Weight systems which are quantum states

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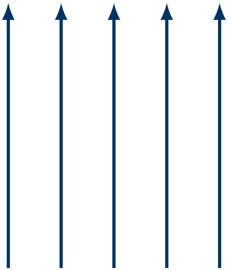
Horizontal chord diagrams



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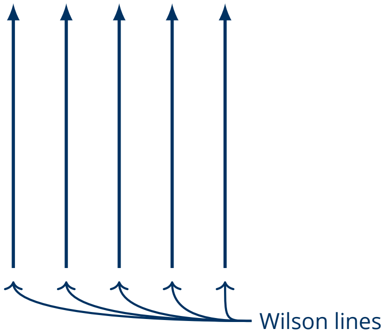

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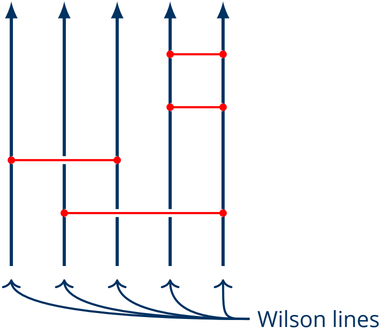

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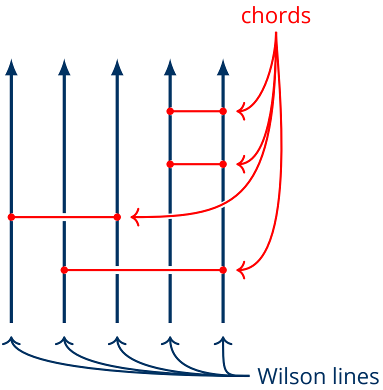
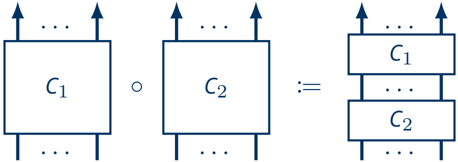


The algebra of horizontal chord diagrams

We can compose horizontal chord diagrams with the same number of Wilson lines



This endows the space

$$\mathscr{D}_n = \mathbb{C}\langle \text{chord diagrams on } n \text{ Wilson lines} \rangle$$

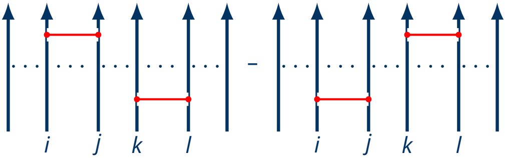
with the structure of non-commutative unital associative $\mathbb{C}$-algebra.

The algebra of horizontal chord diagrams

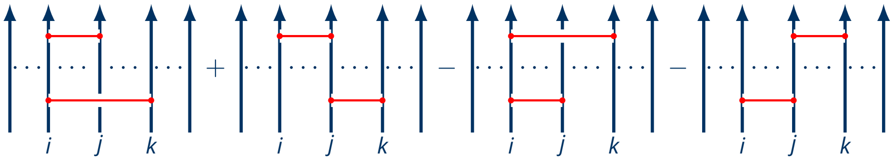
The algebra of horizontal chord diagrams is defined as

$$\mathscr{A}_n = \frac{\mathscr{D}_n}{\mathscr{I}}$$

where $\mathscr{I}$ is the ideal generated by elements of type (2T) and (4T), which encode the so-called infinitesimal braid relations.



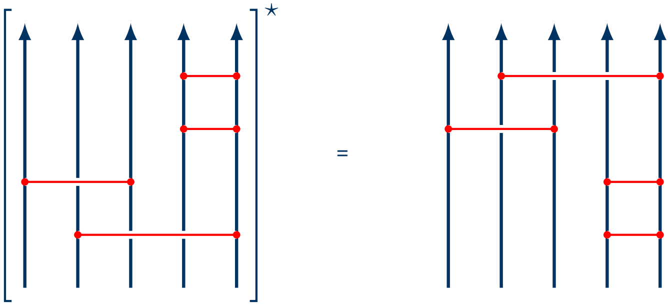
(a) An element of type (2T).



(b) An element of type (4T).

