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A Convenient Category for Geometric Topology

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A Convenient Category for Geometric Topology

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The ∞-topos $\mathbf{Diff}^{\infty}$ of *differentiable stacks*, and its (ordinary) subcategory $\mathbf{Diff}^{\infty}_{\leq 0}$ of 0-truncated objects, the *differential spaces*, contain smooth manifolds as a full subcategory and have excellent formal properties: In both settings there is an intrinsic notion of underlying homotopy type of any object, as well as an intrinsic notion of what it means for an internal hom space to have the correct homotopy type. Extending and modernising work by Cisinski on (∞-)toposes and cofinality, we develop a suite of tools for constructing model structures and variants thereof in $\mathbf{Diff}^{\infty}_{\leq 0}$ and $\mathbf{Diff}^{\infty}$ which may be used to compare more classical constructions in geometric topology – for instance for computing underlying homotopy types – to the canonical constructions provided here, and thus to compare these classical notions with each other. Moreover, these tools are developed in a way so as to be highly customisable, with a view towards future applications.

These model structures moreover allow $\mathbf{Diff}^{\infty}_{\leq 0}$ and $\mathbf{Diff}^{\infty}$ to adopt a second role as a model for the theory of homotopy types. In this latter capacity $\mathbf{Diff}^{\infty}_{\leq 0}$ may be favourably contrasted with quasi-topological spaces: Like the category of quasi-topological spaces, $\mathbf{Diff}^{\infty}_{\leq 0}$ is Cartesian closed and circumvents the construction of complicated topologies, but, additionally, we show that filtered colimits are homotopy colimits, and closed manifolds are compact in the categorical sense. This makes $\mathbf{Diff}^{\infty}_{\leq 0}$ a useful replacement for quasi-topological spaces in applications of the sheaf theoretic h-principle.

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1 Introduction

1.1 Background

A central theme in geometric topology is the interplay between smooth manifolds and their underlying homotopy types. One might try to extract a homotopy type from a smooth manifold in several ways, for instance by using smooth simplices or by considering its underlying topological space; fortunately, one can check that the resulting homotopy types agree due to a smooth approximation argument. This interplay is important for example in the classification of smooth manifolds, as many of the invariants that one assigns to them only depend on their underlying homotopy type, yet their construction often makes crucial use of the presence of a differential structure, such as is the case for de Rham cohomology or the Euler characteristic via the Poincaré-Hopf theorem.

Given two manifolds, a more refined question than whether or not they are diffeomorphic, would be to ask for the totality of ways in which they are diffeomorphic, i.e., one might want to understand the *space* of diffeomorphisms of a given manifold, or more generally even, one might want to construct a space of all maps from one manifold to another. Constructing such a space is delicate, as is the extraction of a suitable underlying homotopy type from it. If one wishes for the space of maps to have a smooth structure, then it is well known that one must venture beyond the realm of finite dimensional smooth manifolds: for example, for any two manifolds M, N with M closed, the set of maps $\mathbf{Mfd}^{\infty}(M,N)$ may be equipped with the structure of an infinite dimensional Fréchet manifold (see [GG73, Th. 1.11]). Again, one might attempt to calculate the underlying homotopy type of the Fréchet manifold $\underline{\mathbf{Mfd}}^{\infty}(M, N)$ using smooth simplices or by considering its underlying topological space, or even the topological space of continuous maps from M to N endowed with the compact-open topology; once more, these homotopy types turn out to be equivalent, but the smooth approximation arguments involved are harder. Also, note that the above shows that if X and Y denote the underlying homotopy types of M and N respectively, then the underlying homotopy type of $\underline{\mathbf{Mfd}}^{\infty}(M, N)$ is equivalent to the homotopy type of maps $\mathcal{S}(X, Y)$.

Such issues abound in the theory invariant sheaves (a.k.a. continuous or equivariant sheaves), a popular framework for studying the sheaf theoretic h-principle (see [Aya09], [RW11], [Dot14], [Kup19]). Let $\mathbf{Emb}_n^{\infty}$ denote the topological category whose objects are the n-dimensional smooth manifolds, and where $\underline{\mathbf{Emb}}_n^{\infty}(M, N)$ is the set of smooth embeddings of M in N, equipped with, equivalently, the underlying topology of the Fréchet manifold $\mathbf{Emb}_n^{\infty}(M, N)$ or the C^{∞}-compact-open topology. Recall that a sheaf F on $\mathbf{Emb}_n^{\infty}$ valued in topological spaces is ***invariant*** if the map $\underline{\mathbf{Emb}}_n^{\infty}(M, N) \times F(M) \to F(N)$ is continuous.

Fixing a smooth manifold N, the following are examples of invariant sheaves:

1. The sheaf $\underline{\mathrm{Imm}}(\_, N)$ sending each manifold M to the space of immersions of M in N.

2. The sheaf $\underline{\mathrm{Emb}}(\_, N)$ sending each manifold M to the space of embeddings of M in N.

3. The sheaf Conf of configuration spaces sending any manifold M to the space of finite subsets of M, topologised in such a way that points may "disappear off to infinity" when M is open (See [RW11, §3]).

An invariant sheaf F is ***microflexible*** ([RW11, Def. 5.1]) if for

(i) any polyhedron K,

(ii) any manifold M,

(iii) compact subsets $A \subseteq B \subseteq M$, and

(iv) subsets $U \subseteq V \subseteq M$ containing A and B, respectively,

the lifting problem

$$\begin{tikzcd}
& \{0\} \times K \arrow[r] \arrow[dl] \arrow[d] & F(V) \arrow[d] \\
{[0,\varepsilon]} \arrow[r] \arrow[urr, dashed] & {[0,1] \times K} \arrow[r] & F(U)
\end{tikzcd} \tag{1}$$

admits a solution for some $0 < \varepsilon < 1$, possibly after passing to a smaller pair $U \subseteq V$ containing A and B, respectively. Of the above examples 1. and 3. are microflexible, while 2. is not.

A microflexible sheaf F satisfies the h-principle on any open manifold M, i.e. the *scanning map* (see [Fra11, Lect. 17])

$$\mathrm{scan} : F(M) \to \Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big) \tag{2}$$

is a homotopy equivalence (see [Fra11, Lect. 20]); the study of $\Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big)$ is often easier than that of $F(M)$. This result relies crucially on the ability to simultaneously view F as a sheaf on the underlying ordinary category of $\mathbf{Emb}_n^\infty$, but also as a homotopy-type-valued presheaf on the ∞-category obtained by replacing the mapping spaces in $\mathbf{Emb}_n^\infty$ with their underlying homotopy types; the microflexibility condition helps mediate between these two aspects of F by exhibiting the sheaf condition as homotopically meaningful.

1.1.1 Summary of issues

Summarising the above discussion:

1. There are numerous subtleties involved in extracting homotopy types from spaces considered in geometric topology.

2. It is difficult to construct suitable topologies on these spaces.

3. Many of these spaces in fact often appear to admit a smooth structure.

Moreover, the statement of the microflexibility condition would be cleaner if we were able to replace the morphism $F(V) \to F(U)$ in (1) with

$$\operatorname*{colim}_{V \supseteq B} F(V) \to \operatorname*{colim}_{U \supseteq A} F(U), \tag{3}$$

but the colimits in (3) are pathological in two ways:

4. They may not retain any of the homotopical information of the spaces $F(U)$.

5. The canonical map

$$\operatorname*{colim}_{U \supseteq A} \operatorname{Hom}(P, F(U)) \to \operatorname{Hom}(P, \operatorname*{colim}_{U \supseteq A} F(U))$$

may not be an isomorphism for any suitably compact space P.

Gromov indeed formulates the microflexibility condition using (3) in [Gro86, §1.4.2], but takes the colimit in a variant of quasi-topological spaces (introduced by Spanier; [Spa63]) rather than topological space with the intent of ensuring that 5. holds (as explained in [Gro86, §1.4.1]).

The construction of the scanning map in (2) involves carefully choosing and then modifying an exponential function $\exp : TM \to M$ (see [RW11, §6]). In order to obtain a scanning map which works for any exponential function, Ayala constructs a variant of the associated bundle (2) using the stalk $\operatorname{colim}_{\delta>0} F\big(\mathring{B}^n{}_\delta(0)\big)$ rather than $F(\mathbf{R}^n)$ (see [Aya09, p. 19]). Like Gromov, Ayala takes the colimit in a variant of quasi-topological spaces. During a scenic drive through the outskirts of Bozeman, Montana, Ayala explained to us that this was to ensure that the colimit is a homotopy colimit (thus addressing 4.), and kindly invited us to think about the example below. To our deep-seated consternation, we discovered that we had, in fact, been provided with a *counter*example to Ayala's claim!

Example 1.1.1. *For each $\delta > 0$ the space $\mathrm{Conf}\big(\mathring{B}^n_\delta(0)\big)$ is weakly equivalent to S^n. In Ayala's variant of quasi-topological spaces the colimit is equivalent to the Sierpinski space, which is contractible. In other variants of quasi-topological spaces one still obtains a contractible two-point space.* ⌟

The issues with Gromov's formulation of microflexibility are often fixed using the formulation in (1). Using the more traditional scanning map (2), it might be possible to recover the results in [Aya09], but at the cost of a considerable increase in technicality.

In the following section we introduce and discuss *differentiable stacks* and *spaces*. These address issues 1.-5. mentioned above, as well as issues 6. & 7. which we come across below. Moreover, replacing quasi-topological spaces with differentiable spaces would allow for Gromov's formulation of microflexibility and the sheaf theoretic h-principle, and we believe that this would also make Ayala's arguments work as written in [Aya09].

Differentiable spaces have already been used as a replacement for (quasi-)topological spaces to solve geometric topological problems in [GTMW09] and [Kup19].

1.2 Differentiable stacks and spaces

Let $\mathbf{Cart}^\infty$ be the category consisting of the spaces $\mathbf{R}^n$ ($0 \leq n < \infty$) and smooth maps between them, then $\mathbf{Diff}^\infty$ is the category of homotopy-type-valued sheaves on $\mathbf{Cart}^\infty$ w.r.t. the usual Grothendieck topology. Objects in $\mathbf{Diff}^\infty$ are called ***differentiable stacks***, and objects in the 0-truncation $\mathbf{Diff}^\infty_{\leq 0}$ of $\mathbf{Diff}^\infty$ are called ***differentiable spaces***. Observe that the restricted Yoneda embedding exhibits the category of smooth manifolds as a full subcategory of $\mathbf{Diff}^\infty_{\leq 0}$.

Cisinski constructs a model structure on $\mathbf{Diff}^\infty_{\leq 0}$ ([Cis03, §6.1]) modelling the theory of homotopy types, whose weak equivalences are maps inducing isomorphisms on cohomology of locally constant sheaves, and whose cofibrations are monomorphisms. Working with $\mathbf{Diff}^\infty_{\leq 0}$ tautologically addresses 3., and also addresses 2., as it is generally easier to define smooth maps into an object than to define a topology. It is possible to deduce from Cisinski's theory that filtered colimits are homotopy colimits, thus addressing 4., but we will recover this statement using a simpler argument in §1.2.1. The availability of continuous (in fact smooth) maps $\underline{\mathbf{Emb}}^\infty_n(M,N) \times F(M) \to F(N)$ often follows formally from some Cartesian closedness argument. Our first contribution in this thesis addresses issue 5.

Theorem A. *Closed manifolds are compact in the categorical sense in* $\mathbf{Diff}^\infty_{\leq 0}$. $\square$

Unfortunately, we are unaware of any explicit characterisation of the fibrations in Cisinski's model structure, making it hard to compute homotopy limits. Moreover, the works [GTMW09] and [Kup19], cited above, use the nerve construction determined by

$$\begin{array}{rccl} \mathbf{A}^\bullet : & \Delta & \to & \mathbf{Cart}^\infty \\ & \Delta^n & \mapsto & \left\{ (x_0, \dots, x_n) \in \mathbf{R}^{n+1} \;\middle|\; x_0 + \cdots + x_n = 1 \right\} \end{array}$$

to extract underlying homotopy types, and to define weak equivalences, called *concordance equivalences*, which, a priori, are different from Cisinski's weak equivalences, raising new questions about issue 1., considered in §1.1.1. In [GTMW09] it is furthermore shown that for any manifold M and any differentiable space F, viewed as a sheaf on the category of manifolds, the canonical map

$$\pi_0(F(M \times \mathbf{A}^\bullet)) \to \pi_0\underline{\mathbf{SSet}}(M(\mathbf{A}^\bullet), F(\mathbf{A}^\bullet))$$

is a bijection, which echoes the discussion in §1.1 on extracting homotopy types of mapping spaces.

Thus, there arise renewed questions about 1. of §1.1.1, as well as two more issues:

6. Having extracted homotopy types from two differentiable stacks X and Y in some suitable manner, how does one calculate their mapping space in terms of smooth maps $X \to Y$.

7. How does one calculate homotopy limits in $\mathbf{Diff}^{\infty}_{\leq 0}$ and $\mathbf{Diff}^{\infty}$?

In our thesis we continue the development of Cisinski's theory using ∞-categorical tools, to address these remaining issues, and in order to provide a navigable theory with a view towards applications in geometric topology and related fields.

Remark 1.2.1. The category $\mathbf{Diff}^{\infty}_{\leq 0}$ contains the diffeological spaces (§5.2), as a full subcategory. These are differentiable spaces which may be characterised as an underlying set together with extra structure. This makes working with them more similar to working with manifolds or topological spaces. One may wonder whether the category of diffeological spaces has the same good properties as $\mathbf{Diff}^{\infty}_{\leq 0}$. Unfortunately, diffeological spaces suffer from many of the same defects as those of quasi-topological spaces discussed in §1.1.1. This extension from sets with structure to sheaves is part of a wider trend, as can be seen for instance with condensed/pyknotic sets and spaces ([Sch19], [BH19]), another replacement for topological spaces and which are the object of intense study. ⌟

1.2.1 Moving $\mathbf{Diff}^{\infty}_{\leq 0}$ to $\mathbf{Diff}^{\infty}$

Working in $\mathbf{Diff}^{\infty}$ both simplifies and illuminates the theory.

Let $\mathcal{X}$ be an ∞-topos, and denote by $\pi : \mathcal{X} \to \mathcal{S}$ the unique geometric morphism to the ∞-topos of homotopy types, then $\mathcal{X}$ is ***locally ∞-connected*** if the constant sheaf functor $\mathcal{X} \leftarrow \mathcal{S} : \pi^*$ admits a left adjoint $\pi_! : \mathcal{X} \to \mathcal{S}$ ([Hoy18, Def. 3.2], [Lur17, Prop. A.1.8]). In this case, $\pi_! : \mathcal{X} \to \mathcal{S}_{/\pi_! 1}$ admits a fully faithful right adjoint, exhibiting $\mathcal{S}_{/\pi_! 1}$ as a localisation of $\mathcal{X}$ ([Lur17, Prop. A.1.11]). For any object X in $\mathcal{X}$ the homotopy type $\pi_! X$ is the ***shape*** of X ([Lur17, Rmk. A.1.10]). A morphism $X \to Y$ in $\mathcal{X}$ is a ***shape equivalence*** if $\pi_! X \to \pi_! Y$ is an equivalence. Shapes offer a canonical notion of underlying homotopy type to which other notions may be compared.

A careful reading of [Cis03, §5] reveals that the weak equivalences in Cisinski's model structure are precisely the shape equivalences (defined by restricting $\pi_! : \mathbf{Diff}^\infty \to \mathcal{S}$ to $\mathbf{Diff}^\infty_{\leq 0}$). We prove that the inclusion $\mathcal{X}_{\leq 0} \hookrightarrow \mathcal{X}$ preserves filtered colimits and pushouts along monomorphisms (see §3.3.2). As $\pi_!$ is a left adjoint, it commutes with colimits, so that one recovers in $\mathbf{Diff}^\infty$ the homotopy colimits made available by Cisinski's model structure.

The fact that the spaces $\mathbf{R}^n$ have contractible shape (see §1.2.3) may be used to show that $\mathbf{Diff}^\infty$ is locally ∞-connected, exhibiting $\mathcal{S}$ as a localisation of $\mathbf{Diff}^\infty$. The embedding $\mathbf{Diff}^\infty \hookrightarrow \underline{\mathrm{Hom}}((\mathbf{Cart}^\infty)^{\mathrm{op}}, \mathcal{S})$ preserves shapes and detects shape equivalences. The shape of any presheaf is given by its colimit, and the functor $\mathbf{A}^\bullet : \Delta \to \mathbf{Cart}^\infty$ is initial, so that concordance equivalences and shape equivalences agree. Moreover, $\mathbf{Cart}^\infty$ is a test category (Theorem 6.1.1) which implies the surprising fact that the restriction of $\pi_!$ to a functor $\mathbf{Diff}^\infty_{\leq 0} \to \mathcal{S}$ is still a localisation.

We have made progress resolving issue 1., but we are still left with issues 6. and 7.

1.2.2 Fibrations in $\mathbf{Diff}^\infty$ and formally cofibrant differentiable stacks

The fact that $\mathbf{Cart}^\infty$ is sifted ([Lur09, Def. 5.5.8.1]) implies that the shape functor of $\mathbf{Diff}^\infty$ preserves finite products (see [Lur09, Prop. 5.5.8.11]), so that for any two differentiable stacks A, X there exists a comparison morphism

$$\pi_! \underline{\mathbf{Diff}}^\infty(A, X) \to \mathcal{S}(\pi_! A, \pi_! X). \tag{4}$$

We say that A is ***formally cofibrant*** if (4) is an equivalence for all differentiable stacks X. Not all differentiable stacks are formally cofibrant, as is the case, for example, for the real line with two origins. The problems of characterising formally cofibrant stacks and calculating homotopy pullbacks are closely related.

Functors such as $\mathbf{A}^\bullet : \Delta \to \mathbf{Diff}^\infty_{\leq 0}$ may be used to transfer model structures (in this case from $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S})$ or $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathbf{Set})$ to $\mathbf{Diff}^\infty$ or $\mathbf{Diff}^\infty_{\leq 0}$ respectively (see Proposition 3.4.4). Ideally, one could use this method to construct model structures which are Cartesian, and in which all objects are fibrant, so that cofibrant objects are formally cofibrant. Faced with a similar problem of constructing a model structure on the category of diffeological spaces, Kihara constructs a new diffeology on the standard simplices ([Kih19, § 1.2]) for which the horn inclusions admit a retract ([Kih19, § 8]). The inclusion $\Delta \to \mathbf{Diff}^\infty$ of these simplices into diffeological spaces produces a model structure in which all objects are fibrant (see Proposition 6.2.3), however, it is not Cartesian (see Proposition 6.2.7).

Fortunately, formally cofibrant stacks are closed under cell-attachment along boundary inclusions of Kihara's simplices, because for any differentiable stack X and any

$n \geq 0$ pullbacks along the maps $X^{\Delta^n} \to X^{\partial\Delta^n}$ are homotopy pullbacks. To show this we define a diagram $\square \to \mathrm{Pro}(\mathbf{Diff}^\infty)$ indexed by the cube category (see Definition 6.3.9) which produces a fibration structure (rather than a full model structure) in which the maps $X^{\Delta^n} \to X^{\partial\Delta^n}$ are fibrations (see Theorem 6.3.19). Formal cofibrancy is stable under $\mathbf{R}$-homotopy equivalence, allowing us to recover the main theorem of [BEBP19] (see Corollary 6.4.8).

Theorem B. *Paracompact Hausdorff smooth manifolds are locally cofibrant.* □

These techniques serve as a paradigm of how it is possible to use the ambient theory developed in our thesis to construct and mix various homotopical calculi tailored to various applications.

1.2.3 $\mathbf{Diff}^\infty$ as a fractured topos

Denote by $\mathbf{Cart}^\infty_{\text{ét}}$ the subcategory of $\mathbf{Cart}^\infty$ containing only the open embeddings, then $\mathbf{Diff}^\infty_{\text{ét}}$ is the ∞-topos of sheaves of homotopy types on $\mathbf{Cart}^\infty_{\text{ét}}$. Objects in $\mathbf{Diff}^\infty_{\text{ét}}$ are called ***étale differentiable stacks***, and the objects of $\mathbf{Diff}^\infty_{\leq 0,\text{ét}}$ are the ***étale differentiable spaces***. The inclusion $\mathbf{Diff}^\infty_{\text{ét}} \to \mathbf{Diff}^\infty$ forms the structure of a *fractured topos* (see §4), and may be shown to be shape preserving. The shape of $\mathbf{R}^n$ in $\mathbf{Diff}^\infty_{\text{ét}}$ is the same as its shape in the category of sheaves on $\mathbf{R}^n$, as a topological space, which by [Lur17, Lms. A.2.2 & A.2.9] is contractible; a fact which is used in §1.2.1.

The ∞-category $\mathbf{Diff}^\infty$ admits an obvious continuous analogue $\mathbf{Diff}^0$. The forgetful functor $\mathbf{Diff}^\infty \to \mathbf{Diff}^0$ sends any smooth manifold to its underlying topological manifold, yielding the following contribution to addressing issue 1.

Theorem C. *The shape of any smooth manifold is equivalent to the shape of its underlying topological manifold.* □

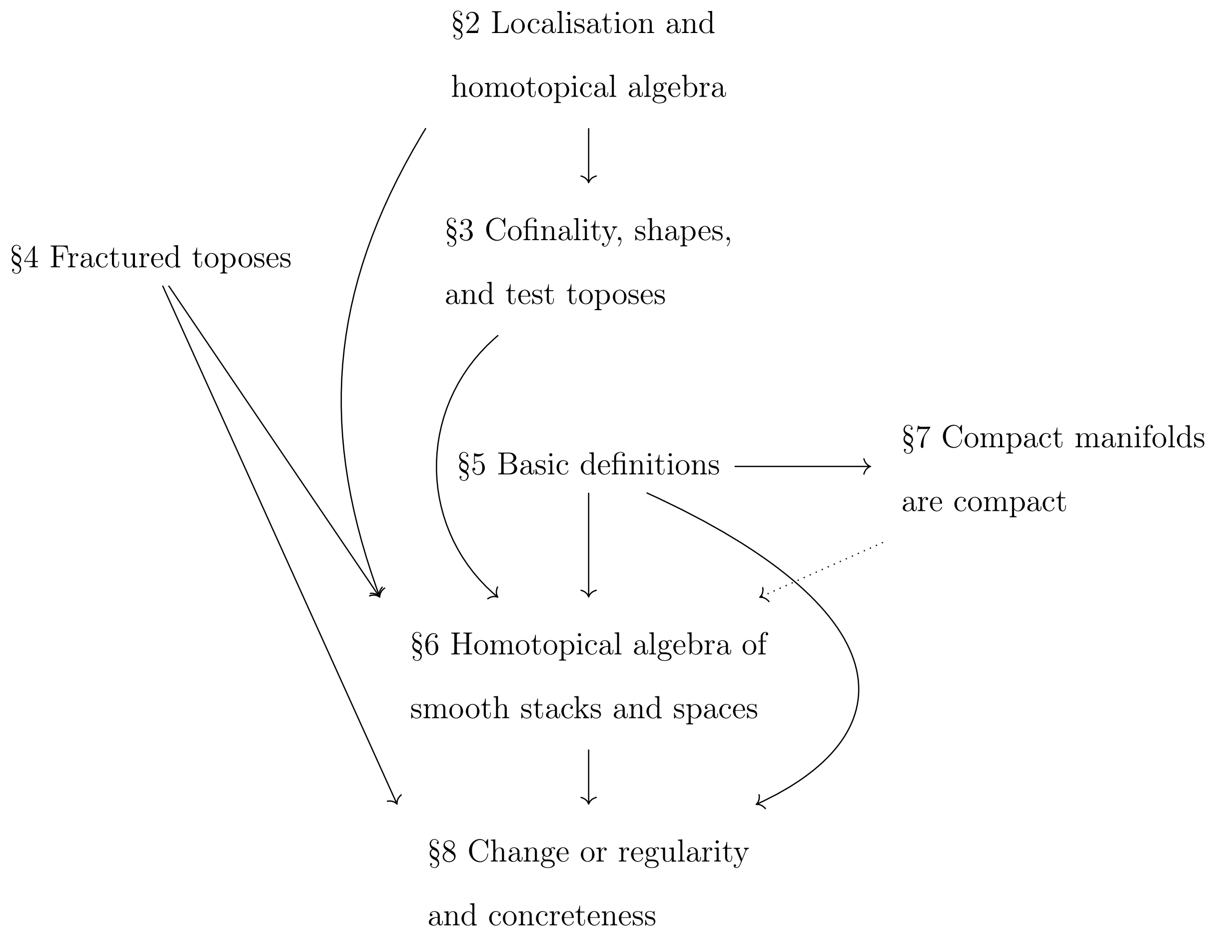
Repeating the process in §1.2.1 we then see that the shape of a topological manifold is given by its total singular complex. Note, that our proof entirely avoids the approximation of continuous maps by smooth ones.

