$$\delta E_m{}^{\alpha}(x,y) = \delta A_m{}^I(x)\ K_I{}^{\alpha}(y) \qquad (V.3.49)$$

Equating (V.3.47) and (V.3.49), and factoring out $K_I{}^{\alpha}$, we obtain $\delta A_m{}^I(x)$:

$$\delta A_m{}^I(x) = \partial_m \varepsilon^I(x) + f^I{}_{JL}\varepsilon^J(x)\ A^L(x) \qquad (V.3.50)$$

i.e. the gauge variation of a non-abelian vector field.

The D = 4 action $S_{D=4}$, obtained after integrating out the G/H harmonics in the $M_4 \times M_K$ compactified action, is invariant under gauge transformations (V.3.50), and must contain therefore a Yang-Mills piece $F^{\mu\nu} F_{\mu\nu}$. This can be directly verified, as was done in the $M_4 \times S^1$ case, and we suggest it as a useful exercise. The relevant orthogonality relations are given in (V.3.22).

The mass of $A_m{}^I(x)$ is of course zero, since otherwise (V.3.50) would not be a symmetry of $S_{D=4}$.

CHAPTER V.4

COMPACTIFYING SOLUTIONS OF D = 11 SUPERGRAVITY

V.4.1 - The D = 4 vacuum: maximal symmetry

The D = 4 vacuum, in order to be a "true vacuum", should have maximal symmetry, i.e. invariance under a maximal set of translations and rotations. This implies, for (+---) D = 4 signature, invariance of the vacuum under SO(1,4), Poincaré, or SO(2,3), corresponding respectively to De Sitter (cosmological constant $\Lambda > 0$), Minkowski ($\Lambda = 0$) or anti De Sitter ($\Lambda < 0$) space. In coset space notation:

i) De Sitter spacetime $\qquad \dfrac{SO(1,4)}{SO(1,3)} \qquad$ (V.4.1a)

ii) Minkowski spacetime $\qquad \dfrac{ISO(1,3) = \text{Poincaré}}{SO(1,3)} \qquad$ (V.4.1b)

iii) anti De Sitter spacetime (AdS^4) $\qquad \dfrac{SO(2,3)}{SO(1,3)} \qquad$ (V.4.1c)

All three cases are important examples of non-compact coset manifolds. We should note, however, that only ii) and iii) admit a positive energy theorem and an S-matrix construction.

Let us examine the allowed vacuum expectation values of the $D=11$ S.G. fields satisfying $D=4$ maximal invariance. In the following, K_I denotes a Killing vector of any of the three groups SO(1,4), Poincaré, SO(2,3).

The $D=11$ vacuum metric must be invariant under the coordinate transformations generated by K_I

$$\delta x^\mu = x^\mu + K_I{}^\mu \qquad (V.4.2)$$

According to (I.6.115) the metric transforms as

$$\delta g_{\Lambda\Pi}(x,y) = \ell_{K_I} g_{\Lambda\Pi} = K_I{}^\mu \partial_\mu g_{\Lambda\Pi}(x,y) + (\partial_\Lambda K_I{}^\mu) g_{\mu\Pi} + + (\partial_\Pi K_I{}^\mu) g_{\mu\Lambda} \qquad (V.4.3)$$

The condition $\delta g_{\Lambda\Pi}(x,y)=0$ ($D=4$ maximal invariance) implies:(*)

$$g_{\mu\nu}(x,y) = g_{\mu\nu}(x)\; f(y) \qquad (V.4.4)$$

$$g_{\mu\alpha}(x,y) = 0 \qquad (V.4.5)$$

$$g_{\alpha\beta}(x,y) = g_{\alpha\beta}(y) \qquad (V.4.6)$$

where $g_{\mu\nu}(x)$ is the metric of (V.4.1); $g_{\mu\alpha}(x,y)$ must vanish, since there are no Poincaré or (anti) De Sitter invariant vectors.

(*) We recall the index convention:
$\mu,\nu,\ldots$: curved $D=4$ indices ; $\alpha,\beta,\ldots$: curved $D=7$ indices ;
$m,n,\ldots$: flat $D=4$ indices ; $a,b,\ldots$: flat $D=7$ indices .

Maximal invariance of the gravitino v.e.v. ψ_Λ implies

$$<\psi_\Lambda> = 0 \qquad (V.4.7)$$

since $<\psi_\Lambda> \neq 0$ would violate the SO(1,3) Lorentz isotropy subgroup of (V.4.1).

Finally, for the "photon" curl $F_{\Lambda_1\ldots\Lambda_4}(x,y)$ we have the condition

$$\ell_{K_I} F_{\Lambda_1\ldots\Lambda_4} = 0 \qquad (V.4.8)$$

with the unique solution:

$$F_{\mu\nu\rho\sigma} = \sqrt{g}\; \varepsilon_{\mu\nu\rho\sigma}\, h(y) \qquad (V.4.9)$$

$$F_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta}(y) \qquad (V.4.10)$$

all other components = 0

Moreover, the Bianchi identity $\partial_{[\alpha} F_{\mu\nu\rho\sigma]} = 0$ implies that $F_{\mu\nu\rho\sigma}$ is y-independent, so that $h(y) = \text{constant}$ in (V.4.9).

Summarizing, the nonvanishing components of the most general background fields with maximal $D=4$ symmetry are given by

$$g_{\mu\nu}(x,y) = g_{\mu\nu}(x)\; f(y) \qquad (V.4.11a)$$

$$g_{\alpha\beta}(y) \qquad (V.4.11b)$$

$$F_{\alpha\beta\gamma\delta}(y) \qquad (V.4.11c)$$

$$F_{\mu\nu\rho\sigma} = e\sqrt{g}\; \varepsilon_{\mu\nu\rho\sigma} \;,\quad e = \text{constant} \qquad (V.4.11d)$$

In flat indices, (V.4.11d) takes the form

$$F_{mnrs} = e\; \varepsilon_{mnrs} \qquad (V.4.12)$$

In what follows, the "warp factor" $f(y)$ will be taken equal to 1. For $f(y) \neq 1$, see Sect. V.9.4.

V.4.2 - $AdS^4 \times M^7$ solutions (Freund-Rubin)

Let us examine the D = 11 field Eqs. (with $\psi_\Lambda = 0$) for $g_{\Lambda\Pi}$, $A_{\Lambda\Pi\Sigma}$:

$$\text{(Einstein)} \quad R_{\Lambda\Pi} - \frac{1}{2} g_{\Lambda\Pi} R = 6\, F_{\Lambda\ldots} F_{\Pi}^{\cdots} - \frac{3}{4} g_{\Lambda\Pi} F_{\cdots\cdot} F^{\cdots\cdot} \qquad (V.4.13)$$

$$\text{(Maxwell)} \quad \mathcal{D}_{\bullet} F^{\bullet\Lambda\Pi\Sigma} = -\frac{1}{96} \varepsilon^{\Lambda\Pi\Sigma\ldots\cdot xxxx} F_{\ldots\cdot} F_{xxxx} \qquad (V.4.14)$$

(cfr. III.8.59 and III.8.53).

Does the maximally symmetric set of fields (V.4.11) satisfy these eqs.?(*) The answer is yes, provided (V.4.11) fulfills some additional conditions.

Separating four- and seven-dimensional indices, and inserting (V.4.11) in the Einstein eqs. yields

$$R_{\mu\nu} = g_{\mu\nu}(-24e^2 - \frac{1}{2} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta}) \qquad (V.4.15a)$$

$$R_{\alpha\beta} = g_{\alpha\beta}(12e^2 - \frac{1}{2} F_{\gamma\delta\epsilon\eta} F^{\gamma\delta\epsilon\eta}) + 6\, F_{\alpha\gamma\delta\varepsilon} F_{\beta}{}^{\gamma\delta\varepsilon} \qquad (V.4.15b)$$

$$R_{\mu\alpha} = 0 \qquad (V.4.15c)$$

Similarly, the Maxwell Eq. (V.4.14) becomes:

$$\mathcal{D}_m F^{mpqr} = 0 \qquad (V.4.16a)$$

$$\mathcal{D}_a F^{abcd} - \frac{1}{2} e\, \varepsilon^{bcdefgh} F_{efgh} = 0 \qquad (V.4.16b)$$

(*) A more practical form of (V.4.13) is

$$R_{\Lambda\Pi} = 6\, F_{\Lambda\ldots} F_{\Pi}^{\cdots} - 1/2\, g_{\Lambda\Pi} F_{\cdots\cdot} F^{\cdots\cdot}$$

Eqs. (V.4.15) - (V.4.16) are satisfied by the following set of curvatures (Freund-Rubin solution)

$$R_{\mu\nu} = -24e^2 g_{\mu\nu} \qquad (V.4.17a)$$

$$R_{\alpha\beta} = 12e^2 g_{\alpha\beta} \qquad (V.4.17b)$$

$$F_{mnrs} = e\, \varepsilon_{mnrs} \qquad (V.4.17c)$$

all other components vanishing.

