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Lax Braidings and the Lax Centre

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ABSTRACT. The purpose of this work is to highlight the notions of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories. Lax centres are lax braided. Generally the centre is a full subcategory of the lax centre, however we show that it is sometimes the case that the two coincide. We identify lax centres of monoidal functor categories in various cases.

Introduction

Braidings for monoidal categories were introduced in [**JS1**] and its forerunners. The centre $\mathcal{Z}\mathscr{X}$ of a monoidal category $\mathscr{X}$ was introduced in [**JS0**] in the process of proving that the free tortile monoidal category has another universal property. The centre of a monoidal category is a braided monoidal category. What we now call lax braidings were considered tangentially by Yetter [**Yet**]. What we now call the lax centre $\mathcal{Z}_l\mathscr{X}$ of $\mathscr{X}$ was considered under the name "weak centre" by P. Schauenburg [**Sch**]. The purpose of this work is to highlight the notions of lax braiding and lax centre for monoidal categories $\mathscr{X}$ and more generally for promonoidal categories $\mathscr{C}$. Lax centres turn out to be lax braided monoidal categories. Generally the centre is a full subcategory of the lax centre, however it is sometimes the case that the two coincide. We have two such theorems under different hypotheses, one in the case sufficient dual objects exist in the additive context, and the other in the cartesian context. For a promonoidal category $\mathscr{C}$, we relate the lax centre of the [**Day**] convolution on $\mathscr{C}$ to the convolution on the lax centre of $\mathscr{C}$. Indeed, sometimes these are equivalent. One reason for being interested in the lax centre of $\mathscr{X}$ is that, if an object X of $\mathscr{X}$ is equipped with the structure of monoid in $\mathcal{Z}_l\mathscr{X}$, then tensoring with X defines a monoidal endofunctor $-\otimes X$ of $\mathscr{X}$; this has applications in cases where the lax centre can be explicitly identified.

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1. Lax braidings for promonoidal categories

Let $\mathscr{V}$ denote a complete cocomplete symmetric closed monoidal category and let $\mathscr{C}$ be a $\mathscr{V}$-enriched category in the sense of [**Kel**]. A *promagmal structure* on $\mathscr{C}$ consists of two $\mathscr{V}$-functors $P : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{V}$ and $J : \mathscr{C} \longrightarrow \mathscr{V}$ (called the *protensor product* and *prounit*). Recall from [**Day**] that a *promonoidal structure* on $\mathscr{C}$ is a promagmal structure equipped further with $\mathscr{V}$-natural isomorphisms

$$\int^U P(U,C;D) \otimes P(A,B;U) \xrightarrow{\text{assoc}} \int^V P(A,V;D) \otimes P(B,C;V)$$

$$\int^U P(U,A;B) \otimes J(U) \xrightarrow{\text{lun}} \mathscr{C}(A,B) \quad \text{and} \quad \int^V P(A,V;B) \otimes J(V) \xrightarrow{\text{run}} \mathscr{C}(A,B)$$

(called the *associativity, left unit* and *right unit constraints*) satisfying two coherence conditions.

The importance of promonoidal structures on $\mathscr{C}$ lies in their equivalence to (left- and -right-) closed monoidal structures on the $\mathscr{V}$-functor category $[\mathscr{C}, \mathscr{V}]$. Given a promonoidal structure on $\mathscr{C}$, we obtain a closed monoidal structure on $[\mathscr{C}, \mathscr{V}]$ where the tensor product $*$ is defined by the *convolution formula*

$$(M * N)C = \int^{X,Y} P(X,Y;C) \otimes MX \otimes NY$$

and the unit is J. Conversely, given a closed monoidal structure on $[\mathscr{C}, \mathscr{V}]$, we obtain a promonoidal structure on $\mathscr{C}$ by defining

$$P(A,B;C) = (\mathscr{C}(A,-) * \mathscr{C}(B,-))C$$

and taking the unit as the prounit.

By way of example, every monoidal structure on $\mathscr{C}$ determines a promonoidal one by defining $P(A,B;C) = \mathscr{C}(B \otimes A, C)$ and $JC = \mathscr{C}(I,C)$. Another example, for any comonoidal $\mathscr{C}$, is defined by $P(A,B;C) = \mathscr{C}(B,C) \otimes \mathscr{C}(A,C)$ and $JC = I$; the comonoidal structure includes $\mathscr{V}$-functors $\mathscr{C} \longrightarrow \mathscr{C} \otimes \mathscr{C}$ and $\mathscr{C} \longrightarrow I$ which are used to make P and J into $\mathscr{V}$-functors in the C variable. These two examples agree in case $\mathscr{V} = \mathbf{Set}$(so that every $\mathscr{C}$ is comonoidal) and the monoidal structure on $\mathscr{C}$ is coproduct.

Symmetries for promonoidal structures were defined by [**Day**] and braidings by [**JS1**]. We generalize this slightly. A *lax braiding* for a promonoidal structure on $\mathscr{C}$ is a $\mathscr{V}$-natural family of morphisms $c_{A,B;C} : P(A,B;C) \longrightarrow P(B,A;C)$ such that the following four diagrams commute.

$$\begin{tikzcd}
\int^U P(U,C;D) \otimes P(A,B;U) \arrow[r, "\int^U c\otimes 1"] \arrow[d, "\text{assoc}"'] & \int^U P(C,U;D) \otimes P(A,B;U) \\
\int^V P(A,V;D) \otimes P(B,C;V) \arrow[d, "\int^V 1\otimes c"'] & \int^W P(W,B;D) \otimes P(C,A;W) \arrow[u, "\text{assoc}"'] \\
\int^V P(A,V;D) \otimes P(C,B;V) \arrow[r, "\text{assoc}^{-1}"] & \int^W P(W,B;D) \otimes P(A,C;W) \arrow[u, "\int^W 1\otimes c"']
\end{tikzcd}$$

$$
\begin{tikzcd}
\int^{V} P(A,V;D)\otimes P(B,C;V) \arrow[r, "\int^{V} c\otimes 1"] \arrow[d, "\mathrm{assoc}^{-1}"'] & \int^{V} P(V,A;D)\otimes P(B,C;V) \\
\int^{U} P(U,C;D)\otimes P(A,B;U) \arrow[d, "\int^{U} 1\otimes c"'] & \int^{W} P(B,W;D)\otimes P(C,A;W) \arrow[u, "\mathrm{assoc}^{-1}"'] \\
\int^{U} P(U,C;D)\otimes P(B,A;U) \arrow[r, "\mathrm{assoc}"] & \int^{W} P(B,W;D)\otimes P(A,C;W) \arrow[u, "\int^{W} 1\otimes c"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\int^{U} P(U,A;B)\otimes JU \arrow[rr, "\int^{U} c\otimes 1"] \arrow[dr, "\mathrm{lun}"'] & & \int^{U} P(A,U;B)\otimes JU \arrow[dl, "\mathrm{run}"] \\
& \mathscr{C}(A,B) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\int^{U} P(A,U;B)\otimes JU \arrow[rr, "\int^{U} c\otimes 1"] \arrow[dr, "\mathrm{run}"'] & & \int^{U} P(U,A;B)\otimes JU \arrow[dl, "\mathrm{lun}"] \\
& \mathscr{C}(A,B) &
\end{tikzcd}
$$

A *braiding* is a lax braiding for which each $c_{A,B;C} : P(A,B;C) \longrightarrow P(B,A;C)$ is invertible. In particular, by regarding a monoidal category as a promonoidal one in the manner described above, we obtain the notion of lax braiding and braiding for a monoidal category; by Yoneda's Lemma in this case, we can regard the lax braiding as a morphism $c_{A,B} : A \otimes B \longrightarrow B \otimes A$ satisfying four conditions; then $c_{A,B;C} : \mathscr{C}(B \otimes A, C) \longrightarrow \mathscr{C}(A \otimes B, C)$ is $\mathscr{C}(c_{A,B}, C)$.