1.2.4 A sample application

It is a commonly invoked fact that the quotient map of a topological principal bundle is a homotopy quotient map. However, the only proof of this that we are aware of, [May75, Thm. 7.6], is very technical and not well known.
Let $E \to B$ be a principal G-bundle, which without loss of generality may be assumed to be Δ-generated ([Dug03]); such spaces form a full subcategory of $\mathbf{Diff}^0_{\leq 0}$. It is then straightforward to prove that the quotient of E by G in $\mathbf{Diff}^0$ coincides with B, so that the given quotient indeed has the correct shape (see Theorem 8.2.6).

1.3 Leitfaden and some remarks on the exposition

The following diagram indicates the logical dependencies of the different sections in this thesis.



One of the central themes of this thesis concerns the extraction of underlying homotopy types of objects in a (∞-)topos, leading to notions of weak equivalences in any (∞-)toposes. Section 2 lays the theoretical foundation for speaking about localisation, which we rely on in the sequel. Given a geometric morphism $\mathcal{X} \to \mathcal{Y}$ between (∞-)toposes we need a good understanding of cofinality in order to compare underlying homotopy types in $\mathcal{X}$ and $\mathcal{Y}$; this is the subject of §3. In §5 we introduce the central characters in this thesis, the ∞-topos $\mathbf{Diff}^\infty$ of differential stacks, and its ordinary subcategory $\mathbf{Diff}^\infty_{\leq 0}$ of differentiable spaces. The ∞-topos $\mathbf{Diff}^\infty$ may be endowed with additional structure, turning it into a *fractured ∞-topos*; fractured ∞-toposes are the subject of §4. Exhibiting $\mathbf{Diff}^\infty$ as a fractured ∞-topos in §6.1 is the first step in showing that we may apply to $\mathbf{Diff}^\infty$ the theory developed in §2 & §3. In §6.3 we construct the *squishy fibration structure*, which we use in §6.4 to show that simplicial complexes built using Kihara's simplices – discussed in §6.2 – and moreover smooth manifolds are *formally cofibrant* (see Definition 6.0.1). In §7 we show that closed manifolds and special intervals are categorically compact. This section is essentially independent from the other sections. One exception is of course that §7 makes use of the definitions introduced in §6.1. The other is in the construction (in a non-essential way; see Remark 6.3.22) in the proof of the existence of the squishy fibration structure. Finally, in §8 we examine how the shape of a differentiable stack is affected by either decreasing its regularity, i.e. turning it into a continuous stack, or increasing its concreteness, by e.g. turning it into a diffeological space, leading to the proof of Theorem C.

We also summarise some of the most important conventions adopted in this thesis. For a detailed account of our conventions, see .

- Following the widely adopted precedent set by Lurie we will refer to quasi-categories as *∞-categories.*
- An ∞-category is however still, strictly speaking, a simplicial set, and thus we will often speak of maps from a simplicial set to an ∞-category, etc.

- For any small category A we adopt the "French" tradition of denoting the ordinary category of presheaves on a small category A by $\widehat{A}$. So the category of simplicial sets is then denoted by $\widehat{\Delta}$.

Part I

Foundations

2 Localisation and homotopical algebra

Throughout this section (C, W) denotes a ***relative*** ∞***-category***, i.e. an ∞-category C together with a subcategory containing all equivalences. It is then natural to study the relationship between C and its localisation $W^{-1}C$; in particular, one may ask which limits in $W^{-1}C$ may be obtained via constructions in C.

Definition 2.0.1. Let K be a simplicial set, then a functor $p : K^{\triangleleft} \to C$ is called a ***homotopy limit*** of $p|_K : K \to C$ if the composition of $K^{\triangleleft} \to C \to W^{-1}C$ is a limit of the composition of $K \xrightarrow{p|_K} C \to W^{-1}C$. A functor $K^{\triangleright} \to C$ is a ***homotopy colimit*** if $(K^{\triangleright})^{\mathrm{op}} \to C^{\mathrm{op}}$ is a homotopy limit. $\lrcorner$

Beginning with the simplest case of a homotopy (co)limit we recall that the localisation functor $\gamma : C \to W^{-1}C$ is both initial and final (see Proposition 3.1.3), so that if x_0 is an initial or final object of C, then $\gamma(x_0)$ is an initial or final object of $W^{-1}C$. Thus, if C has final objects, then $W^{-1}C$ admits all limits iff it admits all pullbacks. The theory that follows is well suited to studying this question.

Definition 2.0.2. A morphism $x' \to x$ in C is ***sharp*** if for every morphism $b \to x$

and every weak equivalence $a \xrightarrow{\sim} b$ there exists a diagram

$$\begin{tikzcd} a' \arrow[r] \arrow[d] & b' \arrow[r] \arrow[d] & x' \arrow[d] \\ a \arrow[r] & b \arrow[r] & x \end{tikzcd}$$

in which all squares are pullbacks and such that $a' \to b'$ is a weak equivalence. ⌟

Definition 2.0.3. An object x in C is called ***right proper*** if the natural functor $W_{/x}^{-1}C_{/x} \to (W^{-1}C)_{/x}$ is an equivalence. The relative category (C, W) is called ***right proper*** if all objects in C are right proper. ⌟

Remark 2.0.4. A model category is right proper in the usual sense iff its underlying relative category is right proper. This may be seen by combining [Rez02, Prop. 2.7] with [Cis19, Cor. 7.6.13][1]. ⌟

Proposition 2.0.5. *Let $f : x' \to x$ be a sharp morphism in C between right proper objects, then any pullback along $f : x' \to x$ is a homotopy pullback.*

Proof. By the sharpness assumption the functor $C_{/x'} \leftarrow C_{/x} : f^*$ preserves weak equivalences, so that [Cis19, Prop. 7.1.14] yields, canonically a commutative diagram

$$\begin{tikzcd} C_{/x'} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] \arrow[d] & C_{/x} \arrow[l, shift left, "f^*"] \arrow[d] \\ W^{-1}C_{/x'} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & W^{-1}C_{/x} \arrow[l, shift left, "f^*"] \end{tikzcd}$$

[1] Rezk's proof of [Rez02, Prop. 2.7] can be interpreted verbatim in model ∞-categories, so that the remark is in fact true for model ∞-categories.

The pullback of any morphism $y \to x$ along f in C thus yields the pullback of $y \to x$ along f in $W^{-1}C$. $\square$

Structures similar to the one in the following condition were first considered in the 1-categorical setting by [And78].

Definition 2.0.6. Assume that C admits a final object, then a ***fibration structure*** on C consists of a subcategory $F \subseteq C$, such that W and F satisfy the following conditions:

(a) F contains all equivalences in C.

(b) W satisfies the 2-out-of-3 property.

The morphisms in W, F, and $F \cap W$ are called *weak equivalences*, *fibrations*, and *trivial fibrations* respectively. An object x for which some (and therefore any) morphism to a final object of C is a fibration is called *fibrant*. Furthermore:

(c) In any diagram

$$\begin{tikzcd} & x' \arrow[d, "f"] \\ y \arrow[r] & x \end{tikzcd}$$

such that x, x', y are fibrant, and such that f is either a fibration, or trivial fibration, the pullback exists and is again a fibration of trivial fibration, respectively.

(d) Any morphism $x \to y$ where y is fibrant admits a factorisation $x \to x' \to y$ such that $x' \to x'$ is a weak equivalence, and $x' \to y$ is a fibration.

An ∞-category equipped with a fibration structure is called a ***fibration category***. $\lrcorner$

Proposition 2.0.7 ([Cis19, Thm. 7.5.18 & Cor. 7.6.13])**.** *In a fibration category any fibrant object is right proper, and any fibration between fibrant objects is sharp.* $\square$

Remark 2.0.8. The above result requires a lot of bootstrapping (from the point of view presented here). In particular, the proof requires the fact (established by independent means) that pullbacks along fibrations between fibrant objects are homotopy pullbacks. $\lrcorner$

Remark 2.0.9. Often, one of the most important consequences of the existence of a fibration structure is the fact that the localisation is finitely complete ([Cis19, Th. 7.5.18]). In this work the homotopy category will always be $\mathcal{S}$, so this aspect is not relevant here. $\lrcorner$

In a fibration category the map from any fibrant object to the terminal object is sharp, and thus finite products of fibrant objects are homotopy products.

Proposition 2.0.10 ([Cis19, Prop. 7.7.4])**.** *If arbitrary products preserve fibrant objects and trivial fibrations between fibrant products, then arbitrary products of fibrant objects are homotopy products.* $\square$

Example 2.0.11. *The classes of weak equivalences and fibrations of any ∞-model category (see [MG14]) form a fibration structure, which moreover satisfies the condition of Proposition 2.0.10 if it admits all limits.* $\lrcorner$

A model structure may be viewed as a fibration structure together with the dual notion of cofibration structure stitched together in a compatible way, or, conversely, a fibration structure may be viewed as "half" a model structure. In the relative ∞-categories considered in this thesis homotopy colimits can be constructed more efficiently by means other than cofibration structures (see §3.3.2). Dually, calculating homotopy limits is hard, which is why we concentrate on identifying sharp maps, using fibration structures. This does not mean that we can dispense with model structures altogether. All (trivial) fibrations considered in this thesis are obtained as morphisms satisfying the right lifting property against sets of (trivial) cofibrations; these latter morphisms will also be used to construct what we call *formally cofibrant objects* (see §6.4), which have similar good formal properties as those enjoyed by cofibrant objects in a model ∞-category.

Remark 2.0.12. Model categories and ∞-categories are frequently viewed as providing competing foundations for homotopy theory (see [MO78400]). In reality, the axioms for model categories can be interpreted without difficulty for all ∞-categories, not just ordinary categories, and model structures constitute tools for studying localisation. Any ∞-category may be obtained as the localisation of an ordinary category (see [Cis19, Prop. 7.3.15], [BK12]), and any presentable ∞-category may be obtained as the localisation of a combinatorial simplicial model category (see [Lur09, Prop. A.3.7.6] & [Lur17, Thm. 1.3.4.20] & [Cis19, Thm. 7.5.18]). Before the work of Joyal, Simpson, Toën, Rezk, Lurie and many others it was simply not practical to present a given ∞-category in any other way than as a relative ordinary category (or a simplicially enriched category). Thus, nowadays, one has a *choice* of whether one wishes to work in a given ∞-category C, or whether one wishes to view C as the localisation of some other (∞-)category D. The optimal choice of D does not necessarily have to be an ordinary category; Mazel-Gee developed the theory model ∞-categories precisely in order to generalise the Goerss-Hopkins obstruction theorem (see [MG16]), and as we show in discussing differentiable stacks in our thesis. $\lrcorner$

3 Cofinality, shapes, and test toposes

Every locally ∞-connected ∞-topos $\mathcal{X}$ (see Definition 3.3.4) comes equipped with a canonical functor $\pi_! : \mathcal{X} \to \mathcal{S}$, assigning to each object in $\mathcal{X}$ its *shape*, which provides a canonical notion of underlying homotopy type or fundamental ∞-groupoid, (see Definition 3.3.1). In this section we establish the necessary background in order to determine when $\pi_!|_{\mathcal{X}_{\leq n}} : \mathcal{X}_{\leq n} \to \mathcal{S}_{\pi_! \mathbf{1}_{\mathcal{X}}}$ is a localisation (§3.3), and how to construct interesting homotopical calculi on $\mathcal{X}$ when this is the case (§3.4). The key example where this is possible is that of a *test topos* (see Definition 3.3.27), due to Cisinski. A test topos is a topos $\mathcal{E}$ which is locally contractible (see Definition 3.3.16), and in which there is an ample supply of monomorphisms. The construction of homotopical calculi on test toposes is achieved by comparing them to presheaf categories on test categories, which are introduced in §3.2. The theory of test categories as well as the comparison rely heavily on notions of cofinality, which we study in §3.1.

The first two subsections of §3 make heavy use of the correspondence between (left) right fibrations and (co)presheaves, which we now briefly recall. For any small simplicial set A there exist two model structures on $\widehat{\Delta}_{/A}$, the ***covariant*** and ***contravariant model structures***, which model, respectively, the theory of left and right fibrations over A (see [Cis19, Thms. 4.1.5 & 4.4.14]), uniquely determined by having as fibrant objects, respectively, the left fibrations and the right fibrations. The weak equivalences are called ***covariant*** and ***contravariant equivalences***, respectively. If $A = \Delta^0$, then the two model structure coincide with the Kan-Quillen model structure, and a left or right fibration over the point is precisely a Kan complex. In this case we refer to the weak equivalences as ***homotopical equivalences***.

For any map $u : A \to B$ between small simplicial sets we obtain the upper adjunction in the following diagram

$$\begin{tikzcd}
\widehat{\Delta}_{/A} \arrow[r, shift left, "u_!"] \arrow[r, phantom, "\perp"] \arrow[d] & \widehat{\Delta}_{/B} \arrow[l, shift left, "u^*"] \arrow[d] \\
\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \arrow[r, shift left, "u_!"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S}) \arrow[l, shift left, "u^*"]
\end{tikzcd}$$

where $u_!$ is given by composing with u and u^* is given by pullback; taking derived functors we obtain the lower adjunction together with (specified) natural transformations, compatible with the adjunction data making the whole diagram commute (see [Cis19, Thm. 7.5.30]) up to homotopy. The lower left adjoint is given by Kan extension, and the lower right adjoint is given by precomposition. Consider a presheaf $X : A^{\mathrm{op}} \to \mathcal{S}$, and denote by $A_{/X}$ its associated right fibration[2], then $u_!X$ may be obtained by taking a fibrant replacement of the composition $A_{/X} \to A \to B$, and then choosing a presheaf $B^{\mathrm{op}} \to \mathcal{S}$ which classifies the resulting right fibration.

In the co(ntra)variant model structure over a point, the fibrant replacement may be given by the Ex^∞-functor. Let X be a simplicial set, then $X \to \mathrm{Ex}^\infty X$ exhibits the localisation of X by all its 1-simplices. (Observe that the Ex^∞ itself is the colimit of the functors Ex^n, which adjoin longer and longer zigzags of 1-simplices to X.) We shall denote any fibrant replacement of a simplicial set X in the Kan-Quillen model structure by $X_{\simeq}$. For $u : A \to \Delta^0$ the Kan extension along u coincides with the

[2]Thanks to the functorial Yoneda lemma ([Cis19, Th. 5.8.13]) the right fibration $A_{/X}$ can, of course, literally be taken to be the pullback of $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})_{/X}$ along the Yoneda embedding $h : A \hookrightarrow \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$.

colimit functor. Thus for any map $X \to A$, the colimit of the presheaf corresponding to X is just $X_{\simeq}$. In particular, one sees that the colimit of the constant presheaf on A with values in 1 produces the homotopy type $A_{\simeq}$.

3.1 Cofinality

Cofinality plays a fundamental role in category theory and appears in many different guises. Final functors were first introduced as functors which interact well with colimits in [SGA 4_{I}, §1.8.1] and this is the aspect which is most well known. In this subsection we give some equivalent characterisations of final functors, before moving on to test categories and test toposes in the following two subsections.

Proposition 3.1.1. *Let $A \to B$ be a morphism of simplicial sets, then the following are equivalent;*

(I) For any ∞-category C and any morphism $f : B \to C$, the colimit of f exists iff the colimit of $f \circ u$ exists, in which case the canonical morphism $\operatorname{colim} f \circ u \to \operatorname{colim} u$ *is an equivalence.*

(II) For any functor $f : B \to \mathcal{S}$ the canonical morphism $\operatorname{colim} f \circ u \to \operatorname{colim} u$ *is an equivalence*[3]*.*

(III) For any left fibration $X \to B$ the induced map $A \times_B X \to X$ is a weak equivalence in the Kan-Quillen model structure.

(IV) For any simplex $b \in B_0$ the induced map $A_{b/} \to B_{b/}$ is a weak equivalence in the Kan-Quillen model structure.

[3]Here $\mathcal{S}$ is constructed using a universe w.r.t. which A and B are small.

$\square$

Definition 3.1.2. A map $A \to B$ of simplicial sets satisfying equivalent conditions of Proposition 3.1.1 is called ***final***, and ***initial*** if $A^{\mathrm{op}} \to B^{\mathrm{op}}$ is final. $\lrcorner$

One of the main advantages of characterisation (IV) of finality is that when A and B are both ordinary categories, then this characterisation involves only categories, despite being an ∞-categorical notion.

We used the following example in the preceding section.

Proposition 3.1.3. *Any localisation is both final and initial.*

Proof. Let (C, W) be a relative ∞-category, and denote by $\gamma : C \to L$ its localisation. As L^{op} is the localisation of C^{op} it is enough to show that $C \to L$ is initial. By the universal property of localisations the map $\gamma^* : \underline{\mathrm{Hom}}(L, \mathcal{S}) \to \underline{\mathrm{Hom}}(C, \mathcal{S})$ is fully faithful so that for any functor $f : L \to \mathcal{S}$ the counit map $\gamma_! \gamma^* f \to f$ is an equivalence. I.e., f is the left Kan extension of $\gamma^* f$, so that they have the same colimit. $\square$

Let A be a small simplicial set, and denote by e the constant presheaf on A with value 1, then the colimit functor $\operatorname{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ factors as

$$\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \simeq \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})_{/e} \to \mathcal{S}_{/\operatorname{colim} e} \simeq \mathcal{S}_{/A_{\simeq}} \to \mathcal{S}.$$

The functor $\operatorname{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{/A_{\simeq}}$ admits a fully faithful right adjoint (see Proposition 3.3.7), which can be shown to be given by the inclusion $\underline{\mathrm{Hom}}(A^{\mathrm{op}}_{\simeq}, \mathcal{S}) \hookrightarrow \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$. Presheaves in the essential image of this inclusion are the ***locally constant presheaves***. A map of presheaves $X \to Y$ such that $\operatorname{colim} X \to \operatorname{colim} Y$

is an equivalence is called a ***locally constant weak equivalence***. The next two sections are concerned with the study of variants of the localisation of $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ along locally constant weak equivalences.

3.2 Test categories

The main ideas discussed in this subsection are essentially all due to Grothendieck, and were first outlined in [Gro83]. A systematic account of Grothendieck's theory is given by Maltsiniotis in [Mal05]. The theory of test categories, and in particular its model categorical aspects, are further developed in [Cis06]. The theory of test toposes is developed in [Cis03].

Given a small simplicial set A, we have shown how $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ models the ∞-category $\mathcal{S}_{/A_{\simeq}}$ in the sense that there is a localisation functor $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{/A_{\simeq}}$. In the special case $A = \Delta$ something rather remarkable happens. The restriction of $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{/\Delta_{\simeq}} \xrightarrow{\sim} \mathcal{S}$ to $\widehat{\Delta} \to \mathcal{S}$ is still a localisation. As the construction of the model category of simplicial sets is quite involved, one might expect this phenomenon to be particular to Δ, but it turns out to be surprisingly common. The starting point for understanding the above phenomenon is the following fact, discussed in the beginning of §3: The classifying space of an ∞-category is nothing but the homotopy type obtained by inverting all its arrows. The classifying space construction is furthermore left adjoint of the inclusion of homotopy types into ∞-categories. Again, surprisingly, and paralleling the situation for $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S})$, the restriction of the classifying space functor to ordinary categories exhibits $\mathcal{S}$ as a localisation of $\mathcal{C}$, and since $\mathcal{C}$ is a localisation of **Cat**, the ∞-category $\mathcal{S}$ is likewise a localisation of **Cat**.

$$\mathbf{Cat} \longrightarrow \mathcal{C} \longrightarrow \mathcal{Q} \underset{\longleftarrow}{\overset{(\_)_{\simeq}}{\underset{\perp}{\longrightarrow}}} \mathcal{S}$$

This fact has been known in essence since [Ill72, Cor. 3.3.1] (specifically, the fact that the category of elements of a simplicial set encodes the same homotopy type as the simplicial set itself is shown in [Ill72, Th. 3.3.ii]. Illusie attributes the ideas presented in [Ill72, §3.3] to Quillen; see also [Qui73]). The relative category **Cat** can be shown to be right proper, by exhibiting a right proper model structure on **Cat** by right transferring the Kan-Quillen model structure (which is right proper) along the functor $\mathrm{Ex}^2 \circ N : \mathbf{Cat} \to \widehat{\Delta}$ (see [Tho80]). Thus for any small category A the category $\mathbf{Cat}_{/A}$ is a model for $\mathcal{S}_{/A_{\simeq}}$.

It is then natural to try to identify ordinary categories other than A for which $\widehat{A}$ canonically models $\mathcal{S}_{/A_{\simeq}}$. Local test categories are precisely such categories, and their characterisations make it feasable to find examples thereof. In a first instance we will focus on the special case when $A_{\simeq} = *$. In this case the the colimit functor $\mathrm{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ factors as $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\int} \mathcal{Q} \xrightarrow{(\_)_{\simeq}} \mathcal{S}$, which restricts to $\widehat{A} \xrightarrow{\int} \mathbf{Cat} \xrightarrow{(\_)_{\simeq}} \mathcal{S}$. Thus a map $X \to Y$ of set-valued presheaves on any small ordinary category A is a locally constant weak equivalence iff the map $A_{/X} \to A_{/Y}$ induces an equivalence on classifying spaces. The functor

$$\begin{array}{rccc} \int : & \widehat{A} & \to & \mathbf{Cat} \\ & X & \mapsto & A_{/X} \end{array}$$

admits a right adjoint $N_A : C \mapsto (a \mapsto \mathrm{Hom}(A_{/a}, C))$. The functor $\int$ models the left adjoint of the adjunction $\mathrm{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{S}$. One may then ask whether this adjunction lifts to a homotopically meaningful adjunction on the level of small categories.

Definition 3.2.1. A small category A is called

1. a ***weak test category*** if $A_{\simeq} \simeq 1$, and if N_A sends homotopical equivalences to locally constant equivalences;

2. a ***local test category*** if $A_{/a}$ is a weak test category for every object a in A;

3. a ***test category***, if A is a local test category, and its classifying space is contractible;

4. a ***strict test category*** if it is test category, and if it is sifted[4].

⌟

Lemma 3.2.2. *Let A be a small category, then for any small category C and any object c in C the canonical functor $\int N_A C_{/c} \to \left(\int N_A C\right)_{/c}$ is an equivalence.* □

Proposition 3.2.3 ([Mal05, Prop. 1.3.9])**.** *Let A be a small ordinary category, then the following are equivalent:*

(I) *A is a weak test category.*

(II) *The counit $\int N_A C \to C$ is a homotopical equivalence for all ordinary small categories C.*

[4]An ∞-category is *sifted* if its diagonal map is initial.

(III) *The counit* $\int N_A C \to C$ *is a homotopical equivalence for all ordinary small categories* C *such that* $C_{\simeq} \simeq 1$.

(IV) *The counit* $\int N_A C \to C$ *is a homotopical equivalence for all ordinary small categories* C *containing a final object.*

(V) *The counit* $\int N_A C \to C$ *is initial for all ordinary small categories* C.

Proof. The implications (V) $\implies$ (II) $\implies$ (III) $\implies$ (IV) are clear. The preceding lemma establishes (IV) $\implies$ (V). We will prove (I) $\implies$ (III) and (II) $\implies$ (I), which completes the proof.

<u>(I) $\implies$ (III)</u>: Let C be category such that $C \to 1$ is a homotopical equivalence, then $\int N_A C \to \int N_A 1 = A$ is a weak equivalence.

<u>(II) $\implies$ (I)</u>: Choosing $C = 1$ shows that $A_{\simeq} \simeq 1$. Let $C \to C'$ be a homotopical equivalence, then, considering the commutative diagram

$$\begin{tikzcd} \int N_A C \arrow[r] \arrow[d] & C \arrow[d] \\ \int N_A C' \arrow[r] & C' \end{tikzcd}$$

we see that $\int N_A C \to \int N_A C'$ is a homotopical equivalence by the 2-out-of-3 property, which precisely says that $N_A C \to N_A C'$ is a weak equivalence. □

Definition 3.2.4. A map of presheaves $X \to Y$ on A is called locally aspherical if $A_{/X} \to A_{/Y}$ is initial. A presheaf X on A is called locally aspherical if $X \to 1$ is locally aspherical. ⌟

The observation that $N_{A_{/a}} = N_A|_{A_{/a}}$ yields the following corollary:

Corollary 3.2.5 ([Mal05, Prop. 1.5.3]). *Let A be a small ordinary category, then the following are equivalent:*

(I) *A is a local test category.*

(II) *The map $N_A C \to N_A C'$ is locally aspherical for any homotopical equivalence $C \to C'$.*

(III) *$N_A C$ is locally aspherical for any small category C such that $C_{\simeq} \simeq 1$.*

(IV) *$N_A C$ is locally aspherical for any small category C admitting a final object*

□

Corollary 3.2.6. *A local test category A, such that $A_{\simeq} \simeq 1$, is a weak test category.*

□

Corollary 3.2.7. *Any strict test category A is a weak test category.*

Proof. Because A is sifted, $A_{\simeq} \simeq 1$, as $\operatorname{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ preserves final objects.

□

We are not aware of any local test categories identified directly using the above criteria. To obtain criteria that can be checked in practice we introduce a piece of extra structure with which we may try to endow a given category (see Corollary 3.2.11).

