The Category of Pointed Topological Spaces

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1 The Category of Pointed Topological Spaces

Definition 1 *A* ***based*** *topological space is a pair $(X, *)$, where X is a nonempty space and $* \in X$ is a chosen* ***basepoint****. We will normally suppress basepoints from notation and write a based space simply X, denoting all basepoints ambiguously by $*$. If X, Y are based spaces, then a continuous map $f : X \to Y$ is said to be* ***based*** *if $f(*) = *$.* $\square$

A based space or map is also said to be **pointed**. We can view a based space as a pair of a nonempty space X and a map $* \to X$ into it from the one-point space. Then a map $f : X \to Y$ is based if and only if it makes the next triangle commute

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y. \end{tikzcd} \tag{1.1}$$

It is for this reason that we sometimes think of based spaces as spaces *under the one-point space*. Although generally unnecessary, this viewpoint can be especially useful when categories and functors are involved.

The collection of all based spaces and maps forms a category Top_*. We denote by $Top_*(X, Y)$ the set of all based maps $X \to Y$. This is a pointed set, with distinguished element given by the **zero map** $X \to * \to Y$. Similarly, we denote by $C_*(X, Y)$ the *space* of all based maps $X \to Y$, topologised as a subspace of $C(X, Y)$ with the compact-open topology. The zero map is a natural basepoint for $C_*(X, Y)$.

There is an obvious functor $Top_* \xrightarrow{U} Top$ which forgets basepoints, and a functor in the opposite direction

$$Top \xrightarrow{(-)_+} Top_*, \qquad K \mapsto K_+ = K \sqcup \{*\} \tag{1.2}$$

which adjoins to an unbased space K a disjoint basepoint[1]. These functors are both faithful, but neither is full. Notice also that they form an adjoint pair. That is, if K is an unbased space and Y is a based space, then there is a bijection

$$Top_*(K_+, Y) \cong Top(K, Y) \tag{1.3}$$

which is natural in both variables.

The category Top_* has all limits and colimits. Limits are *created by* the forgetful functor $Top_* \to Top$[2]. Informally this means that limits in Top_* are formed as in Top, and then given the only sensible basepoint that makes all the canonical maps based. For instance, the product of two based spaces X, Y is the ordinary cartesian product $X \times Y$ given the basepoint $(*_X, *_Y)$. This definition not only makes the projection maps $X \xleftarrow{pr_X} X \times Y \xrightarrow{pr_Y} Y$ based, but means that there are also canonical inclusions

$$X \hookrightarrow X \times Y \hookleftarrow Y. \tag{1.4}$$

Clearly this extends to give products $\prod_{i \in \mathcal{I}} X_i$ of arbitrary families of pointed spaces $(X_i)_{i \in \mathcal{I}}$.

Colimits in Top_* are computed a little differently than in Top. Rather than giving a general discussion we shall define only those which we will need. These being coproducts and pushouts. Telescope diagrams will be discussed at a later point.

The coproduct of two based spaces X, Y is the **wedge**

$$X \vee Y = X \sqcup Y \big/ [*_X \sim *_Y]. \tag{1.5}$$

Adding the maps $X \to X \times Y$ and $Y \to X \times Y$ introduced in (1.4) we easily check:

Lemma 1.1 *There is a canonical basepoint respecting map $X \vee Y \to X \times Y$ which maps the wedge homeomorphically onto the subspace*

$$X \vee Y \cong (X \times *) \cup (* \times Y) \subseteq X \times Y. \quad \blacksquare \tag{1.6}$$

It is frequently useful to identify $X \vee Y$ with its image under this embedding, and we shall frequently do so without mention.

The construction of coproducts extends to arbitrary families of pointed spaces $(X_i)_{i \in \mathcal{I}}$, and we can form arbitrary set-indexed wedges

$$\bigvee_{i \in \mathcal{I}} X_i = \bigsqcup_{i \in \mathcal{I}} X_i \bigg/ [*_i \sim *_{i'}]. \tag{1.7}$$

There is still a continuous bijection

$$\bigvee_{i \in \mathcal{I}} X_i \to \left\{ (x_i) \in \prod_{i \in \mathcal{I}} X_i \,\middle|\, x_i = *_i \text{ for all but at most one index } i \in \mathcal{I} \right\} \subseteq \prod_{i \in \mathcal{I}} X_i \tag{1.8}$$

but this map may fail to be homeomorphism when $\mathcal{I}$ is not finite (consider taking $(X_i, *_i) = (I, 1)$).

