The Gershgorin Circle Theorem

Zack Cramer

University of Waterloo

February 27^{th}, 2017

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

If x is a non-zero vector with $Ax = \lambda x$, then x is an **eigenvector** for A with corresponding **eigenvalue** λ.

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

If x is a non-zero vector with $Ax = \lambda x$, then x is an **eigenvector** for A with corresponding **eigenvalue** λ.

A is **diagonalizable** if $\mathbb{C}^n$ admits a basis of eigenvectors of A.

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

If x is a non-zero vector with $Ax = \lambda x$, then x is an **eigenvector** for A with corresponding **eigenvalue** λ.

A is **diagonalizable** if $\mathbb{C}^n$ admits a basis of eigenvectors of A.

Fact: *Almost* every $n \times n$ matrix with complex entries is diagonalizable.

Theorem (Rayleigh-Ritz)

Let A be an $n \times n$ Hermitian matrix with entries in $\mathbb{C}$.

Theorem (Rayleigh-Ritz)

Let A be an $n \times n$ Hermitian matrix with entries in $\mathbb{C}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A.

Theorem (Rayleigh-Ritz)

Let A be an $n \times n$ Hermitian matrix with entries in $\mathbb{C}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Then

$$\lambda_1 = \inf_{\|x\|=1} x^* A x \quad \textit{and} \quad \lambda_n = \sup_{\|x\|=1} x^* A x.$$

Theorem (Rayleigh-Ritz)

Let A be an $n \times n$ Hermitian matrix with entries in $\mathbb{C}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Then

$$\lambda_1 = \inf_{\|x\|=1} x^* A x \quad \textit{and} \quad \lambda_n = \sup_{\|x\|=1} x^* A x.$$

Corollary

Let $A = (a_{i,j})$ be a complex $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

Theorem (Rayleigh-Ritz)

Let A be an $n \times n$ Hermitian matrix with entries in $\mathbb{C}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Then

$$\lambda_1 = \inf_{\|x\|=1} x^* Ax \quad \text{and} \quad \lambda_n = \sup_{\|x\|=1} x^* Ax.$$

Corollary

Let $A = (a_{i,j})$ be a complex $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

(i) $\lambda_1 \leq a_{j,j} \leq \lambda_n$ *for all* $j = 1, 2, \ldots, n$

Theorem (Rayleigh-Ritz)

Let A be an $n \times n$ Hermitian matrix with entries in $\mathbb{C}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Then

$$\lambda_1 = \inf_{\|x\|=1} x^* A x \quad \text{and} \quad \lambda_n = \sup_{\|x\|=1} x^* A x.$$

Corollary

Let $A = (a_{i,j})$ be a complex $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

(i) $\lambda_1 \leq a_{j,j} \leq \lambda_n$ *for all* $j = 1, 2, \ldots, n$, *and*

(ii) $\lambda_1 \leq \frac{1}{n} \sum_{i,j=1}^{n} a_{i,j} \leq \lambda_n.$

Theorem (Cauchy's Interlacing Theorem)

Let $A \in \mathbb{M}_n(\mathbb{C})$ *be Hermitian, and let* $B \in \mathbb{M}_{n-1}(\mathbb{C})$ *be a principal submatrix of* A*.*

Theorem (Cauchy's Interlacing Theorem)

Let $A \in \mathbb{M}_n(\mathbb{C})$ be Hermitian, and let $B \in \mathbb{M}_{n-1}(\mathbb{C})$ be a principal submatrix of A. If

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

are the eigenvalues of A

Theorem (Cauchy's Interlacing Theorem)

Let $A \in \mathbb{M}_n(\mathbb{C})$ *be Hermitian, and let* $B \in \mathbb{M}_{n-1}(\mathbb{C})$ *be a principal submatrix of* A*. If*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

are the eigenvalues of A *and*

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$$

are the eigenvalues of B

Theorem (Cauchy's Interlacing Theorem)

Let $A \in \mathbb{M}_n(\mathbb{C})$ be Hermitian, and let $B \in \mathbb{M}_{n-1}(\mathbb{C})$ be a principal submatrix of A. If

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

are the eigenvalues of A and

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$$

are the eigenvalues of B, then

$$\lambda_1 \leq \qquad \lambda_2 \leq \qquad \cdots \qquad \lambda_{n-1} \leq \qquad \lambda_n.$$

Theorem (Cauchy's Interlacing Theorem)

Let $A \in \mathbb{M}_n(\mathbb{C})$ be Hermitian, and let $B \in \mathbb{M}_{n-1}(\mathbb{C})$ be a principal submatrix of A. If

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

are the eigenvalues of A and

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$$

are the eigenvalues of B, then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Is A positive semi-definite?

