Dependent choice in Johnstone's topological topos

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1 Statement

The principle of *(countable) dependent choice* can be formulated in the internal logic of any elementary topos with natural numbers object.

Definition 1.1. An elementary topos $\mathcal{E}$ with natural numbers object $\mathbf{N}$ validates *dependent choice* if, for any object X and subobject $R \rightarrowtail X \times X$,

$$\mathcal{E} \models \forall x\!:\!X\, \exists y\!:\!X\, R(x,y) \;\rightarrow\; \forall x\!:\!X\, \exists f\!:\!X^{\mathbf{N}}\, (f(0) = x \wedge \forall n\!:\!\mathbf{N}\, R(f(n), f(n+1))) \;.$$

In the special case of a Grothendieck topos, one can give a simple equivalent formulation avoiding the internal logic.

Proposition 1.2. *A Grothendieck topos $\mathcal{E}$ validates dependent choice if and only if, for every ω^{op}-chain of epimorphisms*

$$\cdots \xrightarrow{e_3} X_3 \xrightarrow{e_2} X_2 \xrightarrow{e_1} X_1 \xrightarrow{e_0} X_0 \;,$$

the limit cone $(L \xrightarrow{l_i} X_i)_{i \geq 0}$ itself consists of epimorphisms.

Johnstone's topological topos $\mathcal{T}$ [Joh79] is the Grothendieck topos given by the site defined below. The generating category is the full subcategory $\mathbb{T}$ of the category of topological spaces on two objects: $\mathbf{1}$, a one point space; and $\mathbf{N}^\infty$, the one point compactification of a discrete countably infinite space. We take the underlying set of

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$\mathbf{N}^\infty$ to be $\mathbb{N} \cup \{\infty\}$. For $i \in \mathbb{N} \cup \{\infty\}$, we write $i \colon \mathbf{1} \to \mathbf{N}^\infty$ for the function whose image is $\{i\}$.

We often consider infinite subsets $L \subseteq \mathbb{N}$ as given by strictly ascending enumerations $\{l_0, l_1, l_2, \dots\}$, and we refer to n as the *index* of l_n in L. Given infinite sets $L \subseteq K \subseteq \mathbb{N}$, we write $\iota_{K \supseteq L} : \mathbf{N}^\infty \to \mathbf{N}^\infty$ for the continuous strictly increasing

$$\iota_{K \supseteq L}(i) \;=\; \begin{cases} j & \text{such that } j \text{ is the index of } l_i \text{ in } K, \text{ if } i < \infty \\ \infty & \text{if } i = \infty \;. \end{cases}$$

Note that if $K \subseteq L \subseteq M$ then $\iota_{M \supseteq K} = \iota_{M \supseteq L} \circ \iota_{L \supseteq K}$. We write ι_K as a shorthand for $\iota_{\mathbb{N} \supseteq K}$. Also, given infinite sets $L, I \subseteq \mathbb{N}$, we write L_I for the infinite subset $\{l_i \mid i \in I\} \subseteq L$. Note that the identities

$$\iota_I = \iota_{L \supseteq L_I} \qquad\qquad \iota_{L_I} = \iota_L \circ \iota_I \tag{1}$$

hold. In fact, the second follows from the first.

The Grothendieck topology consists of all sieves that contain a *basic covering family* of one the following forms.

- The only basic cover of $\mathbf{1}$ is the singleton $\{\mathbf{1} \longrightarrow \mathbf{1}\}$.
- A family $\{B_i \xrightarrow{c_i} \mathbf{N}^\infty\}_{i \in I}$ of maps into $\mathbf{N}^\infty$ is a basic cover if:
    1. the functions $\{c_i\}_{i \in I}$ are jointly surjective, and
    2. there exists a collection $\mathcal{K}$ of infinite subsets of $\mathbb{N}$ satisfying:
        (a) for every infinite subset $M \subseteq \mathbb{N}$, there exist infinite $L \subseteq K \in \mathcal{K}$ such that $L \subseteq M$, and
        (b) for every $K \in \mathcal{K}$, there exists $i \in I$ such that $c_i = \iota_K$.

The above defines the canonical Grothendieck topology $\mathcal{J}_\mathcal{T}$ on the two object generating category $\mathbb{T}$. This is shown in detail in [Joh79], where the covering sieves of the topology are defined directly, avoiding a basis. Johnstone's *topological topos* [Joh79] is the category $\mathcal{T}$ of sheaves on the site $(\mathbb{T}, \mathcal{J}_\mathcal{T})$.

