Lecture 1

Quantum Field Theories: An introduction

The string theory is a special case of a quantum field theory (QFT). Any QFT deals with smooth maps $\gamma : \Sigma \to M$ of Riemannian manifolds, the dimension of Σ is the *dimension* of the theory. We also have an *action function* S defined on the set $\mathrm{Map}(\Sigma, M)$ of smooth maps. A QFT studies integrals

$$\int_{\mathrm{Map}(\Sigma,M)} e^{-\frac{S(\phi)}{\hbar}} V(\phi) D[\phi] \tag{1.1}$$

Here $D[\phi]$ stands for some measure on the space of paths, $\hbar$ is a parameter (usually very small, *Planck constant*) and $V : \mathrm{Map}(\Sigma, M) \to \mathbb{R}$ is an *insertion function*. The number $e^{-S(\phi)/\hbar}$ should be interpreted as the *probability amplitude* of the contribution of the map $\gamma : \Sigma \to M$ to the integral. The integral

$$Z^E = \int_{\mathrm{Map}(\Sigma,M)} e^{-\frac{S(\phi}{\hbar}} d\phi \tag{1.2}$$

is called the *partition function* of the theory. In a relativistic QFT, the space Σ has a Lorentzian metric of signature $(-, +, \ldots, +)$. The first coordinate is reserved for time, the rest are for space. In this case, the integral (1.1) is replaced with

$$Z^M = \int_{\mathrm{Map}(\Sigma,M)} e^{iS(\phi)/\hbar} V(\phi) D[\phi]. \tag{1.3}$$

Let us start with a 0-dimensional theory. In this case Σ is a point, so $\phi : \Sigma \to M$ is a point $x \in M$ and $S : M \to \mathbb{R}$ is a scalar function. The Minkowski partition function of the theory is an integral

$$Z = \int_M e^{iS(x)/\hbar} dx. \tag{1.4}$$

Following the Harvard lectures of C. Vafa in 1999, let us consider the following example:

Example 1.1. Recall the integral expression for the Γ-function:

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt=2\int_0^\infty t^{2s-1}e^{-t^2}dt. \tag{1.5}$$

This integral is convergent for Re $(s)>0$ but can be meromorphically extended to the whole plane with poles at $s\in\mathbb{Z}_{\le 0}$. We have

$$\Gamma(s+1)=s\Gamma(s),\quad \Gamma(1)=1,\quad ,\Gamma(\frac{1}{2})=\sqrt{\pi}.$$

By substituting $t=\sqrt{a}t'$ in (1.5), we obtain the *Gauss integral*:

$$\int\limits_{-\infty}^{\infty} e^{-at^2}dt=\frac{\Gamma(\frac{1}{2})}{a^{1/2}}=\sqrt{\frac{\pi}{a}}. \tag{1.6}$$

Although in the substitution above a is a positive real number, one can show that formula (1.6) make sense, as a Riemann integral, for any complex a with Re $(a)\ge 0$. When Re $(a)>0$ this is easy to see using the Hankel representation of $\Gamma(s)$ as a contour integral in the complex plane. When a is a pure imaginary, it is more delicate and we refer to [Kratzer-Franz], 1.6.1.2.

Taking $a=\pi$, we can use

$$d\mu_G=e^{-\pi t^2}dt$$

to define a probability measure on $\mathbb{R}$. It is called the *Gaussian measure*. Let us compute the integral

$$Z^M(\epsilon)=\int\limits_{-\infty}^{\infty} e^{i\epsilon x^3}d\mu_G=\int\limits_{-\infty}^{\infty} e^{-\pi x^2+i\epsilon x^3}dx.$$

Here $\epsilon=1/\hbar$ We have

$$Z(\epsilon)=\int\limits_{-\infty}^{\infty}\exp(-\pi x^2)\sum_{n=0}^{\infty}(i\epsilon x^3)^n/n!)dx.$$

Obviously,

$$\int\limits_{-\infty}^{\infty} x^{2m-1}e^{-\pi x^2}dx=0.$$

Also

$$\int\limits_{-\infty}^{\infty} x^{2m}e^{-\pi x^2}dx=\Gamma(m+\frac{1}{2})/\pi^{m+\frac{1}{2}}=\frac{1\cdot 3\cdots(2m-1)}{(2\pi)^m}=$$

$$\frac{(2m)!}{\pi^m 2^{2m} m!} = P(2m)/(2\pi)^m,$$

where

$$P(2m) = \frac{(2m)!}{2^m m!} = \frac{1}{m!}\binom{2m}{2} \cdot \binom{2m-2}{2} \cdots \binom{2}{2}$$

is equal to the number of ways to arrange $2m$ objects in pairs. This gives us

$$Z(\epsilon) = 1 + \sum_{n=1}^{\infty} (-1)^n \epsilon^{2n} P(6n)/(2n)!(2\pi)^{3n}. \tag{1.7}$$

Observe that to arrange $6n$ objects in pairs is the same as to make a labelled 3-valent graph Γ with $2n$ vertices by connecting 1-valent vertices of the following disconnected graph:

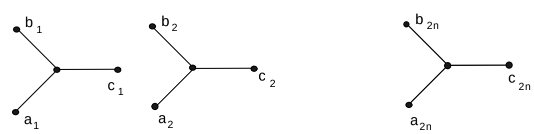


Fig. 1

This graph comes with labeling of each vertex and an ordering of the three edges emanating from the vertex. Let Γ be such a graph, $V(\Gamma)$ be the number of its vertices and $E(\Gamma)$ be the number of its edges. We have $3V(\Gamma) = 2E(\Gamma)$, so that $V(\Gamma) = 2n, E(\Gamma) = 3n$ for some n. Let

$$W(\Gamma) = \frac{(-i\epsilon)^{V(\Gamma)}}{V(\Gamma)!} \frac{1}{(2\pi)^{E(\Gamma)}}.$$

Then

$$Z(\epsilon) = 1 + \sum_{\Gamma} W(\Gamma),$$

where the sum is taken over the set of labeled trivalent graphs. Let $N(\Lambda)$ be the number of labelled trivalent graphs which define the same unlabelled graph when we forget about the labelling. We can write $\bar{W}(\Lambda) = N(\Lambda)W(\Gamma)$, where $N(\Lambda)$ is the number of labelling of the same unlabelled 3-valent graph Λ. Thus

$$Z(\epsilon) = 1 + \sum_{\Lambda} \bar{W}(\Lambda),$$

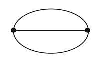
where the sum is taken with respect to the set of all unlabelled 3-valent graphs. It is easy to see that

$$N(\Lambda) = \frac{(2n)!(3!)^{2n}}{\#\mathrm{Aut}(\Lambda)},$$

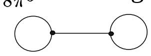
so that

$$\bar{W}(\Lambda) = \frac{(-i\epsilon)^{2n}(2n)!(3!)^{2n}}{(2n)!2\pi)^{3n}\#\mathrm{Aut}(\Lambda)} = \frac{(-6i\epsilon)^{V(\Lambda)}}{(2\pi)^{E(\Lambda)}\#\mathrm{Aut}(\Lambda)}.$$

Given an unlabelled 3-valent graph with $2n$ vertices, we assign to each vertex a factor $(-6i)$, to each edge a factor $1/2\pi$, then multiply all the factors and divide by the number of symmetries of the graph. This gives the *Feynman* rules to compute the contribution of this graph to the coefficient at ϵ^{2n}. For example, the graph



contributes $(-6i)^2\frac{1}{(2\pi)^3}\frac{1}{12} = -\frac{3}{8\pi^3}$ and the graph



contributes $(-6i)^2\frac{1}{(2\pi)^3}\frac{1}{8} = -\frac{9}{16\pi^3}$. The total coefficient at ϵ^2 is $-\frac{15}{16\pi^3}$. This coincides with the coefficient at ϵ^2 in $Z(\epsilon)$ given by the formula (1.7).

Recall that the *Principle of Stationary Phase* says that the main contributions to the integral

$$\int_{\mathbb{R}} e^{i\tau f(x)}\phi(x)dx$$

when τ goes to infinity comes from integrating over the union of small comapct neighborhoods of critical points of $f(x)$. More precisely we have the following lemma:

Lemma 1.1. *Assume $\phi(x)$ has a compact support V and $f(x)$ has no critical points on V. Then, for any natural number n,*

$$\lim_{\tau\to\infty}\tau^n\int_{-\infty}^{\infty} e^{i\tau f(x)}\phi(x)dx = 0.$$

Proof. We use induction on n. The assertion is obvious for $n = 0$. Integrating by parts, we get

$$\frac{i}{\tau}\int_{-\infty}^{\infty} e^{i\tau f(x)}(\frac{\phi(x)}{f(x)'})'dx = \frac{i}{\tau}(\frac{\phi(x)}{f(x)'})e^{i\tau f(x)}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{i\tau f(x)}\phi(x)dx =$$

$$\int_{-\infty}^{\infty} e^{i\tau f(x)}\phi(x)dx.$$

Multiplying both sides by τ^{n+1}, we get

$$\lim_{\tau\to\infty}\tau^{n+1}\int_{-\infty}^{\infty} e^{i\tau f(x)}\phi(x)dx = i\lim_{\tau\to\infty}\tau^n\int_{-\infty}^{\infty} e^{i\tau f(x)}(\frac{\phi(x)}{S(x)'})'dx.$$

Applying the induction to the function $\left(\frac{\phi(x)}{f(x)'}\right)'$ we get the assertion. $\square$

Thus if $f(x)$ has finitely many critical points $x_1, \dots, x_k$, we write our function $f(x)$ as a sum of functions $f_i(x)$ with support on a compact neighborhood K_i of x_i and a function $F(x)$ which has no critical points on the support of $\phi(x)$ and obtain, for any $n > 0$,

$$\int_{-\infty}^{\infty} e^{if(x)/\hbar}\phi(x)dx = \sum_i \int_{K_i} e^{if_i(x)/\hbar}\phi(x)dx + o(\hbar^n).$$

Now let us consider a QFT in dimension 1. Usually we write $D = d+1$, where d is the space-dimension, and 1 is the time-dimension. A QFT in dimension $0+1$ is the quantum mechanics. In this case, we take Σ to be equal to $\mathbb{R}$, $I = [0,1]$ or $S^1 = \mathbb{R}/\mathbb{Z}$ parametrized by t. A map $\gamma : \Sigma \to M$ is *path* in M (infinite, or finite, or a loop). The action is defined by

$$S(\gamma(t)) = \int_\Sigma L(\gamma(t), \dot\gamma(t))dt,$$

where $L : TM \to \mathbb{R}$ is a smooth function defined on the tangent space of M (a *Lagrangian*). The expression $L(\gamma(t), \dot\gamma(t))dt$ is a density on Σ equal to the composition of the differential $d\gamma : T\Sigma \to TM$ and L.

For example, take $M = \mathbb{R}^n$ so that $TM = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(\mathbf{q}, \dot{\mathbf{q}})$. For any $L(\mathbf{q}, \dot{\mathbf{q}})$ and a map $\mathbf{x} : [a,b] \to \mathbb{R}^n$, $L(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ is obtained by replacing $\mathbf{q}$ with $\mathbf{x}(t)$ and $\dot{\mathbf{q}}$ with $\dot{\mathbf{x}}(t)$.

A critical point of the functional $S(\mathbf{x}(t))$ satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial q_i}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \frac{d}{dt}\frac{\partial L}{\partial \dot q_i}(\mathbf{x}(t), \dot{\mathbf{x}}(t)). \tag{1.8}$$

For example, let us take the Lagrangian

$$\sum_{i=1}^n \dot q_i^2 - V(q_1, \dots, q_n) \tag{1.9}$$

Then we get from (1.8)

$$m\frac{d^2x(t)}{dt^2} = -\nabla V(x(t)).$$

Thus a critical path satisfies the Newton Law; it gives the major contribution to the partition function.

Fix $x, x' \in M$ and $t, t' \in \Sigma$. Let $\mathcal{P}(t,x;t',x')$ be the space of smooth maps $\gamma : \Sigma \to M$ such that $\gamma(t) = x, \gamma(t') = x'$. The integral

$$K(t,x;t',x') = \int_{\mathcal{P}(t,x;t',x')} e^{iS(\gamma(t))/\hbar} D[\gamma(t)] \tag{1.10}$$

can be interpreted as the "probability amplitude" that a particle in the position x at the moment of time t moves to the position x' at the time t'.

Let us compute it for the action defined by the Lagrangian (1.9) with $M = \mathbb{R}$. We shall assume that the potential function V is equal to zero.

The space $\mathcal{P}(t, x; t', x')$ is of course infinite-dimensional and the integration over such a space has to be defined. Let us first restrict ourselves to some special finite-dimensional subspaces of $\mathcal{P}(t, x; t', x')$. Fix a positive integer N and subdivide the time interval $[t, t']$ into N equal parts of length $\epsilon = (t' - t)/N$ by inserting intermediate points $t_1 = t, t_2, \dots, t_N, t_{N+1} = t'$. Let us choose some points $x_1 = x, x_2, \dots, x_N, x_{N+1} = x'$ in $\mathbb{R}^n$ and consider the path $\gamma : [t, t'] \to \mathbb{R}^n$ such that its restriction to each interval $[t_i, t_{i+1}]$ is the linear function

$$\gamma_i(t) = x_i + \frac{x_{i+1} - x_i}{t_{i+1} - t_i}(t - t_i).$$

It is clear that the set of such paths is bijective with $(\mathbb{R}^n)^{N-1}$ and so we can integrate a function $F : \mathcal{P}(t, x; t', x') \to \mathbb{R}$ over this space to get a number J_N. Now we can define (1.10) as the limit of integrals J_N when N goes to infinity. However, this limit may not exist. One of the reasons could be that J_N contains a factor C^N for some constant C with $|C| > 1$. Then we can get the limit by redefining J_N, replacing it with $C^{-N} J_N$. This really means that we redefine the standard measure on $(\mathbb{R}^n)^{N-1}$ replacing the measure $d^n x$ on $\mathbb{R}^n$ by $C^{-1} d^n x$. This is exactly what we are going to do. Also, when we restrict the functional to the finite-dimensional space $(\mathbb{R}^n)^{N-1}$ of piecewise linear paths, we shall allow ourselves to replace the integral $\int_t^{t'} L(\gamma, \dot{\gamma}) dt$ by its Riemann sum. The result of this approximation is by definition the right-hand side in (1.10). We should immediately warn the reader that the described method of giving a value to the path integral is not the only possible.

We have

$$K(t, x; t', x') = \lim_{N \to \infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\Big[\frac{im}{2\epsilon} \sum_{i=1}^{N} (x_i - x_{i+1})^2\Big] C^{-N} dx_2 \dots dx_N. \tag{1.11}$$

Here $x_2, \dots, x_N$ are vectors in $\mathbb{R}^n$ and dx_i is the standard measure in $\mathbb{R}^n$. The number C should be chosen to guarantee convergence in (1.11). Using (1.6) we have

$$\int_{-\infty}^{\infty} e^{-a(x_1 - x_2)^2 - a(x_2 - x_3)^2} dx_2 = \int_{-\infty}^{\infty} e^{-2a\left(x_2 - \frac{x_1 + x_3}{2}\right)^2 - \frac{a}{2}(x_1 - x_3)^2} dx_2 =$$

$$= e^{-\frac{a}{2}(x_1 - x_3)^2} \int_{-\infty}^{\infty} e^{-2ax^2} dx = \sqrt{\frac{\pi}{2a}} e^{-\frac{a}{2}(x_1 - x_3)^2}.$$

Next

$$\int_{-\infty}^{\infty} \exp\Big[-\frac{a}{2}(x_1 - x_3)^2 - a(x_3 - x_4)^2\Big] dx_3 =$$

$$= \int_{-\infty}^{\infty} \exp[-\frac{3a}{2}(x_3 - \frac{x_1+x_4}{3})^2 - \frac{a}{3}(x_1-x_4)^2]dx_3 = \sqrt{\frac{2\pi}{3a}}e^{-\frac{a}{3}(x_1-x_4)^2}.$$

Thus

$$\int_{-\infty}^{\infty} \exp[-a(x_1-x_2)^2 - a(x_2-x_3)^2 - a(x_3-x_4)^2]dx_2 =$$

$$\sqrt{\frac{\pi}{2a}}\sqrt{\frac{2\pi}{3a}}\exp[-\frac{a}{3}(x_1-x_4)^2] = \sqrt{\frac{\pi^2}{3a^2}}\exp[-\frac{a}{3}(x_1-x_4)^2].$$

Continuing in this way, we find

$$\int_{-\infty}^{\infty} \exp[-a\sum_{i=1}^{N}(x_i - x_{i+1})^2]dx_2 \ldots dx_N = \sqrt{\frac{\pi^{N-1}}{Na^{N-1}}}\exp[-\frac{a}{N}(x_1-x_{N+1})^2],$$

where $a = m/2i\epsilon$. If we choose the constant C equal to $C = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{1}{2}}$, then we will be able to rewrite (1.11) in the form

$$K(t,x;t',x') = \left(\frac{m}{2\pi iN\epsilon}\right)^{\frac{1}{2}} e^{\frac{mi(x'-x)^2}{2N\epsilon}} = \left(\frac{m}{2\pi i(t'-t)}\right)^{\frac{1}{2}} e^{\frac{mi(x'-x)^2}{2(t'-t)}}. \qquad (1.12)$$

We shall use $K(t,x;t',x')$ to define a certain Hermitian operator in the Hilbert space $L^2(\mathbb{R})$. Recall that for any manifold M with some Lebesgue measure $d\mu$ the space $L^2(M,d\mu)$ consists of square integrable complex valued functions modulo functions equal to zero on the complement of a measure zero set. The hermitian inner product is defined by

$$\langle f,g\rangle = \int_M \bar{f}gd\mu.$$

Example 1.2. An example of an operator in $L^2(M,d\mu)$ is a *Hilbert-Schmidt operator*:

$$Tf(x) = \int_M K(x,y)f(y)d\mu,$$

where $K(x,y) \in L^2(M\times M, \mu\times\mu)$ is the *kernel* of T. In this formula we integrate keeping x fixed. By Fubini's theorem, for almost all x, the function $y \to K(x,y)$ is μ-integrable. This implies that $T(f)$ is well-defined. Using the Cauchy-Schwarz inequality, one can easily checks that

$$||Tf||^2 = \int_M |Tf|^2 d\mu \le ||f||^2 \int_M\int_M |K(x,y)|^2 d\mu d\mu,$$

i.e., T is bounded, and

$$||T||^2 = \sup_{f\neq 0} \frac{||Tf||^2}{||f||^2} \leq \int_M \int_M |K(x,y)|^2 d\mu d\mu.$$

We have

$$(g, Tf) = \int_M \Big(\int_M f(y) K(x,y) d\mu \Big) \overline{g(x)} d\mu = \int_M \int_M K(x,y) f(y) \overline{g(x)} d\mu d\mu.$$

This shows that the Hilbert-Schmidt operator is self-adjoint if and only if

$$K(x,y) = \overline{K(y,x)}$$

outside a subset of measure zero in $M \times M$.

In quantum mechanics one often deals with unbounded operators which are defined only on a dense subspace of a complete separable Hilbert space $\mathcal{H}$. So let us extend the notion of a linear operator by admitting linear maps $D \to \mathcal{H}$ where D is a dense linear subspace of $\mathcal{H}$ (note the analogy with rational maps in algebraic geometry). For such operators T we can define the adjoint operator as follows. Let $D(T)$ denote the domain of definition of T. The adjoint operator T^* will be defined on the set

$$D(T^*) = \{y \in \mathcal{H} : \sup_{0 \neq x \in D(T)} \frac{|\langle T(x), y \rangle|}{||x||} < \infty\}.$$

Take $y \in D(T^*)$. Since $D(T)$ is dense in $\mathcal{H}$ the linear functional $x \to \langle T(x), y\rangle$ extends to a unique bounded linear functional on $\mathcal{H}$. Thus there exists a unique vector $z \in \mathcal{H}$ such that $\langle T(x), y \rangle = \langle x, z \rangle$. We take z for the value of T^* at y. Note that $D(T^*)$ is not necessary dense in $\mathcal{H}$. We say that T is *self-adjoint* if $D(T) = D(T^*)$ and $T = T^*$. We shall always assume that T cannot be extended to a linear operator on a larger set than $D(T)$. Notice that T cannot be bounded on $D(T)$ since otherwise we can extend it to the whole $\mathcal{H}$ by continuity. On the other hand, a self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$ is always bounded. For this reason self-adjoint linear operators T with $D(T) \neq \mathcal{H}$ are called *unbounded* linear operators.

Example 1.3. Let us consider the space $\mathcal{H} = L^2(\mathbb{R}, dx)$ and define the operator

$$Tf = if' = i\frac{df}{dx}.$$

Obviously it is defined on the space of differentiable functions with square integrable derivative. This space contains the subspace of smooth functions with compact support which is known to be dense in $L^2(\mathbb{R}, dx)$. Let us show that the operator $T : D \to \mathcal{H}$ is self-adjoint. Let $f \in D(T)$. Since $f' \in L^2(\mathbb{R}, dx)$,

$$\int_0^t f'(x)\overline{f(x)}dx = |f(t)|^2 - |f(0)|^2 - \int_0^t f(x)\overline{f'(x)}dx$$

is defined for all t. Letting t go to $\pm\infty$, we see that $\lim_{t\to\pm\infty} f(t)$ exists. Since $|f(x)|^2$ is integrable over $(-\infty,+\infty)$, this implies that this limit is equal to zero. Now, for any $f,g\in D$, we have

$$(Tf,g)=\lim_{t\to\infty}\int_0^t if'(x)\overline{g(x)}dx=\lim_{t\to\infty}\Big(if(t)\overline{g(x)}\Big|_{-\infty}^{+\infty}-\int_0^t if(x)\overline{g'(x)}dx\Big)=$$

$$=\lim_{t\to\infty}\int_0^t f(x)\overline{ig'(x)}dx=(f,Tg).$$

This shows that $D\subset D(T^*)$ and T^* is equal to T on D. The proof that $D=D(T^*)$ is more subtle and we omit it.

Let H_1,H_2 be two copies of the space $L^2(M,d\mu)$. Let $T_{t,t'}$ be the Hilbert-Schmidt operator $H_1\to H_2$ defined by a kernel $K(t,x;t'x')$ which has t,t' as real parameters:

$$T_{t,t'}(\phi)(x)=\int_M K(t,x;t',x')\phi(x')d\mu_M.$$

Suppose our kernel has the following properties:

(M)

$$K(t,x;t'',x'')=\int_t^{t''}\int_M K(t,x;t',x')K(t',x';t'',x'')d\mu_M dt',\quad t<t'';$$

(N)

$$\int_M |K(t,x;t',x')|^2 d\mu_M=1;$$

(T)

$$K(t_1,x;t_1',x')=K(t_2,x;t_2',x')\quad\text{if}\quad t_2'-t_2=t_1'-t_1;$$

(C) for any $\phi,\psi\in L^2(M,d\mu)$, the function

$$t\to\Phi(t)=\int_M\overline{\psi(x)}K(t,x;t',x')\phi(x')d\mu_M d\mu_X$$

is continuous for $t'>t$ and $\lim_{t'\to t+}\Phi(t)=\langle\psi,\phi\rangle$.

When K is defined by the path integral, property (M) is taken as one of the axioms of QFT. It expresses the property that any path $\gamma:[t,t'']\to M$ from x to x'' is equal to a sum of paths $\gamma_1:[t,t']\to M$ from x to x' and a path $\gamma_2:[t',t'']\to M$ from x' to x''. Property (N) says that the total probability amplitude of a particle to move from x to somewhere is equal to 1. Notice that property (N) implies that the operator $T_{t,t'}$ is unitary. In fact,

$$\int_M\overline{T_{t,t'}\phi}\cdot T_{t,t'}\psi d\mu_M=$$

$$\int_M \Big(\int_M \overline{K(t,x;t',x')\phi(x)}d\mu_M\Big)\Big(\int_M K(t,x;t',x')\psi(x)d\mu_M\Big)d\mu_M =$$

$$\int_M \overline{\phi(x)}\Big(\int_M \overline{K(t,x;t',x')}K(t,x;t',x')d\mu_M\Big)\psi(x)d\mu_M = \int_M \overline{\phi(x)}\psi(x)d\mu_M.$$

Now we use the following Stone-von Neumann's Theorem:

Theorem 1.1. *Let $U(t), t \in \mathbb{R}_{\geq 0}$ be a family of unitary operators in a Hilbert space $\mathcal{H}$. Assume that*

(i) for all $u, v \in \mathcal{H}$, the function $t \to F(t) = (u, U(t)v)$ is continious for $t > 0$ and $\lim_{t\to 0+} F(t) = (u, v)$;

(ii) for all $t, t' \in \mathbb{R}_{\geq 0}, U(t+t') = U(t) \circ U(t')$.

Then

$$D = \{u \in \mathcal{H} : \lim_{t\to 0+} \frac{U(t) - I}{t}u \quad \textit{exists}\}$$

is dense in $\mathcal{H}$ and the operator defined by

$$Hu = i \lim_{t\to 0+} \frac{U(t) - I}{t}u$$

is self-adjoint. It satisfies

$$U(t) = e^{itH}, \quad t \geq 0.$$

Applying this to our situation, we obtain that

$$T_{t_0,t} = e^{-i(t-t_0)H}, t \geq t_0$$

for some linear operator H. The operator H is called the *Hamiltonian operator* associated to $K(t, x; t', x')$.

We would like to apply the above to our function

$$K(t, x; t', x') = \Big(\frac{m}{2\pi i(t' - t)}\Big)^{\frac{1}{2}} \exp\big(\frac{im(x' - x)^2}{2(t' - t)}\big).$$

Unfortunately we cannot take the function $K(t, x; t', x')$ to be the kernel of a Hilbert-Schmidt operator. Indeed, it does not belong to the space $L^2(\mathbb{R}^2, dxdx')$. In particular property (N) is not satisfied. One can show that (M) is OK, (T) is obviously true and (C) is true if one restricts to functions ϕ, ψ from a certain dense subspace of $L^2(\mathbb{R})$. The way about this is as follows (see [Rauch]).

First let us recall the notion of the Fourier transform in $\mathbb{R}$. It is a linear operator defined on the Schwartz space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ of smooth functions with all derivatives tend to zero faster than any power of $|x|$ as $x \to \infty$. It is given by the formula

$$\mathcal{F}(\phi(x)) := \hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x}\phi(x)dx.$$

Here are some of the properties of this operator:

(i) $\mathcal{F}:\mathcal{S}(\mathbb{R})\to\mathcal{S}(\mathbb{R})$ is an unitary operator;

(ii) $\mathcal{F}^{-1}(\phi(x))=\mathcal{F}(\phi(-x))$;

(iii) $\mathcal{F}(\phi(x)')=i\xi\hat{\phi}(\xi))$;

(iv) $\hat{\phi}(\xi)'=\mathcal{F}(-ix\phi(x))$;

(v) $\mathcal{F}(\phi*\psi)=\sqrt{2\pi}\mathcal{F}(\phi)\mathcal{F}(\psi)$, where

$$\phi*\psi(x)=\int_{-\infty}^{\infty}\phi(x-y)\psi(y)dy.$$

Let us show that our function $K(t,x;t',x')$ is the *propagator* for the *Schrödinger equation*

$$i\frac{\partial}{\partial t}+\frac{m}{2}\frac{\partial^2}{\partial x^2}u(t,x):=iu_t+\frac{m}{2}u_{xx}=0,\quad u(0,x)=f(x)\in L^2(\mathbb{R}).$$

We take for simplicity $m=1$. Supose $f(x)\in\mathcal{S}(\mathbb{R})$. Let us find the solution in $\mathcal{S}(\mathbb{R})$ using the Fourier transform. Using property (iii), we get $\hat{u}_t=-\frac{1}{2}i\xi^2\hat{u}$ (we use the Fourier transform only in the variable x). Integrating this equation with initial condition $\hat{u}(0,\xi)=\hat{f}(\xi)$, we get $\hat{u}(t,\xi)=e^{-it\xi^2/2}\hat{f}(\xi)$.

