Moreover in D=4 we write the explicit form of the duality relation on γ_{ab}

$$\varepsilon_{abcd}\gamma_{cd} = 2i\gamma_5\gamma_{ab} \qquad (II.7.40)$$

and we conclude the chapter with another useful formula valid in every dimension:

$$\Gamma^{c_1..c_q}\Gamma_{a_1..a_n}\Gamma_{c_1..c_q} = I_q^{(n)}\Gamma_{a_1...a_n} \qquad (II.7.41)$$

In (II.7.41) the coefficient $I_n^{(q)}$ is determined by the recurrence relation:

$$I_q^{(n)} = I_1^{(n)}I_{(q-1)}^{(n)} - (q-1)(D-q+2)I_{(q-2)}^{(n)} \qquad (II.7.42a)$$

$$I_0^{(n)} = 1 \qquad (II.7.42b)$$

$$I_1^{(n)} = D - 2n \qquad (II.7.42c)$$

CHAPTER II.8

FIERZ IDENTITIES AND GROUP THEORY

II.8.1 - Introduction

This chapter is very technical but nonetheless very important for all what follows. It deals with a very specific problem which arises in the development of both globally and locally supersymmetric field theories.

As we saw in Chapter II.6, in order to construct the action of a supersymmetric field-theory model we have, in general, to solve exterior form equations on superspace which arise either as Bianchi identities or as field equations associated to a Lagrangian which is itself an exterior form.

A complete cotangent frame on superspace is provided by the vielbein V^a and the gravitino 1-form ψ^A which is a spin 1/2 representation of the Lorentz group SO(1,D-1) and has, moreover, an index A enumerating the supersymmetries $(A=1,2,\ldots,N)$.

Henceforth an arbitrary p-form $\omega^{(p)}$ on superspace can be expanded as follows

$$\omega^{(p)} = \sum_{q=0}^{p} (\omega^{(p)}_{a_1..a_{p-q}A_1\alpha_1...A_q\alpha_q} V^{a_1} \wedge ... \wedge V^{a_{p-q}} \wedge \psi^{A_1\alpha_1} \wedge ... \wedge \psi^{A_q\alpha_q}) \tag{II.8.1}$$

and our exterior-form equations are implemented by requiring that the coefficient of each independent monomial

$$\Omega^{a_1..a_n A_1\alpha_1..A_m\alpha_m} = V^{a_1} \wedge ... \wedge V^{a_n} \wedge \psi^{A_1\alpha_1} \wedge ... \wedge \psi^{A_m\alpha_m} \tag{II.8.2}$$

vanishes independently.

The relevant point is that $\Omega^{a_1...a_n A_1\alpha_1...A_m\alpha_m}$ is a tensor product of irreducible representations of SO(1,d-1) and SO(N) which is antisymmetric in:

$$\{a_1 \leftrightarrow a_2 \leftrightarrow .... \leftrightarrow a_n\} \tag{II.8.3}$$

and symmetric in

$$\{\alpha_1 A_1 \leftrightarrow A_2\alpha_2 \leftrightarrow ... \leftrightarrow A_n\alpha_n\} \tag{II.8.4}$$

This implies that we can decompose $\Omega^{a_1...a_n A_1\alpha_1...A_m\alpha_m}$ into irreducible representations via a Clebsch-Gordan series and that only certain representations occur, the others being ruled out by Eqs. (II.8.3-4). This decomposition, with explicitly calculated coefficients, provides a systematic way to perform the calculations we shall be confronted with. On one side, the absence of certain irreducible representations in the decomposition of the spinor tensor-products is the origin of all the "miraculous" Fierz identities one needs to derive supersymmetric theories; on the other side, using the procedure of projecting every form equation on the irreducible components of $\Omega^{a_1...a_n A_1\alpha_1...A_n\alpha_n}$ one is sure to deal with a set of independent equations and is free from the danger of overcounting.

In principle one should consider tensor products of ψ^A.s with arbitrary number of them but in practice the number of fermions is limited to a maximum of 4. Indeed every supersymmetric Lagrangian for scaling reasons is at most quartic in the gravitino 1-forms. Hence we shall be mainly interested in the decomposition of the product of 2ψ.s (which is rather simple) of 3ψ.s (which is the highest needed in the analysis of Bianchi identities, these latter being 3-forms) and occasionally of 4ψ.s.

II.8.2 - The structure of forms on N-extended D=4 superspace

As we saw in Chapter II.3, rigid anti de Sitter and Minkowski superspaces are the homogeneous supermanifolds (II.3.29-30) possessing four bosonic coordinates x^μ, associated to the translations, and 4N fermionic coordinates $\theta^{A\alpha}$, associated to the N-supersymmetries. The soft version of these manifolds have the same number of coordinates.

The cotangent space to superspace has $4+4N$ dimensions and it is spanned by V^a and ψ^A, as we already remarked. To illustrate the method let us begin with the three-forms and let us call D(3) the linear space spanned by them.

The dimension of D(3) can be easily computed. Let us denote by $z^\Lambda = (x^\mu, \theta^{A\mu})$ the superspace coordinates. The most general 3-form $\Omega^{(3)}$ can be written as:

$$\Omega^{(3)} = \Omega^{(3)}_{\Lambda\Sigma\Pi} dz^\Lambda \wedge dz^\Sigma \wedge dz^\Pi \tag{II.8.5}$$

where the superspace wedge product obeys the standard commutation rule:

$$dz^\Lambda \wedge dz^\Sigma = (-)^{1+\Lambda\Sigma} dz^\Sigma \wedge dz^\Lambda \tag{II.8.6}$$

Instead of the coordinate differentials $dz^{\Lambda} = (dx^{\mu}, d\theta^{A\alpha})$ we can use the intrinsic basis $(V^a, \psi^{\alpha A})$, and this is what we shall do systematically, Hence we find:

$$\Omega^{(3)} = \Omega^{(3)}_{abc} V^a \wedge V^b \wedge V^c + \Omega^{(3)}_{ab(\alpha A)} V^a \wedge V^b \wedge \psi^{\alpha A} +$$
$$+ \Omega^{(3)}_{a(\alpha A)(\beta B)} V^a \wedge \psi^{\alpha A} \wedge \psi^{\beta B} + \Omega^{(3)}_{(\alpha A)(\beta B)(\gamma C)} \psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C} \qquad \text{(II.8.7)}$$

Since the V^a anticommute among themselves and with the $\psi^{\alpha A}$ while the latter commute with each other we easily compute the dimension of the monomials appearing in the expansion (II.8.7). We get:

$$\dim(V^a \wedge V^b \wedge V^c) = 4 \qquad \text{(II.8.8a)}$$

$$\dim(V^a \wedge V^b \wedge \psi^{\beta B}) = \frac{4.3}{2} \cdot 4N = 24N \qquad \text{(II.8.8b)}$$

$$\dim(V^a \wedge \psi^{\alpha A} \wedge \psi^{\beta B}) = 4 \cdot \frac{4N(4N+1)}{2} = 8N(4N+1) \qquad \text{(II.8.8c)}$$

$$\dim(\psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C}) = \frac{2}{3} N(4N+1)(4N+2) \qquad \text{(II.8.8d)}$$

Hence the space of 3-forms in N-extended d=4 superspace has the following dimension:

$$\dim D(3) = \underbrace{(32N^2 + 8N + 4)}_{\text{Bosonic}} \oplus \underbrace{(32N^2 + 24N + 76)\frac{N}{3}}_{\text{Fermionic}} \qquad \text{(II.8.9)}$$

For the first interesting cases we have Table II.8.I. The basic idea of our technique is to write a basis of 3-forms which is composed of irreducible representations of the H=SO(1,3) ⊗ O(N) group. Explicitly we want a decomposition of the following type:

$$\Omega^{(3)} = \Omega^{(3)}_{abc} V^a \wedge V^b \wedge V^c + \Omega^{(3)}_{ab|\alpha A} V^a \wedge V^b \wedge \psi^{\alpha A} +$$
$$+ \Omega^{(3)}_{a|(i)} V^a \wedge X^{(i)} + \Omega^{(3)}_{(i)} \Xi^{(i)} \qquad \text{(II.8.10)}$$

TABLE II.8.I

DIMENSIONS OF 3-FORMS IN D=4 N-EXTENDED SUPERSPACE

N	dim D(3)	dim $\psi^{\alpha A}$	$\dim(\psi^{\alpha A} \wedge \psi^{\beta B})$	$\dim(\psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C})$
1	44 + 44	4	10	20
2	148 + 168	8	36	120
3	316 + 436	12	78	364
4	548 + 912	16	136	816
5	844 + 1660	20	210	1540
6	1204 + 2744	24	300	2600
7	1600 + 4228	28	406	4060
8	2116 + 6176	32	528	6160

where X^i are the 2-form irreducible representations appearing in the decomposition of $\psi^{\alpha A} \wedge \psi^{\beta B}$

$$\psi^{\alpha A} \wedge \psi^{\beta B} = f_i^{\alpha A, \beta B} X^{(i)} \qquad \text{(II.8.11)}$$

and $\Xi^{(i)}$ are the 3-form irreducible representations appearing in the decomposition of $\psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C}$,

$$\psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C} = f_{(i)}^{\alpha A, \beta B, \gamma C} \Xi^{(i)} \qquad \text{(II.8.12)}$$

Since H is a direct product and the fermionic wedge product is symmetric, we use the following procedure. First we classify all the SO(1,3) representations appearing in the tensor product of two or three spin 1/2 representations and all the O(N) representations occurring in the tensor product of two or three O(N) vectors.

TABLE II.8.II

REPRESENTATION OF SO(1,3)

Representation type	Dimension	Corresponding tensor, spinor or spinor tensor
$[1,1]$	6	$X_{ab} = -X_{ba}$ (antisymm. tensor)
$[1,0]^{(+)}$	4	$X_a^{(+)}$ (vector)
$[1,0]^{(-)}$	4	$X_a^{(-)}$ (axial vector)
$[0,0]^{(+)}$	1	$X^{(+)}$ (scalar)
$[0,0]^{(-)}$	1	$X^{(-)}$ (pseudoscalar)
$[3/2,3/2]$	8	$\Xi_{ab} = -\Xi_{ba}$; $\gamma^b\, \Xi_{ab} = 0$ (irred. spinor--tensor)
$[3/2;1/2]^{(+)}$	12	$\Xi_a^{(+)}$; $\gamma^a\, \Xi_a^{(+)} = 0$ (irred. spinor-vector)
$[3/2,1/2]^{(-)}$	12	$\Xi_a^{(-)}$; $\gamma^a\, \Xi_a^{(-)} = 0$ (irred. spinor-axial vector)
$[1/2,1/2]^{(+)}$	4	$\Xi^{(+)}$ (Majorana spinor)
$[1/2,1/2]^{(-)}$	4	$\Xi^{(-)}$ (Majorana pseudospinor)

($\psi^{\alpha A}$ is at the same time an SO(1,3) spinor and an O(N) vector). Then we consider all products of these SO(1,3) and O(N) representations which are completely symmetric in the exchange $(\alpha A \leftrightarrow \beta B \leftrightarrow \gamma C)$. (This procedure is also used for the coupling of spin and isospin in Nuclear Physics.)

We begin by tabulating the relevant representations of the proper Lorentz group SO(1,3) with their dimensions.

In Table II.8.II the numbers on the extreme left are the eigenvalues of the Casimir operators of SO(1,3) whose rank is 2 and the plus and minus superscripts refer to the parity eigenvalues. The dimensions can be calculated using standard formulae in group-theory.

They can be also obtained by more elementary means. For instance, Ξ^{ab} has dimension 8 because it is a spinor-tensor $(4\times 6 = 24)$ satisfying $(4\times 4 = 16)$ conditions $(\gamma^b\, \Xi_{ab} = 0)$.

