Homotopy theory and classifying spaces

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Contents

Lecture 1.

Homotopy theories and model categories

These are **rough notes.** Read at your own risk! The presentation is not necessarily linear, complete, compact, locally connected, orthographically defensible, grammatical, or, least of all, logically watertight. The author doesn't always tell the whole truth, sometimes even on purpose.

This first lecture is deep background: before getting to classifying spaces, I'd like to describe some homotopy theoretic machinery. The first question to ask before trying to understand this machinery is a *very* basic one.

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What is a homotopy theory $\mathbb{T}$?

Equivalent answers

- $\mathbb{T}_{\text{pair}}$ category $\mathcal{C}$ with a subcategory $\mathcal{E}$ of *equivalences*
- $\mathbb{T}_{\text{en}}$ category $\mathcal{R}$ enriched over spaces (simplicial sets)
- $\mathbb{T}_{\text{sc}}$ Segal category
- $\mathbb{T}_{\text{css}}$ complete Segal space
- $\mathbb{T}_{\text{qc}}$ quasi-category
- $\mathbb{T}_{\infty}$ ∞-category (or $(\infty, 1)$ category)

Just like categories

Internal function objects $\mathbb{T}_3 = \mathrm{Cat}^h(\mathbb{T}_1, \mathbb{T}_2) = \mathbb{T}_2^{\mathrm{h}\mathbb{T}_1}$

The main message here is that there has been a remarkable convergence of opinion over the last few years about what a homotopy theory is [11]. All formulations give notions which are equivalent (in a homotopy theoretic sense, see slide 1–10 below), although the objects involved look very different in detail.

A pair $(\mathcal{C}, \mathcal{E})$ is a relative category, and from the point of view of $\mathbb{T}_{\text{pair}}$ homotopy theory is relative category theory. This is the form under which homotopy theories usually show up in nature; $\mathcal{E}$ is usually some collection of morphisms in $\mathcal{C}$ which are not isomorphisms but have some claim to be considered honorary isomorphisms (for instance if $\mathcal{C} = \mathbf{Top}$ is the category of topological spaces, $\mathcal{E}$ might be the collection of homotopy equivalences). But any category $\mathcal{C}$ gives a homotopy theory: take $\mathcal{E}$ to the identity maps or (it turns out equivalently) the isomorphisms in $\mathcal{C}$. Another possibility is to take $\mathcal{E} = \mathcal{C}$.

A category $\mathcal{R}$ enriched over topological spaces is an ordinary category furnished with a topology on each morphism space [9]. From the point of view of $\mathbb{T}_{\text{en}}$ homotopy theory is a continuous form of category theory. (Not *too* continuous: notice that we don't worry about topologies on sets of objects.) The transition $\mathbb{T}_{\text{pair}} \implies \mathbb{T}_{\text{en}}$ involves inverting the arrows in $\mathcal{E}$ in a derived sense [38]. Alternatively, the function spaces in the simplicial category can be view as spaces of zigzags in the original category $\mathcal{C}$, where the backwards-point arrows lie in $\mathcal{E}$ [36].

☡ A Segal category is a simplicial space which is discrete at level 0 and satisfies some homotopical product conditions [89] [10]. The transition $\mathbb{T}_{\text{en}} \implies \mathbb{T}_{\text{sc}}$ amounts to taking the nerve, and treating it as a simplicial space.

☡ A complete Segal space is simplicial space which is not necessarily discrete at level 0 and satisfies some homotopy fibre product conditions and some other homotopical conditions [89] [11]. The transition $\mathbb{T}_{\text{sc}} \implies \mathbb{T}_{\text{css}}$ requires repackaging group-like topological monoids of equivalences into their associated classifying spaces, but leaving the non-invertible morphisms alone. This is an unusually transparent model for a homotopy theory: an equivalence is just a map between simplicial spaces which is a weak equivalence at each level. Internal function objects are also easy to come by here.

☡ A quasi-category (∞-category) is a simplicial set, treated from what classically would be a very peculiar point of view [62] [63]. In some sense this is the most economical model for a homotopy theory.

