$$\omega^{(p)} = d\alpha^{(p-1)} + \delta\beta^{(p+1)} + \gamma^{(p)} \qquad (A.9)$$

with $\gamma^{(p)}$ = harmonic p-form. The proof that α, β, γ exist is difficult, whereas the uniqueness part is easy to settle, using the semipositivity property of the inner product (see for ex. Ref. [6]).

CHAPTER I.2

RIEMANNIAN MANIFOLDS

I.2.1 - Introduction

We already anticipated (Sect. I.1.6) that the Riemannian geometry of a manifold M_n will be developed using the moving frame $\{e_i\}$ and the dual vielbein (co)frame $\{V^i\}$.

We are aware of the fact that a rigorous treatment of differential geometry should be based on the theory of fiber bundles, particularly for what concerns the theory of connections and many global questions. However the essential idea of the Cartan "moving frame" approach to Riemannian geometry is to reduce, as far as possible, problems of Riemannian geometry to problems of linear algebra in vector spaces.

In this way it is possible to give a simple intuitive interpretation to a number of properties which are usually hidden, in the usual tensor approach, under a plethora of indices. There are some subtle points in the derivations of some important formulae which we will neglect; these defects in rigour are however greatly compensated in our opinion by the gain in geometrical intuition. (For rigorous and complete treatments see the books by E. Cartan and H. Flanders).

I.2.2 - Geometry of the linear spaces

To illustrate how the method works we begin with the case of a linear space $\mathbb{R}^n$ and then we extend the procedure to a smooth Riemannian manifold M_n.

Suppose we have curvilinear coordinates $\{x^\mu\}$ on $\mathbb{R}^n$; the tangent vectors at P to the lines $x^\mu = \text{const.}$ span the natural basis. It is convenient to use the symbol $\vec{P}$ to denote the position vector of $\vec{P}$ referred to some origin in $\mathbb{R}^n$. Then the vectors of the natural frame are given by

$$e_\mu = \frac{\partial}{\partial x^\mu} \vec{P} \tag{I.2.1}$$

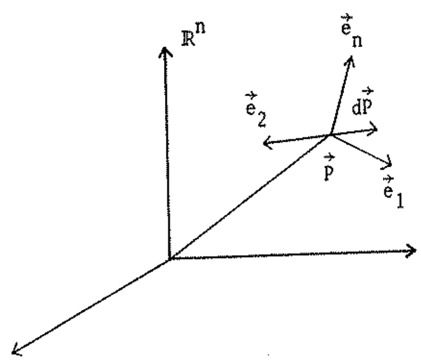

Fig. I.2.1

Each vector at $\vec{P}$ can be expressed in terms of its local components: in particular the displacement vector $d\vec{P}$ is given by

$$d\vec{P} = dx^\mu \frac{\partial}{\partial x^\mu} \vec{P} \tag{I.2.2}$$

Instead of the natural basis (I.2.1) any other frame could work equally well; in particular it is obviously convenient to introduce a set of

vectors $\{\vec{e}_i\}$ which are orthonormal with respect to the n-dimensional Minkowski metric $\eta_{ij} = (1,-1,\ldots,-1)$:

$$\vec{e}_i \cdot \vec{e}_j = \eta_{ij} \tag{I.2.3}$$

(The choice of the signature $(+,-,-,-,-)$ which, from a rigorous point of view corresponds to the choice of pseudo Riemannian rather than Riemannian geometries, is motivated by our final goal which is the theory of gravitation. We omit all the time the "pseudo".s and we use Riemannian for pseudo Riemannian following a by now well established tradition). The frame $\{\vec{e}_i\}$ is called the moving frame: it is related to the natural frame (I.2.1) by a non singular matrix V^μ_i (see Eq. (I.1.182))

$$\vec{e}_i = V^\mu{}_i \, \vec{e}_\mu \quad ; \quad \vec{e}_\mu = V^i{}_\mu \, \vec{e}_i \tag{I.2.4a}$$

$$V^\mu{}_i \, V^i{}_\nu = \delta^\mu_\nu \quad ; \quad V^\mu{}_i \, V^j{}_\mu = \delta^j_i \; . \tag{I.2.4b}$$

Introducting the 1-forms (Eq. (I.1.176-177))

$$V^i = V^i{}_\mu \, dx^\mu \tag{I.2.4c}$$

Eq. (I.2.2) becomes:

$$\vec{dP} = dx^\mu (V^i{}_\mu \, V^\nu{}_i) \frac{\partial}{\partial x^\nu} \vec{P} = V^i e_i(\vec{P}) \; . \tag{I.2.5}$$

According to (I.2.4), the set of 1-forms $\{V^i\}$ is the vielbein frame dual to the moving frame $\{e_i\}$: indeed

$$V^i(\vec{e}_j) = V^i{}_\mu \, V^\nu{}_j dx^\mu(\vec{\partial}_\nu) = \delta^i_j \; . \tag{I.2.6}$$