Table II.8.II exhausts the list of relevant representations because of the following decomposition rules:

$$[1/2,1/2]\otimes[1/2,1/2] = \underbrace{[1,1]\oplus[1,0]^{(+)}}_{\text{symmetric}}\oplus\underbrace{[1,0]^{(-)}\oplus[0,0]^{(+)}\oplus[0,0]^{-}}_{\text{antisymmetric}} \tag{II.8.13a}$$

$$[1/2,1/2]\otimes[1,1] = [3/2,3/2]\oplus[3/2,1/2]\oplus[1/2,1/2] \tag{II.8.13b}$$

$$[1/2,1/2]\otimes[1,0] = [3/2,1/2]\oplus[1/2,1/2] \tag{II.8.13c}$$

$$[1/2,1/2]\otimes[0,0] = [1/2,1/2] \tag{II.8.13d}$$

Equations (II.8.12) say that any 4×4 matrix can be expanded in a complete Dirac basis. Indeed, if we have the wedge product $\psi_A \wedge \bar{\psi}_B$ where ψ_A and ψ_B are Majorana spinor 1-forms:

$$\psi_A = \psi_A^c = C(\bar{\psi}_A)^T \tag{II.8.14}$$

$\psi_A \wedge \bar{\psi}_B$ is a matrix in spinor space and can be expanded as follows:

$$\psi_A \wedge \bar{\psi}_B = \frac{1}{4}\left(\mathbb{1}\, \overset{(+)}{X}_{BA} + \gamma_5\, \overset{(-)}{X}_{BA} + \gamma_5\gamma_a\, \overset{(-)}{X}{}^a_{BA} + \gamma^a\, \overset{(+)}{X}{}^a_{BA}\right) + \frac{i}{4}\,\gamma_{ab}X^{ab}_{BA} \tag{II.8.15}$$

where Eqs. (II.8.13c) and (II.8.13d) correspond to the familiar decomposition of a spinor-vector (or a spinor-tensor) into a traceless part plus a trace. Let us, for instance, consider a spinor-vector ξ_a. We can write:

$$\xi_a = \xi_a^{(12)} + \frac{1}{4}\gamma_a\,\xi^{(4)} \qquad \text{(II.8.16)}$$

where

$$\xi_a^{(12)} = \xi_a - \frac{1}{4}\gamma_a\gamma^b\xi_b \qquad \text{(II.8.17)}$$

is the irreducible [3/2,1/2] part satisfying:

$$\gamma^a\xi_a^{(12)} = 0 \qquad \text{(II.8.18)}$$

TABLE II.8.III

BOSONIC 2-FORMS

Representation	Current	Symmetry	Reality
[1,1]	$X^{ab}_{AB} = \bar{\psi}_A\,\frac{i}{2}\,\gamma_{ab}\psi_B$	$X_{AB} = +X_{BA}$ sym.	$(X^{ab}_{AB})^\dagger = -X^{ab}_{AB}$ im.
$[1,0]^{(+)}$	$\overset{(+)}{X}{}^a_{BA} = \bar{\psi}_A \wedge \gamma^a\psi_B$	$\overset{(+)}{X}{}^a_{AB} = +\overset{(+)}{X}{}^a_{BA}$ sym.	$(\overset{(+)}{X}{}^a_{AB})^\dagger = -\overset{(+)}{X}{}^a_{AB}$ im.
$[1,0]^{(-)}$	$\overset{(-)}{X}{}^a_{AB} = \bar{\psi}_A \wedge \gamma^5\gamma^a\psi_B$	$\overset{(-)}{X}{}^a_{AB} = -\overset{(-)}{X}{}^a_{BA}$ antisym.	$(\overset{(-)}{X}{}^a_{AB})^\dagger = \overset{(-)}{X}{}^a_{AB}$ real
$[0,0]^{(+)}$	$\overset{(+)}{X}_{AB} = \bar{\psi}_A \wedge \psi_B$	$\overset{(+)}{X}_{AB} = -\overset{(+)}{X}_{BA}$ antisym.	$(\overset{(+)}{X}_{AB})^\dagger = \overset{(+)}{X}_{AB}$ real
$[0,0]^{(-)}$	$\overset{(-)}{X}_{AB} = \bar{\psi}_A \wedge \gamma^5\psi_B$	$\overset{(-)}{X}_{AB} = -\overset{(-)}{X}_{BA}$ antisym.	$\overset{(-)}{X}{}^\dagger_{AB} = -\overset{(-)}{X}_{AB}$ im.

and

$$\xi^{(4)} = \gamma\cdot\xi \qquad \text{(II.8.19)}$$

is the [1/2,1/2] part.

Similarly given a spinor-tensor ξ_{ab} we can write:

$$\xi_{ab} = \xi_{ab}^{(8)} - \gamma_{[a}\gamma^m\xi_{b]m} - \frac{1}{12}\gamma_{ab}\xi^{(4)} \qquad \text{(II.8.20)}$$

where

$$\xi_{ab}^{(8)} = \xi_{ab} + \gamma_{[a}\gamma^m\xi_{b]m} + \frac{1}{6}\gamma_{ab}\gamma^m\gamma^n\xi_{mn} \qquad \text{(II.8.21a)}$$

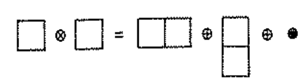
$$\xi_a^{(12)} = \gamma^m\xi_{am} - \frac{1}{4}\gamma^a\gamma^m\gamma^n\xi_{mn} \qquad \text{(II.8.21b)}$$

$$\xi^{(4)} = \gamma^m\gamma^n\xi_{mn} \qquad \text{(II.8.21c)}$$

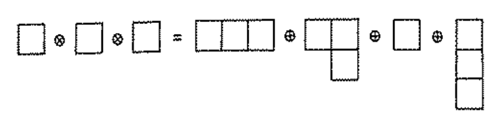
are the irreducible representations [3/2,3/2], [3/2,1/2] and [1/2,1/2], respectively. From this point on we shall define the irreducible representations assuming Eqs. (II.8.16) and (II.8.20) as the standard expansion of any spinor-vector or spinor-tensor.

Coming now to the relevant representations of the group O(N), since $\psi^{\alpha A}$ is an O(N) vector, they are those contained in the tensor product of two or three vectors.

In general we have:



(II.8.22a)



(II.8.22b)

TABLE II.8.IV

O(N) REPRESENTATIONS

Type	$\bullet$	$\begin{array}{\|c\|}\hline ~ \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline ~ \\ \hline ~ \\ \hline ~ \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline ~ \\ \hline ~ \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline ~ & ~ \\ \hline ~ \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline ~ & ~ & ~ \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline ~ & ~ \\ \hline\end{array}$
Dimension	1	N	$N(N^2-3N+2)/6$	$N(N-1)/2$	$N(N^2-4)/3$	$N(N^2+3N)/6$ $-2N/3$	$N(N+1)/2-1$
N = 1	1	1	0	0	0	0	0
N = 2	1	2	0	1	2	2	2
N = 3	1	3	1	3	5	7	5
N = 4	1	4	4	6	16	16	9
N = 5	1	5	10	10	35	30	14
N = 6	1	6	$10 \oplus 10$	15	64	40	20
N = 7	1	7	35	21	105	77	27
N = 8	1	8	56	28	160	112	35

where we have used the standard Young tableaux notation and each of them is meant to represent the corresponding O(N) irreducible (traceless) tensor. The dimensionality of these representations is given by standard formulas.* Our findings are shown in Table II.8.IV. In general given a rank 2 O(N) tensor, like the bosonic 2-forms of Table II.8.III, we shall write its decomposition into irreducible components according to the following conventions:

$$t_{AB} = t_{\begin{array}{|c|c|}\hline A & B \\ \hline\end{array}} + \frac{1}{N}\,\delta_{AB} t_{..} + t_{\begin{array}{|c|}\hline A \\ \hline B \\ \hline\end{array}} \qquad \text{(II.8.23)}$$

where

$$t_{\begin{array}{|c|c|}\hline A & B \\ \hline\end{array}} = \frac{1}{2}\,(t_{AB} + t_{BA}) - \frac{1}{N}\,\delta_{AB} t_{MM} \qquad \text{(II.8.24a)}$$

* See for instance M. Hamermesh, Group Theory (Addison-Wesley, Reading, 1962).

$$t_{..} = t_{MM} \qquad \text{(II.8.24b)}$$

$$t_{\begin{array}{|c|}\hline A \\ \hline B \\ \hline\end{array}} = \frac{1}{2}\,(t_{AB} - t_{BA}) \qquad \text{(II.8.24c)}$$

A rank 3 O(N) tensor will, on the other hand, be decomposed in the following way:

$$t_{ABC} = t_{\begin{array}{|c|c|c|}\hline A & B & C \\ \hline\end{array}} + \delta_{\{AB}t_{C\}} + t_{\begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline\end{array}} + t_{\begin{array}{|c|c|}\hline A & B \\ \hline C \\ \cline{1-1}\end{array}} + t_{\begin{array}{|c|c|}\hline A & C \\ \hline B \\ \cline{1-1}\end{array}} + \left(\text{traces of } \begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ \\ \cline{1-1}\end{array}\right) \qquad \text{(II.8.25)}$$

II.8.3 - Fierz decompositions in the N=1 D=4 superspace

Once every product of ψ_A has been decomposed both with respect to SO(1,3) and O(N), the standard Fierz identities correspond to the exclusion of all representations which are not fully symmetric and to the determination of the decomposition coefficients. We begin by considering the N=1 example explicitly.

Recalling Table II.8.I, we see that, in this case, $\psi^\alpha \wedge \psi^\beta$ has 10 components. Looking at Table II.8.II, on the other hand, we realize that

$$10 = \dim[1,1] + \dim[1,0]^{(+)} = 6 + 4 \qquad \text{(II.8.26)}$$

which are precisely the representations occuring in the symmetric product decomposition. In the N=1 case no contribution from the antisymmetric $[1,0]^{(-)}$, $[0,0]^{(+)}$ and $[0,0]^{(-)}$ representations is allowed. Equation (II.8.15) therefore reduces to

$$\psi \wedge \bar{\psi} = \frac{1}{4}\gamma^a \overset{(+)}{X}_a + \frac{i}{4}\gamma_{ab}X^{ab} \qquad \text{(II.8.27)}$$

Going now to the $\psi^\alpha \wedge \psi^\beta \wedge \psi^\gamma$ sector we read from Table II.8.I that it has 20 components; moreover, from Table II.8.II it is evident that the only way to obtain 20 is by setting

$$20 = 8 + 12 \qquad \text{(II.8.28)}$$

This means that the only representations being completely symmetric in α,β,γ are $[3/2,3/2]$ and $[3/2,1/2]$. This is the origin of all Fierz identities.

The explicit construction is the following. If we have the wedge product of three ψ.s ($\psi \wedge \psi \wedge \psi$) we can start by decomposing two of them according to (II.8.27).

In this way we end up with the following spinor-vector and spinor-tensor:

$$\theta_a = \psi \wedge \bar{\psi} \wedge \gamma_a \psi \qquad \text{(II.8.29a)}$$

$$\theta_{ab} = \frac{i}{2}\psi \wedge \bar{\psi} \wedge \gamma_{ab}\psi \qquad \text{(II.8.29b)}$$

Because of the previous discussion we can set

$$\theta_a = \alpha\, \Xi_a^{(12)} \qquad \text{(II.8.30a)}$$

$$\theta_{ab} = \Xi_{ab}^{(8)} - \gamma_{[a}\Xi_{b]}^{(12)} \qquad \text{(II.8.30b)}$$

where $\Xi_a^{(12)}$ is an irreducible $[3/2,1/2]$ representation (satisfying $\gamma^a \Xi_a^{(12)} = 0$) and $\Xi_{ab}^{(8)}$ is an irreducible $[3/2,3/2]$ representation (satisfying $\gamma_b \Xi_{ab}^{(8)} = 0$). We compute the coefficient by applying Eq. (II.8.27) once again in the definition of θ_a. We get

$$\psi \wedge \bar{\psi} \wedge \gamma^a\psi = \alpha\, \Xi_a^{(12)} = \frac{1}{2}\psi \wedge \bar{\psi} \wedge \gamma_a\psi + \frac{1}{2}\gamma^b\psi \wedge \bar{\psi} \wedge \gamma_{ab}\psi \qquad \text{(II.8.31)}$$

and hence

$$\psi \wedge \bar{\psi} \wedge \gamma_a\psi = \alpha\, \Xi_a^{(12)} = -2i\gamma^b\psi \wedge \bar{\psi} \wedge \frac{i}{2}\gamma_{ab}\psi = -2i\Xi_a^{(12)} \qquad \text{(II.8.32)}$$

TABLE II.8.V

IRREDUCIBLE BASIS OF N=1, d=4 SUPERSPACE

$$\psi \wedge \bar{\psi} = \frac{1}{4}\gamma_a \overset{(+)}{X^a} + \frac{i}{4}\gamma_{ab} X^{ab}$$

$$\psi \wedge \bar{\psi}\gamma^a \psi = \psi \wedge \overset{(+)}{X^a} = -2i\Xi_a^{(12)}$$

$$\psi \wedge \bar{\psi}\frac{i}{2}\gamma_{ab}\psi = \psi \wedge X_{ab} = \Xi_{ab}^{(8)} - \gamma_{[a}\Xi_{b]}^{(12)}$$

With Eq. (II.8.32) we have completed the construction of an irreducible basis for N=1 D=4 superspace.

The results are summarized in Table II.8.V.

II.8.4 - The N=2, D=4 case

We consider now the N=2 superspace. In this case $\psi^{\alpha A} \wedge \psi^{\beta B}$ has dimension 36 and $\psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C}$ has dimension 120 (see Table II.8.I).

