it follows that $\mathbf{I}=(\omega_{\mathbf{K}}^{\flat})^{-1}\circ\omega_{\mathbf{J}}^{\flat}\in\mathrm{End}(TM)$ and similarly we may obtain $\mathbf{J},\mathbf{K}$. These now yield the metric also since e.g. $g(u,v)=\omega_{\mathbf{I}}(u,\mathbf{I}v)$. Thus we can ask for conditions in terms of the triple of forms to determine the hyperkähler structure:

Lemma 6.2 ([Hit87])**.** *Suppose $(M,g,\mathbf{I},\mathbf{J},\mathbf{K})$ is a Riemannian manifold with a triple of skew-adjoint endomorphism of the tangent bundle satisfying the quaternion identities. Then M is hyperkähler iff the corresponding triple of two-forms are* closed.

Proof. Clearly if it is hyperkähler then the forms are closed. Conversely by Theorem 4.1 it is sufficient to show the almost complex structures are integrable. For this we will use the Newlander–Nirenberger theorem in the following form: an almost-complex structure is integrable iff the Lie bracket of any two vector fields of type $(1,0)$ is again of type $(1,0)$ (see [KN69], Theorems 2.8 and 2.7 pp.124-126). Now if u,v are sections of the complexified tangent bundle of M then from (6.1) we see u is of type $(1,0)$ in the complex structure $\mathbf{I}$ (i.e. $\mathbf{I}u=+\sqrt{-1}u$) iff $\iota_u\omega_{\mathbf{J}}=\sqrt{-1}\iota_u\omega_{\mathbf{K}}$ i.e. iff

$$\iota_u\overline{\omega}_{\mathbb{C}}=0$$

where $\overline{\omega}_{\mathbb{C}} = \omega_{\mathbf{J}} - \sqrt{-1}\omega_{\mathbf{K}}$. Thus if v, w are both of type $(1,0)$ for $\mathbf{I}$ we must show $\iota_{[v,w]}\overline{\omega}_{\mathbb{C}} = 0$. This is now straightforward:

$$\begin{aligned}
\iota_{[v,w]}\overline{\omega}_{\mathbb{C}} &= [\mathcal{L}_v, \iota_w]\overline{\omega}_{\mathbb{C}} \\
&= -\iota_w \mathcal{L}_v \overline{\omega}_{\mathbb{C}} && \text{since } w \text{ is } (1,0) \\
&= -\iota_w (d \circ \iota_v)\overline{\omega}_{\mathbb{C}} && \text{by the Cartan formula and } d\overline{\omega}_{\mathbb{C}} = 0 \\
&= 0 && \text{since } v \text{ is } (1,0)
\end{aligned}$$

Thus $\mathbf{I}$ is integrable and similarly for $\mathbf{J}, \mathbf{K}$. □

This will be useful since often it is easier to check if forms are closed than complex structures being covariant constant.

Hyperkähler quotients.

In this section we will define the hyperkähler quotient—a generalization of the symplectic (and Kähler) quotient, enabling lots of new examples of noncompact hyperkähler manifolds to be constructed.

Let $(M, g, \mathbf{I}, \mathbf{J}, \mathbf{K})$ be a hyperkähler manifold. Suppose G is a compact Lie group acting on M preserving the metric and the triple of complex structures. Thus G preserves the triple of Kähler forms and we suppose further that there is a triple of equivariant moment maps $\mu_{\mathbf{I}}, \mu_{\mathbf{J}}, \mu_{\mathbf{K}} : M \to \mathfrak{g}^*$. More invariantly these constitute the components of a single map, the hyperkähler moment map:

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3.$$

Theorem 6.3 ([HKLR87]). *Let $\zeta \in \mathfrak{g}^* \otimes \mathbb{R}^3$ be a G-invariant point (where G acts by the coadjoint action component-wise). Suppose that G acts freely on the subset $\mu^{-1}(\zeta) \subset M$. Then the quotient*

$$M \underset{\zeta}{/\!\!/\!\!/} G := \mu^{-1}(\zeta)/G$$

is a (smooth) manifold with a natural hyperkähler structure induced from that of M. Its real dimension is $\dim_{\mathbb{R}}(M) - 4\dim_{\mathbb{R}}(G)$. If M is complete then so is $M \underset{\zeta}{/\!\!/\!\!/} G$.