Definition 3.2.8. Let M be an ordinary category, admitting a final object 1, then an object I in M with two morphisms $1 \rightrightarrows I$ is called an ***interval*** in M. If M

admits an initial object 0, and the square

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & I \end{array}$$

is a pullback, then I is ***separating interval***. ⌟

Example 3.2.9. *Let $\mathcal{E}$ be an ordinary topos, then the subobject classifier Ω in $\mathcal{E}$ canonically has the structure of a separating interval. The first morphism $1 \to \Omega$ is given by the universal monomorphism, and the second morphism $1 \to \Omega$ classifies the subobject $0 \to 1$.* ⌟

Theorem 3.2.10 ([Mal05, Th. 1.5.6])**.** *Let A be a small category, then the following are equivalent:*

(I) *A is a local test category.*

(II) *The subobject classifier of $\widehat{A}$ is locally acyclic.*

(III) *The category $\widehat{A}$ admits a locally acyclic separating interval.*

□

Corollary 3.2.11. *Let A be a small category admitting finite products and a representable separating interval on $\widehat{A}$, then A is a local test category.* □

Theorem 3.2.12 ([Cis06, Cor. 4.4.20])**.** *If A is a local test category, then the composition of the functors $\int : \widehat{A} \to \mathbf{Cat}_{/A} \to \mathcal{S}_{A_{\simeq}}$ is a localisation of $\widehat{A}$ along the locally constant weak equivalences.* □

Definition 3.2.13. Let A be a small category, then a ***trivial fibration*** is a morphism in $\widehat{A}$ which has the right lifting property w.r.t. all monomorphisms in $\widehat{A}$. ⌟

Theorem 3.2.14 ([Mal05, Th. 1.5.6])**.** *Let A be a small ordinary category, then the following are equivalent:*

(I) *A is a local test category.*

(II) *Any trivial fibration in $\widehat{A}$ is a locally constant weak equivalence.*

(III) *The category $\widehat{A}$ admits a (cofibrantly generated) model structure in which the weak equivalences are the locally constant weak equivalences, and the cofibrations are the monomorphisms.*

Idea of proof. The theorem is proved via the implications (I) $\iff$ (II) $\iff$ (III), which we now sketch:

<u>(II) $\implies$ (I):</u> The map $\Omega \to 1$ is a trivial fibration, and it is closed by pullback, so that we may apply Proposition 3.2.10.

<u>(I) $\implies$ (II):</u> Every trivial fibration is a Ω-homotopy equivalence (see [Cis06, Lm. 1.3.5]).

<u>(III) $\implies$ (II):</u> Obvious.

<u>(II) $\implies$ (III):</u> The Kan-Quillen model structure on $\widehat{\Delta}$ is the Cisinski model structure generated by the interval Δ^1 and the empty generating set. This can be used to show that the class of homotopical equivalences on **Cat** is the class of weak equivalences uniquely characterised by satisfying certain closure properties (i.e. it is a *basic localiser*), generated by the empty set, which in turn can be used to show that the weak equivalences in $\widehat{A}$ are generated by the empty set in an appropriate way. This

generation by a small set (in our case, the empty set), may be used to generate a model structure using a variant Smith's theorem; see [Cis06, Th. 1.4.3]. □

Let $u : A \to B$ be functor between small ∞-categories, such that pulling back along u preserves locally constant weak equivalences. As $u_!$ automatically preserves locally constant weak equivalences, we then obtain an adjunction

$$u_! : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S}) : u^*$$

in which both constituent functors preserve weak equivalences. Unfortunately, even if both A and B are local test categories, then it is no longer true that $u_! : \widehat{A} \to \widehat{B}$ preserves locally constant weak equivalences. The following proposition provides a compatibility condition to fix this defect.

Proposition 3.2.15. *Let $u : A \to B$ be a functor between local test categories such that pulling back along u preserves locally constant weak equivalences, then, if $u_! : \widehat{A} \to \widehat{B}$ preserves monomorphisms, it is the left half of a Quillen adjunction, and thus preserves locally constant weak equivalences. The adjunction is a Quillen equivalence iff $u : A \to B$ is initial.* □

3.3 Test toposes

In the previous subsection we saw how for any test category A the category of presheaves $\widehat{A}$ comes equipped with a canonical model structure exhibiting $\widehat{A}$ as a model for $\mathcal{S}_{/A_{\simeq}}$ (see Theorem 3.2.12). A *test topos* is an ordinary topos $\mathcal{E}$ which admits a canonical model structure, and whose homotopy ∞-category is equivalent to $\mathcal{S}_{/\pi_! 1_\mathcal{E}}$, where $\pi_! 1_\mathcal{E}$ is the *shape* of $\mathcal{E}$ (see Definition 3.3.1); the topos $\widehat{A}$ is an example of a test topos.

We proceed somewhat analogously to the previous subsection: There, for any small ∞-category A, we obtained a localisation $\mathrm{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{/A^{\simeq}}$. A similar localisation exists for any *locally ∞-connected* ∞-toposes, as disucssed in §3.3.1. We take a brief detour in §3.3.2 investigating which colimits preserve 0-truncatedness in an ∞-topos. Then in §3.3.3 we investigate toposic analogues of initial functors, in order to set up the theory of test toposes.

3.3.1 Shape theory

Here we briefly recall how any object in a topos admits a canonical underlying (pro-)homotopy type.

Definition 3.3.1. Let $\mathcal{X}$ be an ∞-topos, and denote by $\pi : \mathcal{X} \to \mathcal{S}$ the unique geometric morphisms to $\mathcal{S}$, then the ***shape*** of $\mathcal{X}$ is the pro-left adjoint of π^*, i.e. the functor $\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$. ⌟

Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism, then the unit $\mathrm{id}_\mathcal{X} \to f_*f^*$ induces a natural transformation $(\pi_\mathcal{Y})_*(\pi_\mathcal{Y})^* \to (\pi_\mathcal{X})_*(\pi_\mathcal{X})^* = (\pi_\mathcal{Y})_* f_* f^* (\pi_\mathcal{Y})^*$, i.e. a morphism from $(\pi_\mathcal{X})_*(\pi_\mathcal{X})^*$ to $(\pi_\mathcal{Y})_*(\pi_\mathcal{Y})^*$ in $\mathrm{Pro}(\mathcal{S})$.

Definition 3.3.2. geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ is a ***shape equivalence*** if it induces a natural isomorphism $(\pi_\mathcal{Y})_*(\pi_\mathcal{Y})^* \to (\pi_\mathcal{X})_*(\pi_\mathcal{X})^*$. A morphism $X \to Y$ in $\mathcal{X}$ is a *shape equivalence* if the geometric morphism $\mathcal{X}_{/X} \to \mathcal{X}_{/Y}$ is a shape equivalence. ⌟

Definition 3.3.3. An ∞-topos $\mathcal{X}$ is called ***∞-connected*** if equivalently:

(I) the inverse image functor π^* of the unique geometric morphism $\pi : \mathcal{X} \to \mathcal{S}$ is fully faithful;

(II) $\pi_*\pi^* = \mathrm{id}_{\mathcal{S}}$.

An ordinary topos is called *∞-connected* if its associated ∞-topos is ∞-connected. ⌟

Definition 3.3.4. An ∞-topos $\mathcal{X}$ is called ***locally ∞-connected*** if the inverse image functor π^* of the unique geometric morphism $\pi : \mathcal{X} \to \mathcal{S}$ admits a left adjoint $\pi_!$. An ordinary topos $\mathcal{E}$ is called *locally ∞-connected* if its associated topos is locally ∞-connected. ⌟

Remark 3.3.5. For locally ∞-connected ∞-topos $\mathcal{X}$, the functor $\pi_! : \mathcal{X} \to \mathcal{S}$ generalises the notion of the connected component functor of a locally connected ordinary topos (see [Ler79, Prop. 1.5]), which goes back to ideas first sketched in [SGA 4_{I}]. ⌟

Proposition 3.3.6. *Let $\mathcal{X}$ be a locally ∞-connected ∞-topos, then for any object X in $\mathcal{X}$ the homotopy type $\pi_!X$ coincides with the shape of the topos $\mathcal{X}_{/X}$. Moreover, for any morphism $X \to Y$ in $\mathcal{X}$, the geometric morphism $\mathcal{X}_{/X} \to \mathcal{X}_{/Y}$ is a shape equivalence iff $\pi_!X \to \pi_!Y$ is an equivalence of homotopy types.* □

Let $\mathcal{X}$ be a locally ∞-connected topos, then the functor $\pi_! : \mathcal{X} \to \mathcal{S}$ factors as $\mathcal{X} \to \mathcal{S}_{/\pi_!\mathbf{1}} \to \mathcal{S}$; I denote the first factor by $\psi_!$.

Proposition 3.3.7 ([Lur17, Prop. A.1.11])**.** *Let $\mathcal{X}$ be a locally ∞-connected topos, then the functor $\psi_! : \mathcal{X} \to \mathcal{S}_{/\pi_!\mathbf{1}}$ admits a fully faithful right adjoint.* □

3.3.2 Homotopy theory of 0-truncated objects in an ∞-topos

Let $\mathcal{X}$ be a fixed ∞-topos. Throughout this section we will discuss the homotopy theory of 0-truncated objects in $\mathcal{X}$.

Lemma 3.3.8. *Consider a pushout square in* $\mathcal{X}$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, hook] & Y' \end{tikzcd}$$

for which the objects X, X', Y *are* 0*-truncated, and in which the top horizontal map (and thus the bottom horizontal map) is a monomorphism, then* Y' *is* 0*-truncated.*

Proof. This is easily checked by hand in the Kan-Quillen model, so that it is true in $\mathcal{S}$. In a presheaf topos a morphism is a monomorphism and a square is a pushout iff it is so pointwise. Left exact reflections preserve both monomorphisms and pullbacks. □

Lemma 3.3.9. *The inclusion* $\mathcal{X}_{\leq 0} \hookrightarrow \mathcal{X}$ *commutes with filtered colimits.*

Proof. Similar to the last proof. □

It would be nice to have proofs of these two facts using descent arguments, similar to the one employed e.g. in [ABFJ20, Prop. 2.26].

Lemma 3.3.10. *A retract of any* 0*-truncated object in* $\mathcal{X}$ *is again* 0*-truncated.*

Proof. Retracts are limits; it is enough to show this on the level of homotopy categories, in which case this is elementary, but it is also true on the level of ∞-categories, which is somewhat more involved to prove (see [Lur09, §4.4.5]). □

Definition 3.3.11 ([Cis03, Def. 3.3.10]). Let $\mathcal{E}$ be an ordinary topos and K, a small category, then a functor $p : K \to \mathcal{E}$ is ***relatively flat*** if the functor $K \to \mathcal{E}_{/\operatorname{colim} p}$ is flat. ⌟

With notation as in the preceding definition, note that $p : K \to \mathcal{E}$ is relatively flat iff the Yoneda extension $\widehat{K} \to \mathcal{E}$ preserves pullbacks. The following proposition is due to Cisinski. We provide a modern (and simpler) proof.

Proposition 3.3.12 ([Cis03, Th. 3.3.9]). *If $\mathcal{X}$ is 1-localic, then the inclusion $\mathcal{X}_{\leq 0} \hookrightarrow \mathcal{X}$ commutes with colimits of relatively flat functors.*

Proof. Let K be a small ordinary category, and $p : K \to \tau_{\leq 0}\mathcal{X}$, a relatively flat functor, then by assumption we obtain a geometric morphism $\widehat{K} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\tau_{\leq 0}\mathcal{X})_{/\operatorname{colim} p}$. Observe that $(\tau_{\leq 0}\mathcal{X})_{/\operatorname{colim} p} = \tau_{\leq 0}(\mathcal{X}_{/\operatorname{colim} p})$ by [Lur09, Lm. 5.5.6.14], so that by [?] the inclusion from ordinary toposes into hypercomplete 1-localic toposes yields a geometric morphism $\underline{\operatorname{Hom}}(K^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{X}_{/\operatorname{colim} p}$. The left adjoint preserve the final object, but this is equivalent to saying that the colimit of $K \to \mathcal{X}_{/\operatorname{colim} p}$ is the final object, which is the identity morphism $\operatorname{colim} p \to \operatorname{colim} p$. □

Example 3.3.13. *If $\mathcal{X}$ is 1-localic, then $p : \Lambda^2_0 \to \mathcal{X}_{\leq 0}$ is relatively flat iff p carries both legs of Λ^2_0 to monomorphisms, so that we recover a special case of Lemma 3.3.8.* ⌟

Example 3.3.14. *If $\mathcal{X}$ is 1-localic topos, then $p : \mathbf{N} \to \mathcal{X}_{\leq 0}$ is relatively flat iff p carries all morphisms in $\mathbf{N}$ to monomorphisms, so that we recover a special case of Lemma 3.3.9.* ⌟

Proposition 3.3.15. *Let $\mathcal{E}$ be an ordinary topos, K, a small ordinary category, and $p : K \to \mathcal{E}$, a relatively flat functor, then for any $X \in \widehat{K}$ the functor $K_{/X} \to \mathcal{E}$ is relatively flat.*

Proof. The colimit of $K_{/X} \to \mathcal{E}$ is given by $p_!X$. As $\widehat{K} \to \mathcal{E}$ preserves pullbacks, $\widehat{K}_{/X} \to \mathcal{E}_{/p_!X}$ preserves all finite limits. □

3.3.3 Test toposes

We saw in Proposition 3.3.7 that for any locally ∞-connected topos $\mathcal{X}$ the category $\mathcal{S}_{/\pi_!\mathbf{1}}$ is a localisation of $\mathcal{X}$; in particular, the ∞-category $\mathcal{S}_{/A_\simeq} = \mathcal{S}_{/\pi_!(\mathbf{1}_{\widehat{A}})}$ is a localisation of $\underline{\hom}(A^{\mathrm{op}}, \mathcal{S})$ for any small ∞-category A. In §3.2 we saw that when A is a local test category, then the restricted functor $\widehat{A} \to \mathcal{S}_{/A_\simeq}$ is still a localisation. Similarly, we may ask when the restricted functor $\mathcal{X}_{\leq 0} \to \mathcal{S}_{/\pi_!1}$ is still a localisation. To get a good theory, we will need to make two additional assumptions on $\mathcal{X}$. The first assumptions is that $\mathcal{X}$ is 1-localic and hypercomplete. The second assumption is explained in the following definition:

Definition 3.3.16. An n-topos is ***locally contractible*** if it is generated by a *set* of objects of contractible shape. ⌟

Example 3.3.17. *Any presheaf topos is locally contractible.* ⌟

Proposition 3.3.18. *Any locally contractible n-topos ($1 \leq n \leq \infty$) is locally ∞-connected.* □

Remark 3.3.19. The converse is true for $n = \infty$. We are unaware whether or not this is the case for $n < \infty$. ⌟

As in the theory of test categories, we need a good notion of initial functors. This will moreover help us reduce the study of locally contractible 1-toposes to the study of test categories.

For the rest of this subsection we fix two n-toposes $\mathcal{X}$ and $\mathcal{Y}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism, and Y, an object of Y, then write $\mathcal{X}_{/Y} := \mathcal{X}_{/f^*Y}$. The morphism

$$\begin{array}{rcl} \mathcal{Y}_{/Y} & \to & \mathcal{X}_{/Y} \\ Y' \to Y & \mapsto & f^*Y' \to f^*Y \end{array}$$

admits a right adjoint given by

$$\begin{array}{rcl} \mathcal{X}_{/Y} & \to & \mathcal{Y}_{/Y} \\ X \to f^*Y & \mapsto & Y \times_{f_*f^*Y} f_*X, \end{array}$$

forming a geometric morphism $f_{/Y} : \mathcal{X}_{/Y} \to \mathcal{Y}_{/Y}$.

Definition 3.3.20. A geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ is called a ***local shape equivalence*** if equivalently:

(I) $f_{/Y} : \mathcal{X}_{/Y} \to \mathcal{Y}_{/Y}$ is a shape equivalence for every $Y \in \mathcal{Y}$.

(II) $\mathcal{X}(f^*Y, (\pi_{\mathcal{X}})^*K) = \mathcal{Y}(Y, (\pi_{\mathcal{Y}})^*K)$ for every $Y \in \mathcal{Y}$ and $K \in \mathcal{S}$.

⌟

Example 3.3.21. *A functor $A \to B$ between small ∞-categories is initial iff the induced geometric morphism* $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S})$ *is a local shape equivalence.* ⌟

Proposition 3.3.22. *If $\mathcal{X}$ and $\mathcal{Y}$ are locally contractible, then a geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ is a local shape equivalence iff f^* preserves contractible objects.*

Proof. If f is a shape equivalence then, by assumption, for any object Y in $\mathcal{Y}$ of contractible shape, f^*Y must also be of contractible shape. Conversely, any object Y in $\mathcal{Y}$ may be written as a colimit $\operatorname{colim}_\alpha Y_\alpha$ of contractible objects, then for any homotopy type we have

$$\begin{aligned}
\mathcal{X}(f^*Y, (\pi_{\mathcal{X}})^*K) &= \mathcal{X}(f^* \operatorname{colim}_\alpha Y_\alpha, (\pi_{\mathcal{X}})^*K) \\
&= \mathcal{X}(\operatorname{colim}_\alpha f^*Y_\alpha, (\pi_{\mathcal{X}})^*K) \\
&= \lim_\alpha \mathcal{X}(f^*Y_\alpha, (\pi_{\mathcal{X}})^*K) \\
&= \lim_\alpha \mathcal{Y}(Y_\alpha, (\pi_{\mathcal{Y}})^*K) \\
&= \mathcal{Y}(\operatorname{colim}_\alpha Y_\alpha, (\pi_{\mathcal{Y}})^*K) \\
&= \mathcal{Y}(Y, (\pi_{\mathcal{Y}})^*K).
\end{aligned}$$

□

Proposition 3.3.23 ([Cis03, Prop.4.1.24])**.** *Let $a : \mathcal{X} \hookrightarrow \mathcal{Y}$ be a geometric embedding which is also a local shape equivalence. If $\mathcal{Y}$ is locally contractible, then so is $\mathcal{X}$.*

Proof. If $U \subseteq \mathcal{Y}_0$ is a subset generating $\mathcal{Y}$ under colimits, then a^*U generates $\mathcal{X}$ under colimits, and consists of contractible objects by the assumption that a is a local shape equivalence. □

Proposition 3.3.24 ([Cis03, Prop.4.1.25])**.** *A topos $\mathcal{X}$ is locally contractible iff there exists a small category C and an aspherical geometric embedding $\mathcal{X} \hookrightarrow \underline{\hom}(C^{\mathrm{op}}, \mathcal{S})$.*

Proof. The "if statement" is clear from the preceding proposition. Conversely, let C be a subcategory of $\mathcal{X}$ spanned by a set of contractible objects generating $\mathcal{X}$ under colimits, then the geometric embedding $\mathcal{X} \hookrightarrow \underline{\mathrm{hom}}(C^{\mathrm{op}}, \mathcal{S})$ is aspherical, because representable objects in any presheaf topos are contractible. □

Proposition 3.3.25. *Let $a : \mathcal{X} \hookrightarrow \mathcal{Y}$ be a geometric embedding which is also a local shape equivalence, then $(\pi_{\mathcal{X}})_! = (\pi_{\mathcal{Y}})_! \circ a_*$.*

Proof. For every $X \in \mathcal{X}$ we have

$$
\begin{aligned}
(\pi_{\mathcal{Y}})_! \circ a_* X &= \mathcal{Y}(a_* X, (\pi_{\mathcal{Y}})^*(\_)) \\
&= \mathcal{X}(a^* a_* X, (\pi_{\mathcal{X}})^*(\_)) \\
&= \mathcal{X}(X, (\pi_{\mathcal{X}})^*(\_)) \\
&= (\pi_{\mathcal{X}})_! X.
\end{aligned}
$$

□

We are now able to state the following generalisation of Theorem 3.2.14.

Theorem 3.3.26 ([Cis03, Th. 4.2.8])**.** *Let $\mathcal{E}$ be a locally contractible ordinary topos, then the following are equivalent:*

(I) *For any object X in $\mathcal{E}$ the projection map $X \times \Omega_{\mathcal{E}} \to X$ is a shape equivalence.*

(II) *Any trivial fibration is a shape equivalence;*

(III) *There exists a local test category C and an aspherical geometric embedding $\mathcal{E} \hookrightarrow \widehat{C}$.*

(IV) *There exists a (necessarily unique as well as cofibrantly generated) model structure on $\mathcal{E}$ in which the weak equivalences are the shape equivalences, and in which the cofibrations are the monomorphisms.*

$\square$

Definition 3.3.27. An ordinary topos satisfying the equivalent conditions of Theorem 3.3.26 is called a ***local test topos***. A local test topos with trivial shape is a ***test topos***. A test topos, whose shape functor commutes with finite products, is a ***strict test topos***. On any topos, the model structure given by Theorem 3.3.26 is referred to as the ***canonical model structure***. ⌟

3.4 Homotopy theory in test toposes

Throughout this subsection $\mathcal{X}$ denotes a hypercomplete 1-localic ∞-topos, such that $\mathcal{X}_{\leq 0}$ is a local test topos. As we saw in §3.3.1 we obtain a localisation functor $\mathcal{X} \to \mathcal{S}_{/\pi_! 1_{\mathcal{X}}}$, and, as this functor is a left adjoint, all colimits are homotopy colimits. The results in §3.3.2 inform us how to construct homotopy colimits in $\mathcal{X}_{\leq 0}$. In contrast, it is not a priori clear how to construct homotopy limits in $\mathcal{X}$. Theorem 3.2.14 furnishes a model structure on $\mathcal{X}_{\leq 0}$, but this model structure is, in general, not that helpful: The homotopy colimits that it makes available for common diagram shapes, are seen in §3.3.2 to be those which commute with the inclusion $\mathcal{X}_{\leq 0} \hookrightarrow \mathcal{X}$, and, in general, the model structure is not useful for calculating homotopy limits because it offers no explicit description of the fibrations. In lucky cases, such as for presheaves on Δ or $\square$, (trivial) fibrations may be characterised via explicit sets of generating (trivial) cofibrations, so that it *is* possible to perform concrete calculations involving homotopy limits. This still leaves the question of calculating homotopy limits in the ∞-topos of homotopy-type-valued presheaves on a nice local test category A. Luckily, it turns out that the canonical model structure on $\widehat{A}$ admits a canonical extension to $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ with the same generating (trivial) cofibrations as for $\widehat{A}$, and whose weak equivalences are the shape equivalences. The goal of this subsection is to develop tools in order to transfer the model structure from $\widehat{A}$ (resp. $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$) to $\mathcal{X}_{\leq 0}$ (resp. $\mathcal{X}$) while keeping the shape equivalences as weak equivalences; if $\widehat{A}$ has explicit generating (trivial) cofibrations, then these produce explicit generating (trivial) cofibrations in $\mathcal{X}_{\leq 0}$ and $\mathcal{X}$.[5] Typically, this will increase the number of fibrant objects in $\mathcal{X}_{\leq 0}$ and $\mathcal{X}$.

We begin by extending the canonical model structure from $\widehat{A}$ to $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$. For this we need to understand how to construct cofibrantly generated model structure on presentable ∞-categories.

Proposition 3.4.1 ([MG14, Th. 3.11]). *Let M be a presentable ∞-category, let $W \subseteq M$ be a subcategory, which is closed under retracts, and satisfies the 2-out-of-3 property. Suppose that I and J are* sets *of homotopy classes of maps. such that*

(a) ${}^{\boxslash}(J^{\boxslash}) \subseteq {}^{\boxslash}(I^{\boxslash}) \cap W$

(b) $I^{\boxslash} \subseteq J^{\boxslash} \cap W$

(c) *and either*

[5]It would be possible to develop these tools in a more general setting, where $\widehat{A}$ is allowed to be any local test topos. However, we are not aware of any local test toposes with explicit generating (trivial) cofibrations for the canonical model structure, which aren't presheaf toposes, which is why we restrict ourselves to the slightly simpler setting described here.

(c$_1$) ${}^{\boxslash}(I^{\boxslash}) \cap W \subseteq {}^{\boxslash}(J^{\boxslash})$, *or*

(c$_1$) $J^{\boxslash} \cap W \subseteq I^{\boxslash}$,

then the I and J define a cofibrantly generated model structure (see [MG14, Def. 3.8]*) on M whose weak equivalences are W.* □

We can now extend the canonical model structure. The following proposition generalises [MG14, Th. 4.4].

Proposition 3.4.2. *Let A be a local test category, then there exists a (necessarily unique) cofibrantly generated model structure on $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ whose weak equivalences are the shape equivalences, and whose trivial fibrations are characterised by having the right lifting property against monomorphisms in $\widehat{A}$.*
Furthermore, if I and J are generating cofibrations and trivial cofibrations, respectively, of the canonical model structure on $\widehat{A}$, then these generate the model structure on $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$.