Solutions with $F_{abcd} \neq 0$ (Englert-type solutions) will be discussed in Sect. V.9.2.

(V.4.17a) describes a four-dimensional Einstein spacetime with negative cosmological constant, identified with anti-De Sitter spacetime in order to have maximal symmetry.

(V.4.17b) corresponds to a seven-dimensional Einstein space M^7 with positive curvature and Euclidean signature (-------). These spaces are always compact, so that (V.4.17) can be rightly called a "spontaneous compactification" of D = 11 S.G. on $AdS^4 \times M^7$. The choice of dimension four for spacetime is here a consequence of the field eqs., and is essentially due to $F_{mnrs} = \partial_{[m} A_{nrs]}$ having rank four. The existence of A_{mnr} is, in turn, due to supersymmetry.

Notice that one could also set F_{abcd} proportional to ε_{abcd} and $F_{mnrs} = 0$, thus obtaining (D = 7 anti De Sitter) × (compact M^4). As Duff, Nilsson and Pope have pointed out, there could be a mathematical way to rule out this unphysical vacuum. They observe that in the 4+7 compactifications, F is set equal to the volume form on a noncompact M^4, and this is consistent with the equation $F = dA$ since d-dimensional noncompact spaces M^d have trivial cohomology group $H^d(M^d, \mathbb{R})$. In 7+4 compactifications, however, F is proportional to the volume of a 4-dimensional compact M_4, and $F = dA$ cannot be globally valid since $H^d(M^d, \mathbb{R})$ is non-trivial.

This argument in favour of the 4+7 Freund-Rubin solutions would not hold if there existed a dual formulation of D = 11 S.G., with a 6-form

$A_{\Lambda_1 \dots \Lambda_6}$ instead of the 3-form $A_{\Lambda\Pi\Sigma}$. If this were the case, F could be set proportional to the volume element of a non-compact D=7 anti De Sitter spacetime.

However, no such formulation does exist (see Chapter III.8).

In conclusion, there seems to be an asymmetry between 4+7 and 7+4 compactifications, with some evidence in favour of $AdS \times M^7$.

V.4.3 - Properties of the internal space M^7: Killing spinors and Weyl holonomy

Einstein spaces

The internal space M^7 must be an Einstein space, as required by Eq. (V.4.17b). Moreover, since we are interested in getting D = 4 gauge fields from the spontaneous compactification on M^7, we restrict our attention to 7-dimensional compact coset spaces G/H.

Every compact $M^7 = G/H$ admitting an Einstein metric corresponds to a solution

$$AdS \times G/H \qquad (V.4.18)$$

of D = 11 S.G. Classifying these solutions is equivalent to classifying all 7-dimensional compact homogeneous Einstein spaces. An exhaustive list is given in Table V.6.1.

The solutions (V.4.17) have the coset space structure

$$\frac{SO(2,3)}{SO(1,3)} \times \frac{G}{H} \qquad (V.4.19)$$

The isometries of these vacua are given by $SO(2,3) \times G$. Since we deal with supergravity, we can ask whether the vacuum (V.4.19) is invariant also under supersymmetry transformations.

First of all, we have to discuss what we mean by a supersymmetric vacuum, and how N = 1, D = 11 supersymmetry is related to D = 4 supersymmetry.

The D = 11 supersymmetry transformations (Table III.8.1), applied on the v.e.v. of the fields

$V_\Lambda^{\ A}$: vielbein of $AdS \times G/H$

$\psi_\Lambda = 0$

$$F_{\Lambda\Pi\Sigma\Omega} = \begin{cases} F_{mnpq} = e\, \varepsilon_{mnpq} \\ F_{abcd} = 0 \\ \text{all other components} = 0 \end{cases} \qquad (V.4.20)$$

will preserve the vacuum structure (V.4.20) if and only if

$$\delta_{SUP}\, \psi_\Lambda = \bar{D}_\Lambda \varepsilon = 0$$

$$\bar{D}_\Lambda = d_\Lambda - \frac{1}{4} B_\Lambda^{\ AB} \Gamma_{AB} - \frac{i}{3} (\Gamma^{ABC}\, V_\Lambda^{\ D} + \frac{1}{8} \Gamma^{ABCDM}\, V_\Lambda^{\ M}) F_{ABCD}$$

(cfr. Table III.8.1, or (V.5.48c)). (V.4.21)

The variations $\delta_{SUP}\, V_\Lambda^{\ A}$, $\delta_{SUP}\, A_{\Lambda\Pi\Sigma}$ are automatically equal to zero for $\psi_\Lambda = 0$.

Consider now Eq. (V.4.21) in the $AdS \times M^7$ background (V.4.20). Splitting the AdS and M^7 indices, and using the D = 11 Γ matrices(*)

$$\Gamma_M = (\gamma_m \otimes \mathbb{1},\ \gamma_5 \otimes \Gamma_a)\ ; \quad \begin{matrix} \gamma_m\text{: D = 4 }\Gamma\text{-matrices} \\ \Gamma_a\text{: D = 7 }\Gamma\text{-matrices} \end{matrix} \qquad (V.4.22)$$

we obtain:

$$\delta_{SUP}\, \psi_\mu(x,y) = (\partial_\mu - \frac{1}{4} B_\mu^{\ mn}\, \gamma_{mn} + 2e\, \gamma_5 \gamma_m\, V_\mu^{\ m})\, \varepsilon(x,y) = 0 \qquad (V.4.23a)$$

$$\delta_{SUP}\, \psi_\alpha(x,y) = (\partial_\alpha - \frac{1}{4} B_\alpha^{\ ab}\, \Gamma_{ab} - e\, \Gamma_a\, V_\alpha^{\ a})\, \varepsilon(x,y) = 0 \qquad (V.4.23b)$$

(*) For notational economy, we use the same symbol Γ for D = 11 and D = 7 Γ-matrices, usually distinguished by their indices. When there is possibility of confusion, we use the symbol $\mathbf{\Gamma}$ for D = 11 Γ-matrices. For ex: $\mathbf{\Gamma}_a = \gamma_5 \Gamma_a$.

where $B_\mu{}^{mn}(x)$ and $B_\alpha{}^{ab}(y)$ are the AdS and M^7 spin connections. ψ and ε can both be expanded in M^7-harmonics:

$$\psi_\mu{}^{\bar{m}\bar{a}}(x,y) = \sum \psi_\mu{}^{\bar{m}I}(x)\ D^{I\bar{a}}(y) \qquad \bar{m} : D=4 \text{ spinor index}$$

$$\psi_\alpha{}^{\bar{m}\bar{a}}(x,y) = \sum \psi_\alpha{}^{\bar{m}I}(x)\ D^{I\bar{a}}(y) \qquad \bar{a} : D=7 \text{ spinor index}$$

$$\varepsilon^{\bar{m}\bar{a}}(x,y) = \sum \varepsilon^{\bar{m}I}(x)\ D^{I\bar{a}}(y) \qquad I : G\text{-irrep index} \tag{V.4.24}$$

We have explicitly written the $D=11$ spinor index as a product of $D=4$ and $D=7$ spinor indices $\bar{m}$ and $\bar{a}$ respectively. The sums are over all the G-irreps containing H-irreducible pieces of the SO(7) index a (see Sect. V.3). For notational convenience, we write the generic term in the sums (V.4.24) as

$$\psi_\mu(x)\ \eta(y)$$

$$\psi_\alpha(x)\ \eta(y)$$

$$\varepsilon(x)\ \eta(y) \tag{V.4.25}$$

and Eqs. (V.4.23) take the form:

$$\bar{D}_m\varepsilon(x) \equiv (\partial_m - \frac{1}{4}B_m{}^{rs}\gamma_{rs} + 2e\ \gamma_5\gamma_m)\varepsilon(x) = 0 \tag{V.4.26a}$$

$$\bar{D}_a\eta(y) \equiv (\partial_a - \frac{1}{4}B_a{}^{cd}\Gamma_{cd} - e\ \Gamma_a)\eta(y) = 0 \tag{V.4.26b}$$

The integrability conditions for (V.4.25) are:

$$\bar{D}^2\varepsilon(x) = [-\frac{1}{4}R^{rs}\gamma_{rs} - 4e^2\ \gamma_{rs}V^r \wedge V^s]\varepsilon(x) = 0 \tag{V.4.27a}$$

$$\bar{D}^2\eta(y) = [-\frac{1}{4}R^{ab}\Gamma_{ab} - e^2\ \Gamma_{ab}\ E^a \wedge E^b]\eta(y) = 0 \tag{V.4.27b}$$

The first is identically satisfied for the AdS $\times M^7$ solution of Eqs. (V.4.20), since the AdS curvature is given by

$$R^{rs}{}_{mn} = -16e^2\ \delta^{rs}_{mn} \tag{V.4.28}$$

Thus we only need to concentrate on Eq. (V.4.27b).