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Is A positive semi-definite? No.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Is A positive semi-definite? No. The principal submatrix

$$B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

has negative determinant, and hence has a negative eigenvalue.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Is A positive semi-definite? No. The principal submatrix

$$B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

has negative determinant, and hence has a negative eigenvalue. Cauchy's interlacing theorem implies that A too has a negative eigenvalue.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Is A positive semi-definite? No. The principal submatrix

$$B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

has negative determinant, and hence has a negative eigenvalue. Cauchy's interlacing theorem implies that A too has a negative eigenvalue.

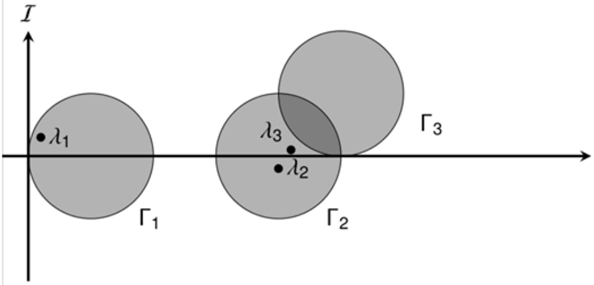
Corollary

If $A \in \mathbb{M}_n(\mathbb{C})$ is positive semi-definite, then every principle submatrix must have non-negative determinant.

Share on Facebook

Zack Cramer

Feb 27th, 3pm.



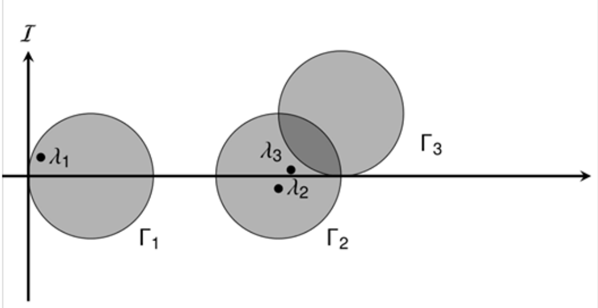
Russian man approximates eigenvalues using this weird old trick. Mathematicians HATE HIM!!!

WWW.UWATERLOO.CA

Share on Facebook

Zack Cramer

Feb 27th, 3pm.



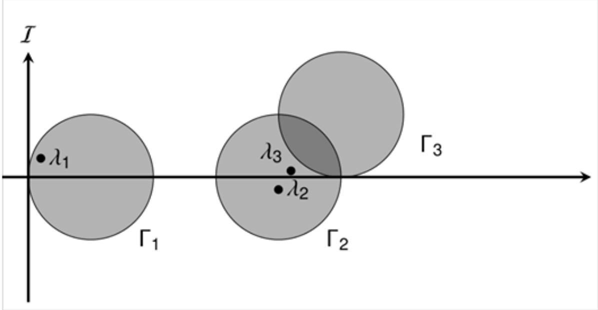
Russian man approximates eigenvalues using this weird old trick. Mathematicians HATE HIM!!!

WWW.UWATERLOO.CA

Share on Facebook

Zack Cramer

Feb 27th, 3pm.



Russian man approximates eigenvalues using this weird old trick. Mathematicians HATE HIM!!!

WWW.UWATERLOO.CA

If $A = (a_{i,j})$ is an $n \times n$ matrix with entries in $\mathbb{C}$, then for each $i = 1, 2, \ldots, n$, we define

If $A = (a_{i,j})$ is an $n \times n$ matrix with entries in $\mathbb{C}$, then for each $i = 1, 2, \ldots, n$, we define

- $R_i := \sum_{j \neq i} |a_{i,j}|$

If $A = (a_{i,j})$ is an $n \times n$ matrix with entries in $\mathbb{C}$, then for each $i = 1, 2, \ldots, n$, we define

- $R_i := \sum_{j \neq i} |a_{i,j}|$ and
- $D_i := \{z \in \mathbb{C} : |z - a_{i,i}| \leq R_i\}$.

If $A = (a_{i,j})$ is an $n \times n$ matrix with entries in $\mathbb{C}$, then for each $i = 1, 2, \ldots, n$, we define

- $R_i := \sum_{j \neq i} |a_{i,j}|$ and
- $D_i := \{z \in \mathbb{C} : |z - a_{i,i}| \leq R_i\}$.

Definition

The sets $D_1, D_2, \ldots, D_n$ are called the **Gershgorin disks** *of A.*

If $A = (a_{i,j})$ is an $n \times n$ matrix with entries in $\mathbb{C}$, then for each $i = 1, 2, \ldots, n$, we define

- $R_i := \sum_{j \neq i} |a_{i,j}|$ and
- $D_i := \{z \in \mathbb{C} : |z - a_{i,i}| \leq R_i\}$.

