Lagrangians for spinor fields

based on S-36

we want to find a suitable lagrangian for left- and right-handed spinor fields.

it should be:

- Lorentz invariant and hermitian
- quadratic in ψ_a and $\psi^\dagger_{\dot a}$

equations of motion will be linear with plane wave solutions (suitable for describing free particles)

terms with no derivative:

$$\psi\psi = \psi^a\psi_a = \varepsilon^{ab}\psi_b\psi_a \qquad + \text{h.c.}$$

terms with derivatives:

$$\cancel{\partial^\mu\psi\partial_\mu\psi}$$

would lead to a hamiltonian unbounded from below

to get a bounded hamiltonian the kinetic term has to contain both ψ_a and $\psi^\dagger_{\dot a}$, a candidate is:

$$i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi$$

is hermitian up to a total divergence

$$\begin{aligned}(i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi)^\dagger &= (i\psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c)^\dagger\\ &= -i\partial_\mu\psi^\dagger_{\dot c}\,(\bar\sigma^{\mu a\dot c})^*\psi_a\\ &= -i\partial_\mu\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a\\ &= i\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\partial_\mu\psi_a - i\partial_\mu(\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a).\\ &= i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - i\partial_\mu(\psi^\dagger\bar\sigma^\mu\psi)\ .\end{aligned}$$

$\bar\sigma^{\mu\dot a a} = (I, -\vec\sigma)$ are hermitian

does not contribute to the action

Our complete lagrangian is:

$$\mathcal{L} = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \tfrac{1}{2}m\psi\psi - \tfrac{1}{2}m^*\psi^\dagger\psi^\dagger$$

the phase of m can be absorbed into the definition of fields

$$m = |m|e^{i\alpha} \qquad \psi = e^{-i\alpha/2}\,\tilde\psi$$

and so without loss of generality we can take m to be real and positive.

Equation of motion:

$$0 = -\frac{\delta S}{\delta\psi^\dagger} = -i\bar\sigma^\mu\partial_\mu\psi + m\psi^\dagger$$

$$0 = -i\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c + m\psi^{\dagger\dot a}$$

Taking hermitian conjugate:

$$\begin{aligned}0 &= +i(\bar\sigma^{\mu a\dot c})^*\,\partial_\mu\psi^\dagger_{\dot c} + m\psi^a\\ &= +i\bar\sigma^{\mu\dot c a}\partial_\mu\psi^\dagger_{\dot c} + m\psi^a\\ &= -i\sigma^\mu_{a\dot c}\,\partial_\mu\psi^{\dagger\dot c} + m\psi_a\ .\end{aligned}$$

$\bar\sigma^{\mu\dot a a} = (I, -\vec\sigma)$ are hermitian

$\bar\sigma^{\mu\dot a a} \equiv \varepsilon^{ab}\varepsilon^{\dot a\dot b}\sigma^\mu_{b\dot b}$

We can combine the two equations:

$$0 = -i\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c + m\psi^{\dagger\dot a}$$

$$0 = -i\sigma^\mu_{a\dot c}\,\partial_\mu\psi^{\dagger\dot c} + m\psi_a$$

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot c}\,\partial_\mu \\ -i\bar\sigma^{\mu\dot a c}\,\partial_\mu & m\delta^{\dot a}{}_{\dot c}\end{pmatrix}\begin{pmatrix}\psi_c\\ \psi^{\dagger\dot c}\end{pmatrix} = 0$$

which we can write using 4x4 gamma matrices:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot c} \\ \bar\sigma^{\mu\dot a c} & 0\end{pmatrix}$$

and defining four-component Majorana field: $\Psi \equiv \begin{pmatrix}\psi_c\\ \psi^{\dagger\dot c}\end{pmatrix}$

as:

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$

Dirac equation

using the sigma-matrix relations:

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma})$$
$$\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$$

$$(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)_a{}^c = -2g^{\mu\nu}\delta_a{}^c$$
$$(\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu)^{\dot{a}}{}_{\dot{c}} = -2g^{\mu\nu}\delta^{\dot{a}}{}_{\dot{c}}$$

we see that

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

and we know that that we needed 4 such matrices;

recall:

$$i\hbar \frac{\partial}{\partial t}\psi_a(x) = \Big(-i\hbar c(\alpha^j)_{ab}\partial_j + mc^2(\beta)_{ab}\Big)\psi_b(x)$$

$$\{\alpha^j, \alpha^k\}_{ab} = 2\delta^{jk}\delta_{ab}\,, \quad \{\alpha^j, \beta\}_{ab} = 0\,, \quad (\beta^2)_{ab} = \delta_{ab}$$

$$\beta = \gamma^0$$
$$\alpha^k = \gamma^0\gamma^k$$
$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$

consider a theory of two left-handed spinor fields:

$$\mathcal{L} = i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \tfrac{1}{2}m\psi_i\psi_i - \tfrac{1}{2}m\psi_i^\dagger\psi_i^\dagger$$

i = 1,2

the lagrangian is invariant under the SO(2) transformation:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

it can be written in the form that is manifestly U(1) symmetric:

$$\chi = \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$
$$\xi = \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$$

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

$$\chi \to e^{-i\alpha}\chi$$
$$\xi \to e^{+i\alpha}\xi$$

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

Equations of motion for this theory:

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot{c}}\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\partial_\mu & m\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = 0$$

we can define a four-component Dirac field: $\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$

Dirac equation

we want to write the lagrangian in terms of the Dirac field:

$$\Psi^\dagger = (\chi^\dagger_{\dot{a}}\,,\ \xi^a)$$

$$\beta \equiv \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$$

Let's define:

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\ \chi^\dagger_{\dot{a}})$$

numerically $\beta = \gamma^0$

but different spinor index structure

Then we find:

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\ \chi^\dagger_{\dot{a}})$$

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$\overline{\Psi}\Psi = \xi^a\chi_a + \chi^\dagger_{\dot{a}}\xi^{\dagger\dot{a}}$$

$$\overline{\Psi}\gamma^\mu\partial_\mu\Psi = \xi^a\sigma^\mu_{a\dot{c}}\partial_\mu\xi^{\dagger\dot{c}} + \chi^\dagger_{\dot{a}}\,\bar{\sigma}^{\mu\dot{a}c}\,\partial_\mu\chi_c$$

$$A\partial B = -(\partial A)B + \partial(AB)$$

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix}$$

$$\xi^a\sigma^\mu_{a\dot{c}}\partial_\mu\xi^{\dagger\dot{c}} = -(\partial_\mu\xi^a)\sigma^\mu_{a\dot{c}}\xi^{\dagger\dot{c}} + \partial_\mu(\xi^a\sigma^\mu_{a\dot{c}}\xi^{\dagger\dot{c}})$$

$$-(\partial_\mu\xi^a)\sigma^\mu_{a\dot{c}}\xi^{\dagger\dot{c}} = +\xi^{\dagger\dot{c}}\sigma^\mu_{a\dot{c}}\partial_\mu\xi^a = +\xi^\dagger_{\dot{c}}\,\bar{\sigma}^{\mu\dot{c}a}\partial_\mu\xi_a$$

$$\bar{\sigma}^{\mu\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^\mu_{b\dot{b}}$$

Thus we have:

$$\overline{\Psi}\gamma^\mu\partial_\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi + \partial_\mu(\xi\sigma^\mu\xi^\dagger)$$

Thus the lagrangian can be written as: $\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger$

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi$$

The U(1) symmetry is obvious:

$$\Psi \to e^{-i\alpha}\,\Psi$$
$$\overline{\Psi} \to e^{+i\alpha}\,\overline{\Psi}$$

The Nether current associated with this symmetry is: $j^\mu(x) \equiv \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\delta\varphi_a(x)$

$$j^\mu = \overline{\Psi}\gamma^\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\chi - \xi^\dagger\bar{\sigma}^\mu\xi$$

later we will see that this is the electromagnetic current

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

There is an additional discrete symmetry that exchanges the two fields, charge conjugation:

$$C^{-1}\chi_a(x)C = \xi_a(x)$$
$$C^{-1}\xi_a(x)C = \chi_a(x)$$

unitary charge conjugation operator

$$C^{-1}\mathcal{L}(x)C = \mathcal{L}(x)$$

we want to express it in terms of the Dirac field:

Let's define the charge conjugation matrix:

$$\mathcal{C} \equiv \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{c}} \end{pmatrix}$$

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\, \chi^\dagger_{\dot{a}})$$

$$\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi^a \\ \chi^\dagger_{\dot{a}} \end{pmatrix}$$

then

$$\Psi^{\mathrm{C}} \equiv \mathcal{C}\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix}$$

and we have:

$$C^{-1}\Psi(x)C = \Psi^{\mathrm{C}}(x)$$

The charge conjugation matrix has following properties:

$$\mathcal{C} \equiv \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{c}} \end{pmatrix}$$

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C}$$

it can also be written as:

$$\mathcal{C} = \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{c}} \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{e\dot{a}} \\ \bar{\sigma}^{\mu\dot{e}a} & 0 \end{pmatrix}$$

and then we find a useful identity:

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = \begin{pmatrix} \varepsilon^{ab} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu_{b\dot{c}} \\ \bar{\sigma}^{\mu\dot{b}c} & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_{ce} & 0 \\ 0 & \varepsilon^{\dot{c}\dot{e}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \varepsilon^{ab}\sigma^\mu_{b\dot{c}}\varepsilon^{\dot{c}\dot{e}} \\ \varepsilon_{\dot{a}\dot{b}}\bar{\sigma}^{\mu\dot{b}c}\varepsilon_{ce} & 0 \end{pmatrix}$$

transposed form of $\bar{\sigma}^{\mu\dot{a}a} \equiv \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{b\dot{b}}$

$$= \begin{pmatrix} 0 & -\bar{\sigma}^{\mu a\dot{e}} \\ -\sigma^\mu_{\dot{a}e} & 0 \end{pmatrix}.$$

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}}$$

Majorana field is its own conjugate:

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix}$$

$$\Psi^{\mathrm{C}} = \Psi$$

similar to a real scalar field
$\varphi^\dagger = \varphi$

Following the same procedure with: $\chi \to \psi$, $\xi \to \psi$

we get:

$$\mathcal{L} = \tfrac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\overline{\Psi}\Psi$$

does not incorporate the Majorana condition

$$\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$$
$$\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C}$$

incorporating the Majorana condition, we get:

$$\mathcal{L} = \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi$$

lagrangian for a Majorana field

If we want to go back from 4-component Dirac or Majorana fields to the two-component Weyl fields, it is useful to define a projection matrix:

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot a}{}_{\dot c} \end{pmatrix}$$

just a name

We can define left and right projection matrices:

$$P_{\rm L} \equiv \tfrac{1}{2}(1-\gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_{\rm R} \equiv \tfrac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot a}{}_{\dot c} \end{pmatrix}$$

And for a Dirac field we find:

$$P_{\rm L}\Psi = \begin{pmatrix} \chi_c \\ 0 \end{pmatrix} \qquad \Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot c} \end{pmatrix}$$

$$P_{\rm R}\Psi = \begin{pmatrix} 0 \\ \xi^{\dagger\dot c} \end{pmatrix}$$

The gamma-5 matrix can be also written as:

$$\begin{aligned} \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\tfrac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \end{aligned} \qquad \varepsilon_{0123} = -1$$

Finally, let's take a look at the Lorentz transformation of a Dirac or Majorana field:

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L(\Lambda)_a{}^c\,\psi_c(\Lambda^{-1}x)$$

$$U(\Lambda)^{-1}\psi^\dagger_{\dot a}(x)U(\Lambda) = R(\Lambda)_{\dot a}{}^{\dot c}\,\psi^\dagger_{\dot c}(\Lambda^{-1}x)$$