The existence of N independent solutions $\eta(y)$ of (V.4.26b) implies N surviving $D=4$ supersymmetries in the AdS $\times M^7$ background.

A necessary condition for (V.4.26b) to hold is given by Eq. (V.4.27b), or

$$C_{ab}\eta \equiv (R^{cd}{}_{ab} - 4e^2\ \delta^{cd}_{ab})\Gamma_{cd}\eta = 0 \tag{V.4.29}$$

The C_{ab} operator defined above contains linear combinations of SO(7) generators Γ_{cd}. If C_{ab} admits N_{MAX} null eigenspinors, it generates a subgroup of SO(7). This subgroup is called the Weyl holonomy group $\mathcal{H}$ of M^7, and is the subgroup of SO(7) which leaves n spinors invariant. In other words, N_{MAX} is equal to the number of singlets appearing in the decomposition of SO(7) under $\mathcal{H}$. From Table V.4.1, where the possible holonomy groups $\mathcal{H}$ and branching rules $SO(7) \rightarrow \mathcal{H}$ are given, we see that N_{MAX} can take the values $N_{MAX} = 0,1,2,4,8$.

We stress that C_{ab} having N_{MAX} null eigenspinors is a necessary but not a sufficient condition for the existence of N_{MAX} solutions to Eq. (V.4.26), i.e. for $N = N_{MAX}$ supersymmetry. The null eigenspinors of (V.4.29) still have to satisfy the first order Eq. (V.4.26b): the number of $D=4$ supersymmetries may be smaller than the maximal N allowed by holonomy.

As discussed in Part III, a supersymmetric gravity theory can be formulated on a supergroup manifold, or in superspace (a super coset manifold). In the case of AdS $\times$ G/H compactifications, the coset structure of a vacuum preserving N supersymmetries can be extended to a super coset space as follows:

$$\frac{SO(2,3)}{SO(1,3)} \times \frac{G}{H} \rightarrow \frac{Osp(4/N) \times G'}{SO(1,3) \times H} \tag{V.4.30}$$

where

$$G = SO(N) \times G' \qquad (V.4.31)$$

Let us justify (V.4.30) and (V.4.31). The rationale of superspace or supergroup manifold formalism is that it allows a geometric interpretation of supersymmetry, seen as part of the superisometry group. Local supersymmetry transformations are just diffeomorphisms in the θ-directions.

Thus a solution of $D=11$ supergravity preserves N-supersymmetry if its extension to superspace (obtained by integration of the rheonomic conditions, see Chapter III.3), admits N fermionic Killing vectors q, or Killing spinors.

The algebra of Killing vectors and Killing spinors closes on a super-Lie algebra. Since q transforms as a Majorana spinor under $SO(2,3) \approx Sp(4)$ (see Chapter II.2), this superalgebra must contain $Osp(4/N)$, the superextension of the bosonic group $Sp(4) \times SO(N)$ (see Sect. V.7.3).

On the other hand the bosonic symmetries of the $AdS \times G/H$ vacuum are given by $SO(2,3) \times G$. Hence, for an N-supersymmetric vacuum, G must have the form $G = G' \times SO(N)$, the $SO(N)$ subgroup being necessary for the superextension to $Osp(4/N) \times G'$.

The Killing spinors q are fermionic tangent vectors to the superspace $M^{4+7/32}$ (the superspace of $D=11$, $N=1$ S.G. has a 32-component Grassman coordinate θ.), so that $q = q(x,y,\theta)$.

To say that $q(x,y,\theta)$ is a fermionic Killing vector of $M^{4+7/32}$ translates into the equations:

$$\ell_q \psi = \frac{1}{4} W^{AB} \Gamma_{AB} \psi \qquad (V.4.32)$$

$$\ell_q V^A = W^A{}_B V^B \qquad (V.4.33)$$

These eqs. are just the supersymmetric extension of Eq. (I.6.119): we have now a supervielbein given by $(V^A(x,y,\theta),\ \psi(x,y,\theta))$ and (V.4.32-V.4.33) are its transformation laws under diffeomorphisms in the θ-directions.

Evaluating (V.4.32) at $\theta = 0$ yields $\ell_q \psi|_{\theta=0} = \delta_{SUP} \psi = 0$, since $\psi(\theta=0) = 0$ and the Lie derivative on ψ gives its supersymmetry variation (see Part III).

Thus the Killing spinors $q(x,y,\theta=0)$ can be interpreted as the supersymmetry parameters $\varepsilon(x)\ \eta(y)$ of (V.4.25). The existence of N Killing spinors, i.e. of an N-supersymmetric background, is equivalent to the existence of N independent solutions of Eqs. (V.4.26), in agreement with our previous results.

Summarizing: the condition $\delta_{SUP} \psi = 0$ for a supersymmetric background can be retrieved from a geometrical point of view by considering the fermionic Killing equation in superspace. The existence of fermionic isometries (i.e. supersymmetries) implies $\delta_{SUP} \psi = 0$ at $\theta = 0$.

V.4.4 - Osp(4/N) formulation

Here we study the $AdS \times M^7$ solutions of $D=11$ supergravity from a more geometrical point of view that elucidates the

$$\frac{Osp(4/N) \times G'}{SO(1,3) \times H} \qquad (V.4.34)$$

structure, already discussed in the previous section in the context of supersymmetry.

We recall the definitions of the $D=11$ curvatures

$$R^{AB} = d\omega^{AB} - \omega^{AC} \wedge \omega_C{}^B \qquad (V.4.35a)$$

$$R^A = dV^A - \omega^A{}_B \wedge V^B - \frac{i}{2} \bar{\psi} \wedge \Gamma^A \psi \qquad (V.4.35b)$$

$$\rho = d\psi - \frac{1}{4} \omega^{AB} \wedge \Gamma_{AB} \psi \qquad (V.4.35c)$$

$$R = dA - \frac{1}{2}\bar{\psi} \wedge \Gamma_{AB}\,\psi \wedge V^A \wedge V^B \qquad \text{(V.4.35d)}$$

the rheonomic equations

$$R^A = 0 \qquad \text{(V.4.36a)}$$

$$R = F_{A_1 \ldots A_4}\, V^{A_1} \wedge \ldots \wedge V^{A_4} \qquad \text{(V.4.36b)}$$

$$\rho = \rho_{AB} V^A \wedge V^B - \frac{i}{3}\left(\Gamma^{A_1 \ldots A_3}\psi \wedge V^{A_4} + \frac{1}{8}\Gamma^{A_1 \ldots A_4 B}\psi \wedge V^B\right) F_{A_1 \ldots A_4} \qquad \text{(V.4.36c)}$$

$$R^{AB} = R^{AB}{}_{CD} V^C \wedge V^D + i\,\rho_{CD}\left(\frac{1}{2}\Gamma^{ABCDE} - \frac{2}{9}\Gamma^{CD[A}\,\delta^{B]E} + 2\,\Gamma^{AB[C}\,\delta^{D]E}\right)\psi \wedge V_E + \bar{\psi}\,\Gamma_{CD}\,\psi\, F^{ABCD} + \frac{1}{24}\bar{\psi}\,\Gamma^{ABC_1 - C_4}\psi\, F_{C_1 - C_4} \qquad \text{(V.4.36d)}$$

and the inner equations

$$\Gamma^{ABC}\rho_{BC} = 0 \qquad \text{(V.4.37a)}$$

$$\mathscr{D}_A F^{AC_1 \ldots C_3} + \frac{1}{96}\varepsilon^{C_1 \ldots C_3 A_1 \ldots A_8} F_{A_1 \ldots A_4} F_{A_5 \ldots A_8} = 0 \qquad \text{(V.4.37b)}$$

$$R^{AC}{}_{BC} = 6\, F^{AC_1 \ldots C_3} F_{BC_1 \ldots C_3} - \frac{1}{2}\delta^A_B F^{C_1 \ldots C_4} F_{C_1 \ldots C_4} \qquad \text{(V.4.37c)}$$