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[1]A slicker definitions is obtained by setting $X_+ = X/\emptyset$.

[2]Reason: The forgetful functor is *monadic*.

Wedges are generally well-behaved. They inherit some topological properties from their factors, but care must be taken with others. For instance $\bigvee_{\mathcal{I}} X_i$ is Hausdorff whenever each X_i is, but $\bigvee_{\mathcal{I}} X_i$ may fail to be locally compact even when each X_i is.

Moving on, the pushout in Top_* of a span

$$Z \xleftarrow{g} X \xrightarrow{f} Y \tag{1.9}$$

is formed exactly as it is in Top. Namely as the quotient $(Y \sqcup Z)/\sim = (Y \vee Z)/\sim$, where $\sim$ is the smallest equivalence relation generated by $f(x) \sim g(x)$, $\forall x \in X$. Notice that the pushout space has a canonical basepoint and all structure maps are pointed.

Definition 2 *For pointed spaces X, Y we define their **smash product** $X \wedge Y$ by means of the pushout*

$$\begin{tikzcd} X \vee Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y. \end{tikzcd} \tag{1.10}$$

□

Thus

$$X \wedge Y = X \times Y \big/ X \vee Y. \tag{1.11}$$

We write $x \wedge y$ for its points, for obvious reasons. The pushout makes clear that the construction is functorial in both variables, and pointed maps $f : X \to X'$, $g : Y \to Y'$ induce a pointed map

$$f \wedge g : X \wedge Y \to X' \wedge Y'. \tag{1.12}$$

Note that although the smash product is important, it is *not* a categorical product. Rather it should be compared to the tensor product in the category of abelian groups. One motivation for its introduction is that it has a natural interaction with the based function spaces.

Proposition 1.2 *Let X, Y, Z be based spaces. If Y is locally compact, then there is a bijection of sets*

$$Top_*(X \wedge Y, Z) \cong Top_*(X, C_*(Y, Z)). \tag{1.13}$$

Proof We apply $Top_*(-, Z)$ to the pushout square (1.10) to get a pullback of sets

$$\begin{tikzcd} Top_*(X \wedge Y, Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Top_*(*, Z) \arrow[d] \\ Top_*(X \times Y, Z) \arrow[r] & Top_*(X \vee Y, Z). \end{tikzcd} \tag{1.14}$$

Then using $Top_*(*, Z) = *$ we can identify

$$Top_*(X \wedge Y, Z) \cong \{f \in Top_*(X \times Y, Z) \mid f|_{X \vee Y} = *\} \subseteq Top_*(X \times Y, Z). \tag{1.15}$$

Since Y is locally compact there is a bijection between the unbased mapping sets $Top(X \times Y, Z) \cong Top(X, Z^Y)$, and we can check that this maps the subset $Top_*(X \wedge Y, Z)$ bijectively onto $Top_*(X, Z^Y)$. ∎

The smash product enjoys the following properties. Note, however, that unlike the cartesian product, the smash product is not always so well behaved (i.e. point 4)).

Proposition 1.3 *The following properties hold for the smash product.*

1) *If X, Y are Hausdorff, then so is $X \wedge Y$.*

2) *For any based spaces X, Y there is a natural homeomorphism $X \wedge Y \cong Y \wedge X$.*

3) *For any based spaces X, Y, Z there is a natural homeomorphism $(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z)$.*

4) *Given based spaces X, Y, Z, if any two are locally compact, then there is a homeomorphism $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.*

5) *For any based space X there are natural homeomorphisms $X \wedge S^0 \cong X \cong S^0 \wedge X$.*

6) *For any based space X there are homeomorphisms $X \wedge * \cong * \cong * \wedge X$.*

7) *For a based space Y and an unbased space K there is a natural homeomorphism $K_+ \wedge Y \cong K \times Y / K \times *$. In particular, for unbased spaces K, L there is a homeomorphism $K_+ \wedge L_+ \cong (K \times L)_+$.*