We can easily adjust the results of [**Day**] on symmetries to obtain the following for lax braidings.

PROPOSITION 1.1. *Let $\mathscr{C}$ be a promonoidal $\mathscr{V}$-category and regard $[\mathscr{C},\mathscr{V}]^{op}$, under the convolution monoidal structure, as promonoidal. Then the Yoneda embedding $Y : \mathscr{C} \longrightarrow [\mathscr{C},\mathscr{V}]^{op}$ preserves promonoidal structures. Moreover, there is a bijection between lax braidings on $\mathscr{C}$ and those on $[\mathscr{C},\mathscr{V}]^{op}$ defined by the requirement that Y should preserve lax braidings; the bijection restricts to braidings and to symmetries.*

EXAMPLE 1.2. Let $\mathscr{V}$ be the monoidal category of vector spaces over the complex number field $\mathbf{k}$. Let $\mathscr{A}$ be an abelian category. We write $\mathscr{A}_g$ for the subcategory of $\mathscr{A}$ with the same objects but with only the invertible morphisms. We write $\mathbf{k}_*\mathscr{A}_g$ for the free $\mathscr{V}$-category on $\mathscr{A}_g$; it has the same objects as $\mathscr{A}_g$ and its hom vector spaces have the homs of $\mathscr{A}_g$ as bases. A promonoidal structure on $\mathbf{k}_*\mathscr{A}_g$ is obtained by defining $P(A,B;C)$ to have basis

$$\{(f,g) \mid 0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0 \text{ is a short exact sequence in } \mathscr{A}\}$$

and defining

$$JC = \begin{cases} \mathbf{k} & \text{for } C = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The associativity constraints come from contemplation of the following 3×3 diagram of short exact sequences.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\ D \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & V \arrow[r] & C \end{tikzcd}$$

A lax braiding is obtained by defining $c_{A,B;C} : P(A,B;C) \longrightarrow P(B,A;C)$ to take the basis element (f,g) to the sum of all those pairs (h,k) such that

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "k"] \arrow[r, shift left, "g"] & B \arrow[l, shift left, "h"] \end{tikzcd}$$

is a direct sum situation; the abelian category $\mathscr{A}$ must be restricted so that this sum is finite. This lax braiding is generally not invertible; however, in the case where $\mathscr{A}$ is the category of finite vector spaces over a fixed finite field, it was shown in [**JS3**] that it is a braiding.

In the presence of duals, various unexpected things can be proved invertible; see [**JS2**, Section 10, Proposition 8], [**Yet**, Proposition 7.1], and [**JS1**, Propositions 7.1 and 7.4].

PROPOSITION 1.3. *If $\mathscr{C}$ is a right autonomous (meaning that each object has a right dual) monoidal category then any lax braiding on $\mathscr{C}$ is necessarily a braiding.*

PROOF. If B has right dual C then the mate of $c_{A,C}$ is an inverse for $c_{A,B}$. While the proof of this is in [**JS2**, Section 10, Proposition 8], we shall repeat it below squeezing out a little more in the form of our Proposition 3.1 below. $\square$

We use the terminology of [**Kel**] so that a monoidal functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is equipped with a natural family of morphisms $FA \otimes FB \longrightarrow F(A \otimes B)$ and a morphism $I \longrightarrow FI$; these morphisms satisfy coherence conditions but are not necessarily invertible; in the case where they are all invertible we say the monoidal functor is *strong*.

PROPOSITION 1.4. *Any lax braiding of a monoidal $\mathscr{V}$-category $\mathscr{C}$ equips the tensor product $\mathscr{V}$-functor $\otimes : \mathscr{C} \otimes \mathscr{C} \longrightarrow \mathscr{C}$ with a monoidal structure. Since monoidal functors preserve monoids, it follows that the tensor product of two monoids in $\mathscr{C}$ is again a monoid.*

2. The lax centre of a promonoidal category

For each promonoidal $\mathscr{V}$-category $\mathscr{C}$, we shall construct a promagmal $\mathscr{V}$-category $\mathscr{Z}_l\mathscr{C}$ which we call the *(left) lax centre* of $\mathscr{C}$. It is quite often canonically promonoidal in which case it is lax braided.

The objects of $\mathscr{Z}_l\mathscr{C}$ are pairs (A, α) where A is an object of $\mathscr{C}$ and α is a $\mathscr{V}$-natural family of morphisms $\alpha_{X;Y} : P(A, X; Y) \longrightarrow P(X, A; Y)$ such that the following two diagrams commute.

$$\begin{tikzcd}
\int^V P(A,V;Z)\otimes P(X,Y;V) \arrow[r, "\int^V \alpha\otimes 1"] \arrow[d, "\mathrm{assoc}^{-1}"'] & \int^V P(V,A;Z)\otimes P(X,Y;V) \\
\int^U P(U,Y;Z)\otimes P(A,X;U) \arrow[d, "\int^U 1\otimes\alpha"'] & \int^W P(X,W;Z)\otimes P(Y,A;W) \arrow[u, "\mathrm{assoc}^{-1}"'] \\
\int^U P(U,Y;Z)\otimes P(X,A;U) \arrow[r, "\mathrm{assoc}"] & \int^W P(X,W;Z)\otimes P(A,Y;W) \arrow[u, "\int^W 1\otimes\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
\int^U P(A,U;X)\otimes JU \arrow[rr, "\int^U \alpha\otimes 1"] \arrow[dr, "\mathrm{run}"'] & & \int^U P(U,A;X)\otimes JU \arrow[dl, "\mathrm{lun}"] \\
& \mathscr{C}(A,X) &
\end{tikzcd}$$

The hom object $\mathscr{Z}_l\mathscr{C}((A,\alpha),(B,\beta))$ is defined to be the equalizer in $\mathscr{V}$ of the two composed paths around the following square.

$$\begin{tikzcd}
\mathscr{C}(A,B) \arrow[r, "P"] \arrow[d, "P"'] & \int_{X,Y}[P(B,X;Y),P(A,X;Y)] \arrow[d, "{[1,\alpha]}"] \\
\int_{X,Y}[P(X,B;Y),P(X,A;Y)] \arrow[r, "{[\beta,1]}"'] & \int_{X,Y}[P(B,X;Y),P(X,A;Y)]
\end{tikzcd}$$

Composition in $\mathscr{Z}_l\mathscr{C}$ is defined so that we have the obvious faithful $\mathscr{V}$-functor $\mathscr{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ taking (A,α) to A.