Taking the inverse Fourier transform, we get

$$u(t,x)=\mathcal{F}^{-1}(e^{-it\xi^2/2}\hat{f}(\xi))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}it\xi^2+ix\xi}\hat{f}(\xi)d\xi. \tag{1.13}$$

Clearly, $u(0,x)=\mathcal{F}^{-1}\mathcal{F}(f)=f(x)$. Of course, we have still to show the existence of a solution. We skip the check that formula (1.13) gives a solution in $\mathcal{S}(\mathbb{R})$. This defines us a linear operator (the propagator)

$$S(t):\mathcal{S}(\mathbb{R})\to\mathcal{S}(\mathbb{R}),\quad f(x)\to u(t,x).$$

We would like to show that it is an integral operator and find its kernel. Let $K(t,x)=\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}e^{-it\xi^2/2})$. Then

$$\int_{-\infty}^{\infty}K(t,x-y)f(y)dy=\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-it\xi^2/2}e^{i(x-y)\xi}d\xi\Big)f(y)dy=$$

$$\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-iy\xi}f(y)dy\Big)e^{ix\xi}e^{-it\xi^2/2}d\xi=\mathcal{F}^{-1}(e^{-it\xi^2/2}\hat{f}(\xi))=u(t,x).$$

Unfortunately, this computation is wrong since the function $e^{-it\xi^2/2}$ does not belong to $\mathcal{S}(\mathbb{R})$. A way about it is to consider this function as a distribution and extend the Forier transform to distributions.

Recall that a *distribution* is a continuous linear functional on the space $C^\infty(\mathbb{R})_0$ of smooth functions with compact support (*test functions*). Any function f which can be integrated over any finite closed interval (but not necessary over the whole $\mathbb{R}$) can be considered as a distribution. Its value at a test function is equal to

$$f(\phi) = \int_{-\infty}^{\infty} \overline{f(x)}\phi(x)dx,$$

where the bar denotes the complex conjugation. Such a distribution is called a *regular distribution* or a *tempered distribution*. The rest are called *singular distributions*. We shall denote the value of a distribution f on a test function ϕ by

$$f(\phi) = \int_{-\infty}^{\infty} \phi(x)\bar{f}(x)dx.$$

If f is a regular distribution defined by a function $f(x)$ from $L^2(\mathbb{R})$, then

$$f(\phi) = \langle f, \phi \rangle.$$

An example of a singular distribution is the *delta-function* $\delta(x-a)$ whose value at a test function ϕ is equal to $\phi(a)$. It is also denoted by δ_a. A linear operator $T : D \to L^2(\mathbb{R})$ with $C^\infty(\mathbb{R})_0 \subset D \cap D(T^*)$ extends to the space of distributions by the formula

$$Tf(\phi) = f(T^*\phi).$$

If $f \in L^2(\mathbb{R})$, viewed as a regular distribution, we have

$$Tf(\phi) = \langle T^*\phi, f \rangle = \langle \phi, Tf \rangle$$

so the two definitions agree.

For example let $T = \frac{d}{dx}$ be defined on the space of functions with square integrable derivative. We have $T^* = -T$ and for any distribution f, $f'(\phi) = f(-\phi)$. If f is a tempered distribution defined by an integrable differential function f such that f' also defines a tempered distribution, then the formula of integration by parts shows that this definition agrees with the usual definition of derivative.

Since the Fourier transform is an example of an operator defined on $\mathcal{S}(\mathbb{R})$ with $\mathcal{F}^* = \mathcal{F}^{-1}$, we can define the Fourier transform of a distribution f by

$$\mathcal{F}(f)(\phi) = f(\mathcal{F}^{-1}(\phi)).$$

All the properties (i)-(v) extend to distributions. In property (v) we define the *convolution* of a regular distribution f and an element ϕ of $\mathcal{S}(\mathbb{R})$ by the formula

$$f * \psi(\phi) = f(\psi * \phi).$$

Lemma 1.2. *For any $a \in \mathbb{C}$ with Re $(a) \geq 0, a \neq 0$,*

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-x^2/4a}.$$

Proof. Assume first that Re $(a) > 0$. Then the function e^{-ax^2} belongs to $\mathcal{S}(\mathbb{R})$ and, using the Gaussian integral, we obtain

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x+\frac{\xi}{2ai})^2} e^{-\xi^2/4a} dx =$$

$$\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{a}} e^{-\xi^2/4a} = \frac{1}{\sqrt{2a}} e^{-\xi^2/4a}.$$

Therefore, for any $\phi \in C^\infty(\mathbb{R})_0$,

$$\mathcal{F}(e^{-ax^2})(\phi) = \frac{1}{\sqrt{2a}} \int_{-\infty}^{\infty} e^{-\xi^2/4\bar{a}} \phi(\xi) d\xi.$$

Consider the both sides as functions of a. When Re $(a) > 0$ each side is a holomorphic function, and, for Re $(a) \geq 0, a \neq 0$, are continuous functions. The unique continuation principle for holomorphic functions implies that the two sides are equal for Re $(a) \geq 0, a \neq 0$. This proves the lemma. $\square$

Now we can use the lemma to set

$$K(t,x) = \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}} e^{-it\xi^2/2}) = \mathcal{F}(\frac{1}{\sqrt{2\pi}} e^{-it(-\xi)^2/2}) = \frac{1}{\sqrt{2\pi it}} e^{i\frac{x^2}{2t}}.$$

Property (v) of Fourier transform gives us

$$S(t)f = K(t,x) * f = \int_{-\infty}^{\infty} K(t, x-y) f(y) dy =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi it}} e^{i\frac{(x-y)^2}{2t}} f(y) dy = \int_{-\infty}^{\infty} K(t,x,y) f(y) dy.$$

Thus we see that the integral operator with the kernel $K(t,x,y) = K(t, x-y), t > 0$, is well-defined as an operator $S(t)$ on the space $\mathcal{S}(\mathbb{R})$. Now observe that

$$\| S(t)f \| = \| u(t,x) \| = \| \mathcal{F}^{-1}(e^{-it\xi^2/2} \hat{f}(\xi)) \| =$$

$$\| e^{-it\xi^2/2} \hat{f}(\xi) \| = \| \hat{f}(\xi) \| = \| f \| .$$

This shows that $S(t)$ is a unitary operator. In particular, $S(t)$ is bounded on $\mathcal{S}(\mathbb{R})$ (of norm 1) and hence continuous. It is known that $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. Thus we can extend $S(t)$ by continuity to an unitary operator on the whole space $L^2(\mathbb{R})$. It satisfies the property

$$\mathcal{F}(S(t)f) = e^{-it\xi^2/2}\mathcal{F}(f).$$

Using property (iii) we get

$$S(t)f = e^{-itH}f, \quad t > 0$$

where $H = -\frac{d^2}{dx^2}$. This justifies the claim that $K(t,x;t'x')$ is the kernel of the operator $S(t'-t) = e^{-i(t'-t)H}$ on $L^2(\mathbb{R})$.

Finally let us try to justify the following formula from physics books:

$$K(t,x;t'x') = \langle x|e^{-i(t'-t)H}|x'\rangle \tag{1.14}$$

First of all for any ϕ,ψ from a Hilbert space $\mathcal{H}$, physicists employ the bra-ket notation

$$\langle\phi|\psi\rangle := \langle\phi,\psi\rangle.$$

If T is a linear operator in $\mathcal{H}$, then

$$\langle\phi|T|\psi\rangle := \langle\phi, T\psi\rangle.$$

Let ϕ_λ be a normalized eigenfunction of an operator T with an eigenvalue λ. Physicists denote it by $|\lambda\rangle$ (although it is defined only up to a factor of absolute value one). To simplify the notation they set

$$\langle\lambda||T||\mu\rangle = \langle\lambda|T|\mu\rangle.$$

Consider an operator (the *position* operator)

$$Q : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R}), f \to xf.$$

It is a self-adjoint operator $\mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R})$. Its eigenfunctions do not belong to the space $L^2(\mathbb{R})$ but rather to the space of distributions.

We have

$$Q\delta_a(\phi) = \delta_a(Q\phi) = \delta_a(x\phi) = a\phi(a) = a\delta_a(\phi).$$

Thus δ_a can be considered as an eigendistribution of Q with eigenvalue a. Thus for any $x \in \mathbb{R}$ we have, according to physicist's notation, $|x\rangle = \delta_x$. Now we have to compute $e^{-i(t'-t)}\delta_x$. Recall that we can view it as an integral operator with kernel $K(t,x;t',x')$ defined on the Schwartz space $\mathcal{S}(\mathbb{R})$ which obviously contains $C^\infty(\mathbb{R})_0$. We have

$$\int_{-\infty}^{\infty} K(t,x;t',x')\delta(x'-b)dx' = K(t,x;t',b),$$

$$\langle\delta_a|e^{-i(t'-t)}|\delta_b\rangle = \langle\delta_b|K(t,x;t',b)\rangle := \delta_a(K(t,x;t',a)) = K(t,a;t',b).$$

Taking $a = x, b = x'$ we get formula (1.14). We have to understand it as

$$\delta_x(e^{-iH(t'-t)}\delta_{x'}) = K(t,x;t',x').$$

For any function $V : M \to \mathbb{C}$ and a point $t \in \Sigma$ one can consider a function on the set $\mathrm{Map}(\Sigma, M)$ defined by

$$V[t](\phi) = V(\phi(t)).$$

Let $V_1, \ldots, V_n$ be functions on M and $t_1, \ldots, t_n \in \Sigma$, one can consider the integral

$$\langle V_1[t_1], \ldots, V_n[t_n]\rangle^M := \int_{\mathrm{Map}(\Sigma,M)} V_1[t_1](\phi)\ldots V_n[t_n](\phi)e^{iS(\phi)}D\phi.$$

The right-hand side is called the path integral with insertion functions $V_1, \ldots, V_n$. The left-hand-side is called the *correlation n-function*. In the example above

$$\langle\delta_x[t], \delta_{x'}[t']\rangle = K(t,x;t',x').$$

Exercises

1.1 Find the Feynman rules to compute

$$Z(\epsilon) = \int_{-\infty}^{\infty} e^{-x^2+i\epsilon x^4}dx.$$

Compute the coefficient at ϵ^2.

1.2 Show that the distribution $G(t,x,y) = K(0,t;x,y)$ (defined to be zero for $t \leq 0$) is a generalized solution of the equation

$$u_t - \frac{1}{2}u_{xx} = \delta(t)\delta(x-y),$$

(you have to give the meaning of the right-hand-side).

1.3 Show that, for any $\lambda \geq 0$, the function $e^{i\lambda x}$ is a generalized eigenfunction of the operator $i\frac{d}{dx}$ in $L^2(\mathbb{R})$ and any generalized eigenfunction coincides with one of these functions.

1.4 Find the Fourier transform and the derivative of the Dirac function $\delta(x-a)$.

Lecture 2

Partition function as the trace of an operator

Recall that the *trace* $\mathrm{Tr}(T)$ of an operator T in a finite dimensional Hilbert space $\mathcal{H}$ is equal to the sum of the diagonal entries of a matrix of T with respect to any basis. If we choose an orthonormal basis $(e_1, \dots, e_n)$, then

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} \langle Te_i, e_i \rangle. \tag{2.1}$$

If T is a normal operator (e.g. Hermitian or unitary), then one can choose an orthonormal basis of V consisting of eigenvectors of T. In this case

$$\mathrm{Tr}(T) = \sum_{\lambda \in \mathrm{Sp}(T)} d(\lambda)\lambda, \tag{2.2}$$

where $\mathrm{Sp}(T)$ is the spectrum of T (the set of eigenvalues) and $d(\lambda)$ is equal to the dimension of the eigensubspace corresponding to the eigenvalue λ. Notice that

$$\det(T) = \prod_{\lambda \in \mathrm{Sp}(T)} \lambda^{d(\lambda)}.$$

This gives

$$\mathrm{Tr}(T) = \ln(\det(e^T)). \tag{2.3}$$

There are several approaches to generalize the notion of the trace to operators in infinite-dimensional Hilbert spaces. We shall briefly discuss them. First assume that T is a bounded operator. First we try to generalize the definition of a trace by using (2.1). One chooses a basis $(e_1, \dots, e_n, \dots)$ and sets

$$\mathrm{Tr}(T) = \sum_{i=1}^{\infty} \langle Te_i, e_i \rangle,$$

if the series convergent. If the convergence is absolute, then this definition does not depend on the choice of a basis. In this case T is called a *trace-class* operator. For example, one can show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ if both A and B are trace-class. An example of a trace-class operator is a Hilbert-Schmidt operator in the space $L^2(M; d\mu)$. If $K(x, y)$ is its kernel, then

$$\mathrm{Tr}(T) = \int_M K(x, x) d\mu.$$

When T is a self-adjoint Hilbert-Schmidt operator, the two definitions coincide. This follows from the Hilbert-Schmidt Theorem.

Example 2.1. Let M be a finite set $\{1, \ldots, n\}$ equipped with the measure $d\mu(A) = \#A$. Then $L^2(M, d\mu) = \mathbb{R}^n$ with inner product

$$\langle \phi, \psi \rangle = \int_M \bar{\phi} \cdot \psi d\mu = \sum_{i=1}^{n} \bar{a}_i b_i,$$

where $\phi = (a_1, \ldots, a_n), \psi = (b_1, \ldots, b_n)$. It is clear that $K(x, y)$ can be identified with a matrix $K = (k_{ij})$ and

$$T\phi(i) = \sum_{j=1}^{n} k_{ij} a_j,$$

so that T is a linear operator defined by the matrix K. Then its trace is equal to

$$\mathrm{Tr}(T) = \sum_{j=1}^{n} k_{jj}.$$

This agrees with definition (2.1) when we take the standard orthonormal basis of $\mathbb{R}^n$.

As we have already mentioned, in physics one deals with unbounded linear operators in $L^2(M, d\mu)$ like a differential operator. One tries to generalize definition (2.3). Notice that

$$\ln \det(T) = \sum_{i=1}^{n} \ln(\lambda_i) = -\frac{d(\sum_{i=1}^{n} \lambda_i^{-s})}{ds}\Big|_{s=0}.$$

Now for any T such that $\mathcal{H}$ has a basis of eigenvectors of T one can define the *zeta-function* of T as follows. Let $0 < \lambda_1 < \lambda_2 \ldots < \lambda_n < \ldots$ be the sequence of positive eigenvalues of T. One sets

$$\zeta_T(s) = \sum_{n=1}^{\infty} \frac{m_n}{\lambda_n^s},$$

where m_n is the *multiplicity* of λ_n, i.e. the dimension of the eigensubspace of eigenvectors with eigenvalue λ_n. When $\mathcal{H} = L^2(M, d\mu)$, where M is a compact manifold of dimension d and T is a positive elliptic differential operator of order r, one can show that $\zeta_T(s)$ is an analytic function for Re $(s) > d/r$ and it can be analytically extended to an open subset containing 0. In this case we define

$$\det(T) = e^{-\zeta_T'(0)}.$$

This obviously agrees with (2.3) when V is finite-dimensional. Also it is easy to see that for any positive number λ

$$\det(\lambda T) = \lambda^{\zeta_T(0)} \det(T). \tag{2.4}$$

This of course agrees with the finite-dimensional case because $\zeta_T(0) = \dim V$.

Example 2.2. Consider the operator $T = -\frac{d^2}{dx^2}$ which acts on the space $L^2(S^1)$, where $S^1 = \mathbb{R}/2\pi r\mathbb{Z}$ with the usual measure dx descended to the factor. Note that in this measure the length of S^1 is equal to $2\pi r$, i.e. S^1 is the circle of radius r. The measure dx corresponds to the choice of metric on the circle determined by its radius. The normalized eigenvectors of T are $\frac{1}{\sqrt{2\pi r}}e^{inx/r}, n \in \mathbb{Z}$. The positive part of the spectrum consists of numbers $(n/r)^2, n \in \mathbb{Z}_{>0}$ with $m_n = 2$. Thus

$$\zeta_T(s) = 2\sum_{n=1}^{\infty}(n/r)^{-2s} = 2r^{2s}\zeta(2s),$$

where $\zeta(s)$ is the Riemann zeta function. It is known to be an analytic function for Re $(s) > 1/2$. This agrees with the above since S^1 is one-dimensional and T is an elliptic operator of second order. We have

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\ln 2\pi.$$

Thus

$$\zeta_T'(0) = -2\ln(2\pi r),$$

and

$$\det{}'(-\frac{d^2}{dx^2}) = (2\pi r)^2. \tag{2.5}$$

Example 2.3. Let us consider the path integral when $\Sigma = \mathbb{R}/2\pi R\mathbb{Z}$ and $M = \mathbb{R}/2\pi r\mathbb{Z}$. We use the action

$$S(\gamma) = \frac{1}{2}\int_0^{2\pi R}\gamma'(t)^2 dt.$$

A map $\gamma : \Sigma \to M$ extends to a map of the universal coverings $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}$. It satisfies $\tilde{\gamma}(t) = \tilde{\gamma}(t + 2\pi R) + 2\pi nr$ for some integer n (equal to the degree of the map of oriented manifolds). Let $\mathrm{Map}(\Sigma, M)_n$ be the set of maps corresponding to the same n. It is clear that each $\gamma \in \mathrm{Map}(\Sigma, M)_n$ can be uniquely written in the form

$$\gamma(t) = \frac{nr}{R}t + \gamma_0(t) = \gamma_n(t) + \gamma_0(t),$$

where $\gamma_0(t)$ satisfies $\gamma_0(t+2\pi R) = \gamma_0(t)$, hence belongs to $L^2(\Sigma)$. The value of S on such γ is equal to

$$S(\gamma) = \frac{1}{2}\int_0^{2\pi R}(\gamma_0'(t)^2 + 2\gamma_0'(t)\gamma_n'(t) + \gamma_n'(t)^2)dt.$$

We have

$$\int_0^{2\pi R}\gamma_0'(t)\gamma_n'(t)dt = \frac{nr}{R}\int_0^{2\pi R}\gamma_0'(t)dt = \frac{nr}{R}(\gamma_0(2\pi R) - \gamma_0(0)) = 0.$$

Thus

$$S(\gamma) = \frac{\pi n^2 r^2}{R} + \frac{1}{2}\int_0^{2\pi R}\gamma_0'(t)^2 dt.$$

The Minkowski partition function is

$$Z^M = \int_{\mathrm{Map}(\Sigma,M)} e^{iS(\gamma)}D\gamma = \sum_{n\in\mathbb{Z}} e^{\pi i\frac{n^2r^2}{R}}\int_{\mathrm{Map}(\Sigma,\mathbb{R})} e^{iS(\gamma)}D\gamma.$$

Observe that we must have

$$\int_{\mathrm{Map}(\Sigma,\mathbb{R})} e^{iS(\gamma)}D\gamma = \int_M K(0,x;2\pi R,x)dx,$$

where

$$K(t,x;t'x') = \frac{1}{\sqrt{2\pi i(t'-t)}}e^{i(x'-x)^2/2(t'-t)},$$

to be consistent with the previous computation of the path integral.

This gives

$$Z = \sum_{n\in\mathbb{Z}} e^{\pi i\frac{n^2r^2}{R}}\frac{1}{2\pi\sqrt{iR}}\int_M dx = \frac{r}{\sqrt{iR}}\sum_{n\in\mathbb{Z}} e^{\pi i\frac{n^2r^2}{R}}.$$

Now we apply the *Poisson summation formula*

$$\sum_{n\in\mathbb{Z}} e^{-\pi x n^2} = \frac{1}{\sqrt{x}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/x}.$$

Taking $x = r^2/iR$, we get

$$Z(r,R) = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} e^{-\pi n^2/x} = \sum_{n\in\mathbb{Z}} e^{-\pi iRn^2/r^2}. \tag{2.6}$$

Let us compute the trace of the operator $e^{-i2\pi RH} = e^{\pi iR\frac{d^2}{dt^2}}$. Its normalized eigenfunctions in $L^2(M)$ are the functions $\psi_n = \frac{1}{\sqrt{2\pi r}}e^{inx/r}$. By (2.1), we have

$$\mathrm{Tr}(e^{-i2\pi RH}) = \sum_{n\in\mathbb{Z}}\langle\psi_n|e^{\pi iR\frac{d^2}{dt^2}}\psi_n\rangle = \sum_{n\in\mathbb{Z}} e^{-\pi iRn^2/r^2}.$$

Comparing this with (2.6), we see that

$$Z = \mathrm{Tr}(e^{-i2\pi RH}),$$

where $H = -\frac{1}{2}\frac{d^2}{dt^2}$.

Remark 2.1. If we repeat the computations for the Euclidean partition function (replacing S with iS) we get

$$Z^E(R,r) = \frac{r}{\sqrt{R}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 r^2/R} = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 R/r^2}.$$

This shows that

$$Z^E(R,r) = \frac{r}{\sqrt{R}} Z^E(R,R/r).$$

If we modify the partition function by inserting the factor $1/\sqrt{r}$, we get

$$Z^E(R,r) = Z^E(R,R/r).$$

This is the first glimpse of the *T-duality*.

Let

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}, \quad \tau \in U = \{x+iy \in \mathbb{C} : y > 0\}.$$

be the modular form associated to the quadratic form $Q(x) = x^2$ (equal to the value at zero of the Riemann theta function in one variable). It satisfies the functional equation

$$\theta(-1/\tau) = (-i\tau)^{\frac{1}{2}}\theta(\tau).$$

(the proof uses the Poisson summation formula). Observe that

$$Z^E(R,r) = \theta(iR/r^2).$$

This is our first encounter with the theory of modular forms.

There is another way to compute the partition function for the action

$$Z = \int_{\mathrm{Map}(\Sigma,\mathbb{R})} \exp\Big(-\frac{1}{2}\int_0^{2\pi R} x'(t)^2 dt\Big) Dx(t),$$

where Σ is the circle of radius R. Notice that

$$\int_0^{2\pi R} x'(t)^2 dt = x(t)x'(t)\big|_0^{2\pi R} - \int_0^{2\pi R} x(t)x''(t)dt = -\langle x(t), x(t)''\rangle_{L^2(S^1)}.$$

Thus

$$Z^E = \int_{\mathrm{Map}(\Sigma,\mathbb{R})} e^{-\langle x(t), Hx(t)\rangle},$$

where $H = -\frac{1}{2}\frac{d^2}{dt^2}$. The integral

$$\int e^{-\langle x(t), Tx(t)\rangle} Dx(t),$$

can be thought as a generalization of the Gaussian integral since $\langle x(t), Tx(t)\rangle$ is a quadratic form in $V = L^2(S^1)$. If $V = \mathbb{R}^n$ and T is a positive-definite self-adjoint operator, we could use the orthogonal change of variables to diagonalize T and write

$$\int_{\mathbb{R}^n} e^{-\langle \mathbf{x}, T\mathbf{x}\rangle} dx_1 \dots dx_n = \int_{\mathbb{R}^n} e^{-\lambda_1 x_1^2 - \dots - \lambda_n x_n^2} dx_1 \dots dx_n =$$

$$\prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-\lambda_i x_i^2} dx_i = \prod_{i=1}^{n} \sqrt{\frac{\pi}{\lambda_i}} = \frac{\pi^{n/2}}{\prod_{i=1}^n \sqrt{\lambda_i}} = \frac{\pi^{n/2}}{\sqrt{\det(T)}} = \det(\frac{1}{\pi}T)^{-\frac{1}{2}}.$$

Here we assumed that all eigenvalues λ_i are positive, or equivalently, that the quadratic form $Q = \langle \mathbf{x}, T\mathbf{x}\rangle$ is positive definite. To get rid of π let us change the measure on $\mathbb{R}$ replacing dx with $\frac{1}{\sqrt{2\pi}}dx$ so that

$$\int_{\mathbb{R}^n} e^{-\langle \mathbf{x}, T\mathbf{x}\rangle} (\sqrt{\pi})^{-n} dx_1 \dots dx_n = \det(T)^{-\frac{1}{2}}.$$

Now, for any normal positive definite operator $T : \mathcal{H} \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$, we can write any element $\phi \in \mathcal{H}$ as a sum $\sum a_n \phi_n$, where (ϕ_n) is an orthonormal basis of eigenvectors of T. The coordinate a_n is an analog of the x_i coordinate from above. This motivates the following definition

$$C \int_V e^{-\langle v, Qv\rangle} Dv = \det{}'(\frac{1}{\pi}Q)^{-1/2}. \tag{2.7}$$

Here the measure $D[v]$ is defined up to some multiplicative constant C. In fact we will be defining the correlation functions by the formula

$$\langle V_1(t_1), \dots, V_n(t_n)\rangle = \frac{\int V_1(t_1)\cdots V_n(t_n) e^{iS(\gamma)} D[\gamma]}{\int e^{iS(\gamma)} D[\gamma]},$$

so the choice of the constant will not matter. We would like to appy this to the operator $H = -\frac{d^2}{dt^2}$ in $L^2(\mathbb{R}/2\pi R)$. However, not all of its eigenvalues are positive. Constant functions form the nullspace of this operator. If we decompose each vector as a sum $\sum_n a_n \psi_n$ of normalized eigenvectors, then coefficients a_n will be analogs of the coordinates in $\mathbb{R}^n$. So, we can write our space as the product of the space of constant functions and functions with $a_0 = 0$. The coefficient of the constant function 1 at $\psi_0 = 1/\sqrt{2\pi R}$ is equal to $\sqrt{2\pi R}$. Thus the integral over the space of constant functions is equal to $\int_0^{2\pi r} \sqrt{2\pi R}(dx/\sqrt{2\pi}) = 2\pi r\sqrt{R}$. So, using (2.5), we obtain

$$Z = (2\pi R)^{-1}(2\pi r\sqrt{R}) = r/\sqrt{R}.$$

This agrees with the computations in example 2.3 if we switch from the Minkowski partition function to the Euclidean one.

Here is another application of the Gaussian integral for quadratic functionals. Consider the action functional $S(\gamma)$ defined by some Lagrangian $L : TM \to \mathbb{R}$. We know that its stationary points are classical solutions. Write $\gamma(t) = \gamma_c(t) + \delta\gamma(t)$, where $\gamma_c(t)$ is a classical solution. Then

$$S(\gamma) = S(\gamma_c) + \frac{1}{2}\frac{\delta^2 S}{\delta\gamma^2}\Big|_{\gamma=\gamma_c}(\delta\gamma)^2 + \text{terms of higher order in} \quad \delta\gamma.$$

This gives a *semi-classical approximation*:

$$\int \exp(iS(\gamma)/\hbar)D[\gamma] \sim \sum_{\text{classical solutions}} \exp(iS(\gamma_c)/\hbar)[\frac{1}{2\pi i}\det(\frac{\delta^2 S}{\delta\gamma^2})\Big|_{\gamma=\gamma_c}]^{-\frac{1}{2}}.$$

This approximation is exact when the action is quadratic in γ.