Proof. First we check it is a smooth manifold. For this it is sufficient to check the derivative of μ is surjective at each point of $Z := \mu^{-1}(\zeta)$ (so that Z is a smooth submanifold) and then the slice theorem implies the quotient, by a free action of a compact group, exists and is smooth.

Choose $m \in Z$. Given $X \in \mathfrak{g}$ let v_X denote the corresponding fundamanetal vector field on M and let $\mathfrak{g}_m \subset T_m M$ denote the tangent space to the orbit through m. Since the action is free $\mathfrak{g} \cong \mathfrak{g}_m; X \mapsto (v_X)_m$. We wish to show that $d\mu_m : T_m M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ is surjective. This follows from the following.

Lemma 6.4. *$d\mu_m$ maps $\mathbf{I}\mathfrak{g}_m \subset T_mM$ onto $(*,0,0) \subset \mathfrak{g}^* \otimes \mathbb{R}^3$, and similarly $d\mu_m(\mathbf{J}\mathfrak{g}_m) = (0,*,0)$ etc. Moreover the four subspaces $\mathfrak{g}_m, \mathbf{I}\mathfrak{g}_m, \mathbf{J}\mathfrak{g}_m, \mathbf{K}\mathfrak{g}_m$ are orthogonal.*

Proof. The second statement follows from the moment map property: Clearly $d\mu_m(v_X) = 0$ for all $X \in \mathfrak{g}$ (since μ maps the orbit though m onto the point ζ). Given $Y \in \mathfrak{g}$ let $\mu^Y = \langle \mu, Y \rangle : M \to \mathbb{R}^3$ be the Y-component of μ. The moment map condition implies that, for all $X, Y \in \mathfrak{g}$:

$$0 = d\mu^Y_m(v_X) = \big(\omega_{\mathbf{I}}(v_X, v_Y), \omega_{\mathbf{J}}(v_X, v_Y), \omega_{\mathbf{K}}(v_X, v_Y)\big)$$

$$= \big(g(\mathbf{I}v_X, v_Y), g(\mathbf{J}v_X, v_Y), g(\mathbf{K}v_X, v_Y)\big).$$

Thus $\mathfrak{g}_m$ is orthogonal to each of $\mathbf{I}\mathfrak{g}_m, \mathbf{J}\mathfrak{g}_m, \mathbf{K}\mathfrak{g}_m$ and the full statement follows by invariance of g.

Now for the first part. Choose $X \in \mathfrak{g}$. We will show $d\mu_m(\mathbf{I}v_X) = (-g(v_{(\cdot)}, v_X), 0, 0) \in \mathfrak{g}^* \otimes \mathbb{R}^3$. Indeed for any $Y \in \mathfrak{g}$ (replacing v_X by $-\mathbf{I}v_X$ above):

$$-d\mu^Y_m(\mathbf{I}v_X) = \big(\omega_{\mathbf{I}}(v_Y, \mathbf{I}v_X), \omega_{\mathbf{J}}(v_Y, \mathbf{I}v_X), \omega_{\mathbf{K}}(v_Y, \mathbf{I}v_X)\big)$$

$$= \big(g(\mathbf{I}v_Y, \mathbf{I}v_X), g(\mathbf{J}v_Y, \mathbf{I}v_X), g(\mathbf{K}v_Y, \mathbf{I}v_X)\big)$$

$$= (g(v_Y, v_X), 0, 0)\,.$$

Now the bilinear form $(X, Y) \mapsto g_m(v_X, v_Y)$ on $\mathfrak{g} \otimes \mathfrak{g}$ is nondegenerate (being the restriction of g to $\mathfrak{g}_m$ composed with the action isomorphism $\mathfrak{g} \to \mathfrak{g}_m$) so the first statement follows. $\square$

Thus $Z = \mu^{-1}(\zeta)$ is a smooth manifold. At $m \in Z$ its tangent space $\operatorname{Ker} d\mu_m$ may be characterised as the orthogonal complement to $\mathbf{I}\mathfrak{g}_m \oplus \mathbf{J}\mathfrak{g}_m \oplus \mathbf{K}\mathfrak{g}_m$ since we have

$$-d\mu^Y_m(v) = \big(\omega_{\mathbf{I}}(v_Y, v), \omega_{\mathbf{J}}(v_Y, v), \omega_{\mathbf{K}}(v_Y, v)\big)$$

$$= \big(g(\mathbf{I}v_Y, v), g(\mathbf{J}v_Y, v), g(\mathbf{K}v_Y, v)\big)$$

for all $v \in T_mM, Y \in \mathfrak{g}$.