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)$$

$$L(1+\delta\omega)_a{}^c = \delta_a{}^c + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^c$$

$$R(1+\delta\omega)_{\dot a}{}^{\dot c} = \delta_{\dot a}{}^{\dot c} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot c}$$

$$D(1+\delta\omega) = 1 + \tfrac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu}$$

$$(S^{\mu\nu}_{\rm L})_a{}^c = +\tfrac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_a{}^c$$

$$(S^{\mu\nu}_{\rm R})^{\dot a}{}_{\dot c} = -\tfrac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot a}{}_{\dot c}$$

$$\tfrac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} +(S^{\mu\nu}_{\rm L})_a{}^c & 0 \\ 0 & -(S^{\mu\nu}_{\rm R})^{\dot a}{}_{\dot c} \end{pmatrix} \equiv S^{\mu\nu}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{e\dot a} \\ \bar\sigma^{\mu\dot e a} & 0 \end{pmatrix} \qquad \Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot c} \end{pmatrix}$$

compensates for ${}^{\dot c}{}_{\dot c} = - {}_{\dot c}{}^{\dot c}$

Canonical quantization of spinor fields I

based on S-37

Consider the lagrangian for a left-handed Weyl field:

$$\mathcal{L} = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger)$$

the conjugate momentum to the left-handed field is:

$$\begin{aligned} \pi^a(x) &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_a(x))} \\ &= i\psi^\dagger_{\dot a}(x)\bar\sigma^{0\dot a a} \end{aligned}$$

and the hamiltonian is simply given as:

$$\begin{aligned} \mathcal{H} &= \pi^a\partial_0\psi_a - \mathcal{L} \\ &= i\psi^\dagger_{\dot a}\bar\sigma^{0\dot a a}\dot\psi_a - \mathcal{L} \\ &= -i\psi^\dagger\bar\sigma^i\partial_i\psi + \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger) \end{aligned}$$

the appropriate canonical anticommutation relations are:

$$\begin{aligned} \{\psi_a(\mathbf{x},t),\psi_c(\mathbf{y},t)\} &= 0\,, \\ \{\psi_a(\mathbf{x},t),\pi^c(\mathbf{y},t)\} &= i\delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned}$$

$$\begin{aligned} \pi^a(x) &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_a(x))} \\ &= i\psi^\dagger_{\dot a}(x)\bar\sigma^{0\dot a a} \end{aligned}$$

or

$$\{\psi_a(\mathbf{x},t),\psi^\dagger_{\dot c}(\mathbf{y},t)\}\bar\sigma^{0\dot c c} = \delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y})$$

using $\bar\sigma^0 = \sigma^0 = I$ we get

$$\{\psi_a(\mathbf{x},t),\psi^\dagger_{\dot c}(\mathbf{y},t)\} = \sigma^0_{a\dot c}\,\delta^3(\mathbf{x}-\mathbf{y})$$

or, equivalently,

$$\{\psi^a(\mathbf{x},t),\psi^{\dagger\dot c}(\mathbf{y},t)\} = \bar\sigma^{0\dot c a}\,\delta^3(\mathbf{x}-\mathbf{y})$$

For a four-component Dirac field we found:

$$\begin{aligned}\mathcal{L} &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m(\chi\xi + \xi^\dagger\chi^\dagger)\\ &= i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi\,.\end{aligned}$$

$$\Psi \equiv \begin{pmatrix}\chi_c\\ \xi^{\dagger\dot{c}}\end{pmatrix}$$

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\ \chi^\dagger_{\dot{a}})$$

and the corresponding canonical anticommutation relations are:

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot{c}}(\mathbf{y},t)\} = \sigma^0_{a\dot{c}}\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot{c}}(\mathbf{y},t)\} = \sigma^0_{a\dot{c}}\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\begin{aligned}\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} &= 0\,,\\ \{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} &= (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\end{aligned}$$

$$\gamma^\mu \equiv \begin{pmatrix}0 & \sigma^\mu_{a\dot{c}}\\ \bar{\sigma}^{\mu\dot{a}c} & 0\end{pmatrix}$$

can be also derived directly from $\partial\mathcal{L}/\partial(\partial_0\Psi) = i\overline{\Psi}\gamma^0$, ...

For a four-component Majorana field we found:

$$\Psi \equiv \begin{pmatrix}\psi_c\\ \psi^{\dagger\dot{c}}\end{pmatrix}$$

$$\begin{aligned}\mathcal{L} &= i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger)\\ &= \tfrac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\overline{\Psi}\Psi\\ &= \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi\,.\end{aligned}$$

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\psi^a,\ \psi^\dagger_{\dot{a}})$$

$$\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C}$$

$$\mathcal{C} \equiv \begin{pmatrix}-\varepsilon^{ac} & 0\\ 0 & -\varepsilon_{\dot{a}\dot{c}}\end{pmatrix}$$

and the corresponding canonical anticommutation relations are:

$$\begin{aligned}\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} &= (\mathcal{C}\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\,,\\ \{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} &= (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\,,\end{aligned}$$

Now we want to find solutions to the Dirac equation:

$$(-i\partial\!\!\!/ + m)\Psi = 0$$

where we used the Feynman slash: $a\!\!\!/ \equiv a_\mu\gamma^\mu$

$$\begin{aligned}a\!\!\!/ a\!\!\!/ &= a_\mu a_\nu\gamma^\mu\gamma^\nu\\ &= a_\mu a_\nu\left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\\ &= a_\mu a_\nu\left(-g^{\mu\nu} + \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\\ &= -a_\mu a_\nu g^{\mu\nu} + 0\\ &= -a^2\,.\end{aligned}$$

then we find:

$$\begin{aligned}0 &= (i\partial\!\!\!/ + m)(-i\partial\!\!\!/ + m)\Psi\\ &= (\partial\!\!\!/\partial\!\!\!/ + m^2)\Psi\\ &= (-\partial^2 + m^2)\Psi\,.\end{aligned}$$

the Dirac (or Majorana) field satisfies the Klein-Gordon equation and so the Dirac equation has plane-wave solutions!

Consider a solution of the form:

$$\Psi(x) = u(\mathbf{p})e^{ipx} + v(\mathbf{p})e^{-ipx}$$

four-component constant spinors

$$p^0 = \omega \equiv (\mathbf{p}^2 + m^2)^{1/2}$$

$$(-i\partial\!\!\!/ + m)\Psi = 0$$

plugging it into the Dirac equation gives:

$$(p\!\!\!/ + m)u(\mathbf{p})e^{ipx} + (-p\!\!\!/ + m)v(\mathbf{p})e^{-ipx} = 0$$

that requires:

$$\begin{aligned}(p\!\!\!/ + m)u(\mathbf{p}) &= 0\\ (-p\!\!\!/ + m)v(\mathbf{p}) &= 0\end{aligned}$$

each eq. has two solutions (later)

The general solution of the Dirac equation is:

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d^\dagger_s(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]$$

$$\widetilde{dp} \equiv \frac{d^3p}{(2\pi)^3 2\omega}$$

Spinor technology

based on S-38

The four-component spinors obey equations:

$$(\not{p} + m)u_s(\mathbf{p}) = 0$$
$$(-\not{p} + m)v_s(\mathbf{p}) = 0$$

s = + or -

In the rest frame, $\mathbf{p} = \mathbf{0}$ we can choose:

for $m \neq 0$

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

$$\not{p} = -m\gamma^0$$
$$\gamma^0 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

convenient normalization and phase

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

this choice corresponds to eigenvectors of the spin matrix:

$$S_z = \tfrac{i}{4}[\gamma^1,\gamma^2] = \tfrac{i}{2}\gamma^1\gamma^2 = \begin{pmatrix}\tfrac{1}{2}\sigma_3 & 0\\ 0 & \tfrac{1}{2}\sigma_3\end{pmatrix}$$

$$S_z u_\pm(\mathbf{0}) = \pm\tfrac{1}{2}u_\pm(\mathbf{0})$$
$$S_z v_\pm(\mathbf{0}) = \mp\tfrac{1}{2}v_\pm(\mathbf{0})$$

$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]$$

this choice results in (we will see it later):

$$[J_z, b^\dagger_\pm(\mathbf{0})] = \pm\tfrac{1}{2}b^\dagger_\pm(\mathbf{0})$$
$$[J_z, d^\dagger_\pm(\mathbf{0})] = \pm\tfrac{1}{2}d^\dagger_\pm(\mathbf{0})$$

creates a particle with spin up (+) or down (-) along the z axis

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

let us also compute the barred spinors:

$$\overline{u}_s(\mathbf{p}) \equiv u_s^\dagger(\mathbf{p})\beta$$
$$\overline{v}_s(\mathbf{p}) \equiv v_s^\dagger(\mathbf{p})\beta$$

$$\beta = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

$$\beta^{\mathrm{T}} = \beta^\dagger = \beta^{-1} = \beta$$

we get:

$$\overline{u}_+(\mathbf{0}) = \sqrt{m}\,(1,\,0,\,1,\,0)\,,$$
$$\overline{u}_-(\mathbf{0}) = \sqrt{m}\,(0,\,1,\,0,\,1)\,,$$
$$\overline{v}_+(\mathbf{0}) = \sqrt{m}\,(0,\,-1,\,0,\,1)\,,$$
$$\overline{v}_-(\mathbf{0}) = \sqrt{m}\,(1,\,0,\,-1,\,0)\,.$$

We can find spinors at arbitrary 3-momentum by applying the matrix that corresponds to the boost:

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)$$

$$D(\Lambda) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$

homework

$$K^j = \tfrac{i}{4}[\gamma^j,\gamma^0] = \tfrac{i}{2}\gamma^j\gamma^0$$
$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]$$
$$\eta \equiv \sinh^{-1}(|\mathbf{p}|/m)$$

we find:

$$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})$$
$$v_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})$$

and similarly:

$$\overline{u}_s(\mathbf{p}) = \overline{u}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$
$$\overline{v}_s(\mathbf{p}) = \overline{v}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$

$$\overline{K^j} = K^j$$
$$\overline{A} \equiv \beta A^\dagger\beta$$

For any combination of gamma matrices we define:

$$\overline{A} \equiv \beta A^\dagger \beta$$

It is straightforward to show:

$$\begin{aligned}
\overline{\gamma^\mu} &= \gamma^\mu \,, \\
\overline{S^{\mu\nu}} &= S^{\mu\nu} \,, \\
\overline{i\gamma_5} &= i\gamma_5 \,, \\
\overline{\gamma^\mu \gamma_5} &= \gamma^\mu \gamma_5 \,, \\
\overline{i\gamma_5 S^{\mu\nu}} &= i\gamma_5 S^{\mu\nu} \,.
\end{aligned}$$

$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\overline{K^j} = K^j$$

homework

For barred spinors we get:

$$\begin{aligned}
\overline{u}_s(\mathbf{p})(\not{p} + m) &= 0 \\
\overline{v}_s(\mathbf{p})(-\not{p} + m) &= 0
\end{aligned}$$

$$\begin{aligned}
(\not{p} + m)u(\mathbf{p}) &= 0 \\
(-\not{p} + m)v(\mathbf{p}) &= 0
\end{aligned}$$

$$\begin{aligned}
\overline{u}_s(\mathbf{p}) &\equiv u_s^\dagger(\mathbf{p})\beta \\
\overline{v}_s(\mathbf{p}) &\equiv v_s^\dagger(\mathbf{p})\beta
\end{aligned}$$

It is straightforward to derive explicit formulas for spinors, but will will not need them; all we will need are products of spinors of the form:

$$\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = \overline{u}_{s'}(\mathbf{0})u_s(\mathbf{0})$$

$$\begin{aligned}
u_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0}) \\
v_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})
\end{aligned}$$

$$\begin{aligned}
\overline{u}_s(\mathbf{p}) &= \overline{u}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K}) \\
\overline{v}_s(\mathbf{p}) &= \overline{v}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})
\end{aligned}$$

which do not depend on p!

we find:

$$\begin{aligned}
\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) &= +2m\,\delta_{s's} \,, \\
\overline{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) &= -2m\,\delta_{s's} \,, \\
\overline{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) &= 0 \,, \\
\overline{v}_{s'}(\mathbf{p})u_s(\mathbf{p}) &= 0 \,.
\end{aligned}$$

Useful identities (Gordon identities):

$$\begin{aligned}
2m\,\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) &= \overline{u}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]u_s(\mathbf{p}) \\
-2m\,\overline{v}_{s'}(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) &= \overline{v}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]v_s(\mathbf{p})
\end{aligned}$$

Proof:

$$\begin{aligned}
\gamma^\mu \not{p} &= \tfrac{1}{2}\{\gamma^\mu, \not{p}\} + \tfrac{1}{2}[\gamma^\mu, \not{p}\,] = -p^\mu - 2iS^{\mu\nu}p_\nu \\
\not{p}'\gamma^\mu &= \tfrac{1}{2}\{\gamma^\mu, \not{p}'\} - \tfrac{1}{2}[\gamma^\mu, \not{p}'] = -p'^\mu + 2iS^{\mu\nu}p'_\nu
\end{aligned}$$

add the two equations, and sandwich them between spinors, and use:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\begin{aligned}
(\not{p} + m)u(\mathbf{p}) &= 0 \\
(-\not{p} + m)v(\mathbf{p}) &= 0
\end{aligned}
\qquad
\begin{aligned}
\overline{u}_s(\mathbf{p})(\not{p} + m) &= 0 \\
\overline{v}_s(\mathbf{p})(-\not{p} + m) &= 0
\end{aligned}$$

An important special case $p' = p$:

$$\begin{aligned}
\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) &= 2p^\mu \delta_{s's} \\
\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) &= 2p^\mu \delta_{s's}
\end{aligned}$$

One can also show:

$$\begin{aligned}
\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) &= 0 \\
\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) &= 0
\end{aligned}$$

homework

Gordon identities with gamma-5:

$$\begin{aligned}
\overline{u}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]\gamma_5 u_s(\mathbf{p}) &= 0 \\
\overline{v}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]\gamma_5 v_s(\mathbf{p}) &= 0
\end{aligned}$$

homework

We will find very useful the spin sums of the form:

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p})$$

can be directly calculated but we will find the correct for by the following argument: the sum over spin removes all the memory of the spin-quantization axis, and the result can depend only on the momentum four-vector and gamma matrices with all indices contracted.

