$|\lambda = 3/2, \underline{6}>, |\lambda = 1, \underline{15}>, |\lambda = \tfrac{1}{2}, 20>, |\lambda = 0, \underline{15}>$

$|\lambda = -\tfrac{1}{2}, \underline{6}>, |\lambda = -1, \underline{1}>$

Consider now the multiplet generated by

$$\lambda'_{MAX} = \frac{N}{2} - \lambda_{MAX} = 3-2 = 1$$

Applying the $B^{\dagger}_{A}$ to $|\lambda=1,\underline{1}>$ we obtain the states

$|\lambda = \tfrac{1}{2}, \underline{6}>, |\lambda = 0, \underline{15}>, |\lambda = -\tfrac{1}{2}, \underline{20}>, |\lambda = -1, \underline{15}>$

$|\lambda = -3/2, \underline{6}>, |\lambda = -2, \underline{1}>$

Putting everything together we see that for each of the SU(6) representations displayed in the N=6 row of Table II.4.VIII we have both the λ=J and λ=-J helicity state which are necessary to build up a spin J massless particle.

Massless spin-one particles are the gauge bosons of Yang-Mills theory, while the massless spin-two particle is the graviton. Correspondingly Table II.4.VI lists the spectra of the N-extended supersymmetric versions of Yang-Mills theory, while Table II.4.VIII displays the field content of the N-extended supergravities. By this name we mean the supersymmetric versions of Einstein gravitational theory which can be identified as N=0 supergravity.

How to construct supersymmetric field theories whose linearization yields the spectra displayed in Tables II.4.VI and II.4.VIII is the question addressed in the sequel of the book. Prior to that, however, we still need to see which modifications are introduced in the supersymmetry algebra (II.2.142) representations by a non vanishing constant $\bar{e}$.

This is the topic of the next chapter.

CHAPTER II.5

SUPERMULTIPLETS IN ANTI DE SITTER SPACE

II.5.1 - Free field equations and the concept of mass in anti de Sitter space

Anti de Sitter space is the bosonic submanifold of the N-extended anti de Sitter superspace (II.3.29). It is the following coset manifold (*)

$$Sp(4,H)/SO(1,3) \sim SO(2,3)/SO(1,3) \qquad (II.5.1)$$

and in the normalization which we use the intrinsic components of its

---

(*) By Sp(4,H) we understand that particular real form of the Sp (4, ℂ) Lie algebra which is defined by the condition $H_{(4)} \Lambda H_{(4)} = -\Lambda^{\dagger}$ (see Eqs. (II.2.121b) and (II.2.133b)).

Riemann tensor are

$$R^{ab}_{\cdot\cdot cd} = -4\bar{e}^2\delta^{ab}_{cd} \equiv -2\bar{e}^2(\delta^a_e\delta^b_d - \delta^a_d\delta^b_c) \qquad \text{(II.5.2)}$$

This result is easily deduced from Eqs. (II.3.55).

In order to study the unitary irreducible representations of the Osp(4/N) superalgebra (II.2.142) we need first some information on the representation of Sp(4,H) ~ SO(2,3). Indeed, as a unitary irreducible representations of Poincaré supersymmetry is composed of a finite number of unitary irreducible representations of the Poincaré subalgebra (each of them infinite dimensional) in the same way, a unitary irreducible representation of Osp(4/N) is made out of a finite number of unitary representations of SO(2,3), also infinite dimensional. In perfect analogy with the Poincaré case, a unitary irreducible representation of SO(2,3) is what one calls a particle in anti de Sitter space. The new features are related with the concept of mass. Indeed the operator P_aP^a is an invariant neither for the full Osp(4/N) algebra nor for the SO(2,3) subalgebra. Hence in AdS a particle is not characterized by the eigenvalue of P_aP^a, rather by the eigenvalue of the true second order Casimir of SO(2,3) which in our normalizations has the following expression:

$$C_2 = -2\,M_{ab}M^{ab} - \frac{1}{4\bar{e}^2}\,P_aP^a \qquad \text{(II.5.3)}$$

This result is easily retrieved by noticing that if we introduce indices Λ=0,1,2,3,4 and we set

$$M_{\Lambda\Sigma} = -M_{\Sigma\Lambda} \quad ; \quad M_{ab} = M_{ab} \quad ; \quad M_{4a} = \frac{1}{4\bar{e}}\,P_a \qquad \text{(II.5.4)}$$