Definition

The sets $D_1, D_2, \ldots, D_n$ are called the **Gershgorin disks** *of A.*

Theorem (Gershgorin Circle Theorem, 1931)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. The eigenvalues of A belong to the union of its Gershgorin disks.

If $A = (a_{i,j})$ is an $n \times n$ matrix with entries in $\mathbb{C}$, then for each $i = 1, 2, \ldots, n$, we define

- $R_i := \sum_{j \neq i} |a_{i,j}|$ and
- $D_i := \{z \in \mathbb{C} : |z - a_{i,i}| \leq R_i\}$.

Definition

The sets $D_1, D_2, \ldots, D_n$ are called the **Gershgorin disks** *of A.*

Theorem (Gershgorin Circle Theorem, 1931)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. The eigenvalues of A belong to the union of its Gershgorin disks.

Proof.

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

$$Ax = \lambda x$$

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

$$Ax = \lambda x \Rightarrow \sum_{j=1}^{n} a_{i,j} x_j = \lambda x_i$$

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

$$Ax = \lambda x \Rightarrow \sum_{j=1}^{n} a_{i,j} x_j = \lambda x_i$$

$$\Rightarrow \sum_{j \neq i} a_{i,j} x_j = (\lambda - a_{i,i}) x_i.$$

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

$$\begin{aligned} Ax = \lambda x &\Rightarrow \sum_{j=1}^{n} a_{i,j} x_j = \lambda x_i \\ &\Rightarrow \sum_{j \neq i} a_{i,j} x_j = (\lambda - a_{i,i}) x_i. \end{aligned}$$

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

$$
\begin{aligned}
Ax = \lambda x &\Rightarrow \sum_{j=1}^{n} a_{i,j} x_j = \lambda x_i \\
&\Rightarrow \sum_{j \neq i} a_{i,j} x_j = (\lambda - a_{i,i}) x_i.
\end{aligned}
$$

Thus,

$$
|\lambda - a_{i,i}| = \left| \sum_{j \neq i} \frac{a_{i,j} x_j}{x_i} \right|
$$

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

$$\begin{aligned} Ax = \lambda x \Rightarrow \sum_{j=1}^{n} a_{i,j} x_j &= \lambda x_i \\ \Rightarrow \sum_{j \neq i} a_{i,j} x_j &= (\lambda - a_{i,i}) x_i. \end{aligned}$$

Thus,

$$|\lambda - a_{i,i}| = \left| \sum_{j \neq i} \frac{a_{i,j} x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{i,j}|$$

Proof.

Choose $x \neq 0$ so that $Ax = \lambda x$. Let x_i be the largest entry of x (in modulus).

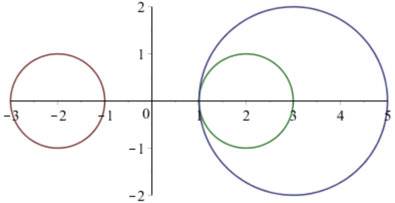
$$\begin{aligned} Ax = \lambda x \Rightarrow \sum_{j=1}^{n} a_{i,j} x_j &= \lambda x_i \\ \Rightarrow \sum_{j \neq i} a_{i,j} x_j &= (\lambda - a_{i,i}) x_i. \end{aligned}$$

Thus,

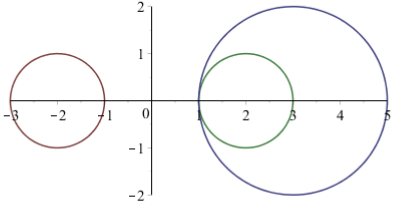
$$|\lambda - a_{i,i}| = \left| \sum_{j \neq i} \frac{a_{i,j} x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{i,j}| = R_i.$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 0 & -2 \end{bmatrix}$$



$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 0 & -2 \end{bmatrix}$$



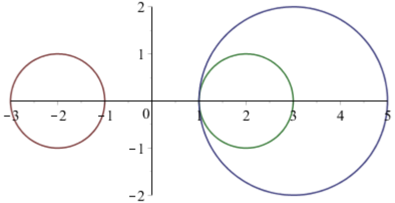
Corollary

Let $A = (a_{i,j})$ be an $n \times n$ complex matrix. If

$$\sum_{j \neq i} |a_{i,j}| < |a_{i,i}|$$

for all i, then A is invertible.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 0 & -2 \end{bmatrix}$$



Corollary

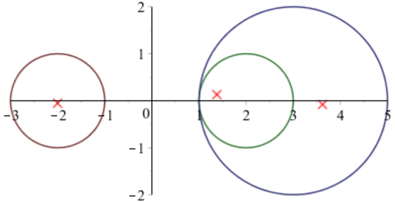
Let $A = (a_{i,j})$ be an $n \times n$ complex matrix. If

$$\sum_{j \neq i} |a_{i,j}| < |a_{i,i}|$$

for all i, then A is invertible.