In the rest frame, $\not{p} = -m\gamma^0$, we have:

$$\sum_{s=\pm} u_s(\mathbf{0})\overline{u}_s(\mathbf{0}) = m\gamma^0 + m$$
$$\sum_{s=\pm} v_s(\mathbf{0})\overline{v}_s(\mathbf{0}) = m\gamma^0 - m$$

Thus we conclude:

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m$$
$$\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p} - m$$

if instead of the spin sum we need just a specific spin product, e.g.

$$u_+(\mathbf{p})\overline{u}_+(\mathbf{p})$$

we can get it using appropriate spin projection matrices:

in the rest frame we have

$$\tfrac{1}{2}(1 + 2sS_z)u_{s'}(\mathbf{0}) = \delta_{ss'}\, u_{s'}(\mathbf{0})$$
$$\tfrac{1}{2}(1 - 2sS_z)v_{s'}(\mathbf{0}) = \delta_{ss'}\, v_{s'}(\mathbf{0})$$

$$S_z u_\pm(\mathbf{0}) = \pm\tfrac{1}{2}u_\pm(\mathbf{0})$$
$$S_z v_\pm(\mathbf{0}) = \mp\tfrac{1}{2}v_\pm(\mathbf{0})$$

the spin matrix $S_z = \frac{i}{2}\gamma^1\gamma^2$ can be written as:

$$S_z = -\tfrac{1}{2}\gamma_5\gamma^3\gamma^0$$

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

in the rest frame we can write γ^0 as $-\not{p}/m$ and γ^3 as $\not{z}$ and so we have:

$$S_z = \frac{1}{2m}\gamma_5\not{z}\not{p}$$

$$z^\mu = (0, \hat{\mathbf{z}})$$
$$z^2 = 1$$
$$z\cdot p = 0$$

we can now boost it to any frame simply by replacing z and p with their values in that frame

frame independent

Boosting to a different frame we get:

$$\tfrac{1}{2}(1 + 2sS_z)u_{s'}(\mathbf{0}) = \delta_{ss'}\, u_{s'}(\mathbf{0})$$
$$\tfrac{1}{2}(1 - 2sS_z)v_{s'}(\mathbf{0}) = \delta_{ss'}\, v_{s'}(\mathbf{0})$$

$$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})$$
$$v_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})$$

$$S_z = \frac{1}{2m}\gamma_5\not{z}\not{p}$$
$$(\not{p} + m)u(\mathbf{p}) = 0$$
$$(-\not{p} + m)v(\mathbf{p}) = 0$$

$$\tfrac{1}{2}(1 - s\gamma_5\not{z})u_{s'}(\mathbf{p}) = \delta_{ss'}\, u_{s'}(\mathbf{p})$$
$$\tfrac{1}{2}(1 - s\gamma_5\not{z})v_{s'}(\mathbf{p}) = \delta_{ss'}\, v_{s'}(\mathbf{p})$$

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m$$
$$\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p} - m$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5\not{z})(-\not{p} + m)$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5\not{z})(-\not{p} - m)$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5\not{z})(-\not{p} + m)$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5\not{z})(-\not{p} - m)$$

Let's look at the situation with 3-momentum in the z-direction:

The component of the spin in the direction of the 3-momentum is called the helicity (a fermion with helicity +1/2 is called right-handed, a fermion with helicity -1/2 is called left-handed.

rapidity

$$\tfrac{1}{m}p^\mu = (\cosh\eta, 0, 0, \sinh\eta)$$
$$z^\mu = (\sinh\eta, 0, 0, \cosh\eta)$$

$$z^2 = 1$$
$$z\cdot p = 0$$

In the limit of large rapidity

$$z^\mu = \frac{1}{m}p^\mu + O(e^{-\eta})$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} + m)$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} - m)$$

In the limit of large rapidity

$$z^\mu = \tfrac{1}{m} p^\mu + O(e^{-\eta})$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) \to \tfrac{1}{2}(1 + s\gamma_5)(-\not{p})$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) \to \tfrac{1}{2}(1 - s\gamma_5)(-\not{p})$$

dropped m, small relative to p

In the extreme relativistic limit the right-handed fermion (helicity +1/2) (described by spinors u+ for b-type particle and v- for d-type particle) is projected onto the lower two components only (part of the Dirac field that corresponds to the right-handed Weyl field). Similarly left-handed fermions are projected onto upper two components (right-handed Weyl field.

Formulas relevant for massless particles can be obtained from considering the extreme relativistic limit of a massive particle; in particular the following formulas are valid when setting $m = 0$:

$$(\not{p} + m)u_s(\mathbf{p}) = 0$$
$$(-\not{p} + m)v_s(\mathbf{p}) = 0$$

$$\overline{u}_s(\mathbf{p})(\not{p} + m) = 0$$
$$\overline{v}_s(\mathbf{p})(-\not{p} + m) = 0$$

$$\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = +2m\,\delta_{s's}\,,$$
$$\overline{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) = -2m\,\delta_{s's}\,,$$
$$\overline{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) = 0\,,$$
$$\overline{v}_{s'}(\mathbf{p})u_s(\mathbf{p}) = 0\,.$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0$$

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m$$
$$\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p} - m$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) \to \tfrac{1}{2}(1 + s\gamma_5)(-\not{p})$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) \to \tfrac{1}{2}(1 - s\gamma_5)(-\not{p})$$

becomes exact

Canonical quantization of spinor fields II

based on S-39

Lagrangian for a Dirac field:

$$\mathcal{L} = i\overline{\Psi}\not{\partial}\Psi - m\overline{\Psi}\Psi$$

canonical anticommutation relations:

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0\,,$$
$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})$$

The general solution to the Dirac equation:

$$(-i\not{\partial} + m)\Psi = 0$$

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]$$

four-component spinors

creation and annihilation operators

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]$$

We want to find formulas for creation and annihilation operator:

$$\int d^3x\; e^{-ipx}\Psi(x) = \sum_{s'=\pm}\left[\tfrac{1}{2\omega}b_{s'}(\mathbf{p})u_{s'}(\mathbf{p}) + \tfrac{1}{2\omega}e^{2i\omega t}d_{s'}^\dagger(-\mathbf{p})v_{s'}(-\mathbf{p})\right]$$

multiply by $\overline{u}_s(\mathbf{p})\gamma^0$ on the left:

$$\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$

$$b_s(\mathbf{p}) = \int d^3x\; e^{-ipx}\,\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)$$

for the hermitian conjugate we get:

$$\left[\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)\right]^\dagger = \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})$$

$$b_s^\dagger(\mathbf{p}) = \int d^3x\; e^{ipx}\,\overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})$$

b’s are time independent!

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

similarly for d:

$$\int d^3x \; e^{ipx} \Psi(x) = \sum_{s'=\pm} \left[\tfrac{1}{2\omega} e^{-2i\omega t} b_{s'}(-\mathbf{p}) u_{s'}(-\mathbf{p}) + \tfrac{1}{2\omega} d_{s'}^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) \right]$$

multiply by $\overline{v}_s(\mathbf{p})\gamma^0$ on the left:

$$\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0$$

$$\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx} \, \overline{v}_s(\mathbf{p}) \gamma^0 \Psi(x)$$

for the hermitian conjugate we get:

$$d_s(\mathbf{p}) = \int d^3x \; e^{-ipx} \, \overline{\Psi}(x) \gamma^0 v_s(\mathbf{p})$$

$$b_s(\mathbf{p}) = \int d^3x \; e^{-ipx} \, \overline{u}_s(\mathbf{p}) \gamma^0 \Psi(x) \qquad d_s(\mathbf{p}) = \int d^3x \; e^{-ipx} \, \overline{\Psi}(x) \gamma^0 v_s(\mathbf{p})$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx} \, \overline{v}_s(\mathbf{p}) \gamma^0 \Psi(x) \qquad b_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx} \, \overline{\Psi}(x) \gamma^0 u_s(\mathbf{p})$$

we can easily work out the anticommutation relations for b and d operators:

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0 \, ,$$
$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta} \, \delta^3(\mathbf{x}-\mathbf{y})$$

$$\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} = 0 \qquad \{b_s^\dagger(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} = 0$$
$$\{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0 \qquad \{d_s^\dagger(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = 0$$
$$\{b_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = 0 \qquad \{b_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0$$

$$b_s(\mathbf{p}) = \int d^3x \; e^{-ipx} \, \overline{u}_s(\mathbf{p}) \gamma^0 \Psi(x)$$

$$b_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx} \, \overline{\Psi}(x) \gamma^0 u_s(\mathbf{p})$$

we can easily work out the anticommutation relations for b and d operators:

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0 \, ,$$
$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta} \, \delta^3(\mathbf{x}-\mathbf{y})$$

$$\begin{aligned}
\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= \int d^3x \, d^3y \, e^{-ipx+ip'y} \, \overline{u}_s(\mathbf{p}) \gamma^0 \{\Psi(x), \overline{\Psi}(y)\} \gamma^0 u_{s'}(\mathbf{p}') \\
&= \int d^3x \, e^{-i(p-p')x} \, \overline{u}_s(\mathbf{p}) \gamma^0 \gamma^0 \gamma^0 u_{s'}(\mathbf{p}') \\
&= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}') \, \overline{u}_s(\mathbf{p}) \gamma^0 u_{s'}(\mathbf{p}) \\
&= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}') \, 2\omega \delta_{ss'} \, .
\end{aligned}$$

$$(\gamma^0)^2 = 1$$

$$\overline{u}_s(\mathbf{p}) \gamma^0 u_{s'}(\mathbf{p}) = 2\omega \delta_{ss'}$$

$$d_s(\mathbf{p}) = \int d^3x \; e^{-ipx} \, \overline{\Psi}(x) \gamma^0 v_s(\mathbf{p})$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx} \, \overline{v}_s(\mathbf{p}) \gamma^0 \Psi(x)$$

similarly:

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0 \, ,$$
$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta} \, \delta^3(\mathbf{x}-\mathbf{y})$$

$$\begin{aligned}
\{d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= \int d^3x \, d^3y \, e^{ipx-ip'y} \, \overline{v}_s(\mathbf{p}) \gamma^0 \{\Psi(x), \overline{\Psi}(y)\} \gamma^0 v_{s'}(\mathbf{p}') \\
&= \int d^3x \, e^{i(p-p')x} \, \overline{v}_s(\mathbf{p}) \gamma^0 \gamma^0 \gamma^0 v_{s'}(\mathbf{p}') \\
&= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}') \, \overline{v}_s(\mathbf{p}) \gamma^0 v_{s'}(\mathbf{p}) \\
&= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}') \, 2\omega \delta_{ss'} \, .
\end{aligned}$$

$$b_s(\mathbf{p}) = \int d^3x\; e^{-ipx}\,\overline{u}_s(\mathbf{p})\gamma^0\Psi(x) \qquad\qquad d_s(\mathbf{p}) = \int d^3x\; e^{-ipx}\,\overline{\Psi}(x)\gamma^0 v_s(\mathbf{p})$$

and finally:

$$\{\Psi_\alpha(\mathbf{x},t),\Psi_\beta(\mathbf{y},t)\} = 0\,,$$
$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\begin{aligned}\{b_s(\mathbf{p}),d_{s'}(\mathbf{p}')\} &= \int d^3x\,d^3y\,e^{-ipx-ip'y}\,\overline{u}_s(\mathbf{p})\gamma^0\{\Psi(x),\overline{\Psi}(y)\}\gamma^0 v_{s'}(\mathbf{p}')\\ &= \int d^3x\,e^{-i(p+p')x}\,\overline{u}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 v_{s'}(\mathbf{p}')\\ &= (2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}')\,\overline{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p})\\ &= 0\,.\end{aligned}$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]$$

