Perturbative Topological Field Theory

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Abstract

We give a review of the application of perturbative techniques to topological quantum field theories, in particular three-dimensional Chern-Simons-Witten theory and its various generalizations. To this end we give an introduction to graph homology and homotopy algebras and the work of Vassiliev and Kontsevich on perturbative knot invariants.

1. Introduction

In these lecture notes we will give a review of some recent mathematical developments in topological field theory following the work of Kontsevich, Axelrod and Singer, Vassiliev, Bar-Natan, Witten and others. Quite remarkable these new ideas are all related to old-fashioned perturbative techniques in field theory. Indeed, it is an interesting comment on the development of the interaction between physics and mathematics that these days pure mathematicians are calculating Feynman diagrams whereas this skill is slowly disappearing among a large fraction of theoretical physics students.

Our starting point in all this will be mostly three-dimensional Chern-Simons-Witten gauge theory [2]. This topological field theory lends itself most conveniently to a perturbative formulation. Actually, one can argue that the model is even more elegant and symmetric in perturbation theory. The reason behind this phenomenon has been pointed out be Witten: Chern-Simons is (an exact low-energy limit of) an open string field theory [1].

The perturbative aspects of Chern-Simons theory have been studied in many papers, starting soon after the (non-perturbative) solution of Witten in [2], see *e.g.*

[3]. A very precise analysis aimed at mathematical rigor has been performed by Axelrod and Singer [4], and we will follow their work closely. The innovations and generalizations we will discuss, in particularly the relations with homotopy algebras, follow the beautiful ideas of Kontsevich [5]. The applications to knot invariants center around the work Vassiliev [6], which has been a great step forward in the classification program of knots. The relation of the work of Vassiliev to perturbative field theory was first pointed out by Bar-Natan [7]. The power of this approach is mainly due to a wonderful theorem proved by Kontsevich [8].

It is well-known how the partition function of Chern-Simons-Witten gauge theory, when considered on a closed three-manifold M, produces a topological invariant $Z(M)$. Although this quantity can be defined non-perturbatively, and the coupling constant $\hbar = 1/k$ is quantized, the resulting expression is analytic in $\hbar$ and thus has a well-defined perturbative expansion

$$Z(M) \sim \exp\left(\sum_{n=0}^{\infty} F_n(M)\hbar^n\right). \tag{1.1}$$

The coefficients $F_n(M)$ define *perturbative* invariants that only depend on calculations of connected Feynman diagrams up to n-loops. They can therefore be considered as finite order invariants. This point of view has some interesting consequences and advantages that we will develop in these notes.

1. Finite order invariants can be calculated evaluating finite-dimensional integrals. They therefore stand a greater chance of being rigorously defined. This is of course a very familiar argument for particle physicists!

2. A perturbative approach greatly facilitates classification, since we now have a natural filtration on the space of invariants by order in perturbation theory. If we are lucky the space of invariants of a given order can turn out be finite and can be analyzed using combinatorical means. This turns out to be the case for knot invariants as we will see in section 4.

3. A perturbative framework can be used to generalize the concept of a topological invariant. If we are somewhat pedantic we can call a (smooth) manifold invariant a function $Z(g)$ that is invariant under smooth changes in the Riemannian metric g, *i.e.*

$$Z(g + \delta g) = Z(g), \quad \text{or} \quad \delta Z = 0, \tag{1.2}$$

where δ is the exterior differential on the space of (isomorphism classes of) Riemannian manifolds $B = \mathrm{Met}(M)/\mathrm{Diff}(M) = B\mathrm{Diff}(M)$. Stated otherwise, Z is locally constant function on B

$$Z \in H^0(B). \tag{1.3}$$

This point of view leads to the natural generalization where topological invariants are higher dimensional cohomology classes on the space of Riemannian structures

$$Z \in H^k(B). \tag{1.4}$$

This idea is actually well-known from string theory, where the partition function Z of the underlying conformal field theory on a Riemann surface is a volume form on the moduli space $\mathcal{M}_g$. The string partition function is then obtained as $\int_{\mathcal{M}_g} Z$.

4. Closely related to the previous remark is the fact that certain generalizations of the action become possible once we have obtained a gauge fixed, perturbative formulation in terms of Feynman diagrams. Stated simply, one can replace the cubic vertex in Chern-Simons theory by quartic, quintic etc. vertices. These higher order vertices typically involve interactions where fields, ghosts and antifields couple in a nontrivial way. This leads to algebraic structures with multilinear operations, so-called homotopy algebras [9]. These structures have recently been described in a very elegant way using the language of operads, see *e.g.* [10], a subject we will not touch upon in these notes.

5. Finally many of the above remarks have a clear interpretation from the poin of view of string (field) theory, where they help us understand the issues of background independence, space-time ghosts etc.

2. Algebras and Feynman Diagrams

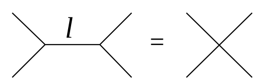
2.1. Graph Cohomology

It is often stated that one of the elegant features of string theory is that one has to consider only one diagram in the topological expansion: a Riemann surface of given genus, this in contrast with ordinary point-particle field theory with its many different Feynman diagrams. Of course, there still remains a very complicated integral to be done over the moduli space $\mathcal{M}_g$ of such a surface.

It is less appreciated that certain types of field theories can be formulated in a very similar flavour [5]. Indeed, let us consider one-dimensional quantum gravity where our 'space-time' is an arbitrary graph Γ. The space of metrics modulo diffeomorphisms on such a singular space is parametrized by an assignment of lengths $l_1, \ldots, l_E \geq 0$ to all the edges of the graph. In 1D quantum gravity we want to sum over all graphs and integrate over the lengths ℓ_i with some particular weights. Of course we can think of the lengths as the Schwinger parameters of the Feynman diagrams of quantum field theory. As such they are completely equivalent to the moduli of Riemann surfaces that appear in string theory. For example the usual field theory propagator takes the form

$$\int dl\, e^{-l(p^2+m^2)} = \frac{1}{p^2+m^2}. \tag{2.1}$$

The Schwinger parameter spaces of the individual graphs can be glued together to form a space that is very analogous to the moduli space $\mathcal{M}_g$ of Riemann surfaces. To every diagram Γ we can associate a cell $c_\Gamma \cong \mathbb{R}_+^E$ where a point $x \in c_\Gamma$ is parametrized as $x = (l_1, \ldots, l_E)$. These cells can be glued together in the following fashion. If one of the lengths, say l_1, becomes zero, the graph will change topology. If the particular propagator connects two different vertices of order n and m, the degenerated diagram will have one vertex instead of two, now of order $n+m-2$. For example in the case of two cubic vertices we will create a quartic vertex

$$\lim_{l\to 0}$$ 

(2.2)

In this process the number of loops will not change, the number of edges and the number of vertices are reduced by one.

Another case is an edge that starts and ends at the same vertex, say of order n. In the limit where the lenth of the edge tends to zero, we are left with a diagram with one loop less and a vertex now of order $n-2$. One verifies that in both cases the Euler number of the graph is reduced by one.

Clearly the boundary of the cell c_Γ will consist of a sum of lower dimensional cells (with signs) associated to graphs $\Gamma' = \Gamma/e$ where one propagator e is contracted to zero

$$\partial c_\Gamma = \sum_{edges\ e} \pm c_{\Gamma/e}. \tag{2.3}$$

We can write this more abstractly as $\partial c_\Gamma = c_{d\Gamma}$ where we introduce the boundary of a graph by

$$d\Gamma = \sum_{edges\ e} \Gamma/e. \tag{2.4}$$

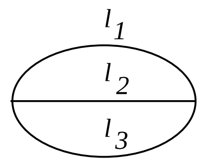
We can now make a moduli space $\mathcal{F}_{g,s}$ of all Feynman diagrams with g loops and s external lines by gluing together the cells that belong to graphs with only cubic vertices with appropriate symmetry factors such that the sum of the boundaries of all cells is zero

$$\mathcal{F}_{g,s} = \sum_{\Gamma\ cubic} \frac{c_\Gamma}{\#\mathrm{Aut}\,\Gamma}. \tag{2.5}$$

This space obviously has (real) dimension

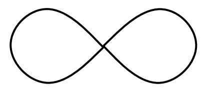
$$\dim \mathcal{F}_{g,s} = 3g - 3 + s. \tag{2.6}$$

We have to decide It is actually an orbifold space, since graphs typically have automorphisms. For example consider the graph



(2.7)

Whenever we degenerate one of the $l_i = 0$ we obtain the quartic diagram



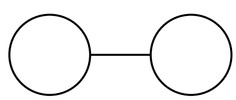
(2.8)

So we have to identify the three faces $l_i = 0$.