Splitting $D = 11$ indices in 4+7 $(A \to (m,a))$ indices, using the familiar basis (V.4.22) for Γ matrices, we now assume

$$\omega^{ma} = 0\ ,\quad \rho_{AB} = 0\ ,\quad F_{m_1 \ldots m_4} = e\,\varepsilon_{m_1 \ldots m_4}$$

$$F = 0 \ \text{ otherwise} \qquad \text{(V.4.38)}$$

and obtain for the rheonomic equations:

$$d\omega^{mn} - \omega^m{}_r \wedge \omega^{rn} = -16e^2\, V^m \wedge V^n + 2ie\,\bar{\psi}\,\gamma_5\gamma^{mn}\,\psi \qquad \text{(V.4.39a)}$$

$$dV^m - \omega^m{}_n \wedge V^n = \frac{i}{2}\bar{\psi} \wedge \gamma^m\,\psi \qquad \text{(V.4.39b)}$$

$$dV^a - \omega^a{}_b \wedge V^b = \frac{i}{2}\bar{\psi} \wedge \gamma_5\,\Gamma^a\,\psi \qquad \text{(V.4.39c)}$$

$$d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} = R^{ab}{}_{cd} \wedge V^c \wedge V^d - i\,\bar{\psi} \wedge \gamma_5\,\Gamma^{ab}\,\psi \qquad \text{(V.4.39d)}$$

$$d\psi - \frac{1}{4}\omega^{mn}\gamma_{mn} \wedge \psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\,\psi = e\,V^a\Gamma_a \wedge \psi + 2e\,V^m \wedge \gamma_5\gamma_m\,\psi \qquad \text{(V.4.39e)}$$

$$dA = \frac{1}{2}\bar{\psi}\,\gamma_{mn} \wedge \psi \wedge V^m \wedge V^n + \bar{\psi}\,\gamma_5\gamma_m\,\Gamma_a\,\psi \wedge V^m \wedge V^a - \frac{1}{2}\bar{\psi}\,\Gamma_{ab} \wedge \psi \wedge V^a \wedge V^b \qquad \text{(V.4.39f)}$$

The inner equations become

$$R^{mr}{}_{nr} = -24e^2\,\delta^m_n \qquad \text{(V.4.40a)}$$

$$R^{ac}{}_{bc} = 12e^2\,\delta^a_b \qquad \text{(V.4.40b)}$$

Suppose now that:

i) there are N orthonormal commuting SO(7) real spinors $\eta_M(y)$ which satisfy the equation (V.4.26b):

$$\mathscr{D}^{SO(7)}\,\eta_M \equiv (d - \frac{1}{4}\,\omega^{ab}\,\Gamma_{ab})\eta_M = e\,V^a\Gamma_a\,\eta_M \qquad (V.4.41a)$$

$$\bar{\eta}_M\eta_N = \delta_{MN} \qquad (V.4.41b)$$

ii) $\mathring{V}^m(x,\theta)$, $\mathring{\omega}^{mn}(x,\theta)$, $\mathring{A}^{MN}(x,\theta)$, $\mathring{\psi}_M(x,\theta)$ are the Osp(4/N) left-invariant one-forms on the coset space

$$M^{4/4N} = \frac{Osp(4/N)}{SO(1,3)\times SO(N)} \qquad (V.4.42)$$

satisfying, by definition, the Osp(4/N) Maurer-Cartan equations:

$$d\mathring{V}^m - \mathring{\omega}^m{}_n \wedge \mathring{V}^n - \frac{i}{2}\,\bar{\mathring{\psi}}_M \wedge \gamma^m\,\mathring{\psi}_M = 0 \qquad (V.4.43a)$$

$$d\mathring{\omega}^{mn} - \mathring{\omega}^m{}_r \wedge \mathring{\omega}^{rn} + 16e^2\,\mathring{V}^m \wedge \mathring{V}^n - 2i\,\bar{\mathring{\psi}}_M \wedge \gamma_5\gamma^{mn}\,\mathring{\psi}_M = 0 \qquad (V.4.43b)$$

$$d\mathring{A}^{MN} + e\,\mathring{A}^{MR} \wedge \mathring{A}^{RN} - 4i\,\bar{\mathring{\psi}}_M \wedge \gamma_5\,\mathring{\psi}_N = 0 \qquad (V.4.43c)$$

$$d\mathring{\psi}_M - \frac{1}{4}\,\gamma^{mn}\,\mathring{\omega}_{mn} \wedge \mathring{\psi}_M - 2e\,\mathring{V}^m \wedge \gamma_5\gamma_m\,\mathring{\psi}_M - e\,\mathring{A}^{MN} \wedge \mathring{\psi}_N = 0 \qquad (V.4.43d)$$

Then we have the

<u>Theorem</u>: a solution of the superspace field equations (V.4.39) is given by the potentials:

$$V^m = \mathring{V}^m(x,\theta) \qquad (V.4.44a)$$

$$\omega^{mn} = \mathring{\omega}^{mn}(x,\theta) \qquad (V.4.44b)$$

$$\psi = \eta_M(y)\,\mathring{\psi}_M(x,\theta) \qquad (V.4.44c)$$

$$V^a = V^a(y) + \frac{1}{8}\,\bar{\eta}_M(y)\,\Gamma^a\eta_N(y)\,\mathring{A}^{MN}(x,\theta) \qquad (V.4.44d)$$

$$\omega^{ab} = \omega^{ab}(y) - \frac{e}{4}\,\bar{\eta}_M(y)\,\Gamma^{ab}\eta_N(y)\,\mathring{A}^{MN}(x,\theta) \qquad (V.4.44e)$$

$$\omega^{ma} = 0 \qquad (V.4.44f)$$

$A = \mathring{A}(x,y,\theta)$: 3-form satisfying Eq. (V.4.45) (V.4.44g)

$$d\mathring{A} = e\,\varepsilon_{mnrp}\,V^m \wedge V^n \wedge V^r \wedge V^p + \frac{1}{2}\,\bar{\mathring{\psi}}_M \wedge \gamma^{mn}\,\mathring{\psi}_M\,V^m \wedge V^n =$$
$$= \bar{\mathring{\psi}}_M \wedge \gamma_5\gamma_m\,\mathring{\psi}_N \wedge V^m\bar{\eta}_M\Gamma_a\eta_N \wedge V^a + \frac{1}{2}\,\bar{\mathring{\psi}}_M \wedge \mathring{\psi}_N\,\bar{\eta}_M\Gamma^{ab}\eta_N \wedge V^a \wedge V^b\,. \qquad (V.4.45)$$

This solution lives in the superspace:

$$M^{4+7/4N} \sim \frac{Osp(4/N)\times G'}{SO(1,3)\times H} \underset{(\text{locally})}{\sim} \underbrace{\frac{Osp(4/N)}{SO(1,3)\times SO(N)}}_{\text{N-extended anti-De Sitter superspace}}\ \underbrace{\frac{G}{H}}_{M^7} \qquad (V.4.46)$$

(recall $G = SO(N)\times G'$)

<u>Proof</u>: insert (V.4.44) into (V.4.39) and use the identities:

$$\bar{\eta}_M\Gamma^{ab}\eta_N\bar{\eta}_R\,\Gamma^b\eta_S A^{MN} \wedge A^{RS} = 4\,\bar{\eta}_M\Gamma^a\eta_S A^{MR} \wedge A^{RS} \qquad (V.4.47a)$$

$$(\eta_M\delta_{NR})A^{MN} = (\frac{1}{16}\,\Gamma_{ab}\eta_R\bar{\eta}_M\Gamma^{ab}\eta_N - \frac{1}{8}\,\Gamma_a\eta_R\bar{\eta}_M\Gamma^a\eta_N)A^{MN} \qquad (V.4.47b)$$

$$\bar{\eta}_M\Gamma^{ac}\eta_N\bar{\eta}_R\Gamma_c{}^b\eta_S\,A^{MN} \wedge A^{RS} =$$
$$= 4\,\bar{\eta}_M\Gamma^{ab}\eta_N\,A^{MR} \wedge A^{RN} - \frac{1}{2}\,\bar{\eta}_M\Gamma^a\eta_N\bar{\eta}_R\Gamma^b\eta_S\,A^{MN} \wedge A^{RS} \qquad (V.4.47c)$$

and

$$R^{ab}{}_{cd}(y)V^c\eta_M\Gamma^d\eta_N \wedge A^{MN} = 4e^2\,V^{[a}\,\bar{\eta}_M\,\Gamma^{b]}\eta_N \wedge A^{MN} \qquad (V.4.48a)$$

$$R^{ab}{}_{cd}\,\bar{\eta}_R\Gamma^c\eta_S\bar{\eta}_M\Gamma^d\eta_N\,A^{RS} \wedge A^{MN} = 2e^2\,\bar{\eta}_R\Gamma^a\eta_S\bar{\eta}_M\,\Gamma^b\eta_N\,A^{RS} \wedge A^{MN} \qquad (V.4.48b)$$