Since the action is free Z/G is a manifold and the tangent space to a point $[G \cdot m] \in Z/G$ is given by the orthogonal complement H_m of $\mathfrak{g}_m$ in T_mZ (for any m in the fibre). This is the same as the orthogonal complement to $\mathfrak{g}_m \oplus \mathbf{I}\mathfrak{g}_m \oplus \mathbf{J}\mathfrak{g}_m \oplus \mathbf{K}\mathfrak{g}_m$ in T_mM. As such it is preserved by $\mathbf{I}, \mathbf{J}, \mathbf{K}$ (so has a quaternionic triple of almost complex structures) and the metric g may be restricted. By Lemma 6.2 it is sufficient now to check that the corresponding triple of two-forms $\omega'_{\mathbf{I}}, \omega'_{\mathbf{J}}, \omega'_{\mathbf{K}}$ are *closed*.

Let $\iota : Z \to M$ be the inclusion and let $\pi : Z \to Z/G$ be the quotient map. Then by definition $\iota^*(\omega_{\mathbf{I}}) = \pi^*(\omega'_{\mathbf{I}})$ so we have

$$\pi^*(d\omega'_{\mathbf{I}}) = d\pi^*(\omega'_{\mathbf{I}}) = d\iota^*(\omega_{\mathbf{I}}) = \iota^*(d\omega_{\mathbf{I}}) = 0$$

by the naturality of the exterior derivative. It follows that $d\omega'_{\mathbf{I}} = 0$ since π is surjective on tangent vectors (by definition). Similarly for $\omega_{\mathbf{J}}, \omega_{\mathbf{K}}$, so we have ideed constructed a hyperkähler manifold.

Finally we briefly discuss completeness. Suppose M is complete and we have a (maximal) geodesic $\gamma : [0, T) \to N := Z/G$ with $\gamma(0) = \pi(m)$. Then we may lift γ to a curve $\widetilde{\gamma} : [0, T) \to Z$ based at m, and tangent to the horizontal subspace H_z (orthogonal to $\mathfrak{g}_z$) at each point. By definition of the metric on N the length along $\widetilde{\gamma}$ is the same as the length along γ. This lifted curve may be extended in M (as M is complete) so has a limit at $t = T$. This limit is in Z since Z is closed in M, so projects to a point of N, so γ can in fact be extended. □

Remark 6.5. Suppose that we write a hyperkähler moment map μ as $(\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$ where $\mu_{\mathbb{R}} = \mu_{\mathbf{I}} : M \to \mathfrak{g}^*$ and $\mu_{\mathbb{C}} = \mu_{\mathbf{J}} + \sqrt{-1}\mu_{\mathbf{K}} : M \to \mathbb{C} \otimes \mathfrak{g}^*$ are built out of μ. Then $\mu_{\mathbb{C}}$ is a holomorphic map on M (in complex structure $\mathbf{I}$), since for all $X \in \mathfrak{g}$ and vector fields Y:

$$\begin{aligned} d\langle \mu_{\mathbb{C}}, X\rangle(\mathbf{I}Y) &= g(\mathbf{J}\mathbf{I}Y, v_X) + \sqrt{-1}g(\mathbf{K}\mathbf{I}Y, v_X) \\ &= -g(\mathbf{K}Y, v_X) + \sqrt{-1}g(\mathbf{J}Y, v_X) \\ &= \sqrt{-1}(g(\mathbf{J}Y, v_X) + \sqrt{-1}g(\mathbf{K}Y, v_X)) \\ &= \sqrt{-1}d\langle \mu_{\mathbb{C}}, X\rangle(Y). \end{aligned}$$