Proof. Let I and J be generating cofibrations and trivial cofibrations, respectively, of the canonical model structure on $\widehat{A}$. By Lemmas 3.3.8-3.3.10, ${}^{\boxslash}(I^{\boxslash})$ (constructed in $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$) coincide with the monomorphisms, so that applying Proposition 3.4.1 to W together with I, J will prove both statements in the proposition. We will now verify (a), and (b), (c$_2$).
Proof of (a): By Lemmas 3.3.8-3.3.10 all colimits involved in constructing the morphisms in ${}^{\boxslash}(J^{\boxslash})$ are homotopy colimits. As all morphisms in J are weak equivalences, the morphisms in ${}^{\boxslash}(J^{\boxslash})$ must be weak equivalences.
Proof of (b): The inclusion $I^{\boxslash} \subseteq J^{\boxslash}$ is clear as $J \subseteq {}^{\boxslash}(I^{\boxslash})$, so we need to show

$I^{\boxslash} \subseteq W$. So, let $X \to Y$ be a morphism in $I^{\boxslash}$.

First, we show that it is enough to prove the statement in the case when Y is representable. For all objects a in A, and all maps $a \to Y$ the morphisms $a \times_Y X \to X$ are in $I^{\boxslash}$. If these morphisms are in W, then $X \to Y$ is in W by faithful descent, as the morphism can be written as a colimit indexed by $A_{/Y} \to A$.

So, assume that Y is representable. As a morphism in $A_{/Y}$ is a monomorphism iff it is a monomorphism in A, we may furthermore assume that A has a final object, and that Y is such a final object.

As the shape of the presheaf represented by the final object in A is contractible, it is enough to show that that the shape of X is contractible. Now, the shape of X is given by $(A_{/X})_{\simeq} \simeq \operatorname{Ex}^{\infty} A_{/X}$, so that any map $S^k \to \pi_! X$ $(k \geq 0)$ may be represented by a map $\operatorname{Sd}^n \partial\Delta^k \to A_{/X}$ for some $n \geq 0$. If $n \geq 1$, then Sd^n is a finite poset, and therefore a finite direct category. We will show that for any finite direct category I and any functor $I \to A_{/Y}$ we obtain a factorisation

$$\begin{tikzcd} I_{\simeq} \arrow[r] \arrow[d] & (A_{/X})_{\simeq} \\ * \arrow[ur, dashed] & \end{tikzcd} \tag{5}$$

Consider the diagram $f : I \to A$, and take a Reedy cofibrant replacement $\tilde{f} \xrightarrow{\sim} f$ in $\widehat{A}$ (see [Cis19, Prop. 7.4.19]), then by an inductive application of [Cis19, Cor. 7.4.4] and Lemmas 3.3.8 & 3.3.9 we see that the colimit of $\tilde{f}$ is 0-truncated. The map $I_{\simeq} \to (A_{/X})_{\simeq}$ corresponds to the map $\pi_! \operatorname{colim} \tilde{f} \to \pi_! X$. Consider a factorisation $\operatorname{colim} \tilde{f} \to c \to 1$ in $\widehat{A}$, where $\operatorname{colim} \tilde{f} \to c$ is a monomorphism, and $c \to 1$ is a trivial fibration, and thus a weak equivalence. By our assumption on X, we obtain a lift

$$\begin{tikzcd} \operatorname{colim} \tilde{f} \arrow[r] \arrow[d, hook] & X \\ c \arrow[ur, dashed] & \end{tikzcd}$$

Taking the shape of this diagram yields the desired lift in (5).

<u>Proof of (c_2):</u> The proof of this fact for $A = \Delta$ is given in [MG14, Prop. 7.9], and may be interpreted verbatim in our setting. $\square$

Next, we recall the following theorem on transferring cofibrantly generated model structures.

Proposition 3.4.3. *Let M be a cofibrantly generated model ∞-category with generating cofibrations I and generating trivial cofibrations J, let N be a presentable ∞-category, and consider an adjunction $f : M \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} N : u$. If the functor u takes relative fJ-cell complexes to weak equivalences, then*

(1) *the ∞-category N admits a cofibrantly generated model structure whose weak equivalences are those morphisms carried to weak equivalences by u, and with generating cofibrations and trivial cofibrations given by fI and fJ respectively, and*

(2) *the adjunction $f : M \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} N : u$ is a Quillen adjunction.*

Sketch of proof. By [DAGX, Prop. 1.4.7], any morphism in N factors into a relative fI-complex (fJ-complex) followed by a morphism with the right lifting property w.r.t. fI (fJ). The rest of the proof now proceeds as in the proof or [Hir03, Th. 11.3.2]. $\square$

Proposition 3.4.4. *Let A be a local test category, and, $u : A \to \mathcal{X}_{\leq 0}$ be a functor. If*

(a) *the objects $\{ua\}_{a \in A_0}$ generate $\mathcal{X}_{\leq 0}$ (and thus $\mathcal{X}$),*

(b) *the objects $\{ua\}_{a \in A_0}$ have contractible shape,*

(c) *for each object a in A the presheaf $u^* u\, a$ has contractible shape, and*

(d) *the functor $u_! : \widehat{A} \to \mathcal{X}_{\leq 0}$ preserves monomorphisms,*

then for any sets I and J of, respectively, generating cofibrations and generating trivial cofibrations for the canonical model structure on $\widehat{A}$, there exist cofibrantly generated model structures on $\mathcal{X}_{\leq 0}$ and $\mathcal{X}$ such that

(1) *the weak equivalences are precisely the shape equivalences,*

(2) *the sets $u_! I$ and $u_! J$ are generating sets for the cofibrations and trivial cofibrations, respectively, and*

(3) *the adjunctions $u_! : \widehat{A} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{X}_{\leq 0} : u^*$ and $u_! : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{X} : u^*$ are Quillen equivalences.*

If moreover

(e) *the inclusions $uk_2 \hookrightarrow ud$ admit retracts for all morphisms $k_2 \hookrightarrow d$ in J,*

then

(4) *all objects in the resulting model structures on $\mathcal{X}_{\leq 0}$ and $\mathcal{X}$ are fibrant.*

Proof. Denote by C the full subcategory of $\mathcal{X}_{\leq 0}$ spanned by the image of u, then the adjunction $u_! : \widehat{A} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{X}_{\leq 0} : u^*$ factors as $\widehat{A} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \widehat{C} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{X}_{\leq 0}$. By (a) and (b) the adjunction $\widehat{C} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{X}_{\leq 0}$ is an aspherical embedding, and thus a local shape equivalence, in which both adjoints preserve shape equivalences by Proposition 3.3.25. Condition (c) says precisely that $A \to C$ is initial. Thus, a map in $\mathcal{X}_{\leq 0}$ is a shape equivalence iff its image under u^* is. This, together with (d) and Lemmas 3.3.8 & 3.3.9 show that the condition in Proposition 3.4.3 is satisfied, which establishes the existence of the model structure as well as (1) & (2). Property (3) follows from the observation that $u_! : \widehat{A} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{X}_{\leq 0} : u^*$ is a Quillen equivalence w.r.t. the *canonical* model structures.

Property (4) is obvious. □

3.4.1 Some tools for applying Proposition 3.4.4

If in Proposition 3.4.4 we take A to be one of the test categories Δ, $\square$ mentioned above, then it is very easy to verify (d).

Proposition 3.4.5. *Let $\mathcal{E}$ be an ordinary topos, then a cocontinuous functor $\widehat{\Delta} \to \mathcal{E}$ preserves monomorphisms iff the inclusion $\Delta^{\{0\}} \sqcup \Delta^{\{1\}} \hookrightarrow \Delta^1$ is sent to a monomorphism.*

Sketch of proof. The proof of [Cis06, Lm. 2.1.10] remains true mutatis mutandis for toposes. □

Proposition 3.4.6. *Let $\mathcal{E}$ be an ordinary topos, then a cocontinuous functor $\widehat{\square} \to \mathcal{E}$*

preserves monomorphisms iff the inclusions $(\delta_i^0, \delta_i^1) : \square^{n-1} \sqcup \square^{n-1} \hookrightarrow \square^n$ *are sent to monomorphisms for all* $n \geq i \geq 1$.

Sketch of proof. The proof of [Cis06, Lm. 8.4.21] remains true mutatis mutandis for toposes. $\square$

In all instances where we apply Proposition 3.4.4, we will check (c) using Proposition 3.4.7, when $A = \Delta$, and, Proposition 3.4.8, when $A = \square$.

Proposition 3.4.7. *Let* (A, I) *and* (B, J) *be pairs consisting of small ordinary categories admitting finite limits together with an interval, and assume that every object in* B *is* J*-contractible. Let* $u : A \to B$ *be a functor carrying* I *to* J *(including the inclusions of the final object, which* u *must then preserve), then* u *is initial.*

Proof. The functor u is initial iff for every object b in B the presheaf u^*b has a contractible colimit in $\mathcal{S}$. Let $J \times b \to b$ be an J-contraction of b, then the unit morphisms produce a diagram

$$\begin{tikzcd}
u^*b \cong e \times u^*b \arrow[r] \arrow[d] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
I \times u^*b \arrow[r] & u^*u_!I \times u^*b \cong u^*(J \times b) \arrow[r] & u^*b, \\
u^*b \cong e \times u^*b \arrow[r] \arrow[u] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

showing that u^*b is I-contractible. $\square$

Proposition 3.4.8. *Let* (B, J) *be a pair consisting of a small ordinary categoriy admitting finite limits together with an interval, and assume that every object in* B

is J-contractible. Let $u : \square \to B$ be a monoidal functor carrying $\square^1$ to J (including the inclusions of the final object, which u must then preserve), then u is initial.

Proof. The functor u is initial iff for every object b in B the presheaf u^*b has a contractible colimit in $\mathcal{S}$. For any two cubical sets X_1, X_2 there are canonical morphisms $X_1 \otimes X_2 \to X_i$ $(i = 1, 2)$. To see this, note that for the for any $k_1, k_2 \in \mathbf{N}$ we have projection maps (in **Set**) $\square^{k_1} \otimes \square^{k_2} \cong \{0,1\}^{k_1} \times \{0,1\}^{k_2} \to \{0,1\}^{k_i}$ for $i = 1, 2$; the canonical morphisms $X_1 \otimes X_2 \to X_i$ $(i = 1, 2)$ are then obtained by extending by colimits. This yields a natural morphism $X_1 \otimes X_2 \to X_1 \times X_2$. Let $J \times b \to b$ be an J-contraction of b, then the unit morphisms produce a diagram

$$\begin{tikzcd}
u^*b \cong e \otimes u^*b \arrow[r] \arrow[d] & e \times u^*b \arrow[r] \arrow[d] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[d] \arrow[dr, bend left, "\mathrm{id}"] & \\
\square^1 \otimes u^*b \arrow[r] & \square^1 \times u^*b \arrow[r] & u^*u_!\square^1 \times u^*b \cong u^*(J \times b) \arrow[r] & u^*b, \\
u^*b \cong e \otimes u^*b \arrow[r] \arrow[u] & e \times u^*b \arrow[r] \arrow[u] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[u] \arrow[ur, bend right, "0"'] &
\end{tikzcd}$$

showing that u^*b is $\square^1$-contractible. $\square$

4 Fractured toposes

4.1 Basic definitions

Definition 4.1.1. A ***fractured* ∞*-topos*** is a triple adjunction

$$\mathcal{X}^{\mathrm{corp}} : \begin{array}{c} \xrightarrow{\quad j_! \quad} \\ \xleftarrow{\quad j^* \quad} \\ \xrightarrow{\quad j_* \quad} \end{array} : \mathcal{X}$$

between ∞-toposes $\mathcal{X}^{\mathrm{corp}}$ and $\mathcal{X}$ satisfying properties (a) - (c) below:

(a) The topos $\mathcal{X}$ is generated by the objects in the image of $j_!$.

(b) For every object X in $\mathcal{X}^{\mathrm{corp}}$ the functors $(j_!)_{/X}$, $(j_*)_{/X}$ in the induced adjunction

$$\mathcal{X}^{\mathrm{corp}}_{/X} : \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} : \mathcal{X}_{/X}$$

are fully faithful, and $(j_!)_{/X}$ preserves finite limits, i.e., $(j^*)_{/X} \vdash (j_!)_{/X}$ is a geometric morphism.

The objects in $\mathcal{X}^{\mathrm{corp}}$ are called ***corporeal objects***. A morphism $U \to X$ in $\mathcal{X}$ is called ***admissible*** if for every pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

in which X' is in $\mathcal{X}^{\mathrm{corp}}$, the morphism $U' \to X'$ is in $\mathcal{X}^{\mathrm{corp}}$. Finally, we require:

(c) For every pullback diagram

$$\begin{tikzcd} \coprod U_\alpha \arrow[r] \arrow[d, "\coprod f_\alpha"] & U \arrow[d, "f"] \\ \coprod X_\alpha \arrow[r, "g"] & X \end{tikzcd}$$

in which $g : \coprod X_\alpha \to X$ is an effective epimorphism and each morphism f_α is admissible, the morphism $f : U \to X$ is admissible.

⌟

Convention 4.1.2. We will typically refer to a fractured ∞-topos by its leftmost adjoint, which we always denote by $j_!$. So with notation as in Definition 4.1.1, we would write $j_! : \mathcal{X}^{\mathrm{corp}} \rightarrowtail \mathcal{X}$. ⌟

We will show that the main character in this thesis, the ∞-topos $\mathbf{Diff}^\infty$ of differentiable stacks (see Definition 5.1.2), admits the structure of a fractured ∞-topos by exhibiting it as the ∞-topos of sheaves on the fractured analogue of a site. This notion relies on the following definition.

Definition 4.1.3 ([Lur18, Def. 20.2.1.1])**.** Let G be an ∞-category, then an ***admissibility structure*** on G is a subcategory G^{ad}, whose morphisms are referred to as *admissible morphisms*, such that:

(a) Every equivalence in G is an admissible morphism.

(b) For any admissible morphism $u \to x$, and any morphism $x' \to x$ there exists a pullback square

$$\begin{tikzcd} u' \arrow[r] \arrow[d] & u \arrow[d] \\ x' \arrow[r] & x, \end{tikzcd}$$

in which $u' \to x'$ is admissible.

(c) For any commutative triangle

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "h"'] & & y \arrow[dl, "g"] \\ & z & \end{tikzcd}$$

in which $g : y \to z$ is admissible, the morphism $f : x \to y$ is admissible iff $h : x \to z$ is.

(d) Admissible morphisms are closed under retracts.

⌟

Example 4.1.4. *The admissible morphisms in a fractured topos form an admissibility structure. Under mild conditions the structure of a fractured ∞-topos may be recovered from its associated admissibility structure (see [Lur18, Rmk. 20.3.4.6]).* ⌟

Definition 4.1.5 ([Lur18, Def. 20.6.2.1])**.** A ***geometric site*** is a triple $(G, G^{\mathrm{ad}}, \tau)$ consisting of

(i) a small ∞-category G,

(ii) an admissibility structure G^{ad} on G, and

(iii) a Grothendieck topology τ on G, such that every covering sieve in τ contains a covering sieve generated by admissible morphisms. $\lrcorner$

Lemma 4.1.6 ([Lur18, Props. 20.6.1.1 & 20.6.1.3])**.** *Let $(G, G^{\mathrm{ad}}, \tau)$ be a geometric site, then there exists a Grothendieck topology on G^{ad} in which a sieve R in G^{ad} is a covering sieve iff the sieve generated by R in G is a covering sieve. Any sheaf on G restricts to a sheaf on G^{ad}.* $\square$

Theorem 4.1.7 ([Lur18, Th. 20.6.3.4])**.** *Let $(G, G^{\mathrm{ad}}, \tau)$ be a geometric site, and denote by $\mathcal{X}$ the ∞-topos of sheaves on G, and, by $\mathcal{X}^{\mathrm{corp}}$ the ∞-topos of sheaves on G^{ad}, then the restriction functor $\mathcal{X}^{\mathrm{corp}} \leftarrow \mathcal{X} : j^*$ admits both a left and a right joint, and the resulting triple adjunction is a fractured ∞-topos.* $\square$

Example 4.1.8. *We briefly present two examples of geometric sites. For A any commutative ring, $\mathbf{Aff}_A^{\mathrm{fp}}$ denote the category of finitely presented affine schemes.*

1. *Consider the triple $(\mathbf{Aff}_A^{\mathrm{fp}}, \mathbf{Aff}_A^{\mathrm{fp,Zar}}, \tau_{\mathrm{Zar}})$, where $\mathbf{Aff}_A^{\mathrm{fp,Zar}}$ consists of the Zariski open embeddings, and τ_{Zar} is the Zariski topology, then the corporeal objects are higher versions of locally finitely presentable schemes; in fact, the 0-truncated objects are precisely locally finitely presentable schemes (see [Car20, Ex. 6.2.1]). Moreover, the ∞-topos of sheaves on $\mathbf{Aff}_A^{\mathrm{fp}}$ is the classifying ∞-topos for local A-algebras. For any ∞-topos $\mathcal{X}$ specifying a geometric morphism $\mathcal{X} \to \mathcal{L}_A$ is equivalent to endowing $\mathcal{X}$ with the structure of a locally ringed topos.*

2. *Consider the triple $(\mathbf{Aff}_A^{\mathrm{fp}}, \mathbf{Aff}_A^{\mathrm{fp,ét}}, \tau_{\mathrm{ét}})$, where $\mathbf{Aff}_A^{\mathrm{fp,ét}}$ are the étale morphisms, and $\tau_{\mathrm{ét}}$ is the étale topology, and the corporeal objects are higher versions of Deligne-Mumford stacks.*

⌟

Remark 4.1.9. In the same way that not every ∞-topos is the category of sheaves on a site, not every fractured ∞-topos is given as in the preceding theorem. However, it *is* true that every fractured ∞-topos may be realised as the localisation of a fractured presheaf ∞-topos, and that this presheaf ∞-topos may be obtained as in the preceding theorem with $\tau = \varnothing$. See [Lur18, Th. 20.5.3.4]. ⌟

4.2 Fractured toposes and local contractibility

Endowing an ∞-topos with the structure of a fractured ∞-topos is useful for calculating the shape of the resulting corporeal objects. We learnt these ideas from [Cis03, §6.1], which relies on a more specialised notion of fractured topos.

Proposition 4.2.1. *Consider a geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ of n-toposes.*

(1) If the inverse image functor $\mathcal{X} \leftarrow \mathcal{Y} : f^*$ is fully faithful, then $f : \mathcal{X} \to \mathcal{Y}$ is locally aspherical, so, in particular, is a shape equivalence.

(2) If $f_* : \mathcal{X} \to \mathcal{Y}$ moreover admits a right adjoint $f^!$, then the resulting geometric morphism $f_* : \mathcal{X} \rightleftarrows \mathcal{Y} : f^!$ is a shape equivalence.

Proof. For every Y in $\mathcal{Y}$ we have $\mathcal{X}(f^*Y, \pi_{\mathcal{Y}}^*(\_)) = \mathcal{X}(f^*Y, f^*\pi_{\mathcal{X}}^*(\_)) = \mathcal{Y}(Y, \pi_{\mathcal{X}}^*(\_))$, which establishes (1). For (2) we simply note that $f^! \dashv f_*$ and $f_* \dashv f^*$ give rise to the maps

$\Pi_\infty(\mathcal{X}) \to \Pi_\infty(\mathcal{Y}) \to \Pi_\infty(\mathcal{X})$, which compose to the identity. As the first map is an equivalence, so is the second. □

Corollary 4.2.2. *Let $j_! : \mathcal{X}^{\mathrm{corp}} \rightarrowtail \mathcal{X}$ be a fractured topos, then for any corporeal object X the toposes $\mathcal{X}^{\mathrm{corp}}_{/X}$ and $\mathcal{X}_{/X}$ have the same shape.* □

Thus, the cohomology of a geometric object such as a scheme with coefficients in a locally constant sheaf is the same when computed in its big or small topos. For us, the technology provides a way of showing that a topos is locally contractible.

Corollary 4.2.3. *Let $j_! : \mathcal{X}^{\mathrm{corp}} \hookrightarrow \mathcal{X}$ be a hypercomplete* 1*-localic topos. If $\mathcal{X}^{\mathrm{corp}}$ is locally contractible, then so is $\mathcal{X}$.* □

Part II

Smooth Stacks and Spaces

5 Basic definitions

5.1 Smooth stacks and spaces

Throughout this subsection we fix some $0 \leq r \leq \infty$.

Notation 5.1.1.

1. $\mathbf{Mfd}^r$ denotes the category of r-differentiable manifolds and r-differentiable maps.

2. $\mathbf{OMfd}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by open subsets of $\mathbf{R}^n$ $(0 \leq n < \infty)$.

3. $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces of $\mathbf{R}^n$ $(0 \leq n < \infty)$.

On each of these small categories we denote by τ the Grothendieck topology in which a sieve on a space M is a covering sieve iff it contains a subset $\{U_\alpha \to M\}$ consisting of jointly surjective open embeddings.

1. $\mathbf{Mfd}^r_{\text{ét}}$ denotes the category of r-differentiable manifolds and r-differentiable open embeddings.

2. $\mathbf{OMfd}^r_{\text{ét}}$ denotes the full subcategory of $\mathbf{Mfd}^r_{\text{ét}}$ spanned by open subsets of $\mathbf{R}^n$ $(0 \leq n < \infty)$.

3. $\mathbf{Cart}^r_{\text{ét}}$ denotes the full subcategory of $\mathbf{Mfd}^r_{\text{ét}}$ spanned by the spaces for $\mathbf{R}^n$ $(0 \leq n < \infty)$.

On each of these small subcategories we denote restriction of τ by $\tau_{\text{ét}}$. ⌟

Definition 5.1.2. An $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r$ is an ***r-differentiable stack***, and the category thereof is denoted by $\mathbf{Diff}^r$; ∞- and 0-differentiable stacks are called ***differentiable*** and ***continuous stacks***, respectively. An object of $\mathbf{Diff}^r_{\leq 0}$ is then an ***r-differentiable space***, and, again for $r = \infty$ and $r = 0$ respectively, a ***differentiable space*** and ***continuous space***.

Similarly, an $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r_{\text{ét}}$ is an ***étale r-differentiable stack***, and the category thereof is denoted by $\mathbf{Diff}^r_{\text{ét}}$; étale ∞- and 0-differentiable stacks are called ***étale differentiable*** and ***étale continuous stacks***, respectively. An object of $\mathbf{Diff}^r_{\text{ét},\leq 0} := (\mathbf{Diff}^r_{\text{ét}})_{\leq 0}$ is then an ***étale r-differentiable space***, and, again for $r = \infty$ and $r = 0$ respectively, a ***étale differentiable space*** and ***étale continuous space***. ⌟

Observe that the restricted Yoneda embedding exhibits $\mathbf{Mfd}^r$ and $\mathbf{OMfd}^r$ as full subcategories of $\mathbf{Diff}^r$, containing $\mathbf{Cart}^r$, and also $\mathbf{Mfd}^r_{\text{ét}}$ and $\mathbf{OMfd}^r_{\text{ét}}$, as full subcategories of $\mathbf{Diff}^r_{\text{ét}}$ containing $\mathbf{Cart}^r_{\text{ét}}$. Thus, $\mathbf{Diff}^r$ may also be characterised as the ∞-category of sheaves on either $\mathbf{Mfd}^r$ or $\mathbf{OMfd}^r$, and $\mathbf{Diff}^r_{\text{ét}}$ may also be characterised as the ∞-category of sheaves on either $\mathbf{Mfd}^r_{\text{ét}}$ or $\mathbf{OMfd}^r_{\text{ét}}$.

Lemma 5.1.3. *The triple* $(\mathbf{OMfd}^r, \mathbf{OMfd}^r_{\text{ét}}, \tau)$ *is a geometric site.*

Proof. Axioms (a) - (c) are clear. To prove (d), consider the diagram

$$\begin{tikzcd} V' \arrow[r, hook] & U' \arrow[r, two heads] & V' \\ V \arrow[u] \arrow[r, hook] & U \arrow[u, hook] \arrow[r, two heads] & V, \arrow[u] \end{tikzcd} \tag{6}$$

where $U \hookrightarrow U'$ is an open subset inclusion. Axiom (d) follows from (b), after proving the following claim:

<u>Claim:</u> The leftmost square in (6) is a pullback.

First, as monomorphisms have the left cancelling property, the map $V \to V'$ is a monomorphism. Let $y' \in V' \cap U$, then y' coincides with its image under $U \to V$, which shows that the leftmost square induces a pullback on underlying sets. Next, consider a commutative square

$$\begin{tikzcd} V' \arrow[r, hook] & U' \\ W \arrow[u] \arrow[r] & U, \arrow[u, hook] \end{tikzcd}$$

then the canonical map of sets $W \to V$ is smooth, as it may be written as the composition of $W \to U \to V$. $\square$

Theorem 5.1.4. *The ∞-category $\mathbf{Diff}^r$ is a fractured topos, whose ∞-topos of corporeal objects is given by $\mathbf{Diff}^r_{\text{ét}}$.* $\square$

Vista 5.1.5. The ∞-topos $\mathbf{Diff}^r$ is the classifying topos of C^r-rings. Let M be a manifold, then $(\mathbf{Diff}^r_{\text{ét}})_{/M}$ is simply the ∞-topos of sheaves on the underlying topological space of M. The r-differentiable structure, or equivalently, local C^r-ring structure, on M is exhibited by the geometric morphism $(\mathbf{Diff}^r_{\text{ét}})_{/M} \to \mathbf{Diff}^r$. There exists a geometric morphism $\mathbf{Diff}^r \to \widetilde{\mathbf{Aff}}^{\mathrm{fp}}_{\mathbf{R}}$ and the composition of $(\mathbf{Diff}^r_{\text{ét}})_{/M} \to \mathbf{Diff}^r \to \widetilde{\mathbf{Aff}}^{\mathrm{fp}}_{\mathbf{R}}$ exhibits M as being locally ringed in $\mathbf{R}$-algebras, and the further composition with the geometric morphism $\widetilde{\mathbf{Aff}}^{\mathrm{fp}}_{\mathbf{R}} \to \widetilde{\mathbf{Aff}}^{\mathrm{fp}}_{\mathbf{Z}}$ exhibits M as a locally ringed space. ⌟

5.2 Diffeological spaces

5.2.1 Concrete objects

Here we collect the necessary background on concrete objects to discuss diffeological spaces. Throughout this subsection $\mathcal{E}$ denotes an ordinary topos.