Eqs. (V.4.47) are derived by a straightforward Fierz rearrangement. Eqs. (V.4.48) are a consequence of the identity

$$R^{ab}{}_{cd}\,\bar{\eta}_{[M}\,\Gamma^{d}\eta_{N]} = 4e^2\,\delta_c^{[a}\,\bar{\eta}_{[M}\,\Gamma^{b]}\eta_{N]} \tag{V.4.49}$$

obtained from the consistency condition of Eq. (V.4.41a):

$$-\frac{1}{4}R^{cd}{}_{ab}\,\Gamma_{cd}\eta_N = -\frac{1}{4}R^{ab}{}_{cd}\,\Gamma^{cd}\eta_N = e^2\,\Gamma^{ab}\eta_N \tag{V.4.50}$$

multiplying by $\bar{\eta}_M\Gamma^c$ and antisymmetrizing in $M \leftrightarrow N$. The 3-form $\overset{\circ}{A}$ has not been written explicitly. There is no need for that because, once Eqs. (V.4.39a-e) are satisfied, Eq. (V.4.45) is an integrable equation and therefore certainly admits a solution.

V.4.4 - Differential operators on M^7

In the linearization of the $D=11$ field equations we will encounter various invariant differential operators on M^7. The masses of the $D=4$ spacetime fields are given in terms of eigenvalues of these invariant operators.

In this section we discuss some of their general properties, such as zero modes or lower bounds on their eigenvalues. Most of the following considerations are independent of the actual choice for the internal manifold M^7.

The relevant operators are:

- the Dirac and Rarita-Schwinger operators for the fermions.
- the Hodge-de Rahm operator acting on scalars, vectors, and antisymmetric tensors of rank ≤ 3.
- the Lichnerowicz operator acting on symmetric rank - two tensors.

These differential operators are invariant, since all of them can be expressed in terms of $\not{\nabla} = \Gamma^a\nabla_a$, the covariant Dirac operator in $D=7$, itself an invariant operator ($\not{\nabla}$ commutes with the covariant Lie derivative, see later).

Dirac and Rarita-Schwinger operators

The relevant SO(7) irreps for the fermions (coming from the $D=11$ Majorana gravitino $\psi_A(x,y)$) are the $[1/2,1/2,1/2]$ and $[3/2,1/2,1/2]$. The first is associated with the $[3/2,1/2]$ of SO(1,3) and hence with the gravitino spectrum, while the second is associated with the $[1/2,1/2]$ of SO(1,3) and hence with the spin 1/2 spectrum.

A $[1/2,1/2,1/2]$ real irrep of SO(7) is an 8-component Majorana spinor η:

$$[1/2,1/2,1/2]:\ \eta = C_{(7)}\,\eta^* \tag{V.4.51}$$

$$C_{(7)} = (D=7)\ \text{charge conjugation matrix}$$

while a $[3/2,1/2,1/2]$ irrep is a Majorana spinor-vector ξ_a satisfying the trace condition $\Gamma^a\xi_a = 0$:

$$[3/2,1/2,1/2]:\ \xi_a = C_{(7)}\,\xi_a^* \quad ; \quad \Gamma^a\xi_a = 0 \tag{V.4.52}$$

See Appendix (V.4.1) for an explicit representation of $D=7$ gamma matrices.

The SO(7) and SO(8) covariant derivatives are respectively defined by

$$\text{SO(7)}:\ \mathcal{D}_a\eta = \partial_a\eta - \frac{1}{4}\omega^{ab}\,\Gamma_{ab}\,\eta \tag{V.4.53a}$$

$$\mathcal{D}_a\xi_b = \partial_a\xi_b - \omega_a{}^c\,\xi_c - \frac{1}{4}\omega_a{}^{cd}\,\Gamma_{cd}\,\xi_b \tag{V.4.53b}$$

$$\text{SO(8)}:\ \nabla_a\eta = \mathcal{D}_a\eta - e\,\Gamma_a\,\eta \tag{V.4.54a}$$

$$\nabla_a\xi_b = \mathcal{D}_a\xi_b - e\,\Gamma_a\,\xi_b \tag{V.4.54b}$$

The invariant operators arising in the linearization of the gravitino field equation are the $D=7$ Dirac operator and the $D=7$ Rarita-Schwinger operator:

$$\nabla\!\!\!/ \equiv \Gamma^a \nabla_a \tag{V.4.55}$$

$$(RS)_a{}^b \equiv \Gamma_a{}^{bc} \nabla_c \tag{V.4.56}$$

$\nabla\!\!\!/$ maps $[1/2,1/2,1/2]$ fields into $[1/2,1/2,1/2]$ fields, while the Rarita-Schwinger operator $(RS)_a{}^b$ maps spinor-vectors into spinor-vectors

$$(RS)_a{}^b \xi_b = \xi'_a = \Gamma_a{}^{cb} \nabla_c \xi_b \tag{V.4.57}$$

$(RS)_a{}^b$, however, does not map $[3/2,1/2,1/2]$ irreps into $[3/2,1/2,1/2]$ irreps, since $\Gamma^a \xi'_a \neq 0$.

This suggests a decomposition of the RS operator into two parts, $\mathcal{R}_a{}^b$ and S^b:

$$(RS)_a{}^b = \mathcal{R}_a{}^b + \Gamma_a S^b \tag{V.4.58}$$

where

$$S^b = \frac{1}{7} \Gamma^a (RS)_a{}^b \tag{V.4.59a}$$

$$\mathcal{R}_a{}^b = (RS)_a{}^b - \frac{1}{7} \Gamma_a \Gamma^c (RS)_c{}^b \tag{V.4.59b}$$

$\mathcal{R}_a{}^b$ now maps $[3/2,1/2,1/2]$ into $[3/2,1/2,1/2]$, and by straightforward algebra we obtain:

$$\mathcal{R}_a{}^b \xi_b = \nabla\!\!\!/ \xi_a + 2e\, \xi_a - \frac{2}{7} \Gamma_a \nabla \cdot \xi \tag{V.4.60}$$

$$S^a \xi_a = -\frac{5}{7} \nabla \cdot \xi \tag{V.4.61}$$

with $\nabla \cdot \xi = \nabla_a \xi^a$.

The operators $\mathcal{R}_a{}^b$, S^b, ∇ and D are easily shown to be invariant, i.e. to commute with the covariant Lie derivative L_A

$$L_A = \ell_A + W_A{}^i C_i{}^{ab} t_{ab} \equiv \ell_A + W_A \tag{V.4.62}$$

with $t_{ab} = SO(7)$ generators in the relevant SO(7) irreps. [Cfr. (I.6.124): in the vector SO(7) irrep $(t_{ab})^{cd} = -\delta^{cd}_{ab}$].

For the $\mathcal{D}$ operator

$$\mathcal{D} = d + \omega^{ab} t_{ab} \equiv d + \omega \tag{V.4.63}$$

the commutator $[L_A, \mathcal{D}]$ is vanishing as an immediate consequence of the equation

$$\ell_A\, \omega = dW_A + [\omega, W_A] \tag{V.4.64}$$

which in turn can be derived by applying ℓ_A to the definition of the torsion-less connection ω^{ab} (I.6.155)

<u>Exercise</u>: prove that $[L_A, \mathcal{R}_a{}^b] = [L_A, S^b] = [L_A, \nabla] = 0$.

According to the discussion in Chapter V.3, we can expand a generic $[3/2,1/2,1/2]$ field $\xi_a(x,y)$ in $[3/2,1/2,1/2]$ - SO(7) harmonics on G/H (cfr. (V.3.38)):

$$\xi_a(x,y) = \sum_I \lambda^I(x)\, \Xi_a{}^{I[3/2,1/2,1/2]}(y) \tag{V.4.65}$$

I: runs on G-irreps containing H-irreducible pieces of $[3/2,1/2,1/2]$

For fermionic SO(7) harmonics we use the notation $\Xi(y)$ instead of $Y(y)$. I is a composite index, labelling the G-irreps together with their indices, i.e. $I = [(\nu),n,\xi]$ of Eq. (V.3.38); the SO(7) spinor index is omitted.