Thus, if we write $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ similarly, then $\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}}) \subset M$ is a complex submanifold (in complex structure $\mathbf{I}$) so inherits the induced Kähler structure, and we may view the hyperkähler quotient as the Kähler quotient of it by the action of G with moment map $\mu_{\mathbb{R}}$:

$$M \mathbin{/\!/\!/}_{\zeta} G = \mu^{-1}(\zeta)/G = (\mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}}))/G = \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}}) \mathbin{/\!/}_{\zeta_{\mathbb{R}}} G.$$

Note also that if the action of G extends to a holomorphic action of the complexification $G_{\mathbb{C}}$ of G then this action will have moment map $\mu_{\mathbb{C}}$. (with respect to the complex symplectic form $\omega_{\mathbb{C}}$). Later on, once we have studied quotients by such (noncompact) complex algebraic groups $G_{\mathbb{C}}$ we will relate this to the complex symplectic quotient $\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/G_{\mathbb{C}}$. For now we will just look at an example where everything can be computed by hand.

The most basic example of a hyperkähler moment map is for $\mathrm{Sp}(n)$ acting on the hyperkähler vector space $\mathbb{H}^n$ (with quaternionic Hermitian form $(\!(\mathbf{p}, \mathbf{q})\!) = \mathbf{p}\mathbf{q}^\dagger \in \mathbb{H}$):

$$g(\mathbf{q}) := \mathbf{q} \cdot g^{-1}, \qquad \mathbf{q} \in \mathbb{H}^n,\ g \in \mathrm{Sp}(n)$$

where as usual we view $\mathbf{q}$ as a row vector (a $1 \times n$ quaternionic matrix), and $\dagger$ denotes the quaternionic conjugate of the transposed quaternionic matrix. This action preserves the hyperkähler structure of $\mathbb{H}^n$ (since, for example the complex structures were defined in terms of *left* multiplication).

Lemma 6.6. *A hyperkähler moment map for the above action is given by*

$$\mu(\mathbf{q})(X) = \frac{1}{2}\,(\!(\mathbf{q}X, \mathbf{q})\!) = \frac{1}{2}\,\mathrm{Im}(\mathbf{q}X\mathbf{q}^\dagger) \in \mathrm{Im}\,\mathbb{H}$$

where $\mathbf{q} \in \mathbb{H}^n, X \in \mathfrak{sp}(n)$ *(so that* $\mu(\mathbf{q}) \in \mathfrak{sp}(n)^* \otimes \mathrm{Im}\,\mathbb{H}$*) and we identify* $\mathbb{R}^3 \cong \mathrm{Im}\,\mathbb{H}$ *via* $(\mu_1, \mu_2, \mu_3) = \mathbf{i}\mu_1 + \mathbf{j}\mu_2 + \mathbf{k}\mu_3$.

Proof. This is clearly equivariant, i.e. $\mu(\mathbf{q}g^{-1})(gXg^{-1}) = \mu(\mathbf{q})(X)$. Write $\widehat{h} = (\!(\cdot,\cdot)\!)$ for the quaternionic form. Since $\widehat{h} = g - \mathbf{i}\omega_1 - \mathbf{j}\omega_2 - \mathbf{k}\omega_3$, the first component of μ is μ_1 where

$$\mu_1(\mathbf{q})(X) = -\frac{1}{2}\omega_1(\mathbf{q}X, \mathbf{q}) \in \mathbb{R}.$$

By Lemma 3.18 this is indeed a moment map for the action of $\mathrm{Sp}(n)$ on $(\mathbb{H}^n, \omega_1)$ since $-\mathbf{q}X$ is the derivative of the flow $\mathbf{q}\exp(-tX)$ generated by X. Similarly for the other components. Observe that for $X \in \mathfrak{sp}(n)$ (i.e. $X = -X^\dagger$) the expression $\mathbf{q}X\mathbf{q}^\dagger \in \mathbb{H}$ is automatically in $\mathrm{Im}\,\mathbb{H}$ (since it equals minus its quaternionic conjugate). □

Note (more abstractly) that the same proof shows $\mu(\mathbf{q})(X) = \frac{1}{2}\,(\!(v_X(\mathbf{q}), \mathbf{q})\!)$ is a moment map for the natural action of the compact symplectic group of any hyperkähler vector space $\mathbb{V}$ with form $(\!(\cdot,\cdot)\!)$.