The promagmal structure on $\mathscr{Z}_l\mathscr{C}$ is defined by taking $P((A,\alpha),(B,\beta);(C,\gamma))$ to be the equalizer of the two composed paths around the following square in which the top and left sides are transforms under the tensor-hom adjunction of the associativity constraint and its inverse.

$$\begin{tikzcd}
P(A,B;C) \arrow[r] \arrow[d] & \left[P(C,Y;Z),\int^X P(A,X;Z)\otimes P(B,Y;X)\right] \arrow[d, "{[1,\int^X \alpha\otimes\beta]}"] \\
\left[P(Y,C;Z),\int^X P(X,A;Z)\otimes P(Y,B;X)\right] \arrow[r, "{[\gamma,1]}"'] & \left[P(C,Y;Z),\int^X P(X,A;Z)\otimes P(Y,B;X)\right]
\end{tikzcd}$$

We take $J(A,\alpha)$ to be the equalizer of the two legs around the following triangle in which the top side and left side come from the unit constraints.

$$\begin{tikzcd}
JA \arrow[rr] \arrow[dr] & & \left[P(A,X;Y),\mathscr{C}(X,Y)\right] \\
& \left[P(X,A;Y),\mathscr{C}(X,Y)\right] \arrow[ur, "{[\alpha_{X;Y},1]}"'] &
\end{tikzcd}$$

It is frequently the case that $\mathcal{Z}_l\mathscr{C}$ is promonoidal in such a way that the forgetful $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong promonoidal. For example, if $\mathscr{C}$ is monoidal then so too is $\mathcal{Z}_l\mathscr{C}$ and $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong monoidal.

The lax braiding on $\mathcal{Z}_l\mathscr{C}$ is defined by taking the unique $c = c_{(A,\alpha),(B,\beta);(C,\gamma)}$ such that the following square commutes.

$$\begin{tikzcd}
P((A,\alpha),(B,\beta);(C,\gamma)) \arrow[r, "\text{equalizer}"] \arrow[d, "c"'] & P(A,B;C) \arrow[d, "\alpha"] \\
P((B,\beta),(A,\alpha);(C,\gamma)) \arrow[r, "\text{equalizer}"'] & P(B,A;C)
\end{tikzcd}$$

The *centre* of $\mathscr{C}$ is the full sub-$\mathscr{V}$-category $\mathcal{Z}\mathscr{C}$ of $\mathcal{Z}_l\mathscr{C}$ consisting of the objects (A,α) for which each $\alpha_{X;Y} : P(A,X;Y) \longrightarrow P(X,A;Y)$ is invertible.

There is a fully faithful $\mathscr{V}$-functor $\Psi : (\mathcal{Z}_l\mathscr{C})^{\mathrm{op}} \longrightarrow \mathcal{Z}_l[\mathscr{C},\mathscr{V}]$ defined by

$$\Psi(A,\alpha) = \Big(\mathscr{C}(A,-), \mathscr{C}(A,-) * F \xrightarrow{\theta_F} F * \mathscr{C}(A,-)\Big)$$

where

$$\theta_F = \left(\int^U P(A,U;-)\otimes FU \xrightarrow{\int^U \alpha_{U;-}\otimes 1_{FU}} \int^U P(U,A;-)\otimes FU\right).$$

In fact, the promagmal structure on $\mathcal{Z}_l\mathscr{C}$ is obtained by restriction along Ψ of the promonoidal (actually monoidal) structure on $\mathcal{Z}_l[\mathscr{C},\mathscr{V}]$. The following diagram of $\mathscr{V}$-functors and $\mathscr{V}$-categories is a pullback.

$$\begin{tikzcd}
(\mathcal{Z}_l\mathscr{C})^{\mathrm{op}} \arrow[r, "\Psi"] \arrow[d] & \mathcal{Z}_l[\mathscr{C},\mathscr{V}] \arrow[d] \\
\mathscr{C}^{\mathrm{op}} \arrow[r, "\text{Yoneda}"'] & [\mathscr{C},\mathscr{V}]
\end{tikzcd}$$

The $\mathscr{V}$-functor Ψ induces an adjunction

$$\mathscr{Z}_l[\mathscr{C},\mathscr{V}] \underset{\tilde{\Psi}}{\overset{\hat{\Psi}}{\rightleftarrows}} [\mathscr{Z}_l\mathscr{C},\mathscr{V}]$$

defined by

$$\widehat{\Psi}(G) = \int^{(A,\alpha)} G(A,\alpha)\otimes\Psi(A,\alpha) \quad \text{and} \quad \widetilde{\Psi}(F,\theta)(A,\alpha) = \mathscr{Z}_l[\mathscr{C},\mathscr{V}](\Psi(A,\alpha),(F,\theta));$$

this last object can be obtained as the equalizer of two morphisms out of $F(A)$. In later sections we shall see that this adjunction can be a lax-braided monoidal equivalence.

3. The lax centre of a monoidal category

Let $\mathscr{C}$ denote a monoidal $\mathscr{V}$-category. The *lax centre* $\mathscr{Z}_l\mathscr{C}$ of $\mathscr{C}$ is the lax centre of $\mathscr{C}$ as a promonoidal category with promonoidal structure defined by

$$JC = \mathscr{C}(I,C) \qquad \text{and} \qquad P(A,B;C) = \mathscr{C}(B\otimes A, C).$$

Using the Yoneda lemma, we identify objects of $\mathscr{Z}_l\mathscr{C}$ with pairs (A,u) where A is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms $u_B : A\otimes B \longrightarrow B\otimes A$ such that the following two diagrams commute.

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "u_{B\otimes C}"] \arrow[dr, "u_B\otimes 1_C"'] & & B\otimes C\otimes A \\
& B\otimes A\otimes C \arrow[ur, "1_B\otimes u_C"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[rr, "u_I"] \arrow[dr, "\cong"'] & & I\otimes A \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

In the case where $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C}$ is monoidal, the lax centre of $\mathscr{C}$, under the name "(left) weak centre", was used in Section 4 of [**Sch**] where it is shown to be related to Yetter-Drinfeld modules.

We shall see that the lax centre can be equal to the centre. As a preliminary to this, we note the following result which implies Proposition 1.3 since every object of a lax braided monoidal category is equipped with a canonical structure of object in the lax centre.

Proposition 3.1. *If (A,u) is an object of the lax centre of a monoidal $\mathscr{V}$-category $\mathscr{C}$ and X is an object of $\mathscr{C}$ with a right dual X^* then the mate of u_{X^*} : $A\otimes X^* \longrightarrow X^*\otimes A$ is an inverse for $u_X : A\otimes X \longrightarrow X\otimes A$.*

Proof. The mate of u_{X^*} is the composite

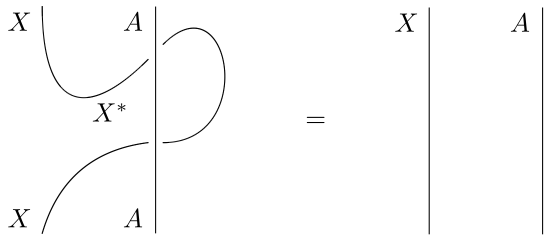
$$X\otimes A \xrightarrow{1_X\otimes 1_A\otimes\eta} X\otimes A\otimes X^*\otimes X \xrightarrow{1_X\otimes u_{X^*}\otimes 1_X} X\otimes X^*\otimes A\otimes X \xrightarrow{\varepsilon\otimes 1_A\otimes 1_X} A\otimes X$$

where η and ε are the unit and the counit for the duality $X \dashv X^*$. The proof that this is a right inverse uses the naturality of u with respect to the morphism