Proof 1) Notice that the quotient map takes the difference $X \times Y \setminus X \vee Y$ homeomorphically onto $X \wedge Y \setminus \{*\}$. Thus to prove the statement it will be sufficient to separate a given non-basepoint $x \wedge y$ from $*$. To do this we use that X, Y are Hausdorff to find open neighbourhoods $U_x \subseteq X$ of x, $V_x \subseteq X$ of $*_X$, $U_y \subseteq Y$ of y, and $V_y \subseteq Y$ of $*_Y$ such that $U_x \cap V_x = \emptyset = U_y \cap V_y$. Then in $X \times Y$, the set $U_x \times U_y$ is an open neighbourhood of (x, y) and

$$V_x \times Y \cup X \times V_y \tag{1.16}$$

is an open neighbourhood of $X \vee Y$. These sets are both saturated and they are disjoint. Hence they give seperating neighbourhoods in $X \wedge Y$ for $x \wedge y$ and $*$.
2) Clear from the definition as a pushout.
3) The spaces are naturally in bijective correspondence and are both quotients of $(X \sqcup Y) \times Z \cong (X \times Z) \sqcup (Y \times Z)$.
4) Using part 2) it will suffice to show the case in which X, Z are locally compact. Then both $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$ are quotients of $X \times Y \times Z$ and we have the solid part of the following diagram

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "id_X \times q_{Y,Z}"'] \arrow[dr, "q_{X,Y} \times id_Z"] & \\ X \times (Y \wedge Z) \arrow[d] & & (X \wedge Y) \times Z \arrow[d] \\ X \wedge (Y \wedge Z) \arrow[rr, dashed] & & (X \wedge Y) \wedge Z \end{tikzcd} \tag{1.17}$$

where the maps $q_{X,Y}, q_{Y,Z}$ as well as the unlabeled vertical arrows are quotients. In the case that X is locally compact, the map $id_X \times q_{X,Y}$ is a quotient and the dotted arrow can be

filled in with a continuous map which is clearly bijective. If also Z is locally compact, then $q_{X,Y} \times id_Z$ is a quotient, and in this case the continuous bijection is a homeomorphism.
5) This follows from part 7) below after noticing that $S^0 \cong (*)_+$. Item 6) follows from 7) similarly, since $* = \emptyset_+$.
7) Clearly $K_+ \vee Y \cong K \sqcup Y$ and $K_+ \times Y \cong K \times Y \sqcup Y$ naturally, where basepoints live in the disjoint copy of Y. Hence $K_+ \wedge Y \cong (K_+ \times Y)/(K_+ \vee Y) \cong K \times Y/K \times *$. ∎

Note that the associativity of the smash product (i.e. item 4)) may fail without some compactness assumptions. For instance the spaces $(\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{Z}$ and $\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{Z})$ are *not* homeomorphic. This example is originally due to D. Puppe, but see [1] §1.7 and Eric Wofsey's math overflow answer [2] for a complete discussion. This is really the same kind of problem we ran into before with function spaces and adjoints. It suggests to us again that really the category of *all* (pointed) topological spaces may not be quite the correct category to be studying.

The following proposition will be useful when we study *cofiber sequences* at a later date.

Proposition 1.4 *Let X, Y be spaces and $A \subseteq X$ a subspace. If Y is locally compact, then there is a homeomorphism*

$$(X \wedge Y)/(A \wedge Y) \cong (X/A) \wedge Y. \tag{1.18}$$

Proof Let $q : X \to X/A$ be the quotient map. Then with no assumptions on Y all the unmarked solid maps in the next diagram are quotients.

$$\begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr, "q\times 1"] & \\ X \wedge Y \arrow[d] & & X/A \times Y \arrow[d] \\ (X \wedge Y)/(A \wedge Y) \arrow[rr, dashed] & & (X/A) \wedge Y \end{tikzcd} \tag{1.19}$$

Thus $X \wedge Y/A \wedge Y$ is a quotient of $X \times Y$ and there is a continuous map $X \wedge Y/A \wedge Y \to (X/A) \wedge Y$ induced which is clearly bijective. If we assume that Y is locally compact, then $q \times 1$ is a quotient map, and in this case the induced bijection is a homeomorphism. ∎

Before closing this section we'll introduce another way of getting into the pointed category other than by adding a *disjoint* basepoint. What we have in mind is the Alexandroff, or one-point, compactification. This construction has some pleasant properties and sheds some light on the relationship between the disjoint union and cartesian product, and the wedge sum and smash product.

