M-theory and Mathematics, NYUAD, 28 January 2020

Supergravity in a Pencil

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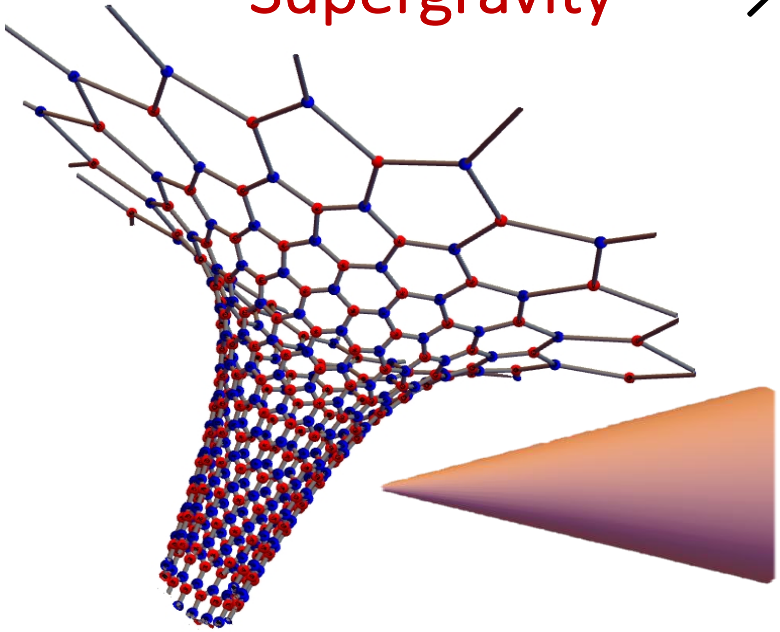
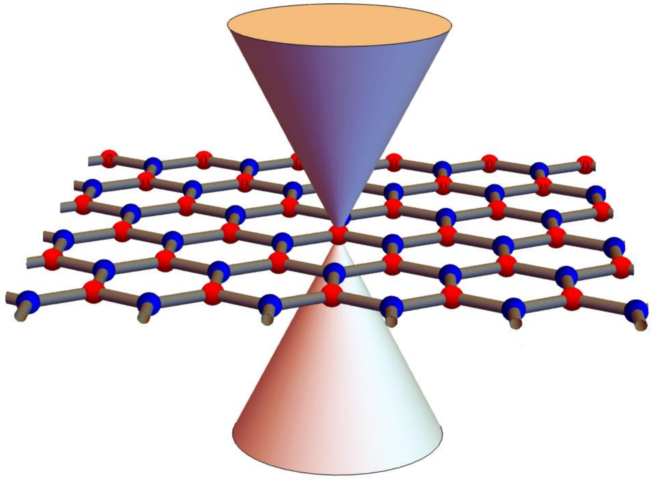
based on L. Andrianopoli, BLC, R. D'Auria, M. Trigiante, JHEP04(2018)007, arXiv:1801.08081;
L.Andrianopoli, BLC, R. D'Auria, G. Gallerati, R. Noris, M. Trigiante, J. Zanelli, JHEP01(2020)084, arXiv:1910.03508

Supergravity → Graphene





Main idea

Objective: Application of the dualities of supergravity to the study of graphene-like 2D materials in condensed matter.

- **The gauge/gravity correspondence relates a strongly coupled gauge theory to a weakly coupled classical gravity theory in one dimension higher.**

bulk

3+1-dim. Supergravity on an anti-de Sitter spacetime

holography →

boundary

Electronic properties of a 2+1-dim. Layer of graphene

- **Top-down approach: Large amout of supersymmetry makes model more predictive**

Relation of the electronic properties of graphene to deformations of the lattice geometry

- **Relevance of supersymmetry in low-energy physics** → **Interdisciplinary approach**

