CHAPTER III.7

SUPERGRAVITY IN 6 DIMENSIONS

III.7.1 - Introduction

In this chapter, we discuss chiral supergravity in six space-time dimensions. A Lagrangian for the theory can be derived utilizing the building rules A)-E). In this example, however, such a Lagrangian is inconsistent because it does not incorporate the additional requirement F) of completeness of the equations of motion. Indeed without F) the rheonomy framework for the construction of the Lagrangian fails, in the sense that the extension of the field equations from x-space to superspace involves new constraints on the x-space fields. As a result of this, we show that the space-time restriction of the action is not supersymmetric.

We stress, however, that the theory could be constructed using only the rheonomic Bianchi identities approach, without any reference to the Lagrangian. In this approach one retrieves the complete set of space-time field equations, including the constraint necessary to extend them to superspace.

Of course, the knowledge of the correct action is necessary for global or quantum considerations. At the end of the chapter we discuss how the building rule F) might be implemented to find a supersymmetric space-time action by adding a Lagrangian multiplier term to the action constructed using the building rules A)-E).

If one counts the bosonic and fermionic degrees of freedom according to the formula (III.5.12) one finds that in D=6 the supergravity multiplet is given by

$$(V^a{}_\mu,\ \psi_\mu,\ B_{\mu\nu})$$

where $V^a{}_\mu$ is the graviton ($\frac{1}{2}\, 6(6-3) = 9$ Bose states), ψ_μ a complex chiral gravitino ($\frac{1}{2}\,(6-3)\cdot 8 = 12$ Fermi states) and $B_{\mu\nu}$ a 2-index photon whose field strength $F_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$ is self-dual ($\frac{1}{2}\,(6-2)(6-3)/2) = 3$ Bose states). This suggests that for D=6 supergravity we could write a complete geometrical action based on a 2-form F.D.A. extension of the D=6 super Poincaré group. This is almost true except for the essential self-duality constraint on the $F_{\mu\nu\rho}$-field, which is the source of troubles.

Calling F^+_{abc} and F^-_{abc} the self-dual and antiself-dual parts of the inner curvature components of the 2-form B, we shall be able to:

i) write a rheonomic parametrization of all the curvatures, consistent with the Bianchis and involving only F^+_{abc} ($F^-_{abc} = 0$ is enforced by the closure of the algebra).

ii) write a geometrical action whose equations of motion in the outer sectors yields the previous rheonomic parametrization (including the constraint $F^-_{abc} = 0$).

However the constraint $F^-_{abc} = 0$ is not a yield of the inner equations of motion of our action. This means that principle F) is violated. As a result our equations of motion do not have a smooth extension from x-space to superspace and, correspondingly, the action is not invariant. The non vanishing variation of the action is proportional to F^-_{abc} as we prove by explicit calculations.

At this point the therapy for this pathological theory is almost evident. What we should do is to add a Lagrangian multiplier Λ^{+}_{abc} whose variation yields $F^{-}_{abc} = 0$ as an inner equation. The terms that do the job are all $V^{a_1} \wedge \dots \wedge V^{a_6}$ terms and can be fixed by requiring explicit supersymmetry invariance of the action. Although we shall not discuss these last steps explicitly the above discussion should convince the reader that rule F) is indeed equivalent to action invariance.

Let us now turn to the construction of our example.

III.7.2 - D=6 Weyl spinors and selfdual tensors

Before proceeding to the explicit construction of our model it is worth establishing the main properties of the Weyl spinor algebra in D=6 which are essential in the sequel. Together with Weyl (or chiral) spinors we shall deal in this theory with 3-index antisymmetric tensors satisfying selfduality or antiselfduality relations. These objects obey a number of identities and relations which play a role analogous to the Fierz identities for Weyl spinors and are of utmost importance in some algebraic manipulations used later on. Therefore in the second part of this section we will establish a number of relations fulfilled by self-dual and/or antiselfdual 3-index tensors. Let us begin with Fierz identities.

In Part II we gave the group theoretical construction of Fierz identities in D=4, 5, 10 and 11, but not in D=6. In this case they are so easily obtained from the 2ψ expansion that it does not pay to set up a group-theoretical machinery. We begin by recalling that from the discussion of Chapter II.7 it follows that in D=6 we may impose on the generic 8-dimensional spinors λ a Weyl condition, namely

$$\Gamma_7 \lambda = \pm \lambda \qquad \text{(III.7.1)}$$

where Γ_7 has been defined in Chapter II.7, together with the other conventions for Γ-matrix algebra. Equation (III.7.1) reduces the dimensionality of the spinor representation down to 4 and since no Majorana

(reality) condition can be imposed on λ we are left with 4 complex (8 real) independent components. From Eq. (III.7.1) it easily follows that the only non-vanishing currents one may construct out of the chiral gravitino are those written below:

$$\bar{\psi} \wedge \Gamma^a \psi \quad ; \quad \bar{\psi} \wedge \Gamma^{abc} \psi \qquad \text{(III.7.2)}$$

while we have:

$$\bar{\psi} \wedge \psi = \bar{\psi} \wedge \Gamma^{ab} \psi = 0 \, . \qquad \text{(III.7.3)}$$

Indeed the chirality can be changed only by an odd number of Γ-matrices and, since $\bar{\psi} \equiv \psi^{\dagger} \Gamma_0$, the identities (III.7.3) follow.

Using now the dualization formula for Γ-matrices

$$\epsilon_{a_1 \dots a_k b_1 \dots b_\ell} \Gamma^{b_1 \dots b_\ell} = (-1)^{\frac{\ell(\ell-1)}{2}} \ell! \, \Gamma_7 \Gamma_{a_1 \dots a_k} \qquad \text{(III.7.4)}$$

one finds for Weyl gravitini (plus sign in (III.7.1)):

$$\bar{\psi} \wedge \Gamma^{abc} \psi = - \bar{\psi} \wedge \Gamma_7 \Gamma^{abc} \psi = \frac{1}{3!} \epsilon^{abcpqr} \bar{\psi} \wedge \Gamma_{pqr} \psi \qquad \text{(III.7.5)}$$

that is, <u>the 3-index current is selfdual</u>.

For an anti-Weyl gravitino (minus sign in (III.7.1)) we would of course have an antiselfdual 3-index current. Since choosing Weyl or anti-Weyl gravitini is just a matter of conventions in the following we shall use only Weyl gravitini, namely:

$$\Gamma_7 \psi = \psi \qquad \text{(III.7.6)}$$

so that (III.7.5) holds. It is now easy deduce the 3ψ and 4ψ Fierz identities relevant to our subsequent discussion.

One first establishes the D=6 2ψ-Fierzing formula:

$$\psi \wedge \bar{\psi} = \left(\frac{1}{4} \bar{\psi} \wedge \Gamma^a \psi \Gamma_a - \frac{1}{48} \bar{\psi} \wedge \Gamma^{abc} \psi \Gamma_{abc}\right) \frac{1-\Gamma_7}{2} \qquad \text{(III.7.7)}$$

To prove it let us start from the general expansion:

$$\psi \wedge \bar{\psi} = A + A'\Gamma_7 + A_a\Gamma^a + A'_a\Gamma^a\Gamma_7 + A_{ab}\Gamma^{ab} + A'_{ab}\Gamma^{ab}\Gamma_7 + \\ + A_{abc}\Gamma^{abc} + A'_{abc}\Gamma^{abc}\Gamma_7 \qquad \text{(III.7.8)}$$

where $A, A', A_a, A'_a, \ldots$ are tensorial structures to be determined. Multiplying both sides of (III.7.7) by $\mathbb{1}$ or Γ_{pq} and tracing, using (III.7.3) and (III.7.6) one finds: $A = A' = A_{ab} = A'_{ab} = 0$.