☡☡ **1.1 Exercise.** Let $\mathcal{C}$ be a category and $\mathcal{E}$ its subcategory of isomorphisms. In this particular case, describe in detail each of the various models for the homotopy theory of $\mathbb{T}(\mathcal{C}, \mathcal{E})$.

The notation $\mathrm{Cat}^h(\mathbb{T}_1, \mathbb{T}_2)$ or $\mathbb{T}_2^{\mathrm{h}\mathbb{T}_1}$ denotes the homotopy theory of functors from the first homotopy theory to the second, but taken in the correct homotopy theoretic way. The notation $\mathbb{T}_2^{\mathrm{h}\mathbb{T}_1}$ is very similar to a notation for homotopy fixed point sets that will come up later on (2.28), but I'll use it anyway. This same ambiguity comes up without the "h": if X and Y are spaces, Y^X is the space of maps from X to Y, but if Y is a space and G is a group Y^G is the fixed-point set of the action of G on Y.

The above internal function objects for homotopy theories are tricky to define correctly in some models, but can have a very familiar feel to them. If $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{C}', \mathcal{E}')$ are two category pairs in which all of the morphisms in $\mathcal{E}$ and $\mathcal{E}'$ are invertible, then the function object $\mathrm{Cat}^h(\mathbb{T}(\mathcal{C}, \mathcal{E}), \mathbb{T}(\mathcal{C}', \mathcal{E}'))$ is equivalent in the sense of homotopy theories to the category in which the objects are functors $\mathcal{C} \to \mathcal{C}'$ and the morphisms are natural transformations. (To promote this to a homotopy theory, pick natural isomorphisms between functors as equivalences.)

★ **1.2 Exercise.** Let G and H be two discrete groups, treated as one-object categories or as homotopy theories (the latter by designating all morphisms as equivalences). Describe the groupoid of functors $G \to H$ in terms of the group structures of G and H. How many components are there to the groupoid? What are the vertex groups?

☡ **1.3 Exercise.** Let $\mathcal{C}$ be a category, and let $\mathcal{E} = \mathcal{C}$. Let $\mathbf{Top}$ be the homotopy theory of topological spaces, where the equivalences are taken to be weak homotopy equivalences. What geometric structures do you think are described by the homotopy theory $\mathrm{Cat}^h(\mathbb{T}(\mathcal{C}, \mathcal{E}), \mathbf{Top})$?

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Examples of homotopy theories ($\mathbb{T}_{\text{pair}}$ presentation)

Geometry

Notation	$\mathcal{C}$	$\mathcal{E}$
$\mathbf{Top}_{\text{he}}$	$\mathbf{Top}$	homotopy equivalences
$\mathbf{Top}$	$\mathbf{Top}$	weak homotopy equivalences
$\mathbf{Top}_R$	$\mathbf{Top}$	R-homology isomorphisms
$\mathbf{Top}_{\leq n}$	$\mathbf{Top}$	iso on π_i for $i \leq n$

Algebra

$\mathbf{Sp}$	simplicial sets	$\lvert f\rvert$ an equivalence in $\mathbf{Top}$
$\mathbf{sGrp}$	simplicial groups	$\lvert f\rvert$ an equivalence in $\mathbf{Top}$
	simp. rings, Lie alg.,etc	(same)
$\mathbf{Ch}_R$	chain complexes over R	homology isomorphisms
	DG algebras	homology isomorphisms

The first four examples illustrate the fact that a category can be associated with many different homotopy theories. For instance, in $\mathbf{Top}_{\leq n}$ the $(n+1)$-sphere is equivalent to a point, but it $\mathbf{Top}$ it isn't.