Such a matrix is called **strictly diagonally dominant**.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 0 & -2 \end{bmatrix}$$



Corollary

Let $A = (a_{i,j})$ be an $n \times n$ complex matrix. If

$$\sum_{j \neq i} |a_{i,j}| < |a_{i,i}|$$

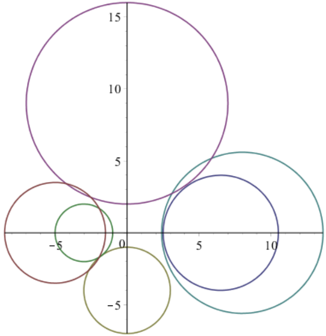
for all i, then A is invertible.

Such a matrix is called **strictly diagonally dominant**.

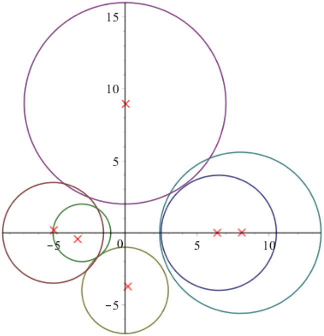
$$A = \begin{bmatrix} -4i & 2/3 & 1 & i/2 & 1/3 & -1/2 \\ 1 & -3 & 0 & 1/2 & 0 & -1/2 \\ 3/5 & 2i & 8 & 1 & -1 & i \\ i/3 & 1 & 0 & 13/2 & -2/3 & 2 \\ 3 & -2 & 1/2 & 0 & 9i & 3i/2 \\ -1 & 5i/4 & 1 & -1/4 & 0 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} -4i & 2/3 & 1 & i/2 & 1/3 & -1/2 \\ 1 & -3 & 0 & 1/2 & 0 & -1/2 \\ 3/5 & 2i & 8 & 1 & -1 & i \\ i/3 & 1 & 0 & 13/2 & -2/3 & 2 \\ 3 & -2 & 1/2 & 0 & 9i & 3i/2 \\ -1 & 5i/4 & 1 & -1/4 & 0 & -5 \end{bmatrix} \qquad \begin{aligned} R_1 &= 3 \\ R_2 &= 2 \\ R_3 &= 5.6 \\ R_4 &= 4 \\ R_5 &= 7 \\ R_6 &= 3.5 \end{aligned}$$

$$A = \begin{bmatrix} -4i & 2/3 & 1 & i/2 & 1/3 & -1/2 \\ 1 & -3 & 0 & 1/2 & 0 & -1/2 \\ 3/5 & 2i & 8 & 1 & -1 & i \\ i/3 & 1 & 0 & 13/2 & -2/3 & 2 \\ 3 & -2 & 1/2 & 0 & 9i & 3i/2 \\ -1 & 5i/4 & 1 & -1/4 & 0 & -5 \end{bmatrix} \qquad \begin{aligned} R_1 &= 3 \\ R_2 &= 2 \\ R_3 &= 5.6 \\ R_4 &= 4 \\ R_5 &= 7 \\ R_6 &= 3.5 \end{aligned}$$

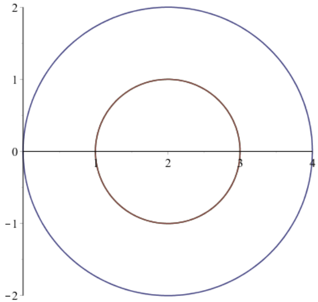


$$A = \begin{bmatrix} -4i & 2/3 & 1 & i/2 & 1/3 & -1/2 \\ 1 & -3 & 0 & 1/2 & 0 & -1/2 \\ 3/5 & 2i & 8 & 1 & -1 & i \\ i/3 & 1 & 0 & 13/2 & -2/3 & 2 \\ 3 & -2 & 1/2 & 0 & 9i & 3i/2 \\ -1 & 5i/4 & 1 & -1/4 & 0 & -5 \end{bmatrix} \qquad \begin{aligned} R_1 &= 3 \\ R_2 &= 2 \\ R_3 &= 5.6 \\ R_4 &= 4 \\ R_5 &= 7 \\ R_6 &= 3.5 \end{aligned}$$