Definition 5.2.1. The topos $\mathcal{E}$ is ***local*** if the right adjoint component π_* of the unique geometric morphism $\pi : \mathcal{E} \to \mathbf{Set}$ admits a further right adjoint $\mathcal{E} \leftarrow \mathbf{Set} : \pi^!$, which is fully faithful[6]. ⌟

From now on we assume that $\mathcal{E}$ is local.

Definition 5.2.2. An object X in $\mathcal{E}$ is ***concrete*** if the canonical morphism $X \to \pi^!\pi_*X$ is a monomorphism. The subcategory of $\mathcal{E}$ spanned by concrete objects is denoted by $\mathcal{E}_{\mathrm{concr}}$. ⌟

Proposition 5.2.3. *The inclusion $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ admits a left adjoint.*

Proof. Recall that in any topos the epimorphisms and the monomorphisms form an orthogonal factorisation system. Let X be an object in $\mathcal{E}$, then $X \to \pi^!\pi_*X$ may be

[6] The right adjoint $\pi^!$ is, in fact, automatically fully faithful, as can be seen from [Joh02, Th. 3.6.1] and the observation that any geometric morphism is indexed over $\mathbf{Set}$.

factored uniquely as $X \twoheadrightarrow X' \hookrightarrow \pi^! \pi_* X$. Consider any map $X \to Y$, where Y is a diffeological space, then the lifting problem

$$\begin{tikzcd}
X \arrow[rr] \arrow[d, two heads] & & Y \arrow[d, hook] \\
X' \arrow[r] & \pi^! \pi_* X \arrow[r] & \pi^! \pi_* Y
\end{tikzcd}$$

admits a unique solution, exhibiting the universality of $X \to X'$. □

Definition 5.2.4. The left adjoint of the inclusion $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ (which exists by the preceding proposition) is called the ***concretisation***, and is denoted by $\mathcal{E}_{\mathrm{concr}} \leftarrow \mathcal{E} : (\_)^\dagger$. ⌟

Proposition 5.2.5. *The category $\mathcal{E}_{\mathrm{concr}}$ is presentable.*

Proof. The pair $(\pi^!, \pi_*)$ is a geometric embedding, so that **Set** is a κ-accessible subcategory of $\mathcal{E}$ for some regular cardinal κ, i.e. $\pi^! : \mathbf{Set} \hookrightarrow \mathcal{E}$ commutes with κ-filtered colimits. We claim that $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ likewise commutes with κ-filtered colimits. Let A be a κ-filtered category, and consider a functor $X : A \to \mathcal{E}_{\mathrm{concr}}$; as filtered colimits, and a fortiori κ-filtered colimits preserve monomorphisms, the canonical map $\operatorname{colim} X \to \operatorname{colim} \pi^! \pi_* X \xrightarrow{\cong} \pi^! \pi_* \operatorname{colim} X$ is a monomorphism, so that $\operatorname{colim} X$ is concrete. □

Example 5.2.6. *The category of simplicial sets $\widehat{\Delta}$ is local, with the functor $\pi^!$ being exhibited by* $\operatorname{cosk}_0 : \mathbf{Set} \hookrightarrow \widehat{\Delta}$*. The concrete objects are then those simplicial sets X such that for any $(n+1)$-tuple $(x_0, \ldots, x_n) \in X_0^{(n+1)}$ there exists at most one n-simplex with precisely these vertices.* ⌟

5.2.2 Diffeological spaces

The topos $\mathbf{Diff}^{\infty}_{\leq 0}$ is local, with the functor $\mathbf{Diff}^{\infty}_{\leq 0} \hookleftarrow \mathbf{Set} : \pi^!$ given by sending any set X to the presheaf $\mathbf{R}^n \mapsto \mathbf{Set}(\pi_* \mathbf{R}^n, X)$.

Definition 5.2.7. A ***diffeological space*** X is a concrete object in $\mathbf{Diff}^{\infty}_{\leq 0}$. A ***plot*** of X is a smooth map $\mathbf{R}^n \to X$. The collection of all plots of X is called the ***diffeology*** of X. ⌟

Definition 5.2.8. Let X be a diffeological space, and $Y \subseteq X$ a subset, then the ***subspace diffeology*** on Y is the diffeology in which a map $\mathbf{R}^n \to Y$ is a plot iff it is a plot viewed as a map to X. ⌟

Example 5.2.9. *The standard simplex Δ^n with the subspace diffeology inherited from $\mathbf{R}^{n+1}$ is denoted by Δ^n_{sub}.* ⌟

Proposition 5.2.10 ([Wat12, Lm. 2.64])**.** *Write* $\mathbf{R}^n_+ := \left\{ (x_1, \dots, x_n) \in \mathbf{R}^n \;\middle|\; x_1, \dots, x_n \geq 0 \right\}$, *and endow this set with the subspace diffeology inherited from $\mathbf{R}^n$. A map $f : \mathbf{R}^n_+ \to \mathbf{R}$ is smooth iff it is the restriction of a smooth map $U \to \mathbf{R}$, where U is an open neighbourhood of $\mathbf{R}^n_+$ in $\mathbf{R}^n$.*

Proof. Write $s : \mathbf{R}^n \to \mathbf{R}^n$, $(x_1, \dots, x_n) \mapsto (x_1^2, \dots, x_n^2)$, then by assumption $f \circ s : \mathbf{R}^n \to \mathbf{R}$ is smooth and is invariant under the action $(\mathbf{Z}^*)^n \times \mathbf{R}^n \to \mathbf{R}^n$, $\big((\sigma_1, \dots, \sigma_n), (x_1, \dots, x_n)\big) \mapsto (\sigma_1 x_1, \dots, \sigma_n x_n)$. By [Sch75] there exists a smooth map $\widetilde{f} : \mathbf{R}^n \to \mathbf{R}$ such that $\widetilde{f} \circ s = f \circ s$. As s restricts to a bijection on the underlying sets of $\mathbf{R}^n_+ \to \mathbf{R}^n_+$, the maps f and $\widetilde{f}$ agree on $\mathbf{R}^n_+$, so that f is a restriction of $\widetilde{f}$. □

Corollary 5.2.11. *Let M be a smooth manifold with corners, and N a smooth manifold* without *corners, then a map $M \to N$ is smooth iff there exists a manifold $\widetilde{M}$ without corners, and an open embedding $M \subseteq \widetilde{M}$ and a smooth map $\widetilde{M} \to N$ which restricts to $M \to N$. In particular, a map $\Delta^n_{\text{sub}} \to N$ is smooth iff there exists an open neighbourhood U of Δ^n_{sub} in $\mathbf{R}^{n+1}$ and a smooth map $U \to N$ which restricts to $\Delta^n_{\text{sub}} \to N$.* □

Let X be a differentiable stack, then no such characterisation of smooth maps $\Delta^n_{\text{sub}} \to X$ need exist when X is not a smooth manifold. One need only consider the identity map $\Delta^n_{\text{sub}} \to \Delta^n_{\text{sub}}$. Despite what the proof of Proposition 5.2.10 might suggest, we are very doubtful that there exists a finite cover of Δ^n_{sub} by affine spaces. This makes it very hard to construct maps $\Delta^n_{\text{sub}} \to X$ even if one understands the maps from $\mathbf{R}^n \to X$ $(n \geq 0)$.

As a special case of Proposition 5.2.5 we obtain the following proposition.

Proposition 5.2.12. *The category of diffeological spaces is presentable.* □

5.2.3 Colimits in $\mathbf{Diff}^{\infty}_{\text{concr}}$

The inclusion $\mathbf{Diff}^{r}_{\text{concr}} \hookrightarrow \mathbf{Diff}^{\infty}$ is closed under at least two types of colimits. To show this we require the following two lemmas. Here D denotes one of the categories $\mathbf{Cart}^r, \mathbf{OMfd}^r, \mathbf{Mfd}^r$.

Lemma 5.2.13. *Let X be a concrete presheaf on D, then the sheafification of X is a diffeological space.* □

This lemma has the following corollary.

Lemma 5.2.14. *Let A be a small category, and consider a diagram $X : A \to \mathbf{Diff}^r_{\mathrm{concr}}$. The colimit of X in the category $\underline{\mathrm{Hom}}(D^{\mathrm{op}}, \mathcal{S})$ is an ordinary concrete presheaf, then taking the colimit of X in $\mathbf{Diff}^r$ yields an object in $\mathbf{Diff}^r_{\mathrm{concr}}$.* □

Proposition 5.2.15. *Consider a span of diffeological spaces*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Z \\ Y & \end{tikzcd}$$

in which both legs are monomorphisms, then the pushout in $\mathbf{Diff}^\infty$ is again a diffeological space.

Proof. By Lemma 3.3.8 the pushout is truncated. If one considers the pushout in $\widehat{\mathbf{Cart}_n}$, then it is easily seen that the resulting presheaf is concrete, so the result follows from Lemma 5.2.14. □

Proposition 5.2.16. *Let α be an ordinal, then for any diagram $p : \alpha \to \mathbf{Diff}^\infty_{\mathrm{concr}}$ in which all the constituent morphisms are monomorphisms, then its colimit in $\mathbf{Diff}^\infty$ is again a diffeological space.*

Proof. Similar to the proof of the preceding lemma. □

Example 5.2.17. *Consider the unique cocontinuous functor $\widehat{\Delta} \to \mathbf{Diff}^\infty_{\leq 0}$ carrying Δ^n to Δ^n_{sub} from Example 5.2.9 then this functor carries the simplicial sets $\partial\Delta^n$ and Λ^n_k to diffeological spaces. These diffeological spaces* are not *equipped with the subspace diffeology of Δ^n_{sub}. Write $\Lambda^2_1 := u_!\Lambda^2_1$ and $\Lambda^2_{1,\mathrm{sub}}$ for the* 1*-horn of Δ^2 with the subdiffeology. Any path passing through the corner of Λ^2_1 must restrict to a constant* *functor which takes values in that corner for a positive amount of time. A retract of* $\Lambda^2_1 \hookrightarrow \Delta^2$ *could then be composed with an injective path passing through the corner of* $\Lambda^2_{1,\mathrm{sub}}$*, yielding a contradiction.* ⌟

6 Homotopical algebra of smooth stacks and spaces

This is the main section in this thesis, in which we apply the theory built in the previous sections in order to construct homotopical calculi on $\mathbf{Diff}^\infty_{\leq 0}$ and $\mathbf{Diff}^\infty$. In §6.1 we show that $\mathbf{Diff}^\infty_{\leq 0}$ is a strict test topos. By the strictness condition the shape functor $\pi_! : \mathbf{Diff}^\infty \to \mathcal{S}$ commutes with finite products so that we obtain a canonical map $\pi_! \underline{\mathbf{Diff}}(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$.

Definition 6.0.1. A differentiable stack A is called ***formally cofibrant*** if for every differentiable stack X the canonical map $\pi_! \underline{\mathbf{Diff}}^\infty(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$ is an equivalence. ⌟

Ideally, we would construct a Cartesian model structure on $\mathbf{Diff}^\infty$ in which the weak equivalences are the shape equivalences, and in which all objects are fibrant; in this case all cofibrant objects would be formally cofibrant. Unfortunately, we are unaware whether such a model structure exists. Examining through the lens of §2 the mechanism by which a (cofibrantly generated) Cartesian model structure produces formally cofibrant objects we observe the following: Assume that we have already shown that a given object A is a formally cofibrant differentiable stack, then if we attach a "cell" $S \hookrightarrow D$ along a map $f : S \to A$, then a natural way of showing that $A \cup_f D$ is also cofibrant is to show that the pullback

$$\begin{array}{ccc}
\underline{\mathbf{Diff}}(A \cup_f D, X) & \longrightarrow & \underline{\mathbf{Diff}}(D, X) \\
\downarrow & & \downarrow \\
\underline{\mathbf{Diff}}(A, X) & \longrightarrow & \underline{\mathbf{Diff}}(S, X)
\end{array}$$

is a homotopy pullback. Thus we would like to

1. show that $\mathbf{Diff}^\infty$ is right proper (see Definition 2.0.3), and to

2. find morphisms $S \to D$ such that the morphism $X^D \to X^S$ is sharp for every differentiable stack X.

Definition 6.0.2. A morphism $S \to D$ in $\mathbf{Diff}^\infty$ is called a ***formal cofibration*** if $X^D \to X^S$ is sharp for every differentiable stack X. ⌟

In §6.2 we introduce an alternative diffeology on the standard simplices, due to Kihara, which produces a model structure on $\mathbf{Diff}^\infty$ in which all objects are fibrant, thus taking care of the first point. While this model structure is not Cartesian its cofibrations are nevertheless formal cofibrations. To show this we introduce in §6.3 yet another homotopical calculus on $\mathbf{Diff}^r$ in the form of a fibration structure, the *squishy fibration structure*. In this fibration structure Kihara's cofibrations *are* formal cofibrations. Any differentiable stack $\mathbf{R}$-homotopy equivalent to a cofibrant object in the Kihara model structure is formally cofibrant, so that many interesting objects such as manifolds are formally cofibrant.

For the rest of this section we fix $0 \leq r \leq \infty$.

6.1 $\mathbf{Diff}^\infty_{\leq 0}$ is a test topos

We will now apply the theory of the preceding sections of this thesis in order to show that $\mathbf{Diff}^r_{\leq 0}$ is a strict test topos. By Corollaray 4.2.3 a first step is to show that the topos $\left(\mathbf{Diff}^r_{\text{ét},\leq 0}\right)_{/\mathbf{R}^d}$ has contractible shape for all $0 \leq d < \infty$. As $\left(\mathbf{Diff}^r_{\text{ét},\leq 0}\right)_{/\mathbf{R}^d}$ is the same for all $0 \leq r \leq \infty$, it suffices to consider the case $r = 0$, and show that the underlying topological space of $\mathbf{R}^d$ has contractible shape. There are myriad ways of achieving this: One could for example use that the shape of a CW complex is given by its singular homotopy type (see [Lur09, §7.1]). Or one could show that

(1) $\mathbf{R}$ has constant shape, and that

(2) $\mathbf{R}$-homotopy equivalences induces shape equivalences.

A modern proof of (1) is given in [Lur17, Lm. A.2.2], and a modern proof of (2), in [Lur17, Lm. A.2.9]. More classically, one could show (1) by first noting that locally constant sheaves of sets on $\mathbf{R}$ are constant; see [Sch14, Lm. 5.1.2]. This shows $\pi_0\pi_!\mathbf{R} = *$ and $\pi_1\pi_!\mathbf{R} = 1$; the latter follows from the fact that for all locally constant sheaves of groups G, the group $H^1(\mathbf{R}; G)$ is in bijection with the equivalence classes of locally constant G-torsors. Finally the higher cohomology groups vanish because $H^i(\mathbf{R}; M) = 0$ for all $i > 0$ and all Abelian sheaves M on $\mathbf{R}$ by [Sch14, Lm. 5.1.1] (i.e. $\mathbf{R}$ has cohomological dimension 1). To see (2) we simply apply the classical theorem that homotopic maps pull back any given locally constant sheaf to isomorphic sheaves; see [Sch14, Thm. 5.2.3].

For the rest of this subsection we fix $0 \leq r \leq \infty$.

Proposition 6.1.1. *The category* $\mathbf{Cart}^r$ *is a strict test category.*

Proof. By Corollary 3.2.11 it is enough to observe that $\mathbf{R}$ together with the inclusions of $\{0\}$ and $\{1\}$ is a separating interval. □

Theorem 6.1.2. *The topos* $\mathbf{Diff}^r_{\leq 0}$ *is a strict test topos.*

Proof. This follows from Corollaries 3.3.24 & 3.3.25, Theorem 3.3.26, Proposition 6.1.1, and the prefatory discussion of this subsection. □

Definition 6.1.3. Consider the cosimplicial object

$$\begin{array}{rcl} \mathbf{A}^\bullet : \ \Delta & \to & \mathbf{Cart}^r \\ \Delta^n & \mapsto & \mathbf{A}^n := \left\{ (x_0, \ldots, x_n) \in \mathbf{R}^{n+1} \,\middle|\, x_0 + \cdots + x_n = 1 \right\}, \end{array}$$

then the spaces $\mathbf{A}^n$ for $n \geq 0$ are referred to as ***extended simplices***. Moreover we write

$$\begin{array}{rcll} \partial \mathbf{A}^n & := & \mathbf{A}^\bullet_! \partial \Delta^n, & n \geq 0 \\ \mathbf{\Lambda}^n_k & := & \mathbf{A}^\bullet_! \Lambda^n_k, & n \geq 1, \ n \geq k \geq 0. \end{array}$$

A morphism $X \to Y$ in $\mathbf{Diff}^r$ is called a ***concordance equivalence*** if $\mathbf{Diff}^r(\mathbf{A}^\bullet, X) \to \mathbf{Diff}^r(\mathbf{A}^\bullet, Y)$ is a locally constant weak equivalence in $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S})$. ⌟

Proposition 6.1.4. *The concordance equivalences and the locally constant equivalences in* $\mathbf{Diff}^r$ *agree. There exist cofibrantly generated model structures on* $\mathbf{Diff}^r$ *and* $\mathbf{Diff}^r_{\leq 0}$ *with the aforementioned weak equivalences and with generating cofibrations and trivial cofibrations given, respectively, by* $\{\partial \mathbf{A}^n \hookrightarrow \mathbf{A}^n\}_{n \geq 0}$ *and* $\{\mathbf{\Lambda}^n_k \hookrightarrow \mathbf{A}^n\}_{n \geq 1,\, n \geq k \geq 0}$.

Proof. We will verify properties (a) - (d) in Proposition 3.4.4. Property (a) is clear. To verify property (b), we observe that $\mathbf{R}^n$ is $\mathbf{R}$-contractible for all $n \geq 0$. Property (c) follows from the previous point and Proposition 3.4.7. Finally, property (d) follows from Proposition 3.4.5. □

The model structures on $\mathbf{Diff}^r_{\leq 0}$ and $\mathbf{Diff}^r$ exhibited in Proposition 6.1.4 are both referred to as the ***concordance model structure***.

Recalling the notation from Example 5.2.9, a morphism $X \to Y$ in $\mathbf{Diff}^r$ is called a ***classical singular equivalence*** if $\mathbf{Diff}^r(\Delta^\bullet_{\mathrm{sub}}, X) \to \mathbf{Diff}^r(\Delta^\bullet_{\mathrm{sub}}, Y)$ is a locally constant weak equivalence in $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S})$.

Proposition 6.1.5. *The classical singular equivalences and the locally constant equivalences in $\mathbf{Diff}^r$ agree. There exist cofibrantly generated model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\le 0}$ with the aforementioned weak equivalences and with generating cofibrations and trivial cofibrations given, respectively, by $\{\partial\Delta^n_{\mathrm{sub}} \hookrightarrow \Delta^n_{\mathrm{sub}}\}_{n\ge 0}$ and $\{(\Lambda_{\mathrm{sub}})^n_k \hookrightarrow \Delta^n_{\mathrm{sub}}\}_{n\ge 1,\, n\ge k\ge 0}$.*

Proof. We will verify properties (a) - (d) in Proposition 3.4.4. To show (a) we observe that $\mathbf{R}^n$ may be obtained as the colimit of $\cdots \hookrightarrow \Delta^n \hookrightarrow \Delta^n \hookrightarrow \cdots$, where each space Δ^n maps concentrically into a strictly larger copy of itself. To verify property (b), we observe that Δ^n is $\mathbf{R}$-contractible for all $n \ge 0$. Property (c) follows from the previous point and Proposition 3.4.7. Finally, property (d) follows from Proposition 3.4.5. $\square$

The model structures on $\mathbf{Diff}^r_{\le 0}$ and $\mathbf{Diff}^r$ exhibited in Proposition 6.1.5 are both referred to as the ***naive simplicially generated model structure***.

Proposition 6.1.6. *Neither the concordance nor the naive simplicially generated model structure is Cartesian.*

Proof. With notation as in Example 5.2.17, Λ^2_1 is not fibrant in the naive simplicially generated model structure. If it were, then the lifting problem

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & \Lambda^2_1 \arrow[d, two heads, "\sim"] \\ \Delta^2_{\mathrm{sub}} \arrow[r] & 1 \end{tikzcd}$$

would admit a solution, contradicting the fact that Λ^2_1 does not have the subspace diffeology as explained in Example 5.2.17. An analogous argument using extended simplices yields an object which is not fibrant in the concordance model structure. $\square$

Propositions do both (individually) have the following corollary:

Corollary 6.1.7. *The shape functor* $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ *preserves finite products.*

Proof. Apply Proposition 3.3.25. $\square$

6.2 Kihara's simplices

It has been a longstanding goal to establish a model structure on diffeological spaces (see e.g. [CW14] and [HS18]). To this end Kihara endows the standard simplices with a new diffeology in [Kih19, § 1.2]. With this diffeology the horn inclusions admit deformation retracts (see Proposition 6.2.1), allowing Kihara to mimic the construction of the model structure on topological spaces in [Qui67, §II.3], and show that the resulting model category is Quillen equivalent to simplicial sets with the Kan-Quillen model structure. We need Kihara's simplices in order to construct formally cofibrant objects in §6.4, but we will also give a simpler proof of the existence of Kihara's model structure, as well as the equivalence to the Kan-Quillen model structure in §6.2.1.

For the convenience of the reader, we repeat the construction of Kihara's simplices: For each $n \geq 1$ and each $0 \leq k \leq n$ we define the set

$$A^n_k := \left\{ (x_0, \ldots, x_n) \in \Delta^n \;\middle|\; x_k < 1 \right\}.$$

We now proceed inductively: On Δ^0 and Δ^1 the diffeology is the subspace diffeology coming from $\mathbf{R}^1$ and $\mathbf{R}^2$, respectively. Let $n > 1$, and assume that the diffeologies on the simplices Δ^m for $m < n$ have been defined, then we define a diffeology on A^n_k by exhibiting this set as the underlying set of the quotient

$$\begin{tikzcd} \Delta^{n-1} \times \{0\} \arrow[r, hook] \arrow[d, two heads] & \Delta^{n-1} \times [0,1) \arrow[d, two heads] \\ 1 \arrow[r, hook] & A^n_k, \end{tikzcd}$$

where $\Delta^{n-1} \times [0,1) \to A^n_n$ is given by $(x_0, \ldots, x_{n-1}; t) \mapsto ((1-t) \cdot x_0, \ldots, (1-t) \cdot x_n, t)$. Finally, the diffeology on Δ^n is determined by the map $\coprod_{k=0}^n A^n_k \to \Delta^n$.

Proposition 6.2.1 ([Kih19, § 8])**.** *The horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for $n = 2$ and $n \geq k \geq 0$ admit a deformation retract.* $\square$

Definition 6.2.2. We write

$$\begin{array}{rccc} \Delta^\bullet : & \Delta & \to & \mathbf{Diff}^r \\ & \Delta^n & \mapsto & \Delta^n \end{array}$$

for the cosimplicial object sending each simplex Δ^n to the standard n-simplex endowed with the diffeology described above. Moreover we write

$$\begin{array}{rcll} \partial\Delta^n & := & \Delta^\bullet_! \partial\Delta^n, & n \geq 0 \\ \Lambda^n_k & := & \Delta^\bullet_! \Lambda^n_k, & n \geq 1,\ n \geq k \geq 0 \end{array}$$

A morphism $X \to Y$ in $\mathbf{Diff}^r$ is called a ***singular equivalence*** if $\mathbf{Diff}^r(\Delta^\bullet, X) \to \mathbf{Diff}^r(\Delta^\bullet, Y)$ is a locally constant weak equivalence in $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S})$. ⌟

Proposition 6.2.3. *The singular equivalences and the locally constant equivalences in $\mathbf{Diff}^r$ agree. There exist cofibrantly generated model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ with the aforementioned weak equivalences and with generating cofibrations and trivial cofibrations given, respectively, by $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ and $\{\Lambda^n_k \hookrightarrow \Delta^n\}_{n\geq 1,\, n\geq k\geq 0}$. Moreover, all objects in these model categories are fibrant.*

Proof. The proof is exactly the same as for Proposition 6.1.5, except that it is not obvious that the simplices Δ^n are Δ^1-contractible, but this is shown in [Kih19, Rmk. 9.3]. $\square$

The model structures on $\mathbf{Diff}^r_{\leq 0}$ and $\mathbf{Diff}^r$ are both referred to as the ***Kihara model structure***.