Graph cohomology can be defined as the cohomology of this moduli space $\mathcal{F}$. Actually following Kontsevich we can formulate graph cohomology just in terms of graphs. First we build a vector space C with a basis spanned by all graphs. We will take these graphs to be closed and with vertices of order ≥ 3. We further give them an orientation. We write $-\Gamma$ for the graph with the opposite orientation. This immediately has important consequences, because some graphs are actually isomorphic to their opposite

$$-\Gamma = \Gamma \;\Rightarrow\; \Gamma = 0. \tag{2.9}$$

If we work over the complex numbers these graphs therefore vanish. An example of such a graph is the quartic diagram above or the dumb-bell diagram



(2.10)

Of course the space of graphs is naturally graded

$$C = \bigoplus_{g=0}^{\infty} C^g \tag{2.11}$$

where C^g are the graphs with g loops, *i.e.*

$$\#\text{vertices} - \#\text{edges} = 1 - g. \tag{2.12}$$

Now the spaces C^g are actually complexes. We can write

$$C^g = \bigoplus_{k=0}^{3g-3} C^g_k, \tag{2.13}$$

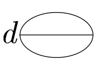
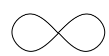
where k is the number of edges. We now have a natural boundary map

$$d : C^g_k \to C^g_{k-1}, \quad \Gamma \to d\Gamma = \sum_{edge\ e} \Gamma/e \tag{2.14}$$

where we sum over the contraction of edges. It is an interesting exercise to verify that indeed $d^2 = 0$. We can now define chains as linear combination of graphs

$$a = \sum_{\Gamma} a_\Gamma \Gamma, \qquad a_\Gamma \in \mathbb{C} \tag{2.15}$$

A closed chain will now be a chain that satisfies $da = 0$. An example of a closed graph is the graph

d  $=$  $= 0.$ (2.16)

We can also define homology classes by

$$da = 0, \qquad a \cong a + db. \tag{2.17}$$

Similarly graph cohomology classes are linear functions $\Phi : C \to \mathbb{C}$ such that $d\Phi = 0$, $\Phi \cong \Phi + d\Psi$, where we simply define

$$d\Phi(\Gamma) = \Phi(d\Gamma) \tag{2.18}$$

Cohomology classes thus vanish on exact graphs. We should think of graph cohomology classes as very special Feynman rules, that respect the degeneration of graphs. That is, any combination of graphs that is a boundary of another graph vanishes according to these rules.

2.2. Lie algebras

An interesting example of such a set of Feynman rules that we will meet many times in these lectures is the one based on Lie algebras and, more general, homotopy Lie algebras. Recall that a Lie algebra is nothing but a vector space V with a bilinear bracket

$$[\cdot,\cdot] : V \otimes V \to V, \qquad u, v \to [u, v], \tag{2.19}$$

that satisfies two conditions: symmetry

$$[u, v] = -[v, u] \tag{2.20}$$

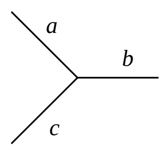
and the Jacobi relation

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0. \tag{2.21}$$

If we choose an explicit basis e_a, then we have structure coefficients

$$[e_a, e_b] = \sum_c f_{ab}{}^c e_c. \tag{2.22}$$

If the Lie algebra is even (bosonic) and simple, it has a unique invariant inner product η_{ab}, and we can introduce completely antisymmetric tensors f_{abc} that we can use as vertices of Feynman graphs



$$= f_{abc}. \tag{2.23}$$

Since this vertex is antisymmetric we should orient it to make it well-defined, for example using blackboard orientation. With this convention the Jacobi identity

$$\sum_e (f_{abe} f_{edc} + f_{dae} f_{ebc} + f_{ace} f_{edb}) = 0, \tag{2.24}$$

reads pictorially

$$\sum_e \qquad\qquad = 0. \tag{2.25}$$

We can now write an interesting graph homology class as follows. For every closed cubic graph Γ let $I(\Gamma)$ be the weight associated to it by the above Feynman rules. We now define the chain by a summation over all cubic g-loop diagrams

$$a_g = \sum_{cubic\ \Gamma \in C_g} \frac{I(\Gamma)\cdot\Gamma}{\#\mathrm{Aut}\,\Gamma}. \tag{2.26}$$

We now claim that $da_g = 0$. This is actual a direct consequence of the Jacobi identity. The diagrams that appear in da_g will have one (and only one) quartic vertex. Such a diagram can be obtained as the boundary of *three* different cubic graphs. That is to say, there are three inequivalent way to resolve the quartic vertex into two cubic vertices. These are of course the s, t and u-channel diagrams, which we will denote as Γ_s, Γ_t and Γ_u. So the total weight associated to this particular quartic diagram will be

$$I(\Gamma_s) + I(\Gamma_t) + I(\Gamma_u) = 0, \tag{2.27}$$

which vanishes precisely by the Jacobi identity. To produce other graph homology classes we have to go to homotopy algebras.

2.3. Homotopy Algebras

The concept of a homotopy algebra is quite general and exists in many different contexts such as commutative, associate, Lie or differential graded algebras. The unifying principle is that the algebraic operations are no longer restricted to be binary [9]. As such the most natural language to describe homotopy structures in algebra is the formalism of operads or trees [10].

However, here we will take a much more down to earth point of view. To be as concrete as possible we also restrict ourselves first to homotopy Lie algebras. We will start from the beginning with graded algebras. That is, our elements might be either commuting or anticommuting. So a Lie algebra can also be a super Lie algebra.

For a homotopy Lie algebra we just introduce generalized brackets on N elements

$$u_1, \ldots, u_N \in V \to [u_1, \ldots, u_N] \in V, \tag{2.28}$$

that again satisfy two relations: symmetry and Jacobi. The symmetry relation now reads

$$[u_1, \ldots, u_i, \ldots, u_j, \ldots, u_N] = \pm[u_1, \ldots, u_j, \ldots, u_i, \ldots, u_N], \tag{2.29}$$

and the generalized Jacobi identity takes the form

$$\sum_{k=1}^{N} \pm[u_1, \ldots, u_k[u_{k+1}, \ldots, u_N]] \pm perm = 0, \tag{2.30}$$

where we refer to the literature [9] for the precise choices of signs. The latter relation can be written much more clearly in terms of Feynman graphs. Hereto we assume that the algebra V also has an invariant inner product by which we can raise and lower indices. We can than define structure coefficients

$$f_{a_0 a_1 \ldots a_N} = \sum_b \eta_{a_0 b} f_{a_1 \ldots a_N}{}^b \tag{2.31}$$

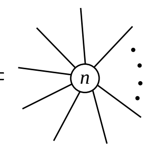
with

$$[e_{a_1}, \ldots, e_{a_N}] = \sum_b f_{a_1 \ldots a_N}{}^b e_a \tag{2.32}$$

We can use these fully graded symmetric tensors as vertices of our Feynman rules.

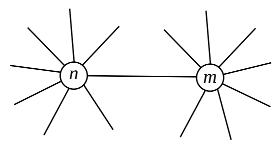
For a vertex of order n we write symbolically

$$f_{a_0 a_1 \ldots a_N} = \tag{2.33}$$



The Jacobi identity then takes the form

$$\sum_{m+n=N} \quad = 0 \tag{2.34}$$



It is now clear how to employ these homotopy algebras to produce graph cycles. We simply sum over *all* graph of a given number of loops (not necessarily only the cubic ones) and associate to each graph Γ the above weight $I(\Gamma)$. The classes

$$a_g = \sum_{\Gamma \in C_g} \frac{I(\Gamma) \cdot \Gamma}{\# \mathrm{Aut}\, \Gamma} \tag{2.35}$$

are easily seen to be closed. The above algebraic structure can be formulated a bit more elegantly as follows: We will consider an algebra A with a multiplication

$$m \in \mathrm{Hom}(A \otimes A, A), \tag{2.36}$$

that satisfy a quadratic relation that we will write symbolically as

$$m \circ m = 0, \tag{2.37}$$

with $m \circ m \in \mathrm{Hom}(A \otimes A \otimes A, A)$. Familiar examples are: (1) Associative algebras with a multiplication that we will write as

$$m(a,b) = a \cdot b, \tag{2.38}$$

and that satisfies the quadratic relation

$$(m \circ m)(a,b,c) = (a \cdot b) \cdot c - a \cdot (b \cdot c) = 0. \tag{2.39}$$

(2) Commutative, associative algebras with the further constrained

$$m(a, b) = m(b, a). \tag{2.40}$$

(3) Lie algebras, where

$$m(a, b) = [a, b] = -[b, a], \tag{2.41}$$

and the quadratic condition is the Jacobi identity

$$(m \circ m)(a, b, c) = [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0. \tag{2.42}$$

We can now try to generalize the algebraic operation m to an operation on n elements of A with $n \neq 2$.

The first nontrivial generalizations are differential graded algebras (DGAs), such as the familiar example of the space $\Omega^*(M)$ of differential forms on a manifold M. This is a (graded) commutative associative algebra, with multiplication the wedge product

$$a \in \Omega^m,\ b \in \Omega^n \to a \wedge b \in \Omega^{m+n}, \qquad b \wedge a = (-1)^{ab} a \wedge b. \tag{2.43}$$

This algebra also has a derivation d, the exterior differential,

$$d : \Omega^n \to \Omega^{n+1}, \qquad d(a \wedge b) = da \wedge b + (-1)^a a \wedge db, \tag{2.44}$$

that is nilpotent: $d^2 = 0$. Generally a DGA has by definition a nilpotent linear operator of degree one

$$d \in \mathrm{Hom}(A, A), \qquad d^2 = 0, \tag{2.45}$$

that is a derivation

$$d(m(a, b)) = m(da, b) + (-)^a m(a, db). \tag{2.46}$$

Conditions (2.37), (2.45) and (2.46) can be combined in writing

$$\mu \circ \mu = 0, \tag{2.47}$$

where μ is now the non-homogeneous linear combination

$$\mu = d + m \in \mathrm{Hom}(A, A) \oplus \mathrm{Hom}(A \otimes A, A) \tag{2.48}$$

and the operation $\circ$ is defined in the obvious way. So we see that DGAs fit in the same framework as ordinary algebras if we generalize the 'multiplication' operation to involve both one and two elements.