1.4 Exercise. Give an example of a map which is an equivalence in $\mathbf{Top}$ but not in $\mathbf{Top}_{\text{he}}$.

Simplicial sets. Simplicial sets [49] [70] are slightly more complicated analogs of simplicial complexes. They have two advantages over simplicial complexes:

- Better colimit properties. In a simplicial complex X a simplex is determined by its set of vertices, so collapsing the vertices of X to a single point in the category of simplicial complexes causes X itself to collapse to a point. But it's hard not to want to collapse the two endpoints of a one-simplex together to get a circle. Simplicial sets possess monolithic simplices of various dimensions; these have vertices but are not determined by them. Collapsing is easy.
- Better limit properties. The relationship between the geometric realization of the (categorical) product of two simplicial complexes and the product of their realizations is obscure. (At best these two spaces have the same homotopy type). There is no such problem with simplicial sets.

There is a short discussion of how to get from simplicial complexes to simplicial sets in [43, §3]. Simplicial complexes are based on the category $\boldsymbol{\Delta}$, which can be described up to isomorphism in (at least) the following three ways. (Note that any (partially) ordered set gives an associated category, in which there is a unique morphism $x \to y$ if $x \leq y$.)

1. $\boldsymbol{\Delta}$ is the category whose objects are the ordered sets $\mathbf{n} = \{0, \ldots, n\}$, $n \geq 0$, and whose maps are the weakly order-preserving maps between these sets ("weakly"= preserves $\leq$).

2. $\mathbf{\Delta}$ is the category whose objects are the finite ordered simplicial complexes Δ_n, $n \geq 0$ and whose maps are the simplicial complex maps between these objects which are weakly order-preserving on the vertices. (Here Δ_n is the space of convex linear combinations of $\{0, \ldots, n\}$.)
3. $\mathbf{\Delta}$ is the category whose objects are the finite categories $\mathbf{n}$, $n \geq 0$, and whose morphisms are the functors between them.

A *simplicial set* X is a contravariant functor from $\mathbf{\Delta}$ to $\mathbf{Set}$, i.e., a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$; maps between simplicial sets are natural transformations of functors. For each $n \geq 0$, X has a set $X_n = X(\mathbf{n}) = X(\Delta_n)$ of n-simplices, and these sets are related by various face maps, degeneracy maps, and their composites. For example, there are two face (vertex) maps $X_1 \to X_0$, corresponding to the two vertex inclusions $\Delta_0 \to \Delta_1$, and one degeneracy map $X_0 \to X_1$ corresponding to the collapse $\Delta_1 \to \Delta_0$.

1.5 Exercise. Any topological space Y has an associated simplicial set $\mathrm{Sing}(Y)$, the singular complex of Y, given by $\mathrm{Sing}(Y)_n = \mathrm{Hom}(\Delta_n, Y)$. Given this and the usual construction of singular homology, how would you define the homology of a simplicial set?

The geometric realization functor for simplicial complexes extends to a geometric realization functor $|(-)|$ for simplicial sets; the realization functor is left adjoint to the singular complex functor (1.5). The realization of X can be obtained explicitly as $|X| = X \times_{\mathbf{\Delta}} \Delta_*$; the notation stands for the cartesian product over $\mathbf{\Delta}$ of the contravariant functor X with the covariant functor Δ_* (slide 2–2). Both of these are functors to $\mathbf{Top}$, where we think of X as taking values in discrete spaces. More formally, $|X|$ is the coend of the functor $(\mathbf{n}, \mathbf{m}) \mapsto X_n \times \Delta_m$ on $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}$.

1.6 Exercise. Draw an analogy between $F \times_{\mathcal{C}} G$, for ($F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Top}$, $G : \mathcal{C} \to \mathbf{Top}$), and $M \otimes_R N$, for (M a right R-module, N a left R-module). Check the formula

$$M \otimes_R N \cong R \otimes_{R^{\mathrm{op}} \otimes_{\mathbb{Z}} R} (M \otimes_{\mathbb{Z}} N)$$

(which seems to relate $\otimes_R$ to Hochschild homology). What's the corresponding formula if any for $F \times_{\mathcal{C}} G$?