$\eta : I \longrightarrow X^* \otimes X$ and the axioms for u_I and $u_{X^*\otimes X}$:

$$\begin{tikzcd}
X \otimes A \arrow[r, "1_X\otimes 1_A\otimes\eta"] \arrow[dr, "1_X\otimes\eta\otimes 1_A"] \arrow[dd, "1_{X\otimes A}"'] & X\otimes A\otimes X^*\otimes X \arrow[d, "1_X\otimes u_{X^*\otimes X}"] \arrow[dr, "1_X\otimes u_{X^*}\otimes 1_X"] & \\
 & X\otimes X^*\otimes X\otimes A \arrow[dl, "\epsilon\otimes 1_X\otimes 1_A"] & X\otimes X^*\otimes A\otimes X \arrow[l, "1_X\otimes 1_{X^*}\otimes u_X"'] \arrow[dl, "\epsilon\otimes 1_A\otimes 1_X"] \\
X\otimes A & A\otimes X \arrow[l, "u_X"] &
\end{tikzcd}$$

Alternatively, we can prove it using string diagrams:



Similarly, the proof that the mate of u_{X^*} is a left inverse uses the naturality of u with respect to the morphism $\varepsilon : X^*\otimes X \longrightarrow I$ and the axioms for u_I and $u_{X\otimes X^*}$. $\square$

Proposition 3.2. *Suppose $\mathscr{F}$ is a monoidal $\mathscr{V}$-category such that, for each object F, the functor $F\otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserves (weighted) colimits. If $K : \mathscr{C} \longrightarrow \mathscr{F}$ is a dense $\mathscr{V}$-functor then, for each object F of $\mathscr{F}$ and endo-$\mathscr{V}$-functor T of $\mathscr{F}$, restriction along K provides a bijection between $\mathscr{V}$-natural transformations*

$$u : F\otimes - \Rightarrow T : \mathscr{F} \longrightarrow \mathscr{F}$$

and $\mathscr{V}$-natural transformations

$$t : F\otimes K- \Rightarrow TK- : \mathscr{C} \longrightarrow \mathscr{F}.$$

The components of u are induced on colimits by the components of the corresponding t; so that, if t is invertible, so is u.

Proof. The density of K means that each M in $\mathscr{F}$ is the $\mathscr{F}(K-,M)$-weighted colimit $\mathrm{colim}(\mathscr{F}(K-,M),K)$ of K. Since $F\otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserves colimits, we have

$$F\otimes M \cong \mathrm{colim}(\mathscr{F}(K-,M), F\otimes K-).$$

It follows that $\mathscr{V}$-natural families of morphisms $u_M : F\otimes M \longrightarrow TM$ are in bijection with $\mathscr{V}$-natural families of morphisms $\mathscr{F}(K-,M) \longrightarrow \mathscr{F}(F\otimes K-, TM)$ which, by Yoneda, are in bijection with $\mathscr{V}$-natural families of morphisms $t_A : F\otimes KA \longrightarrow TKA$. $\square$

Proposition 3.3. *Suppose $\mathscr{F}$ is a monoidal $\mathscr{V}$-category such that, for each object F, the functors $-\otimes F$ and $F\otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserve (weighted) colimits.*

If $K : \mathscr{C} \longrightarrow \mathscr{F}$ is a dense $\mathscr{V}$-functor and $u : F \otimes - \Rightarrow - \otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is a $\mathscr{V}$-natural transformation then, in order for the triangle

$$\begin{tikzcd} F \otimes M \otimes N \arrow[rr, "u_{M\otimes N}"] \arrow[dr, "u_m \otimes 1_N"'] & & M \otimes N \otimes F \\ & M \otimes F \otimes N \arrow[ur, "1_M \otimes u_N"'] & \end{tikzcd}$$

to commute for all M and N in $\mathscr{F}$, it suffices that it commute for all M and N equal to values of K.

Proof. Using the density of K and the colimit preservation properties of the tensor, we have an isomorphism

$$F \otimes M \otimes N \cong \int^{A,B} \mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes F \otimes KA \otimes KB$$

which is $\mathscr{V}$-natural in M and N. There are two similar isomorphisms for the other two vertices of the triangle in the proposition. By $\mathscr{V}$-naturality, the triangle itself transports across the isomorphisms to the triangle

$$\begin{tikzcd} \int^{A,B}\mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes F \otimes KA \otimes KB \arrow[rr, "\int^{A,B} 1 \otimes 1 \otimes u_{KA\otimes KB}"] \arrow[dr, "\int^{A,B} 1\otimes 1\otimes u_{KA}\otimes 1"'] & & \int^{A,B}\mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes KA \otimes KB \otimes F \\ & \int^{A,B}\mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes KA \otimes F \otimes KB \arrow[ur, "\int^{A,B} 1\otimes 1\otimes 1\otimes u_{KB}"'] & \end{tikzcd}$$

which commutes since it is induced on colimits by triangles that commute by hypothesis. So the triangle of the proposition commutes. □

Theorem 3.4. *Suppose $\mathscr{F}$ is a monoidal $\mathscr{V}$-category such that, for each object F, the functor $F \otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserves (weighted) colimits. If the full sub-$\mathscr{V}$-category of $\mathscr{F}$ consisting of the objects with right duals is dense in $\mathscr{F}$ then the lax centre of $\mathscr{F}$ is equal to the centre: $\mathcal{Z}_l\mathscr{F} = \mathcal{Z}\mathscr{F}$.*

Proof. Let $\mathscr{C}$ be the full sub-$\mathscr{V}$-category of $\mathscr{F}$ consisting of the objects with right duals, and let K denote the inclusion. Suppose (F, u) is an object of the lax centre of $\mathscr{F}$. Let t correspond to u under the bijection of Proposition 3.2. By Proposition 3.1, t is invertible. By Proposition 3.2, u is invertible so that (F, u) is in the centre of $\mathscr{F}$. □

Corollary 3.5. *For any Hopf algebra H, the lax centre of the monoidal category* **Comod**H *of left H-comodules is equal to its centre.*

Proof. For any coalgebra H, every comodule is the directed union of its finite dimensional subcomodules (see Section 7, Proposition 1 of [**JS2**]). It follows that the comodules which are finite dimensional (as vector spaces) are dense in the category **Comod**H . The bialgebra structure on H provides the monoidal structure on **Comod**H which is preserved by the underlying functor into vector spaces. Since

H is a Hopf algebra, the objects of $\mathbf{Comod}H$ with right duals are those whose underlying vector spaces are finite dimensional (see Section 9, Proposition 4 of [**JS2**]). So Theorem 3.4 applies. $\square$

COROLLARY 3.6. *For any finite dimensional Hopf algebra H, the lax centre of the monoidal category $\mathbf{Mod}H$ of left H-modules is equal to its centre.*

PROOF. Since Yoneda embeddings are dense, the object H of $\mathbf{Mod}H$ (where the action is the algebra multiplication) is dense in $\mathbf{Mod}H$. Since H is finite dimensional, it has a right dual in $\mathbf{Mod}H$. So the objects of $\mathbf{Mod}H$ with right duals are dense and Theorem 3.4 applies. $\square$

THEOREM 3.7. *Suppose an object F of a monoidal $\mathscr{V}$-category $\mathscr{F}$ is equipped with the structure of monoid in the lax centre $\mathscr{Z}_l\mathscr{F}$ of $\mathscr{F}$. Then $-\otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is equipped with the structure of monoidal $\mathscr{V}$-functor.*

PROOF. Let (F, u) be a monoid in $\mathscr{Z}_l\mathscr{F}$. So we have a monoid structure on F with multiplication $\mu : F\otimes F \longrightarrow F$ and unit $\eta : I \longrightarrow F$ such that the following two diagrams commute.