then the 10 generators satisfy the SO(2,3) Lie algebra in its standard form:

$$[M_{\Lambda\Sigma}, M_{\Gamma\Delta}] = \frac{1}{2}\,(\eta_{\Sigma\Gamma}M_{\Lambda\Delta} + \eta_{\Lambda\Delta}M_{\Sigma\Gamma} - \eta_{\Sigma\Delta}M_{\Lambda\Gamma} - \eta_{\Lambda\Gamma}M_{\Sigma\Delta}) \qquad \text{(II.5.5a)}$$

$$\eta_{\Lambda\Sigma} = (+, -, -, -, +) \qquad \text{(II.5.5b)}$$

and C_2 defined by Eq. (II.5.3) coincides with the standard quadratic invariant

$$C_2 = -2\,M_{\Lambda\Sigma}M^{\Lambda\Sigma} \qquad \text{(II.5.6)}$$

The problem is how to relate the eigenvalues of C_2 to something which we can call the mass and the spin of a particle. The answer to such a question is our present goal. It is mainly a matter of comparison. On one side, as we shall see in the next section, we can construct an irreducible unitary representation of SO(2,3) via the Wigner induced representation method, starting from an irrep of the maximal compact subgroup SO(2) ⊗ SO(3) ⊗ SO(2,3).

The SO(2) quantum number E_0 is the eigenvalue of the hamiltonian operator and, as such, is worth the name of energy (the minimal energy of the representation); the number J labelling the SO(3) irrep is instead what we call the spin. On the other hand an irreducible unitary representation must also be identified with the Hilbert space spanned by the finite norm solutions of a free field equation suitable to the spin S particle we consider.

We know how to write field equations for arbitrary spin fields on an arbitrary curved space-time $\mathcal{M}$. In particular we can choose

$\mathcal{M}$ = AdS = anti de Sitter space

and we have the result we look for, namely an equation

$$\boxtimes_{(s)}\psi_{(s)} = m^2_{(s)}\psi_{(s)} \qquad \text{(II.5.7)}$$

where $\boxtimes_{(s)}$ is a second order invariant differential operator (it commutes with the Lie derivatives along the SO(2,3) Killing vectors) which acts on the space of spin-s wave-functions and whose eigenvalue we can call the mass-squared.

Since there is just one quadratic Casimir operator we must have

$$\boxtimes_{(s)} = \alpha\, C_2 + \beta_s = m_s^2 \qquad \text{(II.5.8)}$$

where α and β_s are constants. Moreover, since C_2 is a function of E_0 and J, that is the labels of the vacuum state in the induced representation procedure, then equation (II.4.8) provides a relation between these labels and $m_s{}^2$. Needless to say we must choose J=S and (II.4.8) becomes a relation between E_0 and $m_s{}^2$, relation which is different for different spins.

The delicate point in this game is the choice of the origin of the $m_{(s)}{}^2$ scale or, in other words, the definition of <u>massless particles</u>. Indeed we know that $m^2=0$ is a singular value for Poincaré representations, corresponding to a reduction of the number of states (multiplet shortening) and the same must be true of anti de Sitter representations. The best way to understand this shortening is from a symmetry point of view. At $m_s{}^2=0$ the wave equation (II.4.7) must acquire a larger symmetry than the Poincaré or anti de Sitter symmetry. This larger symmetry is conformal symmetry for s=0 and s=1/2 while it is a gauge symmetry for s≥1; in anycase it is responsible for a reduction of the dynamical degrees of freedom and the associated particle is worth the name of <u>massless</u>.

Since we are interested in particles of spin s=0,½,1,3/2 and 2, we shall explicitly consider the wave equations of those five kinds of particles.

<u>The s=0 particle</u>

Let $\mathscr{D}_a$ be the Lorentz covariant derivative, defined as follows. Given the covariant differential of a field f(x), belonging to some representation of SO(1,3)

$$\mathscr{D} f(x) = df(x) + \omega^{ab} t_{ab} f(x) \qquad \text{(II.5.9)}$$

(where t_{ab} are the appropriate Lorentz generators) $\mathscr{D}_a f(x)$ is given by:

$$\mathscr{D}_a f(x) = D_a \rfloor \mathscr{D} f(x) \qquad \text{(II.5.10)}$$

D_a being the tangent vector dual to the vierbein V^a:

$$V^a(D_b) = \delta^a_b \qquad \text{(II.5.11)}$$

Having defined the covariant Laplacian by:

$$\Box_{cov} = \mathscr{D}^a \mathscr{D}_a \qquad \text{(II.5.12)}$$

it is well known that the following equation

$$\left(\Box_{cov} + \frac{\mathscr{R}}{3}\right)\varphi(x) = 0 \qquad \text{(II.5.13)}$$

in addition to invariance under whatever isometries the metric $g_{\mu\nu} = V^a_\mu V^b_\nu \eta_{ab}$ may possess, has further invariance under scale transformations which are instead broken by the equation (*)

$$(\Box_{cov} + \frac{\mathscr{R}}{3})\,\varphi(x) = -\,m_0^2\,\varphi(x) \qquad (II.5.14)$$

This means that (II.5.14) is the correct wave-equation for a scalar particle of mass m_0.

If we choose anti de Sitter space as a background we find

$$\mathscr{R}^{ab}{}_{ab} \equiv \mathscr{R} = -\,4\bar{e}^2\,\frac{1}{2}\,(\delta^a_a\delta^b_b - \delta^a_b\delta^b_a) = -\,24\bar{e}^2 \qquad (II.5.15)$$

and (II.5.14) becomes:

$$(\Box_{cov} + m_0^2 - 8\bar{e}^2)\,\varphi(x) = 0 \qquad (II.5.16)$$

The s=1/2 particle

In the spin 1/2 case we have

$$\mathscr{D}\lambda = d\lambda - \frac{1}{4}\,\omega^{ab}\gamma_{ab}\lambda \qquad (II.5.17a)$$

$$[\mathscr{D}_a, \mathscr{D}_b]\lambda = -\,\frac{1}{4}\,\mathscr{R}^{mn}{}_{ab}\gamma_{mn}\lambda \qquad (II.5.17b)$$

and the wave equation

(*) For a short derivation of this result see Part One.

$$i\,\gamma^a\,\mathscr{D}_a\lambda = 0 \qquad (II.5.18)$$

is scale invariant in addition to being invariant under any possible isometry. Hence the correct spin 1/2 equation is

$$i\,\gamma^a\,\mathscr{D}_a\lambda = -\,m_{\frac{1}{2}}\,\lambda \qquad (II.5.19)$$

$m_{1/2}$ being the mass. Squaring the $\not{\mathscr{D}} = \gamma^a\,\mathscr{D}_a$ operator, using

$$\not{\mathscr{D}}^2 = \Box_{cov} - \frac{1}{\,}\,\gamma^{ab}\gamma_{mn}\,\mathscr{R}^{mn}{}_{ab} \qquad (II.5.20)$$

and inserting the explicit form of the anti de Sitter Riemann-tensor (see Eq. (II.5.2)) we get

$$(\Box_{cov} + m_{\frac{1}{2}}^2 - 12\,\bar{e}^2)\lambda = 0 \qquad (II.5.21)$$

The spin 1 particle

The wave equation of a spin 1 particle, which is described by a vector field W_a, is

$$\mathscr{D}^a(\mathscr{D}_a W_b - \mathscr{D}_b W_a) = -\,m_1^2 W_b \qquad (II.5.22)$$

Indeed when $m_1{}^2=0$ Eq. (II.5.22) becomes gauge invariant under the transformation

$$W_a \longmapsto W_a + \mathscr{D}_a\,\varphi(x) \qquad (II.5.23)$$

where $\varphi(x)$ is any scalar function. In the massive case from Eq. (II.5.22) we derive the transversality constraint

$$\mathscr{D}^a W_a = 0 \tag{II.5.24}$$

and on the transverse field Eq. (II.5.22) reduces to

$$\Box_{cov} W_a + 2\mathscr{R}^{am}_{ab} W_m = - m_1^2 W_a \tag{II.5.25}$$

Substituting (II.5.2) into (II.5.25) the spin 1 field equation in anti de Sitter space becomes:

$$(\Box_{cov} + m_1^2 - 12\bar{e}^2) W_a = 0 \tag{II.5.26}$$

The spin 3/2 particle

A spin 3/2 particle is described by a spinor-vector field χ_a. Its wave-equation is the Rarita-Schwinger equation given by:

$$\varepsilon^{abcd}\gamma_5\gamma_b\nabla_c\chi_d = m_{3/2}\,\chi_a \tag{II.5.27}$$

where the anti de Sitter derivative ∇_c is defined as follows:

$$\nabla_c\chi_d = \mathscr{D}_c\chi_d + i\,\bar{e}\,\gamma_c\chi_d \tag{II.5.28}$$