The $[3/2,1/2,1/2]$ - harmonics $\Xi_a^{I[3/2,1/2,1/2]}(y)$ are eigenfunctions of the (irreducible) invariant Rarita-Schwinger operator $\mathcal{R}_a^{\ b}$:

$$\mathcal{R}_a^{\ b}\,\Xi_b^{I[3/2,1/2,1/2]}(y) = M_{3/2}^{\ I}\,\Xi_a^{I[3/2,1/2,1/2]}(y) \qquad (V.4.66)$$

Similarly, a generic $[1/2,1/2,1/2]$ - field $\eta(x,y)$ can be expanded as

$$\eta(x,y) = \sum_I \chi^I(x)\,\Xi^{I[1/2,1/2,1/2]}(y) \qquad (V.4.67)$$

where the harmonics $\Xi^{I[1/2,1/2,1/2]}(y)$ are eigenfunctions of the Dirac operator $\not\nabla$:

$$\not\nabla\,\Xi^{I[1/2,1/2,1/2]}(y) = M_{1/2}\,\Xi^{I[1/2,1/2,1/2]}(y) \qquad (V.4.68)$$

Now we observe that for each $\Xi^{I[1/2,1/2,1/2]}$ appearing in the expansion (V.4.67) it is possible to construct a $[3/2,1/2,1/2]$ - harmonic $\Xi_a^{I[3/2,1/2,1/2]}$ contributing to the expansion (V.4.65):

$$\Xi_a^{I[3/2,1/2,1/2]} = \nabla_a\,\Xi^{I[1/2,1/2,1/2]} - \frac{1}{7}\,\Gamma_a\Gamma^b\nabla_b\,\Xi^{I[1/2,1/2,1/2]}$$
$$= (\nabla_a\,\Xi^{I[1/2,1/2,1/2]})_{3/2} \qquad (V.4.69)$$

where $(\ \ )_{3/2}$ denotes the $[3/2,1/2,1/2]$ irreducible part of the spinor-vector in the brackets.

The eigenvalue of the $[3/2,1/2,1/2]$ - irreducible operator $\mathcal{R}_a^{\ b}$ on $\Xi_a^{I[3/2,1/2,1/2]}$ is easily related to the eigenvalue of $\not\nabla$ on the parent harmonic $\Xi^{I[1/2,1/2,1/2]}$. A little of algebra shows that

$$M_{3/2}^{\ I} = \frac{5}{7}\,M_{1/2}^{\ I} \qquad (V.4.70)$$

Exercise: verify (V.4.70).

Similarly we can compute the value of the operator S^b on $(\nabla_a\,\Xi^{I[1/2,1/2,1/2]})_{3/2}$:

$$S^b(\nabla_b\,\Xi^{I[1/2,1/2,1/2]})_{3/2} = \frac{5}{7}\,\Gamma^{ab}\,\nabla_a(\nabla_b\,\Xi^{I[1/2,1/2,1/2]})_{3/2}$$
$$= -\frac{5}{7}\,M_{1/2}^{\ I}(12e + \frac{6}{7}\,M_{1/2}^{\ I})\,\Xi^{I[1/2,1/2,1/2]} \qquad (V.4.71)$$

To prove this equation we first establish the identity:

$$\nabla^a\nabla_a = \not\nabla\not\nabla + 12e\,\not\nabla \qquad (V.4.72)$$

From (V.4.54b) and the fact that $\mathcal{D}_a$ commutes with Γ^b, we have

$$\nabla_a\,\Gamma^b = \Gamma^b\nabla_a - 2e\,\Gamma_a^{\ b} \qquad (V.4.73)$$

so that

$$\not\nabla\not\nabla = \Gamma^a\nabla_a\Gamma^b\nabla_b = \Gamma^a(\Gamma^b\nabla_a - 2e\,\Gamma_a^{\ b})\nabla_b = \Gamma^a\Gamma^b\nabla_a\nabla_b - 12e\,\not\nabla =$$
$$= \Gamma^{ab}\nabla_a\nabla_b + \nabla^a\nabla_a - 12e\,\not\nabla \qquad (V.4.74)$$

The term $\Gamma^{ab}\nabla_a\nabla_b$ gives a vanishing contribution; indeed

$$\Gamma^{ab}\nabla_a\nabla_b = -\frac{1}{4}\,\Gamma^{ab}\,\Gamma_{cd}\,\mathbb{R}^{cd}_{\ \ ab} =$$
$$= -\frac{1}{4}\,[\Gamma^{ab}_{\ \ cd} - 4\,\delta^{[a}_{[c}\,\Gamma^{b]}_{\ \ d]} - 2\,\delta^{ab}_{cd}]\,\mathbb{R}^{cd}_{\ \ ab} \qquad (V.4.75)$$

where $\mathbb{R}^{cd}_{\ \ ab}$ are the SO(8) - curvature components.*

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* since ∇_a is the SO(8) covariant derivative, $\mathbb{R}^{cd}_{\ \ ab}$ is a subset of the components $\mathbb{R}^{CD}_{\ \ AB}$ of the full SO(8) Riemann tensor, with $A = (a,8)$.

Vanishing of (V.4.75) is due to the cyclic identity on $\mathbb{R}_{abcd}$, implying

$$\Gamma^{abcd}\,\mathbb{R}_{abcd} = 0 \qquad (V.4.76)$$

and to the fact that the SO(8) - curvature is Ricci flat (verify this).

Therefore (V.4.74) indeed yields the relation (V.4.72).

Using now (V.4.61), (V.4.69) and $\nabla^a\Gamma_a = \Gamma_a\nabla^a$ (see V.4.73) we can write:

$$S^b(\nabla_b\,\Xi^{I[1/2,1/2,1/2]})_{3/2} = -\frac{5}{7}\,(\nabla^b\nabla_b - \frac{1}{7}\,\not{\nabla}\not{\nabla})\,\Xi^{I[1/2,1/2,1/2]} \qquad (V.4.77)$$

Substituting (V.4.72) and recalling that $\not{\nabla}\Xi^{I[1/2,1/2,1/2]} = M_{1/2}\Xi^{I[1/2,1/2,1/2]}$ we finally arrive at (V.4.71).

An important thing to observe is that not all the $\Xi_a^{I[3/2,1/2,1/2]}$ harmonics can be obtained as derivatives of the $\Xi^{I[1/2,1/2,1/2]}$ harmonics. Indeed we can distinguish two sets of $\Xi_a^{I[3/2,1/2,1/2]}$ harmonics:

i) the longitudinal harmonics, of the type

$$\Xi_a^{I_L[3/2,1/2,1/2]} = (\nabla_a\,\Xi^{I[1/2,1/2,1/2]})_{3/2} \qquad (V.4.78)$$

The longitudinal representations I_L are therefore those contributing to the expansion of the [1/2,1/2,1/2] spinor (V.4.49). We have

$$\mathcal{R}_a{}^b\,\Xi_b^{I_L[3/2,1/2,1/2]} = \frac{5}{7}\,M_{1/2}{}^{I_L}\,\Xi_a^{I_L[1/2,1/2,1/2]} \qquad (V.4.79)$$

$$S^b\Xi_b^{I_L[3/2,1/2,1/2]} = -\frac{5}{7}\,M_{1/2}{}^{I_L}(12e + \frac{6}{7}\,M_{1/2}{}^{I_L}) \times \Xi^{I_L[1/2,1/2,1/2]} \qquad (V.4.80)$$

ii) the transverse harmonics $\Xi_a^{I_T[3/2,1/2,1/2]}$

These are the [3/2,1/2,1/2] harmonics that cannot be obtained as derivatives of $\Xi^{I[1/2,1/2,1/2]}$, and correspond to the transverse representations, not contributing to the [1/2,1/2,1/2] spinor expansion. We have

$$S^b\,\Xi_b^{I_T[3/2,1/2,1/2]} = 0 \Leftrightarrow \nabla^b\,\Xi_b^{I_T[3/2,1/2,1/2]} = 0 \qquad (V.4.81)$$

Indeed if $S^b\,\Xi_b^{I_T[3/2,1/2,1/2]}$ were nonvanishing, it would also be proportional to a [1/2,1/2,1/2] - harmonic $\Xi^{I_T[1/2,1/2,1/2]}$ (S^b maps [3/2,1/2,1/2] fields into [1/2,1/2,1/2] fields), contrary to the assumption that the transverse representations I_T do not contribute to the [1/2,1/2,1/2] spinor.

The $\mathcal{R}_a{}^b$ eigenvalue $M_{3/2}{}^{I_T}$ on $\Xi^{I_T[3/2,1/2,1/2]}$ cannot be related to an eigenvalue $M_{1/2}{}^{I_L}$ and must be evaluated separately.