★ **1.7 Exercise.** Observe that any category $\mathcal{C}$ has an associated simplicial set $\mathrm{N}(\mathcal{C})$, the nerve of $\mathcal{C}$, with $\mathrm{N}(\mathcal{C})_n = \mathrm{Hom}(\mathbf{n}, \mathcal{C})$ (here Hom denotes the set of functors). Try to determine the homotopy type of $|\mathrm{N}(\mathcal{C})|$ in some simple cases, e.g., if $\mathcal{C}$ is the pushout category (three objects a, b, c, and maps $b \to a$ and $b \to c$), or if $\mathcal{C}$ is the (co-)equalizer category (two objects a, b, and two distinct maps $a \to b$).

★ **1.8 Exercise.** Let $\mathcal{I}$ be the category $0 \to 1$ (i.e., the category $\mathbf{n}$ for $n = 1$). Verify that $\mathrm{N}(\mathcal{I})$ is the simplicial set which corresponds to the ordered simplicial complex $\{\{0\}, \{1\}, \{0, 1\}\}$; its geometric realization is the interval. Observe that a functor $F : \mathcal{C} \to \mathcal{D}$ gives a map $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ of simplicial sets, and that a natural transformation between $F, G \colon \mathcal{C} \to \mathcal{D}$ gives a homotopy $\mathrm{N}(\mathcal{C} \times \mathcal{I}) \cong \mathrm{N}(\mathcal{C}) \times \mathrm{N}(\mathcal{I}) \to \mathcal{D}$.

★ **1.9 Exercise.** Conclude that if a category $\mathcal{C}$ has a terminal object or an initial object, then $\mathrm{N}(\mathcal{C})$ has a contractible geometric realization.

1.10 Exercise. Convince yourself that if K is a finite simplicial complex and $\mathcal{C}$ is the category determined by the poset of simplices of K (ordered by inclusion), then $|\operatorname{N}(\mathcal{C})|$ is homeomorphic to $|K|$. Conclude that any finite complex is weakly homotopy equivalent to the nerve of a category. What about any topological space X? Can you choose the category and the weak equivalence to be natural in X?

1.11 Exercise. If K is a simplicial complex, there is a simplicial set $\operatorname{Sing}(K)$ given by letting $\operatorname{Sing}(K)_n$ be the set of simplicial complex maps $\Delta_n \to K$ (these maps are not required to be monomorphisms on the vertex sets). Check that $|\operatorname{Sing}(K)|$ is *not* necessarily homeomorphic to $|K|$. Are they homotopy equivalent? Check that the situation is substantially nicer if K is an ordered simplicial complex and $\operatorname{Sing}(K)$ is defined in terms of (weakly) ordered simplicial set maps $\Delta_* \to K$.

☡☡ **1.12 Exercise.** The process of passing from (ordered) simplicial complexes to simplicial sets is not totally unrelated to the passage from varieties over a field to objects in $\mathbb{A}^1$-homotopy theory. A simplicial set is a contravariant functor from finite ordered simplicial complexes to sets which takes whatever pushouts exist in the domain category to pullbacks in the range. The first step in constructing $\mathbb{A}^1$-homotopy theory is to consider contravariant functors from varieties to simplicial sets which take certain pushout-like diagrams in the category of varieties to homotopy pullbacks of simplicial sets. Are there any other examples of this kind of construction?

Simplicial objects. A simplicial object in a category $\mathcal{C}$ is a functor $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

1.13 Exercise. Let R be a ring and consider an object in the category $\mathbf{sMod}_R$ of R-modules. There is a normalization functor $N : \mathbf{sMod}_R \to \mathbf{Ch}_R$, which involves dividing out by images of degeneracy maps and taking the alternating sum of the face maps. The functor N establishes an equivalence of categories between $\mathbf{sMod}_R$ and the category of non-negatively graded objects of $\mathbf{Ch}_R$. Familiarize yourself with this [70, Chap. 5] [49, III.2]. What is $N^{-1}(R[n])$, where $R[n]$ is the chain complex which is zero except for a copy of R in degree n?

Non-abelian homological algebra. Simplicial objects can serve as substitutes for chain complexes in categories which are not abelian. For instance, Quillen [84] defined cohomology for a commutative ring R by applying an “indecomposables” functor to a simplicial resolution of R, in much the same way as you might define higher Tor’s for an R-module by applying a tensor product functor to a chain complex resolution of the module.