When the mass $m_{3/2}$ is non-zero, from (II.5.27) one deduces both the irreducibility-transversality constraints

$$\gamma^a\chi_a = 0 \tag{II.5.29a}$$

$$\mathscr{D}^a\chi_a = 0 \tag{II.5.29b}$$

and the Dirac equation:

$$i\not{\mathscr{D}}\chi_a = -(m_{3/2} - 2\bar{e})\chi_a \tag{II.5.30}$$

On the other hand since in anti de Sitter space the derivatives ∇_c are commutative:

$$[\nabla_c,\nabla_d] = -\frac{1}{2}\mathscr{R}^{ab}_{cd}\gamma_{ab} - 2\bar{e}^2\gamma_{cd} = (2\bar{e}^2 - 2\bar{e}^2)\gamma_{cd} = 0 \tag{II.5.31}$$

at $m_{3/2}=0$ the Rarita Schwinger equation (II.5.27) acquires the following gauge invariance:

$$\chi_a \longmapsto \chi_a + \nabla_a\lambda \tag{II.5.32}$$

Indeed we have:

$$\varepsilon^{abcd}\gamma_5\gamma_b\nabla_c\nabla_d\lambda = 0 \tag{II.5.33}$$

Applying $i\not{\mathscr{D}}$ to both sides of Eq. (II.5.30) we find

$$(\boxtimes_{cov} - \frac{1}{4}\gamma^{pq}\mathscr{R}^{rs}_{pq}\gamma_{rs})\chi_a - \gamma^{pq}\mathscr{R}^{am}_{pq}\chi_m = -(m_{3/2} - 2\bar{e})^2\chi_a \tag{II.5.34}$$

and substituting (II.5.2) in (II.5.34) we arrive at

$$\{\Box_{cov} - 16\bar{e}^2 + (m_{3/2} - 2\bar{e})^2\}\chi_a = 0 \qquad \text{(II.5.35)}$$

The spin 2 particle

A spin 2 field is a symmetric tensor $h_{ab}=h_{ba}$ and its wave-equation is the linearized Einstein equation :

$$\frac{1}{2}\Box_{cov} h_{ab} - \mathscr{D}_{\{a}\mathscr{D}^{m} h_{b\}m} + \frac{1}{2}\mathscr{D}_a \mathscr{D}_b h_{mm}$$

$$- 2\mathscr{R}^{bm}_{as} h_{ms} = -\frac{1}{2} m^2_{(2)} h_{ab} \qquad \text{(II.5.36)}$$

When $m_2{}^2 \neq 0$ (II.5.37) yields the irreducibility-transversality constraints:

$$h_{mm} = 0 \quad ; \quad \mathscr{D}^m h_{ma} = 0 \qquad \text{(II.5.37)}$$

and the wave-equation

$$(\Box_{cov} - 8\bar{e}^2)h_{ab} = - m^2_{(2)} h_{ab} \qquad \text{(II.5.38)}$$

On the other hand at $m_{(2)}{}^2=0$ Eq. (II.5.36) acquires the following gauge-invariance

$$h_{mn} \longmapsto h_{mn} + \mathscr{D}_{\{m} t_{n\}} \qquad \text{(II.5.39)}$$

and Eqs. (II.5.37) can be imposed as gauge fixings. The same is true of the spin 3/2 and spin 1 equation. The irreducibility-transversality constraints, become in the massless case gauge fixing choices.

Our results can now be summarized by saying that the second order wave-equation of a massless spin s particle in anti de Sitter space is:

$$(-\Box_{cov} + 4\bar{e}^2\alpha_s)\psi_s = 0 \qquad \text{(II.5.40)}$$

where the numbers α_s are

$$\alpha_0 = 2 \quad ; \quad \alpha_{1/2} = 3 \quad ; \quad \alpha_1 = 3 \qquad \text{(II.5.41a)}$$

$$\alpha_{3/2} = 3 \quad ; \quad \alpha_2 = 2 \qquad \text{(II.5.41b)}$$

This result combined with the results of next section allows to express the C_2 Casimir operator (II.5.6) in terms of the covariant Laplacian $\Box_{cov}$.