Of course we can also take more than two elements which leads us to homotopy algebras with an algebraic operation

$$\mu \in \bigoplus_{k \geq 0} \mathrm{Hom}(A^{\otimes k}, A), \qquad \mu = \sum_k \mu_k \tag{2.49}$$

with relation

$$\mu \circ \mu = 0. \tag{2.50}$$

For example, $\mu_1 = d$, $\mu_2 = m$ for differential graded algebras.

The relations encoded in the above relation actually make more sense if we write them in components. To be concrete we will write them for homotopy Lie algebras where μ_k is denoted by the k-fold bracket. Of course we start in degree one with the map $\mu_1 = [\cdot] = d : A \to A$ that satisfies $d^2 = 0$. Then, in degree two, we find the usual Lie bracket $\mu_2 = [\cdot, \cdot] : A \otimes A \to A$ and the relation

$$d[a,b] = [da, b] \pm [a, db], \tag{2.51}$$

that expresses that d is a derivation of the Lie algebra. At degree three we see the first occurrence of the ternary bracket $\mu_3 = [\cdot,\cdot,\cdot] : A^{\otimes 3} \to A$. It actually modifies the Jacobi relation by

$$[a,[b,c]] \pm [b,[c,a]] \pm [c,[a,b]] = d[a,b,c] \pm [da,b,c] \pm [a,db,c] \pm [a,b,dc]. \tag{2.52}$$

So Jacobi only holds up to d-exact terms. This is the reason one calls these algebras homotopy algebras. At the following degree we find a relation involving the 2 and 3-brackets that is again zero up to exact terms involving the 4-bracket. To find an application of all this we will now turn to Chern-Simons-Witten theory.

3. Chern-Simons-Witten Theory

We will now apply the above ideas to Chern-Simons gauge theory. In this model the fundamental field is a connection A on a three-manifold M. We choose a Lie group G with Lie algebra $\mathfrak{g}$ and consider the gauge field as a Lie algebra valued one-form. We will write

$$A \in \Omega^1(M, \mathfrak{g}) =: \Omega^1. \tag{3.1}$$

Under gauge transformation A transforms as

$$A \to g^{-1}dg + g^{-1}Ag, \qquad g : M \to G, \tag{3.2}$$

or infinitesimally

$$\delta A = d_A \xi = d\xi + [A, \xi], \tag{3.3}$$

where ξ is an element of the Lie algebra of the group of gauge transformations

$$\xi \in \Omega^0(M, \mathfrak{g}) =: \Omega^0. \tag{3.4}$$

Chern-Simons theory is concerned with flat connections, that is connections for which the curvature

$$d_A^2 = F = dA + A^2 \in \Omega^2 \tag{3.5}$$

vanishes. It is a particular feature of three dimensions that the equation of motion $F = 0$ can be seen as the variation of an action, namely

$$S = \frac{1}{4\pi} \int_M \mathrm{Tr}\,(AdA + \frac{2}{3}A^3). \tag{3.6}$$

This action is almost gauge invariant. Under gauge transformations it picks up a term

$$S \to S - \frac{1}{12\pi} \int_M \mathrm{Tr}\,(g^{-1}dg)^3. \tag{3.7}$$

This last term is for topological reasons always 2π times an integer.

The quantum field theory is defined through the path-integral

$$Z = \int [dA]\, e^{ikS}, \tag{3.8}$$

where one integrates over equivalence classes of connections. The coupling constant k, which plays the role of $1/\hbar$, is required to be integer in order to make the path-integral well-defined.

We can now consider this quantum field theory in an (asymptotic) expansion in Planck's constant $1/k$. (Indeed, after a scaling $A \to A/\sqrt{k}$ we have

$$kS = \frac{1}{4\pi} \int_M \mathrm{Tr}\,(AdA + \frac{2}{3\sqrt{k}}A^3), \tag{3.9}$$

so this corresponds with the usual expansion of a field theory with cubic interaction.) For the expansion we pick a classical solution A_0 satisfying, with $d_0 = d_{A_0}$, the classical equation of motion

$$d_0^2 = 0, \tag{3.10}$$

and write

$$S(A_0 + A) = \frac{1}{4\pi} \int_M \mathrm{Tr}\,(A d_0 A + \frac{2}{3} A^3). \tag{3.11}$$

The discussion below greatly simplifies if we assume (not very realistically) that the complex (Ω^*, d_0) is acyclic, *i.e.* , all cohomology groups vanish

$$H^*(M, \mathfrak{g}) = 0. \tag{3.12}$$

This eliminates in particular the possibility of zero modes, *i.e.* a family of classical solutions, since deformations of a classical solution are infinitesimally given by solutions to the equation

$$d_0 a = 0, \tag{3.13}$$

modulo gauge transformations

$$a = d_0 \xi. \tag{3.14}$$

The gauge fixed action can be obtained very directly if one follows the BV quantization scheme [11]. Here we need three ingredients

1. A $(n|n)$-dimensional X of fields and anti-fields with an odd symplectic form ω. This makes the function space $\mathcal{H}^* = \mathcal{C}^\infty(X)$ into a so-called Batalin-Vilkovisky algebra [12]. This is a generalization of a Poisson algebra. $\mathcal{H}^*$ is a graded commutative algebra and a Lie algebra with an odd Lie bracket, the so-called anti-bracket,

$$\{\cdot,\cdot\}:\ \mathcal{H}^n \otimes \mathcal{H}^m \to \mathcal{H}^{n+m-1} \tag{3.15}$$

which reads in local coordinates

$$\{a, b\} = \sum_{i,j} \frac{\partial a}{\partial x^i} \omega^{ij} \frac{\partial b}{\partial x^j} \tag{3.16}$$

It satisfies the relations

$$\begin{aligned}
\{a, b\} &= -(-1)^{(a-1)(b-1)} \{b, a\} \\
\{a, \{b, c\}\} &= \{\{a, b\}, c\} + (-1)^{(a-1)(b-1)} \{b, \{a, c\}\} \\
\{a, bc\} &= \{a, b\} c + (-1)^{a(b-1)} a \{b, c\}
\end{aligned} \tag{3.17}$$

2. Furthermore there is an operator Δ (the BV-Laplacian) defined as

$$\Delta = \sum_{i,j} \omega^{ij} \frac{\partial^2}{\partial x^i \partial x^j} : \mathcal{H}^n \to \mathcal{H}^{n-1}, \tag{3.18}$$

that is nilpotent of order two

$$\Delta^2 = 0, \tag{3.19}$$

and satisfies the compatibility relation

$$\{a, b\} = (-1)^a (\Delta(ab) - (\Delta a)b - a\Delta b) \tag{3.20}$$

(Actually this last relation defines the bracket in terms of the Laplacian, and can be taken as the definition of the bracket.)

3. An action $S : \mathcal{H} \to \mathbb{R}$ that satisfies the master equation

$$\{S, S\} + \tfrac{1}{2}\hbar\Delta S = 0. \tag{3.21}$$

In the case of Chern-Simons theory the space of fields/antifields is taken to be the space of functions on the total space of Lie algebra-valued differential forms.

$$X = \Omega^*(M, \mathfrak{g}). \tag{3.22}$$

Note that we consider the coordinates $f \in \Omega^n$ to be odd/even depending on whether n is even/odd. A general field $\psi \in \Omega^*$ has decomposition

$$\psi = \sum_{n=0}^{3} \psi^{(n)}, \qquad \psi^{(n)} \in \Omega^n. \tag{3.23}$$

Of course $\psi^{(1)} = A$ is the original physical gauge field. We will see that furthermore $\psi^{(0)} = c$ is the usual ghost, $\psi^{(2)} = A^*$ is the anti-field of the connection and $\psi^{(3)} = c^*$ is the anti-field of the ghost (not to be confused with the anti-ghost!).

On $\mathcal{H}$ we define the anti-bracket as follows. With $f, g \in \Omega^*$ we define their anti-bracket as

$$\{f, g\} = \int \mathrm{Tr}\,(f \wedge g) \tag{3.24}$$

The action is taken to be the original Chern-Simons action

$$S = \int_M \mathrm{Tr}\,(\psi d_0 \psi + \frac{2}{3}\psi^3) \tag{3.25}$$

where ψ is now an *arbitrary* differential form. The action S reduces to the classical action we restrict $\psi \in \Omega^1$. One checks that S satisfies separately the (classical) master action

$$\{S, S\} = \int \mathrm{Tr}\,(d_0\psi + \psi^2)^2 = \int d_0(\frac{1}{2}\psi d_0\psi + \frac{1}{3}\psi^3) = 0 \tag{3.26}$$

and the quantum correction $\Delta S = 0$. The recipe is now completed by putting 'half' of the variables to zero. Hereto we impose the Lorentz gauge condition. We choose a metric on M, this gives in particular a Hodge star

$$* : \Omega^k \to \Omega^{3-k}, \tag{3.27}$$

and an adjoint

$$d_0^* = *d_0*, \quad (d_o^*)^2 = 0. \tag{3.28}$$

The gauge condition is now

$$d_0^*\psi = 0. \tag{3.29}$$

In case all cohomology vanishes (no harmonic forms) this precisely eliminates the required degrees of freedom.