Plan

- Main idea
- Graphene and the Dirac equation
  - The graphene honeycomb lattice
  - The graphene Dirac cone
- "Analogue" relativity in condensed matter
  - Geometry in analogue gravity
- Generalized AVZ Model for Graphene
  - Massive Dirac Equation in the AVZ model
- The Generalized AVZ Model at the boundary of AdS_4 Supergravity
  - (Asymptotically) AdS_4 pure N-extended D=4 Supergravity
  - Ultraspinning limit to the locally AdS_3 boundary at $r \to \infty$
  - Achúcarro-Townsend D = 3 Theory
  - The generalized AVZ Ansatz
- The Nieh-Yan-Weyl symmetry
- Quantum Theory of Chern-Simons Supergravity
- Application to Graphene and the K and K' valleys
- Comparison with microscopic models of graphene-like 2D materials
- Final Remarks and Outlook

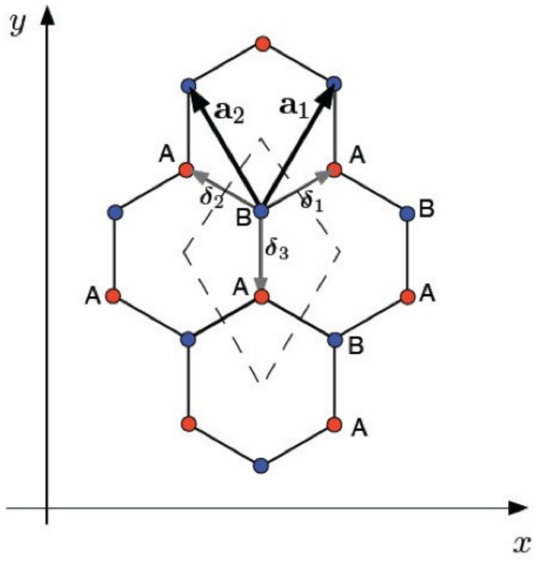
Graphene and the Dirac equation

The graphene honeycomb lattice

Graphene is a two-dimensional layer of carbon atoms (one single layer of graphite).

The carbon atoms in graphene form a honeycomb lattice with a hexagonal structure, due to the sp^2 orbital hybridization.

Bipartite lattice composed by two triangular sublattices (sites A and sites B).

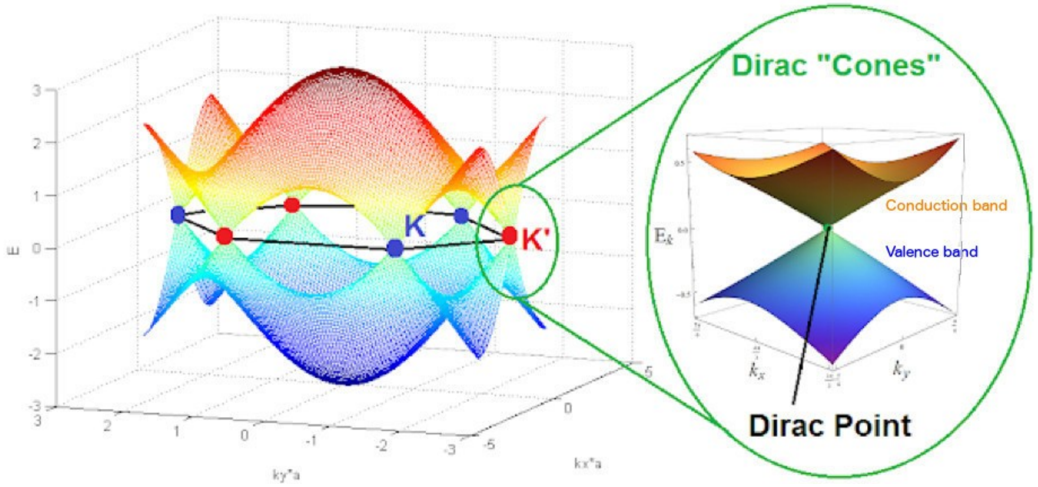


Belonging to site A or B defines an additional

spin-like quantum number: Pseudospin.