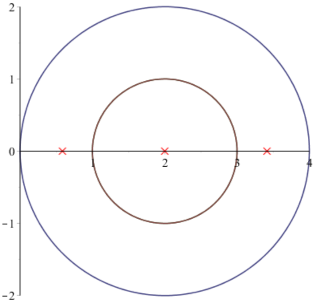


$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

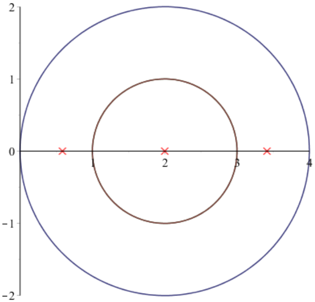
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



The disks did not detect the invertibility of A!

Additional remarks:

Additional remarks:

(1) We could have used the columns of A rather than the rows to make our disks.

Additional remarks:

(1) We could have used the columns of A rather than the rows to make our disks. Why?

Additional remarks:

(1) We could have used the columns of A rather than the rows to make our disks. Why? Because A and A^T have the same eigenvalues!

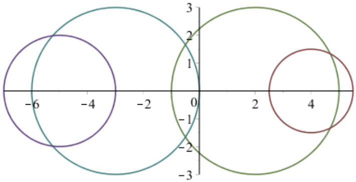
Additional remarks:

(1) We could have used the columns of A rather than the rows to make our disks. Why? Because A and A^T have the same eigenvalues!

(2) More generally, we could have used the disks from SAS^{-1} to approximate the eigenvalues of A.

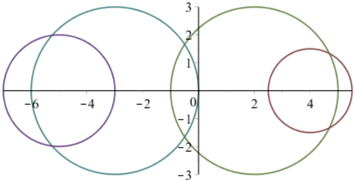
$$A = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -5 & -1 \\ -1/2 & 0 & -1 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -5 & -1 \\ -1/2 & 0 & -1 & 4 \end{bmatrix}$$

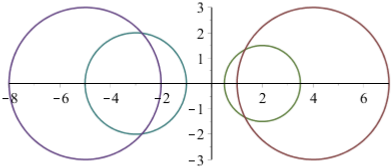


Row radii

$$A = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -5 & -1 \\ -1/2 & 0 & -1 & 4 \end{bmatrix}$$

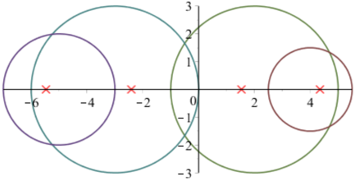


Row radii

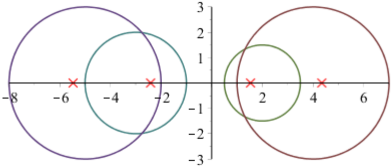


Column radii

$$A = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -5 & -1 \\ -1/2 & 0 & -1 & 4 \end{bmatrix}$$



Row radii



Column radii

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Is it always the case that each disk contains exactly one eigenvalue?

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Is it always the case that each disk contains exactly one eigenvalue?

No...

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Is it always the case that each disk contains exactly one eigenvalue?

No...But we get something almost as good!

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Is it always the case that each disk contains exactly one eigenvalue?

No...But we get something almost as good!

Theorem

Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$.

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Is it always the case that each disk contains exactly one eigenvalue?

No...But we get something almost as good!

Theorem

Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$.

If $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ are k Gershgorin disks of A that are disjoint from the remaining $n - k$ disks

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Is it always the case that each disk contains exactly one eigenvalue?

No...But we get something almost as good!

Theorem

Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$.

If $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ are k Gershgorin disks of A that are disjoint from the remaining $n - k$ disks, then their union contains exactly k eigenvalues of A (counting multiplicities).

Any $n \times n$ matrix A has n Gershgorin disks and n eigenvalues (counting multiplicities).

Is it always the case that each disk contains exactly one eigenvalue?

No...But we get something almost as good!

Theorem

Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$.

If $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ are k Gershgorin disks of A that are disjoint from the remaining $n - k$ disks, then their union contains exactly k eigenvalues of A (counting multiplicities).

In particular, this means that each disk *does* contain exactly one eigenvalue when the disks are disjoint.

Proof.

Proof.

For $t \in [0, 1]$, let A_t be the matrix A with the off-diagonal entries scaled by t

Proof.

For $t \in [0, 1]$, let A_t be the matrix A with the off-diagonal entries scaled by t, so

$$A_0 = \mathrm{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n}),$$
$$A_1 = A.$$

Proof.