This description can also be derived using more conventional techniques, for example as done by Axelrod and Singer [4]. If one chooses again Lorentz gauge

$$d_0^* A = 0, \tag{3.30}$$

we now have to introduce ghost, anti-ghosts, a Lagrange multiplier and a BRST operator satisfying the usual relations

$$QA = -Dc, \quad Qc = \tfrac{1}{2}[c, c] \quad Q\overline{c} = b, \quad Qb = 0. \tag{3.31}$$

The gauge fixed action now reads

$$S_{gf} = S(A_0 + A) + Q \int \mathrm{Tr}\,(\overline{c} d_0^* A) \tag{3.32}$$

If we redefine variables as

$$A^* = *d_0\overline{c}, \;\; \Theta B = *d_0 b \in \Omega^2(\mathfrak{g}) \tag{3.33}$$

these two-forms satisfy by definition

$$d_0^* C = d_0^* B = 0 \tag{3.34}$$

If we introduce the combination

$$\psi = c + A + A^* + c^* \in \Omega^*(\mathfrak{g}), \tag{3.35}$$

then we recover precisely the gauge-fixed action as written above.

The propagator d_0^{-1} can now be defined on $\ker d_0^*$ as

$$d_0^{-1} = d_0^*/\Delta_0, \tag{3.36}$$

with

$$\Delta_0 = [d_0, d_0^*]. \tag{3.37}$$

The propagator has a kernel

$$d_0^{-1}\psi(x) = \int_{y\in M} L(x,y)\psi(y) \tag{3.38}$$

that is a differential form of total degree 2 on $M \times M$,

$$L(x,y) \in \Omega^2(M^2, \mathfrak{g}\otimes\mathfrak{g}), \tag{3.39}$$

and that satisfies

$$d_0 L(x,y) = \delta^3(x,y). \tag{3.40}$$

This kernel has the nilpotency property

$$d_0^2 = 0 \Rightarrow \int_{y\in M} L(x,y)L(y,z) = 0, \tag{3.41}$$

and symmetry

$$L^{ab}(x,y) = L^{ba}(y,x). \tag{3.42}$$

The interactions are given by the three-points vertex

$$V : \Omega^* \otimes \Omega^* \otimes \Omega^* \to \mathbb{R}, \tag{3.43}$$

with

$$V(A,B,C) = \sum_{a,b,c} f_{abc} \int A^a B^b C^c. \tag{3.44}$$

Let us now present the argument of Axelrod and Singer why the n-loop contributions to the partition function are topological invariants. The contribution will have a typical form

$$F_g = \sum_{\Gamma} \frac{I(\Gamma)}{\#\mathrm{Aut}\,\Gamma}, \tag{3.45}$$

where we sum over all connected cubic diagrams Γ with g loops, and $I(\Gamma)$ is the weight of the diagram Γ. This consists of two contributions, a space-time factor $I'(\Gamma)$ and a group theory factor $I''(\Gamma)$. In case that the flat bundle is trivial these two contributions factorizes,

$$I(\Gamma) = I'(\Gamma) \cdot I''(\Gamma), \tag{3.46}$$

which we will now assume for the sake of simplicity. We first concentrate on the space-time factor. It has the general form

$$I'(\Gamma) = \int_{M^V} L^E, \tag{3.47}$$

where $V = 2g - 2$ is the number of vertices and $E = 3g - 3$ the number of edges.

Let us now consider a variation δg of the metric used in the gauge fixing and definition of the propagator L. Since the propagator satisfies by definition

$$d_0 L = \delta^3(x, y), \tag{3.48}$$

and the delta-function does not depend on the choice of metric, the variation of the propagator δL will satisfy

$$d_0(\delta L) = 0 \;\Rightarrow\; \delta L = d_0 K, \tag{3.49}$$

where we assumed acyclicity again. We can therefore write following Axelrod and Singer

$$\begin{aligned}
\delta I'(\Gamma) &= \delta \int_{M^V} L^E \\
&= E \int_{M^V} L^{E-1}(\delta L) \\
&= E \int_{M^V} L^{E-1}(d_0 K) \\
&= E(E-1) \int_{M^V} L^{E-2}(d_0 L) K \\
&= E(E-1) \int_{M^{V-1}} L^{E-2} K,
\end{aligned} \tag{3.50}$$

where in the last line we have used that $d_0 L = \delta^3(x,y)$. This effectively shrinks the propagator to zero, and so reduces our cubic Feynman diagram to a diagram with one quartic vertex, precisely as in our discussion of graph homology. Actually we can write the above manipulation as

$$\delta I'(\Gamma) = I'(d\Gamma). \tag{3.51}$$

Of course to complete the argument we have to add the group theory factors $I^{\mathfrak{g}}(\Gamma)$. But these are precisely the rules related to Lie algebras we were discussing in the previous section. That is, the g-loop contribution to the Chern-Simons partition function is

$$F_g = I'(a_g), \tag{3.52}$$

with a_g the class

$$a_g = \sum_{\Gamma \in C_g,\ cubic} \frac{I''(\Gamma)}{\#\mathrm{Aut}\,\Gamma}\Gamma. \tag{3.53}$$

Therefore we now have the following elegant one line proof of the topological properties of perturbative Chern-Simons theory:

$$da_g = 0 \ \Rightarrow\ \delta F_g = I'(da_g) = 0. \tag{3.54}$$

In light of our discussion above it is now obvious how to generalize this property. Instead of cubic graphs weighted with Lie algebra symmetry factors we now take arbitrary graphs weighted with homotopy Lie algebra Feynman rules. However, things are a bit more complicated. In particular, cubic graphs are special in the following respect. The number of edges is $E = 3g-3$ and the number of vertices $V = 2g-2$. So, since L is a differential form on M^2 of total degree 2, the integrand L^{3g-3} is a differential form on M^{2g-2} of degree $6g-6$. This is a top degree form and can thus be integrated to give a number: $I'(\Gamma)$.

This is clearly a very special property of cubic vertices and three-dimensional manifolds. In the general case the graph Γ has E edges and $V = \sum_{k=3}^{\infty} v_k$ vertices, where v_k denotes the number of vertices of valency k. If the space-time manifold M has dimension d, the degree of the weight $I'(\Gamma)$ will formally be

$$k = (d-1)E - dV = \sum_{k=3}^{\infty}(\frac{d-1}{2}k - d)v_k \geq 0. \tag{3.55}$$

So how to interpret a weight that is not a number but a differential form of positive degreeΓ The idea is to consider not one manifold M but a family of manifolds M_t, where t ranges over some subspace $C \subset B$, the space of Riemannian manifolds. The weight I_Γ now has a natural interpretation as a differential form on B of degree k. Furthermore if we repeat our calculation of the metric dependence of I_Γ we find again

$$\delta I(\Gamma) = I(d\Gamma), \tag{3.56}$$

where δ now has the interpretation of being the exterior differential on B. The g-loop partition function is now closed, and thus gives a cohomology class on B

$$\delta F_g = 0 \;\Rightarrow \Phi_g \in H^k(B). \tag{3.57}$$

4. Perturbative Knot Invariants

It is quite natural to extend the above ideas of perturbative topological gauge theories to include observables, in particular Wilson loops. This is already very interesting for the usual Chern-Simons-Witten model, and we will consequently not discuss generalizations to homotopy structures.

For an embedding $K : S^1 \to M$ and a representation R Wilson loop averages are defined as

$$\langle W_R(K)\rangle = \frac{1}{Z(M)} \int [dA]\, e^{ikS(A)} \mathrm{Tr}_R \left(P \exp \oint_K A \right). \tag{4.1}$$

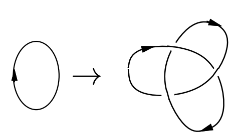
As is well-known these expectation values lead directly to the famous knot polynomials of Jones [16] and their generalizations [17]. These invariants again are analytic in $\hbar = 1/(k+h)$ and consequently have a perturbative expansion

$$\langle W_R(K)\rangle = \sum_{n=0}^{\infty} a_n(K)\hbar^n. \tag{4.2}$$

In this way we are naturally led to also consider knot theory in perturbation theory [3]. Remarkably this perturbative approach to knot invariants was developed more or less independent from physics by Vassiliev [6]. In our exposition we will follow mainly the beautiful and extensive review of Bar-Natan [7] and the seminal paper by Kontsevich [8]. See also the interesting papers [13, 14].

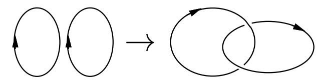
For our purposes a knot will be an embedding of a circle in Euclidean three-space

$$K : S^1 \to \mathbb{R}^3, \tag{4.3}$$



and, more generally, a link will be an embedding of a collection of circles

$$L : S^1 \times \ldots \times S^1 \to \mathbb{R}^3, \qquad (4.4)$$



The important restriction is of course the absence of self-intersection in these maps.

Two knots K_1, K_2 are said to have the same knot type, $K_1 \sim K_2$, if there exists an isotopy of $\mathbb{R}^3$ mapping K_1 into K_2. The fundamental problem in knot theory is to make sense of the classification of knot types. A knot or link invariant $\Phi(K)$ depends by definition only on the knot type

$$K_1 \sim K_2 \Rightarrow \Phi(K_1) = \Phi(K_2). \qquad (4.5)$$

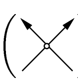
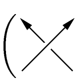
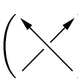
It is clear that a classifications of knot invariants is dual to a classification of knots.