Notice that $\overrightarrow{dP}$ is a vectorial 1-form whose components along the basis $\vec{e}_i \otimes V^j$ are δ^i_j; in other words $\overrightarrow{dP}$ is that vectorial 1-form which gives the identity map of $T_p(M)$ onto itself:

$$\overrightarrow{dP}(\vec{e}_j) = \vec{e}_j \quad . \tag{I.2.7}$$

The relation between two infinitesimally close frames $\{\vec{e}_i\}$ and $\{\vec{e}_i + d\vec{e}_i\}$ is given by

$$d\vec{e}_i = \frac{\partial \vec{e}_i}{\partial x^j} dx^j \tag{I.2.8}$$

and since $d\vec{e}_i$ is a vectorial 1-form we find:

$$d\vec{e}_i = - \vec{e}_j \, \omega^j{}_i \tag{I.2.9}$$

where $\omega^i{}_j$ is an infinitesimal matrix of 1-forms:

$$\omega^j{}_i = \omega^j{}_{i|\mu} \, dx^\mu \quad . \tag{I.2.10}$$

Differentiating the orthonormality relation $\vec{e}_i \cdot \vec{e}_j = \eta_{ij}$ and using (I.2.9) one obtains:

$$\begin{aligned} d(\vec{e}_i \cdot \vec{e}_j) &= d\vec{e}_i \cdot \vec{e}_j + \vec{e}_i \cdot d\vec{e}_j = \\ &= - (\vec{e}_k \cdot \vec{e}_j \, \omega^k{}_i + \vec{e}_i \cdot \vec{e}_k \, \omega^k{}_j) = \\ &= - (\omega_{ij} + \omega_{ji}) = 0 \quad . \end{aligned} \tag{I.2.11}$$

Therefore $\omega^i{}_j$ is an infinitesimal "rotation" matrix of SO(1,n-1); it is called the spin connection.

The Lorentzian group SO(1,n-1) emerges because of our choice of the signature (+,-,-,...,-). In an arbitrary signature (+,...+,-,...-), $\omega^i{}_j$ turns out to be an SO(k,ℓ) Lie algebra element and in particular for the strictly Riemannian case (k=n, ℓ=0) it belongs to SO(n).

We apply the d-operator to both sides of (I.2.5) and (I.2.9); the integrability condition $d^2 \equiv 0$ gives the following 2-form equations:

$$R^i \overset{\text{def}}{\equiv} dV^i - \omega^i{}_j \wedge V^j = 0 \tag{I.2.12a}$$

$$R^i{}_j \overset{\text{def}}{\equiv} d\omega^i{}_j - \omega^i{}_k \wedge \omega^k{}_j = 0 \quad . \tag{I.2.12b}$$

The left-hand sides of these equations, R^i and R^{ij}, are called the torsion 2-form and the curvature 2-form respectively. In the $\mathbb{R}^n$ case they are identically zero. This is so because in Euclidean spaces, such as $\mathbb{R}^n$, it is always possible at each point P to introduce an orthogonal matrix B such that each moving frame $\{\vec{e}_i\}$ can be aligned a given fixed frame $\{\vec{e}^{(0)}_i\}$

$$\vec{e} = \vec{e}^{(0)} B \quad . \tag{I.2.13}$$

Then from (I.2.9) we get

$$\omega = - B^{-1} dB \quad . \tag{I.2.14}$$

Eq. (I.2.8) says that the spin connection associated to the gauge group SO(1,n-1), acting locally on the moving frame, is a pure gauge. Accordingly, Eq. (I.2.12b) expresses the fact that the associated field strength is identically zero.

Let us now consider a vector field $\vec{v}$ defined over a region of $\mathbb{R}^n$; referring it to the moving frame we have

$$\vec{v} = v^i \vec{e}_i \qquad . \tag{I.2.15}$$

Using (I.2.9) we evaluate the change $d\vec{v}$ due to an infinitesimal displacement:

$$d\vec{v} = dv^j \vec{e}_j - v^i \omega^j{}_i \vec{e}_j = (dv^i - \omega^i{}_j v^j)\vec{e}_i \tag{I.2.16}$$

where

$$dv^i - \omega^i{}_j v^j \overset{\text{def}}{\equiv} \mathscr{D} v^i \tag{I.2.17}$$

is called the covariant derivative of v^i.

I.2.3 - The geometry of general Riemannian manifolds in the vielbein basis

We want now to extend the formalism developed for the almost trivial case of $\mathbb{R}^n$ to general manifolds. Suppose we consider an n-dimensional manifold M_n on which a metric $g_{\mu\nu}$ has been defined; according to the general definition given in Chapter I (see considerations following Eq. (I.1.132)) M_n is by definition a Riemannian manifold.