Theorem 1.3. *Johnstone's topological topos $\mathcal{T}$ satisfies dependent choice.*

2 Proof

We begin by introducing notation. We write $X_{\mathbf{1}}$ and $X_{\mathbf{N}^\infty}$ for the sets that make up an object X of the topological topos. Elements of $X_{\mathbf{1}}$ are in one-to-one correspondence with global points of X in $\mathcal{T}$, and we accordingly call such elements *points*. Any element $s \in X_{\mathbf{N}^\infty}$ determines a family of points $(s_i)_{i \leq \infty}$ via restriction along the maps $\mathbf{1} \xrightarrow{i} \mathbf{N}^\infty$ in $\mathbb{T}$, using the presheaf structure of X. Elements $s \in X_{\mathbf{N}^\infty}$ can be understood as specifying *convergences* $(s_n)_{n<\infty} \to s_\infty$; that is, convergent sequences together with their limits. However, there can be distinct $s, t \in X_{\mathbf{N}^\infty}$ for which $s_i = t_i$ for all $i \leq \infty$. As in [Joh79], one can view $X_{\mathbf{N}^\infty}$ as a set of 'proofs' s of convergences $(s_n)_n \to s_\infty$. We say that s *witnesses* the convergence $(s_n)_n \to s_\infty$.

A morphism $X \xrightarrow{f} Y$ in $\mathcal{T}$ is given by a pair of functions $f_{\mathbf{1}} \colon X_{\mathbf{1}} \to Y_{\mathbf{1}}$ and $f_{\mathbf{N}^\infty} \colon X_{\mathbf{N}^\infty} \to Y_{\mathbf{N}^\infty}$ that together give the components of a natural transformation. That is, for any map $c : A \longrightarrow B$ in $\mathbb{T}$ (so $A, B \in \{\mathbf{1}, \mathbf{N}^\infty\}$) and $x \in X_B$, it holds that

$$f_A(x \cdot c) = f_B(x) \cdot c \ ,$$

where we write $x \cdot c$ for the element of X_A obtained by restricting $x \in X_B$ along c using the presheaf structure of X.

Lemma 2.1. *A map $X \xrightarrow{f} Y$ in $\mathcal{T}$ is an epimorphism if and only if $f_{\mathbf{1}}$ is surjective and $f_{\mathbf{N}^\infty}$ satisfies:*

$$\text{for every } t \in Y_{\mathbf{N}^\infty}, \text{ there exists } s \in X_{\mathbf{N}^\infty} \text{ and infinite } K \subseteq \mathbb{N} \text{ s.t. } f_{\mathbf{N}^\infty}(s) = t \cdot \iota_K. \quad (2)$$

Proof. It is standard (see, e.g., [MLM94, Corollary III.7.5]) that epimorphisms in a Grothendieck topos are characterised by the property of *local surjectivity* relative to any defining site. That is, $X \xrightarrow{f} Y$ is an epimorphism if and only if for every object A in the underlying category of the site, it holds that

$$\text{for every } y \in Y_A, \text{ there exists a covering family } \{B_i \xrightarrow{c_i} A\}_{i \in I} \text{ and family } \{x_i \in X_{B_i}\}_{i \in I} \text{ such that } f_{B_i}(x_i) = y \cdot c_i \text{ for every } i \in I. \quad (3)$$

In the case of Johnstone's topological topos $\mathcal{T}$, when A is the object $\mathbf{1}$ of $\mathbb{T}$, it is immediate from the description of the Grothendieck topology $\mathcal{J}_{\mathcal{T}}$ that (3) is equivalent to the surjectivity of $f_{\mathbf{1}}$. Accordingly, we henceforth assume that $f_{\mathbf{1}}$ is surjective and show that (3) is equivalent to (2) when A is $\mathbf{N}^\infty$.

Suppose that A is $\mathbf{N}^\infty$ and (3) holds. To show (2), consider any $t \in Y_{\mathbf{N}^\infty}$. Using (3), let $\{B_i \xrightarrow{c_i} \mathbf{N}^\infty\}_{i \in I}$ be covering (generated by a family $\mathcal{K}$ of infinite subsets) with

$\{x_i \in X_{B_i}\}_{i\in I}$ such that $f_{B_i}(x_i) = t \cdot c_i$ for every $i \in I$. By the definition of covers in $\mathcal{J}_\mathcal{T}$ (one can take $M = \mathbb{N}$), there exists $K \in \mathcal{K}$ such that, for some $i \in I$, we have $c_i = \iota_K$. Thus $s = x_i$ and K are the data required by (2).