Example 2.4. We take $\Sigma = \mathbb{R}/2\pi L$ and $M = \mathbb{R}$. The Lagrangian is

$$L(q,\dot{q}) = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$$

and the Minkowski partition function is

$$Z^M = \int D[\gamma(t)]\exp\Big(\frac{i}{2}\int_\Sigma(\dot{\gamma}(t)^2 - \omega^2\gamma^2(t)\Big)dt.$$

It is called the path integral of the *harmonic oscillator*. The kernel of the operator $e^{-i(t-t_0)H}$ is given by

$$K(t_0,y;t_1,x) = \int_{\gamma(t_0)=y}^{\gamma(t_1)=x} D[\gamma(t)]\exp\Big(\frac{i}{2}\int_\Sigma(\dot{\gamma}(t)^2 - \omega^2\gamma^2(t)\Big)dt.$$

Choose a critical path γ_{cl} for the action defined by our Lagrangian and decompose the action in the Taylor expansion at γ_{cl}.

$$S(\gamma(t)) = S(\gamma_{\text{cl}}) + \frac{1}{2}\frac{\delta^2 S}{\delta\gamma^2}\Big|_{\gamma=\gamma_{\text{cl}}}(\gamma(t) - \gamma_{\text{cl}}). \tag{2.8}$$

The classical path is a solution of the Lagrangian equation:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}} - \frac{\partial L}{\partial q} = \gamma(t)'' + \omega^2\gamma(t) = 0.$$

Its solution satisfying the initial condition $\gamma(t_0) = y, \gamma(t_1) = x$ is

$$\gamma_{\text{cl}}(t) = y\Big(\frac{\sin\omega(t_1 - t)}{\sin\omega(t_1 - t_0)}\Big) + x\Big(\frac{\sin\omega(t - t_0)}{\sin\omega(t_1 - t_0)}\Big).$$

The value of the action functional on the classical solution is

$$S(\gamma_{\text{cl}}) = \frac{1}{2}\int_{t_0}^{t_1} \left(\gamma_{\text{cl}}(t)')^2 - \omega^2\gamma_{\text{cl}}(t)^2\right)dt = \frac{\omega\Big((y^2+x^2)\cos\omega(t_1-t_0) - 2xy\Big)}{2\sin\omega(t_1-t_0)}.$$

The second variation of the action functional is

$$\frac{\delta^2 S}{\delta\gamma^2}(\gamma(t)) = \int_{t_0}^{t_1}\gamma(t)(\frac{d^2}{dt^2}+\omega^2)\gamma(t)dt.$$

Thus we can rewrite (2.8) in the form

$$K(t_0,y;t_1,x) = \exp(iS(\gamma_{\text{cl}}))\times$$

$$\int_{\gamma(t_0)=y}^{\gamma(t_1)=x} D[\gamma(t)]\exp\Big(-\frac{i}{2}\int_{t_0}^{t_1}(\gamma(t)-\gamma_{\text{cl}}(t))(\frac{d^2}{dt^2}+\omega^2)(\gamma(t)-\gamma_{\text{cl}}(t))dt\Big).$$

Now let us make the variable change replacing $\gamma(t)-\gamma_{\text{cl}}(t)$ with $\gamma(t)$. The limits in the path integral change to $\gamma(t_0)=\gamma(t_1)=0$. The paths we integrate over are periodic with the period $T=t_1-t_0$ satisfying $\gamma(t_0)=\gamma(t_1)=0$. Using the generalization of the Gaussian integral to functional integrals we have

$$\int_{\gamma(t_0)=0}^{\gamma(t_1)=0} D[\gamma]\exp\Big(-\frac{i}{2}\int_{t_0}^{t_1}\gamma(t)(\frac{d^2}{dt^2}+\omega^2)\gamma(t)dt\Big) =$$

$$\int_{\text{Map}(\Sigma,\mathbb{R})} D[\gamma]\exp(-\frac{1}{2}\langle\gamma, iD\gamma\rangle_{L^2(\Sigma)}) = \det(\frac{1}{2\pi i}D)^{-\frac{1}{2}},$$

where

$$D = -\frac{d^2}{dt^2}-\omega^2.$$

The eigenfunctions of D satisfying the condition $\gamma(t_0)=\gamma(t_1)=0$ are the functions $\sin(n\pi t/T)$, where $n\in\mathbb{Z}_{>0}$ and $T=t_1-t_0$. The corresponding eigenvalues are equal to $\lambda_n=(n\pi/T)^2-\omega^2$. We know that

$$\det(\frac{1}{2\pi i}D) = \prod_{n=1}^{\infty}\frac{1}{2\pi i}((\pi n/T)^2-\omega^2) = \prod_{n=1}^{\infty}\frac{1}{2\pi i}(n\pi/T)^2\prod_{n=1}^{\infty}(1-\frac{\omega^2T^2}{n^2\pi^2}).$$

Of course here we use a "physicists's argument" since we don't have the right to write the product as the product of two infinite products , one of which is divergent (see the next remark for an attempt to justify the argument). Now we use that the first product corresponds to the action with $\omega=0$. So to be consistent with our previous computation we must have

$$\Big(\prod_{n=1}^{\infty}\frac{1}{2\pi i}(n\pi/T)^2\Big)^{-\frac{1}{2}} = K(t_0,0;t_1,0) = \frac{1}{\sqrt{2\pi iT}}.$$

Note that if we compute the product using the zeta function of the operator $-\frac{1}{2\pi i}\frac{d^2}{dx^2}$ on the sapce of functions $\phi(t)$ on $[t_0,t_1]$ satisfying $\phi(t_0)=\phi(t_1)=0$ we get

$$\prod_{n=1}^{\infty}\frac{1}{2\pi i}(n\pi/T)^2=(2\pi iT)C.$$

The two computations disagree. The way out of this contradiction is the choice of the normalizing constant C which we used to define the Gaussian integral. It shows that we have to choose $C=\sqrt{2\pi}$. Now we use the Euler infinite product expansion for the sine function:

$$\frac{\sin(\omega T)}{\omega T}=\prod_{n=1}^{\infty}(1-\frac{\omega^2T^2}{n^2\pi^2}).$$

From this we deduce that

$$K(t_0,y;t_1,x)=\exp(iS(\gamma_{\mathrm{cl}}))\det(\frac{1}{2\pi i}D)^{-1/2}=$$

$$\exp(iS(\gamma_{\mathrm{cl}}))(\frac{\omega}{2\pi i\sin(T\omega)})^{1/2}=(\frac{\omega}{\pi})^{\frac{1}{2}}\frac{e^{-iT\omega/2}}{\sqrt{1-e^{-2i\omega T}}}\exp(iS(\gamma_{\mathrm{cl}})).$$

Let us rewrite $S(\gamma_{\mathrm{cl}})$ in the following form

$$S(\gamma_{\mathrm{cl}}(t))=\frac{i\omega}{2}\Big((x^2+y^2)\frac{1+e^{-2i\omega T}}{1-e^{-2i\omega T}}-\frac{4xye^{-i\omega T}}{1-e^{-2i\omega T}}\Big)=$$

$$\frac{i\omega}{2}\big((x^2+y^2)(1+2\sum_{n=1}^{\infty}e^{-2in\omega T})-4xye^{-i\omega T}(1+\sum_{n=1}^{\infty}e^{-2in\omega T}\big)=$$

$$\frac{i\omega}{2}\Big(-(x^2+y^2)+\sum_{n=0}^{\infty}(2x^2+2y^2-4xye^{-i\omega T})e^{-2in\omega T}\Big).$$

Using this we obtain

$$K(t_0,y;t_1,x)=(\frac{\omega}{\pi})^{\frac{1}{2}}\frac{e^{-iT\omega/2}}{\sqrt{1-e^{-2i\omega T}}}\times$$

$$\exp\Big[-\frac{\omega}{2}\Big(-(x^2+y^2)+\sum_{n=0}^{\infty}(2x^2+2y^2-4xye^{-i\omega T}e^{-2in\omega T}\Big)\Big]=$$

$$\sum_{n=1}^{\infty}e^{-i\omega(n+\frac{1}{2}))T}A_n(x,y)=e^{-i\omega T/2}\sqrt{\omega/\pi}e^{-\omega(x^2+y^2)/2}+\ldots.$$

Now recall that the kernel of a Hilbert-Schmidt unitary operator can be written in the form

$$G(x,y) = \sum_{n\in\mathbb{Z}} \lambda_n \bar{\psi}_n(x)\psi_n(y),$$

where ψ_n is the normalized eigenfunction with the eigenvalue λ_n. In our case, the eigenvalues of e^{-iTH} must be equal to $e^{-iT(n+\frac{1}{2})\omega}$. Thus the eigenvalues of the Hamiltonian H are $\omega(n+\frac{1}{2})$. We shall see in the lecture that the eigenvectors of H are $\psi_n(x) = (\omega/\pi)^{1/4}H_n(\sqrt{\omega}x)e^{-x^2\omega/2}$, where H_n are the Hermite polynomials. When $n=0$, we get $\psi_0(x)\psi_0(y) = \sqrt{\omega/\pi}e^{-\omega(x^2+y^2)/2} = A_0(x,y)$. This checks the first term.

Exercises

2.1 Let $\Lambda : 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be a non-decreasing sequence of positive real numbers. Define $\zeta_\Lambda(s) = \sum_{n\geq 1}\lambda_n^{-s}$ provided that this sum converges for Re $(s) \gg 0$ and has a meromorphic continuation to the whole complex plane with no pole at $s=0$. Set $\prod_{i=1}^{\infty}\lambda_i = e^{-\zeta'_\Lambda(0)}$.

(i) Prove that for any $N \geq 1$, $\prod_{i=1}^{\infty}\lambda_i = (\lambda_1\cdot\lambda_2\cdots\lambda_N)\prod_{i=N+1}^{\infty}\lambda_i$.

(ii) Give a meaning to the equality $\infty! = \sqrt{2\pi}$.

2.2 Let $T = \mathbb{R}^n/\Gamma$ be a n-torus. Here $\Gamma = \mathbb{Z}\omega_1 + \ldots \mathbb{Z}\omega_n$ and $\omega_1,\ldots,\omega_n$ are linear independent vectors in $\mathbb{R}^n$.

(i) Compute the trace of the Laplace operator $L = -\sum_{i=1}^{n}\partial_{x_i}^2$ in $L^2(T,d\mu)$ where $d\mu$ is induced by the standard volume form on $\mathbb{R}^n$.

(ii) Compute the Euclidian partition function $Z^E(T)$ for the maps from $\Sigma = \mathbb{R}/2\pi R$ to T with the action defined by $S(\gamma) = \frac{1}{2}\int_\Sigma ||\tilde{\gamma}'(t)||^2 dt$, where $\tilde{\gamma}$ is any lift of γ to a smooth map $\mathbb{R}\to\mathbb{R}^n$.

(iii) Let $\Gamma^* = \{y\in\mathbb{R}^n : y\cdot x\in\mathbb{Z} \quad \text{for all} \quad x\in\Gamma\}$. Use the Poisson summation formula

$$\sum_{x\in\Gamma} f(x) = A^{-1}\sum_{y\in\Gamma^*}\hat{f}(y),$$

where $f\in\mathcal{S}(\mathbb{R}^n)$, $\hat{f}$ is its Fourier transform and $A = \det[\omega_1,\ldots,\omega_n]$, to relate the partition functions $Z^E(T)$ and $Z^E(T^*)$.

2.3 Compute the terms $A_1(x,y)$ and $A_2(x,y)$ from Example 2.4 to find the eigenfunctions $\psi_1(x)$ and $\psi_2(x)$.

Lecture 3

Quantum mechanics

The quantum mechanics is a $0+1$ dimension QFT.

Let us recall the main postulates of quantum mechanics. A *quantum state* of a system is a line in a separable Hilbert space $\mathcal{H}$. It can be represented (not uniquely) by a vector ψ of norm 1. To each observable quantity (like position, momentum or energy) one associates a self-adjoint operator A in $\mathcal{H}$ (an *observable*). A measurement of an observable A depends on the given state ψ and is not given precisely but instead there is a probability that the value belongs to a subset $(-\infty, \lambda]$. This probability is equal to $p_A(\lambda;\psi) = ||P_A(\lambda)\psi||^2$, where $P_A(\lambda)$ is the spectral function of A, an operator-valued measure on $\mathbb{R}$. In the case when A is a compact operator, $\mathcal{H}$ has an orthonormal basis (e_n) of eigenvectors of A with eigenvalue λ_n. Then $P_A(\lambda) = \sum_{\lambda_n \leq \lambda} P_{e_n}$, where P_{e_n} is the orthogonal projector operator to the subspace $\mathbb{C}e_n$. Thus, for any simple eigenvalue λ_n of A, $|\langle \psi, e_n\rangle|^2$ can be interpreted as the probablility that the observable A takes value λ_n in the state ψ.

In physics literature one often writes $|\lambda\rangle$ for a norm 1 eigenvector ψ_λ of A with eigenvalue λ and rewrites $\langle \psi, \phi_\lambda\rangle$ in the form $\langle \psi|\lambda\rangle$. Also one writes $\langle \lambda|\mu\rangle$ instead of $\langle\langle\lambda|||\mu\rangle\rangle$.

The *probability amplitude* (a complex number of absolute value ≤ 1) is defined to be $\langle \psi, \lambda\rangle$. The function $\lambda \to \langle \psi, \lambda\rangle$ is called the *wave function* of the state ψ with respect to A. The inner product of two states $\langle \phi, \psi\rangle$ is interpreted as the probability amplitude that the state ϕ changes to the state ψ. Its absolute value is the probablity of this event. Note that by Cauchy-Schwarz inequality, this number is always less or equal to 1 and it is equal to 1 if and only if the two states are equal (as lines in the Hilbert space).

The *expectation value* of A in the state ψ is defined to be

$$\langle A\rangle_\psi = \langle A\psi, \psi\rangle = \int_{-\infty}^{\infty} x d\mu_{A,\psi}, \tag{3.1}$$

where $d\mu_{A,\psi}$ is the measure on $\mathbb{R}$ defined by $d\mu_{A,\psi}(E) = \langle P_A(E)\psi, \psi\rangle$.

Example 3.1. Let $\mathcal{H} = L^2(\mathbb{R}, dx)$ and $A = Q$ is the *position operator* corresponding to the measurement of the coordinate x. It is defined by $Q(f) = xf$. This is an unbounded self-adjoint linear operator. We know from Lecture 1 that $|\lambda\rangle = \delta_\lambda$ (although they do not belong to the space $\mathcal{H}$ but rather to the space of distributions). We have

$$\langle\psi(x),\delta_\lambda\rangle = \int \psi(x)\delta_\lambda dx = \psi(\lambda).$$

The probability amplitude of the value λ of the observable Q in a state ψ is equal to $\psi(\lambda)$. So, in the realization of $\mathcal{H}$ as $L^2(\mathbb{R}, dx)$, a state $\psi(x)$ is interpreted as the wave function of the state ψ with respect to the observable Q. Any $\psi \in \mathcal{H}$ can be written as

$$\psi = \int \psi(\lambda)\delta_\lambda d\lambda.$$

Of course this has to be understood as the equality of distributions. For any test function ϕ we have

$$\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\psi(\lambda)\delta_\lambda d\lambda\Big)\phi(x)dx = \int_{-\infty}^{\infty}\psi(\lambda)\Big(\int_{-\infty}^{\infty}\delta_\lambda\phi(x)dx\Big)d\lambda =$$

$$\int_{-\infty}^{\infty}\psi(\lambda)\phi(\lambda)d\lambda = \psi(\phi).$$

The expectation value of Q is equal to

$$\langle Q\rangle_\psi = \int_{\mathbb{R}}\lambda|\psi(\lambda)|^2 d\lambda.$$

Consider the delta-function δ_a as a state (although it does not belong to $L^2(\mathbb{R}, dx)$). Then the probability of Q to take a value b in the state δ_a is equal to $\langle\delta_a,\delta_b\rangle$. The inner product is of course not defined but we can give it the following meaning. We know that δ_a is equal to the limit of tempered distributions $\frac{1}{\sqrt{2\pi t}}e^{-(x-a)^2/2t}$ when t tends to zero. Thus we can set

$$\langle\delta_a,\delta_b\rangle := \lim_{t\to 0}\delta_b\Big(\frac{1}{\sqrt{2\pi t}}e^{-(x-a)^2/2t}\Big) = \lim_{t\to 0}\frac{1}{\sqrt{2\pi t}}e^{-(b-a)^2/2t}.$$

When $b = x$ is a variable, we get

$$\langle\delta_a,\delta_x\rangle = \lim_{t\to 0}\frac{1}{\sqrt{2\pi t}}e^{-(x-a)^2/2t} = \delta_a = \delta(x-a).$$

Example 3.2. Let $\mathcal{H} = L^2(\mathbb{R}/2\pi R)$ and A equal to the *momentum operator* $P = i\frac{d}{dx}$. Its eigenvectors are the functions $f_n = \frac{1}{\sqrt{2\pi R}}e^{-inx/R}$ with eigenvalue n. We have

$$\langle\psi(x), f_n\rangle = \frac{1}{\sqrt{2\pi R}}\int_0^{2\pi R}\bar{\psi}(x)e^{-inx/R}dx = \bar{a}_n,$$

is the n-th Fourier coefficient of $\bar{\psi}$. So, the wave function of ψ is the function $n \to \bar{a}_n$ on $\mathbb{Z} = \mathrm{Sp}(P)$. The probability that P takes value n at the state ψ is equal to $|a_n|^2$. The expectation value is equal to

$$\langle P \rangle_\psi = \sum_{n \in \mathbb{Z}} n |a_n|^2.$$

The dynamics of a quantum system is defined by a choice of a self-adjoint operator H, called the Hamiltonian operator. In Schrödinger's picture the operators do not change with time, but the states evolve according to the law

$$\psi_t = e^{-\frac{i}{\hbar}Ht}\psi.$$

Here $\hbar$ is a fixed constant, the *Planck constant*. Equivalently, $\psi(t)$ is a solution of the *Schrödinger equation*:

$$i\hbar \frac{d}{dt}\psi_t = H \cdot \psi(t).$$

In *Heisenberg's picture*, the states do not change with time but the observables evolve according to the law

$$A(t) = e^{-\frac{i}{\hbar}Ht} \cdot A \cdot e^{\frac{i}{\hbar}Ht}.$$

We have the *Hamiltonian equation*:

$$\frac{d}{dt}A(t) = [A(t), H]_\hbar, \tag{3.2}$$

where

$$[A, B]_\hbar = \frac{i}{\hbar}(A \circ B - B \circ A).$$

If ψ_t is an eigenvector of H, then $\psi_t = e^{-\frac{i}{\hbar}\lambda t}\psi$ and hence the corresponding state (equal to the line spanned by ψ) does not change with time, i.e. $\psi(x)$ describes a *stationary state*. Usually one measures observables at the stationary states of H.

There are two ways to define a quantum mechanical system. One (due to Feynman) uses the path integral approach. Here we take $\mathcal{H} = L^2(M, d\mu)$ as in Lecture 1 and define the Hamiltonian by means of the path integral. The choice here is the action functional. It is defined in such a way that its stationary paths describe the motions of a classical mechanical system. Another approach is via *quantization* of a classical mechanical system. Recall that the latter is defined by a Lagrangian $L : TM \to \mathbb{R}$ which, in its turn, defines an action functional on the space $\mathcal{F} = \mathrm{Map}([a, b], M)$.

$$S(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))dt.$$

A critical point of this functional defines a motion of the mechanical system. The equations for a critical point are called the *Euler-Lagrange equations*. If one chooses local coordinates $q = (q_1, \dots, q_n)$ in M and the corresponding local coordinates $(q, \dot{q}) = (q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n)$ in TM (so that $\dot{q}_i(\frac{\partial}{\partial q_j}) = \delta_{ij}$), the equations look as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, \dots, n. \tag{3.3}$$

Here the left-hand side is evaluated at a path $t \to (\gamma(t), \dot{\gamma}(t))$ in TM given by $q_i = q_i(t), \dot{q}_i = \frac{dq_i(t)}{dt}$. For example, if we assume that the restriction of L to each tangent space TM_x is a positive-definite quadratic form, we can use L to define a Riemannian metric g on M. A critical path becomes a geodesic.

Another way to define the classical mechanics is via a Hamiltonian function which is a function on the cotangent bundle T^*M.

Recall that any non-degenerate quadratic form Q on a vector space V defines a quadratic form Q^{-1} on the dual space V^*. If we view a quadratic form as a symmetric bilinear form, and hence as a linear map $V \to V^*$, then Q^{-1} is the inverse map. Let us see that, for any $\alpha \in V^*$, $Q^{-1}(\alpha)$ is equal to the maximum (if $Q > 0$) or the minimum (if $Q < 0$) of the function $F_\alpha : V \to \mathbb{R}$ defined by

$$F_\alpha(v) = \alpha(v) - Q(v).$$

If we choose the coordinates so that $V = V^* = \mathbb{R}^n$, and $\alpha(v) = \alpha \cdot v$, then $Q(v) = \frac{1}{2} v \cdot A \cdot v$ for some symmetric matrix A. Thus to find an extremum we must have $\nabla F_\alpha = \alpha - A \cdot v = 0$, hence $v = A^{-1}\alpha$ and we get

$$\max_{v \in V} F_\alpha(v) = \alpha \cdot A^{-1}\alpha - \frac{1}{2}\alpha \cdot A^{-1}\alpha = \frac{1}{2}\alpha \cdot A^{-1}\alpha = Q^{-1}(\alpha).$$

Using this one can generalize the construction of Q^{-1} for any function f on V such that its second differential is non-degenerate. This is called the *Legendre transform* of f. By definition, for any $\alpha \in V^*$,

$$\mathrm{Leg}(f)(\alpha) = \alpha(v) - f(v),$$

where v is the implicit function of α defined by $\alpha = df_v$, where $df_v : V \to \mathbb{R}$ is the differential of f at the point $v \in V$. In order that this function be defined we have to satisfy the conditions of the Implicit Function Theorem: $\det(d^2 f(v)) \neq 0$. In general, the implicit function $v(\alpha)$ is a multivalued function, so the Legendre transform is defined only locally in a neighborhood of an extremum point of the function $\alpha(v) - f(v)$.

We shall apply the Legendre transform to the Lagrangian function L. We denote the local coordinates in the cotangent bundle T^*M by

$$(q, p) = (q_1, \dots, q_n; p_1, \dots, p_n),$$

where the fibre coordinates $(p_1, \dots, p_n)$ are taken to be the dual of the coordinate functions $(\dot{q}_1, \dots, \dot{q}_n)$ in the tangent bundle TM and can be identified with a basis $(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n})$ in TM_x. The Legendre transform of L is equal to

$$H(q, p) = \sum_{i=1}^{n} \dot{q}_i p_i - L(q, \dot{q}),$$

where $\dot{q}_i$ are the implicit functions of $(p_1, \dots, p_n)$ defined by the equation

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}), \quad i = 1, \dots, n$$

The function $H : T^*M \to \mathbb{R}$ is called the *Hamiltonian* associated to the Lagrangian L. As we have explained before, in order it is defined the Lagrangian must satisfy some conditions.

Using the Hamiltonian one can rewrite the Euler-Lagrange equation for a critical path $\gamma(t)$ of the action defined by the Lagrangian in the form:

$$\dot{p}_i := \frac{dp_i}{dt} = -\frac{\partial H(q,p)}{\partial q_i}, \quad \dot{q}_i := \frac{dq_i}{dt} = \frac{\partial H(q,p)}{\partial p_i}.$$

Here a solution is a path $\tilde{\gamma} : I \to T^*M, t \to \tilde{\gamma}(t)$, which satisfies the above equations after we compose it with the coordinate functions $(q,p) : T^*M \to \mathbb{R}$. The projection of the path $\tilde{\gamma}$ to the base M (i.e. the composition with the projection map $T^*M \to M$) is the path $\gamma(t)$ describing the equation of the motion. The difference between the Euler-Lagrange equations and Hamilton's equations is the following. The first equation is a second order ordinary differential equation on TM and the second one is a first order ODE on T^*M which has a nice interpretation in terms of vector fields.

Recall that a (smooth) *vector field* on a smooth manifold X is a (smooth) section ξ of its tangent bundle TX. Let $C^\infty(TX)$ denotes the set of vector fields. It has an obvious structure of a vector space. For each smooth function $\phi \in C^\infty(X)$ one can differentiate ϕ along $\xi \in C^\infty(TX)$ by the formula

$$D_\xi(\phi)(x) = \sum_i \frac{\partial \phi}{\partial x_i} \xi_i,$$

where $x \in X, (x_1, \dots, x_n)$ are local coordinates in a neighborhood of x, and ξ_i are the coordinates of $\xi(x) \in TX_x$ with respect to the basis $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ of TX_x. We also have

$$D_\xi(\phi) = d\phi(\xi),$$

where we consider smooth 1-forms as linear functions on vector fields. This defines a linear map

$$D : C^\infty(TX) \to \mathrm{End}(C^\infty(X)).$$

It is easy to check that $D_\xi(\phi\psi) = D_\xi(\phi)\psi + D_\xi(\psi)\phi$, so that the image of D lies in the subspace of derivations of the algebra $C^\infty(X)$. Given a smooth map $\gamma : [a,b] \to X$, and a vector field ξ we say that γ satisfies the differential equation defined by ξ (or is an *integral curve* of ξ) if

$$\frac{d\gamma}{dt} := (d\gamma)_\alpha(\frac{\partial}{\partial t}) = \xi(\gamma(\alpha)) \quad \text{for all } \alpha \in (a,b).$$

The vector field on the right-hand-side of Hamilton's equations has a nice interpretation in terms of the canonical symplectic structure on the manifold $X = T^*M$.

Recall that a *symplectic form* on a smooth manifold X is a smooth closed 2-form $\omega \in \Omega^2(X)$ which is a non-degenerate bilinear form on each $T(X)_x$. If we view ω_x as a linear map $TX_x \to (TX_x)^* = T^*X_x$, then its inverse defines a linear isomorphism $\imath_\omega(x) : T^*X_x \to TX_x$. Varying x we get an isomorphism of vector bundles $TM \to T^*M$ and by the pull-back of sections an isomorphism of the space of sections

$$\imath_\omega : \Omega^1(X) = C^\infty(T^*X) \to C^\infty(TX).$$

Given a smooth function $F : X \to \mathbb{R}$, its differential dF is a 1-form on X, i.e., a section of the cotangent bundle $T^*(X)$. Thus, applying $\imath_\omega$ we can define a section $\imath_\omega(dF)$ of the tangent bundle, i.e., a vector field. It is called the *Hamiltonian vector field* defined by the function F. We apply this to the situation when $X = T^*(M)$ with coordinates (q,p) and F is the Hamiltonian function $H(q,p)$. We use the symplectic form given in local coordinates by

$$\omega = \sum_i dq_i \wedge dp_i.$$

For any $v, w \in T(X)_x$

$$\omega_x(v,w) = \sum_i dq_i(v)dp_i(w) - \sum_i dq_i(w)dp_i(v).$$

In particular,

$$\omega_x(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}) = \omega_x(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = 0 \quad \text{for all } i,j,$$

$$\omega_x(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_j}) = -\omega_x(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_j}) = \delta_{ij}.$$

This shows that $\imath_\omega(dp_i) = \frac{\partial}{\partial q_i}, \quad \imath_\omega(dq_i) = -\frac{\partial}{\partial p_i}$, hence

$$\imath_\omega(dH) = \imath_\omega\Big(\sum_i \frac{\partial H}{\partial q_i}dq_i + \sum_i \frac{\partial H}{\partial p_i}dp_i\Big) = \sum_i(-\frac{\partial H}{\partial q_i})(\frac{\partial}{\partial p_i}) + \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}.$$

So we see that the ODE corresponding to the Hamiltonian vector field $\Theta(H)$ defined by H is the vector from the right-hand-side of Hamilton's equations. We have

$$\Theta(H)(f) = \{f, H\}.$$

Let (X,ω) be a symplectic manifold. For any two functions $f, g \in C^\infty(X)$ one defines the *Poisson bracket*

$$\{f,g\} = \omega(\imath_\omega(df), \imath_\omega(dg)).$$

By definition of $\imath_\omega$ we have $\omega(\imath_\omega(df), \xi) = df(\xi)$, so that

$$\{f,g\} = df(\imath_\omega(dg)) = \imath_\omega(dg)(f) = -\imath_\omega(df)(g).$$

The Poisson bracket defines a structure of Lie algebra on $C^\infty(X)$ satisfying the additional property :

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

One can show (*Darboux's Theorem*) that it is always possible to choose local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ such that

$$\omega = \sum_i dq_i \wedge dp_i.$$

In these coordinates

$$\{f, g\} = \sum_i (\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}).$$

For example,

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}. \tag{3.4}$$

The corresponding differential equation (= dynamical system, flow) is

$$\frac{df}{dt} = \{f, H\}.$$

A solution of this equation is a path $\gamma : [a, b] \to M$ such that

$$\frac{df(\gamma(t))}{dt} = \{f, H\}(\gamma(t)).$$

This is called the *Hamiltonian dynamical system* on X (with respect to the Hamiltonian function H). If we take f to be coordinate functions q_i, p_i on $X = T^*(M)$, we obtain Hamilton's equations for the critical path $\gamma : [a, b] \to M$ in M.

The flow g^t of the vector field $f \to \{f, H\}$ is a one-parameter group of operators U_t on $\mathcal{O}(M)$ defined by the formula

$$U_t(f) = f_t(x) = f(\gamma(t))$$

where $\gamma(t)$ is the integral curve of the Hamiltonian vector field $f \to \{f, H\}$ with the initial condition $\gamma(0) = x$, The equation for the Hamiltonian dynamical system defined by H is

$$\frac{df_t}{dt} = \{f_t, H\}. \tag{3.5}$$

Here we use the Poisson bracket defined by the symplectic form of M.