Utilizing then the relation:

$$\frac{1+\Gamma_7}{2}\psi \wedge \bar{\psi} = \psi \wedge \bar{\psi} \qquad \text{(III.7.9)}$$

one also finds $A'_a = -A_a$, $A'_{abc} = -A_{abc}$. Finally multiplying both sides of Eq. (III.7.7) by Γ_p or Γ_{pqr} one respectively obtains:

$$A_a = \frac{1}{4}\bar{\psi} \wedge \Gamma^a\psi \qquad \text{(III.7.10)}$$

$$A_{abc} = -\frac{1}{48}\bar{\psi} \wedge \Gamma_{abc}\psi \qquad \text{(III.7.11)}$$

so that (III.7.7) holds true.

The 3ψ (and 4ψ) Fierz identities can now be established. Using formula (III.7.7) we successively get:

$$\Gamma_a\psi \wedge \bar{\psi} \wedge \Gamma^a\psi \equiv \Gamma_a(\psi \wedge \bar{\psi}) \wedge \Gamma^a\psi = \\ = \frac{1}{4}\bar{\psi} \wedge \Gamma^r\psi\, \Gamma_a\Gamma_r \frac{1-\Gamma_7}{2}\Gamma^a\psi - \frac{1}{48}\bar{\psi} \wedge \Gamma^{pqr}\psi \wedge \Gamma_a\Gamma_{pqr}\frac{1-\Gamma_7}{2}\Gamma^a\psi \\ = -\Gamma_a\psi \wedge \bar{\psi} \wedge \Gamma^a\psi\,. \qquad \text{(III.7.12)}$$

where we used Eq. (III.7.6) and the Γ-matrix relations in D=6:

$$\Gamma_a\Gamma_r\Gamma^a = -4\,\Gamma_r \qquad \text{(III.7.13a)}$$

$$\Gamma^a\Gamma_{pqr}\Gamma_a = 0\,. \qquad \text{(III.7.13b)}$$

It follows that

$$\Gamma^a\psi \wedge \bar{\psi} \wedge \Gamma_a\psi = 0\,. \qquad \text{(III.7.14)}$$

In an analogous way we can prove:

$$\Gamma_{abc}\psi \wedge \bar{\psi} \wedge \Gamma^{abc}\psi = 0 \qquad \text{(III.7.15)}$$

but it is more instructive to see how this relation follows from the selfduality relation (III.7.5). Indeed one has:

$$\Gamma_{abc}\psi \wedge \bar{\psi} \wedge \Gamma^{abc}\psi = -\Gamma_7\Gamma_{abc}\psi \wedge \bar{\psi} \wedge \Gamma^{abc}\psi = \\ = \frac{1}{3!}\varepsilon_{abcpqr}\Gamma^{pqr}\psi \wedge \bar{\psi} \wedge \Gamma^{abc}\psi = \\ = \Gamma_{pqr}\psi \wedge \bar{\psi} \wedge \left(\frac{1}{3!}\varepsilon^{abcpqr}\Gamma_{abc}\right)\psi = \\ = -\Gamma_{pqr}\psi \wedge \bar{\psi} \wedge \Gamma^{pqr}\psi \equiv 0\,. \qquad \text{(III.7.16)}$$

Other useful relations may derived from Eq. (III.7.14). Using $\Gamma_{abc} = \Gamma_{bc}\Gamma_a - 2\,\eta_{a[b}\Gamma_{c]}$ one finds:

$$\Gamma_{abc}\psi \wedge \bar{\psi} \wedge \Gamma^c\psi = -2\,\Gamma^{[a}\psi \wedge \bar{\psi} \wedge \Gamma^{b]}\psi \qquad \text{(III.7.17)}$$

and therefore

$$\bar{\psi} \wedge \Gamma_{abc}\psi \wedge \bar{\psi} \wedge \Gamma^a\psi = \bar{\psi} \wedge \Gamma^{[a}\psi \wedge \bar{\psi} \wedge \Gamma^{b]}\psi = 0\,. \qquad \text{(III.7.18)}$$

Finally from the Fierzing formula (III.7.7) one can also derive the following identity

$$\Gamma_a\psi \wedge \bar{\psi} \wedge \Gamma^{abc}\psi = 2\,\Gamma^{[b}\psi \wedge \bar{\psi} \wedge \Gamma^{c]}\psi = \\ = -\Gamma^{abc}\psi \wedge \bar{\psi} \wedge \Gamma_a\psi \qquad \text{(III.7.19)}$$

We now consider selfdual tensors.

First of all we note that a real antisymmetric self-dual or antiself-dual tensor exists in Minkowski space only if $D/2$ is an odd number just as it happens for Weyl spinors. Indeed from the constraint

$$A_{a_1\ldots a_{D/2}} = \alpha\, \varepsilon_{a_1\ldots a_{D/2}\, b_1\ldots b_{D/2}}\, A^{b_1\ldots b_{D/2}} \qquad \text{(III.7.20)}$$

we obtain, taking the dual of both sides:

$$\alpha^2 = -\, (-1)^{D/2}/(D/2!)^2 \; . \qquad \text{(III.7.21)}$$

Therefore for a real tensor $D/2$ is odd and

$$\alpha = \pm \frac{1}{(D/2)!} \; . \qquad \text{(III.7.22)}$$

A tensor satisfying (III.7.20) with $\alpha = +\, 1/(D/2)!$ is said to be self-dual and we will append to it a (+)-subscript. In the opposite case it is anti-selfdual and we will append to it a (−)-subscript. Thus if in $D=6$ A_+^{abc} and B_-^{abc} are selfdual and anti-selfdual tensors, by definition we have:

$$A_+^{abc} = \frac{1}{3!}\, \varepsilon^{abcpqr}\, A_{+pqr} \qquad \text{(III.7.23)}$$

$$B_-^{abc} = -\, \frac{1}{3!}\, \varepsilon^{abcpqr}\, B_{-pqr} \; . \qquad \text{(III.7.24)}$$

Let us now establish a number of useful relations satisfied by selfdual or/and anti-selfdual tensors.