The ⋆-algebra of hcds

The algebra $\mathscr{A}_n$ can be endowed with an anti-linear involution



$\star$-Algebras

Let C be a commutative ring endowed with a ring involution $\bar{\cdot} : C \to C$. A *$\star$-algebra*, or *involutive algebra*, over C is a unital associative C-algebra $\mathscr{O}$ together with an involution $\star : \mathscr{O} \to \mathscr{O}$, such that:

(A1) $(1_{\mathscr{O}})^\star = 1_{\mathscr{O}}$;

(A2) $(z \cdot a + w \cdot b)^\star = \bar{z} \cdot a^\star + \overline{w} \cdot b^\star$, for all $z, w \in C$ and $a, b \in \mathscr{O}$;

(A3) $(ab)^\star = b^\star a^\star$, for all $a, b \in \mathscr{O}$.

A *morphism of $\star$-algebras* is a morphism of algebras which commutes with $\star$.

$\star$-Algebras

Example

Given a group G, the group ring $\mathbb{C}[G]$ has a natural structure of $\star$-algebra given by setting

$$\left(\sum_{i=1}^{k} z_i \cdot g_i\right)^{\star} = \sum_{i=1}^{k} \bar{z}_i \cdot g_i^{-1},$$

for all $z_1, \ldots, z_k \in \mathbb{C}$ and $g_1, \ldots, g_k \in G$.

Remark

The involution $\star$ defined above is the (conjugate of the) antipode of the Hopf algebra $\mathbb{C}[G]$, whose co-multiplication and co-unit are given by

$$\Delta(g) = g \otimes g \qquad \text{and} \qquad \varepsilon(g) = 1 \in \mathbb{C}\,,$$

for each $g \in G$.

$\star$-Algebras

Remark

More generally, given an Hopf algebra H the (conjugate of the) antipode endows H with the structure of $\star$-algebra.

Actually, we have that

$$\mathscr{A}_n \simeq \mathrm{H}_*(\Omega \mathsf{Conf}_n(\mathbb{R}^3))$$

This identification endows $\mathscr{A}_n$ with the structure of Hopf algebra, and $\star$-corresponds to the (conjugate of the) antipode.

Horizontal chord diagrams and observables

Sati and Schreiber observed that, under hypothesis H, the topological sector of the phase space of certain brane intersections is homotopy-equivalent to $\bigsqcup_n \Omega \mathrm{Conf}_n(\mathbb{R}^3)$.

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Thus, the identification

$$\mathscr{A} = \bigoplus_n \mathscr{A}_n \simeq \mathrm{H}_*(\bigsqcup_n \Omega \mathsf{Conf}_n(\mathbb{R}^3))$$

gives an interpretation of $\mathscr{A}$ as quantum observables.

Quantum states

Given a $\star$-algebra of observables $\mathscr{O}$, a *quantum state* (or, simply, *state*) is linear map

$$\varphi : \mathscr{O} \to \mathbb{C}$$

such that $\varphi(x \cdot x^\star) \geq 0$, for all $x \in \mathscr{O}$, and $\varphi(1_\mathscr{O}) > 0$.

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Question: which weight systems are quantum states?

Lie algebra weight systems

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ii. an ad-invariant non-degenerate bi-linear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$;

iii. an ordered collection of finite-dimensional $\mathfrak{g}$-representations $\underline{\rho} = (\rho_1, \ldots, \rho_n)$, called *labelling* where $\rho_i : \mathfrak{g} \to \mathrm{End}(V_i)$ for each $i = 1, \ldots, n$.

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The basic idea is to associate to each horizontal chord diagram $C \in \mathscr{A}_n$ an element in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$, and then take the trace to obtain a complex number.

Lie algebra weight systems

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To each chord $C_{i,j} = [(i,j)] \in \mathscr{A}_n$ we are associating the element

$$\widetilde{W}_{\underline{\rho}}(C_{i,j}) = \sum_{r=1}^{\dim(\mathfrak{g})} \mathrm{Id}_{V_1} \otimes \cdots \otimes \overset{i\text{-th pos.}}{\rho_i(e_r)} \otimes \cdots \otimes \overset{j\text{-th pos.}}{\rho_j(e_r)} \otimes \cdots \otimes \mathrm{Id}_{V_n} \, .$$

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It can be shown that $\widetilde{W}_{\underline{\rho}}$ induces a well-defined morphism of algebras

$$\widetilde{W}_{\underline{\rho}} \colon \mathscr{A}_n \to \mathrm{End}(V^{\otimes n}).$$

The corresponding *Lie algebra weight system* is given by setting

$$W_{\underline{\rho}}(C) = \mathrm{Tr}(\widetilde{W}_{\underline{\rho}}(C)),$$

for each $C \in \mathscr{A}_n$.