A quantization of a mechanical system is defined by assigning to any *observable* $f \in C^\infty(X)$ a self-adjoint operator $A_f \in \mathcal{H}$. This operator may contain a parameter $\hbar$. This assignment must satisfy some natural properties. For example:

$$A_{\{f,g\}} = \lim_{\hbar \to 0}[A_f, A_g]_\hbar \tag{3.6}$$

Under the quantization the Hamiltonian function of the mechanical system becomes a self-adjoint operator H, called the *Hamiltonian operator* of the quantized system. We have

$$A_{\Theta(H)f} = A_{\{f,H\}} = \lim_{\hbar\to 0}[A_f, A_H]_\hbar = \lim_{\hbar\to 0}[A_f, H]_\hbar.$$

Thus the linear map $A \to [A,H]_\hbar$ is interpreted as the quantized action of the Hamiltonian vector field $\Theta(H)$. The analog of the dynamical system (3.5) is the Hamiltonian equation in quantum mechanics (3.2).

For example, when a mechanical system is given on the configuration space $T(\mathbb{R}^n)^*$ with coordinate functions q_i, p_i we need to assign some operators to the coordinate functions:

$$Q_i = A_{q_i}, \quad P_i = A_{p_i}, \quad i = 1, \ldots, n.$$

By analogy with (3.4), we should have

$$[P_i, P_j]_\hbar = [Q_i, Q_j]_\hbar = 0, \quad [Q_i, P_j]_\hbar = \delta_{ij} \tag{3.7}$$

So we have to find an appropriate Hilbert space V and operators $P_i, Q_i \in \mathcal{H}(V)$ satisfying (3.7). We take $V = L^2(\mathbb{R}^n)$ and define

$$Q_i : \phi \to q_i\phi, \quad P_i : \phi \to i\hbar\frac{\partial}{\partial q_i}\phi.$$

Recall that these are unbounded self-adjoint operators.

The operator Q_i (resp. P_i) is called the *position* (resp. *momentum*) operator .

Let us give an example of quantization of a classical mechanical system given by a *harmonic oscillator*. It is given by the Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}(m\dot{q}^2 - m\omega^2 q^2)),$$

where m is the mass and ω is the frequency. The corresponding Hamiltonian function is

$$H(q,p) = p\dot{q} - \frac{1}{2}(m\dot{q}^2 - m\omega^2 q^2) = \frac{1}{2}(\frac{p^2}{m} + m\omega^2 q^2),$$

where we used that $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$ to express $\dot{q}$ via p. The function $H(p,q)$ can be viewed as the *total energy* of the system.

The corresponding Newton equation is

$$\frac{d^2x}{dt^2} = -\omega^2 x.$$

So the motion does not depend on the mass but the total energy does. In the sequel we shall assume for simplicity that $m = 1$. The Hamiltonian operator H can be written in the form

$$H = \frac{1}{2}P^2 + \frac{\omega^2}{2}Q^2 = aa^\dagger - \frac{\omega\hbar}{2} = a^\dagger a + \frac{\hbar\omega}{2}, \tag{3.8}$$

where

$$a = \frac{1}{\sqrt{2}}(\omega Q + iP), \quad a^\dagger = \frac{1}{\sqrt{2}}(\omega Q - iP) \tag{3.9}$$

are the *annihilation* and the *creation* operators. We shall see shortly the reason for these names. They are obviously adjoint to each other. Using the commutator relation $[Q,P] = -i\hbar$, we obtain

$$[a,a] = [a^\dagger, a^\dagger] = 0, \quad [a, a^\dagger] = \hbar\omega, \tag{3.10a}$$

$$[H,a] = -\hbar\omega a, \quad [H, a^\dagger] = \hbar\omega a^\dagger. \tag{3.10b}$$

This shows that the operators $1, H, a, a^\dagger$ form a Lie algebra $\mathcal{H}$, called the *extended Heisenberg algebra*.

So we are interested in the representation of the Lie algebra $\mathcal{H}$ in $L^2(\mathbb{R})$.

Suppose we have an eigenvector ψ of H with eigenvalue λ and norm 1. Since $a^\dagger$ is adjoint to a, we have

$$\lambda||\psi||^2 = \langle\psi, H\psi\rangle = \langle\psi, a^\dagger a\psi\rangle + \langle\psi, \frac{\hbar\omega}{2}\psi\rangle = ||a\psi||^2 + \frac{\hbar\omega}{2}||\psi||^2.$$

This implies that all eigenvalues λ are real and satisfy the inequality

$$\lambda \geq \frac{\hbar\omega}{2}. \tag{3.11}$$

The equality holds if and only if $a\psi = 0$. Clearly any vector annihilated by a is an eigenvector of H with minimal possible absolute value of its eigenvalue. A vector of norm one with such a property is called a *vacuum vector*.

Denote a vacuum vector by $|0\rangle$. Because of the relation $[H,a] = -\hbar\omega a$, we have

$$Ha\psi = aH\psi - \hbar\omega a\psi = (\lambda - \hbar\omega)a\psi.$$

This shows that $a\psi$ is a new eigenvector with eigenvalue $\lambda - \hbar\omega$. Since eigenvalues are bounded from below, we get that $a^{n+1}\psi = 0$ for some $n \geq 0$. Thus $a(a^n\psi) = 0$ and $a^n\psi$ is a vacuum vector. Thus we see that the existence of one eigenvalue of H is equivalent to the existence of a vacuum vector.

Now if we start applying $(a^\dagger)^n$ to the vacuum vector $|0\rangle$, we get, as above, eigenvectors with eigenvalue $\frac{\hbar}{2} + n\hbar\omega$. So we are getting a countable set of eigenvectors

$$\psi_n = a^{\dagger n}|0\rangle$$

with eigenvalues $\lambda_n = \frac{2n+1}{2}\hbar\omega$. It is easy to see, using induction on n that that $||\psi_n||^2 = n!(\hbar\omega)^n$. After renormalization we obtain a countable set of orthonormal eigenvectors

$$|n\rangle = \frac{1}{\sqrt{(\hbar\omega)^n n!}}(a^\dagger)^n|0\rangle, \;\; n = 0,1,2,\ldots. \tag{3.12}$$

One can show that the closure of the subspace of $L^2(\mathbb{R})$ spanned by the vectors $|n\rangle$ is an irreducible representation of the Lie algebra $\mathcal{H}$.

The existence of a vacuum vector is proved by a direct computation. We solve the differential equation

$$\sqrt{2}a\psi = (\omega Q - iP)\psi = \omega q\psi + \hbar\frac{d\psi}{dq} = 0$$

and get

$$|0\rangle = (\frac{\omega}{\hbar})^{\frac{1}{4}}e^{-\omega q^2/\hbar}. \tag{3.13}$$

In fact, we can find all the eigenvectors

$$|n\rangle = \big(\frac{\omega}{\hbar\pi}\big)^{1/4}H_n(\sqrt{\omega}q/\sqrt{\hbar}),$$

where

$$H_n(x) = \frac{1}{\sqrt{2^n n!}}\big(x - \frac{d}{dx}\big)^n e^{-\frac{x^2}{2}}$$

is a *Hermite polynomial* of degree n. It is known also that the orthonormal system of functions $H_n(x)e^{\frac{-x^2}{2}}$ is complete, i.e., forms an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$. Thus we constructed an irreducible representation of $\mathcal{H}$ with unique vacuum vector $|0\rangle$. The vectors $|n\rangle$ are all orthonormal eigenvectors of H with eigenvalues $(n + \frac{1}{2})\hbar\omega$.

The function (3.13) gives the probability amplitude that a particle occupies the position x on the real line in the vacuum state of the system.

According to Example 3.1, the value of the function $|n\rangle$ at λ is equal to the probability that the observable Q takes value λ at the state $|n\rangle$.

Finally let us compute the partition function of the Hamiltonian H. The eigenvalues of H are $\lambda_n = (n + \frac{1}{2})\hbar\omega$ and their multiplicities are equal to 1. So

$$\mathrm{Tr}(e^{itH}) = \sum_{n=0}^{\infty} e^{it(n+\frac{1}{2})\hbar\omega} = \frac{q^{\frac{1}{2}}}{1-q}, \tag{3.14}$$

where $q = e^{it\hbar\omega}$.

Exercises

3.1 Consider the quantum mechanical system defined by the harmonic oscillator. Find the wave function of the moment operator P at a state $|n\rangle$.

3.2 Consider the Lagrangian $L(q,\dot{q}) = \frac{1}{2}(m\dot{q}^2 - U(q))$ on $T(\mathbb{R})$, where $U(q) = 0$ for $q \in (0,a)$ and $U(q) = 1$ otherwise. Quantize this mechanical system, solve the Schrödinger equation and find the stationary states of the Hamiltonian operator.

3.3 Compute the Legendre transform of the function $f(x) = e^x$.

3.4 Let $\Delta(A)_\psi = \langle (A - \langle A\rangle_\psi \mathbf{id})^2\rangle_\psi$ be the expectation value of the operator $(A - \langle A\rangle_\psi \mathrm{id})^2$ (the *dispersion* of an observable A at the state ψ). Prove the *Heisenberg's Uncertainty Principle*

$$\Delta(A)_\psi \Delta(B)_\psi \geq \frac{1}{2}|\langle [A,B]\rangle_\psi|.$$

Lecture 4

The Dirichlet action

Now we shall move to QFT of dimension larger than 1, i.e. $\dim \Sigma > 1$, for example, $\Sigma = T \times N$ where $T \subset \mathbb{R}$ or S^1 and N is a manifold of positive dimension which we shall assume for simplicity to be orientable. A map $\phi : \Sigma \to M$ is given by a function $\phi(t, x), t \in T, x \in N$. Note that when N is 0-dimensional, say $N = \{1, \ldots, n\}$, we can view $\phi(t, x)$ as a vector function $(\phi^1(t), \ldots, \phi^n(t)) : T \to M^n$ and get the quantum mechanics on M^n. For example, if our QFT is a harmonic oscillator, passing from $\mathbb{R} \to \mathbb{R} \times \{1, \ldots, n\}$ corresponds to considering n harmonic oscillators. Replacing $\{1, \ldots, n\}$ by positive-dimensional N means that we consider the whole manifold of harmonic oscillators!

Recall that any QFT is defined by an action functional S on the space of paths. In a one-dimensional theory we defined S by a Lagrangian $L : TM \to \mathbb{R}$. The pull-back of L under the map $(\phi, d\phi) : T\Sigma \to TM$ is a function $F(\phi)$ on $T\Sigma$, so for any *density* $d\mu$ on Σ (i.e. a section of $\Lambda^{\text{top}}(T^*\Sigma)$) we can multiply $F(\gamma)d\mu$ to get a density on Σ which we can integrate. If $\dim \Sigma > 1$ this is not true anymore since $F(\phi)$ is a function on $T\Sigma$ and a density is a function on $\Lambda^{\dim \Sigma}(T\Sigma)$. So the definition of the Lagrangian has to be changed. We are not going into a rigorous mathematical discussion of this definition referring to Deligne-Freed's lectures at the IAS.

Recall that the *jet bundle* of order k of a fiber bundle E over a manifold X is a vector bundle $J^k(E)$ whose local sections are local sections of E together with their partial derivatives up to order k. Let $(e_1, \ldots, e_r)$ be a local frame of E and $x_1, \ldots, x_n$ local coordinates on X. A local frame of $J^k(E)$ is a set $(e_\mu, e_\mu^{i_1 \ldots i_s})$ where $0 < i_1 + \ldots + i_k \leq k, 1 \leq i_1 \leq \ldots \leq i_k \leq n$. Let $(y^\mu, y^\mu_{i_1 \ldots i_s})$ be the corresponding coordinate functions. Any local section $\phi(x) = y^\mu e_\mu$ of E can be uniquely extended to a section $\tilde{\phi}$ of $J^k(E)$ such that

$$y^\mu_{i_1 \ldots i_s}(\tilde{\phi}(x)) = \frac{\partial^{i_1 + \ldots + i_s} \phi^\mu}{\partial x_{i_1} \ldots \partial x_{i_s}}(x).$$

Let $\mathcal{F}$ be the space of sections of $J^k(E)$ (fields, and their partial derivatives). Roughly speaking a Lagrangian of order k is a smooth map from $J^k(E)$ to the space of densities on Σ. We will be usually dealing with Lagrangians of the first order. Then we can write a Lagrangian as

$$\mathcal{L} = L(x_i, y^\mu, y_i^\mu)|d^n x|.$$

So, the action will be defined by a formula

$$S(\phi) = \int_\Sigma \mathcal{L}(x, \phi^\mu, \partial_{x_i}\phi^\mu).$$

One can generalize the Euler-Lagrange equations to the higher-dimensional case:

$$\frac{\partial \mathcal{L}}{\partial y^\mu} - \partial_i \frac{\partial \mathcal{L}}{\partial (y_i^\mu)} = 0, \quad \mu = 1, \dots, \dim M. \tag{4.1}$$

Here we assume that the equality takes place when we evaluate the left-hand side on ϕ. We also assume here that $\mathcal{L}$ is of the first order.

Let us consider an example, which will be very much relevant to the string theory. First, a little of linear algebra. Let V, W be two vector spaces equipped with non-degenerate bilinear forms h and g, respectively. We can define a symmetric bilinear form on the space of linear maps Lin(V, W) by

$$\langle f, \phi \rangle = \mathrm{Tr}(f^* \circ \phi),$$

where $f^* : W \to V$ is the adjoint map with respect to the bilinear forms h and g (i.e. $g(f(v), w) = h(v, f^*(w))$ for any $v \in V$, $w \in W$). Let us explain this definition. Choose a basis $\xi_1, \dots, \xi_n$ in V and a basis $\eta_1, \dots, \eta_m$ in W. Let H be the matrix of h in the first basis and G be the matrix of g in the second basis. Let A be the matrix of f with respect to the bases, and B is the same for ϕ. Then the matrix of f^* is equal to $A^* = H^{-1}A^tG$ so

$$\langle f, \phi \rangle = \mathrm{Tr}(H^{-1}A^tGB) = h^{ij}a_i^s b_j^t g_{st}, \tag{4.2}$$

where $A = (a_i^j), B = (b_i^j), H^{-1} = (h^{ij}), G = (g_{ij})$ and we employ the physics summation notation. Assume that f is the map defined by $f(v) = \xi^i(v)\eta_j$, where ξ^i is an element of the dual basis $(\xi^1, \dots, \xi^n)$. Then $a_n^m = \delta_{mi}\delta_{nj}$. Similarly, take ϕ defined by $\phi(v) = \xi^{i'}(v)\eta_j'$. Then we get $\langle f, \phi \rangle = h^{ii'}g_{jj'}$. If we identify Lin$(V, W)$ with $V^* \otimes W$, we get $f = \xi^i \otimes \eta_j, \phi = \xi^{i'} \otimes \eta_{j'}$ and

$$\langle \xi^i \otimes \eta_j, \xi^{i'} \otimes \eta_{j'} \rangle = h^{ii'}g_{jj'}.$$

From this we deduce that the matrix of the bilinear form on $V^* \otimes W$ with respect to the basis $(\xi^i \otimes \eta_j)$ is equal to the Kronecker product of the matrices H^{-1} and G. The matrix H^{-1} defines an inner product on V^*. So, our inner product on $V^* \otimes W$ could be taken as the definition of the tensor product of the inner product on V^* and on W.

Now we are ready to globalize. Let h be a metric on Σ and g be a metric on M. Define the Lagrangian

$$\mathcal{L}_\Sigma(\phi) = |d\phi|^2 d\mu_X = \mathrm{Tr}(d\phi^* \circ d\phi)d\mu_X. \tag{4.3}$$

Here $d\mu_X$ is the volume form defined by the metric h and the adjoint $d\phi^*$ of $d\phi$ is defined with respect to the metrics h and g.

The corresponding action

$$S_\Sigma(\phi) = \int_\Sigma |d\phi|^2 d\mu_\Sigma \tag{4.4}$$

is called the *Dirichlet action*.

If h is given in local coordinates $x^1, \dots, x^d$ by the matrix $(h_{\alpha\beta})$ and g is given in local coordinates $y^1, \dots, y^D$ by the matrix $(g_{\mu\nu})$ then (4.4) can be rewritten in the form

$$S_\Sigma(\phi) = \int_\Sigma |\det(h)|^{\frac{1}{2}} h^{\alpha\beta} \frac{\partial\phi^\mu}{\partial x_\alpha} \frac{\partial \phi^\nu}{\partial x_\beta} g_{\mu\nu} dx_1 \wedge \dots \wedge dx_d. \tag{4.5}$$

This follows from (4.2) and the fact that $d\mu_X = |\det(h)|^{\frac{1}{2}} dx_1 \wedge \dots \wedge dx_d$. Recall that the volume form on a vector space V is equal to $e^1 \wedge \dots \wedge e^n$, where $e_1, \dots, e_n$ is an orthonormal basis.

Observe the following properties of the Dirichlet action:

(A1) (isometry invariance) For any diffeomorphism $\alpha : \Sigma \to \Sigma'$ preserving the metrics,

$$S_\Sigma(\phi \circ \alpha) = S_{\Sigma'}(\phi);$$

(A2) (locality) if Σ is glued together from Σ_1 and Σ_2 along their boundaries, then

$$S_\Sigma(\phi) = S_{\Sigma_1 \coprod \Sigma_2}(\phi|\Sigma_1 \coprod \Sigma_2)$$

(A3) (conformal scaling) if $h' = e^{c(x)}h$ is a new metric on Σ, and S'_Σ is the new action, then the action functional does not change if and only if $\dim \Sigma = 2$.

Proof. Let $\phi : V = T\Sigma \to W$ be a linear map of inner product spaces and $\alpha : V' \to V$ be an isometry of inner product spaces. Then, for any $v' \in V', w \in W$,

$$\langle \phi(\alpha(v')), w\rangle_W = \langle \alpha(v'), \phi^*(w)\rangle_V = \langle v', \alpha^{-1}(\phi^*(w))\rangle_{V'}.$$

This shows that $(\phi \circ \alpha)^* = \alpha^{-1} \circ \phi^*$, hence

$$\mathrm{Tr}((\phi \circ \alpha)^* \circ (\phi \circ \alpha)) = \mathrm{Tr}(\alpha^{-1} \circ (\phi^* \circ \phi) \circ \alpha) = \mathrm{Tr}(\phi^* \circ \phi).$$

Applying this to the case when ϕ, α are the maps of the tangent spaces, this implies that $\alpha^*(||d\phi||^2 d\mu_{\Sigma'}) = ||d(\phi \circ \alpha)||^2 d\mu_\Sigma$. Property (A1) now follows from the standard properties of integration of differential forms. Property (A2) is obvious from the definition of the action. Property (A3) follows easily from formula (4.5). $\square$

Example 4.1. Let $\Sigma = \mathbb{R}$ with a local coordinate t and $M = \mathbb{R}^n$ with metric defined by a matrix $g_{ij}(x)$ in the canonical basis in $TM_x = \mathbb{R}^n$. Define a metric on Σ by

$||1||_a^2 = h(a)$. Let $f : \mathbb{R} \to \mathbb{R}^n$ be defined by the vector function $(f^1(t), \dots, f^n(t))$. Then

$$||df_a||^2 = \frac{f'(a)G(f(a))f'(a)}{h(a)} = \frac{||f'(a)||^2}{||1||_a^2}.$$

So, if we take $h \equiv 1, G = \frac{1}{2}(\delta_{\mu\nu})$, we obtain the action $S(\gamma) = \frac{1}{2}\int_\Sigma \dot{\gamma}(t)^2 dt$ which we used in the previous lectures.

If $\dim \Sigma > 1$, I do not know any geometric meaning of the Dirichlet action. However, let us see that for a fixed metric g on M one can always choose a metric h on Σ such that the action acquires a very nice meaning. In fact, the metric h is chosen to minimize the action. Let us consider the Dirichlet action as a function of h and compute its variation in the direction δh at $h = h_0$

$$\frac{\delta S}{\delta h}\bigg|_{h=h_0} = (S(h_0 + \epsilon\delta h) - S(h_0))/\epsilon,$$

where $\epsilon^2 = 0$. Note that, for any invertible matrix A and any square matrix B of the same size, we have

$$|A + \epsilon B| = |A| + \epsilon \mathrm{Tr}(A^{-1}B)|A|.$$

Thus

$$|A + \epsilon B|^{\frac{1}{2}} = |A|^{\frac{1}{2}}(1 + \frac{\epsilon}{2}\mathrm{Tr}(A^{-1}B)).$$

Also

$$(A + \epsilon B)^{-1} = A^{-1} - \epsilon A^{-1}BA^{-1}.$$

Let

$$\gamma_{\alpha\beta} = \frac{\partial\phi^\mu}{\partial x_\alpha}\frac{\partial\phi^\nu}{\partial x_\beta}g_{\mu\nu}$$

and $\gamma = (\gamma_{\alpha\beta})$. The matrix γ is the matrix of the metric $\phi^*(g)$. It can be viewed as the metric on the image of Σ under the map ϕ (called the *world-sheet*). Then

$$S(\gamma) = \int_\Sigma |h|^{\frac{1}{2}}\mathrm{Tr}(h^{-1}\gamma)d\mu_\Sigma \tag{4.6}$$

and

$$\epsilon\frac{\delta S}{\delta h}\bigg|_{h=h_0}(\phi) =$$

$$\int_\Sigma [|h_0 + \epsilon\delta h_0|^{\frac{1}{2}}\mathrm{Tr}(h_0 + \epsilon\delta h)^{-1}\gamma)d\mu_\Sigma - \int_\Sigma |h_0|^{\frac{1}{2}}\mathrm{Tr}(h_0^{-1}\gamma)d\mu_\Sigma =$$

$$= \int_\Sigma |h_0|^{\frac{1}{2}}(\frac{1}{2}\mathrm{Tr}(h_0^{-1}\delta h)\mathrm{Tr}(h_0^{-1}\gamma) - \mathrm{Tr}(h_0^{-1}\delta h h_0^{-1}\gamma))d\mu_\Sigma.$$

Set $\delta(h^{-1}) = -h_0^{-1}\delta h h_0^{-1}$. Then

$$\frac{\delta S}{\delta h}\Big|_{h=h_0}(\phi) = \int_\Sigma |h_0|^{\frac{1}{2}}\Big(\frac{1}{2}\mathrm{Tr}(h_0(h_0^{-1}\delta h h_0^{-1}))\mathrm{Tr}(h_0^{-1}\gamma) - \mathrm{Tr}(h_0^{-1}\delta h h_0^{-1}\gamma)\Big)d\mu_\Sigma =$$

$$\int_\Sigma |h_0|^{\frac{1}{2}}\mathrm{Tr}\Big(\delta(h^{-1})(\gamma - \frac{1}{2}h_0\mathrm{Tr}(h_0^{-1}\gamma)\Big)d\mu_\Sigma.$$

Since this must be zero for all possible δh, this implies that a critical metric h_0 satisfies

$$\gamma - \frac{1}{2}h_0\mathrm{Tr}(h_0^{-1}\gamma) = 0. \tag{4.7}$$

This implies

$$|\gamma|^{1/D} = \frac{1}{2}\mathrm{Tr}(h_0^{-1}\gamma)|h|^{1/D},$$

where $D = \dim\Sigma$. Plugging this in formula(4.6) we get

$$S_\Sigma(\phi; h_0) = 2\int_\Sigma |\gamma|^{1/D}|h|^{\frac{2-D}{2D}}d\mu_\Sigma.$$

We get a wonderful fact: if $D = 2$, and the metric on Σ is chosen to be critical, the action has a simple geometric meaning. It is equal to the twice the area of the *world sheet* $\phi(\Sigma)$ in the metric induced from the metric of M.

In physics the latter action is called the *Nambu-Goto* action and the Dirichlet action is called the *Brink-DiVecchia-Howe-Desse-Zumino action*, or the *Polyakov action* for short.

Remark 4.1. In the case when the metric on Σ is Lorentzian, we have to replace $|h|$ with $|-h|$. Also physicists use the metric to "lower the indices". If (g_{ij}) is the matrix of a metric g in a basis $(e_1, \ldots, e_n)$ then for any vector $\Sigma a^i e_i$ the vector $\sum_j(\sum_i a^i g_{ij})e_j$ is denoted by $\sum_i a_i e_i$. In this notation formula (4.5) can be rewritten as

$$S(\phi) = \int_\Sigma |h|^{\frac{1}{2}}h^{ij}\frac{\partial\phi^\alpha}{\partial x_i}\frac{\partial\phi_\beta}{\partial x_j}d\mu_\Sigma.$$

Remark 4.2. The tensor

$$T = (T_{ij})(\phi) = (\gamma_{ij} - \frac{1}{2}h_{ij}h^{\nu\mu}\gamma_{\nu\mu})dx_i dx_j.$$

is called the *energy-momentum tensor*. By (4.7) this tensor is equal to zero if and only if h is a critical metric for the action $S(\phi; h)$. Observe that

$$h^{ij}T_{ij} = |h|^{-\frac{1}{2}}(h^{ij}\gamma_{ij} - \frac{1}{2}h^{ij}h_{ij}h^{\nu\mu}\gamma_{\nu\mu})dx_i \wedge dx_j = |h|^{-\frac{1}{2}}(1 - \frac{D}{2})h^{ij}\gamma_{ij}dx_i dx_j.$$

In particular, T is trace-less if $D = 2$.

Assume $\Sigma = \mathbb{R}^D$, $M = \mathbb{R}^n$ and the metrics h, g are the standard Euclidean metrics. Then

$$T = (\gamma_{ij} - \frac{1}{2}\mathrm{Tr}(\gamma))dx_i dx_j.$$

Let us write down the Euler-Lagrange equations for the Dirichlet action in the case when the metrics h and g are flat (i.e. $h_{\alpha\beta}$ and $g_{\mu\nu}$ are constant functions). We get the equations

$$h^{\alpha\beta} g_{\mu\nu} \frac{\partial^2 \phi^\nu}{\partial x_\alpha \partial x_\beta} = 0, \quad \mu = 1, \dots, D. \tag{4.8}$$

In the case when $\Sigma = \mathbb{R}^n, M = \mathbb{R}$, and $h_{\alpha\beta}$ this is just the *Laplace equation* with respect to the metric h. When $h_{\alpha\beta} = \delta_{\alpha\beta}$, its solutions are *harmonic functions*. In general, the Euler-Lagrange equation for the Lagrangian (4.3) can be written in an invariant form:

$$\mathrm{Tr}(Dd\phi) = 0,$$

where D is the covariant derivative of a section $d\phi$ of $E = T_M^* \otimes \phi^* T_X$ with respect to the natural Riemannian connection defined on the bundle E.

Consider the special case when $\Sigma = \mathbb{R}^2$, with coordinates $(x_1, x_2) = (t, x)$, and $h = (h_{\alpha\beta}) = \mathrm{diag}(1, -1)$. Take $M = \mathbb{R}$. Then the Lagrangian density becomes

$$\mathcal{L} = [(\frac{\partial \phi}{\partial t})^2 - (\frac{\partial \phi}{\partial x})^2] dt dx.$$

The Euler-Lagrange equation is

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0.$$

Notice the analogy with the Lagrangian for a harmonic oscillator (with $m = \omega = 1$)

$$\mathcal{L} = (\frac{dx^2}{dt^2} - x^2) dt.$$

We can view $\phi(t, x)$ as the displacement of the particle located at position x at time t. The Euler-Lagrange equation for the scalar field $\phi(t, x)$ can be thought as the motion equation for infinitely many harmonic oscillators arranged at each point of the straight line.

If h is the flat Lorentzian metric in $\mathbb{R}^n$ defined by the diagonal matrix $\mathrm{diag}[-1, 1, \dots, 1]$ the Euler-Lagrange equation for a scalar field with Dirichlet action is

$$\Box\phi = (\partial_\nu \partial^\nu)\phi = -\frac{\partial^2 \phi}{\partial t^2} - \sum_{\nu=2}^{n} \frac{\partial^2 \phi}{\partial x_\nu^2} = 0.$$

The operator $\Box$ is called the *D'Alembertian operator* or *relativistic Laplacian*.

A little more general, if we take the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \phi \partial^\nu \phi - m^2\phi^2),$$

the Euler-Lagrange equation is the *Klein-Gordon equation*

$$\Box\phi + m^2\phi = 0.$$

In many quantum field theories M is a fibre bundle over Σ and $\phi : \Sigma \to M$ is a section. When M is a G-bundle with some structure group G a map ϕ is called a *classical field*, otherwise ϕ is called a non-linear σ-field. For example, when M is the trivial vector bundle of rank 1, a classical field is called a *scalar field*. Of course any map $\phi : \Sigma \to M$ can be considered a section of a fibre bundle, the trivial bundle $\Sigma \times M \to \Sigma$.