II.5.2 - Unitary irreducible representations of SO(2,3)

We address now the problem of constructing the unitary irreducible representations of the anti de Sitter group from a purely algebraic point of view. Our starting point is a convenient decomposition of the SO(2,3) Lie algebra (II.2.142a) + (II.2.142b) with respect to its maximal compact subgroup

$$G_0 = SO(2) \otimes SO(3) \subset SO(2,3) \qquad \text{(II.5.42)}$$

Since $M^{\dagger}_{ab} = -M_{ab}$ and $P_a = -P_a{}^{\dagger}$ are antihermitean we define

$$H = -\frac{i}{2\bar{e}} P_0 \tag{II.5.43a}$$

$$J_+ = -\sqrt{2}\,(iM_{23} + M_{13}) \tag{II.5.43b}$$

$$J_- = -\sqrt{2}\,(iM_{23} - M_{13}) \tag{II.5.43c}$$

$$J_3 = -2\,i\,M_{12} \tag{II.5.43d}$$

H is hermitean, compact and generates the SO(2) subgroup. It can be identified with the hamiltonian of the system and its eigenvalues are worth the name of energy E:

$$H|\psi> = E|\psi> \qquad (E^* = E) \tag{II.5.44}$$

J_+, J_-, J_3 commute with H and generate the spin subgroup SO(3) ∿ SU(2).

$$[H,J_+] = [H,J_-] = [H,J_3] = 0 \tag{II.5.45a}$$

$$J_+^\dagger = J_- \tag{II.5.45b}$$

$$[J_3,J_\pm] = \pm J_\pm \tag{II.5.45c}$$

$$[J_+,J_-] = J_3 \tag{II.5.45d}$$

The remaining 6 generators spanning the coset SO(2,3)/SO(2) ⊗ SO(3) can be arranged into the combinations

$$\left.\begin{aligned} K_i^+ &= -2M_{0i} + \frac{i}{2\bar{e}} P_i \\ K_i^- &= -2M_{0i} - \frac{i}{2\bar{e}} P_i \end{aligned}\right\} \; i = 1,\, 2,\, 3 \tag{II.5.46a}$$

$$K_i^- = -(K_i^+)^\dagger \tag{II.5.46b}$$

which have the following commutation relations with H (as can be checked from Eqs. (II.2.142):

$$[H,K_i^\pm] = \pm K_i^\pm \tag{II.5.47}$$

Hence the $K_i^\pm$, act as raising and lowering operators for the energy eigenvalues E.

Furthermore we can rearrange the $K_i^\pm$, which under SU(2) transform as vectors, in the following way

$$K^+_{(1+i2)} = \frac{1}{\sqrt{2}} (K_1^+ + iK_2^+) \tag{II.5.48a}$$

$$K^\dagger_{1-i2} = \frac{1}{\sqrt{2}} (K_1^\dagger - iK_2^\dagger) \tag{II.5.48b}$$

$$K_3^+ = K_3^+ \tag{II.5.48c}$$

The commutation relations with J_3 are

$$[J_3,K^+_{1\pm i2}] = \pm K^+_{1\pm i2} \tag{II.5.49a}$$

$$[J_3,K_3^+] = 0 \tag{II.5.49b}$$

This shows that K^+_{1+i2} raises both the energy and the third component of the spin, while K^+_{1-i2} raises E and lowers J_3. Finally $K_3{}^+$ raises E but leaves J_3 unchanged.

In this basis the Casimir invariant (II.5.3) takes the form

$$C_2 = H^2 + \vec{J}^2 + \frac{1}{2}\{K_i^+,K_i^-\} \tag{II.5.50}$$

Let $\mathscr{H}$ be the Hilbert space carrying the typical unitary irreducible representation we are looking for. It is convenient to label the states $|\psi\rangle \in \mathscr{H}$ by the eigenvalues of H, J^2, and J_3:

$$H|(\ldots)Ejm\rangle = E|(\ldots)Ejm\rangle \tag{II.5.51a}$$

$$\vec{J}^2|(\ldots)Ejm\rangle = j(j+1)|(\ldots)Ejm\rangle \tag{II.5.51b}$$

$$J_3|(\ldots)Ejm\rangle = m|(\ldots)Ejm\rangle \tag{II.5.51c}$$

where (...) denotes an as yet unspecified representation label.