Conversely, suppose (2) holds for $A = \mathbf{N}^\infty$. To show (3), consider any $y \in Y_{\mathbf{N}^\infty}$ and define:

$$\mathcal{K} \;=\; \{K \subseteq \mathbb{N} \mid K \text{ infinite, there exists } x_K \in X_{\mathbf{N}^\infty} \text{ s.t. } f_{\mathbf{N}^\infty}(x_K) = y \cdot \iota_K\} \;\; .$$

We show that $\{\mathbf{1} \xrightarrow{i} \mathbf{N}^\infty\}_{i\leq\infty} \cup \{\mathbf{N}^\infty \xrightarrow{\iota_K} \mathbf{N}^\infty\}_{K\in\mathcal{K}}$ is covering. Joint surjectivity is immediate from the left half of the union. Also, since the right-hand part involves a set $\mathcal{K}$ satisfying (2b) in the definition of cover, we just need to show (2a). Accordingly, let $M \subseteq \mathbb{N}$ be infinite. By (2) using $t = y \circ \iota_M$, there exist $s \in X_{\mathbf{N}^\infty}$ and an infinite subset $K' \subseteq \mathbb{N}$ such that $f_{\mathbf{N}^\infty}(s) = y \cdot \iota_M \cdot \iota_{K'}$. So, defining $K = M_{K'}$, we have $K \subseteq M$ and $f_{\mathbf{N}^\infty}(s) = y \cdot \iota_K$, hence also $K \in \mathcal{K}$, establishing (2a) with $L = K$.

We have shown that $\{\mathbf{1} \xrightarrow{i} \mathbf{N}^\infty\}_{i\leq\infty} \cup \{\mathbf{N}^\infty \xrightarrow{\iota_K} \mathbf{N}^\infty\}_{K\in\mathcal{K}}$ is covering. By the surjectivity of $f_\mathbf{1}$, for any $i \leq \infty$, there exists $x_i \in X_\mathbf{1}$ such that $f_\mathbf{1}(x_i) = y \cdot i$. By the definition of $\mathcal{K}$, for every $K \in \mathcal{K}$, we have $x_K \in X_{\mathbf{N}^\infty}$ such that $f_{\mathbf{N}^\infty}(x_K) = y \cdot \iota_K$. This shows that the family $\{x_i \in X_\mathbf{1}\}_{i\leq\infty} \cup \{x_K \in X_{\mathbf{N}^\infty}\}_{K\in\mathcal{K}}$ enjoys the property required by (3). (The use of an uncountable instance of the axiom of choice in the definition of the family $\{x_K \in X_{\mathbf{N}^\infty}\}_{K\in\mathcal{K}}$ can be avoided by taking $\{(K, x) \mid f_{\mathbf{N}^\infty}(x) = y \cdot \iota_K\}$ as the index set instead of $\mathcal{K}$.) □

Proof of Theorem 1.3. Suppose that we have a sequence of epimorphisms in $\mathcal{T}$

$$\cdots \xrightarrow{e^3} X^3 \xrightarrow{e^2} X^2 \xrightarrow{e^1} X^1 \xrightarrow{e^0} X^0 \;\; .$$

Let $(L \xrightarrow{l^k} X^k)_{k\geq 0}$ be the limit of the above diagram. We need to show that every l^k is epimorphic. It suffices to show that l^0 is epimorphic, since then so is every l^k, by the same argument applied to the limit cone $(L \xrightarrow{l^{k'}} X^k)_{k'\geq k}$ of the truncated diagram $\cdots \xrightarrow{e^{k+1}} X^{k+1} \xrightarrow{e^k} X^k$.

Since limits in Grothendieck toposes are pointwise we have

$$L_\mathbf{1} = \left\{ (x^k)_{k\geq 0} \in \prod_{k\in\mathbb{N}} X^k_\mathbf{1} \mid \forall k \; e^k_\mathbf{1}(x^{k+1}) = x^k \right\} \qquad l^k_\mathbf{1}((x^n)_n) = x^k$$

$$L_{\mathbf{N}^\infty} = \left\{ (s^k)_{k\geq 0} \in \prod_{k\in\mathbb{N}} X^k_{\mathbf{N}^\infty} \mid \forall k \; e^n_{\mathbf{N}^\infty}(s^{k+1}) = s^k \right\} \qquad l^k_{\mathbf{N}^\infty}((s^n)_n) = s^k \;\; .$$

We show that l^0 satisfies the conditions of Lenma 2.1. The surjectivity of $l^0_{\mathbf{1}}$ holds by using surjectivity of every $e^k_{\mathbf{1}}$ and applying dependent choice in the meta-theory. It remains to show that $l^0_{\mathbf{N}\infty}$ satisfies property (2).