For $t \in [0, 1]$, let A_t be the matrix A with the off-diagonal entries scaled by t, so

$$A_0 = \mathrm{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n}),$$
$$A_1 = A.$$

As t increases from 0 to 1, two things happen:

Proof.

For $t \in [0, 1]$, let A_t be the matrix A with the off-diagonal entries scaled by t, so

$$A_0 = \mathrm{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n}),$$
$$A_1 = A.$$

As t increases from 0 to 1, two things happen:

(i) the Gershgorin disks inflate to the disks of A

Proof.

For $t \in [0, 1]$, let A_t be the matrix A with the off-diagonal entries scaled by t, so

$$
\begin{aligned}
A_0 &= \mathrm{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n}), \\
A_1 &= A.
\end{aligned}
$$

As t increases from 0 to 1, two things happen:

(i) the Gershgorin disks inflate to the disks of A and

(ii) the eigenvalues vary *continuously* while always remaining in the disks.

Proof.

For $t \in [0,1]$, let A_t be the matrix A with the off-diagonal entries scaled by t, so

$$A_0 = \mathrm{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n}),$$
$$A_1 = A.$$

As t increases from 0 to 1, two things happen:

(i) the Gershgorin disks inflate to the disks of A and

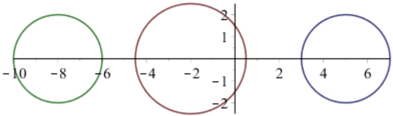
(ii) the eigenvalues vary *continuously* while always remaining in the disks.

Since the disks of A_t that inflate to $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ never intersect the remaining disks, the k eigenvalues in these disks never have a chance to leave!

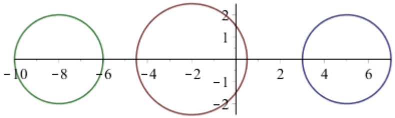
But seeing is believing, am I right??

$$A = \begin{bmatrix} -8 & 0 & -2 \\ 1 & 5 & -1 \\ 3/2 & -1 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} -8 & 0 & -2 \\ 1 & 5 & -1 \\ 3/2 & -1 & -2 \end{bmatrix}$$



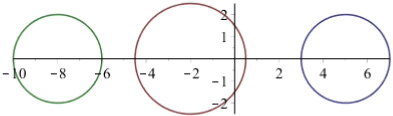
$$A = \begin{bmatrix} -8 & 0 & -2 \\ 1 & 5 & -1 \\ 3/2 & -1 & -2 \end{bmatrix}$$



Corollary

Let $A = (a_{i,j})$ be an $n \times n$ matrix with real entries.

$$A = \begin{bmatrix} -8 & 0 & -2 \\ 1 & 5 & -1 \\ 3/2 & -1 & -2 \end{bmatrix}$$



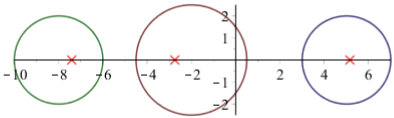
Corollary

Let $A = (a_{i,j})$ be an $n \times n$ matrix with real entries. If

$$|a_{i,i} - a_{j,j}| \geq R_i + R_j$$

for all $i \neq j$, then the eigenvalues of A are real.

$$A = \begin{bmatrix} -8 & 0 & -2 \\ 1 & 5 & -1 \\ 3/2 & -1 & -2 \end{bmatrix}$$



Corollary

Let $A = (a_{i,j})$ be an $n \times n$ matrix with real entries. If

$$|a_{i,i} - a_{j,j}| \geq R_i + R_j$$

for all $i \neq j$, then the eigenvalues of A are real.

Let's see if we can improve the Gershgorin approximation.

Let's see if we can improve the Gershgorin approximation.

Rather than considering each row individually, we can consider two at a time.

Let's see if we can improve the Gershgorin approximation.

Rather than considering each row individually, we can consider two at a time.

Definition

Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. For each $i, j = 1, 2, \ldots, n$ with $i \neq j$, define

$$K_{i,j} := \{z \in \mathbb{C} : |z - a_{i,i}||z - a_{j,j}| \leq R_i R_j\}.$$

Let's see if we can improve the Gershgorin approximation.

Rather than considering each row individually, we can consider two at a time.

Definition

Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. For each $i, j = 1, 2, \ldots, n$ with $i \neq j$, define

$$K_{i,j} := \{z \in \mathbb{C} : |z - a_{i,i}||z - a_{j,j}| \leq R_i R_j\}.$$

The sets $K_{i,j}$ are called **Brauer's ovals of Cassini**.

Theorem (Brauer, 1947)

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

(ii) *The union of the Gershgorin disks contains the ovals of Cassini.*

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

(ii) *The union of the Gershgorin disks contains the ovals of Cassini.*

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

(ii) *The union of the Gershgorin disks contains the ovals of Cassini.*

Proof.