In the same way as we did for $\mathbb{R}^n$ at each point P we set up an orthonormal local reference frame $\{\vec{e}_i\}$ spanning a basis of $T_P(M_n)$:

$$\vec{e}_i \cdot \vec{e}_j = \eta_{ij} \tag{I.2.18}$$

where η_{ij} is the Minkowskian metric on the tangent space. We insist on taking only orthonormal frames since we are going to introduce spinor fields on M_n and since they are SO(1,n-1) representations. Therefore we are forced to restrict the set of affine frames at P, related to each other by elements of $GL(n,\mathbb{R})$

$$\vec{e}'_i = \vec{e}_j A^j{}_i \qquad A \in GL(n,R) \tag{I.2.19}$$

to the subset of orthonormal frames related to each other by elements of $SO(1,n-1)$. In particular spinors cannot be described in the natural frame $\{\vec{\partial}_\mu\}$. Indeed under a coordinate transformation the vectors $\vec{\partial}_\mu$ transform as (see Eq. (I.1.123)):

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \tag{I.2.20}$$

where the Jacobian matrix $(\partial x^\nu/\partial x'^\mu)_P$ is, in general, an element of $GL(n,\mathbb{R})$.

The relation between the moving (orthonormal) frame and the natural one is obviously the same as in the Euclidean case i.e.:

$$\vec{e}_i = V^\mu_i \frac{\partial}{\partial x^\mu} \tag{I.2.21}$$

$$\frac{\partial}{\partial x^\mu} = V^i_\mu \vec{e}_i \tag{I.2.22}$$

with V^i_μ and V^μ_i satisfying (I.2.4b).

From now on we use only moving frames. The relation with the usual tensor formulation which utilizes the natural frame will be given afterwards.

Proceeding now in analogy to the $\mathbb{R}^n$ case, we express an infinitesimal displacement $\vec{đP}$ in terms of the moving frame at $T_p(M)$:

$$\vec{đP} = V^i \vec{e}_i \qquad (I.2.23)$$

where the V^i are the vielbein fields dual to the moving frame:

$$V^i = V^i_\mu \, dx^\mu \quad . \qquad (I.2.24)$$

They are a basis for the 1-forms on the cotangent plane at P.

The notation $\vec{đP}$ for the infinitesimal displacement is due to the fact that (I.2.3) is not in general an exact differential since P, contrary to what happens in the Euclidean case, is not a function of the coordinates. The same remark applies to the evaluation of the change of the moving frame under an infinitesimal translation $\vec{P} \to \vec{P} + \vec{dP}$:

$$đ\,\vec{e}_i = -\vec{e}_j \omega^j{}_i \ , \qquad (I.2.25)$$

where as in the Euclidean case

$$\omega^j{}_i = \omega^j{}_{i|\mu} \, dx^\mu \quad . \qquad (I.2.26)$$

Since $\vec{dP}$ and $d\vec{e}_i$ are not exact differentials we do not expect the torsion 2-form R^i and the curvature 2-form R^{ij} defined by Eqs. (I.2.12) to be zero. Therefore on any manifold M_n we introduce the concept of torsion and the curvature 2-forms by means of the following definitions:

$$R^i \overset{\text{def}}{\equiv} dV^i - \omega^i{}_j \wedge V^j \qquad (I.2.27a)$$

$$R^{ij} \overset{\text{def}}{\equiv} d\omega^{ij} - \omega^i{}_k \wedge \omega^{kj} \quad . \qquad (I.2.27b)$$

where $\omega^{ij} \equiv \omega^i{}_k \eta^{kj}$. In general R^i and R^{ij} have non vanishing values.

The metric tensor on M_n is given in terms of the vielbeins by Eqs. (I.1.187-188), or equivalently it is defined by taking the square of (I.2.23):

$$(\vec{đP})^2 \equiv ds^2 = V^i\vec{e}_i \otimes V^j\vec{e}_j = V^i_\mu \, dx^\mu \otimes V^j_\nu \, dx^\nu \vec{e}_i \cdot \vec{e}_j =$$
$$= \eta_{ij} V^i_\mu V^j_\nu \, dx^\mu \otimes dx^\nu \quad . \qquad (I.2.28)$$

Therefore:

$$g_{\mu\nu} = V^i_\mu V^j_\nu \eta_{ij} \quad . \qquad (I.2.29)$$

We notice that the structure equations (I.2.27) could also be retrieved in the following heuristic manner. Let us take the exterior derivative of both sides of Eqs. (I.1.23) and (I.1.25). We get:

$$d(đP) = dV^i\vec{e}_i - V^i \, đ\vec{e}_i = (dV^i - \omega^i{}_j \wedge V^j)\vec{e}_i \qquad (I.2.30a)$$

$$d(đ\vec{e}_i) = - đ\vec{e}_j \cdot \omega^j{}_i - \vec{e}_j \cdot d\omega^j{}_i =$$
$$= - \vec{e}_j (d\omega^j{}_i - \omega^j{}_k \wedge \omega^k{}_i) \quad . \qquad (I.2.30b)$$

where the 2-form components along the $\{\vec{e}_i\}$ frame define again the torsion and (minus) the curvature.