Corollary 6.2.4. *The shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ commutes with* all *products.* $\square$

To our knowledge this is a new result.

Corollary 6.2.5. *The relative categories $\mathbf{Diff}^r_{\leq 0}$ and $\mathbf{Diff}^r$ proper.* $\square$

Remark 6.2.6. This is established for all local test toposes by Cisinski by more abstract means. [Cis03, Cor. 4.2.12 & Cor. 5.3.20]. The result for $\mathbf{Diff}^r$ is new. $\lrcorner$

Proposition 6.2.7. *The Kihara model structures on $\mathbf{Diff}^r_{\leq 0}$ and $\mathbf{Diff}^r$ are* not *Cartesian.*

Proof. Taking the pushout-product of $\delta : \partial\Delta^1 \hookrightarrow \Delta^1$ with itself produces the inclusion $\delta\Box\delta := (\Delta^1 \times \partial\Delta^1) \sqcup_{\partial\Delta^1\times\partial\Delta^1} \partial\Delta^1 \times \Delta^1 \hookrightarrow \Delta^1 \times \Delta^1$. If this inclusion were a cofibration, then the square

$$\begin{tikzcd}
\delta\Box\delta \arrow[r, hook] \arrow[d, hook] & u_!(\Delta^1 \times \Delta^1) \arrow[d] \\
\Delta^1 \times \Delta^1 \arrow[r] & 1
\end{tikzcd}$$

would admit a lift, producing a similar contradiction as in Proposition 6.1.6. $\square$

Not only is the Kihara model structure not Cartesian, but fibrations are hard to detect, because it is hard to write down maps $\Delta^n \to X$, when X is an arbitrary differentiable stack, essentially for the same reason that it is hard to write down maps $\Delta^n_{\text{sub}} \to X$, as explained in §5.2.2. This issue will be addressed in §6.3.

6.2.1 Kihara's model structure on diffeological spaces

Here we provide a simple proof (using the theory developed in this thesis) of a model structure, originally due to Kihara, on $\mathbf{Diff}^r_{\text{concr}}$, which is Quillen equivalent to the category of simplicial sets together with the Kan-Quillen model structure.

Theorem 6.2.8 ([Kih19, Th. 1.3] [Kih17, Th. 1.1])**.** *There exists a cofibrantly generated model structure on $\mathbf{Diff}^r_{\text{concr}}$ in which the weak equivalences are equivalently the shape equivalences or the singular equivalences, and in which the generating cofibrations and trivial cofibrations are given, respectively, by $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ and $\{\Lambda^n_k \hookrightarrow \Delta^n\}_{n\geq 1,\, n\geq k\geq 0}$. Moreover, all objects in this model categories are fibrant. The adjunction $\widehat{\Delta} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Diff}^r_{\text{concr}}$ is a Quillen equivalence.*

Proof. We shall transfer the model structure from $\widehat{\Delta}$ using Proposition 3.4.3. Transfinite composition of monomorphisms between diffeological spaces taken in $\mathbf{Diff}^r$ remains in $\mathbf{Diff}^r_{\mathrm{concr}}$ by Proposition 5.2.16, so that the transfinite composition of monomorphisms which are also shape equivalences is again a shape equivalence, as shape equivalences in $\mathbf{Diff}^\infty$ are closed under taking colimits. Thus, it is enough to show for any diffeological space X and any map $f : \Lambda^n_k \to X$ $(n \geq 1,\ n \geq k \geq 0)$, that the map $X \to X \cup_f \Delta^n$ is a weak equivalence, which follows from the fact that $X \to X \cup_f \Delta^n$ is a deformation retract, since $\Lambda^n_k \hookrightarrow \Delta^n$ is one.

The fact that all objects are fibrant follows from the fact that all inclusions $\Lambda^n_k \to \Delta^n$ $(n \geq 1,\ n \geq k \geq 0)$ are deformation retracts.

To show that $\widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^r_{\mathrm{concr}}$ is a Quillen equivalence, we observe that by Propositions 5.2.15 & 5.2.16 the Yoneda extension of $\Delta \to \mathbf{Diff}^r_{\leq 0}$, $\Delta^n \mapsto \Delta^n$ factors through the inclusion $\mathbf{Diff}^r_{\mathrm{concr}} \hookrightarrow \mathbf{Diff}^r_{\leq 0}$, thus both the unit and counit of $\widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^r_{\mathrm{concr}}$ are weak equivalences, because this is true for $\widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^r_{\leq 0}$.

$\square$

6.3 The squishy fibration structure on $\mathbf{Diff}^\infty$

In this subsection we construct the squishy fibration structure, which, together with Kihara's simplices, will allow us to construct formally cofibrant objects in the following subsection. In the preceding subsections, using various cosimplicial objects in $\mathbf{Diff}_{\leq 0}\, r$, we considered several model structures before constructing the Kihara model structure, which has good formal properties. Similarly, we shall progress through several model structures induced from cocubical objects, before constructing the squishy fibration structure.

Proposition 6.3.1. *Let* $1 \rightrightarrows I$ *be an interval in* $\mathcal{E}_{\leq 0}$ *(see Definition 3.2.8), and* *assume that the functor $\square \to \mathcal{E}_{\leq 0}$ induced by Proposition A.0.2 (w.r.t. the categorical product on $\mathcal{E}_{\leq 0}$) satisfies conditions* (a) - (d) *of Proposition 3.4.4, then the induced model structures on $\mathcal{E}_{\leq 0}$ and $\mathcal{E}$ are Cartesian.*

Proof. The functor $\widehat{\square} \to \mathcal{E}_{\leq 0}$ is monoidal and commutes with colimits, so that the axioms of a Cartesian model structure are verified by Proposition A.0.4. $\square$

Some basic facts about the cube category are summarised in §A.

6.3.1 Ordinary cubes

By Proposition A.0.2, sending $\square^1$ to either $\mathbf{R}^1$ or Δ^1 produces an adjunction $\widehat{\square} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^{\infty}_{\leq 0}$, with which we may construct a Cartesian closed model structure using Proposition 6.3.1. For reasons completely analogues to the one given in Proposition 6.1.6, it is not the case that all objects are fibrant in either model structure. In order to address this problem, one might be tempted to construct a cocubical object in $\mathbf{Diff}^{r}_{\leq 0}$ whose constituent spaces have an analogues diffeology to the one on Kihara's simplices. Unfortunately, this would destroy the Cartesian closedness of the induced model structure, as this property relies on n-th cube in the cocubical object being the n-fold product of the first cube, i.e. the standard interval. Moreover, one would inherit a defect from the model structures considered in the previous section, namely that the fibrations would be hard to detect.

6.3.2 ε-squishy intervals and cubes

Here we construct a precursor to the squishy fibration structure considered in §6.3.3. Throughout this subsection we fix $0 < \varepsilon < \frac{1}{2}$.

Definition 6.3.2. The pushout of the span

$$\begin{tikzcd} {[0,\varepsilon] \cup [1-\varepsilon,1]} \arrow[r, two heads] \arrow[d, hook] & \{0\} \cup \{1\} \\ \square^1_\varepsilon & \end{tikzcd}$$

(in $\mathbf{Diff}^r$) is called the ε***-squishy interval*** and is denoted by $\mathbb{\square}^1_\varepsilon$. For any $n \in \mathbf{N}$ the n-fold product of $\mathbb{\square}^1_\varepsilon$ is called the ε***-squishy*** n***-cube***, and is denoted by $\mathbb{\square}^n_\varepsilon$. ⌟

Proposition 6.3.3. *The ε-squishy n-cube $\mathbb{\square}^n_\varepsilon$ is 0-truncated for all $n \in \mathbf{N}$.*

Proof. This is an immediate consequence of Lemma 3.3.8. □

Proposition 6.3.4. *For any $n \geq 0$ the differentiable space $\mathbb{\square}^n_\varepsilon$ corepresents the functor sending any differentiable stack X to the summand of $\mathbf{Diff}^r(\square^n, X)$ consisting of those maps $\square^n \to X$ which can be extended (uniquely) to a cocone on $(\{0\} \cup \{1\} \twoheadleftarrow [0,\varepsilon] \cup [1-\varepsilon,1] \hookrightarrow \square^1)^n$.* □

Notation 6.3.5. The above discussion yields a cocubical object, which we denote as follows:

$$\begin{aligned} \mathbb{\square}^\bullet_\varepsilon : \ \square &\to \mathbf{Diff}^r_{\leq 0} \\ \square^n &\mapsto \mathbb{\square}^n_\varepsilon \end{aligned}$$

Moreover, we write

$$\begin{aligned} \partial \mathbb{\square}^n_\varepsilon &:= (\mathbb{\square}_\varepsilon)^\bullet_! \partial \square^n, \quad n \geq 0 \\ \sqcap^n_{k,\xi,\varepsilon} &:= (\mathbb{\square}_\varepsilon)^\bullet_! \sqcap^n_{i,\xi}, \quad n \geq 1,\ n \geq k \geq 0, \xi = 0,1. \end{aligned}$$

⌟

A morphism $X \to Y$ in $\mathbf{Diff}^r$ is called an ***ε-squishy equivalence*** if $\mathbf{Diff}^r(\mathbb{I}^\bullet_\varepsilon, X) \to \mathbf{Diff}^r(\mathbb{I}^\bullet_\varepsilon, Y)$ is a locally constant weak equivalence in $\underline{\mathrm{Hom}}(\square^{\mathrm{op}}, \mathcal{S})$.

Lemma 6.3.6. *The canonical inclusion $\{0\} \cup \{1\} \to \mathbb{I}^1_\varepsilon$ determines a separating interval (see Definition 3.2.8).* $\square$

Proposition 6.3.7. *The ε-squishy equivalences and the locally constant equivalences in $\mathbf{Diff}^r$ agree. There exist cofibrantly generated model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ with the aforementioned weak equivalences and with generating cofibrations and trivial cofibrations given, respectively, by $\{\partial\mathbb{I}^n_\varepsilon \hookrightarrow \mathbb{I}^n_\varepsilon\}_{n\geq 0}$ and $\{\sqcap^n_{k,\varepsilon} \hookrightarrow \mathbb{I}^n_\varepsilon\}_{n\geq 1,\, n\geq k\geq 0}$.*

The model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\leq 0}$ constructed in Proposition 6.3.7 are both referred to as the ***ε-squishy model structure***. Unfortunately, it does not seem to be the case that all objects are fibrant in the ε-squishy model structure. Moreover, the model structure depends on the parameter $0 < \varepsilon < \frac{1}{2}$. Removing this dependence *does* produce a fibration structure, in which all objects are fibrant.

6.3.3 The squishy fibration structure on $\mathbf{Diff}^\infty$

We now finally produce the fibration structure, that will allow us to produce formally cofibrant objects. Throughout this subsection we prove results about pro-∞-categories which are of independent interest.

Proposition 6.3.8. *Let C be an ∞-category admitting finite products, then* $\mathrm{Pro}(C)$ *admits finite products.*

Proof. Let $x_0, \dots, x_n$ be objects in $\mathrm{Pro}(C)$ then for each $0 \leq i \leq n$ there exists a filtered small ordinary category A_i and a functor $x_{i\bullet} : A_i \to C$ such that $x_i \simeq$ "$\lim\limits_{\alpha \in A_i}$"$x_{i\alpha}$ (see [Lur09, Prop. 5.3.1.16]). The category $A_0 \times \cdots \times A_n$ is filtered, and we claim that "$\lim\limits_{\alpha \in A_0 \times \cdots \times A_n}$"$x_{0\bullet} \times \cdots \times x_{n\bullet}$ pro-represents the product of $x_0, \dots, x_n$. To see this, let y be any objects of C, then the isomorphisms

$$\begin{aligned}
\mathrm{Pro}(C)(y\text{``}\lim_{A_0 \times \cdots \times A_n}\text{''}x_{0\bullet} \times \cdots \times x_{n\bullet}, y) &\simeq \lim{}_{A_0 \times \cdots \times A_n} C(x_{0\bullet} \times \cdots \times x_{n\bullet}, y) \\
&\simeq \lim{}_{A_0 \times \cdots \times A_n} C(x_{0\bullet}, y) \times \cdots \times C(x_{n\bullet}, y) \\
&\simeq \lim{}_{A_0} \cdots \lim{}_{A_n} C(x_{0\bullet}, y) \times \cdots \times C(x_{n\bullet}, y) \\
&\simeq \lim{}_{A_0} \cdots \lim{}_{A_{n-1}} C(x_{0\bullet}, y) \times \cdots \times C(x_{n-1\bullet}, y) \times C(x_n, y) \\
&\cdots \\
&\simeq \lim{}_{A_0} C(x_{0\bullet}, y) \times C(x_1, y) \times \cdots \times C(x_n, y) \\
&\simeq C(x_0, y) \times \cdots \times C(x_n, y)
\end{aligned}$$

are natural in y. $\square$

Definition 6.3.9. The pro-differentiable stack

$$\mathbb{\Box}^1 := \text{``}\lim_{\varepsilon > 0}\text{''}\mathbb{\Box}^1_\varepsilon$$

is called the ***squishy interval***. For any $n \in \mathbf{N}$ the n-fold product of $\mathbb{\Box}^1$ is called the ***squishy*** n***-cube***, and is denoted by $\mathbb{\Box}^n$. The resulting cocubical pro-object is denoted as follows:

$$\begin{aligned}
\mathbb{\Box}^\bullet : \ \Box &\to \mathrm{Pro}(\mathbf{Diff}^r) \\
\Box^n &\mapsto \mathbb{\Box}^n
\end{aligned}$$

$\lrcorner$

Proposition 6.3.10. *There is a canonical isomorphism*

$$\mathbb{\square}^n \simeq \text{“}\lim_{\varepsilon>0}\text{”}\mathbb{\square}^n_\varepsilon \quad n \geq 0.$$

Proof. There is an isomorphism $\mathbb{\square}^n \simeq \text{“}\lim_{(\varepsilon_1>0)\times\cdots\times(\varepsilon_n>0)}\text{”}\mathbb{\square}^1_{\varepsilon_1} \times \cdots \times \mathbb{\square}^1_{\varepsilon_n}$ by the proof of Proposition 6.3.8. As the ordered set $\left(0, \frac{1}{2}\right)$ admits products it is sifted, and the diagonal map $\left(0, \frac{1}{2}\right) \to \left(0, \frac{1}{2}\right) \times \cdots \times \left(0, \frac{1}{2}\right)$ is initial so that the induced map $\text{“}\lim_{\varepsilon>0}\text{”}\mathbb{\square}^1_\varepsilon \times \cdots \times \mathbb{\square}^1_\varepsilon \to \text{“}\lim_{(\varepsilon_1>0)\times\cdots\times(\varepsilon_n>0)}\text{”}\mathbb{\square}^1_{\varepsilon_1} \times \cdots \times \mathbb{\square}^1_{\varepsilon_n}$ is an isomorphism. $\square$

Proposition 6.3.11. *For any $n \geq 0$ the pro-differentiable stack $\mathbb{\square}^n$ pro-corepresents the functor sending any differentiable stack X to the summand of $\mathbf{Diff}^\infty(\square^n, X)$ consisting of those maps $\square^n \to X$ which can be extended to a cocone on $(\{0\} \cup \{1\} \twoheadleftarrow [0, \varepsilon] \cup [1-\varepsilon, 1] \hookrightarrow \square^1)^n$ for some $0 < \varepsilon < \frac{1}{2}$.* $\square$

Proof. As pushouts and products preserve epimorphisms, the map $\square^n \to \mathbb{\square}^n_\varepsilon$ is an epimorphisms for all $n \geq 0$ and all $0 < \varepsilon < \frac{1}{2}$, and thus for $0 < \varepsilon' < \varepsilon$ the map $\mathbb{\square}^n_{\varepsilon'} \to \mathbb{\square}_\varepsilon$ is likewise an epimorphism. Let X be any differentiable stack, then $\mathbf{Diff}^\infty(\mathbb{\square}^n, X)$ is the colimit of the diagram $\cdots \to \mathbf{Diff}(\mathbb{\square}^n_\varepsilon, X) \xrightarrow{\varepsilon'<\varepsilon} \mathbf{Diff}(\mathbb{\square}^n_{\varepsilon'}, X) \to \cdots$, where each transition map is a monomorphism, and thus $\mathbf{Diff}(\mathbb{\square}^n, X) \to \mathbf{Diff}(\square^n, X)$ is a monomorphism. The statement then follows from Proposition 6.3.4. $\square$

Lemma 6.3.12. *Let I be a set, and for each element $i \in I$ consider a small filtered category A_i and a functor $X_i : A_i \to \mathcal{S}$, then the canonical morphism*

$$\operatorname*{colim}_{(\alpha_i)\in\prod A_i} \prod_{i\in I} X_{i,\alpha_i} \to \prod_{i\in I} \operatorname*{colim}_{\alpha_i\in A_i} X_{i,\alpha_i} \tag{7}$$

is an equivalence.

Proof. By [KS06, Prop. 3.1.11.ii] the statement is true in **Set**. Then, by [Cis19, Cor. 7.9.9] we may lift the functors $X_i : A_i \to \mathcal{S}$ to functors $A_i \to \widehat{\Delta}$, which we may then compose with the Ex^∞ functor to obtain functors valued in Kan complexes. The morphism in $\widehat{\Delta}$ corresponding to (7) is then an isomorphism, and the statement follows from the fact that Kan complexes as well as weak equivalences are closed under filtered colimits (see [Cis19, Lm. 3.1.24 & Cor. 4.1.17]). □

Proposition 6.3.13. *Let C be an accessible ∞-category admitting finite limits and coproducts, then the ∞-category* $\mathrm{Pro}(C)$ *is cocomplete.*

Proof. We show that $\mathrm{Pro}(C)$ admits pushouts and small coproducts.

$\underline{\mathrm{Pro}(C)\text{ admits pushouts:}}$ Recall that $\mathrm{Pro}(C)$ may be identified with the full subcategory of $\underline{\mathrm{Hom}}(C, \mathcal{S})^{\mathrm{op}}$ spanned by the left exact functors $f : C \to \mathcal{S}$ such that $C_{/f}$ is accessible ([DAGXIII, Prop. 3.1.6]). Consider a pullback square

$$\begin{tikzcd} p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f \arrow[d] \\ g \arrow[r] & h \end{tikzcd}$$

of functors in $\underline{\mathrm{Hom}}(C, \mathcal{S})$ with f, g, h in $\mathrm{Pro}(C)$. As limits of functors are computed pointwise, $p : C \to \mathcal{S}$ commutes with finite limits. Moreover, the above diagram induces a homotopy pullback diagram

$$\begin{tikzcd} C^{\mathrm{op}}_{/p} \arrow[r] \arrow[d] & C^{\mathrm{op}}_{/f} \arrow[d] \\ C^{\mathrm{op}}_{/g} \arrow[r] & C^{\mathrm{op}}_{/h} \end{tikzcd}$$

in $\widehat{\Delta}$ w.r.t. the Joyal model structure. The morphisms $C^{\mathrm{op}}_{/f} \to C^{\mathrm{op}}_{/g}$ and $C^{\mathrm{op}}_{/h} \to C^{\mathrm{op}}_{/g}$ are colimits preserving, so that $C_{/p}$ is accessible by [Lur09, Prop. 5.4.6.6].

<u>$\mathrm{Pro}(C)$ admits small coproducts:</u> Let I be a small set, and consider a family of objects $x_\bullet : I \to \mathrm{Pro}(C)$, then for each i there exists a filtered small ordinary category A_i and a functor $x_{i\bullet} : A_i \to C$ such that $x_i \simeq$ "$\lim\limits_{\alpha \in A_i}$"$x_{i\alpha}$ (see [Lur09, Prop. 5.3.1.16]). We obtain canonical isomorphisms

$$\coprod_{i \in I} x_i \simeq \coprod_{i \in I} \text{“}\lim_{\alpha_i \in A_i}\text{”} x_{i\alpha_i} \simeq \text{“}\lim_{(\alpha_i) \in \prod A_i}\text{”} \coprod_{i \in I} x_{i\alpha_i},$$

in $\underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathcal{S})$, as limits and colimits in presheaf categories are computed pointwise.

□

The functor $\boxdot^\bullet : \square \to \mathrm{Pro}(\mathbf{Diff}^r)$ may thus be extended to a colimit preserving functor $\underline{\mathrm{Hom}}(\square^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Pro}(\mathbf{Diff}^r)$.

Notation 6.3.14. We write

$$\begin{aligned} \partial\boxdot^n &:= \boxdot^\bullet_! \partial\square^n, \quad n \geq 0 \\ \sqcap\!\!\sqcap^n_{k,\xi} &:= \boxdot^\bullet_! \sqcap^n_{i,\xi}, \quad n \geq 1,\ n \geq k \geq 0, \xi = 0, 1. \end{aligned}$$

⌟

Proposition 6.3.15. *There is a canonical isomorphism*

$$\sqcap^n_{k,\xi} \cong \text{“}\lim_{\varepsilon>0}\text{”} \sqcap^n_{k,\xi,\varepsilon} n \geq 1,\ n \geq k \geq 0, \xi = 0, 1.$$

$\square$

Definition 6.3.16. A map $X \to Y$ of differentiable stacks is a ***squishy fibration*** if it has the right lifting property w.r.t. all inclusions $\sqcap^n_{k,\xi} \hookrightarrow \square^n$ $(n \geq 1,\ n \geq k \geq 0, \xi = 0, 1)$, and a ***trivial squishy fibration*** if it has the right lifting property w.r.t. the inclusions $\partial\square^n \hookrightarrow \square^n$ for all $n \geq 0$. ⌟

Notation 6.3.17. Let $0 < \varepsilon' < \varepsilon < \frac{1}{2}$, then $\lambda^{\varepsilon}_{\varepsilon'} : [0,1] \to [0,1]$ denotes any map such that

(a) $\lambda^{\varepsilon}_{\varepsilon'}|_{[0,\varepsilon']} \equiv 0$, $\lambda^{\varepsilon}_{\varepsilon'}|_{[1-\varepsilon',1]} \equiv 1$,

(b) $\lambda^{\varepsilon}_{\varepsilon'}(t) = t$ for all $t \in \left[\frac{1}{2}(\varepsilon + \varepsilon'), 1 - \frac{1}{2}(\varepsilon + \varepsilon')\right]$, and

(c) $\dot{\lambda}^{\varepsilon}_{\varepsilon'}(t) > 0$ for all $t \in (\varepsilon', 1 - \varepsilon')$.

⌟

Lemma 6.3.18. *Let* $0 < \varepsilon' < \varepsilon < \frac{1}{2}$, *then the triangle*

$$\begin{tikzcd} \square^1 \arrow[rr, "\lambda^{\varepsilon}_{\varepsilon'}"] \arrow[dr, ->>] & & \square^1 \arrow[dl, ->>] \\ & \square^1_{\varepsilon} & \end{tikzcd}$$

commutes.

Proof. It is enough to show that composing $[\varepsilon', 1-\varepsilon'] \to \square^1 \to \mathbb{\square}^1_\varepsilon$ yields an epimorphism, then the statement follows from the observation that the triangle

$$\begin{tikzcd}
 & {[\varepsilon', 1-\varepsilon']} \arrow[dl] \arrow[dr] & \\
\mathbb{\square}^1_\varepsilon \arrow[rr, "{[\lambda^{\varepsilon}_{\varepsilon'}]}"] & & \mathbb{\square}^1_\varepsilon
\end{tikzcd}$$

commutes. To see this, let X be any differentiable space, then any map $f : \square^1 \to X$, which descends to a map $\mathbb{\square}^1_\varepsilon \to X$, may be obtained by glueing $f|_{(\varepsilon', 1-\varepsilon')} : (\varepsilon', 1-\varepsilon') \to X$ with $\big[0, \frac{1}{2}(\varepsilon' + \varepsilon)\big) \to 1 \xrightarrow{f(\varepsilon')} X$ and $\big(1 - \frac{1}{2}(\varepsilon' + \varepsilon), 1\big] \to 1 \xrightarrow{f(1-\varepsilon')} X$ along their common intersection. $\square$

Theorem 6.3.19. *Let X be a differentiable stack, then*

$$X^{\Delta^n} \to X^{\partial \Delta^n}$$

is a squishy fibration for any $n \geq 0$.