Now we will give a first example of hyperkähler quotient.

Example 6.7. Let $V = \mathbb{C}^n$ be a complex vector space of dimension n, equipped with the standard Hermitian inner product. Let

$$\mathbb{V} = V \times V^* \cong \mathbb{H}^n.$$

On $V \times V^*$ the structure of hyperkähler vector space is determined by the action of $\mathbf{j}$ given by $\mathbf{j}(v, \alpha) = (\alpha^\dagger, -v^\dagger)$, and the metric, given by $\|(v, \alpha)\|^2 = \|v\|^2 + \|\alpha\|^2 = v^\dagger v + \alpha\alpha^\dagger$. (Equivalently by convention $(v, \alpha) \in V \times V^*$ corresponds to the point of $\mathbb{H}^n$ with components $v_i - \alpha_i\mathbf{j}$.) We consider the action of the circle S^1 on $\mathbb{V}$ as follows:

$$g(v, \alpha) = (gv, \alpha/g)$$

for $(v, \alpha) \in V \times V^*$, and $g \in S^1$. A moment map for this action is $\mu = (\mu_\mathbb{R}, \mu_\mathbb{C})$ where

$$\mu_\mathbb{R}(v, \alpha) = \frac{i}{2}\left(\|v\|^2 - \|\alpha\|^2\right) \in i\mathbb{R} = \mathrm{Lie}(S^1)$$

$$\mu_\mathbb{C}(v, \alpha) = -\alpha(v) \in \mathbb{C} = \mathrm{Lie}(\mathbb{C}^*).$$

This may be checked directly, or deduced from the previous example (it will also follow from more general examples below).

Thus the hyperkähler quotient at the value $\zeta = (i/2, 0, 0)$ of the moment map is thus
$$\left\{(v,\alpha) \in V \times V^* \;\middle|\; \|v\|^2 - \|\alpha\|^2 = 1, \alpha(v) = 0\right\}/S^1.$$
It is easy to see this is equal to the complex quotient
$$\left\{(v,\alpha) \in V \times V^* \;\middle|\; v \neq 0, \alpha(v) = 0\right\}/\mathbb{C}^*,$$
since $\mathbb{C}^* \cong S^1 \times \mathbb{R}^\times_{>0}$, and for any pair (v,α) with v nonzero there is a unique real $t > 0$ such that $t^2\|v\|^2 - t^{-2}\|\alpha\|^2 = 1$. In turn this is the standard description of the cotangent bundle $T^*\mathbb{P}(V)$ of the projective space of V. This follows from the following more general fact:

Lemma 6.8. *Let $M = Gr_k(V)$ be the Grassmannian of k dimensional complex subspaces of V. Let $W \subset V$ be such subspace. Then the tangent space to M at W is naturally $T_W M = \mathrm{Hom}(W, V/W) = W^* \otimes (V/W)$ and in turn*
$$T^*_W M = W \otimes (V/W)^* = W \otimes W^\circ$$
where $W^\circ \subset V^*$ is the annihilator of W: $W^\circ = \{\alpha \in V^* \mid \alpha(W) = 0\}$.

Assuming this, and setting $k = 1$ so $M = \mathbb{P}(V)$, we have
$$T^*\mathbb{P}(V) = \{(L, v \otimes \alpha) \in \mathbb{P}(V) \times V \otimes V^* \mid v \in L, \alpha \in L^\circ\}$$
which agrees with the above description of the hyperkähler quotient (the map $(v,\alpha) \mapsto ([v], v \otimes \alpha)$ is surjective with the $\mathbb{C}^*$ orbits as fibres).

Thus the total space of the holomorphic cotangent bundle to any projective space is a complete hyperkähler manifold. In the case $V = \mathbb{C}^2$ we obtain the cotangent bundle $T^*\mathbb{P}^1$ to the Riemann sphere, of real dimension 4. This hyperkähler four-manifold is the Eguchi–Hanson space, discovered in 1978 [EH78], and was the first nontrivial example of a hyperkähler manifold. The hyperkähler metrics on the higher dimensional cotangent bundles of projective spaces were constructed in 1979 by Calabi [Cal79] (by a different method to that above), who coined the term "hyperkähler".

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