The reason we call this derivation heuristic is that in the differentiation we have substituted the d-operator by $đ$ when acting on the moving frame. Using the same rule for the differentiation of $\vec{e}_i \cdot \vec{e}_j \equiv \eta_{ij}$ we get:

$$d(\vec{e}_i \cdot \vec{e}_j) = đ\vec{e}_i \cdot \vec{e}_j + \vec{e}_i \cdot đ\vec{e}_j =$$
$$= - \vec{e}_k \cdot \omega^k{}_i \vec{e}_j - \vec{e}_i \cdot \vec{e}_k \omega^k{}_j \quad . \qquad (I.2.31)$$

that is:

$$\omega_{ij} = - \omega_{ji} \quad . \tag{I.2.32}$$

Therefore our heuristic arguments tell us that the connection $\omega^i{}_j$ belongs to the Lie algebra of SO(1,n-1) as in the $\mathbb{R}^n$ case. In the sequel we assume the validity of (I.2.32). In this case $\omega^i{}_j$ is called a spin connection. Differentiating both sides of Eqs. (I.2.27), and using $d^2 = 0$, we get the following integrability conditions:

$$dR^i = - d\omega^i{}_j \wedge V^j + \omega^i{}_j \wedge dV^j =$$

$$= - (R^i{}_j + \omega^i{}_k \wedge \omega^k{}_j) \wedge V^j + \omega^i{}_j \wedge (R^j + \omega^j{}_k \wedge V^k) \tag{I.2.33a}$$

$$= - R^i{}_j \wedge V^j + \omega^i{}_j \wedge R^j \tag{I.2.33b}$$

$$dR^i{}_j = d\omega^i{}_k \wedge \omega^k{}_j - \omega^i_k \wedge d\omega^k{}_j = (R^i{}_k + \omega^i{}_\ell \wedge \omega^\ell{}_j) \wedge \omega^k{}_j -$$

$$- \omega^i{}_k \wedge (R^k{}_j + \omega^k{}_\ell \wedge \omega^\ell{}_j) =$$

$$= R^i{}_k \wedge \omega^k{}_j - \omega^i{}_k \wedge R^k{}_j \quad . \tag{I.2.33c}$$

Equations (I.1.33) are referred to as the Bianchi identities obeyed by the curvatures R^i and $R^i{}_j$.

Let us observe explicitly that all the equations introduced so far are exterior equations and as such they are scalars under diffeomorphisms, according to Eq. (I.1.160). Latin indices are inert under diffeomorphisms being indices of the local gauge group SO(1,n-1).

The same is true if we expand $\omega^i{}_j$, R^i and $R^i{}_j$ in a local cotangent basis $\{V^i\}$:

$$\omega^i{}_j = \omega^i{}_{j|k} V^k \tag{I.2.34a}$$

$$R^i = R^i{}_{k\ell} V^k \wedge V^\ell \tag{I.2.34b}$$

$$R^i{}_j = R^i{}_{j|k\ell} V^k \wedge V^\ell \quad . \tag{I.2.34c}$$

Indeed the component fields $\omega^i{}_{j|k}$, $R^i{}_{k\ell}$, $R^i{}_{j|k\ell}$ have indices of the Latin type and hence inert under diffeomorphisms.

Let us collect our results: we started with a Riemannian manifold M_n endowed with a local (orthonormal) moving frame and its dual $\{V^i\}$ in the cotangent plane. The frame $\{V^i\}$ is acted on by the local gauge group SO(1,n-1). We also introduced a local connection 1-form $\omega^i{}_j$.

The local geometry is described by:

i) Structure equations

$$\not{d}\vec{P} = V^i \vec{e}_i \tag{I.2.35a}$$

$$\not{d}\vec{e}_i = - \vec{e}_j \omega^j{}_i \tag{I.2.35b}$$

$$R^i = dV^i - \omega^i{}_j \wedge V^j \tag{I.2.35c}$$

$$R^{ij} = d\omega^{ij} - \omega^i{}_k \wedge \omega^{kj} \tag{I.2.35d}$$

ii) Bianchi identities: i.e. the integrability conditions of the structure equations:

$$dR^i + \omega^i{}_j \wedge R^j + R^i{}_j \wedge V^j = 0 \tag{I.2.36a}$$

$$dR^i{}_j - R^i{}_k \wedge \omega^k{}_j + \omega^i{}_k \wedge R^k{}_j \tag{I.2.36b}$$

iii) The metric postulate

$$\omega^{ij} = - \omega^{ji} \quad . \tag{I.2.37}$$

If one further assumes:

iv) $R^i = 0 \quad .$ (I.2.38)

then M_n is a (Riemannian) manifold with a Riemannian connection.

In this case one can express the spin connection in terms of the vielbein field. Indeed let us expand the 1-forms $\omega^i{}_j$ and the 2-form dV^j along the V^i-basis:

$$dV^i = c^i{}_{jk} V^j \wedge V^k \qquad (I.2.39a)$$

$$\omega^i{}_j = \omega^i{}_{j|k} V^k \quad . \qquad (I.2.39b)$$

Inserting in (I.2.38) we get:

$$c_{ijk} = \frac{1}{2} (\omega_{ik|j} - \omega_{ij|k}) \qquad (I.2.40)$$

where we have lowered the upper index with the metric η_{ij}.