Let us compute the product:

$$\begin{aligned} A_+^{abm}\, B_{\pm abn} &= \pm \frac{1}{(3!)^2}\, \varepsilon^{abmpqr}\, \varepsilon_{abnijk}\, A_{+pqr}\, B_\pm^{ijk} \\ &= \mp \frac{\ell!4!}{(3!)^2}\, \delta^{mpqr}_{nijk}\, A_{+pqr}\, B_\pm^{ijk} \\ &= \mp \frac{1}{3}\, (\delta^m_n\, A_+^{ijk}\, B_{\pm ijk} - 3\, A_{+nij}\, B_\pm^{mij}) \; . \end{aligned} \qquad \text{(III.7.25)}$$

It follows that:

$$A_+^{mij}\, B_{\pm nij} \mp A_+^{nij}\, B_{\pm mij} = \mp \frac{1}{3}\, \delta^m_n\, A_+^{ijk}\, B_{\pm ijk} \; . \qquad \text{(III.7.26)}$$

Symmetrizing or anti-symmetrizing in m, n we obtain

a) $$A_+^{ijk}\, B_{+ijk} = 0 \qquad \text{(III.7.27)}$$

b) $$A_+^{ij[m}\, B_{\pm .ij}^{n]} = 0 \qquad \text{(III.7.28)}$$

c) $$A_+^{ij(m}\, B_{-.ij}^{n)} = \frac{1}{6}\, A_+^{ijk}\, B_{-ijk}\, \eta^{mn} \qquad \text{(III.7.29)}$$

Exchanging + and − we also get:

a') $$A_-^{ijk}\, B_{-ijk} = 0 \qquad \text{(III.7.30a)}$$

b') $$A_-^{ij[m}\, B_{\mp .ij}^{n]} = 0 \qquad \text{(III.7.30b)}$$

Other relations can be obtained by considering the product:

$$\begin{aligned} A_+^{pqa}\, B_{\pm\ \ a}^{rs.}\, \varepsilon_{pqrs\ell m} &= \pm \frac{1}{3!}\, A_+^{pqa}\, B_{\pm ijk}\, \varepsilon^{rsaijk}\, \varepsilon_{pqrs\ell m} = \\ &= \pm\, 2\, A_{+\ \ m}^{pq.}\, B_{\pm pq\ell} \mp 2\, A_{+\ \ \ell}^{pq.}\, B_{\pm pqm} \; . \end{aligned} \qquad \text{(III.7.31)}$$

It follows that:

d) $$A_+^{pqa}\, A_{+\ \ a}^{rs.}\, \varepsilon_{pqrs\ell m} = 0 \qquad \text{(III.7.32)}$$

e) $$A_+^{pqa}\, B_{-\ \ a}^{rs.}\, \varepsilon_{pqrs\ell m} = 4\, A_{+[\ell}^{pq.}\, B_{-m]pq} \qquad \text{(III.7.33)}$$

and exchanging (+) and (−):

d') $$A_-^{pqa}\, A_{-\ \ a}^{rs.}\, \varepsilon_{pqrs\ell m} = 0 \; . \qquad \text{(III.7.34)}$$

Lastly we consider the product

$$A_{+}^{ijk}\, B_{\pm}^{pqa}\, \varepsilon_{ijabcs}\, \varepsilon_{pqk...}{}^{bcm} =$$

$$= \pm \frac{1}{3!} A_{+}^{ijk}\, B_{\pm rst}\, \varepsilon^{pqarst}\, \varepsilon_{ijabcs}\, \varepsilon_{pqk...}{}^{bcm} =$$

$$= \mp 8\, \varepsilon^{ij.bcs}{}_{a}\, \delta_{k}^{[a}\, \delta_{bcm}^{rut]}\, A_{+}^{ijk}\, B_{\pm rut} =$$

$$= \mp 2\, A_{+}^{ija}\, B_{\pm}^{mbc}\, \varepsilon_{ijabbs} \pm 2\, A_{+}^{ijk}\, B_{\pm\ k}^{bc.}\, \varepsilon_{ijmbcs} \qquad \text{(III.7.35)}$$

Using d) and e) we deduce:

f) $$A_{+}^{ijk}\, A_{+}^{pqa}\, \varepsilon_{ijabcs}\, \varepsilon_{pqk...}{}^{bcm} = 12\, A_{+}^{bcs}\, A_{+bcm} \qquad \text{(III.7.36)}$$

g) $$A_{+}^{ijk}\, B_{-}^{pqa}\, \varepsilon_{ijabcs}\, \varepsilon_{pqk...}{}^{bcm} = 2\, A_{+}^{ijk}\, B_{-ijk}\, \delta_{m}^{s} + A_{+[m}^{ij.}\, B_{-s]ij} \qquad \text{(III.7.37)}$$

and exchanging (+) and (-):

f') $$A_{-}^{ijk}\, A_{-}^{pqa}\, \varepsilon_{ijabcs}\, \varepsilon_{pqk...}{}^{bcm} = -\,12\, A_{-}^{bcs}\, A_{-bcm}\,. \qquad \text{(III.7.38)}$$

Finally we note that every anti-symmetric 3-tensor F_{abc} can be uniquely decomposed into a selfdual and an antiselfdual part, namely:

$$F_{abc} = F_{+abc} + F_{-abc}\,. \qquad \text{(III.7.39)}$$

The proof is almost evident and is left to the reader.

After this excursion in the D=6 spinor and tensor algebra we turn to the explicit construction of the model.

III.7.3 - The free differential algebra of D=6 supergravity

The supergravity model we are going to discuss is based on the following free differential algebra:

$$R^{ab} \equiv d\omega^{ab} - \omega^{a}{}_{c} \wedge \omega^{cb} = 0 \qquad \text{(III.7.40a)}$$

$$R^{a} \equiv \mathcal{D}\, V^{a} - \frac{i}{2}\, \bar{\psi} \wedge \gamma^{a}\psi = 0 \qquad \text{(III.7.40b)}$$

$$\rho \equiv \mathcal{D}\, \psi = 0 \qquad \text{(III.7.40c)}$$

$$R^{\otimes} \equiv dB - \frac{i}{2}\, \bar{\psi} \wedge \Gamma^{a}\psi \wedge V_{a} = 0 \qquad \text{(III.7.40d)}$$

where the connection ω^{ab}, the vielbein V^{a} and the Weyl gravitino ψ are the gauge field one-forms of the super Poincaré group in six dimensions and B is a two-form which is a scalar under six dimensional Lorentz-transformations. The left-hand side of Eqs. (III.7.40) defines the curvature two-forms R^{ab}, R^{a}, ρ and the generalized curvature three-form $R^{\otimes}$ of the free differential algebra. As usual $\mathcal{D}$ means the Lorentz covariant exterior derivative:

$$\mathcal{D}\, V^{a} = dV^{a} - \omega^{a}{}_{b} \wedge V^{b} \qquad \text{(III.7.41a)}$$

$$\mathcal{D}\, \psi = d\psi - \frac{1}{4}\, \Gamma^{ab}\omega_{ab} \wedge \psi \qquad \text{(III.7.41b)}$$

To arrive at the F.D.A. (III.5.20) one follows the iterative construction explained in Theorem 2 of Sect. III.6. One starts with the super Poincaré group in D=6 whose M.C. equations are the first three Eqs. (III.7.40a,b,c). One then considers the cohomology classes of the Lie super-algebra which can be built out of the 1-forms V^{a}, ψ, ω^{ab}. Considering the identity representation $D^{(0)}$ in which $\nabla^{(n=0)}$ coincides with the ordinary d-operator, one finds that in this representation there is a non-trivial cohomology class of order 3, namely

$$\Omega(\omega, V, \psi) = \frac{i}{2}\, \bar{\psi} \wedge \Gamma^{a}\psi \wedge V_{a}\,. \qquad \text{(III.7.42)}$$