The representations we are interested in must have an energy spectrum bounded from below. Hence we introduce a multiplet of vacuum states $|(E_0,s)\; E_0\; s\; m\rangle$ which form an SU(2) irreducible representation of spin s

$$\vec{J}^2 = s(s+1) \quad \left(s = \begin{cases}\text{integer} \\ \text{half integer}\end{cases} \right) \tag{II.5.52a}$$

$$-s \le m \le s \tag{II.5.52b}$$

and are eigenstates of H with eigenvalue $E_0>0$

$$H|(E_0,s)E_0 sm\rangle = E_0|(E_0,s)E_0 sm\rangle \tag{II.5.53}$$

Furthermore, by definition, the vacuum is annihilated by all the energy lowering operators:

$$K_i^-|(E_0,s)E_0 sm\rangle = 0 \tag{II.5.54}$$

E_0 and s label the irreducible representation generated by applying to $|(E_0,s)\; E_0\; s\; m\rangle$ the raising operators K_i^+ as many times as we like, and regarding the Hilbert space spanned by such ket vectors as the carrier space. For this reason E_0 and s have been inserted in the slot we had prepared for the representation labels.

Evaluating C_2 on the vacuum $|(E_0,s)\; E_0\; s\; m\rangle$ we get

$$C_2 = E_0(E_0 - 3) + s(s+1) \tag{II.5.55}$$

This result follows from Eqs. (II.5.53), (II.5.54) and from the commutation relation:

$$[K_i^+,K_j^+] = 2\delta_{ij}H + 2i\,\varepsilon_{ijk}J_k \tag{II.5.56}$$

whose validity can be checked by use of Eqs. (II.2.142).

The explicit structure of the Hilbert space $\mathscr{H}$ is then given by

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \tag{II.5.57}$$

where $\mathscr{H}_n$ is the span of all vectors of the form

$$\sum_{n_1+n_2+n_3=n} c_{n_1n_2n_3} (K_1^+)^{n_1} (K_2^+)^{n_2} (K_3^+)^{n_3} |(E_0,s)E_0 sm>$$

$$(c_{n_1n_2n_3} \in \mathbb{C}) \tag{II.5.58}$$

$\mathscr{H}_n$ is the Hilbert subspace of states whose energy E is E_0+n. Indeed for any $|\psi_n> \in \mathscr{H}_n$ we get

$$H|\psi_n> = (E_0+n)|\psi_n> \tag{II.5.59}$$

It is also clear that $\mathscr{H}_n$ is a finite dimensional vector space

$$\dim \mathscr{H}_n < \infty \tag{II.5.60}$$

The crucial point is that, in order for $\mathscr{H}$ to be a true Hilbert space its states must have a positive norm.

$$|\psi> = \bigoplus_{n=0}^{\infty} |\psi_n> \tag{II.5.61a}$$

$$\|\psi\|^2 = \bigoplus_{n=0}^{\infty} \|\psi\|^2 > 0 \qquad \|\psi_n\|^2 = <\psi_n|\psi_n> \tag{II.5.61b}$$

This is guaranteed if the scalar products in all the $\mathscr{H}_n$ subspaces are positive definite

$$\forall\, \psi_n \in \mathscr{H}_n \qquad <\psi_n|\psi_n> > 0 \tag{II.5.62}$$

In this case the Hilbert space $\mathscr{H}$ is composed of those series (II.5.61a) whose norm is convergent

$$\bigoplus_{n=0}^{\infty} \|\psi_n\|^2 < \infty \tag{II.5.63}$$

We may also tolerate the presence of zero-norm states. If these exist we define an Hilbert space $\mathscr{H}_{phys}$ composed of the equivalence classes of all states modulo the zero norm states

$$\mathscr{H}_{phys} = \mathscr{H}/\mathscr{H}^{(0)} \quad ; \quad \mathscr{H}^0 = \{|\psi> \in \mathscr{H},\ \|\psi\|^2 = 0\} \tag{II.5.64}$$

Such a situation is typical of all massless theories and in particular of gauge theories. The zero-norm states which are removed by the standard procedure (II.5.64) are gauge degrees of freedom and their subtraction leads to a shortening of the representation.

What we can never accept is the presence of negative norm states (ghosts).

Hence before declaring that we have found the unitary irreducible representations of SO(2,3) we must ascertain under which conditions the space $\mathscr{H}$ does not contain negative norm states and is therefore a Hilbert space. These conditions are simply expressed as lower bounds on the energy label E_0, relative to the spin s.