Adding and subtracting the two equations analogous to Eq. (I.2.40), but with ijk indices circularly permuted one obtains:

$$\omega_{ij|k} = c_{ijk} + c_{jki} - c_{kij} \qquad (I.2.41)$$

where we have used Eq. (I.2.37).

If one wants to express the spin connection in terms of the space-time derivatives of V^i, it is better to expand Eq. (I.2.38) in the coordinate basis $\{dx^\mu\}$:

$$\partial_{[\mu} V^i_{\nu]} = \frac{1}{2} (\omega^i{}_{j|\mu} V^j_\nu - \omega^i{}_{j|\nu} V^j_\mu) \quad . \qquad (I.2.42)$$

Converting the tangent index into a world index by multiplication with $\eta_{ik} V^k_\rho = V_{i\rho}$, we obtain:

$$\eta_{ik} V^k_\rho \partial_{[\mu} V^i_{\nu]} = \frac{1}{2} (\omega^i{}_{j|\mu} V^j_\nu V^k_\rho - \omega^i{}_{j|\nu} V^j_\mu V^k_\rho) \eta_{ik} \quad . \qquad (I.2.43)$$

This equation can be solved as Eq. (I.2.40) by permuting the $\mu\nu\rho$ indices. We get

$$\omega_{k\ell|\mu} = (f_{\lambda|\mu\nu} + f_{\nu|\lambda\mu} - f_{\mu|\nu\lambda}) V^\lambda_k V^\nu_\ell \qquad (I.2.44)$$

where

$$f_{\lambda|\mu\nu} = V^i_\lambda \partial_{[\mu} V^j_{\nu]} \eta_{ij} \quad . \qquad (I.2.45)$$

Let us now explore the local gauge invariance under the local Lorentz group SO(1,n-1).

Suppose we perform an SO(1,n-1) gauge transformation on the local frames

$$\vec{e}'_i = \vec{e}_j \Lambda^j{}_i \qquad \Lambda \in SO(1,n-1) \quad . \qquad (I.2.46)$$

From

$$d\vec{P} = \vec{e}_i V^i = \vec{e}'_i V'^i \qquad (I.2.47)$$

we obtain:

$$V'^i = (\Lambda^{-1})^i{}_j V^j \quad . \qquad (I.2.48)$$

Then from

$$d\vec{e}' = - \vec{e}' \omega' \qquad (I.2.49)$$

(where we use a matrix notation) using (I.2.25) and (I.2.46), we have:

$$- \vec{e}\omega\Lambda + \vec{e} d\Lambda = - \vec{e}\Lambda\omega' \qquad (I.2.50)$$

and therefore we can write

$$\omega' = -\Lambda^{-1}(d\Lambda - \omega)\Lambda$$

$$\Rightarrow \omega'^{i}{}_{j} = (\Lambda^{-1})^{i}{}_{k}\omega^{k}{}_{\ell}\Lambda^{\ell}{}_{j} - (\Lambda^{-1})^{i}{}_{k}(d\Lambda)^{k}{}_{j} \quad . \tag{I.2.51}$$

The result is that the spin connection ω undergoes an SO(1,n-1) gauge transformation.

One easily finds that the torsion 2-form R^i and the curvature 2-form $R^i{}_j$ transform in the vector and in the adjoint representation of SO(1,n-1) respectively:

$$R'^{i} = (\Lambda^{-1})^{i}{}_{j}\, R^{j} \tag{I.2.52a}$$

$$R^{i}{}_{j} = (\Lambda^{-1})^{i}{}_{k}\, R^{k}{}_{\ell}\, \Lambda^{\ell}{}_{j} \quad . \tag{I.2.52b}$$

Next we compute the change of a vector

$$\vec{v} = v^{i}\,\vec{e}_{i} \tag{I.2.53}$$

under an infinitesimal displacement. Differentiating both sides of (I.2.53) and using (I.2.25), we find:

$$d\vec{v} = \vec{e}_{i}(dv^{i} - \omega^{i}{}_{j}v^{j}) \tag{I.2.54}$$

Hence we define the SO(1,n-1) covariant exterior derivative of v^i by:

$$\mathscr{D}v^{i} \overset{\text{def}}{\equiv} dv^{i} - \omega^{i}{}_{j}v^{j} \quad . \tag{I.2.55}$$

(In the following it will be referred to as the Lorentz covariant derivative).

Indeed taking into account Eq. (I.2.48,51) $\mathscr{D}v^i$ transforms in the same way as v^i.