Indeed using Eqs. (III.7.40b,c) we have:

$$d\Omega = d(\frac{i}{2}\bar{\psi}\wedge\Gamma_a\psi\wedge V^a) = \mathscr{D}(\frac{i}{2}\bar{\psi}\wedge\Gamma_a\psi\wedge V^a) =$$

$$= (\frac{i}{2})^2\,\bar{\psi}\wedge\Gamma_a\psi\wedge\bar{\psi}\wedge\Gamma^a\psi = 0\,. \qquad \text{(III.7.43)}$$

The last equality is nothing else but the Fierz identity (III.7.14). Hence, following the prescription of Theorem 2 of Sect. III.6.3, we can introduce a new 2-form B which satisfies:

$$dB - \frac{i}{2}\bar{\psi}\wedge\Gamma_a\psi\wedge V^a = 0\,. \qquad \text{(III.7.44)}$$

Adding (III.7.44) to the M.C. equations of the super Poincaré group we obtain the F.D.A. (III.7.40). It can be easily verified that no other non-trivial extension of the F.D.A. (III.7.40) can be found.

We assume therefore the given F.D.A. as the fundamental algebraic structure of the theory. The configuration of fields described by Eqs. (III.7.40) corresponds to the physical vacuum with vanishing curvatures. Deviations from the vacuum imply the nonvanishing curvatures:

$$R^{ab} = d\omega^{ab} - \omega^a{}_c\wedge\omega^{cb} \qquad \text{(III.7.45a)}$$

$$R^a = \mathscr{D}V^a - \frac{i}{2}\bar{\psi}\wedge\Gamma^a\psi \qquad \text{(III.7.45b)}$$

$$\rho = \mathscr{D}\psi \qquad \text{(III.7.45c)}$$

$$R^{\otimes} = dB - \frac{i}{2}\bar{\psi}\wedge\Gamma^a\psi\wedge V_a \qquad \text{(III.7.45d)}$$

We have used the same notation for the soft fields $(\omega^{ab}, V^a, \psi, B)$ satisfying (III.7.45) as for the left-invariant ones satisfying (III.7.40). By d-differentiation of (III.7.45) one obtains the (generalized) Bianchi identities:

$$\mathscr{D}R^{ab} = 0 \qquad \text{(III.7.46a)}$$

$$\mathscr{D}R^a + R^{ab}\wedge V_b + \frac{i}{2}(\bar{\rho}\wedge\Gamma^a\psi - \bar{\psi}\wedge\Gamma^a\rho) = 0 \qquad \text{(III.7.46b)}$$

$$\mathscr{D}\rho + \frac{1}{4}\Gamma_{ab}R^{ab}\psi = 0 \qquad \text{(III.7.46c)}$$

$$dR^{\otimes} + \frac{i}{2}(\bar{\rho}\wedge\Gamma^a\psi - \bar{\psi}\wedge\Gamma^a\rho)\wedge V_a + \frac{i}{2}\bar{\psi}\wedge\Gamma^a\psi\wedge R_a = 0\,. \qquad \text{(III.7.46d)}$$

The combined set of equations (III.7.45) ⊕ (III.7.46) exhibits explicitly the gauge invariance under the subgroup $H = SO(1,5)\otimes U(1)$ of $G \equiv ISO(1,5)$, $SO(1,5)$ being the Lorentz group in D=6 and U(1) being associated to the gauge transformations of the B field

$$B \to B + d\varphi^{\otimes} \quad ; \quad R^{\otimes} \to R^{\otimes} \qquad \text{(III.7.47)}$$

where $\varphi^{\otimes}$ is a generic 1-form.

Moreover the system (III.7.45) ⊕ (III.7.46) is invariant under the rigid scale transformations

$$\omega^{ab} \to \omega^{ab} \quad ; \quad R^{ab} \to R^{ab} \qquad \text{(III.7.48a)}$$

$$V^a \to wV^a \quad ; \quad R^a \to wR^a \qquad \text{(III.7.48b)}$$

$$\psi \to w^{1/2}\psi \quad ; \quad \rho \to w^{1/2}\psi \qquad \text{(III.7.48c)}$$

$$B \to w^2 B \quad ; \quad R^{\otimes} \to w^2 R^{\otimes} \qquad \text{(III.7.48d)}$$

To see whether (III.7.45,46) can give rise to a consistent physical theory, we first check the matching of the on-shell bosonic and fermionic degrees of freedom. As we pointed out in the introduction, using formula III.5.12, one finds that V^a_μ has $D(D-3)/2 \equiv 9$ and ψ_μ has $2^{D/2}(D-3)/2 = 12$ degrees of freedom. Since $B_{\mu\nu}$ has $(D-2)\,(D-3)/1.2 = 6$ degrees of freedom we see that we do not have the desired matching. The only way to obtain it is to impose a self-duality (or anti-selfduality) constraint on the field strength of $B_{\mu\nu}$. Namely, calling F_{abc} the intrinsic components of $R^{\otimes}$, we require:

$$F_{abc} = \pm \frac{1}{3!} \varepsilon_{abcpqr} F^{pqr} \tag{III.7.49}$$

or, by using the notation introduced in the previous section:

$$F_{abc} = F_{\pm abc} \tag{III.7.50}$$

When (III.7.50) is satisfied fermions and bosons do match, since the number of on-shell degrees of freedom of B_μ is reduced by a factor 2. As we shall see in the following, the constraint (III.7.50) can be retrieved as a result of projecting the superspace equations of motion in the outer directions or, equivalently, by analyzing Bianchi identities in superspace; instead, space-time equations do not yield this constraint. This is the announced violation of principle F). It will lead to the action non-invariance which can be cured only by the addition of a Lagrangian multiplier (Siegel method: see references at the end of this part).

III.7.4 - Construction of the model

According to building rules A)-E) the action can be written as follows:

$$\mathscr{A} = \int_{M_6} (\Lambda + R^A \wedge \nu_A + R^A \wedge R^B \wedge \nu_{AB}) \tag{III.7.51}$$

where $R^A \equiv (R^{ab}, R^a, \rho, R^{\otimes})$ is the "adjoint" multiplet of the curvatures (III.7.45) and Λ, ν_A, ν_{AB} are polynomials in the soft forms ω^{ab}, V^a, ψ, B of degree 6, 4 and 2 respectively.