Let us first state these bounds and then give a sketch of their derivation

a) For $s \geq 1$ there are no ghosts if and only if

$$E_0 \geq s + 1 \tag{II.5.65}$$

When $E_0 > s+1$ there are no zero-norm states and no representation shortening occurs. The representation is massive. For $E_0 = s+1$ we have zero-norm states which can be decoupled. The corresponding representation is massless and it is described by the appropriate massless wave-equation.

b) For $s=1/2$ there are no ghosts if and only if

$$E_0 \geq 1 \qquad \text{(II.5.66)}$$

Decoupling of zero-norm states takes place for $E_0=3/2$ and $E_0=1$. The first value corresponds to a massless representation described by the massless wave equation (II.5.18), while the limiting value $E_0=1$ is the so called Dirac singleton for which no field-theoretic interpretation has been found and which has no counterpart in Poincaré theory.

c) For $s=0$ there are no ghosts if and only if

$$E_0 \geq \frac{1}{2} \qquad \text{(II.5.67)}$$

The zero-norm states are found for the special values $E_0=2$, $E_0=1$ and $E_0=1/2$. Both values $E_0=1$ and $E_0=2$ yield the standard massless representation described by the conformal invariant wave equation (II.5.13), while the lowest value $E_0=1/2$ is again a Dirac singleton representation with no counterpart in the Poincaré case and no field theory interpretation.

Indeed the best way to convince yourself that $(E_0 = 3/2;\ s = 1/2)$ and $(E_0 = 2,1,;\ s = 0)$ are massless representations is to check that on the corresponding Hilbert spaces one can not only implement the SO(2,3) group but also the full conformal group SO(2,4).

Such a check was done by Fronsdal [14] and we refer the reader to its work.

Finally before sketching the derivation of these results given in [14] and reviewed by Nicolai in [26] we would like to answer a question we are sure the reader is presently concerned with.

If $E_0=2,1$ correspond to the massless $s=0$ case and yield the same value -2 for C_2, what do the other permissible values $2>E_0>1/2$ correspond to? The answer is: to negative but permissible squared-mass values. Indeed in anti de Sitter space m^2 is allowed to be negative provided it is not too negative. This bound which we are going to discuss again in short is of extreme practical importance in supergravity. It says that in anti de Sitter space a saddle point of a potential can still be stable provided the slope of the descent is not too extreme.

The reader will appreciate the value of this fact when he realizes that in extended supergravities no potential is bounded from below and no extrema are found except saddle points. Coming back to the boring task of proving the bounds a), b), c) we just illustrate the procedure with an example which is also the easiest: case a). Let $s \geq 1$ and and consider the action of K^+_{1+i2} on the vacuum $|(E_0,s)\ E_0\ s\ m\rangle$. We can write

$$\begin{aligned} K^+_{1+i2}|(E_0,s)E_0 sm\rangle &= \\ &= R_+ \langle sm1,1|s+1,\ m+1\rangle\,|(E_0,s)E_0+1,\ s+1,\ m+1\rangle \\ &+ R_0 \langle sm1,1|s,\ m+1\rangle\,|(E_0,s)E_0+1,\ s,\ m+1\rangle \\ &+ R_- \langle sm1,1|s-1,\ m+1\rangle\,|(E_0,s)E_0+1,\ s-1,\ m+1\rangle \end{aligned} \qquad \text{(II.5.68)}$$

where $\langle s\ m\ 1,1|\ s',\ m+1\rangle$ are the Clebsch-Gordan coefficients relating the product of a spin (s,m) state with a spin (1,1) state to a spin (s', m+1) state. In our normalizations, which are Nicolai's normalizations [26] we have

$$< sm1,1|s+1,\ m+1 > = \left(\frac{(s+m+1)(s+m+2)}{(2s+1)(2s+2)}\right)^{\frac{1}{2}} \qquad (II.5.69a)$$

$$< sm1,1|s,\ m+1 > = -\left(\frac{(s+m+1)(s-m)}{2s(s+1)}\right)^{\frac{1}{2}} \qquad (II.5.69b)$$

$$< sm1,1|s-1,\ m+1 > = \left(\frac{(s-m)(s-m+1)}{2s(2s+1)}\right)^{\frac{1}{2}} \qquad (II.5.69c)$$

R_+, R_0 and R_- are the reduced matrix elements and equation (II.5.68) is a straightforward application of the Wigner-Eckart theorem of quantum mechanics.