We can extend the notion of covariant derivative to any tensor-valued p-form $\Phi^{i_1 i_2\ldots}{}_{j_1 j_2\ldots}$ as follows:

$$\mathscr{D}\Phi^{i_1 i_2\ldots}{}_{j_1 j_2\ldots} = d\Phi^{i_1 i_2\ldots}{}_{j_1 j_2\ldots} - \omega^{i_1}{}_{k}\Phi^{k i_2\ldots}{}_{j_1 j_2\ldots} - \omega_{j_1}{}^{k}\Phi^{i_1 i_2\ldots}{}_{k j_2} - \ldots \tag{I.2.56}$$

and verify that this is indeed a SO(1,n-1) covariant derivative.

As we have already pointed out one can also introduce p-form fields which are in spinor representations of the tangent group SO(1,n-1). Let σ be one such field in the lowest spinor representation and let

$$\Gamma_{ij} = \frac{1}{2}\,[\Gamma_i,\Gamma_j] \tag{I.2.57}$$

be the Lorentz generators in the spinor representation, with Γ^i Dirac γ-matrices for SO(1,n-1). Then

$$\mathscr{D}\sigma = d\sigma - \frac{1}{4}\,\omega_{ij} \wedge \Gamma^{ij}\sigma \tag{I.2.58}$$

is the covariant derivative of the spinorial p-form σ.

This can be easily checked using (I.2.51) and

$$L\,\Gamma_{ab}\,L^{-1} = \Gamma_{cd}\,\Lambda^{c}{}_{a}\,\Lambda^{d}{}_{b} \tag{I.2.59}$$

where L and Λ are elements of SO(1,n-1) in the spinor and vector representation respectively. Quite generally for a p-form A with indices in any representation D of SO(1,n-1) the covariant derivative is defined by:

$$\mathscr{D}A \overset{\text{def}}{\equiv} dA + \omega^{ij} \wedge D(T_{ij})A \tag{I.2.60}$$

where $D(T_{ij})$ is the representation of the generators T_{ij} of SO(1,n-1). We shall come back to this general formula in Chapter I.5.

Using the Lorentz covariant derivative the torsion 2-form is rewritten as follows:

$$R^{i} = \mathscr{D}V^{i} \tag{I.2.61}$$

and the Bianchi identities (I.2.36) become:

$$\mathcal{D} R^i{}_j = 0 \qquad \text{(I.2.62a)}$$

$$\mathcal{D} R^i + R^i{}_j \wedge V^j = 0 \qquad \text{(I.2.62b)}$$

Let us make the symmetries of the intrinsic curvature tensor $R^i{}_{j|k\ell}$ explicit; from Eq. (I.2.34c) one immediately gets

$$R^i{}_{j|k\ell} = - R^i{}_{j|\ell k} \qquad \text{(I.2.63)}$$

and from the metric postulate (I.2.37):

$$R_{ij|k\ell} = - R_{ji|k\ell} \quad . \qquad \text{(I.2.64)}$$

Furthermore, when $\omega^i{}_j$ is a Riemannian connection, that is when Eq. (I.2.38) holds, from (I.2.62b), we get:

$$R^i{}_j \wedge V^j = 0 \qquad \text{(I.2.65)}$$

Expanding along the vielbein basis we find

$$R^i{}_{j|k\ell} V^j \wedge V^k \wedge V^\ell = 0 \qquad \text{(I.2.66)}$$

which gives the cyclic identity:

$$R^i{}_{j|k\ell} + R^i{}_{k|\ell j} + R^i{}_{\ell|jk} = 0 \qquad \text{(I.2.67)}$$

By repeated use of Eqs. (I.2.63), (I.2.64) and (I.2.67) one also derives:

$$\begin{aligned} R_{ij|k\ell} - R_{k\ell|ij} &= R_{ij|k\ell} + R_{ki|j\ell} + R_{kj|\ell i} = \\ &= R_{ij|k\ell} + R_{ik|\ell j} - R_{jk|\ell i} = - R_{i\ell|jk} - R_{jk|\ell i} = \\ &= R_{\ell i|jk} + (R_{j\ell|ik} + R_{ji|k\ell}) = \\ &= (R_{\ell i|jk} + R_{\ell j|ki}) + R_{ji|k\ell} = \\ &= - R_{\ell k|ij} + R_{ji|k\ell} = \\ &= - R_{ij|k\ell} + R_{k\ell|ij} \quad . \end{aligned} \qquad \text{(I.2.68)}$$

Therefore:

$$R_{ij|k\ell} = R_{k\ell|ij} \quad . \qquad \text{(I.2.69)}$$

From $R^i{}_{j|k\ell}$ one may construct the <u>Ricci tensor</u>

$$R^i{}_{j|ik} \overset{\text{def}}{\equiv} R_{jk} \qquad \text{(I.2.70)}$$

which turns out to be symmetric in j,k, and the <u>curvature scalar</u>

$$\eta^{ij} R_{ij} \overset{\text{def}}{\equiv} R \quad . \qquad \text{(I.2.71)}$$

Because of the aforementioned symmetry properties any other contraction possibility gives at most a change of sign with respect to definitions (I.2.70) and (I.2.71).