$$\begin{tikzcd}
F\otimes F\otimes X \arrow[r, "1\otimes u_X"] \arrow[d, "\mu\otimes 1"'] & F\otimes X\otimes F \arrow[r, "u_X\otimes 1"] & X\otimes F\otimes F \arrow[d, "1\otimes\mu"] \\
F\otimes X \arrow[rr, "u_X"'] & & X\otimes F
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes X \arrow[r, "u_I", "\cong"'] \arrow[d, "\eta\otimes 1"'] & X\otimes I \arrow[d, "1\otimes\eta"] \\
F\otimes X \arrow[r, "u_X"'] & X\otimes F
\end{tikzcd}$$

The monoidal structure on the functor $-\otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is defined as follows: $\phi_0 : I \longrightarrow F$ is equal to η and $\phi_{2;X,Y} : X\otimes F\otimes Y\otimes F \longrightarrow X\otimes Y\otimes F$ is the composite

$$X\otimes F\otimes Y\otimes F \xrightarrow{1\otimes u_Y\otimes 1} X\otimes Y\otimes F\otimes F \xrightarrow{1\otimes 1\otimes\mu\otimes 1\otimes 1} X\otimes Y\otimes F.$$

The following diagrams commute:

$$\begin{tikzcd}
X\otimes F\otimes Y\otimes F\otimes Z\otimes F \arrow[r, "1\otimes u_Y\otimes 1\otimes 1\otimes 1"] \arrow[dd, "1\otimes 1\otimes 1\otimes u_Z\otimes 1"'] & X\otimes Y\otimes F\otimes F\otimes Z\otimes F \arrow[r, "1\otimes 1\otimes\mu\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes 1\otimes u_Z\otimes 1"'] & X\otimes Y\otimes F\otimes Z\otimes F \arrow[dd, "1\otimes 1\otimes u_Z\otimes 1"] \\
& X\otimes Y\otimes F\otimes Z\otimes F\otimes F \arrow[dd, "1\otimes 1\otimes u_Z\otimes 1\otimes 1"] & \\
X\otimes F\otimes Y\otimes Z\otimes F\otimes F \arrow[ur, "1\otimes u_Y\otimes 1\otimes 1\otimes 1"] \arrow[dr, "1\otimes u_{Y\otimes Z}\otimes 1\otimes 1"'] \arrow[dd, "1\otimes 1\otimes 1\otimes 1\otimes\mu"'] & & X\otimes Y\otimes Z\otimes F\otimes F \arrow[dd, "1\otimes 1\otimes 1\otimes\mu"] \\
& X\otimes Y\otimes Z\otimes F\otimes F\otimes F \arrow[ur, "1\otimes 1\otimes 1\otimes\mu\otimes 1"] \arrow[d, "1\otimes 1\otimes 1\otimes 1\otimes\mu"] & \\
X\otimes F\otimes Y\otimes Z\otimes F \arrow[r, "1\otimes u_{Y\otimes Z}\otimes 1"'] & X\otimes Y\otimes Z\otimes F\otimes H \arrow[r, "1\otimes 1\otimes 1\otimes\mu"'] & X\otimes Y\otimes Z\otimes F
\end{tikzcd}$$

$$
\begin{tikzcd}
X \otimes F \otimes F \arrow[rr, "1\otimes\mu"] & & X \otimes F \\
& X \otimes F \arrow[ul, "1\otimes 1\otimes\eta"] \arrow[ur, "1\otimes 1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F \otimes Y \otimes F \arrow[r, "u_Y\otimes 1"] & Y \otimes F \otimes F \arrow[r, "1\otimes\mu"] & Y \otimes F \\
& Y \otimes F \arrow[ul, "\eta\otimes 1\otimes 1"] \arrow[u, "1\otimes\eta\otimes 1"'] \arrow[ur, "1\otimes 1"'] &
\end{tikzcd}
$$

which completes the proof. □

4. The cartesian example

For this section we take $\mathscr{V} = \mathbf{Set}$ and study the lax centre of any category $\mathscr{C}$ equipped with the promonoidal structure defined by $P(A, B; C) = \mathscr{C}(B, C) \times \mathscr{C}(A, C)$ and $JC = 1$. Then the corresponding convolution monoidal structure on the functor category $[\mathscr{C}, \mathbf{Set}]$ is none other than (pointwise cartesian) product.

Consider an object (A, α) of $\mathcal{Z}_l\mathscr{C}$. In order that the natural family of morphisms

$$\alpha_{X;Y} : \mathscr{C}(X, Y) \times \mathscr{C}(A, Y) \longrightarrow \mathscr{C}(A, Y) \times \mathscr{C}(X, Y)$$

should satisfy the second condition for an object of $\mathcal{Z}_l\mathscr{C}$, it must be determined by its second projection; that is,

$$\alpha_{X;Y}(f, g) = (g, \overline{\alpha}_{X;Y}(f, g))$$

for a unique natural family of morphisms

$$\overline{\alpha}_{X;Y} : \mathscr{C}(X, Y) \times \mathscr{C}(A, Y) \longrightarrow \mathscr{C}(X, Y)$$

The first condition on α then follows automatically from naturality. Now we can apply the Yoneda Lemma to see that such families $\overline{\alpha}$ are in bijection with dinatural transformations ϕ (in the sense of [**DuSt**]) from the representable functor $\mathscr{C}(A, -)$, thought of as constant in a contravariant variable, to the hom functor $\mathscr{C}(-, \sim) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathbf{Set}$ of $\mathscr{C}$. In other words, we have a family ϕ of functions $\phi_X : \mathscr{C}(A, X) \longrightarrow \mathscr{C}(X, X)$ such that, for all $f : X \longrightarrow Y$ in $\mathscr{C}$, the following diagram commutes.

$$
\begin{tikzcd}
\mathscr{C}(A, X) \arrow[r, "\phi_X"] \arrow[dd, "\mathscr{C}(1_A, f)"'] & \mathscr{C}(X, X) \arrow[dr, "\mathscr{C}(1_X, f)"] & \\
& & \mathscr{C}(X, Y) \\
\mathscr{C}(A, Y) \arrow[r, "\phi_Y"'] & \mathscr{C}(Y, Y) \arrow[ur, "\mathscr{C}(f, 1_Y)"'] &
\end{tikzcd}
$$

In other words, $f\phi_X(u) = \phi_Y(fu)f$ for all morphisms $f : X \longrightarrow Y$ and $u : A \longrightarrow X$. The bijection is obtained by $\alpha_{X;Y}(f, u) = (u, \phi_Y(u)f)$. We therefore identify objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A, ϕ). A morphism $g : (A, \phi) \longrightarrow (A', \phi')$ in $\mathcal{Z}_l\mathscr{C}$ is a morphism $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\phi_X(vg) = \phi'_X(v)$ for all $v : A' \longrightarrow X$.