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

(ii) *The union of the Gershgorin disks contains the ovals of Cassini.*

Proof.

If $z \in K_{i,j}$, then $|z - a_{i,i}||z - a_{j,j}| \leq R_i R_j$.

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

(ii) *The union of the Gershgorin disks contains the ovals of Cassini.*

Proof.

If $z \in K_{i,j}$, then $|z - a_{i,i}||z - a_{j,j}| \leq R_i R_j$. If $R_i R_j = 0$ then $z = a_{i,i}$ or $z = a_{j,j}$.

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

(ii) *The union of the Gershgorin disks contains the ovals of Cassini.*

Proof.

If $z \in K_{i,j}$, then $|z - a_{i,i}||z - a_{j,j}| \leq R_i R_j$. If $R_i R_j = 0$ then $z = a_{i,i}$ or $z = a_{j,j}$. Otherwise, we have

$$\frac{|z - a_{i,i}|}{R_i} \cdot \frac{|z - a_{j,j}|}{R_j} \leq 1.$$

Theorem (Brauer, 1947)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) *The union of the ovals of Cassini contains the eigenvalues of A.*

(ii) *The union of the Gershgorin disks contains the ovals of Cassini.*

Proof.

If $z \in K_{i,j}$, then $|z - a_{i,i}||z - a_{j,j}| \leq R_i R_j$. If $R_i R_j = 0$ then $z = a_{i,i}$ or $z = a_{j,j}$. Otherwise, we have

$$\frac{|z - a_{i,i}|}{R_i} \cdot \frac{|z - a_{j,j}|}{R_j} \leq 1.$$

Hence $|z - a_{i,i}| \leq R_i$ or $|z - a_{j,j}| \leq R_j$.

The downside...

The downside...

- There are $\binom{n}{2}$ ovals as opposed to n disks.

The downside...

- There are $\binom{n}{2}$ ovals as opposed to n disks.
- The ovals are harder to draw (and harder to code in Maple!)

The downside...

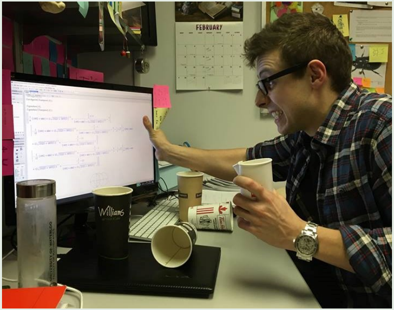
- There are $\binom{n}{2}$ ovals as opposed to n disks.
- The ovals are harder to draw (and harder to code in Maple!)

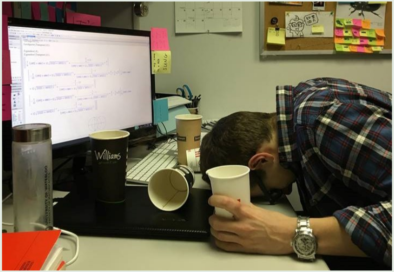
Proof.

The downside...

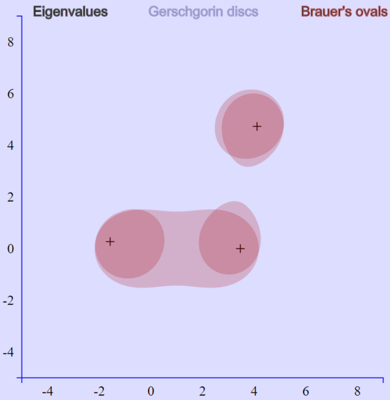
- There are $\binom{n}{2}$ ovals as opposed to n disks.
- The ovals are harder to draw (and harder to code in Maple!)

Proof.





Gerschgorin Disks and Brauer's ovals of Cassini

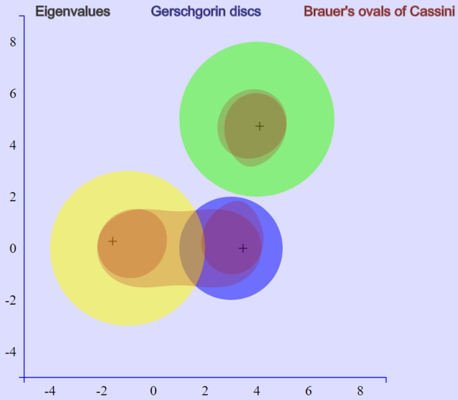


Press the 'Plot' button to produce a plot for the displayed 3x3 matrix. You can edit the values in the matrix by hand, or generate new random values by pressing the button. Press on the plot labels to show or hide corresponding plot elements.