Consider any $t^0 \in X^0_{\mathbf{N}\infty}$. Applying property (2) to $e^0_{\mathbf{N}\infty}$, there exist $t^1 \in X^1_{\mathbf{N}\infty}$ and infinite $L^1 \subseteq \mathbb{N}$ such that $e^0_{\mathbf{N}\infty}(t^1) = t^0 \cdot \iota_{L^1}$. Iteratively, for every $k \geq 1$, given $t^k \in X^k_{\mathbf{N}\infty}$, we apply property (2) to $e^k_{\mathbf{N}\infty}$ to obtain $t^{k+1} \in X^{k+1}_{\mathbf{N}\infty}$ and infinite $L^{k+1} \subseteq \mathbb{N}$, such that $e^k_{\mathbf{N}\infty}(t^{k+1}) = t^k \cdot \iota_{L^{k+1}}$. By dependent choice in the meta-theory, the above gives us a sequence $(t^k)_k \in \prod_{k\in\mathbb{N}} X^k_{\mathbf{N}\infty}$ and a sequence $L^k_{k\geq 1}$ of infinite subsets of $\mathbb{N}$. We define a derived sequence of infinite subsets $(K^k)_k$ by $K^0 = \mathbb{N}$ and $K^{k+1} = K^k_{L^{k+1}}$. Clearly $K^0 \supseteq K^1 \supseteq K^2 \ldots$ is a descending sequence of sets. Also, by (1), we have $\iota_{L^{k+1}} = \iota_{K^k \supseteq K^{k+1}}$ and $\iota_{K^{k+1}} = \iota_{K^k} \circ \iota_{L^{k+1}}$.

We elucidate the above in terms of convergences. The starting convergence t^0 witnesses that $(t^0_n) \to t^0_\infty$ in X^0. Then t^1 witnesses that $(t^1_n)_n \to t^1_\infty$ in X^1, and the preservation of this convergence by e^0 gives us $(e^0_{\mathbf{1}}(t^1_n))_n \to e^0_{\mathbf{1}}(t^1_\infty)$ in X^0 witnessed by $e^0_{\mathbf{N}\infty}(t^1)$, i.e., by $t^0 \cdot \iota_{K^0}$. In general, for any $k \geq 0$, we write $X^k \xrightarrow{d^k} X^0$ for the composite $e^0 \circ e^1 \circ \cdots \circ e^{k-1}$ (so for example $d^0 = \mathsf{id}_{X^0}$ and $d^1 = e^0$). Then t^k witnesses that $(t^k_n)_n \to t^k_\infty$ in X^k, and the preservation of this convergence by d^k gives us a convergence $(d^k_{\mathbf{1}}(t^k_n))_n \to d^k_{\mathbf{1}}(t^k_\infty)$ in X^0 witnessed by $d^k_{\mathbf{N}\infty}(t^k)$ which is equal to $t^0 \cdot \iota_{K^k}$. That is, the convergence associated with $d^k_{\mathbf{N}\infty}(t^k)$ is the subconvergence of $(t^0_n) \to t^0_\infty$ obtained by restricting to the subsequence with indices from K^k.

Let $\{h^k_0, h^k_1, h^k_2, \ldots\}$ enumerate K^k in strictly ascending order. Since $K^{k+1} \subseteq K^k$, we have $h^k_n \leq h^{k+1}_n$, for all n. For each $k \geq 0$, define the diagonal set $D^k = \{h^m_m \mid m \geq k\}$. Then each D^k is an infinite subset of K^k in which h^{k+n}_{k+n} is the element with index n. For later convenience, we note the identity

$$
\begin{aligned}
\iota_{L^{k+1}} \circ \iota_{K^{k+1} \supseteq D^{k+1}} &= \iota_{K^k \supseteq K^{k+1}} \circ \iota_{K^{k+1} \supseteq D^{k+1}} && \text{by (1)} \\
&= \iota_{K^k \supseteq D^{k+1}} \\
&= \iota_{K^k \supseteq D^k} \circ \iota_{D^k \supseteq D^{k+1}} \\
&= \iota_{K^k \supseteq D^k} \circ \iota_{\{n \mid n \geq 1\}} \ , && (4)
\end{aligned}
$$

where the last equality holds because the element with index n in D^{k+1} is h^{k+1+n}_{k+1+n}, which has index $n+1$ in D^k.