For a moment let us look at the special case where $\mathscr{C}$ has finite coproducts. Then, in the above notation, $\overline{\alpha}_{X,Y} : \mathscr{C}(X, Y) \times \mathscr{C}(A, Y) \longrightarrow \mathscr{C}(X, Y)$ is determined by its composite with the natural bijection $\mathscr{C}(X+A, Y) \cong \mathscr{C}(X, Y)\times\mathscr{C}(A, Y)$

so that the Yoneda Lemma can be applied. Thus we have a bijection between the $\overline{\alpha}$ and the natural transformations $\theta : (-) \longrightarrow (-) + A$ defined by the equations

$$\theta_X = \overline{\alpha}_{X;X+A}(\mathrm{copr}_1, \mathrm{copr}_2) = \phi_{X+A}(\mathrm{copr}_2)\mathrm{copr}_1 : X \longrightarrow X + A.$$

We therefore identify objects of $\mathscr{Z}_l\mathscr{C}$ with pairs (A, θ); morphisms $g : (A, \theta) \longrightarrow (A', \theta')$ are morphisms $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\theta'_X = (1_X + g)\theta_X$. For a category $\mathscr{X}$ with finite products, we can take $\mathscr{C} = \mathscr{X}^{\mathrm{op}}$ in the above to see that the lax centre $\mathscr{Z}_l\mathscr{X} = (\mathscr{Z}_l\mathscr{X}^{\mathrm{op}})^{\mathrm{op}}$ of the cartesian monoidal category $\mathscr{X}$ has objects pairs (A, θ) where $\theta : (-) \times A \longrightarrow (-)$ is a natural transformation. The tensor product in $\mathscr{Z}_l\mathscr{X}$ is given by

$$(A, \theta) \otimes (A', \theta') = \Big(A \times A', (-) \times A \times A' \xrightarrow{\theta \times 1_{A'}} (-) \times A' \xrightarrow{\theta'} (-)\Big).$$

The lax braiding $c_{(A,\theta),(A',\theta')} : (A, \theta) \otimes (A', \theta') \longrightarrow (A', \theta') \otimes (A, \theta)$ is the morphism

$$(\theta_{A'}, \mathrm{pr}_1) : (A \times A', \theta'(\theta \times 1_{A'})) \longrightarrow (A' \times A, \theta(\theta' \times 1_A)).$$

The *core* $C_\mathscr{X}$ of the category $\mathscr{X}$ in the sense of [**Fre**] is precisely a terminal object in $\mathscr{Z}_l\mathscr{X}$; it may not exist in general. Although we shall often write $C_\mathscr{X}$ for the underlying object of $\mathscr{X}$, as an object of $\mathscr{Z}_l\mathscr{X}$ it is equipped with a natural transformation $(-) \times C_\mathscr{X} \longrightarrow (-)$; however, it is also a monoid in $\mathscr{X}$ whose multiplication is the morphism $C_\mathscr{X} \times C_\mathscr{X} \longrightarrow C_\mathscr{X}$ into the terminal object in $\mathscr{Z}_l\mathscr{X}$. If the core exists, we have the identification of the lax centre with a slice category:

$$\mathscr{Z}_l\mathscr{X} \cong \mathscr{X}/C_\mathscr{X}.$$

The monoid structure on $C_\mathscr{X}$ defines an obvious monoidal structure on the slice category and the isomorphism is in fact monoidal. If $\mathscr{X}$ is cartesian closed (with internal hom written as $[X, Y]$), we have the formula

$$C_\mathscr{X} \cong \int_X [X, X];$$

but in general this end may not exist either.

Proposition 4.1. *If $\mathscr{X}$ is a complete cartesian closed category and $K : \mathscr{D} \longrightarrow \mathscr{X}$ is a dense functor from a small category $\mathscr{D}$ then $\mathscr{X}$ has a core $C_\mathscr{X} \cong \int_D [KD, KD]$.*

Proof. The denseness of K amounts to the natural isomorphism

$$\mathscr{X}(X, Y) \cong \int_D \mathbf{Set}(\mathscr{X}(KD, X), \mathscr{X}(KD, Y)).$$

Since $\mathscr{D}$ is small and $\mathscr{X}$ is complete, $\int_D [KD, KD]$ exists. We have the calculation:

$$\mathscr{X}\Big(Z, \int_D [KD, KD]\Big) \cong \int_D \mathscr{X}(Z, [KD, KD]) \cong \int_D \mathscr{X}(KD, [Z, KD])$$

$$\cong \int_{X,D} \mathbf{Set}(\mathscr{X}(KD, X), \mathscr{X}(KD, [Z, X])) \cong \int_X \mathscr{X}(X, [Z, X]) \cong \int_X \mathscr{X}(Z, [X, X]),$$

from which it follows that $\int_X [X, X]$ exists and is isomorphic to $\int_D [KD, KD]$. $\square$

We return now to our arbitrary small category $\mathscr{C}$, equipped with the promonoidal structure defined by $P(A, B; C) = \mathscr{C}(B, C) \times \mathscr{C}(A, C)$ and $JC = 1$, so that the corresponding convolution monoidal structure on the functor category $[\mathscr{C}, \mathbf{Set}]$ is the product. Recall that the internal hom for $[\mathscr{C}, \mathbf{Set}]$ is given by the formula

$$[F, G](A) \cong \int_V \mathbf{Set}(\mathscr{C}(A, V) \times FV, GV).$$

Applying Proposition 4.1 with K equal to the Yoneda embedding $\mathscr{C}^{\mathrm{op}} \longrightarrow [\mathscr{C}, \mathbf{Set}]$, we obtain

$$C_{[\mathscr{C},\mathbf{Set}]}(A) \cong \int_{W,V} \mathbf{Set}(\mathscr{C}(A, V) \times \mathscr{C}(W, V), \mathscr{C}(W, V)) \cong \int_V \mathbf{Set}(\mathscr{C}(A, V), \mathscr{C}(V, V))$$

where the second isomorphism uses the Yoneda Lemma. In other words, interpreting the last end and using our previous notation, we have a connection between the core of $[\mathscr{C}, \mathbf{Set}]$ and the lax centre of $\mathscr{C}$:

$$C_{[\mathscr{C},\mathbf{Set}]}(A) \cong \{\phi \mid (A, \phi) \text{ is an object of } \mathcal{Z}_l\mathscr{C}\}$$

The canonical function $C_{[\mathscr{C},\mathbf{Set}]}(A) \times F(A) \longrightarrow F(A)$ takes (ϕ, a) to $F(\phi_A(1_A))(a)$. The monoid structure $*$ on the functor $C_{[\mathscr{C},\mathbf{Set}]}$ is given by $(\phi * \phi')_U(h) = \phi_U(h)\phi'_U(h)$.

Recall from folklore that the *category* $\mathrm{el}F$ *of elements* of a functor $F : \mathscr{C} \to \mathbf{Set}$ has objects pairs (A, a) where A is an object of $\mathscr{C}$ and a is an element of $F(A)$; a morphism $g : (A, a) \longrightarrow (B, b)$ is a morphism $g : A \longrightarrow B$ in $\mathscr{C}$ such that $F(g)(a) = b$. There is an equivalence of categories

$$[\mathscr{C}, \mathbf{Set}]/F \xrightarrow{\sim} [\mathrm{el}F, \mathbf{Set}]$$

taking each object $\rho : T \longrightarrow F$ over F to the functor whose value at (A, a) is the fibre of the component function $\rho_A : T(A) \longrightarrow F(A)$ over $a \in F(A)$. If F is a monoid in $[\mathscr{C}, \mathbf{Set}]$ (that is a functor from $\mathscr{C}$ to the category **Mon** of monoids) then the obvious monoidal structure on $[\mathscr{C}, \mathbf{Set}]/F$ transports to a monoidal structure on $[\mathrm{el}F, \mathbf{Set}]$ which is obtained by convolution from the promonoidal structure on $\mathrm{el}F$ defined by

$$P((A, a), (B, b); (C, c)) = \left\{ A \xrightarrow{u} C \xleftarrow{v} B \mid F(u)(a) * F(v)(b) = c \right\}$$

where $*$ is multiplication in the monoid $F(C)$.