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The homotopy category $\mathrm{Ho}(\mathbb{T})$

The most visible invariant of a homotopy theory

$\mathbb{T}_{\textbf{pair}}$ $\mathrm{Ho}(\mathcal{C},\mathcal{E}) = \mathcal{E}^{-1}\mathcal{C}$

$\mathbb{T}_{\textbf{en}}$ $\mathrm{Ho}(\mathcal{R}) = \pi_0\mathcal{R}$ (i.e. π_0(morphism spaces))

Pluses and minuses

- Elegant (but only a small part of the structure)
- Can be hard to compute in the $\mathbb{T}_{\text{pair}}$ case.

Forming $\mathrm{Ho}(\mathcal{C},\mathcal{E})$ involves taking seriously the idea that the maps in $\mathcal{E}$ are honorary isomorphisms: $\mathrm{Ho}(\mathcal{C})$ is the category which results if the morphisms in $\mathcal{E}$ are forcibly declared to be isomorphisms.

1.14 Exercise. Let K be a finite simplicial complex, $\mathcal{C}$ the poset of simplices of K (ordered by inclusion) and $\mathcal{E} = \mathcal{C}$. The category $\mathrm{Ho}(\mathcal{C})$ is a groupoid (because every morphism has been made invertible). Can you identify this groupoid in some simple cases? In general?

1.15 Exercise. In the above situation, can you guess what the category $\mathcal{R}$ enriched over spaces which corresponds to $\mathbb{T}(\mathcal{C},\mathcal{E})$ looks like? What is lost in this case in passing from $\mathbb{T}(\mathcal{C},\mathcal{E})$ to $\mathrm{Ho}(\mathbb{T}(\mathcal{C},\mathcal{E}))$? Is it always the case that something is lost?

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Examples of homotopy categories

Geometry

($\mathcal{C}$,$\mathcal{E}$)	Maps $X \to Y$ in $\mathrm{Ho}(\mathcal{C},\mathcal{E})$
$\mathbf{Top}_{\text{he}}$	homotopy classes $X \to Y$
$\mathbf{Top}$	homotopy classes $\mathrm{CW}(X) \to Y$
$\mathbf{Top}_R$	homotopy classes $\mathrm{CW}(X) \to L_R(Y)$
$\mathbf{Top}_{\leq n}$	homotopy classes $\mathrm{CW}(X) \to P_n(Y)$

Algebra

$\mathbf{Sp}$	homotopy classes $\lvert X\rvert \to \lvert Y\rvert$
$\mathbf{sGrp}$	pointed homotopy classes $B\lvert X\rvert \to B\lvert Y\rvert$
$\mathbf{Ch}_R$	chain homotopy classes Proj. Res.$(X) \to Y$

Note: $\mathrm{CW}(X)$ denotes a cell complex which is weakly equivalent to the space X.

There is a general theme in the above examples: maps $X \to Y$ in the homotopy category of $(\mathcal{C},\mathcal{E})$ are computed by finding some kind of a nice stand-in X' for X and computing some sort of equivalence classes of maps in $\mathcal{C}$ from X to Y. Clearly, there is extra structure in the categories which makes this possible.

1.16 Exercise. Try to make one of the above calculations by hand, for instance, in the **Top** case. First, construct a category $\mathcal{Q}$ in which the objects are spaces and the maps $X \to Y$ are given by $[\mathrm{CW}(X), Y]$. (Observe that $[\mathrm{CW}(X), Y] \cong [\mathrm{CW}(X), \mathrm{CW}(Y)]$). Then build a functor $\mathbf{Top} \to \mathcal{Q}$. Show that the functor sends weak equivalences to isomorphisms and that it is universal with respect to this property.

One of the most convenient frameworks in which it is possible to make calculations like this is the framework of Quillen model categories.