Recall that for an unpointed space M we denote its one-point compactification M_∞. As a set M_∞ is the disjoint union of M and an additional basepoint, which we'll generically call ∞. The topology on M_∞ is that generated by the open sets of M together with sets of the form $M_\infty \setminus K$, where $K \subseteq M$ is compact. As such M_∞ is compact. Moreover M_∞ is Hausdorff, and hence locally compact, if and only if M is both locally compact and Hausdorff.

The obvious inclusion $M \hookrightarrow M_\infty$ is an open embedding, which is dense if and only if M is not compact. We turn M_∞ into a pointed space by letting the point at infinity act as a basepoint.

Given a map $f : M \to N$ we can define a function of sets

$$f_\infty : M_\infty \to N_\infty \tag{1.20}$$

in the obvious way by letting *i)* $f_\infty|_M = f$, *ii)* $f_\infty(\infty) = \infty$. Then f_∞ is a pointed map, and if we can ensure that it is continuous, then we may be able to make the one-point compactification functorial.

Proposition 1.5 *If $f : M \to N$ is a map between unpointed locally compact Hausdorff spaces, then $f_\infty : M_\infty \to N_\infty$ is continuous if and only if f is proper.*

Proof If $f : M \to N$ is proper, then $f^{-1}(K) \subseteq M$ is compact for each compact $K \subseteq N$. In particular, $f_\infty^{-1}(N_\infty \setminus K) = M_\infty \setminus f^{-1}(K)$ is open in M_∞. Since also $f_\infty^{-1}(V) = f^{-1}(V)$ is open for each open $V \subseteq N$, we see that f_∞ is a continuous pointed map.

Conversely, assume that $f_\infty : M_\infty \to N_\infty$ is continuous. Then for each compact $K \subseteq N$ we have $f_\infty^{-1}(N_\infty \setminus K) = M_\infty \setminus f^{-1}(K)$. This set is open in M_∞ by assumption, and this implies that $f^{-1}(K)$ is compact in M. Because M is Hausdorff and N is locally compact Hausdorff, a sufficient condition for f to be proper is that preimages of compact sets are compact. ∎

A consequence of this proposition is that $M \mapsto M_\infty$ defines a faithful functor

$$\mathcal{H} \to Top_* \tag{1.21}$$

from the category $\mathcal{H}$ of locally compact Hausdorff spaces and proper maps. The functor is clearly not full, but the proposition does tell us that any pointed map $g : M_\infty \to N_\infty$ satisfying $g^{-1}(\infty) = \{\infty\}$ does lie in its image. Notice that the functor takes values in the category of pointed compact Hausdorff spaces.

The category $\mathcal{H}$ has coproducts but it does not have products, since in general the required projection maps will fail to be proper. On the other hand, given locally compact Hausdorff spaces M, N, the cartesian product $M \times N$ is again locally compact Hausdorff. Moreover, if f, g are proper maps, then $f \times g$ is proper. Thus the situation for $\mathcal{H}$ is similar to that for the pointed category with its smash product, in that we have a *monoidal* product which is not categorical. The next proposition states that the one-point compactification functor preserves this monoidal structure.

Proposition 1.6 *If both M, N are locally compact Hausdorff spaces, then there are homeomorphisms of pointed spaces*

1) $(M \sqcup N)_\infty \cong M_\infty \vee N_\infty$

2) $(M \times N)_\infty \cong M_\infty \wedge N_\infty$.

Proof 1) The coproduct $M \sqcup N$ is the pushout in $\mathcal{H}$ of the span $M \leftarrow \emptyset \to N$. Applying the one-point compactification functor to this diagram induces a bijective map $M_\infty \vee N_\infty \to (M \sqcup N)_\infty$ in Top_*. Since the domain is compact and the target is Hausdorff, this map is a homeomorphism.
2) It's not difficult to see that the obvious composition $M \times N \hookrightarrow M_\infty \times N_\infty \to M_\infty \wedge N_\infty$ is an open embedding whose complement is a single point. We claim that its image is dense in the case that at least one of M, N is non-compact. Indeed, in this case $M \times N$ is a saturated non-closed subset of $M_\infty \times N_\infty$, and so its image cannot be closed in the quotient space. Thus, since $M_\infty \wedge N_\infty$ is compact Hausdorff (cf. Pr.1.3) and $M \times N$ is Hausdorff, we can in this case appeal to the uniqueness of the one-point compactification, to get a homeomorphism of $(M \times N)_\infty$ on $M_\infty \wedge N_\infty$ under $M \times N$.