We discuss now some bounds on the eigenvalues λ of $\mathcal{D}$. ($\mathcal{D}\psi = \lambda\psi$). For $R_{ab} = 12e^2\,g_{ab}$ (Freund-Rubin solutions (V.4.17)), the nonnegative quantity:

$$\int d^7y\sqrt{g}\,|\mathcal{D}_a\psi - e\Gamma_a\psi|^2 = \int d^7y\sqrt{g}\,|\nabla_a\psi|^a \geq 0 \qquad (V.4.82)$$

is also equal to

$$(\lambda - 7e)(\lambda + 5e)\int d^7y\sqrt{g}\,|\psi|^2 \qquad (V.4.83)$$

To prove this, use Eq. (V.4.72), valid for $R_{ab} = 12e^2 g_{ab}$:

$$\nabla^a\nabla_a = \not{\nabla}\not{\nabla} + 12e\,\not{\nabla} \qquad (V.4.84)$$

and

$$\not{\nabla} = \not{\mathcal{D}} - 7e \tag{V.4.85}$$

Requiring (V.4.83) to be ≥ 0 yields the bounds

$$\lambda \geq 7e \qquad \text{or} \qquad \lambda \leq -5e \tag{V.4.86}$$

Consider now the quantity

$$\int d^7y \sqrt{g}\, |\mathcal{D}_a\psi + e\Gamma_a\psi|^2 \geq 0 \tag{V.4.87}$$

obtained from (V.4.82) by changing the sign of e. Then (V.4.87) is equal to

$$(\lambda + 7e)(\lambda - 5e) \int d^7y \sqrt{g}\, |\psi|^2 \geq 0 \tag{V.4.88}$$

and the bounds on λ become

$$\lambda \leq -7e \qquad \text{or} \qquad \lambda \geq 5e \tag{V.4.89}$$

Since both (V.4.83) and (V.4.88) are nonnegative, the bounds (V.4.86) and (V.4.89) must hold simultaneously, implying

$$\lambda \leq -7e \qquad \text{or} \qquad \lambda \geq 7e \tag{V.4.90}$$

i.e. a forbidden region $(-7e < \lambda < 7e)$ in the spectrum of $\mathcal{D}$. For Killing spinors η (cfr. V.4.26b) $\nabla_a\eta = 0$, i.e.

$$\not{\mathcal{D}}\eta = 7e\,\eta \tag{V.4.91}$$

Notice that Killing spinors can exist only on Einstein spaces, since

$$\frac{1}{2}[\mathcal{D}_a, \mathcal{D}_b]\eta = -\frac{1}{4} R^{cd}{}_{ab}\, \Gamma_{cd}\, \eta = -e^2\, \Gamma_{ab}\, \eta \tag{V.4.92}$$

Contracting with Γ_b we find

$$(R_{ab} - 12e^2 g_{ab})\, \Gamma^b \eta = 0 \tag{V.4.93}$$

The Einstein condition follows by multiplying on the left by $\bar{\eta}\Gamma^c$.

<u>The Hodge-de Rahm and Lichnerowicz operators</u>

The Hodge-de Rahm operator Δ is defined by

$$\Delta \equiv d\delta + \delta d \tag{V.4.94}$$

and maps p-forms into p-forms; d is the exterior derivative and δ is its adjoint: $\delta = (-1)^p *d*$ (for odd-dimensional spaces)

We recall that a form ω is <u>closed</u> if $d\omega = 0$, <u>coclosed</u> if $\delta\omega = 0$, <u>harmonic</u> if $\Delta\omega = 0$; ω is <u>exact</u> if $\omega = d\alpha$ and <u>coexact</u> if $\omega = \delta\alpha$.

Δ is a nonnegative operator, as may be seen by introducing the norm of ω:

$$(\omega, \omega) \equiv \int \sqrt{g}\, d^7y\, \omega_{a_1 \ldots a_p}\, \omega^{a_1 \ldots a_p} \tag{V.4.95}$$

vanishing only if $\omega = 0$. By definition

$$(\omega, d\omega) = (\delta\omega, \omega) \tag{V.4.96}$$

and hence

$$(\omega, \Delta\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega) \geq 0 \tag{V.4.97}$$

the equality holding if and only if ω is closed and coclosed (i.e. harmonic).

As already discussed in Part I, any p-form ω admits a unique decomposition into its exact, coexact and harmonic pieces:

$$\omega = d\alpha + \delta\beta + \gamma \qquad \gamma\text{: harmonic p-form} \tag{V.4.98}$$

$d\alpha$, $\delta\beta$ and γ are mutually orthogonal with respect to the norm (V.4.95). Eq. (V.4.98) implies the well-known isomorphism between the de Rahm cohomology classes and harmonic forms, so that the number of closed but not exact p-forms on a manifold, given by the p-th Betti number b_p, is also the number of zero modes of Δ (cfr. the discussion after Eq. (I.6.280)).

The action of Δ on p-forms with $0 \le p \le 3$ is given below:

$$\Delta_0\omega = -\Box\omega = -\mathcal{D}^a\mathcal{D}_a\omega \tag{V.4.99}$$

$$\Delta_1\omega_a = -\Box\omega_a - 2R_a{}^b\omega_b = \mathcal{D}^b\mathcal{D}_{[b}\omega_{a]} \tag{V.4.100}$$

$$\Delta_2\omega_{ab} = -\Box\omega_{ab} + 4R_{acbd}\omega^{cd} + 4R^c{}_{[a}\omega_{b]c} = \mathcal{D}^c\mathcal{D}_{[c}\omega_{ab]} \tag{V.4.101}$$

$$\Delta_3\omega_{abc} = -\Box\omega_{abc} + 12R_{[ab}{}^{de}\omega_{c]de} - 6R_{[a}{}^d\omega_{bc]d} \tag{V.4.102}$$

Exercise: prove (V.4.99-102) using the definition (V.4.94). For $R_{ab} = 12e^2g_{ab}$, we mention the following results on the Δ eigenvalues (for more details see [9]).

Zero-forms. For any connected M^7 there is always one zero-mode of Δ_0, namely $\omega = $ constant. Next, considering the nonnegative quantity

$$\int d^7y\sqrt{g}\,|\mathcal{D}_a\mathcal{D}_b\omega + c^2g_{ab}|^2 =$$

$$= [\lambda^2 - \lambda(24e^2 + 2c^2) + 7c^4]\int d^7y\sqrt{g}\,\omega^2 \tag{V.4.103}$$

with $\Delta_0\omega = \lambda\omega$ and $c^2 = 4e^2$ or $c^2 = 0$, one finds that the first non zero Δ eigenvalue satisfies:

$$\lambda \ge 28e^2 \tag{V.4.104}$$

with equality if and only if

$$\mathcal{D}_a\mathcal{D}_b\omega = -4e^2g_{ab}\omega \tag{V.4.105}$$

Such modes occur on the round S^7 (the only M^7 where they occur), and correspond to the existence of conformal Killing vectors $C_a = \mathcal{D}_a\omega$ on S^7 satisfying

$$\mathcal{D}_{(a}C_{b)} = -4e^2g_{ab}\omega \tag{V.4.106}$$

(see Sect. (V.7.1)).

1-forms. From $\Delta_1 = -\Box + 12e^2$ we have immediately

$$\Delta_1 \ge 24e^2 \tag{V.4.107}$$

For transverse vectors $D_mV^m = 0$, there is a stronger bound:

$$\Delta_1 \ge 48e^2 \tag{V.4.108}$$

with equality holding only for Killing vectors.

For Δ_2, Δ_3, similar bounds are more difficult to find, because of the appearance of the uncontracted Riemann tensor R_{abcd} in (V.4.101-102). The same is true for the Lichnerowicz operator Δ_L:

$$\Delta_Lh_{ab} = -\Box h_{ab} - 4R_{acbd}h^{cd} + 4R_{(a}{}^c h_{b)c} \tag{V.4.109}$$

Δ_L is not, in general, positive definite, and lower bounds on the eigenvalue spectrum are difficult to establish, without explicit knowledge of the full Riemann tensor R_{acbd}.