The graphene Dirac cone

The Electron Band Structure of graphene



At the Dirac points (for a range of 1eV) the spectrum (relation between the energy E_k and the momentum k) is linear:

Dirac cone: $E_k = \pm \hbar c |k|$

Electrons in graphene obey the same type of equations as

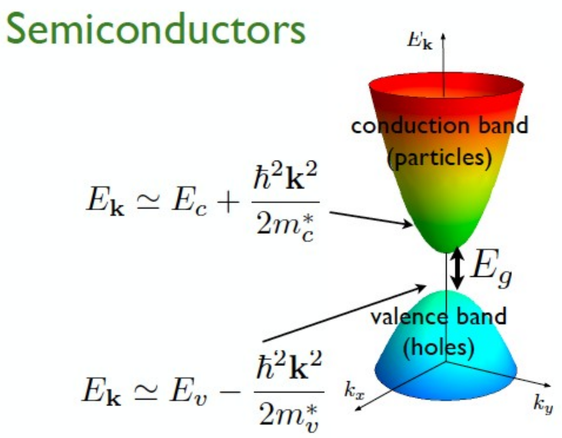
relativistic Dirac massless particles with

Light speed $c \rightarrow v_F = 10^6 \frac{m}{s} = \frac{c}{300}$ **Fermi velocity**

"Analogue" relativity in condensed matter

Relativistic energy: $E_K = \pm\sqrt{(\hbar c|k|)^2 + (mc^2)^2}$

If m finite

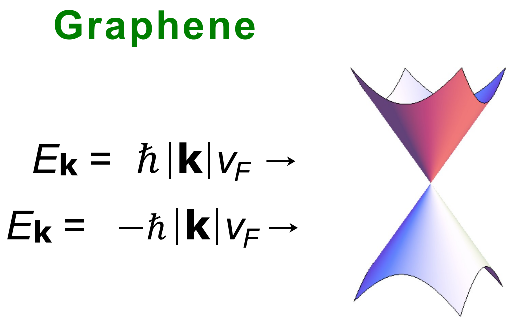


bare mass m → effective mass m^*

Interaction of the electrons with the lattice atoms → $m^* \leq m$.

In graphene $m^*=0$.

Massless case:



Light speed c → Fermi velocity v_F

Geometry in analogue gravity

Lattice disclination → Space Curvature

Missing angle: hexagon substituted by another polygon, e.g. a pentagon

Carbon nanocones

Lattice dislocation → Space Torsion

Glide and shuffle dislocations in graphene, obtained from subsequent disclinations

Generalized AVZ Model for Graphene

[L. Andrianopoli, BLC, R. D'Auria, M. Trigiante, *Unconventional Supersymmetry at the boundary of AdS_4 Supergravity*, JHEP04(2018)007, arXiv:1801.08081;
L. Andrianopoli, B.L. Cerchiai, R. D'Auria, A. Gallerati, R. Noris, M. Trigiante, J. Zanelli, *N-Extended D=4 Supergravity, Unconventional SUSY and Graphene*, JHEP01(2020)084, arXiv: 1910.03508]

In the AVZ model [P.D. Alvarez, M. Valenzuela, J. Zanelli, JHEP 1204 (2012) 058, arXiv:1109.3944] **the fermionic gauge field ψA is a composite field, and the propagating fermion χA originates from the radial component of the gravitino through the Ansatz**: (A=1,..,$\mathcal{N}$, $\mathcal{N}$ number of supersymmetries; i=0,1,2)

Supergravity Gravitino

Propagating fermion (pseudoparticle in graphene)

$$\psi^A = i\,\gamma_i e^i \chi^A .$$

gamma matrices and Vielbein describing the Geometry

Massive Dirac Equation in the AVZ model

The AVZ model can be obtained at the D=2+1 AdS_3 boundary from a supergravity on a (curved) AdS_4 spacetime in D=3+1 through a suitably defined ultraspinning limit.

From the Maurer-Cartan equations of the supersymmetry algebra

Massive Dirac equation

$$\not{\mathfrak{D}}\,\chi = -\frac{3}{2}\, i\, \tau\, \chi$$

$\mathfrak{D}$ covariant derivative

mass $m = \frac{3}{2}\tau$ ← torsion

Killing spinor equation for the boundary supersymmetry

Solutions of the Dirac equation correspond to supersymmetries of the system.

Electronic properties of the pseudoparticles, such as e.g. optical conductivity, are determined by supersymmetry.