Looking at Table II.8.III and at Eq. (II.8.15) we easily obtain the representation content of $\psi^{\alpha A} \wedge \psi^{\beta B}$.

$$\psi_A \wedge \bar{\psi}_B = \frac{1}{8}\,(\mathbb{1}\,\overset{(+)}{X}_{\bullet} + \gamma^5\,\overset{(-)}{X}_{\bullet} + \gamma^5\gamma^a\,\overset{(-)}{X}_{\bullet}{}^{a})\,\epsilon_{BA}$$

$$+ \frac{1}{4}\,\gamma^a(\overset{(+)}{X}{}^{a}_{BA} + \frac{1}{2}\,\overset{(+)}{X}{}^{a}_{\bullet\bullet}\,\delta_{AB})$$

$$+ \frac{i}{4}\,\gamma_{ab}(X^{ab}_{\boxed{B|A}} + \frac{1}{2}\,X^{ab}_{\bullet\bullet}\,\delta_{AB}) \qquad \text{(II.8.33)}$$

where we have used the standard decomposition (II.8.24), the symmetry properties given in Table II.8.III and the fact that, in the O(2) case, we can write

$$t_{\begin{smallmatrix}\boxed{A}\\\boxed{B}\end{smallmatrix}} = \frac{1}{2}\,\epsilon_{AB}\,t_{\bullet} \qquad ; \qquad t_{\bullet} = \epsilon^{RS}\,t_{\begin{smallmatrix}\boxed{R}\\\boxed{S}\end{smallmatrix}} \qquad \text{(II.8.34)}$$

Indeed we have:

$$\overset{(+)}{X}_{\bullet} = [0,0]^{(+)} \otimes \begin{smallmatrix}\Box\\\Box\end{smallmatrix} \quad = \quad 1\times 1 \quad = \quad 1$$

$$\overset{(-)}{X}_{\bullet} = [0,0]^{(-)} \otimes \begin{smallmatrix}\Box\\\Box\end{smallmatrix} \quad = \quad 1\times 1 \quad = \quad 1$$

$$\overset{(-)}{X}{}^{a}_{\bullet} = [1,0]^{(-)} \otimes \begin{smallmatrix}\Box\\\Box\end{smallmatrix} \quad = \quad 4\times 1 \quad = \quad 4$$

$$\overset{(+)}{X}{}^{a}_{\boxed{A|B}} = [1,0]^{(+)} \otimes \Box\Box \quad = \quad 4\times 2 \quad = \quad 8$$

$$\overline{14}$$

$$\overset{(+)}{X}{}^{a}_{\bullet\bullet} = [1,0]^{(+)} \otimes \bullet \quad = \quad 4\times 1 \quad = \quad 4$$

$$X^{ab}_{\boxed{B|A}} = [1,1] \otimes \Box\Box \quad = \quad 6\times 2 \quad = \quad 12$$

$$X^{ab}_{\bullet\bullet} = [1,1] \otimes \bullet \quad = \quad 6\times 1 \quad = \quad \underline{6} \qquad \text{(II.8.35)}$$

$$22$$

The possible representations appearing in the decomposition of $\psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C}$ are now obtained by looking at the product of $\psi^{\alpha A}$, which is $[1/2,1/2] \otimes \Box$ times each of the X we have so far introduced. Let us define:

$$\theta^{ab}_{ABC} = \frac{i}{2}\,\psi_A \wedge \bar{\psi}_B \wedge \gamma^{ab}\psi_C \qquad \text{(II.8.36a)}$$

$$\overset{(+)}{\theta}{}^{a}_{ABC} = \psi_A \wedge \bar{\psi}_B \wedge \gamma^a\psi_C \qquad \text{(II.8.36b)}$$

$$\overset{(-)}{\theta}{}^{a}_{ABC} = \psi_A \wedge \bar{\psi}_B \wedge \gamma^5\gamma^a\psi_C \qquad \text{(II.8.36c)}$$

$$\overset{(+)}{\theta}_{ABC} = \psi_A \wedge \bar{\psi}_B\psi_C \qquad \text{(II.8.36d)}$$

$$\overset{(-)}{\theta}_{ABC} = \psi_A \wedge \bar{\psi}_B \wedge \gamma^5\psi_C \qquad \text{(II.8.36e)}$$

Starting from the top we find:

$$\theta^{ab}_{ABC} = ([1/2,1/2] \otimes \Box\,) \otimes ([1,1] \otimes \Box\Box \otimes [1,1] \otimes \bullet) =$$

$$= \{[1/2,1/2] \otimes [1,1]\} \otimes \{\,\Box \otimes (\,\Box\Box \otimes \bullet)\} =$$

$$= \{[3/2,3/2] \otimes [3/2,1/2]^{(+)} \otimes [3/2,1/2]^{(-)} + [1/2,1/2]^{(+)} +$$

$$\otimes \left\{ \square\square\square \otimes \begin{smallmatrix}\square\square\\ \square\end{smallmatrix} \otimes \square \right\} \tag{II.8.37}$$

At this point we recall that the representations $[3/2,3/2]$ and $[3/2,1/2]^{(+)}$ are fully symmetric in $(\alpha \longleftrightarrow \beta \longleftrightarrow \gamma)$ so that they can be patched together only with $\square\square\square$ and $\square$ (the latter being the trace of the former).

In this way we have singled out four elements of the irreducible basis, namely

$$\Xi^{ab}_{\boxed{A|B|C}} = [3/2,3/2] \otimes \square\square\square = 8\times 2 = 16 \tag{II.8.38a}$$

$$\Xi^{ab}_{A} = [3/2,3/2] \otimes \square = 8\times 2 = 16 \tag{II.8.38b}$$

$$\Xi^{a}_{ABC} = [3/2,1/2]^{(+)} \otimes \square\square\square = 12\times 2 = 24 \tag{II.8.38c}$$

$$\overset{(+)}{\Xi}{}^{a}_{A} = [3/2,1/2]^{(+)} \otimes \square = 12\times 2 = \underline{24} \tag{II.8.38d}$$

$$80$$

On the other hand the representations $[3/2,1/2]^{(-)}$, $[1/2,1/2]^{(+)}$ and $[1/2,1/2]^{(-)}$ are not fully symmetric in spinor space and therefore can be patched together only with the $\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ representation of $O(2)$. This latter, however, for $N=2$ is equivalent to $\square$ since the anti-symmetric pair AB can be replaced by an ϵ_{AB} tensor. Hence the re-remaining elements of the irreducible basis are:

$$\overset{(-)}{\Xi}{}^{a}_{A} = [3/2,1/2]^{(-)} \otimes \begin{smallmatrix}\square\square\\ \square\end{smallmatrix} = 12\times 2 = 24 \tag{II.8.39a}$$

$$\overset{(+)}{\Xi}_{A} = [1/2,1/2]^{(+)} \otimes \begin{smallmatrix}\square\square\\ \square\end{smallmatrix} = 4\times 2 = 8 \tag{II.8.39b}$$

$$\overset{(-)}{\Xi}_{A} = [1/2,1/2]^{(-)} \otimes \begin{smallmatrix}\square\square\\ \square\end{smallmatrix} = 4\times 2 = \underline{8} \tag{II.8.39c}$$

$$40$$

Indeed $40+80=120$, and therefore (II.8.38) and (II.8.39) exhaust the list of representations contained in $\psi^{\alpha A}\wedge\psi^{\beta B}\wedge\psi^{\gamma C}$. All the θ's introduced in Eqs. (II.8.36) can be expanded in the basis provided by the Ξ's. The coefficients are obtained via a lengthy but straightforward algebra. This will be omitted; the result is shown in Table II.8.IV.

TABLE II.8.VI

IRREDUCIBLE BASIS FOR N=2, d=4 SUPERSPACE

$$\psi_A\wedge\bar\psi_B = \frac{1}{8}\epsilon_{BA}(\mathbb{1}\overset{(+)}{X}{}_{\cdot}^{\cdot} + \gamma^5\overset{(-)}{X}{}_{\cdot}^{\cdot} + \gamma^5\gamma^a\overset{(-)}{X}_a)$$
$$+\frac{1}{4}\gamma^a(\overset{(+)}{X}{}^{a}_{\boxed{B|A}} + \frac{1}{2}\overset{(+)}{X}{}^{a}_{\cdot\cdot}\delta_{AB})$$
$$+\frac{i}{4}\gamma^{ab}(X^{ab}_{\boxed{B|A}} + \frac{1}{2}X^{ab}_{\cdot\cdot}\delta_{AB})$$

$$\psi_A\wedge\bar\psi_B\wedge\psi_C = \frac{1}{2}\epsilon_{BC}\overset{(+)}{\Xi}_A$$

$$\psi_A\wedge\bar\psi_B\wedge\gamma_5\psi_C = -\frac{i}{2}\epsilon_{BC}\overset{(-)}{\Xi}_A$$

$$\psi_A\wedge\bar\psi_B\wedge\gamma^5\gamma^a\psi_C = \frac{1}{2}\epsilon_{BC}\{\overset{(-)}{\Xi}{}^{a}_{A} - \frac{1}{4}\gamma^a(\gamma^5\overset{(+)}{\Xi}_A + i\,\overset{(-)}{\Xi}_A)\}$$

$$\psi_A\wedge\bar\psi_B\wedge\gamma^a\psi_C = -i\,\overset{(+)}{\Xi}{}^{a}_{\boxed{A|B|C}} - \frac{3}{4}i\,\delta_{\{AB}\overset{(+)}{\Xi}{}^{a}_{C\}} + \frac{1}{3}\epsilon_{A\{B}\gamma^5\overset{(-)}{\Xi}{}^{a}_{C\}} -$$
$$-\frac{1}{4}\gamma^a[\epsilon_{A\{B}\overset{(+)}{\Xi}_{C\}} + i\,\gamma^5\epsilon_{A\{B}\overset{(-)}{\Xi}_{C\}}]$$

TABLE II.8.VI (cont'd)

$$\frac{i}{2}\,\psi_A\wedge\bar{\psi}_B\wedge\gamma^{ab}\psi_C = -i\,\Xi^{ab}_{\begin{array}{|c|c|c|}\hline A&B&C\\\hline\end{array}} - i\,\delta_{\{AB}\Xi^{ab}_{C\}} - \frac{1}{2}\gamma^{[a}(\overset{(\pm)}{\Xi}{}^{b]}_{\begin{array}{|c|c|c|}\hline A&B&C\\\hline\end{array}}$$

$$+\frac{3}{4}\,\delta_{\{AB}\overset{(+)}{\Xi}{}^{b]}_{C\}}) - \frac{i}{3}\,\epsilon_{A\{B}\gamma^5\gamma^{[a}\overset{(-)}{\Xi}{}^{b]}_{C\}} -$$

$$-\frac{i}{12}\,\epsilon_{A\{B}\gamma^{ab}(\overset{(+)}{\Xi}_{C\}} - i\,\gamma^5\overset{(-)}{\Xi}_{C\}})$$

Note: All spinors and spinor-tensors are Majorana.

II.8.5 - The N=3, D=4 case

With the same techniques we can obtain the decomposition into irreducible representations of the 3ψ-sector of N=3 superspace. This decomposition is utilized in deriving the results of Chapter IV.7. From Table II.8.I we see that the dimension of the N=3 3ψ-sector is 364. The irreducible basis is found to be the following

$$\overset{(+)}{\Xi}_A = [1/2,1/2]^{(+)}\otimes\square \quad = \quad 4\times3 \quad = \quad 12 \qquad \text{(II.8.40a)}$$

$$\overset{(-)}{\Xi}_A = [1/2,1/2]^{(-)}\otimes\square \quad = \quad 4\times3 \quad = \quad 12 \qquad \text{(II.8.40b)}$$

$$\overset{(+)}{\Xi}_{\begin{array}{|c|c|}\hline A&B\\\hline C\\\cline{1-1}\end{array}} = [1/2,1/2]^{(+)}\otimes\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array} \quad = \quad 4\times5 \quad = \quad 20 \qquad \text{(II.8.40c)}$$

$$\overset{(-)}{\Xi}_{\begin{array}{|c|c|}\hline A&B\\\hline C\\\cline{1-1}\end{array}} = [1/2,1/2]^{(-)}\otimes\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array} \quad = \quad 4\times5 \quad = \quad 20 \qquad \text{(II.8.40d)}$$

$$\Xi_{\begin{array}{|c|}\hline A\\\hline B\\\hline C\\\hline\end{array}} = [1/2,1/2]\otimes\begin{array}{|c|}\hline \ \\\hline \ \\\hline \ \\\hline\end{array} \quad = \quad 4\times1 = 4 \qquad \text{(II.8.40e)}$$

$$\overset{(+)}{\Xi}{}^a_{\begin{array}{|c|}\hline A\\\hline\end{array}} = [3/2,1/2]^{(+)}\otimes\square \quad = 12\times3 = 36 \qquad \text{(II.8.40f)}$$

$$\overset{(-)}{\Xi}{}^a_{\begin{array}{|c|}\hline A\\\hline\end{array}} = [3/2,1/2]^{(-)}\otimes\square \quad = 12\times3 = 36 \qquad \text{(II.8.40g)}$$

$$\overset{(-)}{\Xi}{}^a_{\begin{array}{|c|c|}\hline A&B\\\hline C\\\cline{1-1}\end{array}} = [3/2,1/2]^{(-)}\otimes\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array} \quad = 12\times5 = 60 \qquad \text{(II.8.40h)}$$

$$\Xi^a_{\begin{array}{|c|c|c|}\hline A&B&C\\\hline\end{array}} = [3/2,1/2]\otimes\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline\end{array} \quad = 12\times7 = 84 \qquad \text{(II.8.40i)}$$

$$\Xi^{ab}_{\begin{array}{|c|c|c|}\hline A&B&C\\\hline\end{array}} = [3/2,1/2]\otimes\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline\end{array} \quad = \quad 8\times7 = 56 \qquad \text{(II.8.40j)}$$

$$\Xi^{ab}_A = [3/2,3/2]\otimes\square \quad = \quad 8\times3 = 24 \qquad \text{(II.8.40k)}$$

$$\overline{364}$$

and the coefficients of the decomposition are displayed in Table II.8.VII.