Example 4.2. An example of a classical field is a *gauge field* or a *connection* on a principal G-bundle over Σ. It is defined by a 1-form $\sum A_i dx_i$ on Σ with values in the adjoint affine bundle $\mathrm{Ad}(G)$. In other words, it is a section of the bundle $T\Sigma \otimes \mathrm{Ad}(G)$. For example when $G = \mathrm{GL}(n)$, a gauge field is a map of vector bundles $A : T\Sigma \to \mathrm{End}(E)$, where E is a smooth vector bundle of rank r over Σ. It satisfies

$$A(\xi)(fs) = \xi(f)s + fA(\xi)(s),$$

where f is a local smooth function and s is a local section of E. It is clear that the difference of two connections is a morphism of vector bundles and thus a connection is a section of an affine bundle. Each connection A defines the $\mathrm{Lie}(G)$-valued 2-form, the *curvature form*,

$$F_A = dA + \frac{1}{2}[A, A] = \sum F_{ij} dx_i \wedge dx_j.$$

Here F_{ij} is a local function on X with values in $\mathrm{Lie}(G)$. We define the Lagrangian density on $\mathcal{A}_X$ by setting

$$\mathcal{L}(A) = F_A \wedge *F_A.$$

Here $*$ is the star-operator on the space of differential forms with values in a vector bundle E equipped with a metric g. It is determined by the property $\alpha\wedge*\alpha = \langle\alpha, \beta\rangle d\mu$, where $\langle , \rangle$ is a natural bilinear form on the space of such forms (determined by the Riemannian metric on TM and the metric on E) and $d\mu$ is the volume form defined by the metric on M.

The Euler-Lagrange equation for the gauge fields is the *Yang-Mills equation*:

$$\sum_{i=1}^{d}(\frac{\partial F_{ij}}{\partial x_i} + [A_j, F_{ij}]) = 0, \quad j = 1, \dots, d. \tag{4.9}$$

There are two approaches to quantization in higher-dimensional QFT. First uses *functional integrals* which generalize the path integrals.

Consider a space of fields $\mathcal{F}_\Sigma = \mathrm{Map}(\Sigma, M)$ on a D-dimensional manifold Σ. We assume that $\Sigma = T \times \Sigma'$, where $\dim T = 1$ (a "time factor"). For each $t \in T$ we denote by ϕ_t the restriction of a field $\phi \in \mathcal{F}_\Sigma$ to $\Sigma_t = \{t\} \times \Sigma'$. Let $\mathcal{F}_{\Sigma_t}$ be the space of fields on Σ_t obtained by restrictions of fields from $\mathcal{F}_\Sigma$. Fix two fields $f_i \in \mathcal{F}_{\Sigma_{t_i}}, i = 1, 2$. Consider an action $S : \mathcal{F}_\Sigma \to \mathbb{R}$ and set

$$R(f_1, f_2) = \int_{f_1}^{f_2} e^{iS(\phi)} D[\phi], \tag{4.10}$$

where we integrate over the space of fields ϕ on Σ such that $\phi_{t_i} = f_i$. We use some measure $D[\phi]$ on $\mathcal{F}_\Sigma$.

Observe the obvious analogy with our previous definition where we take $\Sigma' = \{\text{point}\}$ and $\mathcal{F}_\Sigma$ is the set of maps $\Sigma \to M$.

Now if we consider some Hilbert space $\mathcal{H}_t$ of functions on $\mathcal{F}_{\Sigma_t}$, the integral operator with kernel (4.10) defines a linear map

$$T_{t_1 t_2} : \mathcal{H}_{t_1} \to \mathcal{H}_{t_2}, \quad T_{t_1 t_2}(\Psi)(f) = \int_{\mathrm{Map}(\Sigma', M)} R(f, g)\Psi(g) D[g]$$

It also defines a self-adjoint Hamiltonian operator $H : \mathcal{H}_0 \to \mathcal{H}_0$ such that

$$T_{t_1 t_2} = e^{-i(t_2 - t_1)H}$$

which can be used to define a Hermitian map $\mathcal{H}_{t_1} \times \mathcal{H}_{t_2} \to \mathbb{C}$,

$$\langle \Psi_2, \Psi_1 \rangle = \langle \Psi_2 | e^{-i(t_2 - t_1)H} \Psi_1 \rangle_{\mathcal{H}_{t_2}}.$$

The kernel R_{f_1, f_2} has special meaning for $f_1 = f_2$. The integral

$$\mathrm{Tr}(e^{-i(t_2 - t_1)H}) = \int_{\mathrm{Map}(\Sigma', M)} Z_{f,f} D[f]$$

is the trace of the operator $e^{i(t_2 - t_1)H}$. It is called the *partition function* of the theory.

More generally, let $O_i(P_i), i = 1, \dots, n$, be a *local quantum field* at a point P_i on Σ (a local field is a functional on $\mathcal{F}_\Sigma$ which depends only on $\phi(P)$ and derivatives of ϕ at P). An example of a local field is the functional $\phi \to l(\phi(P))$, where $l : M \to \mathbb{R}$ is a function on M. We set

$$Z_{f_1, f_2}(O_i(P_i); \Sigma) = \int_{f_1}^{f_2} e^{iS(\phi)} \prod_i O_i(P_i)(\phi) D[\phi] \tag{4.11}$$

This leads to the *correlation function*

$$\langle O_1(P_1) \dots O_n(P_n) \rangle = \frac{Z_{f_1, f_2}(O_i(P_i); \Sigma)}{Z_{f_1, f_2}(\Sigma)}. \tag{4.12}$$

We can use (4.11) to define a Hermitian form on the space $\mathcal{H}$ of functions on $\mathcal{F}_{\Sigma'} = \mathrm{Map}(\Sigma', M)$

$$R(O(P);t_1,t_2)(\psi_2,\psi_1) = \int_{\mathcal{F}_{\Sigma_{t_2}}} \psi_2(f_2)^* \int_{\mathcal{F}_{\Sigma_{t_1}}} \psi_1(f_1) Z_{f_1,f_2}(O(P);\Sigma) D[f_1]D[f_2].$$

This is still linear in ψ_1 and half-linear in ψ_2. Thus it defines a linear operator $\check{O}(P)$ such that

$$R(O(P);t_1,t_2)(\psi_2,\psi_1) = \langle \psi_2 | e^{-i(t_2-t_1)H} \check{O}(P) e^{i(t_2-t_1)H} \psi_1 \rangle_{\mathcal{H}}.$$

We can get rid of the parameter $t = t_2 - t_1$ by letting it go to infinity, i.e. define

$$\lim_{t\to\infty} R(O(P);\psi_2,\psi_1) = \lim_{t\to\infty} \langle \psi_2 | e^{-itH} \check{O}(P) e^{itH} \psi_1 \rangle.$$

In this way we get a local operator $\check{O}(P)$ in the Hilbert space $\mathcal{H}$. It is called the *vertex operator* associated to a functional $O(P)$.

Another approach to quantization generalizes the one we used for the harmonic oscillator. Again we assume that $\Sigma = T \times \Sigma'$. For any field $\phi : \Sigma \to M$ we denote by ∂_0 the partial derivative in the time variable. By analogy with classical mechanics we introduce the conjugate *momentum field*

$$\pi(t,x) = \frac{\delta \mathcal{L}}{\delta \partial_0}(\phi).$$

For example, when $\mathcal{L}(\phi) = \frac{1}{2}((\frac{\partial \phi}{\partial t})^2 - (\frac{\partial \phi}{\partial x})^2)$ we obtain $\pi = \frac{\partial \phi}{\partial t}$. We can also introduce the Hamiltonian functional :

$$H(\phi) = \frac{1}{2}\int_{\Sigma'} (\pi\dot{\phi} - \mathcal{L}) dx.$$

Then the Euler-Lagrange equation is equivalent to the *Hamiltonian equations* for fields

$$\dot{\phi}(t,x) = \frac{\delta H}{\delta \pi(t,x)}, \quad \dot{\pi}(t,x) = -\frac{\delta H}{\delta \phi(t,x)},$$

where the dot means the derivative with respect to the time variable. Here we consider π and ϕ as independent variables in the functional H and use the partial derivatives of H.

To quantize the fields ϕ and π we have to reinterpret them as Hermitian operators in some Hilbert space $\mathcal{H}$ which satisfy the commutator relations (remembering that ϕ is an analog of q and π is an analog of $\dot{q}$).

$$[\Phi(t,x), \Pi(t,y)] = \frac{i}{\hbar}\delta(x-y), \tag{4.13}$$

$$[\Phi(t,x), \Phi(t,y)] = [\Pi(t,x), \Pi(t,y)] = 0. \tag{4.14}$$

Here we have to consider Φ, Π as operator valued distributions, i.e. a continuous linear functionals on the space of test functions on Σ equipped with some measure μ with values in the space of operators in a Hilbert space $\mathcal{H}$. Any function on Σ with values in the space of operators in $\mathcal{H}$ which is integrable with respect to some operator-valued measure $d\mu_O$ defines a distribution

$$\phi \to \int_\Sigma T(x)\phi(x)d\mu_O.$$

The commutator of two operator valued distributions is a bilinear form on the space of test functions:

$$(\phi, \psi) \to [T_1(\phi), T_2(\psi)].$$

Thus the meaning of (4.13) is

$$[\Phi(\phi), \Pi(\psi)] = \int_\Sigma \phi(x)\psi(x)d\mu.$$

Example 4.3. Assume that $\Sigma' = \mathbb{R}$ and $\mathcal{L}$ is the Klein-Gordon Lagrangian $\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - m^2\phi^2$. A solution of the Klein-Gordon equation can be written as a Fourier integral

$$\Phi(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} a(k)e^{i(kx-\omega_k t)} + \overline{a(k)}e^{i(-kx+\omega_k t)}dk,$$

where

$$\omega_k^2 = k^2 + m^2.$$

Similarly we have a Fourier integral for $\Pi(t,x)$:

$$\Pi(t,x) = \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} \omega_k(-e^{ik\cdot x}a(k)e^{-i\omega_k t} + e^{-ik\cdot x}\overline{a(k)}e^{i\omega_k t})dk.$$

To quantize we replace $a(k)$ with an operator a_k and $\overline{a(k)}$ with the adjoint operator $a_k^\dagger$ and consider the above expansions as operator integrals. This implies that the operators $\Psi(t,x)$ and $\Pi(t,x)$ are Hermitian. The commutator relations (4.13) will be satisfied if we require the commutator relations

$$[a_k, a_{k'}^\dagger] = \delta(k-k'),$$

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0.$$

This is in complete analogy with the case of the harmonic oscillator, where we had only one pair of operators $a, a^\dagger$ satisfying $[a, a^\dagger] = 1, [a,a] = [a^\dagger, a^\dagger] = 0$ (or n operators $a_i, a_i^\dagger$ satisfying $[a_i, a_j^\dagger] = \delta_{i,j}, [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$). There is a big difference however. In our case the Heisenberg Lie algebra generated by $1, a_k, a_k^\dagger$ is infinite-dimensional.

Exercises

4.1 Let $\mathcal{L} = \mathcal{L}(\phi, d\phi)$ be a Lagrangian on Σ with a metric h defined on the space of maps $\Sigma \to \mathbb{R}^n$. Define the *energy-momentum tensor* by

$$T_{\alpha\beta}(\phi) = \sum_{\mu} h_{\beta\nu} \partial_\alpha \frac{\delta \mathcal{L}}{\delta(\partial_\nu)}(\phi) - h_{\alpha\beta}\mathcal{L}(\phi)$$

(i) Show that this definition agrees with the one given for the Dirichlet action;

(ii) Prove that $\sum \frac{\partial T_{\alpha\beta}(\phi)}{\partial x_\beta} = 0$ if ϕ satisfies the Euler-Lagrange equations.

4.2 Consider a system of N harmonic oscillators viewed as a finite set of masses arranged on a segment $[a, b]$, each connected to the next one via springs of length ϵ. Write the Lagrangian $L(\epsilon)$ describing this system. Show that the limit of $L(\epsilon)$ when ϵ goes to zero is equal to

$$\int_a^b c_1(\frac{\partial\phi(t,x)}{\partial t})^2 - c_2(\frac{\partial\phi(t,x)}{\partial x})^2]dx.$$

where c_1, c_2 are some positive constants and $\phi(t, x)$ is the function which measures the displacement of the particle located at position x at time t.

4.3 Let (X, g) be a Riemannian manifold of dimension n and E be a vector bundle over X equipped with a Riemannian metric. Show that there exists a unique linear isomorphism $* : \Lambda^k(T^*X) \otimes E \to \Lambda^{n-k}(T^*X) \otimes E$ such that, for any $\alpha, \beta \in \Gamma(X, \Lambda^k(T^*X) \otimes E)$, one has $\alpha \wedge *\beta = g^{-1}(\alpha, \beta)\mu_{\mathrm{vol}}$. Here $\alpha \wedge \beta$ is defined locally by extending via linearity the product $(a \otimes e) \wedge (b \otimes e') = (e, e')a \wedge b$. Also g^{-1} is the inverse metric on $T^*(X)$ extended to $\Lambda^k(T^*X) \otimes E$ by the formula $g^{-1}(a \otimes e, b \otimes e') = b(e, e') \wedge^k (g^{-1})(a, b)$.

4.4 Using the star-operator defined in the previous problem show that the Dirichlet action can be rewriten in the form

$$S(\phi) = \int_\Sigma d\phi \wedge *d\phi,$$

where $d\phi$ is considered as a section of the bundlle $T^*\Sigma \otimes \phi^*(TM)$.

Lecture 5

Bosonic strings

From now on we stick with dimension $D = 2$ of our QFT. This is where strings appear. Our manifold Σ will be a smooth 2-manifold with a pseudo-Riemannian metric h. It could be the plane $\mathbb{R}^2$ or a cylinder $S^1 \times \mathbb{R}$, or a torus $S^1 \times S^1$, or a sphere S^2, or a compact Riemann surface Σ_g of genus $g > 1$. Of course each time we should specify a metric on Σ.

We shall begin with the case when Σ is a cylinder $S^1 \times \mathbb{R}$ (*closed strings*) or $\Sigma = \mathbb{R} \times [0, \beta]$ (an open string). We use the coordinate θ in the circle direction and the coordinate t (time) in the $\mathbb{R}$-direction. A map $\phi(t, \theta) : \Sigma \to M$ can be considered as a map

$$\tilde{\phi}(t) : \mathbb{R} \to \mathcal{L}(M), \quad t \to (\theta \to \phi(t, \theta)),$$

where $\mathcal{L}(M)$ is the *loop space* of M, i.e. the space of smooth maps from a circle to M. In the case of open strings $\mathcal{L}(M)$ must be replaced with the space $\mathcal{P}(M)$ of paths in M. We shall consider only closed strings, however occasionally we state the corresponding results for open strings.

We shall also assume in the beginning that $M = \mathbb{R}^n$ with the Lorentzian flat metric $g = (g_{\mu\nu}) = (\eta_{\mu\nu})$, where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. We will write vectors in M as $(x^1, \ldots, x^n)$ and denote by $(x_1, \ldots, x_n)$ the vector $(-x^1, x^2, \ldots, x^n)$ equal to $(x^\nu g_{\nu 1}, \ldots, x^\nu g_{\nu n})$. Later on we will of course consider more general target spaces M. We consider the Dirichlet action

$$S(\phi; h) = \frac{T}{2} \int_\Sigma |d\phi|^2 d\mu_\Sigma = \frac{T}{2} \int_\Sigma \sqrt{|h|} h^{\alpha\beta} \partial_\alpha \phi^\mu \partial_\beta \phi_\nu d\mu_\Sigma. \tag{5.1}$$

Here T is a certain parameter of a string (the *string tension*). It is equal to $1/2\pi\alpha'$ for open strings, where α' is a certain other constant called the *Regge slope*. For closed strings $T = 1/\pi\alpha'$. We use the subscript h to emphasize the dependence of the action on h.

$$S(\alpha \circ \phi; \alpha^*(h)) = S(\phi; h), \tag{5.2}$$

where α is a diffeomorphism of Σ. This means that the action is invariant with respect to smooth reparametrizations of the maps. Also, for any $f \in C^\infty(\Sigma)$, we have

$$S(\phi; e^f h) = S(\phi; h). \tag{5.3}$$

This means that the action is conformally invariant.

It is known (see, for example, [Modern Geometry] by Dubrovin, Fomenko and Novikov) that there exists a unique diffeomeorphism α such that $\alpha^*(h) = e^f h_0$, where f is a smooth function and h_0 is a flat metric given locally by the diagonal matrix $\mathrm{diag}(-1, 1)$. By (5.2) and (5.3)

$$S(\phi; h) = S(\alpha \circ \phi; h_0), \tag{5.4}$$

We shall fix the metric on Σ by equipping $\mathbb{R}$ with the metric $-dx^2$ and taking $S^1 = \mathbb{R}/2\pi r\mathbb{Z}$ with the metric induced by the standard metric dt^2 on $\mathbb{R}$. Then we have two constraints on ϕ. One comes from the Euler-Lagrange equation for the action S_{h_0} and another comes from the vanishing of the energy-momentum tensor. Since the Lagrangian function for the action S_{h_0} is equal to

$$\mathcal{L} = \frac{T}{2}(\partial_t \phi^\mu \partial_t \phi_\mu - \partial_x \phi^\mu \partial_x \phi_\mu) \tag{5.5}$$

the Euler-Lagrange equation for the action S_{h_0} is

$$(\partial_t^2 - \partial_x^2)\phi^\mu = 0, \quad \mu = 1, \ldots, D. \tag{5.6}$$

So our field (ϕ^μ) satisfies the Klein-Gordon massless equation.

The value of the energy-momentum tensor $T_{\alpha\beta}$ at h_0 is equal to

$$T_{10} = T_{01} = \partial_t \phi^\mu \partial_x \phi_\mu = 0; \quad T_{00} = T_{11} = \frac{1}{2}(\partial_t \phi^\mu \partial_t \phi_\mu + \partial_x \phi^\mu \partial_x \phi_\mu) = 0. \tag{5.7}$$

To solve the *wave equation* (5.6) we introduce the *light-cone coordinates*

$$\sigma^+ = t + x, \quad \sigma^- = t - x.$$

Let ∂_+, ∂_- denote the partial derivatives with respect to these coordinates. We have

$$\partial_+ = \frac{1}{2}(\partial_x + \partial_t), \quad \partial_- = \frac{1}{2}(\partial_t - \partial_x).$$

Thus we can rewrite (5.6) in the form

$$\partial_+ \partial_- \phi^\mu = 0.$$

This easily implies that a general solution of (**??**)EQ) can be written as as sum

$$\phi^\mu = \phi_L^\mu(\sigma^+) + \phi_R^\mu(\sigma^-).$$

Using the boundary conditions, we see that, in the case of a closed string, the functions $\phi_L^\mu + \phi_R^\mu$, $\frac{\partial \phi_L^\mu}{\partial \sigma^+}$ and $\frac{\partial \phi_R^\mu}{\partial \sigma^-}$ are periodic with period $2\pi r$, so that we can use the Fourier expansion to write

$$\phi_L(t,x)^\mu = \frac{1}{2}x^\mu + l\alpha_0^\mu \sigma^+ + ilr\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^+/r}, \tag{5.8a}$$

$$\phi_R(t,x)^\mu = \frac{1}{2}x^\mu + l\tilde{\alpha}_0^\mu \sigma^- + ilr\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^-/r}, \tag{5.8b}$$

where $l = \frac{1}{\sqrt{2\pi r T}}$ and $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. We shall see in a moment a reason for the choice of the constant l. Also, since we want ϕ^μ to be real,

$$\alpha_n^\mu = \overline{\alpha}_{-n}^\mu, \quad \tilde{\alpha}_n^\mu = \overline{\tilde{\alpha}}_{-n}^\mu.$$

The field $\phi_L(t,x)^\mu$ (resp. $\phi_R(t,x)^\mu$) describes the *"left-moving" modes* (resp. *"right-moving" modes*) of a closed string.

Note that

$$\partial_+\phi_L(t,x)^\mu = l\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\sigma^+/r}, \tag{5.9a}$$

$$\partial_-\phi_R(t,x)^\mu = l\sum_{n\in\mathbb{Z}}\tilde{\alpha}_n^\mu e^{-in\sigma^-/r}. \tag{5.9b}$$

It is clear that

$$x^\mu = \int_0^{2\pi r}\phi^\mu(0,x)dx, \quad \mu = 1,\dots,D$$

and can be interpreted as the *center-of-mass* coordinates.

By analogy with $D = 1$ QFT the momentum field is defined to be

$$P^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t\phi^\mu)} = T\partial_t\phi^\mu.$$

The expression

$$p^\mu = T\int_0^{2\pi r}\frac{d\phi^\mu(0,x)}{dt}dx = 2\pi r T(2l\alpha_0^\mu) = \frac{2}{l}\alpha_0^\mu. \tag{5.10}$$

is the *total momentum coordinate* of the string at $t = 0$. Now we can rewrite the equations (5.11) in the form

$$\phi_L(t,x)^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l^2p^\mu\sigma^+ + ilr\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^+/r}, \tag{5.11a}$$

$$\phi_R(t,x)^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l^2p^\mu\sigma^- + ilr\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^-/r}, \tag{5.11b}$$

Remark 5.1. If we choose the Riemannian metric on Σ instead of pseudo-Riemannian, we will be able to identify the cylinder $\Sigma = \mathbb{R} \times S^1$ with the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by means of the transformation

$$(t, x) \to z = e^{(-t+ix)/r}.$$

The Euler-Lagrange equation (3.7) gives

$$\partial_z \partial_{\bar{z}} X(z, \bar{z}) = 0$$

The equation of a string becomes

$$\phi_L(z)^\mu = \frac{1}{2}x^\mu + \frac{1}{2}il^2 p^\mu \ln z + ilr \sum_{k \neq 0} \frac{1}{k}\alpha_k^\mu z^k, \tag{5.12a}$$

$$\phi_R(\bar{z})^\mu = \frac{1}{2}x^\mu + \frac{1}{2}il^2 p^\mu \ln \bar{z} + ilr \sum_{k \neq 0} \frac{1}{k}\tilde{\alpha}_k^\mu \bar{z}^k \tag{5.12b}$$

The stress-tensor $(T_{\alpha\beta})$ can be rewritten in the new coordinates too. We have

$$T_{zz} = \partial_z \phi^\mu \partial_z \phi_\mu = \sum_{m \in \mathbb{Z}} L_m z^{m-2}, \quad T_{\bar{z}\bar{z}} = \partial_{\bar{z}} \phi^\mu \partial_{\bar{z}} \phi_\mu = \sum_{m \in \mathbb{Z}} \tilde{L}_m \bar{z}^{m-2}.$$

This is a familiar expression from the conformal field theory.

The Hamiltonian of our theory is equal to

$$H = \int_0^{2\pi r} (\partial_t \phi^\mu P_\mu - \mathcal{L}) dx = \frac{T}{2} \int_0^{2\pi r} (\partial_t \phi^\mu \partial_t \phi_\mu + \partial_x \phi^\mu \partial_x \phi_\mu) dx. \tag{5.13}$$

Observe that it vanishes on a string which satisfies the constraint that the energy-momentum tensor vanishes. Plugging in the expressions for ϕ^μ in terms of $\alpha_n^\mu, \tilde{\alpha}_n^\mu$, we obtain

$$H = \frac{1}{2} \sum_{n \in \mathbb{Z}} (\alpha_{-n}^\mu \alpha_{n\mu} + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu}) \tag{5.14}$$

Now it is clear the introduction of the constant l. It made our formulas not depend on T. Observe that we could simplify the sum by getting rid of $\frac{1}{2}$ but we don't do it, since in a moment the coefficients $\alpha_n, \tilde{\alpha}_n$ will become operators.

Using the new coordinates we can also rewrite the constraints (5.7) in the form

$$\partial_+ \phi^\mu \partial_+ \phi_\mu = \frac{1}{2}(T_{00} + T_{01}) = 0,$$

$$\partial_- \phi^\mu \partial_- \phi_\mu = \frac{1}{2}(T_{00} - T_{01}) = 0.$$

This immediately gives

$$\partial_t (\phi_R)^\mu \partial_t (\phi_R)_\mu = \partial_t (\phi_L)^\mu \partial_t (\phi_L)_\mu = 0.$$

This can be restated in terms of the Fourier coefficients as follows:

$$L_m = \frac{T}{2}\int_0^\pi (\partial_t(\phi_R)^\mu \partial_t(\phi_R)_\mu e^{-2imx} dx = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{m-n}\alpha_{n\mu} = 0 \tag{5.15a}$$

$$\tilde{L}_m = \frac{T}{2}\int_0^\pi (\partial_t(\phi_L)^\mu \partial_t(\phi_L)_\mu e^{-2imx} dx = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}^\mu_{m-n}\tilde{\alpha}_{n\mu} = 0. \tag{5.15b}$$

Observe that

$$H = L_0 + \tilde{L}_0 \tag{5.16}$$

Now we quantize ϕ^μ as in the previous lecture by taking α^μ_n as operators in some Hilbert space. Since we want ϕ^μ to be Hermitian we require

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^\dagger, \quad \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^\dagger.$$

We need

$$[P^\mu(t,x), X^\nu(t,x')] = -i\delta(x-x')\eta^{\nu\mu},$$

Plugging in the mode expansions, we see that this is equivalent to the following commutator relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{5.17a}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \tag{5.17b}$$

all other commutators between, $x^\mu, \alpha^\mu_n, \tilde{\alpha}^\mu_n$ are equal to zero.

We can also quantize the expressions for $H, L_m, \tilde{L}_m$. In order they make sense as operators in some Hilbert spaces we will require (by analogy with the harmonic oscillator) that in our representation α_n kills any state provided that n is large enough. Thus the sum $\sum_{n>0}\alpha_{m-n}\alpha_n$ makes sense. Let us set, for any operators A_i, A_j with indices in an ordered set I,

$$: A_i A_j := \begin{cases} A_i A_j, & \text{if } i \geq j; \\ A_j A_i & \text{otherwise.} \end{cases} \tag{5.18}$$

It is called the *normal order* for the composition of operators. Since $[\alpha_{m-n}, \alpha_n] = 0$, we can rewrite $L_m, m \neq 0$, in the form

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha^\mu_{m-n}\alpha_{n\mu} :$$

The situation with L_0 is more complicated since $\alpha^\mu_{-n}\cdot\alpha^\mu_n$ do not commute. Of course we know that $\alpha^\mu_n \cdot \alpha^\mu_{-n} = n + \alpha^\mu_n \cdot \alpha^\mu_{-n}$ so we can write

$$L_0 = \frac{1}{2}\alpha^\mu_0\alpha_{0\nu} + \sum_{n>0}\alpha^\mu_{-n}\cdot\alpha_{n\mu} + \frac{1}{2}(D-2)\sum_{n\geq 1} n$$

Here we get $D-2$ because when we sum with respect to μ, the contributions corresponding to $\mu = 0, 1$ cancel each other. We shall deal with the last sum later. Now we define L_0 and $\tilde{L}_0$ by dropping out the infinite sum.

Similarly we define the operators $\tilde{L}_m$. The operators $L_m, \tilde{L}_m$ are called the *Virasoro operators*. Notice that L_m and L_{-m} (resp. $\tilde{L}_m$ and $\tilde{L}_{-m}$) are adjoint of each other.