The Generalized AVZ Model at the boundary of AdS_4 Supergravity

ADS_4 vacuum of pure $\mathcal{N}$-extended D=4 supergravity

← symmetry: $Osp(\mathcal{N}|4)$

broken to

Suitable ultraspinning limit in the Feffermann-Graham parametrization: AdS_3 slicing of AdS_4 (black string)

Achúcarro-Townsend Chern-Simons supergravity on conformal locally AdS_3 boundary located at radial infinity ($r \to \infty$)

← symmetry $OSp(p|2) \times Osp(q|2)$; $p+q=\mathcal{N}$

AVZ Ansatz

AVZ model for propagating spin ½ particle, instrumental in describing the electronic properties of graphene-like 2D materials at the Dirac points

(Asymptotically) AdS$_4$ pure $\mathcal{N}$-extended D=4 Supergravity

Connection: $\mathbb{A} = \frac{1}{2}\omega^{\mathcal{AB}} L_{\mathcal{AB}} + \frac{1}{2}A^{CD} T_{CD} + \overline{\Psi}^{A}_{\alpha} Q^{\alpha}_{A},$

SO(2,3) generators: $L_{\mathcal{AB}}$ $(\mathcal{A}, \mathcal{B} = 0, \ldots, 4)$; connection: $\omega^{\mathcal{AB}}$

SO($\mathcal{N}$) generators: T_{AB} $(A, B = 1, \ldots, \mathcal{N})$; connection: A^{AB}

Majorana supersymmetry generators: Q^{α}_{A} $(\alpha = 1, \ldots, 4)$; gravitino: $\overline{\Psi}^{A}_{\alpha}$

The simplest action for this super-connection is a Yang-Mills theory for the smallest superalgebra that extends the AdS$_4$ symmetry and yields a spin-1/2 field minimally coupled to Einstein gravity and the Maxwell field [P. D. Alvarez, P. Pais, and J. Zanelli, Phys. Lett. **B735** (2014) 314–321, arXiv:1306.1247].

Symmetry: Osp($\mathcal{N}$|4)

In turn, this is shown to correspond to the boundary theory of a Chern-Simons theory for a super-connection in D=5 [Y. M. P. Gomes, J. A. Helayel-Neto, Phys. Lett. B777 (2018) 275–280, arXiv:1711.0322].

Covariant decomposition w.r.t. the spatial boundary

Maurer Cartan Equations: $d\mathbb{A} + \mathbb{A} \wedge \mathbb{A} = 0$

Rewrite in a form which is covariant with respect to the Lorentz group at the spatial boundary SO(1; 2):

SO(2,3) ⊃ SO(1,1) x SO(1,2)

Accordingly, split the indices:

$\mathcal{A}$=(0,1,2,3,4) = (a,4), a=0,1,2,3 ⟶ a=(0,i) , i=1,2 and 3 radial component

so that the SO(1,1) grading of the fields becomes manifest:

$$\begin{cases} E^i_\pm &= \pm\frac{1}{2}\left(V^i \mp \ell\omega^{3i}\right) \ , \\ \omega^{i3} &= \frac{1}{\ell}\left(E^i_+ + E^i_-\right) \ , \\ V^i &= E^i_+ - E^i_- \ , \end{cases}$$

where $V^a = \ell\omega^{a4}$ is the $D = 4$ Vielbein;

ℓ cosmological constant

Decompose the gravitini in their chiral components with respect to SO(1,1), represented by Γ^3:

$$\Psi^{A\alpha} = \Psi^A_+ + \Psi^A_- \,, \qquad \Gamma^3\Psi^A_\pm = \pm i\,\Psi^A_\pm \,.$$

Ultraspinning limit to the locally AdS$_3$ boundary at $r \to \infty$

[R. Emparan, G. T. Horowitz, R. C. Myers, JHEP **0001** (2000) 021, hep-th/9912135]

$$E^i_+(r,x) = \frac{1}{2}\left(\frac{r}{\ell}\right) E^i(x) + O\left(\frac{\ell^2}{r^2}\right), \qquad E^i_-(r,x) = -\frac{1}{2}\left(\frac{\ell}{r}\right) E^i(x) + O\left(\frac{\ell^2}{r^2}\right),$$

with $x = (x^\mu)$, $\mu = 0, 1, 2$ boundary coordinates.