TABLE II.8.VII

THE IRREDUCIBLE BASIS FOR N=3, D=4 SUPERSPACE

$$\psi_A\wedge\bar{\psi}_B\wedge\psi_C = \Xi_{\begin{array}{|c|}\hline A\\\hline B\\\hline C\\\hline\end{array}} + \overset{(+)}{\Xi}_{\begin{array}{|c|c|}\hline B&A\\\hline C\\\cline{1-1}\end{array}} + \delta^{(BC)}_{(AM)}\overset{(+)}{\Xi}_M$$

$$\psi_A\wedge\bar{\psi}_B\wedge\gamma^5\psi_C = -\gamma^5\Xi_{\begin{array}{|c|}\hline A\\\hline B\\\hline C\\\hline\end{array}} + i\,\overset{(-)}{\Xi}_{\begin{array}{|c|c|}\hline B&A\\\hline C\\\cline{1-1}\end{array}} + i\,\delta^{(BC)}_{(AM)}\overset{(-)}{\Xi}_M$$

TABLE II.8.VII (cont'd)

$$\psi_A \wedge \bar{\psi}_B \wedge \gamma^a \psi_C = \Xi^a_{\begin{smallmatrix}A&B&C\end{smallmatrix}} + \delta^{\{AB}\overset{(+)}{\Xi}{}^{C\}}_a - \frac{1}{4}\gamma^a\Big(\overset{(+)}{\Xi}_{\begin{smallmatrix}A&B\\C\end{smallmatrix}} + \overset{(+)}{\Xi}_{\begin{smallmatrix}A&C\\B\end{smallmatrix}}\Big) - \frac{i}{4}\gamma^5\gamma^a\Big(\overset{(-)}{\Xi}_{\begin{smallmatrix}A&B\\C\end{smallmatrix}} + \overset{(-)}{\Xi}_{\begin{smallmatrix}A&C\\B\end{smallmatrix}}\Big)$$
$$- \frac{1}{4}\gamma^a\big(\delta^{(AC)}_{(BM)} + \delta^{(AB)}_{(CM)}\big)\overset{(+)}{\Xi}_M - \frac{i}{4}\gamma^5\gamma^a\big(\delta^{(AC)}_{(BM)} + \delta^{(AB)}_{(CM)}\big)\overset{(-)}{\Xi}_M +$$
$$+ i\gamma^5\Big(\overset{(-)}{\Xi}{}^a_{\begin{smallmatrix}A&B\\C\end{smallmatrix}} + \overset{(-)}{\Xi}{}^a_{\begin{smallmatrix}A&C\\B\end{smallmatrix}}\Big) + i\gamma^5\big(\delta^{(AC)}_{(BM)} + \delta^{(AB)}_{(CM)}\big)\overset{(-)}{\Xi}{}^a_M$$

$$\psi_A \wedge \bar{\psi}_B \wedge \gamma^5\gamma^a\psi_C = 3i\,\overset{(-)}{\Xi}{}^a_{\begin{smallmatrix}B&A\\C\end{smallmatrix}} + 3i\,\delta^{BC}_{AM}\,\Xi^a_M - \gamma_5\gamma_a\Xi_{\begin{smallmatrix}A\\B\\C\end{smallmatrix}}$$
$$+ \frac{i}{4}\gamma^a\,\overset{(-)}{\Xi}_{\begin{smallmatrix}B&A\\C\end{smallmatrix}} + \frac{1}{4}\gamma^5\gamma^a\,\overset{(+)}{\Xi}_{\begin{smallmatrix}B&A\\C\end{smallmatrix}} +$$
$$+ \frac{i}{4}\delta^{(BC)}_{(AM)}\overset{(-)}{\Xi}_M + \frac{1}{4}\gamma^5\gamma^a\,\delta^{(BC)}_{(AM)}\overset{(+)}{\Xi}_M$$

$$\psi_A \wedge \bar{\psi}_B \wedge \gamma_{ab}\psi_C = \Xi^{ab}_{\begin{smallmatrix}A&B&C\end{smallmatrix}} + \delta_{\{AB}\Xi^{ab}_{C\}} -$$
$$- \gamma_{[a}\Xi_{b]}^{\begin{smallmatrix}A&B&C\end{smallmatrix}} - \gamma_{[a}\delta^{\{AB}\Xi^{C\}}_{b]} -$$
$$- 2i\gamma^5\gamma_{[a}\Big(\overset{(-)}{\Xi}{}_{b]}^{\begin{smallmatrix}A&B\\C\end{smallmatrix}} + \overset{(-)}{\Xi}{}_{b]}^{\begin{smallmatrix}A&C\\B\end{smallmatrix}}\Big) -$$
$$- 2i\gamma^5\gamma_{[a}\overset{(-)}{\Xi}{}^M_{b]}\big(\delta^{(AC)}_{(BM)} + \delta^{(AB)}_{(CM)}\big) -$$

TABLE II.8.VII (cont'd)

$$- \frac{1}{6}\gamma_{ab}\Big(\overset{(+)}{\Xi}_{\begin{smallmatrix}A&C\\B\end{smallmatrix}} + \overset{(+)}{\Xi}_{\begin{smallmatrix}A&B\\C\end{smallmatrix}}\Big) - \frac{1}{6}\gamma_5\gamma_{ab}\Big(\overset{(-)}{\Xi}_{\begin{smallmatrix}A&C\\B\end{smallmatrix}} +$$
$$+ \overset{(-)}{\Xi}_{\begin{smallmatrix}A&B\\C\end{smallmatrix}}\Big) - \frac{1}{6}\gamma_{ab}\big(\delta^{(AC)}_{(BM)} + \delta^{(AB)}_{(CM)}\big)\overset{(+)}{\Xi}_M -$$
$$- \frac{1}{6}\gamma_5\gamma_{ab}\overset{(-)}{\Xi}_M\big(\delta^{(AC)}_{(BM)} + \delta^{(AB)}_{(CM)}\big) \quad .$$

II.8.6 - The N=2,D=5 case

As we discussed in Chapter II.7 Majorana spinors do not exist in five dimensions. Rather we can introduce a doublet of pseudo-Majorana spinor 1-forms ψ_A which transform as 4-component $SO(1,4) \sim Sp(4)$ spinors and obey the conjugation rule:

$$(\psi_A)^c = C(\bar{\psi}_A)^T = \varepsilon_{AB}\,\psi_B \tag{II.8.41}$$

The rigid D=5 N=2 superspace can be identified with the homogeneous space:

$$G/H = \overline{SU(2,2/1)}/SO(1,4) \otimes U(1) \tag{II.8.42}$$

which has five bosonic coordinates x^a and eight fermionic ones $\theta^{\alpha A}$ (Pseudo-Majorana). Its cotangent space is spanned by V^a and ψ_A. As in the D=4, N=2 case $\psi^{\alpha A} \wedge \psi^{\beta B}$ has 36 components and $\psi^{\alpha A} \wedge \psi^{\beta B} \wedge \psi^{\gamma C}$ 120 components. We must arrange these latter into irreducible representations of $H = SO(1,4) \otimes O(2)$ rather than into irreducible representations of $H = SO(1,3) \otimes O(2)$.

To do this we list the relevant $SO(1,4)$ representations. Our task is simplified by the local isomorphism $SO(1,4) \sim Usp(2,2)$ which is the spinor representation of the Lorentz group.

$Usp(2,2)$ has rank 2 and we can characterize each of its representations by two numbers directly related to the symmetry of the spinor indices.

TABLE II.8.VIII

IRREDUCIBLE REPRESENTATIONS OF USP(2,2) ⊗ SO(1,4)

Rep. type	Dimension	Corresponding spinor, tensor, spinor-tensor
[0,0]	1	scalar $X^{\otimes} \sim C_{\alpha\beta}$ (charge conjug. mat.)
[1,0]	4	spinor ψ^{α}
[2,0]	10	symmetric bispinor $\lambda^{\alpha\beta} = + \lambda^{\beta\alpha} \sim$ antisymmetric tensor X^{ab}
[1,1]	5	antisymmetric (traceless) bispinor $\lambda^{\alpha\beta} = - \lambda^{\beta\alpha} (\lambda^{\alpha\beta} C_{\alpha\beta} = 0) \sim$ vector $X^a = \Gamma^a_{\alpha\beta} \lambda^{\alpha\beta}$
[2,1]	16	Irreducible spinor-vector Ξ^a; $\Gamma^a \Xi_a = 0$
[3,0]	20	Irreducible spinor tensor $\Xi^{ab} = - \Xi^{ba}$; $\Gamma^b \Xi_{ab} = 0$

We now decompose the product

$$\psi_A \wedge \bar{\psi}_B = \varepsilon_{BC} \psi_A \wedge \psi_C^T C = \varepsilon_{BC} (\psi_A \wedge \psi_C^T) C \qquad \text{(II.8.43)}$$

where C is the charge conjugation matrix. We want the fully symmetric part of the tensor product

$$([1,0] \otimes \square) \otimes ([1,0] \otimes \square) =$$

$$= ([1,0] \otimes [1,0]) \otimes (\square \otimes \square) =$$

$$= ([2,0] \oplus [0,0] \oplus [1,1]) \otimes (\square\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix} \oplus \bullet) \qquad \text{(II.8.44)}$$

$[0,0]$ and $[1,1]$ are antisymmetric in spinor indices and can be patched together only with the antisymmetric $O(2)$ representation. Hence two elements of the irreducible basis for the $\psi_A{}^{\alpha} \wedge \psi_B{}^{\beta}$ space are given by:

$$X^{\otimes} = \psi_A \wedge \bar{\psi}_A = \varepsilon_{AB} \psi_B^T C \psi_A = [0,0] \otimes \begin{smallmatrix}\square\\ \square\end{smallmatrix} = 1 \qquad \text{(II.8.45a)}$$

$$X^a = \bar{\psi}_A \wedge \Gamma^a \psi_A = \varepsilon_{AB} \psi_B^T C \Gamma^a \psi_A = [1,1] \otimes \begin{smallmatrix}\square\\ \square\end{smallmatrix} = 5 \times 1 = 5 \qquad \text{(II.8.45b)}$$

The representation $[2,0]$ is symmetric and can be coupled only with $\square\square$ or with its trace "$\bullet$". Hence the third element of the $\psi_A{}^{\alpha} \wedge \psi_B{}^{\beta}$ irreducible basis is given by:

$$X^{ab}_{AB} = \bar{\psi}_A \frac{i}{2} \Gamma^{ab} \psi_B = \varepsilon_{AC} \psi_C^T C \frac{i}{2} \Gamma^{ab} \psi_B = [2,0] \otimes (\square\square \oplus \bullet) =$$

$$= 10 \times (2+1) = 30 \qquad \text{(II.8.46)}$$

$1 \oplus 5 \oplus 30 = 36$ is then the dimension this superspace sector. The decomposition coefficients follow from standard Γ-matrix algebra. We obtain:

$$\psi_A \wedge \bar{\psi}_B = \frac{i}{4} \Gamma_{ab} X^{ab}_{AB} + \frac{1}{8} \delta_{AB} (\Gamma^a X_a + \mathbb{1} X^{\otimes}) \qquad \text{(II.8.47)}$$