The requirements of homogeneous scaling of the Lagrangian $\mathscr{L}$ under (III.7.48) and of SO(1,5)-gauge invariance enable us to write the following general ansatz:

$$\mathscr{L} = \Lambda + R^{ab} \wedge \nu_{ab} + R^a \wedge \nu_a + (\bar\rho \wedge n - \bar n \wedge \rho) + + R^{\otimes} \wedge \nu_{\otimes} + R^A \wedge R^B \wedge \nu_{AB} \tag{III.7.52}$$

where

$$\Lambda = 0 \tag{III.7.53a}$$

$$\nu_{ab} = \frac{1}{4} \varepsilon_{abc_1-c_4} V^{c_1} \wedge \ldots \wedge V^{c_4} + \alpha_1 V^a \wedge V^b \wedge B \tag{III.7.53b}$$

$$\nu_a = i\,\alpha_2\, \bar\psi \wedge \Gamma_a \psi \wedge B + i\,\alpha_3\, \bar\psi \wedge \Gamma^b \psi \wedge V_b \wedge V_a + + i\,\alpha_4\, \bar\psi \wedge \Gamma_{abc}\psi \wedge V^b \wedge V^c \tag{III.7.53c}$$

$$n = \alpha_5\, \Gamma_{abc}\psi \wedge V^a \wedge V^b \wedge V^c + \alpha_6\, \Gamma_a \psi \wedge V^a \wedge B \tag{III.7.53d}$$

$$\nu_{\otimes} = i\,\alpha_7\, \bar\psi \wedge \Gamma_a \psi \wedge V^a \tag{III.7.53e}$$

and

$$R^A \wedge R^B \wedge \nu_{AB} = \eta\, R^{\otimes} \wedge R^a \wedge V_a + \gamma\, R^a \wedge R_a \wedge B + + \delta\, R^{\otimes} \wedge R^{\otimes} \wedge B \tag{III.7.53f}$$

where all the numerical constants are real, except α_5, α_6 which may be complex.

To justify Eqs. (III.7.53) we merely observe that the corresponding terms of the Lagrangian all scale under (III.7.48) homogeneously with the Einstein term $1/4\, R^{ab} V^{c_1} \ldots V^{c_4} \varepsilon_{abc_1 \ldots c_4}$, namely as $[w^4]$ (the 1/4 factor is just a normalization), and are SO(1,5)-gauge invariant since no term contains the bare gauge field ω^{ab}. Moreover (III.7.53a) holds since the only two terms which are allowed, namely $\bar\psi \wedge \Gamma^{abc}\psi \wedge \bar\psi \wedge \Gamma_a \psi \wedge V_b \wedge V_c$ and $\bar\psi \wedge \Gamma_{abc}\psi \wedge \bar\psi\, \Gamma_{abd}\psi \wedge V^c \wedge V^d$ are identically zero because of the Fierz identities (III.7.16 and 18). Before imposing the further U(1) gauge invariance, namely invariance under (III.7.47), let us utilize the freedom of adding an exact differential to $\mathscr{L}$ in order to eliminate some of the terms appearing in

The quickest way to arrive at the determination of ξ and at the explicit parametrization of the rheonomic curvatures is to combine the information given by the equations of motion with the Bianchi identities in the outer directions. We can write down the following ansatz for the rheonomic parametrization of the remaining curvatures:

$$R^{ab} = R^{ab}{}_{cd}\, V^c \wedge V^d + (\bar{\theta}^{ab}{}_c \psi \wedge V^c + \text{h.c}) + i(a_1 F_+^{mab} + a_2 F_-^{mab})\bar{\psi} \wedge \Gamma_m \psi + (a_3 F_{+pq.}{}^{[a} + a_4 F_{-pq.}{}^{[a})\bar{\psi} \wedge \Gamma^{b]pq}\psi \qquad \text{(III.7.72a)}$$

$$\rho = \rho_{ab}\, V^a \wedge V^b + (b_1 F_+^{abc} + b_2 F_-^{abc})\Gamma_{ab}\psi \wedge V_c \qquad \text{(III.7.72b)}$$

where $\theta^{ab}{}_c$ is a spinor constructed out of ρ_{ab}. The parameters $a_1,\dots,a_4$; b_1, b_2 are real and complex, respectively, to be fixed by equations of motion and/or Bianchi's. F_{abc} is decomposed into its self-dual and anti-self-dual components, according to (III.7.39). These Ansätze are the most general expressions we may write which are rheonomic, $SO(1,5)\times U(1)$ gauge invariant and homogeneous in the scaling parameter w (observe that ρ_{ab} and F_{abc} scale as $[w^{-3/2}]$ and $[w^{-1}]$ respectively).

To determine the values of b_1, b_2 and ξ, we collect the information given by the $1\psi-4V$'s projection of the gravitino equation (III.7.70) and the $2\psi-2V$'s projection of the Maxwell-Bianchi identity (III.7.46d).

The $1\psi-4V$ projection of gravitino equation (III.7.70) is given by:

$$\{-(-3+\tfrac{1}{2}\xi)\tfrac{\xi}{2}(F_{+apq}-F_{-apq}) - (3+\tfrac{\xi}{2})(F_{+apq}+F_{-apq})\}\varepsilon^{abpqmn}\,\Gamma_b\psi - 2(b_1 F_{+pqr}+b_2 F_{-pqr})\Gamma_{abc}\Gamma^{pq}\psi\,\varepsilon^{rabcmn} - \tfrac{3}{2}\xi(F^a_{+.pq} - F^a_{-.pq})\Gamma_{abc}\psi\,\varepsilon^{pqbcmn} = 0\,. \qquad \text{(III.7.73)}$$

From this equation it is evident that b_1, b_2 are real numbers. Using this information we can write the $2\psi-2V$'s projection of Maxwell-Bianchi equation (III.7.46d) as follows:

$$\tfrac{i}{2}\bar{\psi} \wedge \Gamma^a\psi\{3(F_{+abc}+F_{-abc}) - \tfrac{1}{2}\xi(F_{+abc}-F_{-abc})\} \wedge V^b \wedge V^c - \tfrac{i}{2}\bar{\psi} \wedge [\Gamma^{pq},\Gamma_b]\psi \wedge V^b \wedge V^c(b_1 F_{+pqc} + b_2 F_{-pqc}) - i\tfrac{\xi}{4}\bar{\psi} \wedge \Gamma^a\psi(F_{+abc}-F_{-abc}) \wedge V^b \wedge V^c = 0\,. \qquad \text{(III.7.74)}$$

Using $[\Gamma^{pq},\Gamma_b] = -4\,\delta^{[p}_b\,\Gamma^{q]}$ we obtain the following constraint:

$$(3-\tfrac{1}{2}\xi - 4b_1)F_{+abc} + (3+\tfrac{1}{2}\xi + 4b_2)F_{-abc} = 0\,. \qquad \text{(III.7.75)}$$

Performing now the Γ-matrix product in (III.7.73) (namely $\Gamma_{abc}\cdot\Gamma_{pq}$) and annihilating the coefficient of the Γ_{abc} ψ-structure we obtain

$$\{\tfrac{3}{2}\xi(F^a_{+.pq} - F^a_{-.pq}) + 12(b_1 F^a_{+.pq} + b_2 F^a_{-.pq})\}\Gamma_{abc}\psi\,\varepsilon^{pqbcrs} = 0\,. \qquad \text{(III.7.76)}$$

Now since $\Gamma_{abc}\psi$ behaves like a selfdual tensor, the identity (III.7.32) implies

$$F_{+apq}\,\Gamma^{abc}\psi\,\varepsilon^{pq}{}_{..bcrs} = 0 \qquad \text{(III.7.77)}$$

and therefore Eq. (III.7.76) implies

$$(\tfrac{3}{2}\xi + 12\,b_2)F_{-abc} = 0\,. \qquad \text{(III.7.78)}$$

The annihilation of the $\Gamma_a\psi$ coefficient in Eq. (III.7.73) gives:

$$(8\,b_1 + \xi - \tfrac{1}{4}\xi^2 - 3)F_{+abc} + (16\,b_2 + 2\xi - \tfrac{1}{4}\xi^2 + 3)F_{-abc} = 0\,. \qquad \text{(III.7.79)}$$