We complete the proof that $l^0_{\mathbf{N}\infty}$ satisfies property (2) by constructing $(s^k)_k \in L_{\mathbf{N}\infty}$ such that $l^0_{\mathbf{N}\infty}((s^k)_k) = t^0 \cdot \iota_{D^0}$. Accordingly, define $s^0 = t^0 \cdot \iota_{D^0}$. It remains to extend s^0 to a sequence $(s^k)_k \in L_{\mathbf{N}\infty}$.

To help orientate the reader, we first give an informal description of the construction of $(s^k)_k$, and then follow with the formal treatment. We already have $s^0 \in X^0_{\mathbf{N}\infty}$

which witnesses the convergence $(t^0_{h^n_n})_n \to t^0_\infty$, which can be equivalently written as $\left(t^0_{\iota_{K^0 \supseteq D^0}(n)}\right)_n \to t^0_\infty$. Given s^k for $k \geq 0$, we define s^{k+1} so that the associated convergence $\left(s^k_n\right)_n \to s^k_\infty$ satisfies: for $n \leq k$, the point s^{k+1}_n is some chosen $x^{k+1}_n \in X^{k+1}_{\mathbf{1}}$ such that $e^k_{\mathbf{1}}(x^{k+1}_n) = s^k_n$ (such an element exists by the surjectivity of $e^k_{\mathbf{1}}$); for n with $k < n < \infty$, we have $s^{k+1}_n = t^{k+1}_{\iota_{K^{k+1} \supseteq D^{k+1}}(n-(k+1))}$; and $s^{k+1}_\infty = t^{k+1}_\infty$. The above properties imply that $e^k_{\mathbf{1}}(s^{k+1}_i) = s^k_i$, for all $i \leq \infty$. The formal definition of s^{k+1} below, which is given via the sheaf structure, implies the stronger property that $e^k_{\mathbf{N}^\infty}(s^{k+1}) = s^k$ holds. This ensures that the resulting sequence $(s^k)_k$ resides in $L_{\mathbf{N}^\infty}$.

Formally, we iteratively, for $k = 0, 1, \ldots$, define s^k together with $\{x^k_n \in X^k_{\mathbf{1}}\}_{n<k}$ such that: (i) $s^k_n = x^k_n$ for all $n < k$, and (ii) $s^k \cdot \iota_{\{n|n \geq k\}} = t^k \cdot \iota_{K^k \supseteq D^k}$. Note that (i) and (ii) together determine s, because they express that s is the amalgamation of $\{x^k_n \in X^k_{\mathbf{1}}\}_{n<k} \cup \{t^k \cdot \iota_{K^k \supseteq D^k} \in X^k_{\mathbf{N}^\infty}\}$, which is a matching family for the cover $\{\mathbf{1} \xrightarrow{n} \mathbf{N}^\infty\}_{n<k} \cup \{\mathbf{N}^\infty \xrightarrow{\iota_{\{n|n\geq k\}}} \mathbf{N}^\infty\}$. (As the cover is disjoint the matching property is vacuous, as will also be the case in all subsequent applications of the sheaf property in this proof.) In the case $k = 0$, the family $\{x^0_n \in X^0_{\mathbf{1}}\}_{n<0}$ is empty, and (i) and (ii) hold for the $s^0 = t^0 \cdot \iota_{D^0}$ because $K^0 = \mathbb{N}$ and $\iota_{\mathbb{N}}$ is the identity. In the case $k > 0$, using the surjectivity of $e^k_{\mathbf{1}}$, let $x^k_n \in X^k_{\mathbf{1}}$ be such that $e^{k-1}_{\mathbf{1}}(x^k_n) = s^{k-1}_n$, for every $n < k$. Define s^k to be the amalgamation of the family $\{x^k_n \in X^k_{\mathbf{1}}\}_{n<k} \cup \{t^k \cdot \iota_{K^k \supseteq D^k} \in X^k_{\mathbf{N}^\infty}\}$ with respect to the cover $\{\mathbf{1} \xrightarrow{n} \mathbf{N}^\infty\}_{n<k} \cup \{\mathbf{N}^\infty \xrightarrow{\iota_{\{n|n\geq k\}}} \mathbf{N}^\infty\}$. Then (i) and (ii) are satisfied by construction.