The representations appearing in the $\psi_A^{\alpha} \wedge \psi_B^{\beta} \wedge \psi_C^{\gamma}$ sector are now easily identified by considering the product of the highest representation χ_{BC}^{ab} with ψ_A^{α}. We have

$$\psi_A \wedge \bar{\psi}_B \wedge \frac{i}{2}\, \Gamma^{ab}\psi_C = \psi_A \wedge \chi^{ab}_{BC}$$

$$= ([1,0] \otimes \square\,) \otimes ([2,0] \otimes (\,\square\square \oplus \bullet\,))$$

$$= ([1,0] \otimes [2,0]) \otimes [\,\square \otimes (\,\square\square \oplus \bullet\,)]$$

$$= ([3,0] \oplus [2,1] \oplus [1,0]) \otimes (\,\square\square\square \oplus \begin{smallmatrix}\square\square\\ \square\,\end{smallmatrix} \oplus \square\,) \qquad \text{(II.8.48)}$$

Since $[3,0]$ is symmetric it can be coupled only with $\square\square\square$ and its trace $\square$. Therefore the irreducible basis contains the following two elements:

$$\Xi^{ab}_{\boxed{A\,|\,B\,|\,C}} \quad (\Gamma_b \Xi^{ab}_{\boxed{A\,|\,B\,|\,C}} = 0);\ [3,0] \otimes \square\square\square = 20 \times 2 = 40 \qquad \text{(II.8.49a)}$$

$$\Xi^{ab}_{A}\ ,\ (\Gamma_b \Xi^{ab}_{A} = 0);\ [3,0] \otimes \square = 20 \times 2 = 40 \qquad \text{(II.8.49b)}$$

The representations $[2,1]$ and $[1,0]$ are not fully symmetric and therefore can couple only with

$$\begin{smallmatrix}\square\square\\ \square\,\end{smallmatrix} \sim \square$$

The remaining elements of the irreducible basis are then:

$$\Xi^a_A\ ;\ (\Gamma_a \Xi^a_A = 0);\ [2,1] \otimes \begin{smallmatrix}\square\square\\ \square\,\end{smallmatrix} = 16 \times 2 = 32 \qquad \text{(II.8.50a)}$$

$$\Xi_A\ ; \qquad [1,0] \otimes \begin{smallmatrix}\square\square\\ \square\,\end{smallmatrix} = 4 \times 2 = 8 \qquad \text{(II.8.50b)}$$

and we have

$$40 + 40 + 32 + 8 = 120 \qquad \text{(II.8.51)}$$

as expected.

The actual decomposition coefficients can be calculated with a somewhat lengthy but straightforward algebra. They are given in Table II.8.IV.

TABLE II.8.IX

IRREDUCIBLE BASIS OF N=2, D=5 SUPERSPACE

$$\psi_A \wedge \bar{\psi}_B = \frac{i}{4}\, \Gamma_{ab}\chi^{ab}_{BA} + \frac{1}{8}\, \delta_{AB}(\Gamma_a \chi^a + 1\!\!1 \chi^{\otimes})$$

$$\psi_A \wedge \bar{\psi}_B \wedge \psi_C = \frac{1}{2}\, \delta_{BC}\psi_A \wedge \chi^{\otimes} = -\frac{i}{2}\, \delta_{BC}\Xi_A$$

$$\psi_A \wedge \bar{\psi}_B \wedge \Gamma^a\psi_C = \frac{1}{2}\, \delta_{BC}\psi_A \wedge \chi^a = -\frac{i}{2}\, \delta_{BC}(\Xi^a_A + \frac{1}{5}\, \Gamma^a \Xi_A)$$

$$\frac{i}{2}\, \psi_A \wedge \bar{\psi}_B \wedge \Gamma^{ab}\psi_C = \psi_A \wedge \chi^{ab}_{BC} = \Xi^{ab}_{\boxed{A\,|\,B\,|\,C}} - \delta_{AB}\Xi^{ab}_C$$

$$- \delta_{BC}\Xi^{ab}_A + 3\delta_{CA}\Xi^{ab}_B$$

$$- \delta_{AB}(\frac{1}{3}\, \Gamma^{[a}\Xi^{b]}_C - \frac{1}{10}\, \Gamma^{ab}\Xi_C)$$

$$+ \frac{1}{2}\, \delta_{BC}(\frac{1}{3}\, \Gamma^{[a}\Xi^{b]}_A - \frac{1}{10}\, \Gamma^{ab}\Xi_A)$$

For paedagogical purposes we just give a sketch of the derivation of this Table.

Since

$$\bar{\psi}_B \wedge \psi_A = \frac{1}{2}\,\delta_{BA}\bar{\psi}_M \wedge \psi_M \tag{II.8.52}$$

the second equation of Table II.8.IX is just a definition

$$\Xi_A = i\;\psi_A \wedge \bar{\psi}_B \wedge \psi_B \tag{II.8.53}$$

Considering now the third equation of the same Table, since

$$\bar{\psi}_A \wedge \Gamma^a\psi_B = \frac{1}{2}\,\delta_{AB}\bar{\psi}_M \wedge \Gamma^a\psi_M \tag{II.8.54}$$

we can set

$$\psi_A \wedge \bar{\psi}_B \wedge \Gamma^a\psi_C = \frac{1}{2}\,\delta_{BC}\theta^a_A \tag{II.8.55}$$

where

$$\theta^a_A = \psi_A \wedge \bar{\psi}_M \wedge \Gamma^a\psi_M \tag{II.8.56}$$

is a spinor-vector. In general it can be decomposed as:

$$\theta^a_A = \overset{(16)}{\theta}{}^a_A + \frac{1}{5}\,\Gamma^a\,\overset{(4)}{\theta}_A \tag{II.8.57}$$

By definition we set

$$\Xi^a_A = i\;\overset{(16)}{\theta}{}^a_A \tag{II.8.58}$$

then, in order to obtain the third equation of Table II.8.IX we just have to prove $\theta_A^{(4)} = -i\Xi_A$, which is indeed true since

$$\Gamma^a\psi_A \wedge \bar{\psi}_M \wedge \Gamma_a\psi_M = \psi_A \wedge \bar{\psi}_M \wedge \psi_M \tag{II.8.59}$$

The fourth equation of Table II.8.IX is more complicated.

Setting

$$\theta^{ABC}_{ab} = \frac{i}{2}\,\psi^A \wedge \bar{\psi}^B \wedge \Gamma^{ab}\psi^C \tag{II.8.60}$$

$$\theta^{ABC}_a = \Gamma^b\theta^{ABC}_{ab} \tag{II.8.61}$$

$$\theta^{ABC} = \Gamma^a\theta^{ABC}_a \tag{II.8.62}$$

and using the Fierz rearrangement formula given by the first equation of Table II.8.IX we can easily prove

$$\theta^{ABC}_a = -\frac{1}{2}\,\theta^{CBA}_a + \frac{3i}{8}\,\delta^{AB}\psi^C \wedge \bar{\psi}_M \wedge \Gamma^a\psi_M - $$

$$-\frac{i}{8}\,\delta^{AB}\Gamma_a\Gamma^r\psi^C \wedge \bar{\psi}_M \wedge \Gamma_r\psi_M - \frac{i}{4}\,\delta^{AB}\Gamma_a\psi^C \wedge \bar{\psi}_M \wedge \Gamma^a\psi_M \tag{II.8.63}$$

Now from the representation analysis done in the text we know that $\theta_{ab}{}^{ABC}$ must contain only the representations

$$\Xi^{ab}_{\boxed{A\,|\,B\,|\,C}}\;,\quad \Xi^{ab}_A\;,\quad \Xi^a_A\;,\quad \Xi_A \tag{II.8.64}$$

Hence we can write a decomposition of the type:

$$\theta^{ABC}_{ab} = \Xi_{ab}^{\boxed{A\,|\,B\,|\,C}} + \alpha\delta^{AB}\Xi^C_{ab} + \beta\delta^{BC}\Xi^A_{ab} + \gamma\delta^{CA}\Xi^B_{ab} +$$

$$+\left(a\delta^{AB}\Gamma_{[a}\Xi^C_{b]} + b\delta^{BC}\Gamma_{[a}\Xi^A_{b]} + c\Gamma_{[a}\Xi^B_{b]}\delta^{CA}\right) +$$

$$+ \left(a'\delta^{AB}\,\frac{i}{2}\,\Gamma^{ab}\Xi^{C} + b'\delta^{BC}\,\frac{i}{2}\,\Gamma^{ab}\Xi^{A} + c'\delta^{CA}\,\frac{i}{2}\,\Gamma^{ab}\Xi^{B}\right) \qquad \text{(II.8.65)}$$

The coefficient in front of $\Xi_{ab}^{\boxed{A|B|C}}$ is just a definition because this representation appears only once. The coefficients in front of $\Xi_{ab}{}^{A}$ are now fixed by the fact that

$$\delta^{BC}\theta^{ABC}_{ab} = 0 \qquad \text{(II.8.66)}$$

and the requirement that

$$\varepsilon^{SB}\left(\alpha\delta^{AB}\Xi^{C}_{ab} + \beta\Xi^{A}_{ab}\delta^{BC} + \gamma\delta^{CA}\Xi^{B}_{ab}\right) \qquad \text{(II.8.67)}$$

be fully symmetric in (S,A,C). Indeed the representation $[3,0]$ is symmetric in $(\alpha \leftrightarrow \beta \leftrightarrow \gamma)$ and hence must be also symmetric in (S,A,C). At this point two of the coefficients (α,β,γ) are determined. The remaining one is absorbed in the definition of $\Xi_{A}{}^{ab}$ which appears nowhere else. The remaining coefficients are determined by Eq. (II.8.66) and by comparison with (II.8.63). In fact from (II.8.63) we get

$$\theta_{a}^{\boxed{A|B|C}} = 0 \qquad \text{(II.8.68)}$$

and the remaining δ^{AB}, δ^{CA}, δ^{BC} terms of θ^{ABC}_{a} appear to be given in terms of $\psi_{C} \wedge \bar{\psi}_{M} \wedge \Gamma_{a}\psi_{M}$ and $\psi_{C} \wedge \bar{\psi}_{M} \wedge \psi_{M}$, which is precisely what we need.

The result we have been discussing will be utilized in Chapter III.5 where D=5 supergravity is explicitly constructed.

II.8.7 - Systematics of Fierz identities in eleven dimensions

The theory of supergravity in eleven dimensions occupies a special and privileged position among all the other supergravities. It is the maximally extended theory, it has a simple and beautiful structure and it spontaneously compactifies to 4-dimensions, giving rise to N=8 supergravity and to several other interesting models. Because of this it will be discussed over and over in this book. Chapter III.7 is devoted to its explicit construction and the whole of Part Five deals with the spontaneous compactifications of this theory.

As a necessary technical preparation for its development, in this section we undertake the decomposition into irreducible components of the 2, 3 and 4 sectors of the D=11 superspace.

In D=11 we have at most N=1 supersymmetry which is associated to a Majorana 32-component spinor 1-form ψ^{α}. All decompositions are therefore decompositions of tensor products of SO(1,10) irreducible representations, SO(1,10) being the Lorentz group in eleven dimensions.

We start by giving the dimensionality of the SO(1,10) representations appearing in the symmetric product of two, three and four gravitino 1-forms ψ (ψ is a spin 1/2 Majorana 1-form).

The eleven-dimensional Lorentz group SO(1,10) has, like SO(1,9), rank 5 and therefore its irreducible representations are labeled by 5 integer of half-integer numbers.

In the integer case we are dealing with a bosonic representation and the 5-numbers λ_1, λ_2, λ_3, λ_4, λ_5 labeling it can be identified with the number of boxes in each row of a Young tableau. In this way the representation $(1)^2(0)^3$ corresponds, for instance, to the tableau $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline\end{array}$, namely to an antisymmetric tensor $T_{a_1a_2}$. Analogously $(2)^2(0)^3$ corresponds to the tableau $\begin{array}{|c|c|}\hline a_1 & a_3 \\ \hline a_2 & a_4 \\ \hline\end{array}$ that is to the tensor $T_{\substack{a_1a_2\\a_3a_4}}$ while (1^5) is a skew-symmetric 5-index tensor $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array} \sim T_{a_1 \cdots a_5}$.

TABLE II.8.X

DIMENSION OF SO(1,10) IRREPS APPEARING IN THE SYMMETRIC PRODUCTS OF 2,3,4 IRREPS $(1/2)^5$

Bose irreps Type	Dimension	Fermi irreps Type	Dimension
$(0)^5$	1	$(1/2)^5$	32
$(1)(0)^4$	11	$(3/2)(1/2)^4$	320
$(1)^2(0)^3$	55	$(3/2)^2(1/2)^3$	1408
$(1)^3(0)^2$	165		
$(1)^4(0)$	330		
$(1)^5$	462	$(3/2)^5$	4224
$(2)(0)^4$	65		
$(2)(1)(0)^3$	429		
$(2)^2(0)^3$	1144		
$(2)(1)^4$	4290		
$(2)^2(1)^3$	17160		
$(2)^5$	32604		

In the half-integer case the representation is of the Fermi type. The corresponding object is a spinor tensor having in its vectorial indices the symmetry of the Young tableau $\lambda_1 - 1/2$, $\lambda_2 - 1/2$, $\lambda_3 - 1/2$, $\lambda_4 - 1/2$, $\lambda_5 - 1/2$. Moreover, it is irreducible in the sense that whatever trace can be obtained by contracting it with Γ-matrices is zero.