Proof. In this proof we use the following notation:

$$\begin{aligned}
\square^k \star_{i,\xi} \Delta^n \quad &:= \quad \left(\sqcap^k_{i,\xi} \times \Delta^n\right) \sqcup_{\square^k \times \Delta^n} \left(\square^k \times \partial \Delta^n\right); \\
\mathbb{\square}^k \star_{i,\xi} \Delta^n \quad &:= \quad \left(\mathbb{\sqcap}^k_{i,\xi} \times \Delta^n\right) \sqcup_{\mathbb{\square}^k \times \Delta^n} \left(\mathbb{\square}^k \times \partial \Delta^n\right); \\
\mathbb{\square}^k_\varepsilon \star_{i,\xi} \Delta^n \quad &:= \quad \left(\mathbb{\sqcap}^k_{i,\xi,\varepsilon} \times \Delta^n\right) \sqcup_{\mathbb{\square}^k_\varepsilon \times \Delta^n} \left(\mathbb{\square}^k_\varepsilon \times \partial \Delta^n\right) \qquad \left(0 < \varepsilon < \tfrac{1}{2}\right).
\end{aligned}$$

We must show that for every $n \geq 1$, $n \geq k \geq 0$ and $\xi = 0, 1$

$$\begin{tikzcd}
\mathbb{\square}^k \star_{i,\xi} \Delta^n \arrow[r] \arrow[d, hook] & X \\
\mathbb{\square}^n \times \Delta^n &
\end{tikzcd}$$

admits a lift. The horizontal map is represented by a map

$$\square^k \star_{i,\xi} \Delta^n \to X$$

which factors through $\mathbb{\square}^k_\varepsilon \star_{i,\xi} \Delta^n$ for some $0 < \varepsilon < \frac{1}{2}$. Fix $0 < \varepsilon' < \varepsilon$, and write $\lambda := \lambda_{\varepsilon'}$. To prove the statement we define maps $\mu, \nu : \square^k \times \Delta^n \to \square^k \times \Delta^n$ such that the digram

$$\begin{tikzcd}
\square^k \star_{i,\xi} \Delta^n \arrow[rrrr, "\left(\lambda^k \times \mathrm{id}_{\Delta^n} |_{\square^k \star_{i,\xi} \Delta^n}\right)^2"] \arrow[d, hook] & & & & \square^k \star_{i,\xi} \Delta^n \arrow[d, hook] \\
\square^k \times \Delta^n \arrow[r, "\lambda^k \times \mathrm{id}_{\Delta^n}"] & \square^k \times \Delta^n \arrow[r, "\mu"] & \square^k \times \Delta^n \arrow[r, "\nu"] & \square^k \times \Delta^n \arrow[r, "\lambda^k \times \mathrm{id}_{\Delta^n}"] & \square^k \times \Delta^n
\end{tikzcd}$$

commutes and admits a (necessarily unique) diagonal lift. (Qualitatively, the first instance of $\lambda^k \times \mathrm{id}_{\Delta^n}$ ensures that the resulting lift factors through $\mathbb{\square}^k_{\varepsilon'} \times \Delta^n$, μ is a first approximation to the desired retract, then ν completes the retraction in the "Δ^n-direction", and the second instance of $\lambda^k \times \mathrm{id}_{\Delta^n}$ completes the retract in the "$\square^i$-direction".) Recall, that by Lemma 6.3.18 the map $\lambda^k \times \mathrm{id}_{\Delta^n} : \square^k \times \Delta^n \to \square^k \times \Delta^n$ descends to the identity map $\mathrm{id} : \mathbb{\square}^k_\varepsilon \times \Delta^n \to \mathbb{\square}^k_\varepsilon \times \Delta^n$, so that the triangle obtained by postcomposing with $\square^k \star_{i,\xi} \Delta^n \to X$ in

$$\begin{tikzcd}
\square^k \star_{i,\xi} \Delta^n \arrow[r] \arrow[d] & \square^k \star_{i,\xi} \Delta^n \arrow[r] & X \\
\square^k \times \Delta^n \arrow[ur] & &
\end{tikzcd}$$

commutes, yielding a commutative diagram

$$\begin{tikzcd}
\square^k_{\varepsilon'} \star_{i,\xi} \Delta^n \arrow[r] \arrow[d] & X \\
\square^k_{\varepsilon'} \times \Delta^n. \arrow[ur] &
\end{tikzcd}$$

This lift factors through $\square^k \times \Delta^n \twoheadrightarrow \square^k_{\varepsilon'} \times \Delta^n$, because of the first instance of $\lambda^k \times \mathrm{id}_{\Delta^n}$ in the sequence of morphisms in the bottom of the above diagram.

<u>Construction of μ and ν:</u> In order to ease the notational burden we will only define μ and ν for $i = k$ and $\xi = 1$.

To define μ, I require an auxiliary smooth function $\rho : \square^{k-1} \times \Delta^n \to \square^1$, such that

(a) $\rho(t_1, \ldots, t_k, s_0, \ldots, s_n) = 1$ if $t_1, \ldots, t_k > \frac{2}{3} \cdot \varepsilon'$ or $s_0 + \cdots + s_n > \frac{2}{3}$;

(b) $\rho(t_1, \ldots, t_k, s_0, \ldots, s_n) = 0$ if $t_1, \ldots, t_k < \frac{1}{3} \cdot \varepsilon'$ and $s_0 + \cdots + s_n < \frac{1}{3}$.

Then, we define

$$\begin{array}{rrcl}
\mu : & \square^k \times \Delta^n & \to & \square^k \times \Delta^n \\
& ((t_1, \ldots, t_k), s) & \mapsto & ((t_1, \ldots, t_{k-1}, \rho(t_1, \ldots, t_{k-1}, s) \cdot t_k), s).
\end{array}$$

Using partition of unity one can patch together the retractions $\Delta^n \to \Lambda^n_{k_2}$, $1 \leq k_2 \leq n$

to obtain a retract $\sigma : \left\{ (s_0, \ldots, s_n) \in \Delta^n \;\middle|\; s_0 + \cdots + s_n > \frac{1}{3} \right\} \to \partial\Delta^n$. Now, let $\tau : \square^1 \to \square^1$ be a smooth map such that

(a) $\tau(t) = 1$ for $t > \frac{2}{3} \cdot \varepsilon'$, and

(b) $\tau(t) = 0$ for $t < \frac{1}{3} \cdot \varepsilon'$.

Then, we define

$$\begin{array}{rrcl} \nu : & \square^k \times \Delta^n & \to & \square^k \times \Delta^n \\ & ((t_1, \ldots, t_k), s) & \mapsto & (\,(t_1, \ldots, t_k), \operatorname{id}_{\Delta^n} + (\operatorname{id}_{\Delta^n} + \tau(t_k) \cdot (\sigma - \operatorname{id}_{\Delta^n})\,)(s)\,). \end{array}$$

Proof of continuity of lift: By construction, it is clear that the lift is continuous at any point which gets mapped to $\square^k \times \Delta^n \setminus \left(\square^{k-1} \times \{0\}\right) \times \partial\Delta^n$. Points which get mapped to $\left(\square^{k-1} \times \{0\}\right) \times \partial\Delta^n$ admit a neighbourhood which gets mapped to $\left(\square^{k-1} \times \{0\}\right) \times \Delta^n$, which concludes the proof. $\square$

Remark 6.3.20. It does not seem to be the case that the inclusion of pro-objects $\sqcap^n_{k,\delta} \hookrightarrow \square^n$ admits a retract for $n \geq 2$, $n \geq k \geq 0$. ⌟

Theorem 6.3.21. *The locally constant weak equivalences and the squishy fibrations determine a fibration structure on* $\mathbf{Diff}^\infty$*, in which all objects are fibrant.*

Proof. We must verify that weak equivalences and fibrations described in the statement of the theory satisfy conditions (a) - (d) of Definition 2.0.6. Conditions (a) & (b) are clear.

Proof of (c): Let $X \to Y$ be a morphism of differentiable stacks. Denote by $X \to X_3 \to Y$ a factorisation of $X \to Y$ into a trivial cofibration followed by a fibration in the $\frac{1}{3}$-squishy model structure, then denote by $X_3 \to X_4 \to Y$ a factorisation of $X_3 \to Y$ into a fibration followed by a fibration in the $\frac{1}{4}$-squishy model structure and so on; finally, denote by $X \to X' \to Y$ the factorisation of $X \to Y$ obtained by taking the transfinite composition of the maps $X \to X_3 \to X_4 \to \cdots$. The map $X \to X'$ is a shape equivalence, as these are preserved by colimits. We claim that $X' \to X$ is a squishy fibration. To see this, consider a lifting problem

$$\begin{tikzcd}
\sqcap^n_{k,\xi} \arrow[r] \arrow[d] & X' \arrow[d] \\
\square^n \arrow[r] & Y
\end{tikzcd}$$

for some $n \geq 1,\ n \geq k \geq 0, \xi = 0, 1$, then by Proposition 7.3.3 and the fact that compact objects are closed under finite colimits, we see that $\sqcap^n_{k,\xi} \to X'$ must factor through $X_n \to X'$ for some $n \geq 3$. We then obtain a new lifting problem

$$\begin{tikzcd}
\sqcap^n_{k,\xi,\varepsilon} \arrow[r] \arrow[d] & X_n \arrow[d] \\
\square^n_\varepsilon \arrow[r] & Y
\end{tikzcd}$$

for some $0 < \varepsilon < \frac{1}{n}$, so that we obtain a lift.

Proof of (d): Let us call a map of differentiable stacks which has the right lifting property w.r.t. the inclusions $\partial\square^n \hookrightarrow \square^n$ for $n \geq 0$ a *squishy acyclic fibration.* As squishy fibrations and squishy acyclic fibrations are defined via a right lifting property they are both closed under pullback. It thus remains to show that the squishy acyclic fibrations are precisely the squishy fibrations which are also shape equivalences, but this is completely formal: Let $X \to Y$ be a map of differentiable stacks, then $X \to Y$ is an squishy acyclic fibration iff $(\boxdot^\bullet)^*X \to (\boxdot^\bullet)^*Y$ is a trivial fibration in $\underline{\mathrm{Hom}}(\square^{\mathrm{op}}, \mathcal{S})$ iff $(\boxdot^\bullet)^*X \to (\boxdot^\bullet)^*Y$ is both a fibration and a weak equivalence iff $X \to Y$ is both a squishy fibration and a shape equivalence.

Finally, all differentiable stacks are seen to be fibrant by applying Theorem 6.3.19 for $n = 0$. $\square$

Remark 6.3.22. In the proof (c) in Theorem 6.3.21 one could alternatively factor any morphism of differentiable stacks through a path fibration constructed using Δ^1. ⌟

Definition 6.3.23. The fibration structures on $\mathbf{Diff}^\infty_{\leq 0}$ and $\mathbf{Diff}^\infty$ are both referred to as the ***squishy fibration structure***. ⌟

Corollary 6.3.24. *The cofibrations in the Kihara model structure are formal cofibrations.* $\square$

While not being a model structure, the squishy fibration structure does enjoy some of the nice properties of a (Cartesian) model structure.

Proposition 6.3.25. *The squishy (trivial) fibrations are closed under arbitrary products.*

Proof. This follows from the fact that the squishy (trivial) fibrations are determined via right lifting property. $\square$

Proposition 6.3.26. *For any differentiable stack X*

(1) *the map $X^{\boxdot^n} \to X^{\partial\boxdot^n}$ is a squishy fibration for $n \geq 0$, and*

(2) *the map $X^{\boxdot^n} \to X^{\sqcap^n_{k,\varepsilon}}$ is a squishy trivial fibration for $n \geq 1$, $n \geq k \geq 0$.*

$\square$

6.4 Formally cofibrant objects

Recall that a differentiable stack A is *formally cofibrant* if for every differentiable stack X the canonical map $\pi_!\underline{\mathbf{Diff}}^\infty(A,X) \to \mathcal{S}(\pi_!A, \pi_!X)$ is an equivalence. The full subcategory of $\mathbf{Diff}^\infty$ spanned by formally cofibrant stacks is denoted by $\mathbf{C}$.

6.4.1 Closure properties of formally cofibrant stacks

Formally cofibrant objects are closed under various operations.

Proposition 6.4.1. *The subcategory $\mathbf{C} \subseteq \mathbf{Diff}^\infty$ of formally cofibrant stacks is closed under arbitrary coproducts.*

Proof. All objects in $\mathbf{Diff}^\infty$ are fibrant in the squishy fibration structure. □

Proposition 6.4.2. *Let $A : \mathbf{N} \to \mathbf{Diff}^\infty$ be a diagram such that $A_i \to A_{i+1}$ is a cofibration in the Kihara model structure for all $i \in \mathbf{N}$ then* $\operatorname{colim} A$ *is formally cofibrant.*

Proposition 6.4.3. *The subcategory $\mathbf{C} \subseteq \mathbf{Diff}^\infty$ of formally cofibrant stacks is closed under finite products.*

Proof. Let A, B be formally cofibrant stacks, and let X be any stack, then one obtains the following series of canonical equivalences:

$$\begin{aligned}
\pi_! \underline{\mathbf{Diff}}^\infty(A \times B, X) &\simeq \pi_! \underline{\mathbf{Diff}}^\infty(A, \underline{\mathbf{Diff}}^\infty(B, X)) \\
&\xrightarrow{\simeq} \mathcal{S}(\pi_! A, \pi_! \underline{\mathbf{Diff}}^\infty(B, X)) \\
&\xrightarrow{\simeq} \mathcal{S}(\pi_! A, \mathcal{S}(\pi_! B, \pi_! X)) \\
&\simeq \mathcal{S}(\pi_! A \times \pi_! B, \pi_! X) \\
&\simeq \mathcal{S}(\pi_!(A \times B), \pi_! X).
\end{aligned}$$

□

Proposition 6.4.4. *The ∞-category of formally cofibrant objects is closed under* $\mathbf{R}$-*,* Δ^1-*, and* $\square^1$*-homotopy equivalence.*

6.4.2 Partitions unity

Here we describe a more refined closure property of formally cofibrant objects than the ones discussed in §6.4.1, which will let us exhibit many interesting spaces as being formally cofibrant.

Let X be a diffeological space, and let $U = \{U_\alpha\}_{\alpha \in A}$ be a cover of X then there exists a $\mathbf{Diff}^\infty_{\mathrm{concr}}$-enriched category X_U with

$$\begin{aligned}
\mathrm{Obj}\, X_U &= \textstyle\coprod_\sigma U_\sigma \\
\mathrm{Mor}\, X_U &= \textstyle\coprod_{\sigma \supseteq \tau} U_\sigma
\end{aligned}$$

where σ, τ denote non-empty finite subsets of A such that $U_\sigma := \bigcap_{\alpha \in \sigma} U_\alpha \neq \varnothing$. The geometric realisation of (the nerve of) X_U is denoted by BX_U. The space BX_U may be constructed in stages using the pushouts

$$\begin{tikzcd}
\coprod_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} U_{\sigma_n} \times \partial\Delta^n \arrow[r, hook] \arrow[d, hook] & BX_U^{(n-1)} \arrow[d, hook] \\
\coprod_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} U_{\sigma_n} \times \Delta^n \arrow[r, hook] & BX_U^{(n)}
\end{tikzcd} \tag{8}$$

At each stage one can construct inductively an obvious commutative square obtained by replacing $BX_U^{(n)}$ by X in (8), thus producing a canonical map $BX_U \to X$. As the legs in each pushout are monomorphisms each stage $BX_U^{(n)}$ is a diffeological space by Proposition 5.2.15; the object BX_U is then a diffeological space by Proposition 5.2.16, as it is a filtered colimit of diffeological spaces along monomorphisms.

Definition 6.4.5. A covering on a diffeological space is called ***numerable*** if it admits a subordinate partition of unity. ⌟

The original formulation of the following lemma in the setting of topological spaces is due to Segal [Seg68, §4] and tom Dieck [tD71, Th. 4]. Translating these results into the smooth setting is very technical, and is carried out by Kihara in [Kih20, §9].

Lemma 6.4.6 ([Kih20, Prop. 9.5])**.** *Let X be a diffeological space, and let U be a numerable cover of X, then the canonical map $BX_X \to X$ is a Δ^1-homotopy equivalence.*

Theorem 6.4.7. *Let X be a diffeological space, and let U be a numerable cover of X. If each member of U is formally cofibrant, then so is X.*

Proof. By Lemma 6.4.6 and Proposition 6.4.4 the space X is formally cofibrant iff BX_U is. We will show that each stage $BX_U^{(n)}$ is formally cofibrant, and then conclude that BX_U is formally cofibrant by Proposition 6.4.2. The diffeological space $BX_U^{(0)}$ is formally cofibrant by Proposition 6.4.1. Applying $\underline{\mathbf{Diff}}^\infty(\_, X)$ to the square (8) yields the pullback

$$\begin{tikzcd}
\underline{\mathbf{Diff}}^\infty(BX_U^{(n)}, X) \arrow[r] \arrow[d] & \prod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} \underline{\mathbf{Diff}}^\infty(U_{\sigma_n}, X)^{\Delta^n} \arrow[d] \\
\underline{\mathbf{Diff}}^\infty(BX_U^{(n-1)}, X) \arrow[r] & \prod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} \underline{\mathbf{Diff}}^\infty(U_{\sigma_n}, X)^{\partial\Delta^n}
\end{tikzcd}$$

in which the vertical morphism to the right is sharp as it is a squishy fibration by Theorem 6.3.19 and Proposition 6.3.25. □

Corollary 6.4.8. *Any Hausdorff paracompact manifold is formally cofibrant.*

Proof. Such manifolds admit numerable covers in which all intersections are either empty or diffeomorphic to $\mathbf{R}^n$ for some $n \geq 0$. □

Remark 6.4.9. The above corollary may be extended to infinite dimensional manifolds, as studied in [KM97]; see [Kih20, Th. 11.1]. ⌟

6.4.3 Counterexamples

There are many directions in which it is not possible to extend Corollary 6.4.8. One cannot extend to non-0-truncated stacks:

Example 6.4.10. $B\mathbf{Z} = \pi_! \underline{\mathbf{Diff}}(1, S^1) = \pi_! \underline{\mathbf{Diff}}(\pi^* B\mathbf{Z}, S^1) \neq \mathcal{S}(B\mathbf{Z}, B\mathbf{Z}) = \mathbf{Z}$. ⌟

One must be careful when dropping the Hausdorfness requirement:

Example 6.4.11. *Denote by* $\mathbf{R}_{\bullet\bullet}$ *the real line with two origins, then*

$$\begin{aligned} B\mathbf{Z} &= \pi_! \underline{\mathbf{Diff}}(\mathbf{R}, S^1) \\ &= \pi_! \underline{\mathbf{Diff}}(\mathbf{R}_{\bullet\bullet}, S^1) \\ &\neq \mathcal{S}(\pi_! \mathbf{R}_{\bullet\bullet}, \pi_! S^1) \\ &= \mathcal{S}(B\mathbf{Z}, B\mathbf{Z}) \\ &= \mathbf{Z}. \end{aligned}$$

⌟

Example 6.4.12. *Denote by* $\mathbf{R}_{||}$ *the space obtained by glueing two copies of* $\mathbf{R}$ *along the subspace* $(-\infty, -1) \cup (1, \infty)$, *then* $\mathbf{R}_{||}$ *is* $\mathbf{A}^1$*-homotopy equivalent to* S^1*, so that it is formally cofibrant. In particular,*

$$\pi_! \underline{\mathbf{Diff}}(\mathbf{R}_{||}, S^1) = \pi_! \underline{\mathbf{Diff}}(S^1, S^1) = \mathcal{S}(\pi_! S^1, \pi_! S^1) = \mathcal{S}(\pi_! \mathbf{R}_{||}, \pi_! S^1).$$

⌟

Non-paracompact manifolds may not be formally cofibrant:

Example 6.4.13. *Let* $\mathbf{L}$ *denote the long line. It has trivial shape but is not contractible. Thus* $\mathcal{S}(\pi_! \mathbf{L}, \pi_! \mathbf{L}) = \mathcal{S}(1, 1) = 1$*, while* $\underline{\mathbf{Diff}}(\mathbf{L}, \mathbf{L})$ *has at least two path components.* ⌟

7 Compact manifolds are compact

Here we show that closed manifolds and modified notions of closed intervals are both compact in the categorical sense.

The only general paradigm that we are aware of for proving categorical compactness in (∞-)toposes involves n-coherent ∞-toposes (see [Lur18, §A.2.3.]). These are ∞-toposes satisfying strong boundedness conditions. Applying this theory to the case at hand would amount to showing that $\mathbf{Diff}^\infty$ is generated by sufficiently compact objects (in the topological sense), and that these are closed under pullback. This has no hope of being true; while, for instance, closed manifolds *do* indeed generate $\mathbf{Diff}^\infty$, it is possible to obtain the Cantor set (with the discrete differential structure) as a pullback of closed manifolds. Fortunately, the pullback stability of compact manifolds is close enough to being true by virtue of all manifolds being locally compact. In order to exploit this property we must acquire a good understanding of the sheafification procedure, which we do in the following subsection.

7.1 Sheafification in one step

Throughout this subsection C denotes a small (ordinary) site. Let $X : C^{\mathrm{op}} \to \mathcal{S}_{\leq n}$ be a presheaf, then for $n = 0$ the *plus-construction* (recalled below), first introduced in [SGA 4_{I}, §4.ii.3], is a functor $\widehat{C} \to \widehat{C}$ such that a double application thereof to X produces the sheaf universally associated to X. For arbitrary $n < \infty$ the plus-construction must be applied $n + 1$ times in order to obtain the stack universally associated to X (see [Lur09, §6.5.3]). The plus-construction is built using Čech coverings; an analogous construction using hypercovers produces the universally associated stack in one step. Considering again the case $n = 0$, such a construction, using a *variant* of hypercovers, has been part of the folklore for decades, and first appeared in writing in [Yuh07], following ideas of Dubuc; this is the construction that we will use in this section.

We will first recall the plus-construction so that we may better contrast it with the sheafification construction in one step. For more details, we refer the reader to [Yuh07, §3.2]. From now on the word *presheaf* refers to a presheaf valued in sets.

Let X be a presheaf on C, let c be an object of C, and let $U = \{u \to c\}$ be a covering of c. A ***matching family of*** X ***at*** U is a family of maps $\{u \to X\}$, satisfying the following property: For any pair of morphisms $a \to u$, $a \to u'$, if the square

$$\begin{tikzcd} & u \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & c \\ & u' \arrow[ur] & \end{tikzcd}$$

commutes, then so does

$$\begin{tikzcd} & u \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & X \\ & u' \arrow[ur] & \end{tikzcd}$$

Given another covering $V = \{v \to c\}$, then a matching family of X at V is a ***refinement*** of the matching family of X at U if for each span $c \leftarrow v \to X$ in the matching family at V there exists a span $c \leftarrow u \to X$ in the matching family at U

together with a morphism $v \to u$ such that the resulting diagram

$$\begin{tikzcd}
& v \arrow[dl] \arrow[dr] \arrow[dd] & \\
c & & X \\
& u \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes. The ***plus-construction*** applied to X yields a presheaf X^+, which sends any object c in C to the set of matching families of X at coverings of c modulo the equivalence relation generated by refinement.

Dubuc and Yuhjtman's construction is obtained by modifying the notion of matching family. For a covering $U = \{u \to c\}$ a ***hyper-matching family of X at U*** is a family of maps $\{u \to X\}$ satisfying the following property: For any pair of morphisms $a \to u$, $a \to u'$, if the square

$$\begin{tikzcd}
& u \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & c \\
& u' \arrow[ur] &
\end{tikzcd}$$

commutes, there exists a covering $\{v \to a\}$ such that the squares

$$\begin{tikzcd}
& u \arrow[dr] & \\
v \arrow[ur] \arrow[dr] & & X \\
& u' \arrow[ur] &
\end{tikzcd}$$

obtained from composing the morphisms $v \to a$ in the covering with $a \to u$ and u' commute. Refinement of hyper-matching families is defined as for matching families. The assignment of equivalence classes of hyper-matching families to objects in C is functorial. We denote resulting presheaf on C by $X^{\ddagger}$.

Proposition 7.1.1 ([Yuh07, §3.2])**.** *The presheaf $X^{\ddagger}$ is the associated sheaf of X.* $\square$

7.2 Closed manifolds are compact

Throughout this subsection we fix $0 \leq r \leq \infty$. We now apply Proposition 7.1.1 to prove the following theorem.

Theorem 7.2.1. *Let M be a closed manifold, then $\mathbf{Diff}^r_{\leq 0}(M, \_)$ commutes with filtered colimits.*

Proof. Let A be a filtered category, and consider a functor $X : A \to \mathbf{Diff}^{\infty}_{\leq 0}$, then we must show that the canonical morphism

$$\operatorname*{colim}_{\alpha} \mathbf{Diff}^{\infty}_{\leq 0}(M, X_\alpha) \to \mathbf{Diff}^{\infty}_{\leq 0}(M, \operatorname*{colim}_{\alpha} X_\alpha) \tag{9}$$

is invertible. Recall, that the category $\mathbf{Diff}^{\infty}_{\leq 0}$ is equivalently given by the category of sheaves on $\mathbf{Mfd}^{\infty}$. Denote by F the colimit of X in $\widehat{\mathbf{Mfd}}_{\infty}$, then $\mathbf{Diff}^{\infty}_{\leq 0}(M, \operatorname{colim}_\alpha X_\alpha) = F^{\ddagger}(M)$.