In the case that both M, N are compact, we have $M_\infty = M_+$ and $N_\infty = N_+$, and Pr. 1.3 tells us that $M_+ \wedge N_+ \cong (M \times N)_+ = (M \times N)_\infty$. ∎

Example 1.1 It can be shown that if X is compact Hausdorff and $x \in X$ is any point, then there is a homeomorphism

$$X \cong (X \setminus \{x\})_\infty. \tag{1.22}$$

Consider the following. Let $I = [0, 1]$ be the interval given the basepoint 1 and, $J = [0, 1]$ the unit interval based at basepoint $\frac{1}{2}$. Then the observation (1.22) in tandem with Propositions 1.6 give us

$$I \wedge I \cong [0, 1)_\infty \wedge [0, 1)_\infty \cong ([0, 1) \times [0, 1))_\infty \tag{1.23}$$

and clearly this is homeomorphic to $[0, 1]^2$. On the other hand, the same line of reasoning shows that

$$J \wedge J \cong \bigvee^4 [0, 1]^2. \tag{1.24}$$

Note that although I, J have the same underlying space, they are not homeomorphic as pointed spaces. □

Using stereographic projection we get for any $n \geq 0$ a homeomorphism

$$(\mathbb{R}^n)_\infty \cong S^n \tag{1.25}$$

between the one-point compactification of $\mathbb{R}^n$ and the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$. We will always assume that we have chosen the homeomorphism which identifies the point at infinity with the unit vector $e_1 = (1, 0, \ldots, 0) \in S^n$, which in particular will be our preferred basepoint for S^n.

Proposition 1.5 now gives a way to construct pointed maps $S^m \to S^n$. Namely by compactifying proper maps $\mathbb{R}^m \to \mathbb{R}^n$. In general it is not too practical to actually study maps between spheres in this way, but since it can be useful for defining maps, and will appear from time to time, we introduce some special notation. For a real vector space V we write

$$S^V = V_\infty \tag{1.26}$$

for its one-point compactification. The following is then a direct corollary of Proposition 1.6.

Corollary 1.7 *If V, W are finite dimensional real vector spaces and S^V, S^W their respective one-point compactifications, then there is a homeomorphism*

$$S^V \wedge S^W \cong S^{V \oplus W}. \tag{1.27}$$

In particular, for any $n, m \geq 0$ there is a homeomorphism

$$S^m \wedge S^n \cong S^{m+n}. \quad \blacksquare \tag{1.28}$$

Remark In more detail, the map (1.27) is that induced by

$$V \times W \to V \oplus W, \qquad (v, w) \mapsto v + w \tag{1.29}$$

where we understand $v + \infty = \infty + w = \infty + \infty = \infty$. $\blacksquare$

1.1 Exercises

The Smash Product

1) Show that the canonical bijection $(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$ is a homeomorphism when both Y, Z are compact.

Locally Compact Hausdorff Spaces

1) Let M be locally compact Hausdorff. Assume that $A \subseteq M$ is closed and locally closed. Show that the inclusion $A_\infty \subseteq M_\infty$ is a closed embedding, and that there is a basepoint respecting homeomorphism $M/A \cong M_\infty/A_\infty$.

2) Let X be compact Hausdorff. Assume $U \subseteq X$ is an open set with compact closure $\overline{U}$. Show that there is a homeomorphism $U_\infty \cong X/(X \setminus U)$. Conclude that if $x \in X$, then $(X \setminus \{x\})_\infty \cong X$.

3) Let X, Y be pointed spaces. Let H^* be an ordinary cohomology theory with coefficients in a field. Show that $\widetilde{H}^*(X \wedge Y) \cong \widetilde{H}^*X \otimes \widetilde{H}^*Y$.

4) Assume that K is locally compact and consider the following diagrams

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} W \wedge K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \wedge K \arrow[d] \\ Y \wedge K \arrow[r] & Z \wedge K \end{tikzcd} \tag{1.30}$$

Show that if the left-hand square is a pushout in Top_*, then so is the right-hand square.

References

[1] J. May, J. Sigurdsson, *Parametrized Homotopy Theory*, American Mathematical Society, (2006).

[2] E. Wofsey, Math Overflow Answer, https://mathoverflow.net/questions/196084/counterexample-for-associativity-of-smash-product.