We conclude by giving a list of the SO(7) harmonics $Y^{(\nu)[\lambda]}(y)$ on G/H (cfr. (V.3.37)) relevant to compactified $D = 11$ supergravity:

harmonic	SO(7) irrep	dimension	
$Y(y)$	$[0,0,0]$	1 : scalar harmonic	(V.4.110a)
$Y_a(y)$	$[1,0,0]$	7 : transverse vector harmonic $\mathscr{D}^a Y_a = 0$	(V.4.110b)
$Y_{[ab]}(y)$	$[1,1,0]$	21 : transverse two-form $\mathscr{D}^a Y_{[ab]} = 0$	(V.4.110c)
$Y_{[abc]}(y)$	$[1,1,1]$	35 : transverse 3-form $\mathscr{D}^a Y_{[abc]} = 0$	(V.4.110d)
$Y_{(ab)}(y)$	$[2,0,0]$	27 : transverse symmetric traceless $Y_{(aa)} = 0,\ \mathscr{D}^a Y_{(ab)} = 0$	(V.4.110e)
$\Xi(y)$	$[\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	8 : spinor	(V.4.110f)
$\Xi_a(y)$	$[\frac{3}{2},\frac{1}{2},\frac{1}{2}]$	48 : irreducible transverse vector-spinor $\Gamma^a \Xi_a = 0;\ \mathscr{D}^a \Xi_a = 0$	(V.4.110g)

where we have omitted the G-irrep labels and indices $\binom{(\nu)}{n}$; $[\lambda_1,\lambda_2,\lambda_3]$ classifies the SO(7) irrep by specifying the corresponding Young tableau, and the underlined numbers give its dimension.

The eigenvalues of the invariant M^7 operators on the harmonics (V.4.110) are denoted as follows(*):

$$-\Delta_0 Y = \mathscr{D}^a \mathscr{D}_a Y = M_{(0)^3} Y \qquad \text{(V.4.111a)}$$

$$-\Delta_1 Y_a = 2\mathscr{D}^b \mathscr{D}_{[b} Y_{a]} = (\Box + 24) Y_a = M_{1(0)^2} Y_a \qquad \text{(V.4.111b)}$$

(*) M^7 is taken to be Einstein with $R_{ab} = 12\, g_{ab}$, (e = 1 units).

$$-\Delta_2 Y_{[ab]} = 3\mathscr{D}^c \mathscr{D}_{[c} Y_{ab]} = \left[(\Box + 48)\ \delta^{de}_{ab} - 4R_{a\cdot b\cdot}^{\cdot d \cdot e}\right] Y_{[de]} =$$

$$= M_{(1)^2(0)} Y_{[ab]} \qquad \text{(V.4.111c)}$$

$$^{*}d\, Y_{[abc]} = \frac{1}{24} \epsilon_{abc}{}^{e_1 \ldots e_4} D_{[e_1} Y_{e_2 e_3 e_4]} = M_{(1)^3} Y_{[abc]} \qquad \text{(V.4.111d)}$$

$$-\Delta_L Y_{(ab)} = \left[(\Box + 48)\ \delta^{(de)}_{(ab)} - 4R_{a\cdot b\cdot}^{\cdot d \cdot e}\right] Y_{(de)} = M_{(2)(0)^2} Y_{(ab)} \qquad \text{(V.4.111e)}$$

$$\not{\mathscr{D}}\Xi = \Gamma^a(\mathscr{D}_a - \Gamma_a)\Xi = (\not{\mathscr{D}} - 7)\Xi = M_{(1/2)^3}\Xi \qquad \text{(V.4.111f)}$$

$$\mathcal{R}_a{}^b \Xi_b = (\not{\mathscr{D}} - 5)\Xi_a = M_{(3/2)(1/2)^2} \Xi_a \qquad \text{(V.4.111g)}$$

APPENDIX V.4.1: SO(7) Γ-MATRICES

The SO(7) Clifford algebra is normalized as

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \qquad \eta_{ab} = \text{diag}\,(-------)$$

Explicit representation:

$$\Gamma_m = i\, \mathbb{1}_4 \otimes \tau_m \qquad \Gamma_3 = \mu_8 \otimes \tau_3 \qquad \Gamma_A = \mu_A \otimes \tau_3$$

$(m = 1,2 \quad , \quad A = 4,5,6,7)$

where τ_1, τ_2, τ_3 are the usual 2×2 Pauli matrices, and

$$\{\mu_A, \mu_B\} = -2\delta_{AB}$$

$$\mu_4 = -i\begin{pmatrix} 0 & \tau_1 \\ \tau_1 & 0 \end{pmatrix} \qquad \mu_5 = -i\begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix}$$

$$\mu_6 = i\begin{pmatrix} 0 & \tau_3 \\ \tau_3 & 0 \end{pmatrix} \qquad \mu_7 = \begin{pmatrix} 0 & -\mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}$$

$$\mu_8 = \begin{pmatrix} -i\,\mathbb{1}_{2\times2} & 0 \\ 0 & i\,\mathbb{1}_{2\times2} \end{pmatrix}$$

Hence:

$$\Gamma_1 = \begin{pmatrix} 0 & -i\,\mathbb{1}_{4\times4} \\ -i\,\mathbb{1}_{4\times4} & 0 \end{pmatrix} \qquad \Gamma_2 = \begin{pmatrix} 0 & -\mathbb{1}_{4\times4} \\ \mathbb{1}_{4\times4} & 0 \end{pmatrix}$$

$$\Gamma_3 = \begin{pmatrix} -\mu_8 & 0 \\ 0 & \mu_8 \end{pmatrix} \qquad \Gamma_A = \begin{pmatrix} -\mu_A & 0 \\ 0 & \mu_A \end{pmatrix}$$

This representation is antihermitian:

$$\Gamma_a^{\dagger} = -\Gamma_a$$

The 7-dimensional charge conjugation matrix $C_{(7)}$ is defined by

$$C_{(7)}\Gamma_a C_{(7)}^{-1} = -\Gamma_a^{T}$$

and in our representation takes the form:

$$C_{(7)} = \left(\begin{array}{cc|cc} 0 & 0 & i\tau_2 & 0 \\ 0 & 0 & 0 & -i\tau_2 \\ \hline -i\tau_2 & 0 & 0 & 0 \\ 0 & i\tau_2 & 0 & 0 \end{array}\right)$$

with the properties

$$C_{(7)}^2 = \mathbb{1} \quad , \quad C_{(7)}^T = -C_{(7)}$$

A Majorana spinor η in $D=7$ satisfies the condition

$$C_{(7)}\,\eta^* = \eta$$

TABLE V.4.1

HOLONOMY GROUPS FOR 7-MANIFOLDS

Holonomy	Branching rule for 8 of SO(7)	N_{MAX}
Spin (7)	8	0
G_2	1 + 7	1
SU(4)	$4 + \bar{4}$	0
SU(2) × SU(2) × SU(2)	(1,2,2) + (2,2,1)	0
SU(3)	$1 + 1 + 3 + \bar{3}$	2
Sp(2)	4 + 4	0
SU(2) × SU(2)	(2,2) + (1,3) + (1,1)	1
SU(2) × SU(2)	(2,2) + (1,2) + (1,2)	0
SU(2) × SU(2)	(1,2) + (1,2) + (2,1) + (2,1)	0
SU(2) × SU(2)	(2,2) + (2,2)	0
SU(2)	1 + 1 + 1 + 1 + 2 + 2	4
SU(2)	1 + 1 + 3 + 3	2
SU(2)	1 + 2 + 2 + 3	1
SU(2)	1 + 7	1
SU(2)	4 + 4	0
SU(2)	2 + 2 + 2 + 2	0
1	1 + 1 + 1 + 1 + 1 + 1 + 1 + 1	8

CHAPTER V.5

THE D = 4 MASS SPECTRUM IN $AdS^4 \times M^7$ BACKGROUNDS

V.5.1 – The linearized field equations of D=11 supergravity

In this chapter we derive the D = 4 mass operators arising from the Freund-Rubin compactification on an arbitrary manifold M^7.

As already discussed in the previous chapters, the D = 4 spacetime fields are the x-dependent coefficients in the harmonic expansion of the D = 11 field fluctuations. Here we define the fluctuations $h^A{}_B(x,y)$, $\psi_\Lambda(x,y)$, $a_{\Lambda\Pi\Sigma}(x,y)$ around an arbitrary background by:

$$V_\Lambda{}^A(x,y) = \overset{\circ}{V}_\Lambda{}^A(x,y) + h^A{}_B(x,y)\overset{\circ}{V}_\Lambda{}^B(x,y) \qquad \text{(V.5.1a)}$$

$$\Psi_\Lambda(x,y) = 0 + \psi_\Lambda(x,y) \qquad \text{(V.5.1b)}$$

$$A_{\Lambda\Pi\Sigma}(x,y) = \overset{\circ}{A}_{\Lambda\Pi\Sigma}(x,y) + a_{\Lambda\Pi\Sigma}(x,y) \qquad \text{(V.5.1c)}$$