I.2.4 - Relation with the standard world-tensor formalism

Up to now all the fields defined on M_n have been expressed in terms of their components along the vielbein basis so that all the indices transform linearly with respect to $SO(1,n-1)$. Of course it is also possible to use the natural frame $\{\vec{\partial}_\mu\}$ in the tangent plane and its dual $\{dx^\mu\}$. In that case we get the description of the classical tensor calculus where all the indices transform covariantly with respect to a change of local charts.

We give briefly the expression of all the geometrical tensors in the natural frame and their relation to the same objects in the intrinsic frame. In the natural frame $\{\vec{\partial}_\mu \equiv \vec{e}_\mu\}$ the relation between two infinitesimally close frames is given by

$$\not{d}\vec{e}_\nu = \vec{e}_\mu \Gamma^\mu{}_\nu \tag{I.2.72}$$

where*

$$\Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu|\rho}\, dx^\rho \tag{I.2.73}$$

is called the affine connection since it makes the transition between two frames $\{\partial_\mu\}$ and $\{\partial'_\mu\}$ related by an element of $GL(n,\mathbb{R})$.

Proceeding as before we can define the torsion 2-form R^μ and the curvature 2-form $R^\mu{}_\nu$ by the replacement

$$V^i \to dx^\mu \tag{I.2.74}$$

$$\omega^i{}_j \to -\Gamma^\mu{}_\nu \tag{I.2.75}$$

Accordingly in the natural frame the torsion is:

$$R^\mu = d(dx^\mu) + \Gamma^\mu{}_\nu \wedge dx^\nu = -\Gamma^\mu{}_{\nu|\rho}\, dx^\nu \wedge dx^\rho \quad . \tag{I.2.76}$$

* Notice the change of sign in (I.2.72) with respect to (I.2.25) in order to adhere to the usual conventions in the world tensor formalism (see e.g. Ref. 17).

The antisymmetry condition (I.2.37) becomes:

$$dg_{\mu\nu} \equiv d(\vec{e}_\mu \cdot \vec{e}_\nu) = \not{d}\vec{e}_\mu \cdot \vec{e}_\nu + \vec{e}_\mu \cdot \not{d}\vec{e}_\nu =$$

$$= + \vec{e}_\rho \cdot \Gamma^\rho{}_\mu \vec{e}_\nu + \vec{e}_\mu \cdot \vec{e}_\rho \Gamma^\rho{}_\nu \tag{I.2.77}$$

that is:

$$dg_{\mu\nu} - \Gamma^\rho{}_\mu g_{\rho\nu} - \Gamma^\rho{}_\nu g_{\mu\rho} = 0 \tag{I.2.78}$$

which is the metric postulate in the coordinate basis.

On the other hand the condition $R^\mu = 0$ defining the Riemannian connection, upon use of Eq. (I.2.76), yields the symmetry properties of the Christoffel symbol

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu} \equiv \left\{ {\mu \atop \nu\rho} \right\} \tag{I.2.79}$$

where

$$\left\{ {\mu \atop \nu\rho} \right\} = \frac{1}{2}\, g^{\sigma\mu}(-\partial_\sigma g_{\nu\rho} + \partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho}) \quad . \tag{I.2.80}$$

Eq. (I.2.80) is obtained from (I.2.78) in the same way as we obtained (I.2.41) from (I.2.39).

Finally the curvature 2-form becomes

$$R^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu \quad . \tag{I.2.81}$$

Expanding along the natural basis we retrieve the definition of the Riemann curvature tensor

$$R^\mu{}_{\nu|\rho\sigma} = \frac{1}{2}\{\partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\rho}\} \quad . \tag{I.2.82}$$

Let us now perform a change of local chart; in local coordinates

$$x'^i = x^i(x^1,\dots,x^n) \quad . \tag{I.2.83}$$

Recalling our discussion of vector fields and forms, the natural frames and coframes transform as follows (Eqs. (I.1.123) and (I.1.161)):

$$\partial'_\mu = (J^{-1})^\nu{}_\mu \, \partial_\nu \tag{I.2.84a}$$

$$d'x^\mu = J^\mu{}_\nu \, dx^\nu \tag{I.2.84b}$$

where

$$J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \in GL(n,\mathbb{R}) \quad . \tag{I.2.85}$$

Eq. (I.2.84) are analogous to Eqs. (I.2.46) and (I.2.48). Therefore a coordinate transformation induces a local gauge transformation of $GL(n,\mathbb{R})$ on the basis frames. With the same procedure used in deriving Eq. (I.2.51) from Eq. (I.2.49) one finds the transformation of $\Gamma^\mu{}_\nu$ under a change of local chart:

$$(\Gamma')^\mu{}_\nu = + J(d\mathfrak{J} + \Gamma)J^{-1} \quad . \tag{I.2.86}$$

Correspondingly we have:

$$R'^\mu = J^\mu{}_\nu \, R^\nu \tag{I.2.87a}$$

$$R'^\mu{}_\nu = J^\mu{}_\rho \, R^\rho{}_\sigma (J^{-1})^\sigma{}_\nu \tag{I.2.87b}$$