$$\Psi^A_{+\mu}(r,x)\, dx^\mu = \sqrt{\frac{r}{2\ell}}\begin{pmatrix}\psi^A(x)\\ \mathbf{0}\end{pmatrix} + O\left(\frac{\ell}{r}\right), \quad \Psi^A_{-\mu}(r,x)\, dx^\mu = \sqrt{\frac{\ell}{2r}}\begin{pmatrix}\mathbf{0}\\ \eta^{AB}\psi_B(x)\end{pmatrix} + O\left(\frac{\ell}{r}\right).$$

Bulk supesymmetry algebra broken at the boundary through the symmetric metric $\eta^{AB} = \begin{pmatrix}\mathbf{1}_{pxp} & \mathbf{0}_{pxq}\\ \mathbf{0}_{qxp} & -\mathbf{1}_{qxq}\end{pmatrix}$

OSp($\mathcal{N}$|4) ⟶ OSp(p|2)x OSp(q|2), p+q=$\mathcal{N}$

Maurer-Cartan Equations at the boundary

$$\begin{cases} d\omega^{ij} + \omega^i{}_k \wedge \omega^{kj} - \frac{1}{\ell^2} E^i \wedge E^j - \frac{1}{2\ell}\left(\overline{\psi}^A \wedge \gamma^{ij}\eta_{AB}\psi^B\right) = 0\,, \\ dE^i + \omega^i{}_j \wedge E^j - \frac{i}{2}\left(\overline{\psi}^A \wedge \gamma^i \psi_A\right) = 0\,, \\ dA^{CD} + A^C_M \wedge A^{MD} + \frac{1}{\ell}\overline{\psi}^{[C} \wedge \eta^{D]B}\psi_B = 0\,, \\ d\psi^A + \frac{1}{4}\omega^{ij} \wedge \gamma_{ij}\psi^A + \frac{i}{2\ell}E^i \wedge \gamma_i \eta^{AB}\psi_B + A^{AB} \wedge \psi^B = 0\,. \end{cases}$$

Achúcarro-Townsend D = 3 Theory

[A. Achúcarro, P. K. Townsend, Phys. Lett. B 229 (1989) 383]

With the definitions:

$$\text{Torsionful connection: } \Omega^i_{(\pm)} := \omega^i \pm \frac{E^i}{\ell} , \quad (\omega^i := \frac{1}{2}\epsilon^{ijk}\,\omega_{jk}) ,$$

$$\psi_+ := (\psi^{a_1}) , \quad \psi_- := (\psi^{a_2}) ,$$

$$A_+ := (A^{a_1 b_1}) , \quad A_- := (A^{a_2 b_2}) , \quad (A^{a_1 b_2} = A^{a_2 b_1} = 0 \text{ for consistency}) ,$$

$$\text{Covariant Derivatives: } \mathcal{D}[\Omega_+, A_+]\psi_+ := \left(d\psi^{a_1} + \frac{i}{2}\Omega^i_+ \wedge \gamma_i \psi^{a_1} + A^{a_1 b_1} \wedge \psi_{b_1}\right) ,$$

$$\mathcal{D}[\Omega_-, A_-]\psi_- := \left(d\psi^{a_2} + \frac{i}{2}\Omega^i_- \wedge \gamma_i \psi^{a_2} + A^{a_2 b_2} \wedge \psi_{b_2}\right) .$$

the Maurer-Cartan equations at the boundary can be recovered from the topological Achúcarro-Townsend supergravity in D=2+1:

Chern-Simons Lagrangian for OSp(p|2) | Chern-Simons Lagrangian for OSp(q|2) | Gibbons-Hawking term (total derivative)

$$\mathcal{L} = \mathcal{L}_{(+)} - \mathcal{L}_{(-)} - \frac{1}{2} d(\Omega_{+k} \wedge \Omega^k_-) ,$$

$$\mathcal{L}_{(\pm)} := \frac{1}{2}\left(\Omega_{\pm i} d\Omega^i_\pm - \frac{1}{3}\epsilon_{ijk}\,\Omega^i_\pm \wedge \Omega^j_\pm \wedge \Omega^k_\pm\right) + \mathrm{Tr}\left(A_\pm \wedge dA_\pm + \frac{2}{3}A_\pm \wedge A_\pm \wedge A_\pm\right) \pm \frac{2}{\ell}\overline{\psi}_\pm \wedge \mathcal{D}[\Omega_\pm, A_\pm]\psi_\pm .$$

The generalized AVZ Ansatz

AVZ Ansatz: $\psi^A = i\,\gamma_i\, e^i\, \chi^A$

- The spin 3/2 component of the gravitino is projected out, while the spin ½ component yields a propagating fermion, suitable to describe pseudoparticles in graphene like 2D materials.

- The matrix γ_i plays the role of an intertwiner, allowing an identification of the graphene worldvolume with the supergravity target space-time:

$$E^i = \mathrm{f}\, e^i$$

where e^i is a supersymmetry invariant dreibein on the graphene worldsheet.

- The torsion has a trace part β and an antisymmetric part τ :

$$\mathcal{D}[\omega]e^i = \beta \wedge e^i + \tau\, \epsilon^{ijk} e_j \wedge e_i.$$

The Nieh-Yan-Weyl symmetry

- The AVZ Ansatz features a local scale invariance, the Nieh-Yan-Weyl (NYW) symmetry [H. T. Nieh and M. L. Yan, Annals Phys. 138 (1982) 237]:

$$e^i \to \lambda(x) e^i , \quad \chi^A \to \frac{1}{\lambda(x)} \chi^A, \quad \lambda(x) \neq 0.$$

- It is the breaking of this conformal invariance that turns an originally topological Chern-Simons theory into a system with a propagating spin-1/2 field.

- Under a NYW transformation, one can always set $\beta = 0$ locally.

- In a local patch $\overline{\chi_\pm}\chi_\pm$ are constants

- The quantities $\eta_{AB}\overline{\chi}^A\chi^B = \overline{\chi}_+\chi_+ - \overline{\chi}_-\chi_-$, $\overline{\chi}\chi = \overline{\chi}_+\chi_+ + \overline{\chi}_-\chi_-$ play the role of topological index.

Difference n_B-n_A between the occupation numbers of graphene electrons in sites A and B

Some Properties of the model

- There are no bosonic propagating degrees of freedom:

  Number of bosons $\neq$ Number of fermions

- Supersymmetry is implemented purely as a gauge symmetry (adjoint representation)

→ Unconventional Supersymmetry

- The propagating fermion satisfies a massive Dirac equation:

$$\not{D}[\Omega_\pm, A_\pm]\,\chi_\pm = -3\,i\,\tau_\pm\,\chi_\pm$$

  obtained as the Killing spinor equations of the boundary supersymmetry.

- Mass is generated by the geometric properties of supergravity, such as torsion.

Quantum Theory of Chern-Simons Supergravity

[L. Andrianopoli, B.L. Cerchiai, P.A. Grassi, M. Trigiante, The Quantum Theory of Chern-Simons Supergravity, JHEP 1906 (2019) 036, arXiv:1903.04431].

- The AVZ Ansatz corresponds to an (unconventional) gauge fixing (with vorticity) of supersymmetry in the framework of the BRST quantization.
- The supersymmetry parameter ϵ_A is proportional to the propagating fermion:

$$\epsilon_A \propto (\bar{\chi}\chi)\, \chi_A$$

- The identification of the graphene worldvolume Lorentz group with the supergravity target space-time symmetries defines a topological twist, along the lines of [Kapustin, Saulina, Nucl.Phys. B 823 (2009) 403]:

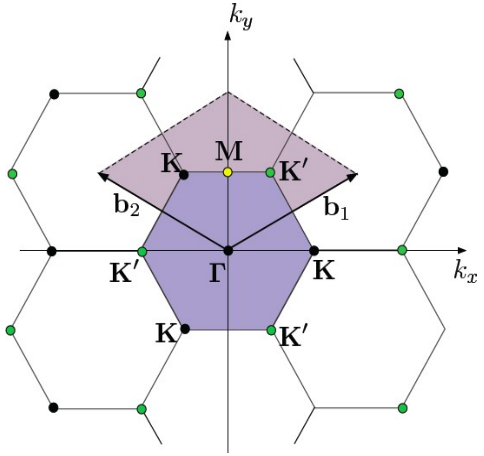
Chern-Simons theory on a SuperGroup with a gauge fixing of fermion gauge symmetries

$\cong$

topologically twisted super-Chern-Simons theory coupled to SUSY matter fields.