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Model category: $(\mathcal{C}, \mathcal{E})$ **with extras**

Routine axioms

- **MC0** *equivalences* ($\sim$), *cofibrations* ($\hookrightarrow$), *fibrations* ($\twoheadrightarrow$)
- **MC1-3** composites, retracts, 2 out of 3, limits, colimits

Lifting

- **MC4**

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "f"'] & X \arrow[d, two heads, "g"] \\
B \arrow[r] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

$\exists h$ if f or g is $\sim$

Factorization

- **MC5** any map factors $\overset{\sim}{\hookrightarrow} \cdot \twoheadrightarrow$ and $\hookrightarrow \cdot \overset{\sim}{\twoheadrightarrow}$

For more information on model categories, see for instance [58], [49, II], [41], or [54]. The ur-reference is Quillen [86].

Axioms **MC1-3** guarantee that $\mathcal{C}$ has limits and colimits, that all three distinguished classes of maps are closed under composites and retracts, and that the class of equivalences has the "2 out of 3" property (given composable arrows f, and g, if two of the three maps f, g, fg are equivalences, so is the third). In recent treatments the factorizations from **MC5** are usually assumed to be functorial.

1.17 Exercise. What does it mean to say, for instance, that the class of cofibrations is closed under retracts?

A map which is a (co-)fibration and an equivalence is called an acyclic (co)fibration. Axiom MC4 is reminiscent of the homotopy lifting property or the homotopy extension property. This axiom is sometimes expressed as the statement that cofibrations have the left lifting property (LLP) with respect to acyclic fibrations, while fibrations have the right lifting property (RLP) with respect to acyclic cofibrations.

1.18 Exercise. Observe that the model category axioms are self-dual; if $\mathcal{C}$ is a model category, so is $\mathcal{C}^{\mathrm{op}}$.

1.19 Exercise. Use **MC4** in combination with the retract property to show that in a model category $\mathcal{C}$ a map f is a cofibration if and only if f has the LLP with respect to acyclic fibrations, or an acyclic cofibration if and only if it has the LLP with respect to fibrations. (By duality, there are parallel characterizations of fibrations and acyclic fibrations.) Conclude that, given the equivalences, the fibrations and cofibrations determine one another.

Suppose that $\mathcal{C}$ is a model category with initial object ϕ and terminal object $*$ (why do such objects always exist in a model category?). An object X of $\mathcal{C}$ is *cofibrant* if $\phi \hookrightarrow X$ and *fibrant* if $X \twoheadrightarrow *$. A cofibrant replacement X^{c} for X is obtained from the **MC5** factorization $\phi \hookrightarrow X^{\mathrm{c}} \overset{\sim}{\twoheadrightarrow} X$ and a fibrant replacement X^{f} from the factorization $X \overset{\sim}{\hookrightarrow} X^{\mathrm{f}} \twoheadrightarrow *$. (More loosely, a cofibrant replacement is a cofibrant object mapping to X by an equivalence, and a fibrant replacement is a fibrant object receiving an equivalence from X.)

☡ **1.20 Exercise.** If $(\mathcal{C}, \mathcal{E})$ is a model category, argue that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y)$ can be computed as the set of "homotopy classes" of maps in $\mathcal{C}$ from X^{c} to Y^{f}. The homotopy classes are constructed as follows. Define $X \times \Delta_1$ (a notation, not a product!) by factoring the fold map: $X^{\mathrm{c}} \amalg X^{\mathrm{c}} \hookrightarrow X \times \Delta_1 \overset{\sim}{\twoheadrightarrow} X$. Now declare two maps $f, g : X^{\mathrm{c}} \to Y^{\mathrm{f}}$ to be homotopic if $f + g : X^{\mathrm{c}} \amalg X^{\mathrm{c}} \to Y^{\mathrm{f}}$ extends over $X \times \Delta_1$. (Hint: manipulate cofibrations and fibrations to show that homotopy is an equivalence relation on maps $X^{\mathrm{c}} \to Y^{\mathrm{f}}$ and that homotopy respects compositions. Then construct a category $\mathcal{C}'$ with the same objects as $\mathcal{C}$, but in which the maps $X \to Y$ are the homotopy classes of maps $X^{\mathrm{c}} \to Y^{\mathrm{f}}$, and argue that an appropriate functor $\mathcal{C} \to \mathcal{C}'$ is universal with respect to functors on $\mathcal{C}$ which send equivalences to isomorphisms.)