For instance the irrep $(3/2)(1/2)^4$ is a spinor tensor with the symmetry $(1)(0)^4$ in its Bose indices, namely Ξ_a. The irreducibility means $\Gamma^a \Xi_a = 0$. Analogously $(3/2)^2(1/3)^3$ is a spinor tensor with Bose indices of the type $(1)^2(0)^3$, namely $\Xi_{a_1a_2}$ (skew symmetric). The irreducibility condition is $\Gamma^{a_2}\Xi_{a_1a_2} = 0$.

The use of numerology provides an easy tool to work out the representations appearing in each symmetric product. We find

$$\{(1/2)^5 \otimes (1/2)^5\}_{sym} = (1)(0)^4 \oplus (1)^2(0)^3 \oplus (1)^5 \quad ,$$

$$(32 \times 33/2 = 528 = 11 + 55 + 462) \quad ; \tag{II.8.69a}$$

$$\{(1/2)^5 \otimes (1/2)^5 \otimes (1/2)^5\}_{sym} = (1/2)^5 \oplus (3/2)(1/2)^4 \otimes \otimes (3/2)^2(1/2)^3 \otimes (3/2)^5 \quad ,$$

$$((32 \times 33 \times 34)/(3 \times 2) = 32 + 320 + 1408 + 4224) \quad ; \tag{II.8.69b}$$

$$\{(1/2)^5 \otimes (1/2)^5 \otimes (1/2)^5 \otimes (1/2)^5\}_{sym} = (0)^5 \oplus (1)^3(0)^2 \oplus (1)^4(0) \oplus (1)^5 \oplus (2)(0)^4 \oplus (2)(1)(0)^3 \oplus (2)(1)^4 \oplus (2)^2(0)^3 \oplus (2)^2(1)^3 \oplus (2)^5 \quad ,$$

$$((32 \times 33 \times 34 \times 35)/(4 \times 3 \times 2) = 1 + 165 + 330 + 462 + 65 + + 429 + 4290 + 1144 + 17160 + 32604) \quad , \tag{II.8.69c}$$

These decompositions are made explicit in the following way. Let ψ be the Majorana gravitino 1-form and $\bar{\psi} = \psi^\dagger\Gamma_0 = \psi^T C$ be its bar conjugate. Then we can write the Fierz decompositions given in Table II.8.XI, where $\Xi^{(32)}$, $\Xi^{(320)}_a$, $\Xi^{(1408)}_{a_1a_2}$, $\Xi^{(4224)}_{a_1\dots a_5}$ are, respectively, the irreducible representations $(1/2)^5$, $(3/2)(1/2)^4$, $(3/2)^2(1/2)^3$, $(3/2)^5$ listed in Table II.8.X. Similarly, $X^{(1)}$, $X^{(65)}{}_{a,b}$, $X^{(165)}_{a_1a_2a_3}$, $X^{(330)}_{a_1\dots a_4}$, $X^{(462)}_{a_1\dots a_5}$, $X^{(429)}_{a_1a_2a_3}$, $X^{(1144)}_{a_1a_2a_3a_4}$, $X^{(4290)}_{b_1\dots b_5 a_a}$, $X^{(17160)}_{a_1\dots a_5 b_1b_2}$ are, respectively, the bosonic irreducible representations $(0)^5$, $(2)(0)^4$, $(1^3)(0)^2$, $(1^4)(0)$, $(1)^5$, $(2)(1)(0)^3$, $(2)^2(0)^3$, $(2)(1)^4$, $(2)^2(1)^3$, also listed in Table II.8.X. Moreover, we have

$$\tilde{X}^{(462)}_{a_1\dots a_6} = \varepsilon_{a_1\dots a_6 b_1\dots b_5} X^{(462)}_{b_1\dots b_5} \tag{II.8.70}$$

TABLE II.8.XI

EXPLICIT FIERZ DECOMPOSITION OF D=11 SUPERSPACE

$$\psi\wedge\bar{\psi} = \frac{1}{32}\left(\Gamma_a\bar{\psi}\wedge\Gamma^a\psi - \frac{1}{2}\Gamma_{a_1a_2}\bar{\psi}\wedge\Gamma^{a_1a_2}\psi + \frac{1}{5!}\Gamma_{a_1\ldots a_5}\bar{\psi}\wedge\Gamma^{a_1\ldots a_5}\psi\right)$$

$$\psi\wedge\bar{\psi}\wedge\Gamma_a\psi = \Xi^{(320)}_a + \frac{1}{11}\Gamma_a\Xi^{(32)}$$

$$\psi\wedge\bar{\psi}\wedge\Gamma_{a_1a_2}\psi = \Xi^{(1408)}_{a_1a_2} - \frac{2}{9}\Gamma_{[a_1}\Xi^{(320)}_{a_2]} + \frac{1}{11}\Gamma_{a_1a_2}\Xi^{(32)}$$

$$\psi\wedge\bar{\psi}\wedge\Gamma_{a_1\ldots a_5}\psi = \Xi^{(4224)}_{a_1\ldots a_5} + 2\Gamma_{[a_1a_2a_3}\Xi^{(1408)}_{a_4a_5]} + \frac{5}{9}\Gamma_{[a_1\ldots a_4}\Xi^{(320)}_{a_5]} - \frac{1}{77}\Gamma_{a_1\ldots a_5}\Xi^{(32)}$$

$$\bar{\psi}\wedge\Gamma_{a_1}\psi\wedge\bar{\psi}\wedge\Gamma_{a_2}\psi = \chi^{(65)}_{\substack{a_1\\a_2}} + \frac{1}{11}\delta_{a_1a_2}\chi^{(1)}$$

$$\bar{\psi}\wedge\Gamma_{a_1a_2}\psi\wedge\bar{\psi}\wedge\Gamma_{a_3}\psi = \chi^{(429)}_{\substack{a_1a_2\\a_3}} + \chi^{(165)}_{a_1a_2a_3}$$

$$\bar{\psi}\wedge\Gamma_{a_1a_2}\psi\wedge\bar{\psi}\wedge\Gamma_{a_3a_4}\psi = \chi^{(1144)}_{\substack{a_1a_2\\a_3a_4}} + \chi^{(330)}_{a_1a_2a_3a_4} + \frac{4}{9}\delta_{\substack{[a_1\\ [a_3}}\chi^{(65)}_{\substack{a_2]\\ a_4]}} - \frac{2}{11}\delta^{a_1a_2}_{a_3a_4}\chi^{(1)}$$

$$\bar{\psi}\wedge\Gamma_{a_1\ldots a_5}\psi\wedge\bar{\psi}\wedge\Gamma_{a_6}\psi = \varepsilon_{a_1\ldots a_6b_1\ldots b_5}\chi^{(462)}_{b_1\ldots b_5} + \chi^{(4230)}_{\substack{a_1\ldots a_5\\a_6}} + \frac{15}{7}\delta_{a_6[a_1}\chi^{(330)}_{a_2\ldots a_5]}$$

$$\bar{\psi}\wedge\Gamma_{a_1\ldots a_5}\psi\wedge\bar{\psi}\wedge\Gamma_{a_6a_7}\psi = \frac{i}{56}\varepsilon_{a_1\ldots a_7b_1\ldots b_4}\chi^{(330)}_{b_1\ldots b_4} -$$

TABLE II.8.XI (cont'd)

$$- \frac{i}{300}\varepsilon_{b_1\ldots b_5a_1\ldots a_5[a_6}\chi^{(4290)}_{\substack{b_1\ldots b_5\\a_7]}} + \chi^{(17160)}_{\substack{a_1\ldots a_5\\a_6a_7}} - \frac{180}{21}\delta^{a_6a_7}_{[a_1a_2}\chi^{(167)}_{a_3a_4a_5]}$$

$$- i\,1200\,\delta^{[a_6}_{[a_1}\tilde{\chi}^{(462)\,a_7]}_{a_2\ldots a_5]}$$

The decomposition of Table II.8.XI is a substitute for all Fierz identities which correspond to the appearance of the same irreps in several different products of fermionic currents.

II.8.8 - Irreducible representations of SO(1,9) and the irreducible basis of the D=10 superspace

Supergravity and super Yang-Mills theory in ten space dimensions are as important as D=11 supergravity. Indeed ten are the critical dimensions of superstring theory and the matter coupled supergravity in D=10 is the field theoretic limit of this non local theory. Superstring generated supergravities are treated in Part Six; D=10 super Yang-Mills will be presented in Chapter II.9. Here we study the irreducible representations of the 10-dimensional Lorentz group SO(1,9) and the decomposition into an irreducible basis of the D=10 superspace.

In ten dimensions we have Majorana Weyl spinors. We shall deal with fermionic Majorana 0-forms which are respectively Weyl and anti-Weyl.

$$C(\bar{\lambda})^T = \lambda \quad ; \quad C(\bar{\chi})^T = \chi \qquad \text{(II.8.71a)}$$

$$\frac{1}{2}(\mathbb{1} + \Gamma_{11})\lambda = \lambda \qquad \text{(II.8.71b)}$$

$$\frac{1}{2}(\mathbb{1} - \Gamma_{11})\chi = \chi \qquad \text{(II.8.71c)}$$

The spinor λ (called the gaugino) will turn out to be the supersymmetric partner of the Yang-Mills field $A = A_\mu dx^\mu$ and because of that carries, in general, an index running in the adjoint representation of some internal symmetry group G. χ which has the opposite chirality and which we name the gravitello sits in the graviton multiplet: it is part of the pure supergravity theory and accordingly it carries no internal symmetry index.

The SO(1,9) gamma matrices:

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \qquad (a = 0,1,\ldots,9) \qquad \text{(II.8.72)}$$

are 32×32 as the 11-dimensional ones. The charge conjugation matrix is antisymmetric

$$C^2 = -1 \quad ; \quad C^* = C \quad ; \quad C^T = -C \qquad \text{(II.8.73a)}$$

$$C\Gamma_a C^{-1} = -\Gamma_a^T \qquad \text{(II.8.73b)}$$

and Γ_{11} is "symmetric" in the C-sense:

$$C\Gamma_{11}C^{-1} = -\Gamma_{11}^T \Leftrightarrow (C\Gamma_{11})^T = C\Gamma_{11} \qquad \text{(II.8.74)}$$

This is what allows the definition of Majorana-Weyl spinors. In particular the D=10 superspace has 10 bosonic coordinates $\{x^a\}$ and 16 fermionic coordinates $\{\theta^\alpha\}$ corresponding to the independent components of a Majorana-Weyl spinor θ:

$$\frac{1}{2}(\mathbb{1} + \Gamma_{11})\theta = \theta \qquad \text{(II.8.75)}$$

It follows that the cotangent space to superspace is spanned by the zenhbein V^a and gravitino 1-form ψ^α which is also Majorana-Weyl:

$$C(\bar\psi)^T = \psi \quad ; \quad \frac{1}{2}(\mathbb{1} + \Gamma_{11})\psi = \psi \qquad \text{(II.8.76)}$$

Setting

$$\Gamma^{a_1\ldots a_n} = \Gamma^{[a_1}\Gamma^{a_2}\ldots\Gamma^{a_n]} \qquad \text{(II.8.77)}$$

we find

$$C\Gamma^{a_1\ldots a_n}C^{-1} = (-)^{S_n}(\Gamma^{a_1\ldots a_n})^T \qquad \text{(II.8.78)}$$

where

$$S_n = 0 \qquad n = 3,4,7,8 \qquad \text{(II.8.79a)}$$

$$S_n = 1 \qquad n = 1,2,5,6,9,10 \qquad \text{(II.8.79b)}$$

As usual (see Chapter II.7) if $S_n = 0$, we say that $\Gamma^{a_1\ldots a_n}$ is antisymmetric, while if $S_n = 1$, we say that it is symmetric.