It is now easy to verify that the three equations (III.7.75), (III.7.78) and (III.7.79) admit a non-trivial solution if and only if

$$F_{-abc} = 0 \,. \qquad \text{(III.7.80)}$$

In that case we also get

$$b_1 = \frac{3}{4}\left(1 - \frac{\eta}{\sqrt{3}}\right) \qquad ; \qquad \xi = 2\sqrt{3}\,\eta \qquad \text{(III.7.81a)}$$

where

$$\eta = \pm 1 \,. \qquad \text{(III.7.81b)}$$

The reversed situation where $F_+^{abc} = 0$ gives instead an inconsistency due to the identity (III.7.33). If we had chosen from the beginning to work with anti-Weyl gravitinos then we would have obtained $F_+^{abc} = 0$; indeed $\Gamma_{abc}\psi$ would have been anti-selfdual so that the identity (III.7.34) would now apply. The important thing to stress is that we have obtained a space-time equation of motion for B, namely the self duality constraint, by analyzing the theory only in the outer directions of superspace. This space time equation of motion $(F_-^{abc} = 0)$ does not follow from the $V^{a_1} \wedge V^{a_2} \wedge V^{a_3} \wedge V^{a_4} \wedge V^{a_5}$ projection of Eq. (III.7.69). This is the announced violation of principle F) which makes, at the very end, the Lagrangian formulation of the theory inconsistent as it stands and the mechanism of rheonomy meaningless.

The theory can of course be cured by introducing a new field (a Lagrangian multiplier) which enforces $F_-^{abc} = 0$ as a space-time equation.

Let us forget about this, however, and go on with the analysis of the remaining sectors of the field equations. All the other outer projections of Eqs. (III.7.68-70) are consistent with the results (III.7.80) and (III.7.81). Furthermore they determine uniquely the values of a_1, a_2, and the explicit form of $\theta^{ab}{}_c$. Let us briefly see how these informations are obtained also from the Bianchi identities the rheonomic ansatz (III.7.71-72). From the $1\psi - 2V$ projection of the torsion-Bianchi one finds

$$\bar{\Sigma}^{abc}\psi \wedge V_b \wedge V_c + \bar{\theta}^{ab|}{}_c\psi \wedge V^c \wedge V_b + \frac{i}{2}\bar{\rho}_{bc}\Gamma_a\psi \wedge V^b \wedge V^c$$
$$+ \text{h.c.} = 0 \qquad \text{(III.7.82)}$$

where Σ_{abc} is a spinor defined by

$$\mathscr{D} F_+^{abc} = \mathscr{D}_\ell F_+^{abc} V^\ell + \bar{\Sigma}^{abc}\psi \,. \qquad \text{(III.7.83)}$$

Σ_{abc} can be determined in turn from the $3V - 1\psi$ projection of the Maxwell-Bianchi (III.7.46d):

$$(\bar{\Sigma}_{abc}\psi + \frac{i}{2}\bar{\rho}_{ab}\Gamma_c\psi + \text{h.c.}) \wedge V^a \wedge V^b \wedge V^c = 0 \qquad \text{(III.7.84)}$$

from which it follows:

$$\bar{\Sigma}_{abc} = -\frac{i}{2}\bar{\rho}_{[ab}\Gamma_{c]} \,. \qquad \text{(III.7.85)}$$

Inserting this result into (III.7.82) we find

$$\bar{\theta}^{ab}{}_c = -\frac{i}{2}(\bar{\rho}_{c[a}\Gamma_{b]} + \eta\sqrt{3}\,\bar{\rho}_{[bc}\Gamma_{a]}) \,. \qquad \text{(III.7.86)}$$

Finally the value of a_1 can be easily calculated from the $2\psi - 1V$ projection of (III.7.46b). After some spinor algebra one finds

$$a_1 = -\frac{3}{2}(1 + \eta\sqrt{3}) \,. \qquad \text{(III.7.87)}$$

The final result for the rheonomic parametrization of the curvature is the following

$$R^{ab} = R^{ab}{}_{mn} V^m \wedge V^n - [\frac{1}{2} i(\bar{\rho}^{c[a}\Gamma^{b]} + \eta\sqrt{3}\,\bar{\rho}^{[ca}\Gamma^{b]}) \wedge \psi +$$
$$+ \text{h.c}] \wedge V_c - \frac{3}{2} i(1 + \frac{\eta}{\sqrt{3}})\bar{\psi} \wedge \Gamma_c\psi\, F_+^{abc} \qquad \text{(III.7.88a)}$$

$$R^a = -\eta\sqrt{3}\, F_+^{abc}\, V_b \wedge V_c \qquad \text{(III.7.88b)}$$

$$R^{\otimes} = F_{+abc}\, V^a \wedge V^b \wedge V^c \qquad \text{(III.7.88c)}$$

$$\rho = \rho_{ab}\, V^a \wedge V^b + \frac{3}{4}\left(1 - \frac{\eta}{\sqrt{3}}\right) F_+^{abc}\, \Gamma_{ab}\psi \wedge V_c\ . \qquad \text{(III.7.88d)}$$

In conclusion, we have found that the equations of motion on the whole superspace imply the self-duality of F^{abc}: $F^{abc} = F^{abc}_{(+)}$. Vice-versa, the set of purely space-time equations of motion $[V \wedge \dots \wedge V$ projection of Eqs. (III.7.68-70)$]$:

$$R^{\ell q}{}_{mq}(\omega) - \frac{1}{2}\delta^{\ell}_{m} R^{pq}{}_{pq}(\omega) = 3(F_+^{\ell qr} F_{+mqr} + F_-^{\ell qr} F_{-mqr} -$$

$$- \frac{1}{3}\delta^{\ell}_{m} F_+^{pqr} F_{-pqr}) \qquad \text{(III.7.89a)}$$

$$\varepsilon_{pqabrs} R^{ab|rs}(\omega) + 8 F_+^{\ell m[p} F_-^{q]}{}_{\ell m} = 0 \qquad \text{(III.7.89b)}$$

$$\Gamma^{abc} \rho_{abc} = 0 \qquad \text{(III.7.89c)}$$

do not imply any restriction on $F^{abc} = F^{abc}_{(+)} + F^{abc}_{(-)}$; therefore, for a purely space-time observer, all the configurations described by (III.7.89) are on-shell, while for a superspace observer the shell is described by (III.7.89), plus the extra condition $F^{abc}_{(-)} = 0$.

Consequently, only the self-dual on-shell configurations can be lifted from space-time to the whole superspace through the Lie derivative lifting (or supersymmetry transformations), along the tangent vector $\bar{\varepsilon}\tilde{D}$ with $\tilde{D}$ = supersymmetry generator. This discrimination among configurations which, from the space-time point of view, are equally good on-shell states deprives the rheonomy mechanism of its very justification. Indeed all what this means is just that the space-time Eqs. (III.7.89) are not supersymmetric. The supersymmetric set of field equations is given by (III.7.89) $\oplus$ $F^-_{abc} = 0$.