- This paves the way to the investigation of the embedding of the model in string theory [Gaiotto, Witten, JHEP 1006 (2010) 097] and its understanding within the rich web of dualities existing in 2+1 dimensions.

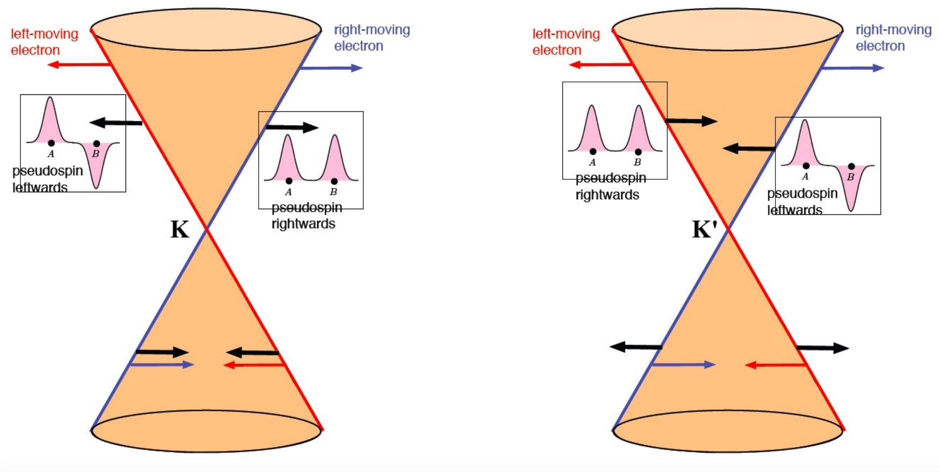
Application to Graphene and the **K** and **K'** valleys



The reciprocal lattice of graphene is also a honeycomb lattice, rotated by an angle of $\pi/2$, featuring two inequivalent types of Dirac points: **K** and **K'**.

The corresponding Dirac equations are mapped to each other by a reflection symmetry

Around the **K**-point the pseudospin direction is opposite to the **K'**-point.



Reflection symmetry for $p = q$

OSp(p|2)×OSp(q|2) symmetry of the Achucarro-Townsend model:

The fermionic fields χ_A split into two sets, (χ_{a1}, χ_{a2}), $a_1 = 1, \ldots, p;\ a_2 = 1, \ldots, q$.

In the special case $p = q$ a manifest parity symmetry emerges in the model, under which the fermions in the two sets are interchanged.

Correspondingly, for the torsion:

$$\beta_+ = \beta_- = \beta\,, \quad \tau_+ + f/\ell = \tau_- - f/\ell = \tau.$$

In the absence of global obstructions, it is possible to set $\beta = 0$.

The masses generated by the torsion in the two sectors are:

$$m_\pm = \frac{3}{2}\tau_\pm = \frac{3}{2}\tau \mp 3\frac{f}{\ell}\,,$$ with τ parity odd, f even.

The $\pm$ sectors, being related by the reflection symmetry in one spatial axis, can be naturally associated with the **K**, **K'** valleys of graphene.

Comparison with microscopic models of graphene-like 2D materials

Adding more supersymmetry allows to describe the K and K' valleys in the first Brillouin zone of the reciprocal lattice, by identifying the sector + with the K valley and the sector – with the K' valley, respectively, in the case p=q.

Mechanisms for opening mass gaps in graphene-like 2D msterials include

1) Breaking sublattices equivalence generating a parity odd mass term *M* (Semenoff model [G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984)]), e.g. by depositing a graphene monolayer on a suitable substrate, for instance of boron nitride or silicon carbide.