Proceeding in the same way as from Eqs. (I.2.53) to (I.2.55) one finds for the $GL(n,\mathbb{R})$ covariant derivative of a vector field:

$$(\nabla v)^\mu = dv^\mu + \Gamma^\mu{}_\nu \, v^\nu \quad . \tag{I.2.88}$$

The extension to a general (k,ℓ)-tensor field is:

$$\nabla A^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell} = A^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell} + \Gamma^{\mu_1}{}_\rho \, A^{\rho\mu_2\dots\mu_k}{}_{\nu_1\dots\nu_\ell} + \\ - \Gamma^\rho{}_{\nu_1} \, A^{\mu_1\dots\mu_k}{}_{\rho\nu_2\dots\nu_\ell} + \dots \quad . \tag{I.2.89}$$

We stress that we cannot introduce spinor fields in the $GL(n,\mathbb{R})$-covariant basis $\{\partial_\mu\}$.

Let us now observe that the formula (I.2.24) giving the change of frame from the natural to the intrinsic basis can be thought of as induced by a coordinate transformation from a general basis dx^μ to an orthonormal basis V^i, V^i_μ being an element of $GL(n,\mathbb{R})$. Consequently the relation between the spin connection $\omega^i{}_j$ and the affine connection $\Gamma^\mu{}_\nu$ is given by the law (I.2.51):

$$\Gamma^\mu{}_\nu = - V^\mu{}_i \omega^i{}_j V^j{}_\nu + V^\mu{}_i \, dV^i{}_\nu \tag{I.2.90a}$$

$$\omega^i{}_j = - V^i{}_\mu \Gamma^\mu{}_\nu V^\nu{}_j - V^i{}_\nu \, dV^\nu{}_j \quad . \tag{I.2.90b}$$

Multiplying (I.2.90a) by $V^k{}_\mu$ we find:

$$dV^i{}_\nu - \omega^i{}_j V^j{}_\nu - V^i{}_\mu \Gamma^\mu{}_\nu = 0 \quad . \tag{I.2.91}$$

Taking into account Eqs. (I.2.55) and (I.2.88), Eq. (I.2.91) can be interpreted as the vanishing of the combined Lorentz and $GL(n,\mathbb{R})$ covariant derivative on $V^i{}_\mu$.

An analogous equation follows from (I.2.90b) for the inverse vielbein.

Finally we observe that the relation between objects evaluated in the coordinate basis and in the intrinsic basis is given by the intertwining vielbein matrix $V^i{}_\mu$: every coordinate index can be transformed into an intrinsic one by $V^i{}_\mu$:

$$A^{i\dots}_{\dots\dots} = V^i{}_\mu \, A^{\mu\dots}_{\dots\dots} \qquad \text{(I.2.92)}$$

and vice versa

$$A^{\mu\dots}_{\dots\dots} = V^\mu{}_i \, A^{i\dots}_{\dots\dots} \qquad \text{(I.2.93)}$$

In particular

$$A^iA_i \equiv \eta_{ij}A^iA^j = \eta_{ij}V^i{}_\mu V^j{}_\nu A^\mu A^\nu = g_{\mu\nu}A^\mu A^\nu = A^\mu A_\mu \,. \qquad \text{(I.2.94)}$$

where we have used Eq. (I.2.29).

Therefore coordinate scalars are also Lorentz scalars and vice versa.

Other useful relations are the following ones:

$$\mathcal{D}\, A^i = V^i{}_\mu \, \nabla A^\mu \qquad \text{(I.2.95a)}$$

$$\nabla A^\mu = V^\mu{}_i \, \mathcal{D} A^i \qquad \text{(I.2.95b)}$$

where $\mathcal{D}$ and ∇ are the covariant derivatives in the tangent or natural frames respectively.

Eqs. (I.2.95) can be proved by direct computation using Eqs. (I.2.90). Notice that the affine connection entering (I.2.95) is symmetric in its lower indices which implies that the torsion R^a is zero.

Therefore (I.2.95) is <u>not true</u> in presence of a <u>nonvanishing torsion</u>.

CHAPTER I.3

GROUP MANIFOLDS AND MAURER-CARTAN EQUATIONS

I.3.1 - Introduction

In this chapter we discuss Lie groups from a differential geometric point of view. As in previous chapters we just give those main definitions and properties which are relevant for the subsequent developments; previous knowledge of group theory is required.

The chapter is divided in two parts; in the first (Section 1 to 6) we concentrate on the study of those properties which are peculiar to group manifolds, like the existence of left- and right-invariant vector fields or 1-forms. This leads to the discussion of the Lie algebra associated to Lie groups and to the dual concept of Maurer-Cartan equations. Within the same framework we shortly discuss the adjoint and co-adjoint representations of groups and algebras and the Killing metric; finally a short account is given of the Riemannian geometry of semi-simple group manifolds.

The second part of this chapter is devoted to the study of manifolds which are locally diffeomorphic to group manifolds. They are