If we want a good action functional what we have to do is to introduce a Lagrangian multiplier whose variation yields the missing field equation. Indeed, in the next section we show that the space-time restriction of the Lagrangian in (III.7.66) is not invariant under supersymmetry. We give here the supersymmetry transformations on the fields:

$$\ell_\varepsilon \omega^{ab} \equiv \delta_\varepsilon \omega^{ab} = -\frac{1}{2} i(\bar{\rho}^{c[a}\Gamma^{b]} + \eta\sqrt{3}\,\bar{\rho}^{[ca}\Gamma^{b]})\,\varepsilon V_c -$$

$$- \frac{3}{2} i\left(1 + \frac{\eta}{\sqrt{3}}\right)\bar{\psi}\,\Gamma_c\,\varepsilon F_+^{abc} + \text{h.c.} \qquad \text{(III.7.90a)}$$

$$\ell_\varepsilon V^a \equiv \delta_\varepsilon V^a = \frac{1}{2} i(\bar{\varepsilon}\Gamma^a\psi - \bar{\psi}\,\Gamma^a\varepsilon) \qquad \text{(III.7.90b)}$$

$$\ell_\varepsilon B \equiv \delta_\varepsilon B = \frac{1}{2} i(\bar{\varepsilon}\Gamma^a\psi - \bar{\psi}\,\Gamma^a\varepsilon) \wedge V_a \qquad \text{(III.7.90c)}$$

$$\ell_\varepsilon \psi \equiv \delta_\varepsilon \psi = D\varepsilon + \frac{3}{2}\left(1 - \frac{\eta}{\sqrt{3}}\right) F^+_{abc}\,\Gamma^{ab} V^c \varepsilon \qquad \text{(III.7.90d)}$$

obtained with the Lie derivative formula, using the rheonomic parametrization (III.7.88) and $F_-^{abc} = 0$.

III.7.5 - Non-invariance of the space-time action and how to cure it

In computing $\delta_\varepsilon \mathscr{A}$ we work in second order formalism, that is we assume Eqs. (III.7.71a) to hold.

Recalling that the space-time variation of the Lagrangian (III.7.66) coincides with the space-time restriction of the Lie derivative we begin to compute the exterior differential $d\mathscr{L}$. Trading d for $\mathscr{D}$ and using several times Eqs. (III.7.46) one finds:

$$- i d\mathscr{L} = (3 - \sqrt{3}\eta)(R^a \wedge V_a + R^{\otimes}) \wedge R^b \wedge \bar{\psi} \wedge \Gamma_b\psi +$$

$$+ \left[(3 + \eta\sqrt{3})R^{\otimes} + (3 - \eta\sqrt{3})R^a \wedge V_a\right] \wedge (\bar{\rho} \wedge \Gamma_b\psi -$$

$$- \bar{\psi} \wedge \Gamma_b\rho) \wedge V^b + 3R^a \wedge (\bar{\rho} \wedge \Gamma_{abc}\psi +$$

$$+ \bar{\psi} \wedge \Gamma_{abc}\rho) \wedge V^b \wedge V^c -$$

$$- 2 \bar{\rho} \wedge \Gamma_{abc}\rho \wedge V^a \wedge V^b \wedge V^c - 3 R^a \wedge R_a \wedge \bar{\psi} \wedge \Gamma_b\psi \wedge V^b +$$

$$+ 2 i \eta \sqrt{3} R^a \wedge R_a \wedge R^{\otimes} . \qquad \text{(III.7.91)}$$

In simplifying (III.7.91) one makes use of the torsion equation (III.7.71) and of the fact that all the terms with 3 or 4 ψ's are zero by the Fierz identities (III.7.17-18-19). Notice that the R^{ab}-curvature does not appear in (III.7.91).

Contracting with $\varepsilon \equiv \bar{\varepsilon}\tilde{D}$ and taking into account that the 2nd-order equations (III.7.71) contain only inner components, we obtain:

$$- i \underline{\varepsilon}\rfloor d\mathscr{L} = - (3 - \eta\sqrt{3})(R^a \wedge V_a + R^{\otimes}) \wedge R^b \bar{\varepsilon} \Gamma_b \wedge \psi -$$

$$- \left[(3 - \eta\sqrt{3})R^a \wedge V_a + \right.$$

$$\left. + (3 + \eta\sqrt{3})R^{\otimes}\right] \wedge V^b \wedge (\underline{\varepsilon}\rfloor\bar{\rho} \wedge \Gamma_b\psi +$$

$$+ \rho \wedge \Gamma_b\varepsilon) - 3 R^a \wedge R_a\bar{\varepsilon} \wedge \Gamma_b\psi \wedge V^b +$$

$$+ 3 R^a \wedge V^b \wedge V^c \wedge (\underline{\varepsilon}\rfloor\bar{\rho} \wedge \Gamma_{abc}\psi +$$

$$+ \bar{\rho} \Gamma_{abc}\varepsilon) - 2 \underline{\varepsilon}\rfloor\bar{\rho} \wedge \Gamma_{abc}\rho \wedge V^a \wedge V^b \wedge V^c + [\text{h.c.}] . \qquad \text{(III.7.92)}$$

This expression must be restricted to space-time M_6 if it is to represent the space-time variation $\delta\mathscr{L}|_{M_6}$: this means that the curvatures ρ and $\underline{\varepsilon}\rfloor\rho$ have to be restricted to their inner components only. Let us now observe that, since the Lie derivatives (III.7.90) are already an exact symmetry of the selfdual on-shell configurations, a possible symmetry of the action must differ from (III.7.90) by terms which are proportional to the space-time equations of motion namely Eqs. (III.7.89) and $F_-^{abc} = 0$).

In second order formalism the only possible change in Eqs. (III.7.90) can occur in Eq. (III.7.90d). Indeed δV^a and δB cannot change since $R^{\otimes}$ and R^a have just inner components, while $\delta\omega^{ab}$ is not independent being given in terms of δV^a, δB and $\delta\psi$ by the chain rule. This is confirmed by the explicit expression (III.7.92) where the only unknown object is $\underline{\varepsilon}\rfloor\rho$. Therefore we modify Eq. (III.7.90d) as follows:

$$\ell_\varepsilon \psi \equiv \mathscr{D}\varepsilon + \underline{\varepsilon}\rfloor\rho^{(0)} \rightarrow \mathscr{D}\varepsilon + \underline{\varepsilon}\rfloor\rho = \mathscr{D}\varepsilon + \underline{\varepsilon}\rfloor\rho^{(0)} + k\varepsilon \qquad \text{(III.7.93)}$$

where

$$\underline{\varepsilon}\rfloor\rho^{(0)} = \frac{3}{4}\left(1 - \frac{\eta}{\sqrt{3}}\right)F_+^{abc}\,\Gamma_{ab}\,V_c\varepsilon \qquad \text{(III.7.94)}$$

and k represents the modification from the on-shell to the off-shell transformation law of the gravitino. Taking into account the scaling properties and the fact that it must be a bosonic 1-form, we see that k can be only proportional to the l.h.s. of the selfduality equation. Therefore we write