In general we have

$$\Gamma_{11}\Gamma_{a_1\ldots a_n} = \mathrm{cost}(n)\ \epsilon_{a_1\ldots a_n b_{n+1}\ldots b_{10}}\Gamma^{b_{n+1}\ldots b_{10}} \qquad \text{(II.8.80)}$$

where cost(n) is a number depending on n. Formula (II.8.80) implies that, if we consider the bilinear forms

$$I^{a_1\ldots a_n} = \bar\lambda\Gamma^{a_1\ldots a_n}\lambda \qquad \text{(II.8.81a)}$$

$$\chi^{a_1\ldots a_n} = \bar\psi\wedge\Gamma^{a_1\ldots a_n}\psi \qquad \text{(II.8.81b)}$$

where λ is a Majorana Weyl 0-form and ψ a Majorana Weyl 1-form, then $I^{a_1\ldots a_n}$ is non-vanishing only when both $\Gamma^{a_1\ldots a_n}$ and $\Gamma_{11}\Gamma^{a_1\ldots a_n}$ are antisymmetric, while $X^{a_1\ldots a_n}$ is non-vanishing only when both $\Gamma^{a_1\ldots a_n}$ and $\Gamma_{11}\Gamma^{a_1\ldots a_n}$ are symmetric. A look at formulae (II.8.79) is sufficient to conclude that the only non-vanishing I-current is

$$I^{a_1a_2a_3} = \bar{\lambda}\Gamma^{a_1a_2a_3}\lambda \qquad \text{(II.8.82)}$$

while the only non-vanishing X-currents are

$$X^{a} = \bar{\psi} \wedge \Gamma^{a}\psi \qquad \text{(II.8.83a)}$$

$$X^{a_1\ldots a_5} = \bar{\psi} \wedge \Gamma^{a_1\ldots a_5}\psi = -\frac{1}{5!}\,\varepsilon_{a_1\ldots a_5 b_1\ldots b_5}X^{b_1\ldots b_5} \qquad \text{(II.8.83b)}$$

This result is understood by recalling that $I^{a_1a_2a_3}$ has $(10\cdot 9\cdot 8)/(3\cdot 2) = 120$ components, which is precisely the number of components of the antisymmetric object $\lambda^{\alpha}\lambda^{\beta}$: $16 \cdot 15/2 = 120$. On the other hand X_a has 10 components and the antiselfdual $X_{a_1\ldots a_5}$ has 1/2 $(10\cdot 9\cdot 8\cdot 7\cdot 6)/(5\cdot 4\cdot 3\cdot 2) = 126$ components which together make the 136 of the symmetric object

$$\dim \psi^{\alpha} \wedge \psi^{\beta} = \frac{1}{2}\, 16 \times 17 = 136 \qquad \text{(II.8.84)}$$

On the other hand the dimension of the 3ψ-sector is easily calculated:

$$\dim \psi^{\alpha} \wedge \psi^{\beta} \wedge \psi^{\gamma} = \frac{16\cdot 17\cdot 18}{3\cdot 2} = 816 \qquad \text{(II.8.85)}$$

Since the gravitino does not carry SO(N) indices the representations relevant to our analysis are only those of the Lorentz group as in the D=11 case of the previous section.

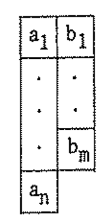
SO(1,9) has rank 5: hence its representations are labelled by 5 numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, where $\lambda_i \geq \lambda_{i+1}$. For bosonic representations λ_i are all integer and stand for the number of boxes in the rows of a Young tableau. The representation is therefore a tensor and $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$ give the symmetry of its indices. In the fermionic case λ_i are all half-integer and the representation is a spinor-tensor whose bosonic indices have the symmetry $[\lambda_1 - 1/2, \lambda_2 - 1/2, \lambda_3 - 1/2, \lambda_4 - 1/2, \lambda_5 - 1/2]$. The spinor-tensor also fulfills a convenient trace condition, obtained by contraction with a Γ-matrix, which guarantees its irreducibility. In both the bosonic and the fermionic case, λ_i are also the eigenvalues of a complete set of Casimir operators.

We have computed the dimensionality of the relevant representations using standard formulae in group theory and our results are summarized in Table II.8.XII where, for writing convenience we have arranged the indices as in a Young tableau rotated of 90°.

For instance, when we write a tensor of the following type:

$$T_{a_1\ldots a_n \atop b_1\ldots b_m}$$

its symmetry is that of the following tableau:



and it is traceless

$$T_{a_1\ldots a_{n-1}x \atop b_1\ldots b_{m-1}x} = 0 \qquad \text{(II.8.86)}$$

The spinor-tensor $\Xi_{\substack{a_1\ldots a_n\\ b_1\ldots b_m}}$ have, in their bosonic indices the same properties as $I_{\substack{a_1\ldots a_n\\ b_1\ldots b_m}}$; moreover, in order to be irreducible they satisfy a trace condition with Γ-matrices:

$$\Gamma^{a_n}\,\Xi_{\substack{a_1\ldots a_n\\ b_1\ldots b_m}} = 0 \qquad \text{(II.8.87)}$$

Now we have to explain why the Γ-matrix trace conditions do indeed convert a spinor tensor into an irreducible representation. This is simply a counting argument. On one hand we have the dimension of the irreducible representation which was computed from group theory. On the other hand we have a spinor tensor. If we do not impose any Γ-trace condition it has 16 × (dimension of the boson rep.) components. We just have to show that the Γ-trace condition subtracts the correct number of components. To do that for the cases listed above we introduce the following recurrence relations.

Let $\theta_{a_1\ldots a_n}$ be an antisymmetric tensor spinor. We write

$$\theta_{a_1\ldots a_{n-1}} = \Gamma^{a_n}\,\theta_{a_1\ldots a_{n-1}a_n} \qquad \text{(II.8.88)}$$

Let $\Pi_{\substack{a_1\ldots a_n\\ b}}$ be a tensor spinor with $\begin{array}{|c|c|}\hline a_1 & b\\ \hline \vdots & \\ \cline{1-1} a_n & \\ \cline{1-1}\end{array}$ traceless symmetry in bosonic indices.

We set

$$\Gamma^{a_n}\,\Pi_{\substack{a_1\ldots a_{n-1}a_n\\ b}} = -\frac{(n+1)}{(n-1)}\,\Pi_{\substack{a_1\ldots a_{n-1}\\ b}} - \theta_{a_1\ldots a_{n-1}b} \qquad \text{(II.8.89)}$$

Eq. (II.8.89) is justified by the fact that after elimination of the index a_n the remaining tensor is, as far as the bosonic indices are concerned, the sum of the following two tableaux:

$$\begin{array}{|c|c|}\hline a_1 & b\\ \hline \cdot & \\ \cdot & \\ \cdot & \\ \cdot & \\ \cline{1-1} a_{n-1} & \\ \cline{1-1}\end{array} \;\oplus\; \begin{array}{|c|}\hline a_1\\ \hline \cdot\\ \cdot\\ \cdot\\ \hline a_{n-1}\\ \hline b\\ \hline\end{array} \qquad \text{(II.8.90)}$$

The normalization factors in Eq. (II.8.89) are obviously arbitrary and have been chosen in a particular way only for later convenience. Now using Eq. (II.8.89) with a little algebra one can show the following identity:

$$\begin{aligned}\Gamma^m\,\Pi_{\substack{a_1\ldots a_{n-1}\\ m}} &= -(n-1)\Gamma^m\,\theta_{a_1\ldots a_{n-1}m}\\ &= -(n-1)\theta_{a_1\ldots a_{n-1}}\end{aligned} \qquad \text{(II.8.91)}$$

Eqs. (II.8.88) and (II.8.91) are the fundamental tools of our counting argument. Let us for instance consider the spinor tensor $\theta_{a_1\ldots a_5}$. It has (126 + 126) × 16 = 4032 components. The condition $\Gamma^{a_5}\theta_{a_1\ldots a_5} = 0$ corresponds to 16 × 210 = 3360 constraints and indeed we find 4032 - 3360 = 672. The same argument goes through for all the remaining [3/2,3/2,..., 1/2] representations.

Coming now to the [5/2,3/2,...,1/2,...] representations we start by considering, for instance, the spinor tensor $\Pi_{\substack{a_1\ldots a_4\\ m}}$. It has 16 × 1728 = 27648 components. If we impose the condition $\Gamma^{a_4}\Pi_{\substack{a_1\ldots a_4\\ b}} = 0$, it would seem from Eq. (II.8.89) that we subtract a spinor tensor

$\Pi_{a_1\ldots a_3 \atop b}$ and a spinor tensor $\theta_{a_1\ldots a_3 b}$, namely $16\times 945 + 16\times 210$ components. Because of the identity (II.8.91) however, this overcounts the constraints and we have to take 16×120 of them (the components of $\theta_{a_1 a_2 a_3}$) back. We get

$$27648 - 16 \times 945 - 16 \times 210 + 16 \times 120 = 11088 \qquad \text{(II.8.92)}$$

which is the correct dimension of the representation [5/2,3/2,3/2,3/2, 1/2]. In the same way we can check all the remaining numbers of the Table. Now that we have classified the irreducible representations, every spinor tensor will be decomposed into irreducible components. What we just need are the Clebsch-Gordan coefficients which were obtained via an iterative procedure starting from the recursion relations (II.8.88) and (II.8.89). We omit the extremely long but straightforward computations. The result is summarized in Table II.8.XIII.

Equipped with this lore we can now derive the irreducible basis of the 2ψ and 3ψ-sectors.

We have already pointed out that $\psi^\alpha \wedge \psi^\beta$ has 136 components corresponding to the two currents X^a, $X^{a_1\ldots a_5}$ defined in Eqs. (II.8.83). The exact decomposition coefficients are very easily computed and are given by:

$$\psi \wedge \bar{\psi} = \frac{1}{16}\,\Gamma_a X^a + \frac{1}{32\cdot 5!}\,\Gamma_{a_1\ldots a_5} X^{a_1\ldots a_5} \qquad \text{(II.8.93)}$$

The $\psi^\alpha \wedge \psi^\beta \wedge \psi^\gamma$ form, instead, has 816 components which are distributed among the two spinor-tensors:

$$\Xi_{a_1\ldots a_5} = \psi \wedge \bar{\psi} \wedge \Gamma_{a_1\ldots a_5}\psi \qquad \text{(II.8.94a)}$$

$$\Xi_a = \psi \wedge \bar{\psi} \wedge \Gamma_a \psi \qquad \text{(II.8.94b)}$$

TABLE II.8.XII

REPRESENTATIONS OF SO(1,9)

Rep. type	Dimension	Corresponding tensor/spinor-tensor	
[2,2,2,1,1]	6930 ⊕ 6930	$T_{a_1\ldots a_5 \atop b_1\ldots b_3}$	self- or antiself-dual in $a_1\ldots a_5$
[2,2,2,1,0]	10560	$T_{a_1\ldots a_4 \atop b_1\ldots b_3}$	
[2,2,2,0,0]	4125	$T_{a_1\ldots a_3 \atop b_1\ldots b_3}$	
[2,2,1,1,1]	3696 ⊕ 3696	$T_{a_1\ldots a_5 \atop b_1 b_2}$	self- or antiself-dual in $a_1\ldots a_5$
[2,2,1,1,0]	5940	$T_{a_1\ldots a_4 \atop b_1 b_2}$	
[2,2,1,0,0]	2970	$T_{a_1 a_2 a_3 \atop b_1 b_2}$	
[2,2,0,0,0]	770	$T_{a_1 a_2 \atop b_1 b_2}$	
[2,1,1,1,1]	1050 ⊕ 1050	$T_{a_1\ldots a_5 \atop b}$	self- or antiself-dual in $a_1\ldots a_5$
[2,1,1,1,0]	1728	$T_{a_1\ldots a_4 \atop b}$	
[2,1,1,0,0]	945	$T_{a_1 a_2 a_3 \atop b}$	
[2,1,0,0,0]	320	$T_{a_1 a_2 \atop b}$	

TABLE II.8.XII (cont'd)

Rep. type	Dimension	Corresponding tensor/spinor-tensor	
[2,0,0,0,0]	54	$T_{\substack{a\\b}}$	
[1,1,1,1,1]	126 ⊕ 126	$T_{a_1\ldots a_5}$	self- or antiself-dual in $a_1\ldots a_5$
[1,1,1,1,0]	210	$T_{a_1\ldots a_4}$	
[1,1,1,0,0]	120	$T_{a_1a_2a_3}$	
[1,1,0,0,0]	45	$T_{a_1a_2}$	
[1,0,0,0,0]	10	T_a	
$[\frac{5}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}]$	5280	$\Xi_{\substack{a_1\ldots a_5\\b}}$; $\Gamma^{a_5}\Xi_{\substack{a_1\ldots a_5\\b}}=0$	
$[\frac{5}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2}]$	11088	$\Xi_{\substack{a_1\ldots a_4\\b}}$; $\Gamma^{a_4}\Xi_{\substack{a_1\ldots a_4\\b}}=0$	
$[\frac{5}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}]$	8800	$\Xi_{\substack{a_1\ldots a_3\\b}}$; $\Gamma^{a_3}\Xi_{\substack{a_1\ldots a_3\\b}}=0$	
$[\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	3696	$\Xi_{\substack{a_1a_2\\b}}$; $\Gamma^{a_2}\Xi_{\substack{a_1a_2\\b}}=0$	
$[\frac{5}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	720	$\Xi_{\substack{a\\b}}$; $\Gamma^{a}\Xi_{\substack{a\\b}}=0$	
$[\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}]$	672	$\Xi_{a_1\ldots a_5}$; $\Gamma^{a_5}\Xi_{a_1\ldots a_5}=0$	
$[\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2}]$	1440	$\Xi_{a_1\ldots a_4}$; $\Gamma^{a_4}\Xi_{a_1\ldots a_4}=0$	
$[\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}]$	1200	$\Xi_{a_1\ldots a_3}$; $\Gamma^{a_3}\Xi_{a_1a_2a_3}=0$	