By dependent choice in the meta-theory, the above defines a sequence $(s^k)_k$. To see that this indeed lies in $L_{\mathbf{N}^\infty}$, we must show that $e^k_{\mathbf{N}^\infty}(s^{k+1}) = s^k$, for all k. By the characterisation of s^k via (i) and (ii), it is enough to show that $e^k_{\mathbf{N}^\infty}(s^{k+1})$ is an amalgamation of the matching family $\{x^k_n \in X^k_{\mathbf{1}}\}_{n<k} \cup \{t^k \cdot \iota_{K^k \supseteq D^k} \in X^k_{\mathbf{N}^\infty}\}$ for the cover $\{\mathbf{1} \xrightarrow{n} \mathbf{N}^\infty\}_{n<k} \cup \{\mathbf{N}^\infty \xrightarrow{\iota_{\{n|n\geq k\}}} \mathbf{N}^\infty\}$. When $n < k$, we have:

$$
\begin{aligned}
(e^k_{\mathbf{N}^\infty}(s^{k+1}))_n &= e^k_{\mathbf{1}}(s^{k+1}_n) && \text{naturality of } e^k \\
&= e^k_{\mathbf{1}}(x^{k+1}_n) && \text{property (i)} \\
&= s^k_n && \text{choice of } x^{k+1}_n \\
&= x^k_n \ .
\end{aligned}
$$

It remains to verify $e^k_{\mathbf{N}^\infty}(s^{k+1}) \cdot \iota_{\{n|n\geq k\}} = t^k \cdot \iota_{K^k \supseteq D^k}$. This holds because both sides restrict along the cover $\{\mathbf{1} \xrightarrow{0} \mathbf{N}^\infty\} \cup \{\mathbf{N}^\infty \xrightarrow{\iota_{\{n|n\geq 1\}}} \mathbf{N}^\infty\}$ to the same matching family;

that is, the two identities below hold.

$$(e^k_{\mathbf{N}^\infty}(s^{k+1}))_{\iota_{\{n|n\geq k\}}(0)} = t^k_{\iota_{K^k\supseteq D^k}(0)} \tag{5}$$

$$e^k_{\mathbf{N}^\infty}(s^{k+1}) \cdot \iota_{\{n|n\geq k\}} \cdot \iota_{\{n|n\geq 1\}} = t^k \cdot \iota_{K^k\supseteq D^k} \cdot \iota_{\{n|n\geq 1\}} \quad . \tag{6}$$

Indeed, (5) holds because

$$\begin{aligned}
(e^k_{\mathbf{N}^\infty}(s^{k+1}))_{\iota_{\{n|n\geq k\}}(0)} &= (e^k_{\mathbf{N}^\infty}(s^{k+1}))_k \\
&= e^k_{\mathbf{1}}(s^{k+1}_k) && \text{naturality of } e^k \\
&= e^k_{\mathbf{1}}(x^{k+1}_k) && \text{property (i)} \\
&= s^k_k && \text{choice of } x^{k+1}_k \\
&= t^k_{\iota_{K^k\supseteq D^k}(0)} && \text{property (ii)} \quad ,
\end{aligned}$$

and (6) because

$$\begin{aligned}
e^k_{\mathbf{N}^\infty}(s^{k+1}) \cdot \iota_{\{n|n\geq k\}} \cdot \iota_{\{n|n\geq 1\}} &= e^k_{\mathbf{N}^\infty}(s^{k+1}) \cdot \iota_{\{n|n>k\}} \\
&= e^k_{\mathbf{N}^\infty}(s^{k+1} \cdot \iota_{\{n|n>k\}}) && \text{naturality of } e^k \\
&= e^k_{\mathbf{N}^\infty}(t^{k+1} \cdot \iota_{K^{k+1}\supseteq D^{k+1}}) && \text{property (ii)} \\
&= e^k_{\mathbf{N}^\infty}(t^{k+1}) \cdot \iota_{K^{k+1}\supseteq D^{k+1}} && \text{naturality of } e^k \\
&= t^k \cdot \iota_{L^{k+1}} \cdot \iota_{K^{k+1}\supseteq D^{k+1}} && \text{definition of } t^{k+1} \\
&= t^k \cdot \iota_{K^k\supseteq D^k} \cdot \iota_{\{n|n\geq 1\}} && \text{by (4)} \quad .
\end{aligned}$$

□

References

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