We want to calculate the hamiltonian in terms of the b and d operators; in the four-component notation we would find:

$$H = \int d^3x\;\overline{\Psi}(-i\gamma^i\partial_i + m)\Psi$$

let's start with:

$$\begin{aligned}(-i\gamma^i\partial_i + m)\Psi &= \sum_{s=\pm}\int\widetilde{dp}\,\Big(-i\gamma^i\partial_i + m\Big)\Big(b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx}\\ &\qquad\qquad + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\Big)\\ &= \sum_{s=\pm}\int\widetilde{dp}\,\Big[b_s(\mathbf{p})(+\gamma^i p_i + m)u_s(\mathbf{p})e^{ipx}\\ &\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^i p_i + m)v_s(\mathbf{p})e^{-ipx}\Big]\\ &= \sum_{s=\pm}\int\widetilde{dp}\,\Big[b_s(\mathbf{p})(\gamma^0\omega)u_s(\mathbf{p})e^{ipx}\\ &\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^0\omega)v_s(\mathbf{p})e^{-ipx}\Big].\end{aligned}$$

$$(\not{p} + m)u_s(\mathbf{p}) = 0$$
$$(-\not{p} + m)v_s(\mathbf{p}) = 0$$

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]$$

$$H = \int d^3x\;\overline{\Psi}(-i\gamma^i\partial_i + m)\Psi$$

$$(-i\gamma^i\partial_i + m)\Psi = \sum_{s=\pm}\int\widetilde{dp}\,\Big[b_s(\mathbf{p})(\gamma^0\omega)u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})(-\gamma^0\omega)v_s(\mathbf{p})e^{-ipx}\Big]$$

thus we have:

$$\begin{aligned}H &= \sum_{s,s'}\int\widetilde{dp}\,\widetilde{dp}'\,d^3x\,\Big(b_{s'}^\dagger(\mathbf{p}')\overline{u}_{s'}(\mathbf{p}')e^{-ip'x} + d_{s'}(\mathbf{p}')\overline{v}_{s'}(\mathbf{p}')e^{ip'x}\Big)\\ &\qquad\times\omega\,\Big(b_s(\mathbf{p})\gamma^0 u_s(\mathbf{p})e^{ipx} - d_s^\dagger(\mathbf{p})\gamma^0 v_s(\mathbf{p})e^{-ipx}\Big)\\ &= \sum_{s,s'}\int\widetilde{dp}\,\widetilde{dp}'\,d^3x\,\omega\,\Big[\,b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p})\;\overline{u}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{-i(p'-p)x}\\ &\qquad - b_{s'}^\dagger(\mathbf{p}')d_s^\dagger(\mathbf{p})\;\overline{u}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{-i(p'+p)x}\\ &\qquad + d_{s'}(\mathbf{p}')b_s(\mathbf{p})\;\overline{v}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{+i(p'+p)x}\\ &\qquad - d_{s'}(\mathbf{p}')d_s^\dagger(\mathbf{p})\;\overline{v}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{+i(p'-p)x}\Big]\end{aligned}$$

$$\begin{aligned}H &= \int d^3x\;\overline{\Psi}(-i\gamma^i\partial_i + m)\Psi\\ &= \sum_{s,s'}\int\widetilde{dp}\,\widetilde{dp}'\,d^3x\,\omega\,\Big[\,b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p})\;\overline{u}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{-i(p'-p)x}\\ &\qquad - b_{s'}^\dagger(\mathbf{p}')d_s^\dagger(\mathbf{p})\;\overline{u}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{-i(p'+p)x}\\ &\qquad + d_{s'}(\mathbf{p}')b_s(\mathbf{p})\;\overline{v}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{+i(p'+p)x}\\ &\qquad - d_{s'}(\mathbf{p}')d_s^\dagger(\mathbf{p})\;\overline{v}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{+i(p'-p)x}\Big]\\ &= \sum_{s,s'}\int\widetilde{dp}\;\tfrac{1}{2}\Big[\,b_{s'}^\dagger(\mathbf{p})b_s(\mathbf{p})\;\overline{u}_{s'}(\mathbf{p})\gamma^0 u_s(\mathbf{p})\\ &\qquad - b_{s'}^\dagger(-\mathbf{p})d_s^\dagger(\mathbf{p})\;\overline{u}_{s'}(-\mathbf{p})\gamma^0 v_s(\mathbf{p})\;e^{+2i\omega t}\\ &\qquad + d_{s'}(-\mathbf{p})b_s(\mathbf{p})\;\overline{v}_{s'}(-\mathbf{p})\gamma^0 u_s(\mathbf{p})\;e^{-2i\omega t}\\ &\qquad - d_{s'}(\mathbf{p})d_s^\dagger(\mathbf{p})\;\overline{v}_{s'}(\mathbf{p})\gamma^0 v_s(\mathbf{p})\Big]\\ &= \sum_s\int\widetilde{dp}\;\omega\,\Big[\,b_s^\dagger(\mathbf{p})b_s(\mathbf{p}) - d_s(\mathbf{p})d_s^\dagger(\mathbf{p})\Big].\end{aligned}$$

$$\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$$
$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0$$

$$H = \sum_s \int \widetilde{dp}\ \omega \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - d_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) \right]$$

$$\{ d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}') \} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')\, 2\omega \delta_{ss'}$$

finally, we find:

$$H = \sum_{s=\pm} \int \widetilde{dp}\ \omega \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) + d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \right] - 4\mathcal{E}_0 V$$

$$V = (2\pi)^3 \delta^3(\mathbf{0}) = \int d^3x$$

$$\mathcal{E}_0 = \tfrac{1}{2} (2\pi)^{-3} \int d^3k\ \omega$$

four times the zero-point energy of a scalar field and opposite sign!