As a particular case, we see that the category of elements of $C_{[\mathscr{C},\mathbf{Set}]}$ is $\mathcal{Z}_l\mathscr{C}$ and the monoid structure on $C_{[\mathscr{C},\mathbf{Set}]}$ corresponds to the promagmal structure on $\mathcal{Z}_l\mathscr{C}$.

Putting all this together, we have proved the following result.

THEOREM 4.2. *For any small category $\mathscr{C}$ equipped with the promonoidal structure whose convolution gives the cartesian monoidal structure on $[\mathscr{C}, \mathbf{Set}]$, there is an equivalence and an isomorphism of categories:*

$$[\mathcal{Z}_l\mathscr{C}, \mathbf{Set}] \xrightarrow{\simeq} [\mathscr{C}, \mathbf{Set}]/C_{[\mathscr{C},\mathbf{Set}]} \xrightarrow{\cong} \mathcal{Z}_l[\mathscr{C}, \mathbf{Set}].$$

The promagmal category $\mathcal{Z}_l\mathscr{C}$ is lax-braided promonoidal resulting in a lax-braided convolution monoidal structure on $[\mathcal{Z}_l\mathscr{C}, \mathbf{Set}]$ for which the above composite equivalence is lax-braided monoidal.

The objects of $[\mathscr{C}, \mathbf{Set}]/C_{[\mathscr{C},\mathbf{Set}]}$ can also be interpreted in terms of dinatural transformations. A natural transformation $F \longrightarrow C_{[\mathscr{C},\mathbf{Set}]}$ has components

$$FA \longrightarrow \int_U \mathbf{Set}(\mathscr{C}(A, U), \mathscr{C}(U, U))$$

which are in natural bijection with families of morphisms

$$\mathscr{C}(A,U) \longrightarrow \mathbf{Set}(FA, \mathscr{C}(U,U))$$

natural in A and dinatural in U. By Yoneda, these families are in natural bijection with families of morphisms

$$\rho_U : FU \longrightarrow \mathscr{C}(U,U)$$

dinatural in U. Write $\mathrm{Hom}_{\mathscr{C}}$ for the set-valued hom functor of the category $\mathscr{C}$.

PROPOSITION 4.3. *For any small category $\mathscr{C}$, the lax centre $\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}]$ of the cartesian monoidal category $[\mathscr{C}, \mathbf{Set}]$ is equivalent to the category of dinatural transformations $\rho : F \longrightarrow \mathrm{Hom}_{\mathscr{C}}$ over $\mathrm{Hom}_{\mathscr{C}}$. Given such a dinatural ρ, the corresponding object of $\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}]$ is (F, u) where*

$$u_M : F \times M \longrightarrow M \times F$$

is defined by $(u_M U)(x, m) = (M(\rho_U(x))(m), x)$ for all x in FU and m in MU.

THEOREM 4.4. *If $\mathscr{C}$ is a category in which every endomorphism is invertible then the lax centre $\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}]$ of the cartesian monoidal category $[\mathscr{C}, \mathbf{Set}]$ is equal to the centre $\mathcal{Z}[\mathscr{C}, \mathbf{Set}]$.*

PROOF. Notice in Proposition 4.3 that each $\rho_U(x)$ is an endomorphism, so under the present hypotheses, an inverse for u_M is defined by

$$(u_M^{-1}U)(m, x) = \Big(x, M(\rho_U(x)^{-1})(m)\Big).$$

$\square$

Before closing this section, let us consider the case where $\mathscr{C}$ is a groupoid. Then the equation $f\phi_X(u) = \phi_Y(fu)f$ can be rewritten $f\phi_X(u)f^{-1} = \phi_Y(fu)$ so that

$$\phi_X(f) = f\phi_A(1_A)f^{-1}.$$

In other words, objects of $\mathcal{Z}_l\mathscr{C}$ can be identified with automorphisms $s : A \longrightarrow A$; the corresponding ϕ is defined by the conjugation formula $\phi_X(f) = fsf^{-1}$. So $\mathcal{Z}_l\mathscr{C} = \mathscr{C}^{\mathbf{Z}}$ is the category of automorphisms in $\mathscr{C}$. As described in Example 9 of [**DaSt**], the promonoidal structure is defined by

$$P((A,s),(B,t);(C,r)) = \left\{ A \xrightarrow{\ u\ } C \xleftarrow{\ v\ } B \ \mid\ {}^u s\, {}^v t = r \right\}.$$

The lax braiding $P((A,s),(B,t);(C,r)) \longrightarrow P((B,t),(A,s);(C,r))$ takes (u,v) to $({}^u sv, u)$. The family of morphisms $\alpha_{X;Y} : \mathscr{C}(X,Y) \times \mathscr{C}(A,Y) \longrightarrow \mathscr{C}(A,Y) \times \mathscr{C}(X,Y)$ corresponding to the ϕ corresponding to s is then defined by $\alpha_{X;Y}(f,u) = (u, usu^{-1}f)$ which is obviously invertible (the inverse takes (u,g) to $(us^{-1}u^{-1}g, u)$). This implies that the lax centre of $\mathscr{C}$ is equal to the centre of $\mathscr{C}$ and that the lax braiding is a braiding. It also follows that $C[\mathscr{C}, \mathbf{Set}] = \mathrm{Aut}_{\mathscr{C}}$ where $\mathrm{Aut}_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathbf{Set}$ is the functor taking the object A to $\mathscr{C}(A,A)$ and the morphism f to conjugation by f.

THEOREM 4.5. *If $\mathscr{C}$ as in Theorem 4.2 is a groupoid then*

$$\mathcal{Z}_l\mathscr{C} = \mathcal{Z}\mathscr{C} = \mathscr{C}^{\mathbf{Z}}, \quad \mathcal{Z}_l[\mathscr{C}, \mathbf{Set}] = \mathcal{Z}[\mathscr{C}, \mathbf{Set}], \quad C_{[\mathscr{C}, \mathbf{Set}]} = \mathrm{Aut}_{\mathscr{C}}$$

and there is a braided monoidal equivalence

$$\mathcal{Z}[\mathscr{C}, \mathbf{Set}] \xrightarrow{\ \sim\ } [\mathscr{C}^{\mathbf{Z}}, \mathbf{Set}].$$

5. The central cohypomonad

The lax centre of a monoidal $\mathscr{V}$-category $\mathscr{X}$ can be, in very special cases, monadic over $\mathscr{X}$ or comonadic over $\mathscr{X}$. However, with the mere assumption of left closedness, we find that the lax centre $\mathscr{Z}_l\mathscr{X}$ is the $\mathscr{V}$-category of coalgebras for a "cohypomonad", a concept we shall now define.

Let Δ denote the category whose objects are finite ordinals $\langle n\rangle = \{1, 2, \ldots, n\}$ and whose morphisms are order-preserving functions. It becomes strict monoidal under the tensor product defined by ordinal sum: $\langle m\rangle + \langle n\rangle = \langle m+n\rangle$. Recall that a comonad on the $\mathscr{V}$-category $\mathscr{X}$ can be identified with a strict monoidal functor $\mathbf{G} : \Delta^{\mathrm{op}} \longrightarrow [\mathscr{X}, \mathscr{X}]$ where the endo-$\mathscr{V}$-functor category $[\mathscr{X}, \mathscr{X}]$ is monoidal under composition. A monad on $\mathscr{X}$ is a strict monoidal functor $\Delta \longrightarrow [\mathscr{X}, \mathscr{X}]$; a mere monoidal functor is something less, so we call it a *hypomonad*.