3	i	1
-1	4+5i	2
2	1	-1

Plot
Random entries

Gerschgorin Disks and Brauer's ovals of Cassini



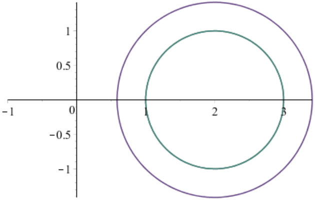
Press the 'Plot' button to produce a plot for the displayed 3x3 matrix. You can edit the values in the matrix by hand, or generate new random values by pressing the button. Press on the plot labels to show or hide corresponding plot elements.

3	i	1
-1	4+5i	2
2	1	-1

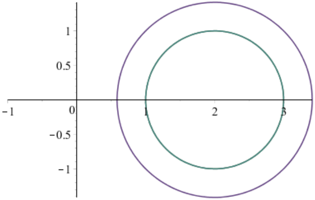
Plot   Random entries

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

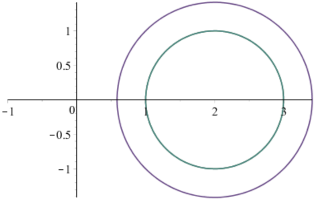


$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

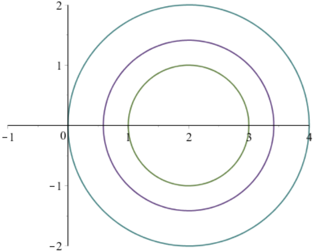


$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

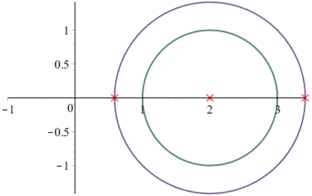
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



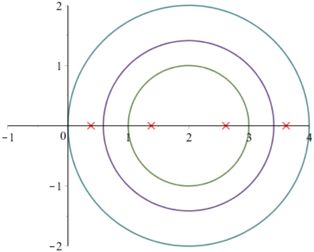
$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$



$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$



Remarkably, considering more than two rows at a time doesn't work!

Remarkably, considering more than two rows at a time doesn't work! The union of the sets

$$K_{i,j,k} := \{z \in \mathbb{C} : |z - a_{i,i}||z - a_{j,j}||z - a_{k,k}| \leq R_i R_j R_k\}$$

for i, j, k distinct may not even contain the eigenvalues of A.

Remarkably, considering more than two rows at a time doesn't work! The union of the sets

$$K_{i,j,k} := \{z \in \mathbb{C} : |z - a_{i,i}||z - a_{j,j}||z - a_{k,k}| \le R_i R_j R_k\}$$

for i, j, k distinct may not even contain the eigenvalues of A.

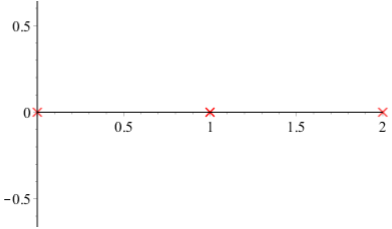
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remarkably, considering more than two rows at a time doesn't work! The union of the sets

$$K_{i,j,k} := \{z \in \mathbb{C} : |z - a_{i,i}||z - a_{j,j}||z - a_{k,k}| \leq R_i R_j R_k\}$$

for i, j, k distinct may not even contain the eigenvalues of A.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

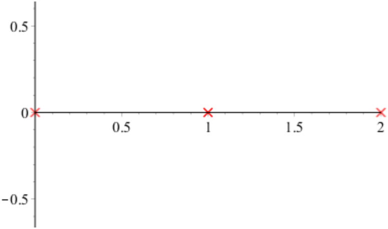


Remarkably, considering more than two rows at a time doesn't work! The union of the sets

$$K_{i,j,k} := \{z \in \mathbb{C} : |z - a_{i,i}||z - a_{j,j}||z - a_{k,k}| \leq R_i R_j R_k\}$$

for i, j, k distinct may not even contain the eigenvalues of A.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} R_1 &= 1 \\ R_2 &= 1 \\ R_3 &= 0 \\ R_4 &= 0 \end{aligned}$$



Final remarks:

Final remarks:

- A slight improvement on the Gershgorin theorem can be used to show that *all* matrices of the form

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n \times n}$$

  are invertible.

Final remarks:

- A slight improvement on the Gershgorin theorem can be used to show that *all* matrices of the form

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n\times n}$$

  are invertible.

- Versions of Gershgorin's theorem hold for partitioned matrices and for matrices of operators.

Thank you!