Example

Consider $\mathfrak{g} = \mathfrak{gl}_2$, $\langle A, B \rangle = \mathrm{Tr}(AB)$, and take $\rho_1 = \rho_2 \colon \mathfrak{gl}_2 \to \mathrm{End}(\mathbb{C}^2)$.

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$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix of $\langle \cdot, \cdot \rangle$ with respect to the basis id_2, x, y, h is

$$M_{\langle \cdot, \cdot \rangle} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

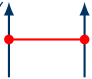
Example

An orthonormal basis for $\langle \cdot, \cdot \rangle$ is given by

$$e_1 = \frac{\mathrm{id}_2}{\sqrt{2}} \qquad e_2 = \frac{(x+y)}{\sqrt{2}} \qquad e_3 = \frac{\mathrm{i}(x-y)}{\sqrt{2}} \qquad e_4 = \frac{h}{\sqrt{2}}.$$

Thus, we can explicitly compute $\widetilde{W}_{\rho,\mathbb{C}^2} : \mathscr{A}_2 \to \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ with respect to the basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ given by $e_i \otimes e_j$, with the lexicographic order.

Example

$\widetilde{W}_{\mathfrak{gl}_2,\mathbb{C}^2}\Bigg($  $\Bigg)$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The standard $\mathfrak{gl}_n$-weight system

More in general, one can show that the construction of Lie algebra weight systems with $\mathfrak{g} = \mathfrak{gl}_N$, $\langle A, B \rangle = \mathrm{Tr}(AB)$, and $\rho_1 = \rho_2 = \ldots = \rho_n$ the defining representation, assigns to the chord $C_{i,j} \in \mathscr{A}_n$ the transposition $\tau_{ij} \in \mathfrak{S}_n \subset \mathrm{End}((\mathbb{C}^N)^{\otimes n})$.
It follows that for $C \in \mathscr{A}_n$

$$W_{\mathfrak{gl}_N, \mathbb{C}^N}(C) = N^{\#\text{ number of cycles in } \sigma(C)},$$

where $\sigma(C) \in \mathfrak{S}_n$ is the permutation associated to C obtained by associating to each chord the corresponding transposition.

Theorem [Corfield, Sati and Schreiber, '21]:

The $(\mathfrak{gl}_N, \mathbb{C}^N)$-weight systems are quantum states.

Proof:

$$\begin{array}{ccc}
\mathscr{A}_n & \xrightarrow{\;W_{\mathfrak{gl}_N,\mathbb{C}^N}\;} & \mathbb{C} \\
{\scriptstyle \sigma}\big\downarrow & \nearrow & \\
\mathbb{C}[\mathfrak{S}_n] & &
\end{array}
\qquad w_N(-) = N^{\#\,\text{cycles}(-)}$$

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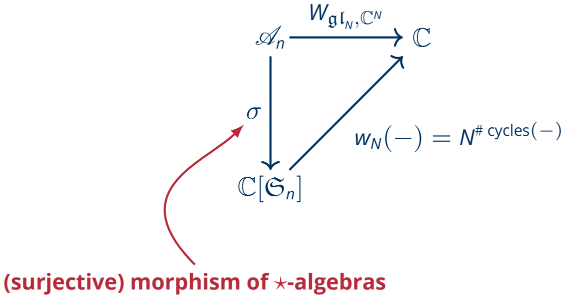

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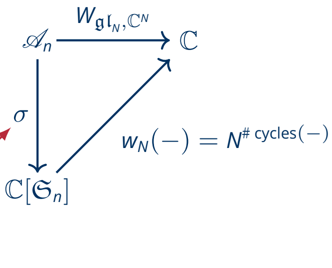

(surjective) morphism of $\star$-algebras

$\Longrightarrow$ $W_{\mathfrak{gl}_N, \mathbb{C}^N}$ **is a state iff** w_N **is a state**

Thus we want to study the function

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where d_C is the shortest path metric in the Cayley graph of $(\mathfrak{S}_n, \text{transpositions})$.