The map (9) is injective: Let $f : M \to X_\alpha$, $g : M \to X_{\alpha'}$ be two maps which get mapped to the same equivalence class by (9), then there exists a covering of M by open subsets $\{U_i\}_{i \in I}$, such that for each $i \in I$ there exists an object α_i and maps

$\alpha \to \alpha_i$, $\alpha' \to \alpha_i$ such that $f|_{U_i} = g|_{U_i} : U_i \to X_{\alpha_i}$. Because M is compact, the covering $\{U_i\}_{i \in I}$ admits a finite subcovering (which is a refinement of $\{U_i\}_{i \in I}$); choose an object β in A with morphisms $\alpha_i \to \beta$ for all $i \in I'$, then f and g determine the same matching family of X_β at M, which must descend to a morphism $M \to X_\beta$, because X_β is a sheaf, and this morphism is in the same equivalence class as f and g, so that f and g are equivalent.

The map (9) is surjective: Let $\{U_i\}_{i \in I}$ be a covering of M, and let $\{f_i : U_i \to F\}_{i \in I}$ be a hyper-matching family of F, then one may again restrict to a finite subcovering $\{U_i\}_{i \in I'}$. For each $i, j \in I'$ choose a covering $\{W_{ijk}\}_{k \in K_{ij}}$ of $U_i \cap U_j$. The covering $\{U_i\}_{i \in I'}$ can be further refined to a covering $\{V_i\}_{i \in I'}$ such that $\overline{V}_i \subseteq U_i$ for each $i \in I'$. Each intersection $\overline{V}_i \cap \overline{V}_j$ lies in $\bigcup_{k \in K'_{ij}} W_{ijk}$ for a finite subset $K'_{ij} \subseteq K_{ij}$. The intersection $V_i \cap V_j$ is then covered by $\{W_{ijk} \cap V_i \cap V_j\}_{k \in K'}$. We then obtain a hyper-matching family $\{f|_{V_i} : V_i \to F\}_{i \in I'}$; as I' is finite all maps in the hyper-matching family may be represented by maps $f|_{V_i} : V_i \to X_\alpha$ for some fixed α in A. For each pair i, j' in I we can then find an object α_{ij} in A and a map $\alpha \to \alpha_{ij}$ such that for each $k \in K'_{ij}$ the resulting square

$$\begin{tikzcd} & V_i \arrow[dr] & \\ W_{kij} \arrow[ur] \arrow[dr] & & X_{\alpha_{ij}} \\ & V_j \arrow[ur] & \end{tikzcd}$$

commutes. Chose an object β in A and a morphism $\alpha_{ij} \to \beta$ for each pair i, j in I', then the resulting compositions of $f|_{V_i} : V_i \to X_\alpha \to X_\beta$ for all $i \in I'$ form a hyper-matching family of M at X_β, and thus a map $M \to X_\beta$, because X_β is a sheaf. $\square$

7.3 Special intervals are compact

Throughout this subsection we fix $0 \leq r \leq \infty$. As discussed in §5.2.2, ordinary intervals seem to be determined by infinitely many plots, and are thus unlikely to be categorically compact. However, this also means that for any differentiable stack X which is not a manifold without boundary it is very hard to construct maps $[0,1] \to X$, even if one has a good grasp of the maps $\mathbf{R}^n \to X$. It thus stands to reason that the *extendable* and *squishy intervals* considered here are likely to be more useful in practice, and luckily these *are* categorically compact.

Let M be a smooth manifold (without boundary), then by Proposition 5.2.10, maps $[0,1] \to M$ are precisely the maps which are the restriction of maps $(-\varepsilon, 1+\varepsilon) \to M$. This motivates the following definition.

Definition 7.3.1. The functor

$$\begin{array}{rcl} \mathbf{Diff}^\infty_{\leq 0} & \to & \mathbf{Set} \\ X & \mapsto & \left\{ f : [0,1] \to X \;\middle|\; \exists\, \varepsilon > 0, \exists\, \tilde{f} : (-\varepsilon, 1+\varepsilon) \to X, \text{ s.t. } f = \tilde{f}|_{j[0,1]} \right\} \end{array}$$

is called the ***extendable interval***, and is denoted by $\mathbf{Diff}^\infty([\![0,1]\!], \_)$. ⌟

Proposition 7.3.2. *The extendable interval preserves filtered colimits.*

Proof. The comparison map

$$\operatorname*{colim}_{\alpha} \mathbf{Diff}^{\infty}_{\leq 0}(\llbracket 0,1 \rrbracket, X_\alpha) \to \mathbf{Diff}^{\infty}_{\leq 0}(\llbracket 0,1 \rrbracket, \operatorname*{colim}_{\alpha} X_\alpha) \tag{10}$$

is given by the restriction of

$$\operatorname*{colim}_{\alpha} \mathbf{Diff}^{\infty}_{\leq 0}([0,1], X_\alpha) \to \mathbf{Diff}^{\infty}_{\leq 0}([0,1], \operatorname*{colim}_{\alpha} X_\alpha).$$

The proof is very similar to the proof of Proposition 7.2.1. Again, denote by F the colimit of X in $\widehat{\mathbf{Mfd}}_\infty$.

<u>The map (10) is injective:</u> Let $f : \llbracket 0,1 \rrbracket \to X_\alpha$, $g : \llbracket 0,1 \rrbracket \to X_{\alpha'}$ be two maps which get mapped to the same equivalence class by (10). W.l.o.g. we may assume that there exists a single value $\varepsilon > 0$, such that the maps f, g are represented by maps $\widetilde{f} : (-\varepsilon, 1+\varepsilon) \to X_\alpha$, $\widetilde{g} : (-\varepsilon, 1+\varepsilon) \to X_{\alpha'}$. There exists some $0 < \varepsilon' < \varepsilon$ and a covering of $(-\varepsilon', 1+\varepsilon')$ by open subsets $\{U_i\}_{i\in I}$, such that for each $i \in I$ there exists an object α_i and maps $\alpha \to \alpha_i$, $\alpha' \to \alpha_i$ such that $f|_{U_i} = g|_{U_i} : U_i \to X_{\alpha_i}$. For any $0 < \varepsilon'' < \varepsilon'$ there exists a finite subset $I' \subseteq I$ such that $(-\varepsilon'', 1+\varepsilon'') \subseteq \bigcup_{i \in I'} U_i$, and this part of the proof proceeds precisely as in the corresponding part of the proof Proposition 7.2.1.

<u>The map (10) is surjective:</u> Consider $\varepsilon > 0$, and let $\{U_i\}_{i\in I}$ be a covering of $(-\varepsilon, 1+\varepsilon)$, and let $\{f_i : U_i \to F\}_{i \in I}$ be a hyper-matching family of F. By restricting to $(-\varepsilon', 1+\varepsilon')$ for some $0 < \varepsilon' < \varepsilon$ one may again restrict to a finite subcovering $\{U_i\}_{i\in I'}$ and proceed as in the second half of the proof of Proposition 7.2.1. $\square$

Proposition 7.3.3. *The squishy interval* $\boxdot^1$ *is compact.*

Proof. Similar to the proof of Proposition 7.3.2. □

Corollary 7.3.4. *Squishy (trivial) fibrations are closed under filtered colimits.* □

Vista 7.3.5. It is possible to define extendable and squishy variants of compact manifolds with boundary (and possibly even manifolds with corners), which are then also compact. ⌟

8 Change of regularity and concreteness

Consider the following diagram, in which every functor is the left adjoint of either a pullback functor or an inclusion:

$$
\begin{tikzcd}
{} & & \mathbf{TSpc} \\
& \mathbf{Diff}^{\infty}_{\mathrm{concr}} \arrow[r] & \mathbf{Diff}^{0}_{\mathrm{concr}} \arrow[u] \\
\text{“more concrete”} \arrow[uu, no head] & \mathbf{Diff}^{\infty}_{\leq 0} \arrow[u] \arrow[r] & \mathbf{Diff}^{0}_{\leq 0} \arrow[u] \\
& \mathbf{Diff}^{\infty} \arrow[u] \arrow[r] & \mathbf{Diff}^{0} \arrow[u] \\
\bullet \arrow[uu, no head] \arrow[rrr, "\text{smooth to continous}"] & & & {}
\end{tikzcd}
\tag{11}
$$

It is natural to ask how applying these functors to a given space affects its homotopy type. Denote by $u : \mathbf{Cart}^{\infty} \to \mathbf{Cart}^{0}$ the forgetful functor, then both constituent functors in the adjunction $u_* : \underline{\mathrm{Hom}}((\mathbf{Cart}^{\infty})^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\mathrm{Hom}}((\mathbf{Cart}^{0})^{\mathrm{op}}, \mathcal{S}) : u^*$ preserve sheaves, thus one obtains a commutative triangle of ∞-connected geometric morphisms

$$
\begin{tikzcd}
\mathbf{Diff}^{\infty} \arrow[rr, shift left=3] \arrow[rr, phantom, shift left=1.5, "\perp"] \arrow[rr, phantom, shift right=1.5, "\perp"] \arrow[rr, shift right=3] \arrow[dr, shift right=3] \arrow[dr, phantom, shift right=1.5, "\vdash"] \arrow[dr, phantom, shift left=1.5, "\vdash"] \arrow[dr, shift left=3] & & \mathbf{Diff}^{0} \arrow[ll] \arrow[dl, shift left=3] \arrow[dl, phantom, shift left=1.5, "\dashv"] \arrow[dl, phantom, shift right=1.5, "\dashv"] \arrow[dl, shift right=3] \\
& \mathcal{S} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

so that the functor $u_! : \mathbf{Diff}^\infty \to \mathbf{Diff}^0$ in (11) is shape preserving for *all* differentiable stacks. The other functors in (11) do not preserve the shape of all objects in their respective domains. In §8.1 we show that the functor $\mathbf{Diff}^\infty_{\leq 0} \to \mathbf{Diff}^0_{\leq 0}$ preserves the shape of smooth manifolds. For the other functors we are not aware of any systematic method for determining the objects for which they preserve their shape.

The topmost left adjoint $\mathbf{Diff}^0_{\mathrm{concr}} \to \mathbf{TSpc}$ sends any diffeological space to its underlying set together with the coarsest topology making all plots into it continuous. The respective subcategories on which the unit and counit morphisms are equivalences may be identified with the category $\Delta\mathbf{TSpc}$ of Δ-generated spaces [SYH18, Prop. 3.2]. Thus, $\Delta\mathbf{TSpc}$, a popular convenient category for doing algebraic topology, is completely contained in $\mathbf{Diff}^0_{\leq 0}$. We give a sample algebro-topological application of $\mathbf{Diff}^0$ in §8.2.

Remark 8.0.1. By the above discussion all Δ-generated spaces are of singular shape in the sense of [Lur17, Def. A.4.9]. ⌟

8.1 Change of regularity

Let M be a smooth manifold, then the following is a well known folk theorem.

Theorem 8.1.1. *The map of simplicial sets* $\mathbf{Mfd}_\infty(\Delta^\bullet, M) \to \mathbf{Mfd}_0(\Delta^\bullet, M)$ *is a weak homotopy equivalence.*

Proof. Let $\{U\}$ be a good cover of M, then M may be written as the colimit of the

Čech nerve

$$\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \coprod U \times_M U' \rightrightarrows \coprod U,$$

but each stage of the above diagram is the coproduct of copies of smooth Euclidean space, so that $u_! : \mathbf{Diff}^\infty \to \mathbf{Diff}^0$ sends the diagram to the analogous diagram obtained by replacing each of copy of smooth Euclidean space by topological Euclidean space. As $u_!$ commutes with colimits, we deduce that $u_!$ sends M to its underlying topological manifold.

The functor $u : \mathbf{Cart}^\infty \to \mathbf{Cart}^n$ is initial, as its composition with $\mathbf{A}^\bullet : \Delta \to \mathbf{Cart}^\infty$ is initial, and initial functors are right cancelative. Thus the unit map $M \to u_! u^* M$ is a weak equivalence, and applying $(\mathbf{A}^\bullet)^*$ produces the map of simplicial sets in the statement of the theorem. □

Remark 8.1.2. In the proof of Theorem 8.1 it is not necessary to use a good cover of M; the same argument goes through using a hypercover which at each stage is diffeomorphic to a coproduct of Euclidean space. ⌟

8.2 A sample application to algebraic topology: principal bundles

It is often taken for granted that the base space of a principal bundle in **TSpc** is a homotopy quotient of the total space (in a sense which we make precise in course of the discussion below)[7]. While this statement is true, it is not straightforward to

[7] In fact, this is often claimed, incorrectly, to be true of the quotient of *any* free group action. To see that this is not the case, consider any non-trivial ordinary group G, and equip it with the discrete topology. Let G act on copy of itself equipped with the trivial topology, then the quotient

show using classical methods. For CGWH spaces a proof may be obtained by combining [May75, Thm. 7.6] and [Shu09, Lm. 12.4], where the first reference relies on technical pointset topological arguments. Alternatively, one can exhibit **TSpc** as a model topos (in the sense of [Rez10, §6]), and use a descent argument, which, when performed with rigour, is again technically demanding. Both of these approaches however do not adequately address the relationship between the geometry and homotopy theory of principal bundles.

We now exhibit how in the theory developed in this thesis this statement admits a natural proof.

Definition 8.2.1. Let C be an ∞-category with a final object, and let $G : \Delta^{\mathrm{op}} \to C$ be a group object, then a ***G-object*** in C is a Cartesian natural transformation $\Delta^1 \times \Delta^{\mathrm{op}} \to C$ with target G. ⌟

Convention 8.2.2. In $\mathcal{S}$, **Set**, $\mathbf{Mfd}^{\infty}$ etc. G-objects will, respectively, be referred to as *G-homotopy types, G-sets, G-smooth manifolds* etc. ⌟

Example 8.2.3. *Let G be an ordinary group, and let $P_\bullet$ be a G-set, then P_0 may be equipped with both a left G-action and a right G-action in a canonical way. To obtain the left G-action, observe that for any map $\alpha : \Delta^0 \to \Delta^n$ in Δ the square*

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, "\alpha^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \times \overset{n\times}{\cdots} \times G \arrow[d] \\ P_0 \arrow[r] & 1 \end{tikzcd}$$

is a point. If the quotient were a homotopy quotient, it would have to model the classifying space of G.

is again a pullback diagram. For each $n \geq 1$, choosing $\alpha : 0 \mapsto n$ one obtains a bijection between P_n and the Cartesian *product $G \times \overset{n\times}{\cdots} \times G \times P_0$. By transferring the remaining face and degeneracy maps along these isomorphisms one obtains a simplicial object encoding a left G-action. The right G-action on P_0 is obtained similarly and, writing $(g,x) \mapsto g \cdot x$ for the induced left action, the right action is given by $(x,g) \mapsto g^{-1} \cdot x$, and the isomorphisms between the left G-set and right G-set over $G_\bullet$ is given by*

$$\begin{aligned} G \times \overset{n\times}{\cdots} \times G \times P_0 &\;\to\; P_0 \times G \times \overset{n\times}{\cdots} \times G \\ (g_1, \ldots, g_n, x) &\;\mapsto\; (g_1 \cdots g_n \cdot x, g_1, \ldots, g_n). \end{aligned}$$

⌟

Definition 8.2.4. Let G be a group object in either $\mathbf{Diff}^\infty_{\mathrm{concr}}$, $\mathbf{Diff}^0_{\mathrm{concr}}$, or $\mathbf{TSpc}$, then a ***classical***[8] *smooth, continuous, or topological* ***principal** G**-bundle*** is, respectively, a G-object P in $\mathbf{Diff}^\infty_{\mathrm{concr}}$, $\mathbf{Diff}^0_{\mathrm{concr}}$, or $\mathbf{TSpc}$ such that $B := \operatorname{colim} P$ admits a cover by open subsets U for which $P_0|_U \cong U \times G$ as G-objects. ⌟

Definition 8.2.5. Let (C, W) be a relative ∞-category, then an augmented simplicial object $(\Delta^{\mathrm{op}})^{\triangleright} \to C$ is called a ***homotopy quotient*** if it is a homotopy colimit of its restriction $\Delta^{\mathrm{op}} \to C$ (in the sense of Definition 2.0.1). ⌟

Thus the theorem that we wish to prove is the following:

Theorem 8.2.6. *Let G be a topological group, and let $P \to B$ be a classical principal G-bundle, then B is a homotopy quotient of P, viewed as a G-space (Definition*

[8] This qualifier reflects the fact that, in a precise sense, *any* G-object in an ∞-topos may be viewed as principal bundle.

8.2.1).

The proof of Theorem 8.2.6 relies on the following two results.

Lemma 8.2.7. *The quotient map of a classical principal bundle in* $\mathbf{Diff}^r_{\mathrm{concr}}$ *(*$0 \leq r \leq \infty$*) is an effective epimorphism.*

Proof. Let G be a diffeological group, and let $P \to B$ be a classical principal G-bundle. For any cover $\{U\}$ of B such $P|_U \cong U \times G$ for all U in $\{U\}$ we obtain a pullback square

$$\begin{array}{ccc} \coprod U \times G & \longrightarrow & P \\ \downarrow & & \downarrow \\ \coprod U & \longrightarrow & B. \end{array}$$

Each projection $U \times G \to U$ admits a section, and is thus an epimorphism, so that $\coprod U \times G \to \coprod U$ is an epimorphism. The morphism $E \to B$ is then an epimorphism by [Lur09, Prop. 6.2.3.15]. $\square$

Proposition 8.2.8. *Let* G *be a diffeological group, and let* $P \to B$ *be a classical principal bundle in* $\mathbf{Diff}^r_{\mathrm{concr}}$ *(*$0 \leq r \leq \infty$*), the* B *is a homotopy quotient of* P*, viewed as a* G*-space.*

Proof. This follows from the preceding lemma and from combining [Lur09, Cor. 6.2.3.5] with the classical fact that

$$\cdots \;\substack{\longrightarrow\\[-2pt] \longrightarrow\\[-2pt] \longrightarrow}\; P \times G \rightrightarrows P$$

is canonically isomorphic to the Čech nerve

$$\cdots \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; P \times_B P \; \substack{\longrightarrow \\ \longrightarrow} \; P$$

of $P \to B$. □

Proof of Theorem 8.2.6. Replacing a topological space with its associated Δ-generated space does not affect its homotopy type in the Quillen model structure, and it is easily checked that this procedure preserves the property of being a classical principal bundle, so we may assume w.l.o.g. that both P and B are Δ-generated. We may now view $P \to B$ as a classical principal bundle in $\mathbf{Diff}^0_{\mathrm{concr}}$, so that the proof follows from Proposition 8.2.8. □

Appendix

A The cube category

Here we discuss some background material on the cube category needed in §6.3.

Definition A.0.1. The ***cube category*** $\square$ is the subcategory of **Set** whose objects are given by $\{0,1\}^n$ $(n \geq 0)$, and whose morphisms are generated by the maps

$$\begin{array}{rrcl} \delta_i^\xi : & \square^{n-1} & \to & \square^n \\ & (x_1, \dots, x_{n-1}) & \mapsto & (x_1, \dots, x_{i-1}, \xi, x_i, \dots, x_{n-1}) \end{array}$$

for $n \geq i \geq 1$ and $\varepsilon = 0, 1$, and

$$\begin{array}{rrcl} \sigma_i : & \square^{n+1} & \to & \square^n \\ & (x_1, \dots, x_{n+1}) & \mapsto & (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \end{array}$$

for $n \geq 0$ and $n \geq i \geq 1$. The category of ***cubical sets*** is the category $\widehat{\square}$ of presheaves on $\square$. ⌟

The cube category $\square$ admits a (strict) monoidal structure given by $(\square^m, \square^n) \mapsto \square^{m+n}$ which extends to cubical sets via Day convolution. This monoidal structure is denoted by $\otimes$.

We denote by $\square^{\leq 1}$ the full subcategory of $\square$ spanned by $\square^0, \square^1$.

Proposition A.0.2 ([Cis06, Prop. 8.4.6])**.** *Let M be a monoidal category, then the* *restriction functor*

$$\underline{\mathrm{Hom}}(\square, M) \to \underline{\mathrm{Hom}}(\square^{\leq 1}, M)$$

induces an equivalence of categories between the full subcategory of $\mathbf{Cat}(\square, M)$ *spanned by monoidal functors, and the full subcategory of* $\mathbf{Cat}(\square^{\leq 1}, M)$ *spanned by functors sending* $\square^0$ *to the monoidal unit of* M. □

Definition A.0.3. For every $n \geq 0$ the ***boundary of*** $\square^n$ is the subobject $\partial\square^n := \cup_{(j,\zeta)}\mathrm{Im}_{\delta_j^\zeta} \subset \square^n$, and for every $n \geq i \geq 1$ and $\xi = 0, 1$ the (i, ξ)***-th horn of*** $\square^n$ is the subobject $\sqcap^n_{i,\xi} := \cup_{(j,\zeta)\neq(i,\xi)}\mathrm{Im}_{\delta_j^\zeta} \subset \square^n$. ⌟

Proposition A.0.4 ([Cis06, Lm. 8.4.36])**.** *For* $m \geq 1$, $n \geq k \geq 1$ *and* $\varepsilon = 0, 1$ *the universal morphisms determined by the pushouts of the spans contained in the commutative squares*

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \otimes \partial\square^m \arrow[r] \arrow[d] & \square^n \otimes \partial\square^m \arrow[d] \\ \sqcap^n_{i,\varepsilon} \otimes \square^m \arrow[r] & \square^n \otimes \square^m \end{tikzcd} \qquad \begin{tikzcd} \partial\square^m \otimes \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d] & \partial\square^n \otimes \square^m \arrow[d] \\ \square^m \otimes \sqcap^n_{i,\varepsilon} \arrow[r] & \square^n \otimes \square^m \end{tikzcd}$$

recover the canonical inclusions $\sqcap^{n+m}_{i,\varepsilon} \hookrightarrow \square^{n+m}$ *and* $\sqcap^{n+m}_{i+m,\varepsilon} \hookrightarrow \square^{n+m}$ *and the universal morphism determined by the pushout of the span contained in the commutative square*

$$\begin{tikzcd} \partial\square^m \otimes \partial\square^n \arrow[r] \arrow[d] & \partial\square^m \otimes \square^n \arrow[d] \\ \square^m \otimes \partial\square^n \arrow[r] & \square^m \otimes \square^n \end{tikzcd}$$

recovers the inclusion $\partial\square^{m+n} \hookrightarrow \square^{m+n}$. □

Theorem A.0.5 ([Cis06, Cor. 8.4.13 or Prop. 8.4.27])**.** *The cube category* $\square$ *is a test* *category.* $\square$

Theorem A.0.6 ([Cis06, Th. 8.4.38]). *The maps*

(i) $\partial\square^n \hookrightarrow \square^n$ *(*$n \geq 0$*), and*

(ii) $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n$ *(*$n \geq i \geq 1$, $\varepsilon = 0, 1$*)*

generate, respectively, the cofibrations and acyclic cofibrations of the test model structure on $\widehat{\square}$*.* $\square$

Theorem A.0.7 ([Cis06, Th. 8.4.38]). *The test model structure together with the monoidal structure on* $\widehat{\square}$ *form a monoidal model structure.* $\square$

Conventions and notation

Linguistic conventions In order to facilitate readability we use the following contractions:

- We write "iff" instead of "if and only if".
- We write "w.l.o.g." instead of "without loss of generality".
- We write "w.r.t." instead of "with respect to".

Editorial conventions

- Propositions stated without proof are marked with the symbol "$\square$".

Category theory

- We identify ordinary categories with their nerves, and consequently do not notationally distinguish between ordinary categories and their nerves.
- ∞-categories (including ordinary categories) are denoted by C, D, ...
- Let C be an ∞-category and let $x, y \in C$ be two objects, then the homotopy type of morphisms from x to y is denoted by $C(x, y)$.
- For any enriched or cartesian closed ∞-category C and any to objects x, y in C the enriched or internal hom object in C is denoted by $\underline{C}(x, y)$, while y^x is used only for internal hom objects. If C is enriched *and* Cartesian closed, then

$\underline{C}(x, y)$ refers to the enriched hom objects and $\underline{\underline{C}}(x, y)$ refers to the internal hom object.

- For any ∞-category C we denote it subcategory of n-truncated objects by $C_{\leq n}$.
- For A any small ordinary category $\widehat{A}$ denotes the category of (set-valued) presheaves on A.
- For any two categories C, D, an arrow $C \hookrightarrow D$ denotes a fully faithful functor.
- For any two categories C, D, an arrow $C \rightarrowtail D$ denotes a faithful functor.
- We use the following notation for various categories:
  - **Set** denotes the category of sets.
  - **TSpc** denotes the category of topological spaces.
  - Δ**TSpc** is the full subcategory of **TSpc** spanned by the Δ-generated topological spaces.
  - $\mathbf{Cart}^r$ denotes the category of r-times differentiable smooth manifolds and smooth maps.
  - $\mathbf{OMfd}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by open subsets of $\mathbf{R}^n$ $(0 \leq n < \infty)$.
  - $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces of $\mathbf{R}^n$ $(0 \leq n < \infty)$.
- $\underline{\mathrm{Hom}}$ denotes the internal hom in $\widehat{\Delta}$, the category of simplicial sets.

- Let X be a simplicial set, then $X_{\simeq}$ denotes the classifying space of X, given e.g. by $\mathrm{Ex}^{\infty}A$.

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