$$k = \beta\, F_-^{abc}\,\Gamma_{ab}\,V_c \qquad \text{(III.7.95)}$$

where β is a parameter to be fixed by the requirement of action invariance. Let us now make the substitution

$$\underline{\varepsilon}\rfloor\rho = \underline{\varepsilon}\rfloor\rho^{(0)} + k\varepsilon =$$

$$= \left\{\frac{3}{4}\left(1 - \frac{\eta}{\sqrt{3}}\right)F_+^{abc} + \beta\, F_-^{abc}\right\}\Gamma_{ab}\,V_c\varepsilon \qquad \text{(III.7.96)}$$

into Eq. (III.7.91) and let us separately annihilate the terms proportional to $6 - V$'s and those proportional to $1\psi - 5V$'s. One finds the following two equations respectively:

$$\left[-(3 - \eta\sqrt{3})R^m \wedge V_m - (3 + \eta\sqrt{3})R^{\otimes}\right] \wedge \bar{\rho}^{(0)} \wedge \Gamma_a \varepsilon V^a +$$

$$+ \left[3 R^a \wedge V^b \wedge V^c \wedge \bar{\rho}^{(0)}\,\Gamma_{abc}\varepsilon - \right.$$

$$\left. - 2\,\bar{\varepsilon}\,\tilde{k}\,\Gamma_{abc}\,\rho^{(0)}\right] \wedge V^a \wedge V^b \wedge V^c + \text{h.c.} = 0 \qquad \text{(III.7.97a)}$$

$$[-\eta\sqrt{3}\,R^m \wedge V_m + (3-\eta\sqrt{3})R^{\otimes}] \wedge R^a \wedge \bar{\varepsilon}\,\Gamma_a\,\psi\, +$$
$$+ [(3-\eta\sqrt{3})R^m \wedge V_m + (3+\eta\sqrt{3})R^{\otimes}] \wedge V^a \wedge \bar{\varepsilon}\,\tilde{k}\,\Gamma_a\,\psi$$
$$+ 3\,R^a \wedge V^b \wedge V^c\,\bar{\varepsilon}\,\tilde{k}\,\Gamma_{abc}\,\psi + \text{h.c.} = 0 \qquad \text{(III.7.97b)}$$

where

$$\tilde{k} = \Gamma_0\,k^{\dagger}\,\Gamma_0 = k^{\dagger} = -\,k\,. \qquad \text{(III.7.98)}$$

Let us first consider Eq. (III.7.97a): using (III.7.71), the explicit form of k and remembering that the Eqs. (III.7.97) are restricted to space time so that:

$$\rho^{(0)}\big|_{M_6} = \rho^{(0)}{}_{ab}\,V^a \wedge V^b \qquad \text{(III.7.99)}$$

one finds that the annihilation of the terms proportional to the two structures

$$F_+^{apq}\,\bar{\rho}^{(0)}_{st}\,\Gamma_b\varepsilon \quad ; \quad F_-^{apq}\,\bar{\rho}^{(0)}_{st}\,\Gamma_b\varepsilon \qquad \text{(III.7.100)}$$

gives a single condition on the parameter β, namely:

$$\beta = \frac{\sqrt{3}}{4}\,\eta\,. \qquad \text{(III.7.101)}$$

With the choice (III.7.101) also the coefficients of the structures $F_{\pm}^{pqr}\,\bar{\rho}\,\Gamma_{abc}\varepsilon$ vanish identically.

Coming now to Eq. (III.7.97b) and performing the same substitutions as in (III.7.97a) one finds several terms containing the following different kinds of structures

$$F_+\,F_+\,\bar{\varepsilon}\,\Gamma_a\psi \quad ; \quad F_+\,F_-\,\bar{\varepsilon}\,\Gamma_{abc}\psi$$
$$F_+\,F_-\,\bar{\varepsilon}\,\Gamma_a\psi \quad ; \quad F_+\,F_-\,\bar{\varepsilon}\,\Gamma_{abc}\psi \qquad \text{(III.7.102)}$$
$$F_-\,F_-\,\bar{\varepsilon}\,\Gamma_a\psi \quad ; \quad F_+\,F_-\,\bar{\varepsilon}\,\Gamma_{abc}\psi$$

Let us concentrate on the cancellation of the $F_+\,F_+\,\bar{\varepsilon}\,\Gamma^a\psi$ and $F_-\,F_-\,\bar{\varepsilon}\,\Gamma_a\psi$ structures. After some lengthy tensor algebra one finds that they all cancel except for the two terms:

$$\text{const} \times \{F_+^{pqa}\,F_{+ijk} + F_-^{pqa}\,F_{-ijk}\}\,(\varepsilon^{dij}{}_{\ldots abc} + 6\,\delta^{ijd}_{abc}) \times$$
$$\times\,\varepsilon_{pqd\ldots}{}^{bkm}\,\bar{\varepsilon}\,\Gamma_m\psi\,. \qquad \text{(III.7.103)}$$

Using now the identities (III.5.36) and (III.5.38) one finds that while the term containing the purely self dual part vanishes identically the corresponding term containing the anti-self dual part does not.

Therefore the conclusion is that

$$\delta_{\varepsilon}\mathscr{A} \neq 0 \qquad \text{(III.7.104)}$$

as anticipated.

We leave to the reader to verify that the other purely self dual structures occurring in (III.7.97) also vanish identically: this implies that if we put $F_-^{abc} = 0$ the Lagrangian becomes invariant.

At this point it should be clear how an invariant Lagrangian for the above theory could be devised. It suffices to introduce a Lagrangian multiplier 0-form Λ_+^{abc} which is a self dual antisymmetric 3-tensor.

Adding to the action $\mathscr{A}$ of Eq. (III.7.66) the following term

$$\Delta\mathscr{A} = \int_{M_6} \Lambda_+^{abc}\,R^{\otimes} \wedge V_a \wedge V_b \wedge V_c \qquad \text{(III.7.105)}$$

we gain a new equation of motion, namely the variation δA_{+}^{abc} which yields

$$F_{-}^{abc} = 0 \qquad \text{(III.7.106)}$$

after projecting on six vielbeins. Devising a suitable supersymmetry transformation for A_{+}^{abc} one, can cancel the F_{-} terms which previously did not cancel and in this way the action $\mathcal{A}' = \mathcal{A} + \Delta\mathcal{A}$ becomes supersymmetric.

CHAPTER III.8

D=11 SUPERGRAVITY

III.8.1 - Introduction

Since the beginning of Supergravity it was realized that its framework naturally leads to the idea of a multidimensional space-time with $D = 4+n$ dimensions. This is so because the Lagrangian can be constructed only in certain dimensions and has specific properties depending on D: in particular various arguments, already advocated in Part II, indicate that only $D \leq 11$ is allowed. Therefore the D=11 case is of special interest since, in such a field theory, the number of space-time dimensions is not a "fitted" parameter, rather it has an intrinsic justification (it is the maximum one allowed by local supersymmetry). On the other hand higher space-time dimensions is not a new idea. Since the classical work of Kaluza-Klein* it is known that gravity on a higher dimensional manifold M_D which splits into

* Th. Kaluza, Sitzungsber. Prens. Akad. Wis. Berlin, Kl (1921) 966; O. Klein, Z. Phys. 37 (1926) 895.