The expression of the Hamiltonian operator is now straightforward:

$$H = L_0 + \tilde{L}_0 + (D-2)\sum_{n=1}^{\infty} n. \tag{5.19}$$

Since the last sum obviously does not make sense, we regularize it by setting

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}.$$

So, finally we get

$$H = L_0 + \tilde{L}_0 - \frac{D-2}{12}, \tag{5.20}$$

From now on

$$L_0 = \frac{1}{2}\alpha_0^{\mu}\alpha_{0\mu} + \sum_{n\geq 1}\alpha_{-n}^{\mu}\alpha_{n\mu}$$

$$\tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^{\mu}\tilde{\alpha}_{0\mu} + \sum_{n\geq 1}\tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_{n\mu}.$$

Let us find the commutator relations between the operators L_n. First we use the following well-known identity:

$$[AB, CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B.$$

This gives

$$[\alpha_{m-k}\cdot\alpha_k, \alpha_{n-l}\alpha_l] = \alpha_{m-k}\cdot[\alpha_k,\alpha_{n-l}]\alpha_l + \alpha_{m-k}\alpha_{n-l}[\alpha_k,\alpha_l] +$$

$$[\alpha_{m-k},\alpha_{n-l}]\alpha_l\alpha_k + \alpha_{n-l}[\alpha_{m-k}\alpha_l]\alpha_k = k\delta_{k,l-n}\alpha_{m-k}\alpha_l + k\delta_{k,-l}\alpha_{m-k}\alpha_{n-l} +$$

$$(m-k)\delta_{m-k,l-n}\alpha_l\alpha_k] + [(m-k)\delta_{m-k,-l}\alpha_{n-l}\alpha_k.$$

Here we skip the upper index μ. This easily implies

$$[L_m, L_n] = \frac{1}{2}\sum_k (k\alpha_{m-k}^{\mu}\alpha_{n+k,\mu} + (m-k)\alpha_{m-k+n}^{\mu}\alpha_{k\mu}).$$

Changing k to $k-n$ in the first sum we obtain for $m+n\neq 0$

$$[L_m,L_n]=(m-n)L_{m+n},\quad \text{if}\quad m+n\neq 0.$$

For $m+n=0$ we have a problem since $\sum_{k\in\mathbb{Z}}\alpha^\mu_{-k}\alpha_{k\mu}$ is not defined. Since $[\alpha^\mu_{-k},\alpha^\mu_k]=-k$, we see that the difference

$$[L_m,L_{-m}]=2mL_0+A(m)\mathrm{id}$$

for some scalar $A(m)$. Using the Jacobi identity, we find that, for $k+n+m=0$,

$$(n-m)A(k)+(m-k)A(n)+(k-n)A(m)=0.$$

Setting $k=1$ and $m=-n-1$ gives

$$A(n+1)=\frac{(n+2)A(n)-(2n+1)A(1)}{n-1}.$$

This shows that $A(m)=am^3+bm$ for some constants a,b. We will fix the constants when we consider the representation of the Lie algebra generated by $a^\mu_m,\tilde{a}^\mu_m$ in some Hilbert space.

Exercises

5.1 Let Vect(S^1) be the Lie algebra of complex vector fields on the circle. Each field is given by a convergent series $\sum_{n\in\mathbb{Z}}a_ne^{in\theta}\frac{d}{d\theta}$, where $a_n\in\mathbb{C}$. Let $L_n=e^{in\theta}\frac{d}{d\theta}$.

(i) Show that $[L_n,L_m]=i(m-n)L_{n+m}$.

(ii) Let Vir $=$ Vect$(S^1)\oplus\mathbb{C}$. Set

$$[(\xi,a),(\eta,b)]=([\xi,\eta],B(\xi,\eta)),$$

where B is a bilinear form on Vect(S^1). Show that this defines a structure of a Lie algebra on Vir if and only if B satisfies

$$B([\xi,\eta],\zeta)+B([\eta,\zeta],\xi)+B([\zeta,\xi],\eta)=0.$$

(iii) Let $B(n,m)=B(L_n,L_m)$. Show that $B(n,m)=0$ unless $n+m=0$ and $A(n,-n)=an^3+bn$ for some $a,b\in\mathbb{C}$.

(iv) Prove that two bilinear forms B and B' define isomorphic Lie algebras if and only if $B(n,-n)-B'(n,-n)$ is a linear function in n.

5.2 Let $\mathbb{C}[t,t^{-1}]$ be the algebra of Laurent polynomials in one variable. For any $P(t)=\sum_k a_kt^k\in\mathbb{C}[t,t^{-1}]$ let Res$(P(t))=a_{-1}$. Define a bilinear form on $\mathbb{C}[t,t^{-1}]$ by $\phi(P(t),Q(t))=\mathrm{Res}(Q\frac{dP}{dt})$.

(i) Show that ϕ is skew-symmetric and satisfies

$$\phi(PQ, R) + \phi(QR, P) + \phi(RP, Q) = 0.$$

(ii) Let $\mathcal{D} = \mathrm{Der}(\mathbb{C}[t, t^{-1}]) = \mathbb{C}[t, t^{-1}]\frac{d}{dt}$ be the Lie algebra of derivations of $\mathbb{C}[t, t^{-1}]$. Show that $\mathcal{D} \oplus \mathbb{C}$ is a Lie algebra with respect to the Lie bracket

$$[(P\frac{dP}{dt}, a), (Q\frac{dP}{dt}, b)] = ([P\frac{dP}{dt}, Q\frac{dP}{dt}], \phi(P, Q)).$$

(iii) Let $L_m = (-t^{m+1}\frac{d}{dt}, 0), c = (0, 1)$. Show that

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}c.$$

(iv) Show that any central extension of the Lie algebra Vect(S^1) with one-dimensional center is isomorphic to the Lie algebra defined by the commutator relations as in (iii).

Lecture 6

Fock space

Let $\mathfrak{g}$ be a Lie algebra over a field F with the Lie bracket $[a, b]$. Recall the construction of the *envelopping* algebra $\mathfrak{U}(\mathfrak{g})$. It is an associative algebra over F which is universal with respect to homomorphisms $f : \mathfrak{g} \to A$ of associative algebras such that $f([a, b]) = f(a)f(b) - f(b)f(a)$. It is constructed as the quotient of the tensor algebra

$$\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/I = (\oplus_{n=0}^{\infty} T^n(\mathfrak{g}))/I,$$

where I is the ideal generated by elements $a \otimes b - b \otimes a - [a, b]$, where $a, b \in \mathfrak{g}$. For example, if $\mathfrak{g}$ is a commutative Lie algebra (i.e. $[a, b] = 0$ for all $a, b \in \mathfrak{g}$) $\mathfrak{U}(\mathfrak{g})$ is isomorphic to the symmetric algebra Sym($\mathfrak{g}$), i.e. a free commutative algebra generated by the vector space $\mathfrak{g}$ (isomorphic to the polynomial algebra in variables indexed by a basis of $\mathfrak{g}$). In general, $\mathfrak{U}(\mathfrak{g})$ has a basis consisting of ordered products $e_{j_1} \ldots e_{j_k}, j_1 \leq \ldots \leq j_k$, where $(e_i)_{i \in I}$ is an ordered basis of the vector space $\mathfrak{g}$.

Recall that a *linear representation* of $\mathfrak{g}$ in a vector space V is a homomorphism of the Lie algebras $\rho : \mathfrak{g} \to \mathrm{End}(V)$, where the latter is equipped with a structure of a Lie algebra by setting $[A, B] = A \circ B - B \circ A$. We say that V is a $\mathfrak{g}$-module. By definition of the envelopping algebra, this is equivalent to equipping V with a structure of a left module over $\mathfrak{U}(\mathfrak{g})$. This allows us to extend the terminology of the theory of modules over associative rings to modules over Lie algebras.

An example of a linear representation is the *adjoint representation* $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by $\mathrm{ad}_{\mathfrak{g}}(a)(x) = [x, a]$. The fact that it is a linear representation follows from the *Jacobi identity* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

An *ideal* $\mathfrak{l}$ in a Lie algebra $\mathfrak{g}$ is a linear subspace such that $[a, b] \in \mathfrak{l}$ for any $a \in \mathfrak{g}$ and any $b \in \mathfrak{l}$, or, equivalently, a submodule in the adjoiont representation. An example of an ideal in $\mathfrak{g}$ is the *commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$ generated by the commutators $[a, b], a, b \in \mathfrak{g}$. A noncommutative Lie algebra without non-trivial ideals is called a *simple* Lie algebra.

We will be mostly dealing with infinite-dimensional Lie algebras. An example of such an algebra is a *Heisenberg algebra*. It is characterised by the condition that its center (the set of elements commuting with all elements in the algebra) is one-dimensional and coincides with the commutator. Let $\mathfrak{l}$ be a Heisenberg algebra and let

z be a basis of its center $\mathfrak{l}'$. We define a bilinear alternating form $(,)$ on $\mathfrak{l}$ by

$$[a,b] = (a,b)z.$$

Its kernel is equal to $\mathfrak{l}'$, and the induced bilinear form on $\bar{\mathfrak{l}} = \mathfrak{l}/\mathfrak{l}'$ is nondegenerate. For example, if $\mathfrak{l}$ is finite-dimensional, $\dim_F \bar{\mathfrak{l}} = 2k$ and $\bar{\mathfrak{l}}$ has a basis $(e_1, \ldots, e_k; e_{-1}, \ldots, e_{-k})$ such that

$$(e_i, e_{-j}) = \delta_{ij}, \quad (e_i, e_j) = (e_{-i}, e_{-j}) = 0, \quad 1 \leq i, j \leq k.$$

Thus $\mathfrak{l}$ is completely determined by the commutator relations

$$[e_i, e_{-j}] = \delta_{i,j} z. \tag{6.1}$$

So all Heisenberg Lie algebras of the same dimension are isomorphic.

If $\mathfrak{l}$ is infinite-dimensional, we assume additionally that $\mathfrak{l}$ is $\mathbb{Z}$-graded, i.e.

$$\mathfrak{l} = \oplus_{n \in \mathbb{Z}} \mathfrak{l}_n,$$

where each linear subspace $\mathfrak{l}_n$ is finite-dimensional, and

$$\mathfrak{l}_0 = \mathfrak{l}', \quad [\mathfrak{l}_n, \mathfrak{l}_m] \subset \mathfrak{l}_{n+m}.$$

Let

$$\mathfrak{l}_+ = \oplus_{n>0} \mathfrak{l}_n, \quad \mathfrak{l}_- = \oplus_{n<0} \mathfrak{l}_n.$$

It follows that

$$[\mathfrak{l}_+, \mathfrak{l}_+] = [\mathfrak{l}_-, \mathfrak{l}_-] = 0$$

and the bilinear form on $\bar{\mathfrak{l}}$ restricts to a non-degenerate alternating bilinear form on each $\mathfrak{l}_n \oplus \mathfrak{l}_{-n}$. Thus we can choose a basis $(e_i)_{i \in \mathbb{Z}_+}$ in $\mathfrak{l}_+$ and a basis $(e_{-i})_{i \in \mathbb{Z}_-}$ in $\mathfrak{l}_-$ such that $\mathfrak{l}$ is determined by the commutator relations as in (6.1). Together with $[z, e_i] = [z, e_{-i}] = [z, z] = 0$ these are called the *Heisenberg commutator relations*.

Notice that

$$\mathfrak{b}_\pm = \mathfrak{l}_0 \oplus \mathfrak{l}_\pm$$

are maximal abelian Lie subalgebras of $\mathfrak{l}$. Consider a linear representation of $\mathfrak{b}_+$ in the one-dimensional linear space F defined by

$$\rho_a(z)(1) = a, \quad \rho_a(\mathfrak{l}_+)(1) = 0.$$

Here $a \in F$ is a fixed parameter of the representation. Now we can define a linear representation of the whole Lie algebra $\mathfrak{l}$ by taking the *induced representation*:

$$V(a) = \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{l}}(\rho_a) := \mathfrak{U}(\mathfrak{l}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} F.$$

Recall that for any left module M over an associative F-algebra B the extension of scalars of M to a B-algebra A is a left A-module $A \otimes_B M$ defined as the quotient linear space $A \otimes M/T$, where T is the linear subspace spanned by tensors $ab \otimes m - a \otimes bm$ with $b \in B, a \in A, m \in M$ and multiplication $a \cdot (a' \otimes m + T) = aa' \otimes m + T$.

We can identify $\mathfrak{U}(\mathfrak{b}_+)$ with the algebra of polynomials $F[t_0, t_1, t_2, \dots]$ in variables t_i corresponding to the basis $(z, e_1, \dots, e_n, \dots)$ of $\mathfrak{b}$. Similarly we identify $\mathfrak{U}(\mathfrak{l})$ (as a linear space) with the linear space of Laurent polynomials $F[\dots, t_{-1}, t_0, t_1, \dots]$. However the multiplication is different:

$$t_i t_{-i} = t_{-i} t_i + t_0, \quad i > 0, \tag{6.2}$$

and any other pair of variables commutes. The $F[t_0, t_1, t_2, \dots]$-module corresponding to the representation ρ_a is the quotient algebra of $F[t_0, t_1, t_2, \dots]$ modulo the ideal generated by $t_0 - a, t_1, t_2, \dots$. Let us first describe the induced module $V(a)$ as a linear space. A monomial $t_{j_1} \cdots t_{j_k}$ of degree k in $F[\dots, t_{-1}, t_0, t_1, \dots]$ is called *normally ordered* if $j_1 \leq j_2 \leq \dots \leq j_k$. For any monomial $t_{i_1} \cdots t_{i_k}$ write

$$: t_{i_1} \cdots t_{i_k} := t_{j_1} \cdots t_{j_k},$$

where $j_1 \leq j_2 \leq \dots \leq j_k$ and $(i_1, \dots, i_k) = (j_{\sigma(1)}, \dots, j_{\sigma(k)})$ for some permutation $\sigma \in S_k$. Using the relations (6.2), we can write

$$t_{i_1} \cdots t_{i_k} =: t_{i_1} \cdots t_{i_k} : + \quad \text{normally ordered monomials of degree less than } k.$$

We call the above the *normal ordering decomposition* of the monomial $t_{i_1} \cdots t_{i_k}$ and write it as n.o.d$(t_{i_1} \cdots t_{i_k})$. For example, the normal ordering decomposition of $t_{-2} t_1 t_{-1}$ is equal to

$$t_{-2} t_1 t_{-1} = t_{-2} t_{-1} t_1 + t_{-2} t_0 =: t_{-2} t_1 t_{-1} : + t_{-2} t_0.$$

Observe now that if in a normally ordered monomial $t_{i_1} \cdots t_{i_k}$ the index i_k is positive, the coset of $t_{i_1} \cdots t_{i_k} \otimes 1$ in the induced module $V(a)$ is equal to $t_{i_1} \cdots t_{i_k} \otimes t_{i_k} \cdot 1$ and hence is zero. If the monomial is equal to $t_{i_1} \cdots t_{i_k} t_0$ then $t_{i_1} \cdots t_{i_k} t_0 \otimes 1 = a t_{i_1} \cdots t_{i_k} \otimes 1$. This shows that $V(a)$ has a basis consisting of elements $1 \otimes 1$ and $t_{i_1} \cdots t_{i_k} \otimes 1$, where $i_1 \leq \dots \leq i_k < 0$. This allows us to identify $V(a)$ with the linear space $F[t_{-1}, t_{-2}, \dots]$. We have an isomorphism of linear spaces

$$V(a) = \mathrm{Sym}(\mathfrak{l}_-). \tag{6.3}$$

The vector

$$|0\rangle := 1 \otimes 1$$

is called the *vacuum vector*. The structure of a $\mathfrak{U}(\mathfrak{l})$-module on $V(a)$ is given by

$$t_{j_1} \cdots t_{j_k} \cdot (t_{-i_1} \cdots t_{-i_k} \otimes 1) = \text{n.o.d}(t_{j_1} \cdots t_{j_k} \cdot t_{-i_1} \cdots t_{-i_k}) \otimes 1 =$$

$$\text{n.o.d}(t_{j_1} \cdots t_{j_k} \cdot t_{-i_1} \cdots t_{i_{-k}}) |0\rangle.$$

Note that $V(a)$ carries a natural grading defined by

$$\deg(l_{i_1} \cdots l_{i_k}) = i_1 + \dots + i_k,$$

where $l_i \in \mathfrak{l}_{-i}$ and $\deg |0\rangle = 0$.

Remark 6.1. More explicitly the representation $V(a)$ of $\mathfrak{l}$ can be described as follows. We identify $V(a)$ with the polynomial algebra $\mathbb{C}[x_1, x_2, \dots]$ and assign to $e_i, i > 0$, the operator $\frac{\partial}{\partial x_i}$, to $e_{-i}, i > 0$, the operator $x_i : p(x) \to x_i p(x)$, and to z the scalar operator $a$id. Then $[\frac{\partial}{\partial x_i}, x_j] = \text{id}$ and hence we get a representation obvioulsy isomorphic to $V(a)$.

There is an inner product on the space $V(a)$ defined as follows. Let E be any linear space over a field of characteristic 0 equipped with a symmetric bilinear form g. First we define the bilinear form in $T^n(E)$ by

$$g^{\otimes n}(v_1 \otimes \dots \otimes v_n, w_1 \otimes \dots \otimes w_n) = g(v_1, w_1) \cdots g(v_n, w_n)$$

and then extend it to the whole $T(E) = \oplus_n T^n(E)$ by requiring that $T^n(E)$ and $T^m(E)$ are mutually orthogonal. Using the polarization process, we identify $S^n(E)$ with $S^n(E^*)^*$ equal to the subspace of symmetric tensors in $T^n(E) = E^{\otimes n}$ and then restrict $g^{\otimes n}$ to $S^n(E^*)^*$ to get a symmetric bilinear form $\text{sym}^n(g)$. One can show that this inner product is non-degenerate if g is non-degenerate. Recalling the polarization isomorphism we see that

$$\text{sym}^n(g)(e_{i_1} \dots e_{i_n}, e_{j_1} \dots e_{j_n}) = \frac{1}{n!} \sum_{\sigma} g(e_{i_1}, e_{\sigma(j_1)}) \dots g(e_{i_n}, e_{\sigma(j_n)}), \tag{6.4}$$

where the sum is taken with respect to all permutations of n letters. Here we identify $\text{Sym}(E)$ with the space of polynomials in a basis (e_i) of E. This defines a symmetric bilinear form $\text{sym}(g)$ on $\text{Sym}(E)$. Following the physics agreement we shall drop $\frac{1}{n!}$ in this formula. A similar construction can be given for any hermitian bilinear form. In fact, if we choose a positive definite hermitian form on E we can complete the tensor algebra $T(E)$ with respect to the corresponding norm and obtain a Hilbert space $\hat{T}(E)$. This space is called the *Fock space* associated to the unitary space E. The completion of the subspace $\text{Sym}(E)$ is called the *bosonic Fock space*. Similarly we can restrict ourselves with the exterior algebra $\Lambda(E)$ identified with the subspace of alternating tensors in $T(E)$. Its completion is called the *fermionic Fock space*. We will deal with it later.

We apply the construction of the Fock space to the Heisenberg algebra over $\mathbb{C}$ by taking $E = \mathfrak{l}_-$, where $\mathfrak{l}_-$ is equipped with a structure of a unitary space.

Let us consider the Lie algebra $\mathfrak{g}$ with a linear basis $1, \alpha_n^\mu, n \in \mathbb{Z}, \mu = 1, \dots, D$ with Lie bracket defined by commutator relations (5.17b):

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+k,0}\eta^{\mu\nu}, \tag{6.5}$$

Let $\mathfrak{l}$ be the graded Heisenberg algebra over $\mathbb{C}$ with $\dim \mathfrak{l}_n = D$ for all $n \neq 0$. Let $(e_n^\mu), \mu = 1, \dots, D$, be a basis in $\mathfrak{l}_n$ such that, for any $n > 0$,

$$[e_n^\mu, e_m^\nu] = \delta_{m,n}\eta^{\mu\nu}z.$$

Consider the direct sum of Lie algebras $\mathfrak{l}^D = \mathfrak{l} \oplus \mathfrak{a}$, where $\mathfrak{a} = \mathbb{R}^D$ is viewed as an abelian Lie algebra. Let (e_0^μ) be a basis of $\mathfrak{a}$. I claim that $\mathfrak{g}$ is isomorphic to $\mathfrak{l}^D$. To see this we define the linear map $f : \mathfrak{l}^D \to \mathfrak{g}$ as follows.

$$f(e_n^\mu) = \begin{cases} \frac{i}{\sqrt{n}}\alpha_n^0 & \text{if } n \neq 0; \\ \frac{1}{\sqrt{n}}\alpha_n^\mu & \text{if } n \neq 0, \mu \neq 0; \\ \alpha_0^\mu & \text{if } n = 0, \end{cases} \tag{6.6}$$

and $f(z) = 1$. It is clear that this is an isomorphism of Lie algebras. We call $\mathfrak{l}^D$ the *oscillator algebra* of $\mathbb{R}^D$. Let $V(a)$ be the linear representation of the subalgebra $\mathfrak{l}$ generated by $z, e_i, i \neq 0$ described above. We can extend the representation to the whole $\mathfrak{l}^D$ by setting

$$\rho(e_0^\mu)|0\rangle = \lambda_\mu|0\rangle$$

for some $\lambda^\mu \in \mathbb{R}$. Since e_0^μ belong to the center, it defines the representation. We denote the obtained representation by $V(a;\lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_D) \in \mathbb{R}^D$. It is more natural to consider λ as a linear function on the ideal $\mathfrak{a}$ of $\mathfrak{l}^D$ generated by e_0^μ's so that $\lambda_\mu = \lambda(e_0^\mu)$. We will be interested only in representations corresponding to $a = 1$ so that we set $V(\lambda) = V(1;\lambda)$. Its vacuum state is denoted by $|\lambda\rangle$. Recall that we can write any element of $V(\lambda)$ as

$$|\epsilon(\lambda)\rangle := \epsilon_{\mu_1,\ldots,\mu_n}^{k_1,\ldots,k_n}\alpha_{-k_1}^{\mu_1}\ldots\alpha_{-k_n}^{\mu_n}|\lambda\rangle,$$

where $\epsilon = (\epsilon_{\mu_1,\ldots,\mu_n}(k_1,\ldots,k_n))$ is a tensor symmetric in lower and upper indices defining a linear map $S^n(\mathfrak{l}_-) \to S^n(\mathbb{R}^D)$ with finite-dimensional support. It is called a *Lorentz polarization tensor*. Fix $\mathbf{k} = (k_1,\ldots,k_n) \in \mathbb{Z}_{>0}$ and set

$$|\epsilon(\lambda;\mathbf{k})\rangle = \epsilon(\lambda;k_1,\ldots,k_n) = \sum_{1\leq\mu_1\leq\ldots\leq\mu_n=D}\epsilon_{\mu_1,\ldots,\mu_n}^{k_1,\ldots,k_n}\alpha_{-k_1}^{\mu_1}\ldots\alpha_{-k_n}^{\mu_n}|\lambda\rangle.$$

Let us define the inner product in $V(\lambda)$. We may assume that $|\lambda\rangle$ is of norm 1. Recall that we want the operators α_i^μ and α_{-i}^μ to be adjoint to each other. Then

$$\langle\alpha_{-i}^\mu|\lambda\rangle, \alpha_{-j}^\nu|\lambda\rangle\rangle = \langle\alpha_j^\nu\alpha_{-i}^\mu|\lambda\rangle, |\lambda\rangle\rangle = \langle(\alpha_{-i}^\mu\alpha_j^\nu + j\delta_{ij}\nu^{\mu\nu})|\lambda\rangle, |\lambda\rangle\rangle = j\delta_{ij}\nu^{\mu\nu}.$$

So we see that $\frac{1}{\sqrt{n}}\alpha_{-n}^\mu|\lambda\rangle$ form an othonormal basis in Minkowski sense. In particular, the vectors $\alpha_{-n}^0|\lambda\rangle$ have squared norm equal to $-n$. Following the discussion above we can extend the inner product to the whole $V(\lambda)$. Two different monomials in α_{-i}^μ's are orthogonal and

$$||\alpha_{-k_1}^{\mu_1}\ldots\alpha_{-k_n}^{\mu_n}|\lambda\rangle|| = k_1\ldots k_n,$$

if all $k_1,\ldots,k_n$ are distinct. We leave to the reader to deal with the general case. Note that, for any $k \in \mathbb{Z}_{>0}$, and $\epsilon = (\epsilon_1,\ldots,\epsilon_D) \in \mathbb{R}^D$,

$$|||\epsilon(\lambda;k\rangle|| = k||\epsilon|| = \epsilon_\mu\epsilon^\mu. \tag{6.7}$$

Remark 6.2. One can define the Lie algebra $\mathfrak{l}^D$ and the Fock space $V(\lambda)$ in a coordinate-free way. Let E be a vector space over a field K equipped with a non-degenerate symmetric bilinear form (x, y). An element of $\mathfrak{l}(E) = K[t] \otimes E$ can be interpreted as a finite linear combination of tensors $v_n \otimes t^n, v_n \in E, n \in \mathbb{Z}$. Consider the Lie algebra with generators $v_n \otimes t^n, z$, where z is central, satisfying the commutator relations

$$[v_m \otimes t^m, v_n \otimes t^n] = m(v_n, v_m)\delta_{m,-n} z. \tag{6.8}$$

If $E = \mathbb{R}^D$ is the Euclidean vector space, by choosing an orthonormal basis in E, we see that $\mathfrak{l}(E) \cong \mathfrak{l}^D$. One defines the Fock space $\mathcal{F}(E) = \mathrm{Sym}(t^{-1}K[t^{-1} \otimes E)$. Its elements are finite linear combinations of tensors $v_{-n_1} \dots v_{-n_k} \otimes t^{-n}, n_1 + \dots + n_k = n > 0, v_{-n_1} \dots v_{-n_k} \in \mathrm{Sym}^k(E)$. The inner product on $\mathcal{F}(E)$ is defined by extending the bilinear form $(v_n \otimes t^n, v_m \otimes t^m) = (v_n, v_m)$ on $t^{-1}K[t^{-1} \otimes E$ to the symmetric product. The Lie algebra $\mathfrak{l}(E)$ has a representation in $\mathcal{F}(E)$ by defining $v_m \otimes t^m$ to be the adjoint of $v_{-m} \otimes t^{-m}$ for $m > 0$ and letting $v_{-m} \otimes t^{-m}$ act by multiplication:

$$v_{-m} \otimes t^{-m}(v_{-n_1} \dots v_{-n_k} \otimes t^{-n}) = v_{-m}v_{-n_1} \dots v_{-n_k} \otimes t^{m-n}.$$

Also we let $v_0 \otimes 1$ act by

$$v_0 \otimes 1(v_{-n_1} \dots v_{-n_k} \otimes t^{-n}) = \lambda(v_0),$$

where $\lambda : E \to K$ is a fixed linear form. It is easy to see that, in the case $E = \mathbb{R}^D$, we get a representation isomorphic to $V(\lambda)$.

There is one more important requirement on the spaces $V(\lambda)$. The Lie algebra of the Poincarè group $\mathcal{P}$ of the Minkowski space $V = \mathbb{R}^{1,D-1}$ must have a linear representation in these spaces. Recall that $\mathcal{P}$ is the semi-product of the translation group V and the orhogonal group $O(D-1, 1)$. The Lie algebra of $\mathcal{P}$ is the direct product of the abelian algebra $V \cong \mathbb{R}^D$ and the algebra of matrices $A = (a^{\mu\nu}) \in M_D(\mathbb{R})$ satisfying $\eta_{\mu\nu}a^{\nu\tau} + a^{\nu\mu}\eta_{\nu\tau} = 0$. It has a set of generators $e_i, i = 1, \dots, D$ and $e_{ij}, 1 \le i < j \le D$, satisfying the commutator relations

$$[e_i, e_j] = 0, i, j = 1, \dots, D, \tag{6.9a}$$

$$[e_{jk}, e_i] = \eta_{ji}e_k - \eta_{ki}e_j \tag{6.9b}$$

$$[e_{ij}, e_{kl}] = \eta_{jk}e_{il} - \eta_{ik}e_{jl} - \eta_{jl}e_{il} + \eta_{il}e_{jk} \tag{6.9c}$$

Here e_{ij} corresponds to the matrices $E_{ij} - eE_{ij}$, where $e = 1$ if $i \neq 1$ and -1 if $i = 1$. Define the operators

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu) \tag{6.10}$$

Then one checks that

$$[p^\mu, J^{\nu\rho}] = -i\eta^{\mu\nu}p^\rho + i\eta^{\mu\rho}p^\nu$$

$$[J^{\mu\nu}, J^{\rho\lambda}] = -i\eta^{\nu\rho}J^{\mu\lambda} + i\eta^{\mu\rho}J^{\nu\lambda} + i\eta^{\nu\lambda}J^{\mu\rho} - i\eta^{\mu\lambda}J^{\nu\rho}.$$

This shows that the correspondence $e_\mu \to p^\mu, e_{\mu\nu} \to J^{\mu\nu}$ is a representation of the Poincaré Lie algebra in the Fock spaces $V(\lambda)$ (one has only to replace the commutators $[,]$ with $i[,]$).

Here the operators $x^\mu p^\nu - x^\nu p^\mu$ correspond to the matrices e_{ij} in the natural linear representation of $\mathfrak{so}(D-1,1)$ in the space of vector fields in $\mathbb{R}^d$ as the vector fields $\xi^{\mu\nu} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu}$. We can view a state $|\epsilon(\lambda;\mathbf{k})\rangle \in V(\lambda)$ as a polynomial function on $\mathrm{Sym}^n(\mathfrak{l}_-)$ with values in $\mathrm{Sym}^n(\mathbb{R}^d)$ so that the vector field $\xi^{\mu\nu}$ acts naturally on such states. The translation part of the Lie algebra of the Poincaré group acts via the operators p^μ.