4.1. Vassiliev invariants

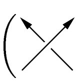
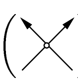
A fruitful way to think about knot invariants is to consider the space B of all embeddings. This is a very nontrivial infinite-dimensional space, with the topological complications coming from the fact that we have to excluded self-intersections. In particular B is not connected, and the connected components of B correspond precisely to the different knot classes. From this point of view a knot invariant is a locally constant function on B

$$\Phi \in H^0(B). \qquad (4.6)$$

We can think of B as a subspace of the space of all (smooth) maps $S^1 \to \mathbb{R}^3$. In this space we have different strata B_n of codimension n, where we allow n distinct normal intersections, and $B_0 = B$. We can inductively extend Φ to these singular knots in B_n through the local definition

Φ () $=$ Φ () $-$ Φ () . (4.7)

We will also write

$\nabla\Phi$ () $:=$ Φ () . (4.8)

This defines the 'derivative' of a knot invariant. Notice it is a 'partial derivative' since its definition depends on which crossing we have picked in a particular blackboard projection of the knot. It is the wonderful idea of Vassiliev that this point of view produces a natural filtration on the space of knot invariants. This is best explained by an analogy:

Consider the space A of analytic functions in z. We have a natural filtration

$$P_0 \subset P_1 \subset \cdots \subset P_n \subset \cdots \subset A, \tag{4.9}$$

where P_n is the space of polynomials of degree n,

$$p \in P_n \Rightarrow p(z) = a_n \frac{z^n}{n!} + \ldots + a_1 z + a_0,\ \ a_i \in \mathbb{C}. \tag{4.10}$$

With $\nabla = d/dz$, we have

$$p \in P_n \Leftrightarrow \nabla^{n+1} p = 0. \tag{4.11}$$

Furthermore the quotient spaces $P_n/P_{n-1} \cong \mathbb{C}$ are one-dimensional and correspond to the leading coefficient of $p \in P_n$

$$\nabla^n p = a_n \in P_n/P_{n-1}. \tag{4.12}$$

With this analogy in mind, we now try to define 'polynomial' knot invariants (not to be confused with knot polynomials!). These will be the invariants satisfying $\nabla^k \Phi = 0$ for large enough k.

More precisely, we call a knot invariant a Vassiliev invariant of order n, and we will write $\Phi \in V_n$, if Φ vanishes on B_{n+1}. (and consequently on all B_k with $k > n$.) Stated otherwise, Φ is a Vassiliev invariant of order n if *all* $(n = 1)$th derivatives vanish

$$\nabla^{n+1} \Phi = 0, \tag{4.13}$$

or, equivalently

$$\Phi \left(\underbrace{\times \cdots \times}_{n+1} \right) = 0. \tag{4.14}$$

We will see in a moment that

$$\dim V_n / V_{n-1} < \infty, \tag{4.15}$$

and that the 'leading coefficient'

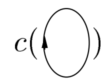
$$\nabla^n \Phi \in V_n / V_{n-1} \tag{4.16}$$

has a natural interpretation in terms of (Feynman) diagrams. Actually this relation makes the description of the spaces V_n algorithmically speaking completely straightforward. There is now a universal classification of Vassiliev invariants. The classification of all knot invariants can only differ from this by 'nonperturbative' terms.

The simplest examples of Vassiliev invariants are the coefficients of the Conway-Alexander polynomial $c(K)$ [15]. Recall that this famous knot invariant associates to each knot a polynomial in the formal variable z

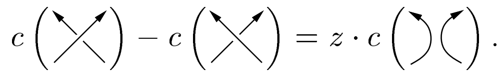
$$c(K) = c_0(K) + c_1(K)z + \ldots + c_n(K)z^n, \tag{4.17}$$

defined recursively through



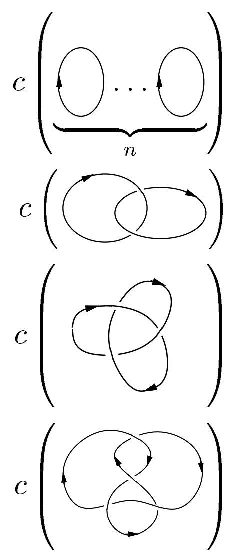
$= 1,$ (4.18)

and the skein relation



(4.19)

The first few nontrivial examples are

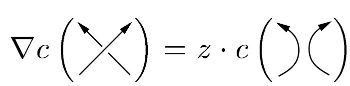


$$\begin{aligned} &= 0, \quad \text{if} \quad n > 1, \\ &= z, \\ &= 1 + z^2, \\ &= 1 - z^2. \end{aligned}$$

It is an easy exercise to verify that the coefficients c_n are Vassiliev invariants of order n

$$c_n \in V_n. \tag{4.20}$$

We simply use the skein relation



(4.21)

$n+1$ times, which implies that

$$\nabla^{n+1} c = z^{n+1}(\ldots) \tag{4.22}$$

and consequently

$$\nabla^{n+1} c_n = 0. \tag{4.23}$$

The invariants c_0 and c_1 have an elementary interpretation

$$c_0(K) = \begin{cases} 1, & \text{if } K \text{ has one component,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.24}$$

and

$$c_1(K) = \begin{cases} lk(K), & \text{if } K \text{ has two components,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.25}$$

where lk is the linking number. The higher order invariants are much more mysterious. Analogous results hold for the Jones, HOMFLY and Kauffman polynomials. For example, the HOMFLY polynomial H_n, $n \in \mathbb{Z}$, is defined through the skein relation

$$t^{\frac{n+1}{2}} H_n \left(\quad \right) - t^{-\frac{n+1}{2}} H_n \left(\quad \right) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) H_n \left(\quad \right). \tag{4.26}$$

(The Jones invariant is given by $n = -3$.) If we substitute $t = e^z$, and expand

$$H_n = \sum h_{n,k} z^k, \tag{4.27}$$

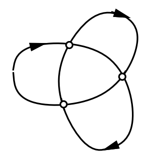
one verifies that Vassiliev invariants are produced: $h_{n,k} \in V_k$.

4.2. Weight systems and chord diagrams

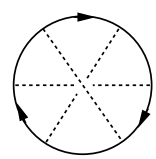
The relation with perturbative field theory emerges in the following remarkable way. To each Vassiliev invariant we can associate a so-called weight system. Weight systems are basically generalizations of Feynman rules.

Consider a Vassiliev invariant $\Phi \in V_n$ and its derivative $\nabla^n \Phi$. The latter is a knot invariant for knots with n distinct normal self-intersections. It has the important property that it vanishes as soon as we create another intersection, since by definition

$\nabla^{n+1}\Phi = 0$. So we can assume that this knot is unknotted except for the n self-intersections. An example would be the following knot with three self-intersections

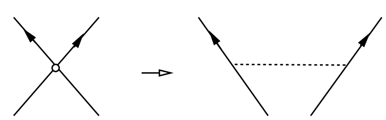
K :  (4.28)

To such a configuration we can associate in a unique way a graph, a so-called chord diagram. For convenience let us assume that our knot has only one component. The map $K : S^1 \to \mathbb{R}^3$ is one-to-one apart from $2n$ special points on the circle where the knot intersects itself. Let us label these points consecutively $P_1, \ldots, P_{2n} \in S^1$. The points P_j and P_j are pairwise related if $K(P_i) = K(P_j) \in \mathbb{R}^3$. In the corresponding diagram the points P_i and P_j will be connected with a (dashed) line. A diagram Γ with n of these internal lines will be called a chord diagram of order n. In our example we obtain the graph

Γ :  (4.29)

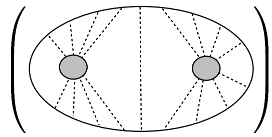
where the intersections of the dashed lines are meaningless.

In physical terminology chord diagrams are just fermion loops with exchange of gauge bosons. These interactions are 'abelian', we ignore the self-interactions of the bosons for the moment. In this intuitive physical picture the above procedure can be regarded as replacing an effective four-fermion (self)interaction by a propagating intermediate gauge boson

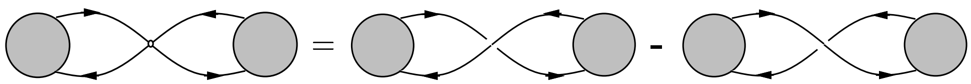
$(\psi^\dagger\psi)^2 \to \psi^\dagger A\psi + A^2$:  (4.30)

Clearly a knot invariant $\Phi \in V_n$ leads to a function $\nabla^n\Phi$ on the space of chord diagrams. The question is now what are the conditions that the functions $\nabla^n\Phi$ that come from knot invariants satisfy. There are two rather obvious constraints that we will discuss in a moment, but it is a powerful result of Kontsevich (which we will describe in the next section) that these two conditions are sufficiently characterize all the Vassiliev invariants.

The first condition states that Φ vanishes whenever there is an isolated chord

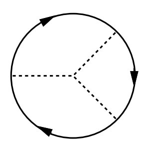
condition (1): Φ  $= 0$ (4.31)

This relation is easily proved by inspection of the following figure

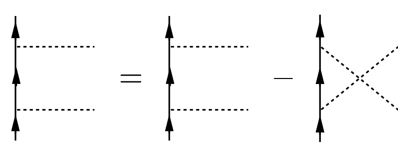
 $= 0$ (4.32)

since the two terms on the RHS are simply related by a 2π rotation of one of the two 'blobs.' We do remark though that in the case of knot invariants that are framing dependent, such as is the case with many invariants coming from conformal field theory, this identity will not be satisfied. That is why in actual quantum field theory realizations we will sometimes discard condition (1).