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where d_C is the shortest path metric in the Cayley graph of $(\mathfrak{S}_n, \text{transpositions})$. The eigenvalues of the above function are well-known:

1. these are parametrised by partitions of n;
2. the eigenvalue associated to λ has multiplicity $(\chi^{(\lambda)}(\mathrm{id}))^2$;
3. the eigenvalue corresponding to λ can be computed explicitly and is

$$\frac{n!}{N^n\chi^{(\lambda)}(\mathrm{id})}\cdot\#\text{ssYT}_\lambda(N)\geq 0$$

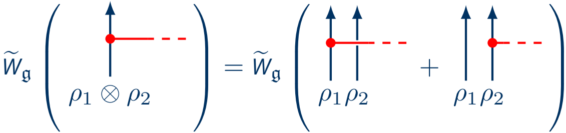
□

General $\mathfrak{gl}_N$–weight systems

The tensor product of two representations of a Lie algebra is defined as

$$(\rho \otimes \rho')(g)[v_1 \otimes v_2] = \rho(g)[v_1] \otimes v_2 + v_1 \otimes \rho'(g)[v_2]\,.$$

Thus



General $\mathfrak{gl}_N$—weight systems

The $\underline{i}$-tensor splitting

$$\Delta_{\underline{i}} : \mathscr{A}_n \to \mathscr{A}_{\sum_j i_j}$$

with $\underline{i} = (i_1, ..., i_n) \in \mathbb{N}^n$, is obtained by replacing the r-th strand with i_r parallel strands, and replacing each chord with the sum of all possible “lifts” of said chord in the new horizontal chord diagram.

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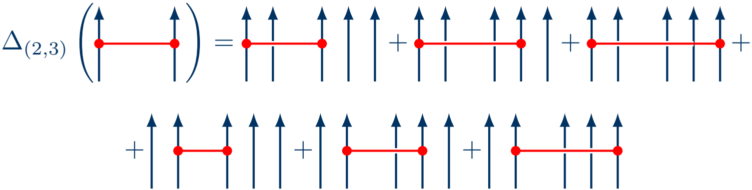

General $\mathfrak{gl}_N$−weight systems

Theorem [C., '22]:

Let $\underline{\rho} = (\rho_1, ..., \rho_n)$ be a $\mathfrak{gl}_N$-label where $\rho_i \in \{\mathrm{Alt}^k(\mathbb{C}^N), \mathrm{Sym}^k(\mathbb{C}^N)\}_k$. Then, $W_{\mathfrak{gl}_N,\underline{\rho}}$ is a quantum state.

Proof:

We can decompose $W_{\mathfrak{gl}_N,\underline{\rho}}$ as follows

$$\mathscr{A}_n \xrightarrow{\Delta_{\underline{\rho}}} \mathscr{A}_{|\underline{\rho}|} \xrightarrow{\sigma} \mathbb{C}[\mathfrak{S}_{|\underline{\rho}|}] \xrightarrow{\cdot c_{\underline{\rho}}} \mathbb{C}[\mathfrak{S}_{|\underline{\rho}|}] \xrightarrow{w_N} \mathbb{C}\,.$$

Where $c_{\underline{\rho}}$ encodes the action of Young symmetrisers.

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While the map $\sigma \circ \Delta_{\underline{\rho}}$ is a morphism of $\star$-algebras, the map $c_{\underline{\rho}}$ is not – for any choice of $\underline{\rho}$ which is not $\mathbb{C}^N$.

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Where $c_{\underline{\rho}}$ encodes the action of Young symmetrisers.
While the map $\sigma \circ \Delta_{\underline{\rho}}$ is a morphism of $\star$-algebras, the map $c_{\underline{\rho}}$ is not – for any choice of $\underline{\rho}$ which is not $\mathbb{C}^N$. For every possible label $c_{\underline{\rho}}^2 = c_{\underline{\rho}}$. Under our hypothesis on the ρ_is we have that $c_{\underline{\rho}}^\star = c_{\underline{\rho}}$. These facts ensure us that the composition $w_N \circ \cdot c_{\underline{\rho}}$ is a state. $\square$

Thank you!