2) Introducing a suitable periodic local magnetic flux (Haldane model [F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988)]), inducing an Aharonov-Bohm phase, breaking time-invariance.

In our model a Semenoff-type mass is identified with the parity odd trace part of the torsion, while a Haldane-type mass with the antisymmetric part.

Final remarks

- In the framework of holography, we have obtained a description of 2D graphene-like materials in a suitable AdS_3 patch at the boundary of an extended supergravity in one dimension higher with N supersymmetries.

- This top-down approach is more predictive than the common bottom-up one, because it is strongly constrained from the supersymmetry properties of the gravity theory.

↓

- Construction of explicit solutions of the Dirac equation and of their properties.

- The model features supersymmetry, and it can be viewed as a top-down approach to understand the origin of the observed super-symmetric phenomenology in graphene [S.-S. Lee, "Emergence of super-symmetry at a critical point of a lattice model", Phys. Rev. B76 (2007) 075103, cond-mat/0611658: M. Ezawa, "Supersymmetry and unconventional quantum Hall effect in graphene", Phys. Lett. A372 (2008) 924, cond-mat/0606084].

Outlook

- We have ongoing discussions with the condensed matter groups, both theoretical and experimental, at the Politecnico di Torino.

- Holographic renormalization: In collaboration with O.Mišković and R.Olea, we want to apply the holographic renormalization scheme to our AdS_4/graphene correspondence. In this framework the counterterms in the holographic renormalization should sum up to topological invariants [Aros, Contreras, Olea, Troncoso, Zanelli, Phys.Rev.Lett. 84 (2000) 1647-1650; Olea, JHEP 0506 (2005) 023].

- Topological properties of the $D = 2 + 1$ theory: In collaboration with R. Olea and J. Zanelli we are studying the topological properties of the theory in $D=2+1$, particularly at the boundary of a $1+1$ interface with the aim of characterizing boundary currents in the presence of domain walls.

Outlook

- Originating from a different (unconventional) gauge fixing, the AVZ model should be a topologically non-trivial inequivalent corner of the theory, defined on a curved AdS worldvolume, rather than ordinary Minkowski.

- Addition of spin: Study of the spin-orbit interaction and the quantum spin Hall effect, first postulated in graphene in 2005 [C.L. Kane, E.J. Mele, PRL 95 (22), 226081, arXiv:cond-mat/0411737], but more easily testable in small gap semiconductors like Hg Te/Cd Te (mercury-, cadmium-telluride) [M. König et al., Science Express Research Articles. 318 (5851): 766770, arXiv:0710.0582 [cond-mat.mes-hall]] with very strong spin-orbit coupling.

- The Haldane and Semenoff-type masses are identified with geometric properties of the model.

Outlook

- Role of the Fermi velocity: A graphene sheet is “relativistic” in the sense of the Fermi velocity v_F playing the role of analogue speed of light. However, in our top-down approach, the speed of light, as coming from the D=4 supergravity, is naturally associated with the true speed of light c.
  Two different possible interpretations:

  1) The D = 4 supergravity is already analogue.

  2) Postulate a more general relation between the geometry of the supergravity space-time and the graphene worldsheet [Noris, Fatibene, arXiv:1910.04634]

- Different model for graphene like materials: In [A. Iorio and P. Pais, Annals Phys. **398** (2018) 265–286, arXiv:1807.0876] a different model is constructed, starting from a superalgebra of the form

  $$A(1,1) = SU(2|1,1),$$

  whose bosonic subgroup contains SU(1,1) x SU(2).

  The doublet labeling the **K** and **K'** valleys is here naturally gauged by construction.

  Can describe topological features of graphene such as *grain boundaries*.

  In our construction we can reproduce a similar case by starting from $\mathcal{N}$=4 and choosing p=4, q=0, with supergroup

  OSp(4|2) x SO(2,1) , containing SO(4) x SO(2,1)=SU(2) x SU(2) x SO(2,1).

- Application to more general Weyl semimetals, topologically non trivial materials in higher dimensions?

Thank you!