Remark 6.3. Recall that an irreducible linear representation V of the Poincaré group is described by the following data. First one restricts the representation to the translation subgroup T. Since the latter is an abelian group the linear space V decomposes into the direct sum of eigensubspaces $V_\chi, \chi \in T^*$,, where $V_\chi = \{v \in V : t\cdot v = \chi(t)v, \forall t \in T\}$. The Lorentz group $G = \mathrm{SO}(n-1,1)$ acts on T^*. It is easy to see that the set $\{\chi \in T^* : V_\chi \neq \{0\}\}$ is an orbit $\mathcal{O}$ of G. Let H be the isotropy subgroup of some $\chi_0 \in \mathcal{O}$. Then the restriction of the representation to H defines an irreducible representation of H in V_{χ_0}. Now the natural action of G on $\mathcal{O} = G/H$ lifts to an action on the vector bundle $E = G \times^H V_{\chi_0} = G \times V_{\chi_0}/H$, where H acts on the product by $h\cdot(g,v) = (gh, h^{-1}v)$. There is a natural action of G on the space $\Gamma(E)$ of sections of this bundle and the representation V is realized as an irreducible subrepresentation of $\Gamma(E)$.

For example, consider the irreducible representation V which contains a vacuum vector $|\lambda\rangle$. The translation group T acts via the operators α_0^μ. This shows that $|\lambda\rangle$ is an eigenvector corresponding to the character $\lambda \in (\mathbb{R}^D)^*$. This shows that the fibres of the vector bundle E are one-dimensional and the group $H = SO(n-1,1)_\lambda$ acts identically on the fibre over $\lambda \in \mathcal{O} = SO(n-1,1)\cdot\lambda$. Thus the data describing the representation consists of the orbit of λ determined by $||\lambda||^2$ (if the norm is positive then the group $\mathrm{SO}(n-1,1)_\lambda \cong \mathrm{SO}(n-1)$ is compact) and the trivial representation of $\mathrm{SO}(n-1,1)$. Physicists say that $|\lambda\rangle$ transforms like a scalar.

We can define the similar space $\tilde{V}(\lambda)$ corresponding to right movers. Its vacuum state is denoted by $|\lambda\rangle_R$. Then we consider the tensor product $V(\lambda)\otimes\tilde{V}(\lambda)$. Its vacuum state is $|\lambda\rangle_L \otimes |\lambda\rangle_R$.

Its vectors look like this

$$\epsilon^{k_1,\ldots,k_n;l_1,\ldots,l_m}_{\mu_1,\ldots,\mu_n;\nu_1,\ldots,\nu_m}\alpha^{\mu_1}_{-k_1}\ldots\alpha^{\mu_n}_{-k_n}|\lambda\rangle_L \otimes \tilde{\alpha}^{\nu_1}_{-s_1}\ldots\tilde{\alpha}^{\nu_m}_{-s_m}|\lambda\rangle_R,$$

where $k_n \le k_{n-1} \le \ldots \le k_1$ and $s_m \le s_{n-1} \le \ldots \le s_1$ and the polarization tensor

$$\epsilon = (\epsilon^{k_1,\ldots,k_n;l_1,\ldots,l_m}_{\mu_1,\ldots,\mu_n;\nu_1,\ldots,\nu_m})$$

is symmetric in μ and ν (resp. in k_i and l_j) separately.

One defines the norm on $\tilde{V}(\lambda)$ similar to the norm on $V(\lambda)$ and then gets a non-degenerate inner product on $V(\lambda)\otimes\tilde{V}(\lambda)$. Finally we complete this space to get the

Fock space of the closed bosonic string theory

$$\mathcal{F}_{\text{closed}} = \bigoplus_{\lambda \in \mathbb{R}^D} \hat{V}(\lambda) \hat{\otimes} \hat{\tilde{V}}(\lambda).$$

Finally, let us see the representation of the Virasoro algebra generated by the operators L_n (resp. $\tilde{L}_m$) in the space $V(\lambda)$ (resp. $\tilde{V}(\lambda)$). Recall that

$$L_0 = \frac{1}{2}\alpha_0^\mu \alpha_{0\mu} + N, \quad \tilde{L}_0 = \frac{1}{2}\alpha_0^\mu \alpha_{0\mu} + \tilde{N},$$

where

$$N = \sum_{n \geq 1} \alpha_{-n}^\mu \alpha_{n\mu}, \quad \tilde{N} = \sum_{n \geq 1} \alpha_{-n}^\mu \alpha_{n\mu}.$$

$$L_m = \sum_{n \geq 1} \alpha_{m-n}^\mu \alpha_{n\mu}, \quad \tilde{L}_m = \sum_{n \geq 1} \alpha_{m-n}^\mu \alpha_{n\mu}, \quad m \neq 0$$

The operators N and $\tilde{N}$ are called *level operators*. It is easy to check that

$$N \cdot \alpha_{-k_1}^{\mu_1} \dots \alpha_{-k_n}^{\mu_n} |\lambda\rangle_L = (k_1 + \dots + k_n) \alpha_{-k_1}^{\mu_1} \dots \alpha_{-k_n}^{\mu_n} |\lambda\rangle_L$$

and a similar formula holds for $\tilde{N}$.

Recall that $[L_m, L_n] = (m-n)L_{m+n} + A(m)\text{id}$. We have to find the constant $A(m) = am^3 + bm$.

One applies $[L_m, L_{-m}]$ to some ground states to compute these constants. Notice that

$$L_m|\lambda\rangle = \frac{1}{2}(\sum_{n \in \mathbb{Z}} \alpha_{1-n}^\mu \alpha_{n\mu})|\lambda\rangle = 0, \quad m > 0 \tag{6.11}$$

$$L_0|\lambda\rangle = \frac{1}{2}\alpha_0^\mu \alpha_{0\mu}|\lambda\rangle = \frac{1}{2}||\lambda||^2|\lambda\rangle \tag{6.12}$$

Also

$$L_{-1}|\lambda\rangle = \frac{1}{2}\sum_{n \in \mathbb{Z}} \alpha_{-1-n}^\mu \alpha_{n\mu}|\lambda\rangle = \frac{1}{2}(\alpha_{-1}^\mu \alpha_{0\mu} + \alpha_0^\mu \alpha_{-1\mu})|\lambda\rangle = \lambda_\mu \alpha_{-1}^\mu |\lambda\rangle.$$

Here we used that the operators L_m and L_{-m} are adjoint to each other. Thus

$$\langle\lambda|L_1 L_{-1}|\lambda\rangle\rangle = \langle L_{-1}|\lambda\rangle, L_{-1}|\lambda\rangle = \frac{1}{4}||2\alpha_{-2}^\mu \alpha_{0\mu}|\lambda\rangle||^2 = ||\lambda_\mu \alpha_{-1}^\mu |\lambda\rangle||^2 = ||\lambda||^2,$$

and we obtain

$$\langle\lambda|[L_{-1}, L_1]|\lambda\rangle = \langle\lambda|L_{-1}L_1\lambda\rangle = ||\lambda||^2 =$$

$$\langle\lambda|(2L_0 + A(1)\mathrm{id})|\lambda\rangle = ||\lambda||^2(1 + A(1)^2).$$

This gives $A(1) = a + b = 0$. Also

$$\langle\lambda|L_2L_{-2}|\lambda\rangle = ||L_{-2}|\lambda\rangle||^2 = \frac{1}{4}||2\alpha^{\mu}_{-2}\alpha_{0\mu} + \alpha^{\mu}_{-1}\alpha_{-1\mu}|\lambda\rangle||^2 =$$

$$||\alpha^{\mu}_{-2}\alpha_{0\mu}|\lambda\rangle||^2 + \frac{1}{4}||\alpha^{\mu}_{-1}\alpha_{-1\mu}|\lambda\rangle||^2 = 2||\lambda||^2 + \frac{1}{2}D.$$

Here we used (6.7) and

$$||\alpha^{\mu}_{-1}\alpha_{-1\mu}|\lambda\rangle||^2 = \langle\lambda|\alpha_{1\mu}\alpha^{\mu}_{1}\alpha^{\mu}_{-1}\alpha_{-1\mu}|\lambda\rangle = \langle\lambda|\alpha_{1\mu}(\alpha^{\mu}_{-1}\alpha^{\mu}_{1} + \eta^{\mu\mu})\alpha_{-1\mu}|\lambda\rangle =$$

$$\langle\lambda|\eta^{\mu\mu}\alpha_{1\mu}\alpha_{-1\mu} + \alpha_{1\mu}\alpha^{\mu}_{-1}\alpha^{\mu}_{1}\alpha_{-1\mu}|\lambda\rangle = \langle\lambda|2\eta_{\mu\mu}\alpha_{1\mu}\alpha_{-1\mu}|\lambda\rangle = 2\eta_{\mu\mu}\eta^{\mu\mu} = 2D.$$

Thus

$$\langle\lambda|[L_2, L_{-2}]|\lambda\rangle = \langle\lambda|L_2L_{-2}|\lambda\rangle = \langle\lambda|4L_0 + A(2)\mathrm{id}|\lambda\rangle =$$

$$2||\lambda||^2 + A(2) = 2||\lambda||^2 + \frac{1}{2}D.$$

This gives $A(2) = 8a + 2b = 6a = \frac{1}{2}D$, hence $a = D/12$. Finally we obtain that for all $m, n \in \mathbb{Z}$, we have the following commutator relation for the Virasoro operators acting in the space $V(\lambda)$.

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}D(m^3 - m)\delta_{m,-n}. \tag{6.13}$$

Exercises

6.1 Let $\mathfrak{l} = \oplus_n\mathfrak{l}_n$ be a graded Heisenberg Lie algebra. Let Sp($\mathfrak{l}$) be the symplectic group of linear automorphisms of $\mathfrak{l}$ which preserve the alternating from (x, y). Construct a linear projective representaion of Sp($\mathfrak{l}$) in the space $\mathbb{P}(V(a))$ which is compatible with the representation of $\mathfrak{l}$ in $V(a)$.

6.2 Compute the norm of the state $L_3|\lambda\rangle$.

6.3 Compute the norm of any state $\alpha^{\mu_1}_{-k_1}\ldots\alpha^{\mu_n}_{-k_n}|\lambda\rangle$.

6.4 Let $p(n)$ denote the dimension of the space of eigenvectors of the level operator N with eigenvalue n. Compute the generating function $\mathrm{Tr}(q^N) = \sum_{n=0}^{\infty} p(n)q^n$. Explain the notation.

Lecture 7

Physical states for bosonic string

The expression for the Hamiltonian provides the *mass-squared formula* . Recall that in the special relativity theory the mass is defined as the negative of the norm of the moment vector in the Minkowski space-time. Let us explain it. We use the metric in the space-time $\mathbb{R}^4$ with coordinates $(x_0, x_1, x_2, x_3) = (x_0, \mathbf{x})$ defined by $dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$. Here $x_0 = ct$, where t is time and c is a constant equal to the speed of light. To describe the motion we use the Lagrangian density

$$\mathcal{L} = -mc\sqrt{c^2 - |\mathbf{x}'|^2}dt = mc^2\sqrt{1 - \frac{v^2}{c^2}}dt,$$

where $\mathbf{x}' = \frac{d\mathbf{x}}{dt}, \mathbf{x} = (x_1, x_2, x_3)$ and m is a constant called the *mass*. The energy and the moment for this Lagrangian are equal to

$$p^i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = mc^2/\sqrt{1 - \frac{v^2}{c^2}}, \quad i = 1, 2, 3,$$

$$E = p^i \dot{x}_i - L = mc^2/\sqrt{1 - \frac{v^2}{c^2}}.$$

We have

$$E^2 - c^2|\mathbf{p}|^2 = m^2c^4,$$

so if we set

$$\mathbf{P} = (E/c^2, p^1/c, p^2/c, p^3/c),$$

we obtain that

$$m^2 = -||\mathbf{P}||^2,$$

where we use the Minkowski norm defined by the matrix diag$[-1, 1, 1, 1]$. The vector $\mathbf{P}$ is called the *total momentum vector*. In our situation $\mathbf{P} = (p^1, \ldots, p^D)$ so we can define the *quantum mass-square operator* by

$$M^2 = -p^\mu p_\mu \otimes 1 - 1 \otimes p^\mu p_\mu = -\frac{4}{l^2}(\alpha_0^\mu \alpha_{0\mu} \otimes 1 + 1 \otimes \tilde{\alpha}_0^\mu \tilde{\alpha}_{0\mu}).$$

We shall scale the masses to assume that $l^2 = 8$ so that

$$M^2 = (-L_0 + N) \otimes 1 + 1 \otimes (-\tilde{L}_0 + \tilde{N}).$$

Thus the mass-square of the ground state $|\lambda\rangle_L \otimes |\lambda\rangle_R$ is equal to

$$-|\lambda|^2 = \lambda_1^2 - \lambda_2^2 + \ldots - \lambda_D^2.$$

Recall that in the pre-quantized theory we had Virasoro constraints $T_{\alpha\beta} = 0$. It follows from (5.15) that the analogs of these constraints in the quantum string theory are the conditions that $L_m\psi = 0$ for any element ψ of the Fock space. However, because $[L_n, L_{-n}] = 2nL_0 + c\mathrm{id}$, this would imply that $\psi = 0$. Thus we have to require that $L_m\psi = 0$ only for positive m and $L_0\psi = a\psi$ for some ψ, and similarly for the operators $\tilde{L}_m$. We set

$$\mathcal{F}_{\text{phys}}(\lambda) = \{\psi \in \mathcal{F}(\lambda) : L_m\psi = 0, \quad m > 0, (L_0 - a)\psi = 0\} \tag{7.1a}$$

$$\tilde{\mathcal{F}}_{\text{phys}}(\lambda) = \{\psi \in \tilde{\mathcal{F}}(\lambda) : \tilde{L}_m\psi = 0, \quad m > 0, (\tilde{L}_0 - a)\psi = 0\} \tag{7.1b}$$

$$\mathcal{F}_{\text{phys}}^{\text{closed}} = \oplus_{\lambda \in (\mathbb{R}^D)^*} \mathcal{F}_{\text{phys}}(\lambda) \otimes \tilde{\mathcal{F}}_{\text{phys}}(\lambda). \tag{7.1c}$$

A state satisfying these conditions is called *physical*. Also for any state ξ and a physical state ϕ, we have

$$\langle \phi | L_{-n}\chi \rangle = \langle L_n(\phi) | \chi \rangle = 0, \quad n > 0$$

Thus the intersection of $\mathcal{F}_{\text{phys}}$ with the sum $\mathcal{F}_{\text{spur}} = \oplus_{n>0} L_{-n}(\mathcal{F})$ belongs to the null-space of $\mathcal{F}_{\text{phys}}$. The elements of this space are called *spurious states*. The Hilbert space which we want will be the quotient

$$\mathcal{F}_+^{\text{closed}} = \mathcal{F}_{\text{phys}}^{\text{closed}} / \mathcal{F}_{\text{spur}} \cap \mathcal{F}_{\text{phys}}^{\text{closed}}.$$

Note that the operators $J^{\mu\nu}$ defined in (6.9) commute with Virasoro operators, so that the Poincaré Lie algebra acts in the spaces of physical states.

Remark 7.1. An abstract Lie algebra is called a *Virasoro algebra* if it can be defined by generators $z, l_n, n \in \mathbb{Z}$ with commutator relations

$$[l_m, l_n] = (m-n)l_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n} z, \quad [z, l_n] = 0.$$

It can be shown that any any Lie algebra obtained as a central extension with one-dimensional center of the algebra of vector fields on a circle is isomorphic to a Virasoro algebra. A representation of the Virasoro algebra in a vector space V is called a representation with highest weight a and charge c if z acts as a scalar operator $c\ \mathrm{id}_V$ and there exists a vector v_0 (called a *highest weight vector*) such that

$$l_m v_0 = 0, m > 0, \quad l_0 v_0 = a v_0.$$

A universal representation with this property is called a *Verma module* and is denoted by $V(a,c)$. It can be constructed by using a similar construction as the representations $V(a)$ we constructed for a Heisenberg algebra. One considers the subalgebra Vir_+ generated by the operators $L_m, m \geq 0$, then defines a one-dimensional representation by $L_0 \cdot 1 = a, L_m \cdot 1 = 0, m > 0$ and finally takes the induced representation $\mathfrak{U}(\mathrm{Vir}) \otimes_{\mathrm{Vir}} \mathbb{C}$. Its elements are linear combinations of monomials $L_{-n_1} \ldots L_{-n_k} v_0$ with positive n_i's. Any irreducible representation with highest weight a and charge c is isomorphic to a quotient of $V(a,c)$. So, we see that each nonzero $\psi \in \mathcal{F}_{\mathrm{phys}}(\lambda)$ generates a representation space V_ψ for the Virasoro algebra with highest weight a and charge $c = D$. Its highest weight vector is ψ. As we have seen before any physical state ξ belongs to $V_\psi^\perp$.

Since all physical states are eigenvalues of L_0 with eigenvalue a, we obtain the *mass-formula* for physical states:

$$M^2 = -(2L_0 - 2N) = 2N - 2a. \tag{7.2}$$

Let us see which ground states in $\mathcal{F}^{\mathrm{closed}}$ are physical. Since

$$N|\lambda\rangle = 0, \quad L_m \cdot |\lambda\rangle = 0, \quad m > 0, \quad L_0|\lambda\rangle = \frac{1}{2}|\lambda|^2,$$

and the same is true for the right mode operators $\tilde{L}_n$, we see that the ground state $|\lambda\rangle = |\lambda\rangle_L \otimes |\lambda\rangle_R$ is physical if and only if

$$|\lambda|^2 = 2a. \tag{7.3}$$

For this vacuum state

$$M^2|\lambda\rangle = -2a$$

We shall see from the next discussion that a must be equal to 1. Thus the vacuum vectors have negative mass. Such states are called *tachyons* (they travel faster than light!). The existence of such states will force us to abandon bosonic strings and consider superstrings.

Let us look for $\psi \in \mathcal{F}_{\mathrm{phys}}(\lambda)$ of level 1. Each such ψ has the form $\epsilon_\mu \alpha_{-1}^\mu |\lambda\rangle$. We have

$$L_{-1}\psi = \alpha_{-1}\alpha_0\psi = \epsilon_\mu \lambda^\mu \psi, \quad L_m\psi = 0, m > 1, \quad L_0\psi = (\frac{1}{2}|\lambda|^2 + 1)$$

Thus ψ is physical if and only if

$$\epsilon_\mu \lambda^\mu = 0, \quad |\lambda|^2 = 2a - 2. \tag{7.4}$$

If $a > 1$, we may choose $\lambda = (0,1,0,\ldots,0)$, and then $\epsilon = (1,0,\ldots,0)$ satisfies (7.4) but $||\psi|| = |\epsilon|^2 = -1$. This means that we have *ghosts*, i.e. states of negative norm. This should be avoided since the the quantum mechanics deals only with Hilbert spaces with unitary inner product. This forces us to take $a \leq 1$.

If $a < 1$, we may take $\lambda = (1, 0, \ldots, 0)$ and $\epsilon = (0, \epsilon_1, \ldots, \epsilon_D)$ so we have $D - 1$-dimensional space of physical states of positive norm and no states of non-positive norm.

If $a = 1$, we may take $\lambda = (1, -1, 0, \ldots, 0)$ and hence $\epsilon_0 = \epsilon_1$. This shows that we have a $(D-2)$-dimensional space of states of positive norm and a one-dimensional space of states of norm 0. The state $L_{-1}|\lambda\rangle = \lambda_\mu \alpha^\mu_{-1}$ is spurious and is physical if $|\lambda|^2 = \lambda_\mu \lambda^\mu = 0$. Thus, if $a = 1$, $\mathcal{F}_{\text{phys}}(\lambda)$ contains a one-dimensional space of spurious states of norm 0. Factoring this space out we get a $(D-2)$-dimensional space $\mathcal{F}_{\text{phys}}(\lambda)/\mathcal{F}_{\text{spur}}(\lambda) \cap \mathcal{F}_{\text{phys}}(\lambda)$, each element of which can be represented by a state of positive norm. So far, we find that $a \leq 1$ and no restriction on D appears.

Let $\chi = \sum_i \psi_i \otimes \tilde{\psi}_i \in \mathcal{F}_{\text{phys}}(\lambda) \otimes \tilde{\mathcal{F}}_{\text{phys}}(\lambda)$, where $\psi_i \in \mathcal{F}_{\text{phys}}(\lambda), \tilde{\psi}_i \in \tilde{\mathcal{F}}_{\text{phys}}(\lambda)$. Applying $L_0 \otimes 1$ we see that $L_0 \psi_i = a$ and applying $1 \otimes \tilde{L}_0$ we see that $\tilde{L}_0 \tilde{\psi}_i = a$. This implies that $N\psi_i = \tilde{N}\tilde{\psi}_i$. Hence

$$\chi = \epsilon^{\mu_1, \ldots, \mu_n; \nu_1, \ldots, \nu_m}_{k_1, \ldots, k_n; s_1, \ldots, s_m} \alpha^{\mu_1}_{-k_1} \ldots \alpha^{\mu_n}_{-k_n} \tilde{\alpha}^{\nu_1}_{-s_1} \ldots \tilde{\alpha}^{\nu_n}_{-s_n} |\lambda\rangle,$$

where $k_1 + \ldots + k_n = s_1 + \ldots + s_m$. Let us look at the physical states in $\mathcal{F}^{\text{closed}}_{\text{phys}}(\lambda)$ of level 1, i.e. $N\psi = \tilde{N}\psi = \psi$. They are of the form

$$\psi = \epsilon_{\mu\nu} \alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1} |\lambda\rangle.$$

We have

$$L_1 \psi = \alpha^\mu_0 \alpha_{1\mu} (\epsilon_{\mu\nu} \alpha^\mu_{-1} \tilde{\alpha}^\mu_{-1} |\lambda\rangle) = \epsilon_{\mu\nu} \lambda^\mu \tilde{\alpha}^\nu_{-1} |\lambda\rangle, \quad L_m \psi = 0, m > 1. \tag{7.5}$$

Similarly,

$$\tilde{L}_1 \psi = \epsilon_{\mu\nu} \lambda^\nu \alpha^\mu_{-1} |\lambda\rangle, \quad \tilde{L}_m \psi = 0, m > 1. \tag{7.6}$$

Also $L_0 \psi = \tilde{L} \psi = (\frac{1}{2}|\lambda|^2 + 1)\psi$, so that

$$|\lambda|^2 = 2a - 2. \tag{7.7}$$

In view of (7.5) and (7.6), we get

$$\epsilon_{\mu\nu} \lambda^\mu = \epsilon_{\mu\nu} \lambda^\nu = 0.$$

The norm of the state ψ is equal to

$$\mathcal{N} = \epsilon_{\mu\nu} \epsilon_{ij} \eta^{i\mu} \eta^{j\nu} = \epsilon_{\mu\nu} \epsilon^{\mu\nu}.$$

If $\lambda = 0$, we have no restriction on $\epsilon_{\mu\nu}$ and hence we have physical states of negative norm. So $\lambda \neq 0$. If $a < 1$, we may assume that $\lambda = (1, 0, \ldots, 0)$ so that $\epsilon_{00} = 0$ guarantees that ψ is physical. But if we take $\epsilon_{ij} = 0$ for $i, j \neq 0$, we get a state of negative norm, a ghost. So

$$a = 1.$$

If $\lambda \neq 0$, we may assume that $\lambda = (1, -1, 0, \ldots, 0)$, and then the condition is $\epsilon_{i0} = \epsilon_{i1}, \epsilon_{0i} = \epsilon_{1i}, i = 0, \ldots, D-1$ so that the norm is equal to $\sum_{\mu,\nu \geq 2} \epsilon^2_{\mu\nu}$. We see that all physical states are of nonnegative norm. The states of zero norm satisfy $\epsilon_{\mu\nu} = 0$ if $\mu, \nu \geq 2$.

Note that the states $L_{-1}\epsilon_\nu \tilde{\alpha}^\nu_{-1}|\lambda\rangle = \lambda_\mu \epsilon_\nu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1}|\lambda\rangle$ and $\tilde{L}_{-1}\eta_\mu \alpha^\mu_{-1}|\lambda\rangle = \lambda_\nu \epsilon_\mu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1}|\lambda\rangle$ are spurious. Since $|\lambda|^2 = 2a - 2 = 0$, these states are also physical because $|\lambda|^2 \sum_\mu \epsilon_\mu = |\lambda|^2 \sum_\mu \eta_\mu = 0$. It is easy to see now that any physical state of norm 0 is spurious and we can factor it out. Thus we obtain that for any non-zero light-like λ the space $\mathcal{F}^{\text{closed}}_{\text{phys}}(\lambda)$ is of dimension $D^2 - 2D - 2D = D(D-4)$ and all its elements can be represented by physical states of positive norm.

For any λ of norm 0, the space of solutions $\epsilon = (\epsilon_{\mu\nu})$ of (7.5) and (7.6) is the direct sum of one-dimensional space of matrices with nonzero trace, the $\frac{1}{2}D(D-1) -$ 1-dimensional space of trace-less symmetric matrices and $\frac{1}{2}D(D-3)$-dimensional space of antisymmetric matrices. The corresponding physical states are called *dilatons*, *gravitons* and *anti-symmetric tensors*. These are massless particles (i.e. $M^2\psi = 0$).

Let us go to the second level, i.e. consider the physical states in $\mathcal{F}_{\text{phys}}(\lambda)$ of the form

$$\psi = \epsilon_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|\lambda\rangle + \eta_\mu \alpha^\mu_{-2}|\lambda\rangle,$$

where $\epsilon_{ij} = \epsilon_{ji}$. We have

$$L_1 \cdot \psi = (\alpha^\mu_0 \alpha_{1\mu} + \alpha^\mu_{-1}\alpha_{2\mu})\psi = 2(\epsilon_{\mu\nu}\lambda^\nu + \eta_\mu)\alpha^\mu_{-1}|\lambda\rangle = 0,$$

$$L_2 \cdot |\epsilon, \lambda\rangle = (\alpha^\mu_0 \alpha_{2\mu} + \frac{1}{2}\alpha^\mu_1 \alpha_{1\mu})\psi = (\epsilon^\mu_\mu + 2\lambda_\mu \eta^\mu)\alpha^\mu_{-1}|\lambda\rangle = 0,$$

$$L_0\psi = \frac{1}{2}|\lambda|^2\psi + 2 = (\frac{1}{2}|\lambda|^2 + 2)\psi.$$

This implies

$$\epsilon_{\mu\nu}\lambda^\nu + \eta_\mu = 0, \quad \epsilon^\mu_\mu + 2\lambda_\mu \eta^\mu = 0, \tag{7.8}$$

$$|\lambda|^2 = 2a - 4 = -2.$$

The norm of this state is equal to $2\mathcal{N}$, where

$$\mathcal{N} = \epsilon_{\mu\nu}\epsilon^{\mu\nu} + \eta_\mu \eta^\mu = \epsilon^2_{00} - 2\sum_{\nu>0}\epsilon^2_{0\nu} + \sum_{\mu,\nu>0}\epsilon^2_{\mu\nu} - \eta^2_0 + \sum_{\mu>0}\eta^2_\mu.$$

Choose a system of coordinates in $\mathbb{R}^D$ such that $\lambda = (c, 0, \ldots, 0), c^2 = 2$. Using (7.8) we can eliminate η_i's and ϵ_{00} so that

$$\mathcal{N} = -\frac{1}{25}\Big(\sum_{i=1}^{D-1}\epsilon_{ii}\Big)^2 + \sum_{i,j=1}^{D-1}\epsilon^2_{ij}. \tag{7.9}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\mathcal{N} \geq \sum_{i,j=1}^{25} \epsilon_{ij}^2 - \frac{D-1}{25} \sum_{i=1}^{D-1} \epsilon_{ii}^2 = \sum_{i,j=1,i\neq j}^{25} \epsilon_{ij}^2 - \frac{D-26}{25} \sum_{i=1}^{D-1} \epsilon_{ii}^2. \tag{7.10}$$

Thus, if $D \leq 26$, all states are of non-negative norm. If $D > 26$, the state ψ with $\epsilon_{ii} = 1, i \geq 1$ and $\epsilon_{ij} = 0, i, j \neq 0, i \neq j$ has the norm equal to $(D-1)(-D+26)/25 < 0$. Now if we take a physical state $\tilde{\psi}$ from $\tilde{\mathcal{F}}_{\text{phys}}(\lambda)$ of level 2 and of positive norm, then $\psi \otimes \tilde{\psi}$ is an element of $\mathcal{F}_{\text{phys}}^{\text{closed}}(\lambda)$ of negative norm. So we have ghosts. If $D = 26$, a state of zero norm must satisfy $\epsilon_{ij} = 0, i, j \geq 1, i \neq j$. The states

$$L_{-1}(\epsilon_\mu \alpha_{-1}^\mu ||\lambda\rangle) + cL_{-2}|\lambda\rangle = (\epsilon_\mu \lambda_\nu + \frac{c}{2}\delta_{\mu\nu})\alpha_{-1}^\mu \alpha_{-1}^\nu ||\lambda\rangle + c\lambda_\mu \alpha_{-2}^\mu |\lambda\rangle$$

are spurious. It is easy to see that any norm 0 physical state is equal to a physical spurious state. So we can factor them out and obtain only the space with only postive norms. Thus we have shown that $D \leq 26$.