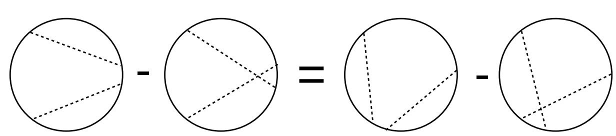
Condition (2) can be explained best by considering a diagram with an internal vertex of valency three

 (4.33)

where we did not indicate all other chords that are of the usual type, *i.e.* just connecting two boundary points. Note that at present this is not a well-defined chord diagram. We can now 'resolve' this third order vertex using the rule

 (4.34)

This resolution can now be done in three inequivalent ways, depending on which of the three edges emanating from the vertex we choose. Condition (2) says that all three are equivalent. Diagrammatically this gives a relation with four terms

condition (2):  (4.35) and therefore this relation has been baptized the four-term relation. The topological proof of this relation follows quite naturally by considering a degenerated self-intersection where three points $P_i, P_j, P_k \in S^1$ are mapped to one point in $\mathbb{R}^3$

$$K(P_i) = K(P_j) = K(P_k) \tag{4.36}$$

and the various resolutions in two distinct crossings, see [8, 7].

Before we continue our discussion of knot invariants, let us first make some further remarks about the space of chord diagrams. We will write

$$C = \bigoplus_{n=0}^{\infty} C_n \tag{4.37}$$

for the free graded module of all chord diagrams factored by the relations (1) and (2). That is, we consider linear combinations of chord diagrams, identify combinations through the four-term relation and put all diagrams with an isolated chords identically to zero. A weight system is now simply a map $C_n \to \mathbb{C}$. One should think of such a map as a very special set of Feynman rules. To every diagram we assign a number respecting the relations (1) and (2). The beautiful result of Kontsevich is that

$$V_n/V_{n-1} \cong C_n^*. \tag{4.38}$$

That is, every chord diagram of order n determines a Vassiliev invariant of order n and does so uniquely up to terms of lower order. This is a very nontrivial result. In the original work of Vassiliev it was not immediately clear that no other conditions than (1) and (2) would emerge by considerations of higher codimension. In particular the space V_n is finite-dimensional.

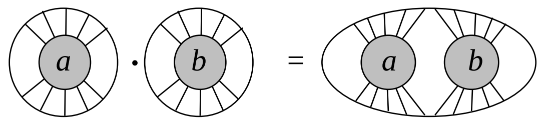
It is actually more convenient to (temporarily) forget about condition (1) and only restrict ourselves to condition (2) — the 4-term relation. Let

$$F = \bigoplus_{n=0}^{\infty} F_n \tag{4.39}$$

denote the module of chord diagrams where we only imposed the condition (2). Remarkably the space of such diagrams forms a Hopf algebra. That is, we can both multiply and comultiply such diagrams. The multiplication

$$\cdot : \ F \otimes F \to F \tag{4.40}$$

is essentially the connected sum of the two graphs. If $\Gamma_a \in F_n$ and $\Gamma_b \in F_m$ then $\Gamma_a \cdot \Gamma_b \in F_{n+m}$ is defined as



(4.41)

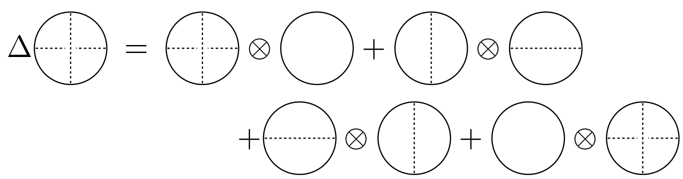
It is a remarkable application of condition (2) that this connected sum is independent of the points on the graphs Γ_a and Γ_b that we choose to connected the two. The comultiplication is a map

$$\Delta : F \to F \otimes F \tag{4.42}$$

defined as

$$\Delta(\Gamma_a) = \sum_{b \cup c = a} \Gamma_b \otimes \Gamma_c \tag{4.43}$$

where we partition the set of chords a into two subsets b and c. For example



(4.44)

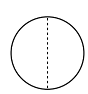
With unit O and counit $\epsilon(\Gamma) = \delta_{\Gamma,O}$ one shows that the above defines a (co)commutative, (co)associative Hopf algebra. Such simple Hopf algebras have a very canonical structure. They are always a free polynomial algebra generated by the primitive elements. A primitive element x satisfies by definition

$$\Delta(x) = x \otimes 1 + 1 \otimes x. \tag{4.45}$$

If we denote these elements as $x_1, x_2, \ldots$ then we consequently have

$$F = \mathbb{C}[x_1, x_2, x_3, \ldots]. \tag{4.46}$$

The most important example of a primitive element is the generator

$x_1 =$  $\in F_1.$ (4.47)

This is typically the diagram that we should put to zero if we also impose condition (1), *i.e.* go from F to C. Actually it is enough to just put $x_1 = 0$ in the ring generated by the primitive elements of F. That is, we have

$$C = \mathbb{C}[x_2, x_3, \ldots]. \tag{4.48}$$

The (co)multiplicative structure on the space of chord diagrams is also interesting from the point of Feynman rules or knot invariants. Since we can comultiply diagrams, we can multiply invariants using

$$(\Phi_1 \cdot \Phi_2)(a) = (F_1 \otimes F_2)(\Delta(\Gamma_a)) = \sum_{b \cup c = a} \Phi_1(\Gamma_b)\Phi_2(\Gamma_c). \tag{4.49}$$

4.3. Lie algebras

There are a natural set of Feynman rules coming from Lie algebras that realize the weight systems in F^*. Let $\mathfrak{g}$ be a semi-simple Lie algebra. We will write e_a for an orthonormal basis for $\mathfrak{g}$, and have structure coefficients

$$[e_a, e_b] = \sum_c f_{abc} e_c \tag{4.50}$$

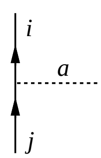
with f_{abc} fully antisymmetric. Let R be a representation $R : \mathfrak{g} \to \mathrm{End}(V)$. We write $R_a = R(e_a)$ and clearly have by definition

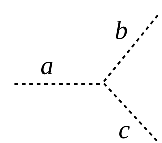
$$[R_a, R_b] = \sum_c f_{abc} R_c. \tag{4.51}$$

If we now introduce the Feynman rules

$= \delta_{ij},$


$= \delta_{ab},$ (4.52)

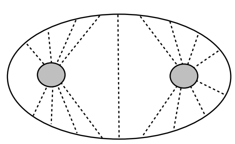
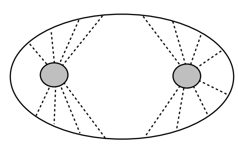


$= R_a^{ij},$


$= f_{abc},$ (4.53)

then equation (4.51) corresponds precisely to the relation (4.34). Condition (2) is therefore satisfied.

In general condition (1) will not be satisfied in these Lie algebra Feynman rules. In fact, one easily computes that


$= c_R \cdot$

(4.54)

with c_R the value of the quadratic Casimir

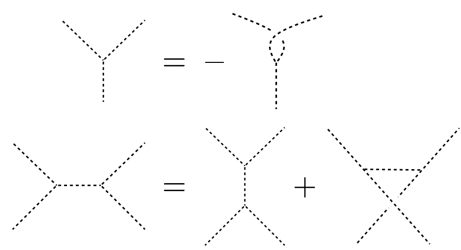
$$C = \sum_a e_a e_a \tag{4.55}$$

in the representation R. We obtain real (framing independent) knot invariants from these rules by factoring out the ideal generated by x_1. One may wonder whether all weight systems are related to (classical) Lie algebras. Numerical calculations have not yet lead to the opposite conclusion.

Actually the structure of Lie algebras with their cubic vertex is quite naturally as becomes clear from a third description of the space F that is much closer related to Chern-Simons perturbation theory and that also includes the cubic gauge boson vertex. Let

$$D = \bigoplus_{n=0}^{\infty} D_n \tag{4.56}$$

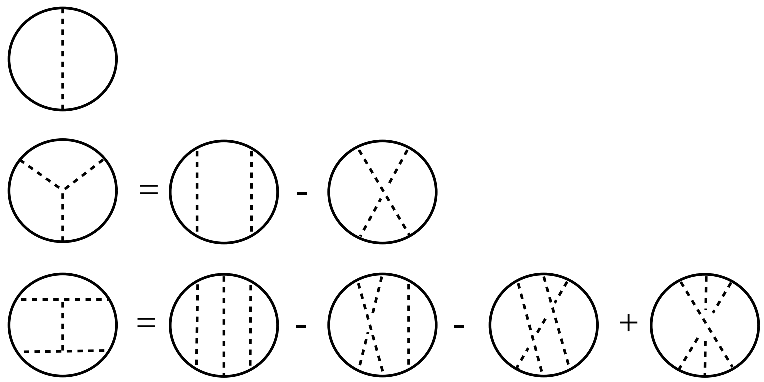
be the space of chord diagrams where we also allow cubic interactions of the gauge fields. That is we have one fermion loop, a fermion-boson interaction, plus the cubic vertex. We further impose the relation (4.34). One can now show that (1) $D \cong F$, and the graphical representations of (2) antisymmetry of the structure coefficients and (3) the Jacobi relation



(4.57)

Of course to show that Lie algebras exhaust the set of knot invariants it remains to be shown that the weight systems in D^* are actually Feynman rules, *i.e.* based on the composition principle where general weights are built out of elementary weights of the individual trivalent vertices.