A *cohypomonad* on $\mathscr{X}$ is a monoidal functor $\mathbf{G} : \Delta^{\mathrm{op}} \longrightarrow [\mathscr{X}, \mathscr{X}]$. More explicitly, it is an augmented simplicial endo-$\mathscr{V}$-functor

$$\begin{tikzcd}
G_0 & G_1 \arrow[l, "\epsilon"'] \arrow[r, "\delta" description] & G_2 \arrow[l, shift right=2, "\epsilon_0"'] \arrow[l, shift left=2, "\epsilon_1"] \arrow[r, "\delta_0", shift left=1.5] \arrow[r, "\delta_1"', shift right=1.5] & \cdots \arrow[l, shift right=3, "\epsilon_0"'] \arrow[l, "\epsilon_1" description] \arrow[l, shift left=3, "\epsilon_2"]
\end{tikzcd}$$

on $\mathscr{X}$ together with $\mathscr{V}$-natural transformations $\gamma_{2;m,n} : G_m \circ G_n \longrightarrow G_{m+n}$ and $\gamma_0 : 1_{\mathscr{X}} \longrightarrow G_0$ satisfying naturality of $\gamma_{2;m,n}$ in $\langle m\rangle$ and $\langle n\rangle$, plus associativity and unit conditions. A cohypomonad is called *normal* when $\gamma_0 : 1_{\mathscr{X}} \longrightarrow G_0$ is invertible.

A *coalgebra* for $\mathbf{G}$ is an object A of $\mathscr{X}$ together with a morphism $\alpha : A \rightarrow G_1 A$ (called the *coaction*) such that the following two diagrams commute.

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\gamma_{0;A}"'] & G_1 A \arrow[d, "\epsilon_A"] \\
& G_0 A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & G_1 A \arrow[dr, "\delta_A"] & \\
& & G_2 A \\
G_1 A \arrow[r, "G_1\alpha"'] & G_1 G_1 A \arrow[ur, "\gamma_{2;1,1;A}"'] &
\end{tikzcd}$$

Such a coalgebra gives rise to an extended simplicial diagram on the value of $\mathbf{G}$ at A; we omit the details. A *coalgebra morphism* is a morphism in $\mathscr{X}$ which commutes with the coactions. We obtain a $\mathscr{V}$-category $\mathscr{X}^{\mathbf{G}}$ of $\mathbf{G}$-coalgebras by taking the obvious equalizer in $\mathscr{V}$ to define the $\mathscr{V}$-valued homs

We now turn to our principal example of a cohypomonad. Suppose $\mathscr{X}$ is a left-closed monoidal $\mathscr{V}$-category. For each natural number n, define the endo-$\mathscr{V}$-functor G_n of $\mathscr{X}$ by the end formula

$$G_n A = \int_{X_1,\ldots,X_n} [X_1 \otimes \cdots \otimes X_n, X_1 \otimes \cdots \otimes X_n \otimes A],$$

where the square brackets denote the left internal hom. The end exists when, for example, we assume $\mathscr{X}$ is complete, right closed, and has a small dense full sub-$\mathscr{V}$-category. (Alternatively, we could avoid the internal homs and these size problems by looking at modules (= distributors) from $\mathscr{X}$ to $\mathscr{X}$ rather than functors.)

The functor $\mathbf{G} : \Delta^{\mathrm{op}} \longrightarrow [\mathscr{X}, \mathscr{X}]$ is defined as follows. The value at the object $\langle n\rangle$ is of course G_n. Let $\xi : \langle m\rangle \longrightarrow \langle n\rangle$ be an order-preserving function and suppose the fibre of ξ over $k \in \langle n \rangle$ has cardinality m_k. The $\mathscr{V}$-natural transformation $G_\xi : G_n \longrightarrow G_m$ has its component at A defined by commutativity of the triangle

$$\begin{tikzcd}
G_n A \arrow[rr, "G_\xi A"] \arrow[drr, "\mathrm{proj}_{Y_1 \otimes \cdots \otimes Y_{m_1}, \ldots, Y_{m-m_n+1} \otimes \cdots \otimes Y_{m_1}}"'] & & G_m A \arrow[d, "\mathrm{proj}_{Y_1, \ldots, Y_m}"] \\
& & {[Y_1 \otimes \cdots \otimes Y_m, Y_1 \otimes \cdots \otimes Y_m \otimes A]}
\end{tikzcd}$$

for all choices of objects $Y_1, \ldots, Y_m$.

We now describe the monoidal structure on the functor $\mathbf{G}$. In fact, it is normal; there is an obvious canonical $\mathscr{V}$-natural isomorphism $\gamma_0 : 1_{\mathscr{X}} \longrightarrow G_0$. The component of the $\mathscr{V}$-natural transformation $\gamma_{2;m,n} : G_m \circ G_n \longrightarrow G_{m+n}$ at A is defined by commutativity of the diagram

$$\begin{tikzcd}
\int_{\mathbf{Y}} \Big[\bigotimes_m \mathbf{Y}, \bigotimes_m \mathbf{Y} \otimes \int_{\mathbf{X}} \Big[\bigotimes_n \mathbf{X}, \bigotimes_n \mathbf{X} \otimes A\Big]\Big] \arrow[r, "\gamma_{2;m,n} A"] \arrow[d, "\int_{\mathbf{Y}}[1, 1 \otimes \mathrm{proj}_{\mathbf{X}}]"'] & \int_{\mathbf{Y}, \mathbf{X}} \Big[\bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X}, \bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X} \otimes A\Big] \arrow[d, "\mathrm{proj}_{\mathbf{Y}, \mathbf{X}}"] \\
\int_{\mathbf{Y}} \Big[\bigotimes_m \mathbf{Y}, \bigotimes_m \mathbf{Y} \otimes \Big[\bigotimes_n \mathbf{X}, \bigotimes_n \mathbf{X} \otimes A\Big]\Big] \arrow[d, "\mathrm{proj}_{\mathbf{Y}}"'] & \Big[\bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X}, \bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X} \otimes A\Big] \arrow[d, "\cong"] \\
\Big[\bigotimes_m \mathbf{Y}, \bigotimes_m \mathbf{Y} \otimes \Big[\bigotimes_n \mathbf{X}, \bigotimes_n \mathbf{X} \otimes A\Big]\Big] \arrow[r, "{[1, \mathrm{canon}]}"'] & \Big[\bigotimes_m \mathbf{Y}, \Big[\bigotimes_n \mathbf{X}, \bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X} \otimes A\Big]\Big]
\end{tikzcd}$$

for all lists $\mathbf{Y} = (Y_1, \ldots, Y_m)$ and $\mathbf{X} = (X_1, \ldots, X_n)$ of objects, where the map canon: $Y \otimes [X, Z] \longrightarrow [X, Y \otimes Z]$ corresponds, under the tensor-hom adjunction to $1 \otimes \mathrm{eval} : Y \otimes [X, Z] \otimes X \longrightarrow Y \otimes Z$.

PROPOSITION 5.1. *Let $\mathscr{X}$ be a complete closed monoidal $\mathscr{V}$-category with a small dense sub-$\mathscr{V}$-category. The structure just defined on $\mathbf{G} : \Delta^{op} \longrightarrow [\mathscr{X}, \mathscr{X}]$ makes it a normal cohypomonad for which $\mathscr{X}^{\mathbf{G}}$ is equivalent to the lax centre of $\mathscr{X}$.*

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