we will assume that the zero-point energy is cancelled by a constant term

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

spin-1/2 states:

vacuum:

$$|0\rangle$$

$$b_s(\mathbf{p})|0\rangle = d_s(\mathbf{p})|0\rangle = 0$$

b-type particle with momentum $\mathbf{p}$, energy $\omega = (\mathbf{p}^2 + m^2)^{1/2}$, and spin $S_z = \frac{1}{2}s$:

$$|p, s, +\rangle = b_s^\dagger(\mathbf{p})|0\rangle$$

labels the charge of a particle

d-type particle with momentum $\mathbf{p}$, energy $\omega = (\mathbf{p}^2 + m^2)^{1/2}$, and spin $S_z = \frac{1}{2}s$:

$$|p, s, -\rangle = d_s^\dagger(\mathbf{p})|0\rangle$$

b- and d-type particles are distinguished by the value of the charge:

$$Q = \int d^3x\, j^0$$

$$j^\mu = \overline{\Psi} \gamma^\mu \Psi$$

$$\Psi \to e^{-i\alpha} \Psi, \quad \overline{\Psi} \to e^{+i\alpha} \overline{\Psi}$$

very similar calculation as for the hamiltonian; we get:

$$\begin{aligned} Q &= \int d^3x\, \overline{\Psi} \gamma^0 \Psi \\ &= \sum_{s=\pm} \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) + d_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) \right] \\ &= \sum_{s=\pm} \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \right] + \text{constant} \end{aligned}$$

counts the number of b-type particles - the number of d-type particles

(later, the electron will be a b-type particle and the positron a d-type particle)

For a Majorana field:

$$\mathcal{L} = \tfrac{i}{2} \Psi^{\mathrm{T}} \mathcal{C} \partial\!\!\!/ \Psi - \tfrac{1}{2} m \Psi^{\mathrm{T}} \mathcal{C} \Psi$$

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

we need to incorporate the Majorana condition:

$$\Psi = \mathcal{C} \overline{\Psi}^{\mathrm{T}}$$

$$\overline{\Psi}(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p}) \overline{u}_s(\mathbf{p}) e^{-ipx} + d_s(\mathbf{p}) \overline{v}_s(\mathbf{p}) e^{ipx} \right]$$

$$\mathcal{C} \overline{\Psi}^{\mathrm{T}}(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p})\, \mathcal{C} \overline{u}_s^{\mathrm{T}}(\mathbf{p}) e^{-ipx} + d_s(\mathbf{p})\, \mathcal{C} \overline{v}_s^{\mathrm{T}}(\mathbf{p}) e^{ipx} \right]$$

$$\mathcal{C} \overline{u}_s(\mathbf{p})^{\mathrm{T}} = v_s(\mathbf{p})$$

$$\mathcal{C} \overline{v}_s(\mathbf{p})^{\mathrm{T}} = u_s(\mathbf{p})$$

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$$\mathcal{C} \overline{\Psi}^{\mathrm{T}}(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} + d_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} \right]$$

$$d_s(\mathbf{p}) = b_s(\mathbf{p})$$

we have just used:

$$\mathcal{C}\overline{u}_s(\mathbf{p})^{\mathrm{T}} = v_s(\mathbf{p})$$
$$\mathcal{C}\overline{v}_s(\mathbf{p})^{\mathrm{T}} = u_s(\mathbf{p})$$

Proof:

$$\mathcal{C} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{aligned} \overline{u}_+(\mathbf{0}) &= \sqrt{m}\,(1,\,0,\,1,\,0)\,, \\ \overline{u}_-(\mathbf{0}) &= \sqrt{m}\,(0,\,1,\,0,\,1)\,, \\ \overline{v}_+(\mathbf{0}) &= \sqrt{m}\,(0,\,-1,\,0,\,1)\,, \\ \overline{v}_-(\mathbf{0}) &= \sqrt{m}\,(1,\,0,\,-1,\,0)\,. \end{aligned}$$

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \quad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

by direct calculation:

$$\mathcal{C}\overline{u}_s(\mathbf{0})^{\mathrm{T}} = v_s(\mathbf{0})$$
$$\mathcal{C}\overline{v}_s(\mathbf{0})^{\mathrm{T}} = u_s(\mathbf{0})$$

boosting to any frame we get:

$$\begin{aligned} \overline{u}_s(\mathbf{p}) &= \overline{u}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K}) \\ \overline{v}_s(\mathbf{p}) &= \overline{v}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K}) \\ u_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0}) \\ v_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0}) \end{aligned}$$

$$\beta\mathcal{C} = -\mathcal{C}\beta$$
$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}}$$
$$K^j = \tfrac{i}{4}[\gamma^j,\gamma^0] = \tfrac{i}{2}\gamma^j\gamma^0$$
$$\mathcal{C}^{-1}K^j\mathcal{C} = -(K^j)^{\mathrm{T}}$$

$$\mathcal{C}\overline{u}_s(\mathbf{p})^{\mathrm{T}} = v_s(\mathbf{p})$$
$$\mathcal{C}\overline{v}_s(\mathbf{p})^{\mathrm{T}} = u_s(\mathbf{p})$$

We have found that a Majorana field can be written:

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + b_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]$$

canonical anticommutation relations:

$$\{\Psi_\alpha(\mathbf{x},t),\Psi_\beta(\mathbf{y},t)\} = (\mathcal{C}\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\,,$$
$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\,,$$

translate into:

$$\{b_s(\mathbf{p}),b_{s'}(\mathbf{p}')\} = 0\,,$$
$$\{b_s(\mathbf{p}),b_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\delta_{ss'}\,,$$

calculation the same as for the Dirac field

The hamiltonian for the Majorana field is:

$$\begin{aligned} H &= \tfrac{1}{2}\int d^3x\;\Psi^{\mathrm{T}}\mathcal{C}(-i\gamma^i\partial_i + m)\Psi \\ &= \tfrac{1}{2}\int d^3x\;\overline{\Psi}(-i\gamma^i\partial_i + m)\Psi\,, \end{aligned}$$

and repeating the same manipulations as for the Dirac field we would find:

$$H = \tfrac{1}{2}\sum_{s=\pm}\int \widetilde{dp}\;\omega\left[b_s^\dagger(\mathbf{p})b_s(\mathbf{p}) - b_s(\mathbf{p})b_s^\dagger(\mathbf{p})\right]$$

$$\{b_s(\mathbf{p}),b_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\delta_{ss'}\,,$$

$$H = \sum_{s=\pm}\int \widetilde{dp}\;\omega\,b_s^\dagger(\mathbf{p})b_s(\mathbf{p}) - 2\mathcal{E}_0V.$$

two times the zero-point energy of a scalar field and opposite sign!

$$V = (2\pi)^3\delta^3(\mathbf{0}) = \textstyle\int d^3x$$
$$\mathcal{E}_0 = \tfrac{1}{2}(2\pi)^{-3}\textstyle\int d^3k\;\omega$$

we will assume that the zero-point energy is cancelled by a constant term