TABLE II.8.XII (cont'd)

Rep. type	Dimension	Corresponding tensor/spinor-tensor
$[\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	560	$\Xi_{a_1a_2}$; $\Gamma^{a_2}\Xi_{a_1a_2}=0$
$[\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	144	Ξ_a; $\Gamma^a\Xi_a=0$
$[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	16	Ξ (Majorana Weyl spinor)

TABLE II.8.XIII

DECOMPOSITION COEFFICIENTS FOR FERMIONIC REP.

$$\theta = \theta^{(16)}$$

$$\theta_a = \theta_a^{(144)} + \frac{1}{10}\Gamma_a\theta^{(16)}$$

$$\theta_{ab} = \theta_{ab}^{(560)} - \frac{1}{4}\Gamma_{[a}\theta_{b]}^{(144)} - \frac{1}{90}\Gamma_{ab}\theta^{(16)}$$

$$\theta_{a_1a_2a_3} = \theta_{a_1a_2a_3}^{(1200)} + \frac{1}{2}\Gamma_{[a_1}\theta_{a_2a_3]}^{(560)} - \frac{3}{56}\Gamma_{[a_1a_2}\theta_{a_3]}^{(144)} - \frac{1}{720}\Gamma_{a_1a_2a_3}\theta^{(16)}$$

$$\theta_{a_1\ldots a_4} = \theta_{a_1\ldots a_4}^{(1440)} - \Gamma_{[a_1}\theta_{a_2\ldots a_4]}^{(1200)} - \frac{1}{5}\Gamma_{[a_1a_2}\theta_{a_3a_4]}^{(560)} + \frac{1}{84}\Gamma_{[a_1a_2a_3}\theta_{a_4]}^{(144)} + \frac{1}{5040}\Gamma_{a_1\ldots a_4}\theta^{(16)}$$

$$\theta_{a_1\ldots a_5} = \theta_{a_1\ldots a_5}^{(672)} + \frac{5}{2}\Gamma_{[a_1}\theta_{a_2\ldots a_5]}^{(1440)} - \frac{5}{6}\Gamma_{[a_1a_2}\theta_{a_3\ldots a_5]}^{(1200)} - \frac{1}{12}\Gamma_{[a_1\ldots a_3}\theta_{a_4a_5]}^{(560)} + \frac{1}{336}\Gamma_{[a_1\ldots a_4}\theta_{a_5]}^{(144)} + \frac{1}{30240}\Gamma_{a_1\ldots a_5}\theta^{(16)}$$

TABLE II.8.XIII (cont'd)

$$\theta_{\substack{a\\b}} = \theta^{(720)}_{\substack{a\\b}} - \frac{1}{6}\Gamma_{[a}\theta^{(144)}_{b]}$$

$$\theta_{\substack{a_1a_2\\b}} = \theta^{(3696)}_{\substack{a_1a_2\\b}} + \frac{3}{4}\Gamma_{[a_1}\theta^{(720)}_{\substack{a_2]\\b}} - \frac{1}{6}\left(\Gamma_b\theta^{(560)}_{a_1a_2} - \Gamma_{[a_1}\theta^{(560)}_{a_2]b}\right) - \frac{3}{80}\left(\Gamma_{a_1a_2}\theta^{(144)}_b - \Gamma_{b[a_1}\theta^{(144)}_{a_2]} - \frac{1}{3}\eta_{b[a_1}\theta^{(144)}_{a_2]}\right)$$

$$\theta_{\substack{a_1a_2a_3\\b}} = \theta^{(8800)}_{\substack{a_1a_2a_3\\b}} - \Gamma_{[a_1}\theta^{(3696)}_{\substack{a_2a_3]\\b}} - \frac{9}{28}\Gamma_{[a_1a_2}\theta^{(720)}_{a_3]} - \frac{1}{4}\left(\Gamma_b\theta^{(1200)}_{a_1\ldots a_3} + \Gamma_{[a_1}\theta^{(1200)}_{a_2\ldots a_3]b}\right) - \frac{1}{9}\left(\Gamma_{[a_1a_2}\theta^{(560)}_{a_3]b} + \Gamma_{b[a_1}\theta^{(560)}_{a_2a_3]} + \frac{1}{2}\eta_{b[a_1}\theta^{(560)}_{a_2a_3]}\right) + \frac{3}{280}\left(\Gamma_{a_1\ldots a_3}\theta^{(144)}_b + \Gamma_{b[a_1a_2}\theta^{(144)}_{a_3]} + \eta_{b[a_1}\Gamma_{a_2}\theta^{(144)}_{a_3]}\right)$$

$$\theta_{\substack{a_1\ldots a_4\\b}} = \theta^{(11088)}_{\substack{a_1\ldots a_4\\b}} + \frac{5}{3}\Gamma_{[a_1}\theta^{(8800)}_{\substack{a_2\ldots a_4]\\b}} - \frac{2}{3}\Gamma_{[a_1a_2}\theta^{(3696)}_{\substack{a_3a_4]\\b}} - \frac{5}{42}\Gamma_{[a_1\ldots a_3}\theta^{(720)}_{\substack{a_4]\\b}} - \frac{3}{8}\left(\Gamma_b\theta^{(1440)}_{a_1\ldots a_4} - \Gamma_{[a_1}\theta^{(1440)}_{a_2\ldots a_4]b}\right) + \frac{7}{24}\left(\Gamma_{b[a_1}\theta^{(1200)}_{a_2\ldots a_4]} - \Gamma_{[a_1a_2}\theta^{(1200)}_{a_3a_4]b} + \frac{5}{7}\Gamma_{b[a_1}\theta^{(1200)}_{a_2\ldots a_4]}\right) + \frac{7}{135}\left(\Gamma_{b[a_1a_2}\theta^{(560)}_{a_3a_4]} - \Gamma_{[a_1\ldots a_3}\theta^{(560)}_{a_4]b} + \frac{10}{7}\eta_{b[a_1}\Gamma_{a_2}\theta^{(560)}_{a_3a_4]}\right) - \frac{1}{360}\left(\Gamma_{b[a_1\ldots a_3}\theta^{(144)}_{a_4]} - \Gamma_{a_1\ldots a_4}\theta^{(144)}_b + \frac{15}{7}\eta_{b[a_1}\Gamma_{a_2a_3}\theta^{(144)}_{a_4]}\right)$$

TABLE II.8.XIII (cont'd)

$$\theta_{\substack{a_1\ldots a_5\\b}} = \theta^{(5280)}_{\substack{a_1\ldots a_5\\b}} - \frac{15}{4}\Gamma_{[a_1}\theta^{(11088)}_{\substack{a_2\ldots a_5]\\b}} - \frac{25}{12}\Gamma_{[a_1a_2}\theta^{(8800)}_{\substack{a_3\ldots a_5]\\b}} + \frac{5}{12}\Gamma_{[a_1\ldots a_3}\theta^{(3696)}_{\substack{a_4a_5]\\b}} + \frac{25}{500}\Gamma_{[a_1\ldots a_4}\theta^{(720)}_{\substack{a_5]\\b}} - \frac{5}{12}\left(\Gamma_b\theta^{(672)}_{a_1\ldots a_5} + \Gamma_{[a_1}\theta^{(672)}_{a_2\ldots a_5]b}\right) - \frac{205}{96}\left(\Gamma_{b[a_1}\theta^{(1440)}_{a_2\ldots a_5]} + \Gamma_{[a_1a_2}\theta^{(1440)}_{a_3\ldots a_5]b} + \eta_{b[a_1}\theta^{(1440)}_{a_2\ldots a_5]}\right) + \frac{25}{96}\left(\Gamma_{b[a_1a_2}\theta^{(1200)}_{a_3\ldots a_5]} + \Gamma_{[a_1\ldots a_3}\theta^{(1200)}_{a_4a_5]b} + 2\eta_{b[a_1}\Gamma_{a_2}\theta^{(1200)}_{a_3\ldots a_5]}\right) + \frac{5}{216}\left(\Gamma_{b[a_1\ldots a_3}\theta^{(560)}_{a_4a_5]} + \Gamma_{[a_1\ldots a_4}\theta^{(560)}_{a_5]b} + 3\eta_{b[a_1}\Gamma_{a_2a_3}\theta^{(560)}_{a_4a_5]}\right) - \frac{1}{1344}\left(\Gamma_{b[a_1\ldots a_4}\theta^{(144)}_{a_5]} + \Gamma_{a_1\ldots a_5}\theta^{(144)}_b + 4\eta_{b[a_1}\Gamma_{a_2\ldots a_4}\theta^{(144)}_{a_5]}\right)$$

In principle, $\Xi_{a_1\ldots a_5}$ contains the representations (672), (1440), (560), (144) and (16). Many of them, however, have to be zero because we have, at most, 816 components. Actually the only way to obtain 816 with the above numbers is by summing (672) and (144). Therefore, we can conclude

$$\psi \wedge \bar\psi \wedge \Gamma_{a_1\ldots a_5}\psi = \Xi^{(672)}_{a_1\ldots a_5} + \frac{1}{336}\Gamma_{[a_1\ldots a_4}\Xi^{(144)}_{a_5]} \qquad \text{(II.8.95a)}$$

$$\psi \wedge \bar\psi \wedge \Gamma_a\psi = \alpha\,\Xi^{(144)}_a \qquad \text{(II.8.95b)}$$

where α is a coefficient to be determined.

A further justification of (3.10) (Fierz identities) is the following. As we already know $\psi \wedge \psi \wedge \Gamma_{a_1 \ldots a_5} \psi$ is antiselfdual and therefore has 126 components. Hence $\Xi_{a_1 \ldots a_5}$ is an antiselfdual spinor tensor which, therefore, has at most $16 \times 126 = 2016$ components. Considering the decomposition of the Table we see that:

$$\begin{aligned} 4032 &= 672 + 1440 + 1200 + 560 + 144 + 16 \\ &= (672 + 1200 + 144) + (1440 + 560 + 16) \\ &= 2016 + 2016 \end{aligned} \tag{II.8.96}$$

Hence we conclude that the most general antiselfdual spinor tensor $\theta^{(antidual)}_{a_1 \ldots a_5}$ is a superposition either of (672), (1200) and (144) or of (1440), (560) and (16). By explicit inversion of Table II.8.XIII we can show that, for an antiselfdual spinor tensor the components (1440), (560) and (16) are zero so that the first is the correct expansion. It is then sufficient to note that the representations $(672) = [3/2, 3/2, 3/2, 3/2, 3/2]$ and $(144) = [3/2, 1/2, 1/2, 1/2, 1/2]$ are certainly fully symmetric in $(\alpha \leftrightarrow \beta \leftrightarrow \gamma)$ because they correspond to the highest spins in the products

$$[1/2, 1/2, 1/2, 1/2, 1/2] \otimes [1, 1, 1, 1, 1] \quad ,$$

$$[1/2, 1/2, 1/2, 1/2, 1/2] \otimes [1, 0, 0, 0, 0] \quad .$$

Hence (672) and (144) are indeed the two irreducible components of $\psi^\alpha \wedge \psi^\beta \wedge \psi^\gamma$.

The coefficient α of Eq. (II.8.95b) is easily computed using Eq. (II.8.93) once again in

$$\Gamma^{a_5} \psi \wedge \bar{\psi} \wedge \Gamma_{a_1 \ldots a_5} \psi = \frac{1}{336} \Gamma^{a_5} \Gamma_{[a_1 \ldots a_4} \Xi_{a_5]} \tag{II.8.97}$$

TABLE II.8.XIV

IRREDUCIBLE BASIS FOR N=1, D=10 SUPERSPACE

$\psi \wedge \psi = 1/16\, \Gamma_a X^a + 1/32 \cdot 5!\, \Gamma_{a_1 \ldots a_5} X^{a_1 \ldots a_5}$
$\psi \wedge \psi \wedge \Gamma_a \psi = \psi \wedge X_a = 1/336\, \Xi^{(144)}_a$
$\psi \wedge \psi \wedge \Gamma_{a_1 \ldots a_5} \psi = \psi \wedge X_{a_1 \ldots a_5} = \Xi^{(672)}_{a_1 \ldots a_5} + 1/336\, \Gamma_{[a_1 \ldots a_4} \Xi^{(144)}_{a_5]}$

The result is

$$\alpha = \frac{1}{336} \tag{II.8.98}$$

Our discussion is summarized in Table II.8.XIV.