The space D is a Hopf algebra too. In this formulation there is a simple criterion whether an element is primitive or not, $a \in D$ is primitive iff a is connected. This allows us to write down easily the first three primitive elements



(4.58)

This shows in particular

$$\begin{aligned} \dim C_0 &= 1, \qquad & \dim C_1 &= 0, \\ \dim C_2 &= 1, \qquad & \dim C_3 &= 1. \end{aligned} \tag{4.59}$$

We are now left with the task to present the proof of Kontsevich that the space of chord diagrams C_n is dual to the quotient space V_n/V_{n-1}. Hereto we have to integrate a weight system $\Psi = \nabla^n \Phi \in C_n^*$ to a knot invariant $\Phi \in V_n$. The idea is based on our experience with conformal field theory, or if one wishes with the Hamiltonian formulation of Chern-Simons theory.

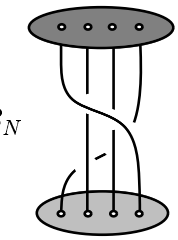
5. The Knizhnik-Zamolodchikov connection

In conformal field theory knot invariant naturally are constructed out of braid group representation. Recall that the braid group B_N is generated by the elements $\sigma_1, \ldots, \sigma_{N-1}$ with the relations

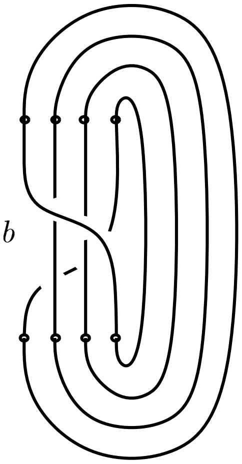
$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \qquad |i-j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{aligned} \tag{5.1}$$

Given a braid on N strands, $b \in B_N$, we can produce a knot out of it by identifying the ends

$b \in B_N$



$\Rightarrow K = \mathrm{Tr}\, b$



(5.2)

Given a braid group representation

$$R : B_N \to \mathrm{End}(V), \tag{5.3}$$

we can try to define a knot invariant by

$$\Phi(\mathrm{Tr}\, b) = \mathrm{Tr}\, R(b). \tag{5.4}$$

This knot invariant is only well-defined, *i.e.* only depends on the knot type, if the trace is invariant under the (second) Markov move

$$\mathrm{Tr}\, R(\sigma_N b) = \mathrm{Tr}\, R(b), \qquad b \in B_N, \sigma_N \in B_{N+1}. \tag{5.5}$$

(The first Markov move $\mathrm{Tr}\, R(b_1 b_2) = \mathrm{Tr}\, R(b_2 b_1)$ is obvious.) The above property is indeed satisfied by the braid matrices of conformal field theory. A good conceptual explanation of this remarkable fact was given by Witten's proof that the CFT braid representations are identical to the representations that appear in the canonical quantization of Chern-Simons theory, which is a truly covariant three-dimensional relativistic field theory.

The CFT braid matrices are typically obtained as holonomies of a connection Ω, the Khnizhik-Zamolodchikov connection [18], on the configuration space $\mathbb{C}_N$ of N distinct unmarked points $z_1, \ldots, z_N$ in the complex plane. Let us briefly recall the origin of the Khnizhik-Zamolodchikov equation in conformal field theory.

The Wess-Zumino-Witten model for compact Lie group G with Lie algebra $\mathfrak{g}$ has (chiral) primary fields

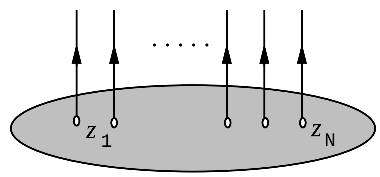
$$\phi(z) \in V. \tag{5.6}$$

that carries a representation $R : \mathfrak{g} \rightarrow \mathrm{End}(V)$. Consider the chiral correlation functions or holomorphic blocks

$$\xi = \langle \phi(z_1) \cdots \phi(z_N) \rangle \in V^{\otimes N}. \tag{5.7}$$

As is well-known these are holomorphic sections of a holomorphic vector bundle over configuration space. They have an alternative interpretation as the wave-functions of Chern-Simons theory in the presence of external charges.

We denote such a correlator or wave function by a graphical representation

$\xi(z_1, \ldots, z_N) =$ 

(5.8)

The holomorphic blocks satisfy the famous Knizhnik-Zamolodchikov equation, which can be elegantly written as

$$(d - \hbar\Omega)\xi = 0, \tag{5.9}$$

with

$$d = \sum_i \frac{\partial}{\partial z_i} dz_i, \tag{5.10}$$

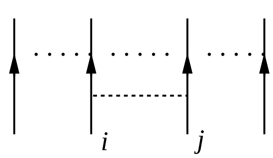
and Ω the KZ connection

$$\Omega = \sum_{i \neq j} C_{ij} \frac{dz_i - dz_j}{z_i - z_j}. \tag{5.11}$$

Here

$$C_{ij} = \sum_a R_a^{(i)} R_a^{(j)}, \tag{5.12}$$

and $R^{(i)}$ denotes the action of $\mathfrak{g}$ on the i^{th} factor in $V^{\otimes N}$. We will write symbolically

$C_{ij} =$ 

(5.13)

Let us briefly consider the derivation of this equation. One starts with an insertion of the stress tensor

$$\oint_{z_j} \frac{dz}{2\pi i} \langle T(z) \phi(z_1) \cdots \phi(z_N) \rangle, \tag{5.14}$$

then uses the OPE

$$T(z)\phi(w) \sim \frac{h_R\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}, \tag{5.15}$$

and

$$J_a(z)\phi(w) \sim \frac{R_a\phi(w)}{z-w}, \tag{5.16}$$

together with the Sugawara construction

$$T = \hbar\sum_a J_aJ_a, \qquad \hbar = \frac{1}{k+h}. \tag{5.17}$$

In this way one finds

$$\begin{aligned} \frac{\partial}{\partial z_j}\langle\phi(z_1)\cdots\phi(z_N)\rangle &= \oint_{z_j}\frac{dz}{2\pi i}\langle T(z)\phi(z_1)\cdots\phi(z_N)\rangle \\ &= \oint_{z_j}\frac{dz}{2\pi i}\langle\hbar\sum_a J_a(z)J_a(z)\phi(z_1)\cdots\phi(z_N)\rangle \\ &= \sum_{i\neq j}\hbar\frac{C_{ij}}{z_i-z_j}\langle\phi(z_1)\cdots\phi(z_N)\rangle. \end{aligned} \tag{5.18}$$

In the abelian case, $\mathfrak{g} = \mathbb{C}^n$, all representations are one-dimensional, $R_a^{(i)} = p_i \in \mathbb{C}^n$, and the sections are given by the familiar vertex operator correlation functions

$$\xi = \prod_{i<j}(z_i-z_j)^{\hbar(p_i\cdot p_j)}. \tag{5.19}$$

One of the beautiful characteristic properties of the KZ connection $D = d - \hbar\Omega$ is that it is flat (for all values of $\hbar$). More precisely, one has $D^2 = 0$ or

$$d\Omega = \Omega\wedge\Omega = 0. \tag{5.20}$$

For the proof of this statement it is convenient to write

$$\Omega = \sum_{i<j}\Omega_{ij}, \tag{5.21}$$

with

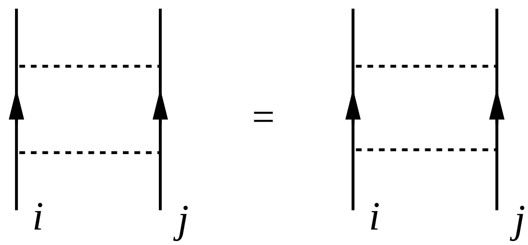
$$\Omega_{ij} = C_{ij}\frac{dz_i - dz_j}{z_i - z_j} = C_{ij}d\log(z_i-z_j) \tag{5.22}$$

(no summation). We clearly have by the above relation $d\Omega_{ij} = 0$, so we only have to prove $[\Omega, \Omega] = 0$. Hereto we have to distinguish three separate cases.