$|R_+|^2$, $|R_0|^2$ and $|R_-|^2$ are easily calculated choosing in sequence m=s, m=s-1, m=s-2 and utilizing the commutator

$$[K^-_{1-i2}, K^+_{1+i2}] = -2(H + J_3) \qquad (II.5.70a)$$

$$K^-_{1-i2} = -(K_{1+i2})^+ \qquad (II.5.70b)$$

in the evaluation of $||K_{1+2}|(E_0,s)\ E_0\ s\ m>||^2$.

The result is

$$|R_+|^2 = 2(E_0 + s) \qquad (II.5.71a)$$

$$|R_0|^2 = 2(E_0 - 1) \qquad (II.5.71b)$$

$$|R_-|^2 = 2(E_0 - s - 1) \qquad (II.5.71c)$$

So we see that (II.5.65) is a necessary condition for the absence of ghosts. To prove that it is also sufficient is a much harder story and we refer the reader to the literature [26]. From (II.5.71c) however the appearance of null-norm states, characteristic of massless representations is immediately evident in the case

$$E_0 = s + 1$$

The other bounds are proved with similar techniques.

Let us now call $D(E_0,s)$ the unitary irreducible representations of SO(2,3) with the proper bounds on E_0 implemented and let us finally come to the relation between this energy label and the mass-squared m^2. Taking into account Eq. (II.5.55), the second order field equation of a spin s field must be of the following form:

$$\{C_2(\Box_{cov}) - E_0(E_0 - 3) - s(s + 1)\}\psi_s = 0 \qquad (II.5.72)$$

where $C_2(\Box_{cov})$ is the expression of the C_2 Casimir (II.5.3) in terms of second order differential operators on the coset manifold

$$AdS = SO(2,3)/SO(1,3) \qquad (II.5.73)$$

Since $\Box_{cov}$ is a second order differential operator which is invariant and since there is no more than one quadratic Casimir, we must have

$$C_2(\Box_{cov}) = a\Box_{cov} + b_s \qquad (II.5.74)$$

where a and b_s are constants. Now since P_a can be identified with the tangent vector D_a dual to the vierbein the normalization coefficient a is fixed by inspection of (II.5.3):

$$C_2(\Box_{cov}) = -\frac{1}{4\bar{e}^2}\Box_{cov} + b_s \qquad (II.5.75)$$

The constant b_s can now be fixed by comparison of (II.5.72) with the field equation (II.5.40).

We take the appropriate massless value $E_0 = s+1$ and we obtain

$$-\frac{1}{4\bar{e}^2}\Box_{cov} - 2(s+1)(s-1) + b_s = -\frac{1}{4\bar{e}^2}\Box_{cov} + \frac{1}{4\bar{e}^2}4\bar{e}^2\alpha_S \qquad (II.5.76)$$

Hence we deduce

$$b_s = 2(s^2-1) + \alpha_S \qquad (II.5.77)$$

so that

$$C_2(\Box_{cov}) = -\frac{1}{4\bar{e}^2}\Box_{cov} + 2(s^2-1) + \alpha_s = E_0(E_0-3) + s(s+1) \qquad (II.5.78)$$

We can now derive the E_0/m relation for each spin. Comparing successively (II.5.78) with (II.5.26), (II.5.35) and (II.5.38) by use of (II.5.41) we find:

$$\frac{m_0^2}{4\bar{e}^2} = (E_0-2)(E_0-1) \qquad (II.5.79a)$$

$$\frac{m_{1/2}^2}{4\bar{e}^2} = E_0(E_0-3) + \frac{9}{4} = (E_0-\frac{3}{2})^2 \qquad (II.5.79b)$$

$$\frac{m_1^2}{4\bar{e}^2} = E_0(E_0-3) + \frac{9}{4} = (E_0-\frac{3}{2})^2 \qquad (II.5.79c)$$

$$\frac{(m_{3/2}-2\bar{e})^2}{4\bar{e}^2} = E_0(E_0-3) + \frac{9}{4} = (E_0-\frac{3}{2})^2 \qquad (II.5.79d)$$

$$m_2^2/4\bar{e}^2 = E_0(E_0-3) \qquad (II.5.79e)$$

These relations are summarized for the reader's benefit in Table II.5.I.

As we are going to see each Osp(4/N) representation decomposes into a certain number of SO(2,3) representations with energy labels related by integral or half-integral shifts: these relations inserted into Table II.5.I reflect into mass relations among the fields belonging to the same multiplet.