The proof that $D = 26$ consists of analyzing states of the next level. We skip it. The result that $\mathcal{F}_{\text{phys}}^{\text{closed}}$ does not contain ghosts if and only if $a = 1, D = 26$ is called the *No Ghost Theorem*. It was proven by R. Brower, P. Goddard and C. Thorn.

Observe that $a = 1$ and $D = 26$ agrees with the definition of the Hamiltonian operator $H = L_0 + \tilde{L}_0 - \frac{D-2}{24}$ using the regularization of the sum $\sum_{n=1}^\infty n$. So physical states satisfy $H\psi = 0$.

Remark 7.2. One can show that one can choose a subspace in $\mathcal{F}_{\text{phys}}^{\text{closed}}$ invariant with respect to $\mathrm{SO}(24)$ such that its states represent all states of positive norm modulo spurious states. This is achieved by a "light-cone gauge" which consists of fixing the first and the last coordinate ϕ^μ of the string. The group $\mathrm{SO}(1,25)$ acts in the space in $\mathcal{F}^{\text{closed}}$ via its induced representation. Thus $\mathcal{F}_+^{\text{closed}}$ defines a linear representation of the group $SO(24)$. It also leaves the homogeneous parts invariant and hence defines a finite dimensional representation in each space of given level. Elements of this space which belong to an irreducible component are interpreted as *elementary particles*. For example, the anti-symmetric tensors of level 1 define the adjoint representation of $\mathrm{SO}(24)$. The dilatons define the trivial representation and gravitons define the standard representation of $\mathrm{SO}(4)$ on the space $S^2(\mathbb{R}^{24})$.

Exercises

7.1 Find physical states of level 2 in the Fock space of a closed bosonic string.

7.2 By analyzing physical states of level 3 in $\mathcal{F}(\lambda)$ finish the proof of the No Ghost Theorem.

Lecture 8

BRST-cohomology

We shall discuss another approach to defining physical states which is called the *BRST-quantization*. In this approach one introduces an operator Q in a Fock space $\mathcal{F}$ of a given string theory such that $Q^2 = 0$ and

$$\mathcal{F}^{\text{phys}} = \text{Ker}(Q)/\text{Im}(Q) \tag{8.1}$$

We shall start with reminding the definition of the cohomology group of a Lie algebra $\mathfrak{g}$ with coefficients in its linear representation V. Let $\mathcal{F} = \Lambda(\mathfrak{g}^*) \otimes V$ be the tensor product of the exterior algebra of the space $\mathfrak{g}^*$ and the space V. This will be an analog of our Fock space. If $\mathfrak{g} = \text{Lie}(G)$ for some Lie group G, then $\mathcal{F}$ can be identified with the space Ω_L^{inv} of left-invariant smooth differential forms on G with values in the trivial vector bundle defined by the space V. Let (e_i) be a basis of $\mathfrak{g}$ and (f_i) be the dual basis of $\mathfrak{g}^*$. Let $C^n(\mathfrak{g}, V) = \Lambda^n(\mathfrak{g}^*) \otimes V$ so that $\mathcal{F} = \oplus_{n=0}^{\infty} C^n(\mathfrak{g}, V)$. Elements of $C^n(\mathfrak{g}, V)$ are linear combinations of the decomposable tensors

$$f_{i_1} \wedge \ldots \wedge f_{i_n} \otimes v, \quad i_1 < \ldots < i_k, \quad v \in V$$

Let us assign to f_i the operator $a^i = f_i \wedge$ and to e^i the contraction operator $a_i = \iota_{e^i}$ such that $\iota_{e^i}(f_{i_1} \wedge \ldots \wedge f_{i_n} \otimes v) = (-1)^{k+1} f_{i_1} \wedge \ldots \hat{f_i} \ldots \wedge f_{i_n} \otimes v$ if $i = i_k$ for some k and 0 otherwise. If $\mathfrak{g} = \text{Lie}(G)$ the operator a^i is the exterior product with the differential dx_i and $a_i = \frac{\partial}{\partial x_i}$. Here $x_1, \ldots, x_{\dim \mathfrak{g}}$ are local coordinates on G corresponding to the basis (b_i). We have

$$\{a^i, a_j\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a^i, a^j\} = 0.$$

Here, for any associative algebra A and $x, y \in A$,

$$\{x, y\} = xy + yx$$

It is called the *anti-commutator* or the *Poisson bracket*.

The structure of $\mathfrak{g}$ is determined by the constants c_{ij}^k such that

$$[e_i, e_k] = c_{ij}^k e_k.$$

Let $K_i = \rho(e_i)$. Define the BRST-operator in $\mathcal{F}$ by

$$Q = \sum_i a^i \otimes K_i - \frac{1}{2}\sum_{i,j,k} c_{ij}^k a^i a^j a_k \otimes 1 \in \mathrm{End}(\mathcal{F} \otimes V).$$

Lemma 8.1.

$$Q^2 = 0$$

Proof. Let $A_1 = a^i \otimes K_i$, $A_2 = c_{ij}^k a^i a^j a_k \otimes 1$, where we skip the summation sign. We have

$$A_1^2 = a^i a^j \otimes K_i K_j + a^j a^i \otimes K_j K_i =$$

$$\sum_{i<j} a^i a^j \otimes (K_i K_j - K_j K_i) = \sum_{i<j} a^i a^j \otimes c_{ij}^s K_s.$$

Also

$$A_1 A_2 + A_2 A_1 = -\frac{1}{2}(a^i a^j a^k a_t + a^j a^k a_t a^i) \otimes c_{jk}^t K_i.$$

Using the anti-commutator relations we see that $a^i a^j a^k a_t + a^j a^k a_t a^i = 0$ unless $t = i$. In the latter case $a^i a^j a^k a_i + a^j a^k a_i a^i = a^j a^k (a^i a_i + a_i a^i) = a^j a^k$ so that

$$A_1 A_2 + A_2 A_1 = -\frac{1}{2}(c_{jk}^i a^j a^k + c_{kj} a^k a^j) \otimes K_i = -\sum_{i<j} a^j a^k \otimes c_{jk}^i K_i.$$

Here we used that $c_{ij}^k = -c_{ji}^k$ and $a^j a^k = -a^k a^j$. This shows that $A_1^2 + A_1 A_2 + A_2 A_1 = 0$. It remains to show that $A_2^2 = 0$. We have

$$4A_2^2 = f_{ij}^k f_{st}^m a^i a^j a_k a^s a^t a_m \otimes 1 = f_{ij}^k f_{st}^m (a^i a^j a_k a^s a^t a_m + a^s a^t a_m a^i a^j a_k) \otimes 1.$$

If $k \neq s, t, m \neq i, j$, the expression in the bracket is equal to zero. So

$$4A_2^2 = f_{ij}^k f_{kt}^m (a^i a^j a_k a^k a^t a_m + a^k a^t a_m a^i a^j a_k) \otimes 1+$$

$$f_{ij}^k f_{sk}^m (a^i a^j a_k a^s a^k a_m + a^s a^k a_m a^i a^j a_k) \otimes 1+$$

$$+ f_{mj}^k f_{st}^m (a^m a^j a_k a^s a^t a_m + a^s a^t a_m a^m a^j a_k) \otimes 1+$$

$$f_{im}^k f_{st}^m (a^i a^m a_k a^s a^t a_m + a^s a^t a_m a^i a^m a_k) \otimes 1.$$

Using the Jacoby identity

$$f_{ij}^m f_{mk}^l + f_{jk}^m f_{mi}^l + f_{ki}^m f_{mj}^l = 0,$$

it is easy to see that each of the four sums is equal to zero. $\square$

Applying the previous lemma we can define the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in V as follows:

$$H^*(\mathfrak{g}, V) = \mathrm{Ker}(Q)/\mathrm{Im}(Q).$$

Example 8.1. Let $\mathfrak{g}$ be an abelian Lie algebra of dimension n. Its linear representation is a module over $\mathfrak{U}(\mathfrak{g}) \cong \mathbb{C}[e_1, \dots, e_n]$. Let $V = C^\infty(\mathbb{R}^n)$ with the action of $\mathfrak{g}$ defined by $\rho(e_i) = K_i : f \to \frac{\partial f}{\partial t_i}$. Then $C^n(\mathfrak{g}, V)$ can be identified with the space Ω^n of smooth differential forms of degree n

$$\omega = f_{i_1 \dots i_n}(x) dx_{i_1} \wedge \dots \wedge dx_{i_n}.$$

The BRST-operator

$$Q = \sum_i a^i \otimes K_i$$

coincides with the exterior derivative d. We know that $d^2 = 0$ and

$$H^n(\mathfrak{g}, V) = H^n_{\mathrm{DR}}(\mathbb{R}^n) = 0, n > 0.$$

Example 8.2. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Assume that G is a complex semi-simple group and let $V = \mathbb{C}$ be its trivial representation. Then

$$H^n(\mathfrak{g}, \mathbb{C}) \cong H^n(G, \mathbb{C}),$$

where G is considered as a smooth manifold.

In our situation we want to take for $\mathfrak{g}$ the Virasoro algebra Vir and V its representation in the Fock space $\mathcal{F}$ of bosonic string. The space $\Lambda(\mathfrak{g}^*)$ is called the space of *ghost fields*.

We will be dealing with a version of the BRST complex which uses the *semi-infinite* cohomology. Instead of differential forms $f_{i_0} \wedge \dots \wedge f_{i_k}$ we consider semi-infinite forms. Let $I = (i_0, i_1, \dots)$ be any strictly decreasing sequence of integers such that the sets $I \cap \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{<0} \setminus (I \cap \mathbb{Z}_{<0})$ are finite. A *semi-infinite form* is a formal expression of the form

$$\psi_I = f_{i_0} \wedge f_{i_1} \wedge \dots .$$

The number $N = \#I \cap \mathbb{Z}_{>0} - \#\mathbb{Z}_{\leq 0} \setminus (I \cap \mathbb{Z}_{\leq 0})$ is called the *degree* of ψ_I.

Let

$$\psi_N = \psi_I,$$

where $I = (N, N-1, \dots)$. Its degree is equal to N. We extend the operators a^k and a_k to semi-definite forms in the obvious manner. Note that any form of degree N can be obtained from ψ_N by applying operators $a^k a_n$. Also observe that

$$a_k \psi = 0, \quad k > \deg \psi,$$

$$a^k\psi = 0, \quad k < -\deg\psi.$$

Let Vir be the abstract Virasoro algebra with generators l_n. We want to construct the representaion of Vir on the space of semi-infinite forms $\Omega^{\infty/2}(N)$ of degree N. Let ad$'$ be the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. It is defined by $\mathrm{ad}'(x)(f)(y) = f([x,y])$. Let us identify (f_i) with the dual basis (l_i) of Vir. We have

$$\mathrm{ad}'(l_n)(f_i)(l_m) = f_i([l_m, l_n]) = f_i((m-n)l_{n+m}) =$$

$$(m-n)f_i(l_{n+m}) = (m-n)\delta_{i,n+m} = (m-n)\delta_{m,i-n}.$$

This shows that

$$\mathrm{ad}'(l_n)(f_i) = (i-2n)f_{i-n} \tag{8.2}$$

If $n \neq 0$, we can set

$$\rho(l_n)f_{i_0} \wedge f_{i_1} \wedge \ldots = \sum_{k\geq 0} f_{i_0} \wedge \ldots \wedge \rho(l_n)(f_{i_k}) \wedge \ldots =$$

$$\sum_{k\geq 0} f_{i_0} \wedge \ldots \wedge f_{i_{k-1}} \wedge (i-2n)(f_{i_k-n}) \wedge f_{i_{k+1}} \wedge \ldots .$$

Observe that the sum is finite because $n \neq 0$. However it is not defined for $n = 0$. We easily check that, for $n, m, n+m \neq 0$,

$$[\rho(l_n), \rho(l_m)] = (n-m)\rho(l_{n+m}).$$

So our problem is to define $\rho(l_0)$ such that all Virasoro commutators work. Next observe that

$$\rho(l_n) = \sum_{i\in\mathbb{Z}}(i-2n)a^{i-n}a_i = \sum_{k\in\mathbb{Z}}(k-n)a^k a_{n+k} = \sum_{k\in\mathbb{Z}}(n-k)a_{n+k}a^k.$$

We use this formula to set

$$\rho(l_0) = \sum_{k\in\mathbb{Z}} k : a^k a_k :$$

Here $:\Phi_1 \ldots \Phi_k:$ denotes the *normal order* of a composition of operators defined by putting on the right the operator annihilating the vector ψ_N and inserting the sign of the permutation which has been made. If no such operators occurs among the factors we do nothing. Note that $-a_k a^k = a^k a_k - 1$ so changing $a^k a_k$ to $-a_k a^k$ differs from the usual product. It is easy to see now that each $\rho(l_n)$ is well-defined.

Let us compute $[\rho(l_n), \rho(l_m)]$. We have

$$[\rho(l_n), \rho(l_m)] = (n-i)(m-j)[a_{n+i}a^i, a_{m+j}a^j].$$

Assume $n, m \neq 0$, $i \neq m+j, j \neq n+i$. Then, its is easy to see that $[: a_{n+i}a^i :, : a_{m+j}a^j :] = 0$. Assume $n, m \neq 0$, $i = m+j, j \neq n+i$. Then

$$[: a_{n+i}a^i :, : a_{m+j}a^j :] = a_{n+i}a^i a_i a^j - a_i a^j a_{n+i}a^i =$$

$$(a^i a_i + a_i a^i)a_{n+i}a^j = a_{n+i}a^j.$$

Similarly we get

$$[: a_{n+i}a^i :, : a_{m+j}a^j :] = -a_{m+j}a^i$$

if $n, m \neq 0, j = n+i, i \neq m+j$. Note that $j = n+i, i = m+j$ implies $m = -n$. Thus, if $m \neq -n$, we get

$$[\rho(l_n), \rho(l_m)] = \sum_j (n-m-j)(m-j)a_{n+m+j}a^j -$$

$$\sum_i (m-n-i)a_{m+n+i}a^i = (n-m)a_{n+m+j}a^j = (n-m)\rho(l_{m+n}).$$

Assume $n, m \neq 0$ and $n = -m > 0$. Then

$$[\rho(l_n), \rho(l_{-n})] = (2n-j)(-n-j)([: a_j a^{-n+j} :, : a_{-n+j}a^j :].$$

Now, for any i, j such that $i > N, j \leq N$ we have

$$[a_i a^j, a_j a^j] = a_i(1 - a_j a^j)a^i - a_j(1 - a_i a^i)a^j = a_i a^i - a_j a^j = 1 + : a_i a^i : - : a_j a^j :.$$

Similarly, if $i \leq N, j > N$, we have

$$[a_i a^j, a_j a^j] = -1 + : a_i a^i : - : a_j a^j :.$$

Now

$$[\rho(l_n), \rho(l_m)] = \sum_{N < j \leq n+N} (2n-j)(-n-j)(1 + : a_j a^j : - : a^{-n+j}a^{-n+j} :=$$

$$2n\rho(l_0) - \frac{13n^3}{6} - (N^2 - N + \frac{n}{6}).$$

Finally

$$[\rho(l_n), \rho(l_m)] = (n-m)\rho(l_{n+m}) + -\frac{13}{6}n^3 + (N^2 + N + \frac{n}{6})\delta_{n,-m}. \tag{8.3}$$

If we fix the vacuum state ψ_1 (i.e. take $N = 1$) the central charge is equal to $\frac{26}{12}$. Also, $\rho(l_0)|\psi_{-1} = \psi_{-1}$. Recall that the representation of Vir in $\mathcal{F}$ has the charge $\frac{D}{12}$ and the vacuum vector $|\lambda\rangle$ is the eigenvector of L_0 with eigenvalue $a = 1$.

We define the *BRST operator* (Bechi-Rouet-Stora-Tyutin) on $\Omega^{\infty/2}(N) \otimes \mathcal{F}(\lambda)$ by

$$Q = a^n \otimes L_n - \frac{1}{2}(n-m) : a^n a^m a_{n+m} : \otimes 1.$$

Theorem 8.1. *If $D = 26$, then $Q^2 = 0$.*

Proof. Let $\rho(l_n) = M_n$. We have

$$A_2^2 = \frac{1}{4}\sum_{k<n}(a^n M_n a^k M_k + a^k M_k a^n M_n).$$

We use that

$$M_n a^k = (n-m)a^m a_{n+m} a^k = a^k M_n - (2n-k)a^{k-n}.$$

Using this we get

$$a^n M_n a^k M_k + a^k M_k a^n M_n = a^n a^k [M_n, M_k] - (2n-k)a^n a^{k-n} M_k.$$

Changing the index k to $k+n$ in the second sum, and applying (8.3), we get,

$$a^n M_n a^k M_k + a^k M_k a^n M_n = a^n a^k([M_n, M_k] - (n-k)M_{n+k}) =$$

$$\frac{26}{12}(-n^3 + n)a^n a^{-n}.$$

On the other hand,

$$A_1^2 = a^n \otimes L_n \circ a^m \otimes L_m - \frac{1}{2}a^m a^m \otimes [L_n, L_m] =$$

$$\frac{1}{2}(n-m)a^n a^m \otimes L_{n+m} + \frac{D}{24}(n^3 - n)a^n a_{-n}.$$

Finally,

$$A_1 A_2 + A_2 A_1 = -\frac{k-m}{2}(a^n : a^k a^m a_{k+m} : \otimes L_n + : a^k a^m a_{k+m} : a^n \otimes L_n +$$

$$\frac{k-m}{2}(a^n : a^m a^k a_{k+m} : \otimes L_n + : a^m a^k a_{k+m} : a^n \otimes L_n = -\frac{k-m}{2} \otimes L_{k+m}.$$

So $(A_1 + A_2)^2 = 0$ if $D = 26$. ☐

Let $\Lambda^{\infty/2}(n)$ be the linear space of semi-infinite forms of degree n. It is clear that Q maps $\Lambda^{\infty/2}(n) \otimes \mathcal{F}$ to $\Lambda^{\infty/2}(n+1) \otimes \mathcal{F}$. Let $H^n(\mathrm{Vir}; \mathcal{F}) = \mathrm{Ker}(Q_n)/\mathrm{Im}\,(Q_{n-1})$, where $Q_n = Q|\Lambda^{\infty/2}(n) \otimes \mathcal{F}$. We set $H^n_{\mathrm{rel}}(\mathrm{Vir}; \mathcal{F})_0$ be the subspace of $H^n(\mathrm{Vir}; \mathcal{F})$ generated by the cosets of forms which do not contain f_0 and which are annihilated by $\rho(l_0) \otimes 1 + 1 \otimes L_0$.

For any graded vector space $V = \oplus_{n=0}^{\infty} V_n$ with $\dim V_n < \infty$ we set

$$\mathrm{char}(V)(q) = \sum_{n=0}^{\infty} \dim V_n q^n.$$

If $T : V \to V$ such that $V_n = \{v \in V : T(v) = nv\}$, then $\dim V_n = \mathrm{Tr}(T|V_n)$ and

$$\mathrm{char}(V)(q) = \mathrm{Tr}(q^T).$$

We shall apply this to the case when $V = \mathcal{F}(\lambda)$ and $T = L_0$. Recall that

$$L_0 a_{-n_1}^{\mu_1} \dots a_{-n_k}^{\mu_k}|\lambda\rangle = \frac{1}{2}|\lambda|^2 + (n_1 + \dots + n_k).$$

So it is easy to see that

$$\mathrm{char}(\mathcal{F}(\lambda)) := \mathrm{Tr}(q^{L_0}) = q^{\frac{1}{2}|\lambda|^2}\phi(q)^{-D}.$$

where

$$\phi(q) = \prod_{k>0}(1 - q^k).$$

We have already noticed that the representation of Vir in $\mathcal{F}(\lambda)$ is reducible. Let us try to decompose it into irreducible modules. First we write

$$\mathcal{F}(\lambda) = \mathcal{F}(\lambda') \otimes \mathcal{F}(\lambda''),$$

where $\lambda = (\lambda_1, \dots, \lambda_{24}), \lambda'' = \lambda_0$. Let us assume that $\lambda' \neq 0, \lambda'' \neq 0$. Let $M(h,c)$ denote the *Verma module* for the representation of Vir with central charge c and character h (see the previous Lecture). The Verma module $M(h,c)$ has the universal property with respect to all representations of Vir with central charge c and character h. Any such representations is a quotient of the Verma module. One can show that $M(h,c)$ is spanned by the elements $L_{-n_1} \dots L_{-n_k}|0\rangle$, $n_i > 0$. The grading of $M(h,c)$ is defined by taking $M(h,c)_n$ to be the subspace spanned by the monomials as above with $n_1 + \dots + n_k = n$. We have

$$\mathrm{char}(M(h,c)) = \phi(q)^{-1}.$$

It is known that the Verma module $M(h,1)$ is irreducible if $h < 0$ and irreducible and unitary for $c > 1, h > 1$. Considering $\mathcal{F}(\lambda'')$ as a representation of Vir with character $h = -\frac{1}{2}\lambda''^2$ and central charge $c = 1$. Comparing the characters, we find that

$$\mathcal{F}(\lambda'') \cong M(-\frac{1}{2}|\lambda''|^2, 1).$$

The charge a of the representation $\mathcal{F}(\lambda')$ is equal to 25 and the character is equal to $\frac{1}{2}|\lambda'|^2$. We have

$$\mathrm{char}(\mathcal{F}(\lambda')) = q^{\frac{1}{2}|\lambda'|^2}\phi^{-25}.$$

The character of the irreducible module $M(\frac{1}{2}|\lambda'|^2 + k, 25)$ is equal to $q^{\frac{1}{2}|\lambda'|^2 + k}\phi(q)^{-1}$. This shows that

$$\mathrm{char}(\mathcal{F}(\lambda')) = \sum_{k \geq 0} p_{(24)}(k) M(\frac{1}{2}|\lambda'|^2 + k, 25),$$

where

$$\sum_n p_{(d)}(n)q^n = \phi(q)^{-d}.$$

We conclude that

$$\mathcal{F}(\lambda) = M(-\frac{1}{2}|\lambda''|^2, 1) \otimes \sum_{k\geq 0} p_{(24)}(k)M(\frac{1}{2}|\lambda'|^2 + k, 25).$$

Let v_0 be the vacuum vector of $M(-\frac{1}{2}|\lambda''|^2, 1)$. Set

$$T = \oplus_{k\geq 0}p_{(24)}(k)\{u \otimes v_0 : u \in M(\frac{1}{2}|\lambda'|^2 + k, 25), L_n(u) = 0, n > 0.$$

We have the following result due to I. Frenkel, Garland, and Zuckerman:

Theorem 8.2. *Assume $\lambda \neq 0$. Then $H^n_{rel}(Vir, \mathcal{F}(\lambda)) = 0$ for $n \neq 0$ and*

$$\dim H^0_{rel}(Vir, \mathcal{F}(\lambda)) = p_{(24)}(1 - \frac{1}{2}|\lambda|^2)$$

if $1 - \frac{1}{2}|\lambda|^2$ is an integer and zero otherwise.

Define the map $s : \mathcal{F}_{\text{phys}}(\lambda) \to C^0(\text{Vir}, \mathcal{F}(\lambda)$ by $s(v) = \psi_{-1} \otimes v$, where $\psi_{-1} = f_{-1} \wedge f_{-2} \wedge \ldots$. We have $L_n(v) = 0, n > 0$, $a^n(\psi_{-1}) = 0, n < 0$, $a_{n+m}\psi_{-1} = 0, n + m \geq 0$. Thus $: a^n a^m a_{n+m} : \psi_{-1} = 0$ unless $n + m \leq 0$ and n or $m \geq 0$. Assume $n \geq 0$. Then $m \leq 0$. If $m < 0, : a^n a^m a_{n+m} : \psi_{-1} = -a_{n+m}a^n a^m \psi_{-1} = 0$. So, $m = n = 0$ and $: a^0 a^0 a_0 : \psi_{-1} = a_0(-a_0 a^0 + 1)\psi_{-1} = a_0\psi_{-1}$. Therefore, we obtain

$$Q(\psi_{-1} \otimes v) = a_0\psi_{-1} \otimes L_0 v - 2\frac{1}{2} : a^0 a^0 a_0 : \psi_{-1} \otimes v = a_0\psi \otimes (Lv - v) = 0.$$

This defines a map from $\mathcal{F}_{\text{phys}} \to H^0_{\text{rel}}(\text{Vir}, \mathcal{F}(\lambda))$. If $s(v) \in \text{Im}\,(Q)$, then $v \in \mathcal{F}^{\text{spur}} \cap \mathcal{F}_{\text{phys}}$ and we get an injective map

$$\mathcal{H}_{\text{phys}} \cong H^0_{\text{rel}}(\text{Vir}, \mathcal{F}(\lambda)).$$

On the other hand, $\dim \mathcal{F}_{\text{phys}} \cap T = p_{(24)}(k)(1 - \frac{1}{2}|\lambda|^2)$ and $T \cap \text{rad}(\mathcal{F}_{\text{phys}}) = 0$. Thus $\mathcal{F}_{\text{phys}} \cap T$ is mapped isomorphically to $H^0_{\text{rel}}(\text{Vir}, \mathcal{F}(\lambda))$ and hence $\mathcal{H}_{\text{phys}} \cong H^0_{\text{rel}}(\text{Vir}, \mathcal{F}(\lambda))$.

Exercises

8.1 Show that the equivalent definition of the cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in a linear representation $\rho : \mathfrak{g} \to V$ can be given as follows. Let $C^n(\mathfrak{g}; V)$ be the space of anti-symmetric n-multilinear maps from $\mathfrak{g}$ with coefficients in V. Define the coboundary map $\delta : C^n(\mathfrak{g}; V) \to C^n(\mathfrak{g}; V)$ by the formula

$$(\delta f)(x_1, \ldots, x_{n+1}) = \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{n+1}) +$$

$$\sum_{i=1}^{n+1}(-1)^{i+1}\rho(x_i)(f(x_1,\ldots,\hat{x_i},\ldots,x_{n+1})).$$

Check that $d^2=0$ and set $H^n(\mathfrak{g};V)=\mathrm{Ker}(d|C^n(\mathfrak{g};V))/\mathrm{Im}(d|C^{n-1}(\mathfrak{g};V))$.

8.2 Consider the trivial representaion of $\mathfrak{g}$ in a vector space V. Show that $H^0(\mathfrak{g};V)=V$, $H^1(\mathfrak{g};V)=Hom(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}],V)$.

8.3 A central extension of a Lie algebra $\mathfrak{g}$ with help of a vector space V is a Lie algebra $\mathfrak{g}'$ containing V as a central abelian subalgebra such that $\mathfrak{g}'/V\cong\mathfrak{g}$. Show that such central extensioncs can be classified by the space $H^2(\mathfrak{g};V)$, where $\mathfrak{g}$ acts trivially on V.

8.4 Prove that $H^2(\mathrm{Vir};\mathbb{R})\cong\mathbb{R}$.

8.5 Let Q be the BRST-operator defined for the Virasoro algebra with coefficients in a representation $\mathcal{F}(\lambda)$, $\lambda\in\mathbb{R}^D$. Show that the exists a constant a such that $(Q+a)^2=0$.