(1) First the obvious result

$$[\Omega_{ij}, \Omega_{ij}] = 0, \tag{5.23}$$

with corresponding picture

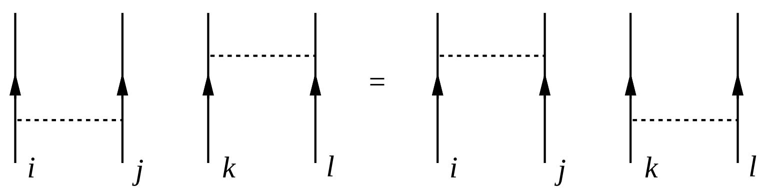


(5.24)

(2) Secondly, the equally obvious case

$$[\Omega_{ij}, \Omega_{kl}] = 0, \qquad i, j \neq k, l, \tag{5.25}$$

with a pictorial interpretation



(5.26)

(3) The interesting case is however

$$\begin{aligned} [\Omega_{ij}, \Omega_{jk}] &= [C_{ij}, C_{jk}] \frac{dz_i - dz_j}{z_i - z_j} \wedge \frac{dz_j - dz_k}{z_j - z_k} \\ &= C_{ijk} \frac{dz_i - dz_j}{z_i - z_j} \wedge \frac{dz_j - dz_k}{z_j - z_k}. \end{aligned} \tag{5.27}$$

The important observation is now that the prefactor

$$C_{ijk} = \sum_{a,b,c} f^{abc} R_a^{(i)} R_b^{(j)} R_c^{(k)} \tag{5.28}$$

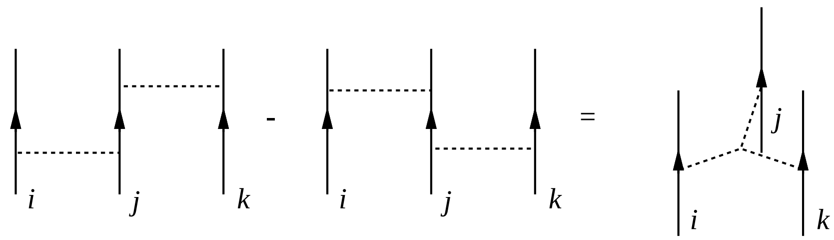
is completely anti-symmetric in i, j, k. Therefore we can use Arnol'd's identity [20]

$$\frac{dz_i - dz_j}{z_i - z_j} \wedge \frac{dz_j - dz_k}{z_j - z_k} + cyclic = 0, \tag{5.29}$$

to show that

$$[\Omega_{ij}, \Omega_{jk}] + cyclic = 0. \tag{5.30}$$

Of course again we can draw a picture (as symmetric as possible)



(5.31)

From this last picture it will be clear that the abstract KZ connection is flat precisely when we consider weight diagrams that satisfy the four-term relation. Indeed the only thing that we needed were the relations

$$[C_{ij}, C_{kl}] \;=\; 0, \qquad i,j \neq k,l \tag{5.32}$$

$$[C_{ij}, C_{ik}+C_{jk}] \;=\; 0, \tag{5.33}$$

which are the infinitesimal pure braid relations of Kohno [19]. The last relation is certainly equivalent to the four-term relation, now applied to strands instead of knots, and so will be valid in our formalism of chord diagrams.

Given the integrable KZ connection the braid representations are obtained as follows. For a closed path in $s : [0,1] \to \mathbb{C}_N$ we define the holonomy

$$b = b(1) = P \exp \int_0^1 ds\, \Omega, \tag{5.34}$$

by the equation

$$\frac{db}{dt} = \Omega \cdot b. \tag{5.35}$$

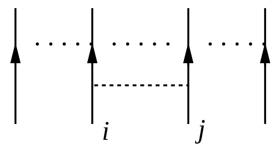
(We write $\Omega(t)$ as an abbreviation for $s^*\Omega(s(t))$.) This can be integrated perturbatively using Dyson's formula as

$$b = 1 + \sum_{m=1}^{\infty} \int_{0\leq t_1 \leq \ldots \leq t_m \leq 1} dt_1 \cdots dt_m\, \Omega(t_1)\cdots\Omega(t_m). \tag{5.36}$$

We will now take this definition of knot invariants and generalize it the setting of chord diagrams. More precisely, if $K = \mathrm{Tr}\, b$ is a knot obtained from a braid, we can define an nth order knot invariant as

$$\Phi(K) = \int_{0\leq t_1 \leq \ldots \leq t_n \leq 1} dt_1 \cdots dt_n\, \Omega(t_1)\cdots\Omega(t_n) \in C^n, \tag{5.37}$$

with the diagrammatic KZ connection

$$\Omega = \sum_{1 \leq i < j \leq N} \frac{dz_i - dz_j}{z_i - z_j} \cdot$$ 

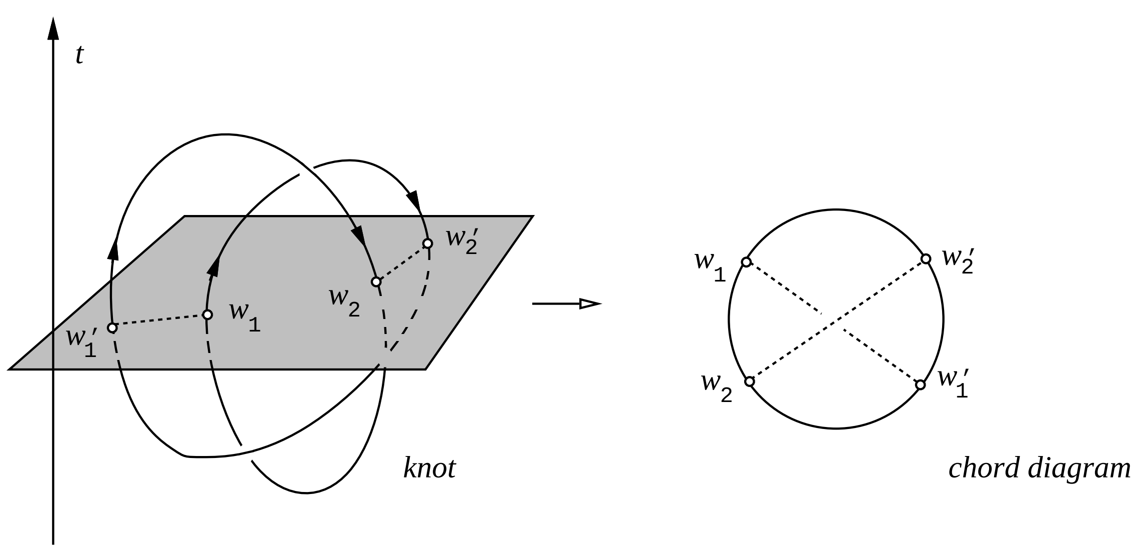
, (5.38)

and with the obvious graphical definition of the trace as in (5.2). Equivalently,

$$\Phi(K) = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} dt_1 \cdots dt_n \sum_{pairings\ P} \bigwedge_{i=1}^{n} \frac{dw_i - dw_i'}{w_i - w_i'} \cdot \Gamma_P \in C_n, \tag{5.39}$$

where we sum over all choices of pairs $w_i, w_i' \in \{z_1, \ldots, z_N\}$ and where Γ_P is the chord diagram that corresponds to the pairing P. Proving that this defines a knot invariant uses of course the fact that the KZ connection Ω is flat. Therefore we can smoothly deform the braid.

The above definition is however not general enough, since we can also have knots that are not traces of braids. (Every knot class can be written as a trace of a braid, but that is not the same.) In that case we cannot a priori apply the above prescription. Kontsevich's formula basically takes a leap of faith and applies it anyhow! To give the formula we have to use a time direction and parametrize the space-time points $x \in \mathbb{R}^3$ as $x = (t, z) \in \mathbb{R} \times \mathbb{C}$, t being time and z space. Now our knot K we will be 'created' at time $t = a$ and 'disappear' at $t = b$, *i.e.* the time slices $X_s = \{t = s\} \cong \mathbb{C}$ satisfy $K \cap X_s \neq \ \Leftrightarrow s \in [a, b]$. Now for such a time $t \in [a, b]$ the set $K \cap X_t$ will consist of a number of points. We can define with the following picture in mind



(5.40)

the perturbative knot invariant

$$\widehat{\Phi}(K) = \int\limits_{a \leq t_1 \leq \ldots \leq t_n \leq b} dt_1 \cdots dt_n \sum_{pairings\ P} (-1)^{\#} \bigwedge_{i=1}^{n} \frac{dw_i - dw_i'}{w_i - w_i'} \cdot \Gamma_P \in C_n. \tag{5.41}$$

Here # indicates the number of down-going strands after a choice of orientation of the knot. (A new feature that we didn't have to deal with in the case of braid knots.)

It turns out that as it stands this is not yet a true knot invariant. There is a correction due to the fact that a (Morse) knot can have critical points, and the deformation argument that follows from the flatness of the KZ connection can never change the number of critical points. This effect is already observed for the figure eight (un)knot which gives the expression

$$\widehat{\Phi}(\infty) = \bigcirc + \cdots \tag{5.42}$$

This gives a corrective factor for each critical point. So the final formula for the *Universal Vassiliev Invariant* of the knot K with c critical points reads [8]

$$\Phi(K) = \frac{\widehat{\Phi}(K)}{\widehat{\Phi}(\infty)^{\frac{c}{2}-1}} \in C_n \tag{5.43}$$

This is a universal invariant in the sense that it takes value in the module of chord diagrams. We can thus pair it with a weight system $\Psi \in C_n^*$ to produce a number $\langle \Psi, \Phi(k) \rangle \in \mathbb{C}$.

We will not be in a position to proof here in full detail the fact that this invariant is well-defined and indeed independent of smooth deformations, see *e.g.* [7]. It is also clear that many questions abound: How to generalize these invariants along the lines of the Chern-Simons-Witten theoryΓ How to describe non-perturbative effectsΓ Are Lie algebras, or even just the classical Lie algebras sufficient to exhaust the Vassiliev invariantsΓ if anything, I hope to have given the reader at least the idea that these questions might not be as hopeless as one might think at first sight!

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