☡ ☡ **1.21 Exercise.** [37] Expand on the above idea to get objects $X \times \Delta_n$, $n \geq 0$, all equivalent to X, which fit into a cosimplicial object $X \times \Delta_*$ in $\mathcal{C}$. To begin, $X \times \Delta_0$ is X^{c} and $X \times \Delta_1$ is as above. Construct $X \times \partial\Delta_2$ by gluing 3 copies of $X \times \Delta_1$ together along the "vertex" copies of $X \times \Delta_0$. Build $X \times \Delta_2$ by noticing that there is a natural map $X \times \partial\Delta_2 \to M$, where

$$M = (X \times \Delta_1) \times_{X \times \Delta_0} (X \times \Delta_1),$$

and factoring this map into the a cofibration followed by an acyclic fibration. Where did M come from? (Consider the two collapses of an ordered 2-simplex onto an ordered 1-simplex.) Proceed by induction $\ddot\smile$

Remark. The simplicial set $\mathrm{Hom}_{\mathcal{C}}(X \times \Delta_*, Y^{\mathrm{f}})$ is equivalent to $\mathrm{Hom}_{\mathcal{R}}(X, Y)$, where $\mathcal{R}$ is the category enriched over spaces (simplicial sets) representing the homotopy theory $(\mathcal{C}, \mathcal{E})$.

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Examples of model categories $(\mathcal{C}, \mathcal{E})$

Geometry

- $\mathbf{Top}_{\text{he}}$, Hurewicz fibrations, closed NDR-pair inclusions
- $\mathbf{Top}$, Serre fibrations, retracts of relative cell inclusions
- $\mathbf{Top}^{\mathcal{D}}$, objectwise Serre fibrations, retracts of relative diagram cell inclusions.

Algebra

- $\mathbf{Sp}$, Kan fibrations, monomorphisms
- $\mathbf{Ch}_R^+$, surjections in degrees > 0, monomorphisms such that the cokernel in each degree is projective

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Here $\mathbf{Ch}_R^+$ is the category of nonnegatively graded chain complexes over R, with homology isomorphisms as equivalences.

1.22 Exercise. Verify that the indicated choices produce a model category structure on $\mathbf{Ch}_R^+$ (this is one way to build up homological algebra).

1.23 Exercise. Produce another model category structure on $\mathbf{Ch}_R^+$ in which the cofibrations are the monomorphisms and the fibrations are maps which are surjective in positive degrees and in each degree have an injective R-module as cokernel. Conclude that there are sometimes options available when it comes to putting a model category structure on $(\mathcal{C}, \mathcal{E})$.

Remark. It's not too complicated to produce the model category structure on $\mathbf{Top}$ (and actually pretty interesting, since the usual approach depends on a widely applicable trick due to Quillen called the small object argument). The verifications I've seen for the model category structure on $\mathbf{Sp}$ are messier and less satisfying.

Diagrams give interesting model categories.

1.24 Exercise. Suppose that $(\mathcal{C}, \mathcal{E})$ has a model category structure. Let $\mathcal{D}$ be the pushout category (two-source category) $\{a \leftarrow b \rightarrow c\}$, and consider the category $\mathcal{C}^{\mathcal{D}}$ whose objects are the functors $\mathcal{D} \rightarrow \mathcal{C}$ and whose morphisms are the natural transformations; this provides a homotopy theory in which the equivalences are the natural transformations which for each object of $\mathcal{D}$ give a morphism in $\mathcal{E}$. Consider a morphism

$$\begin{tikzcd}
X \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\
X' & Y' \arrow[l] \arrow[r] & Z'
\end{tikzcd}$$

in $\mathcal{C}^{\mathcal{D}}$ and call it

- a fibration, if each of the vertical maps is a fibration in $\mathcal{C}$, and
- a cofibration, if $Y \rightarrow Y'$, $X \amalg_Y Y' \rightarrow X'$ and $Z \coprod_Y Y' \rightarrow Z'$ are cofibrations in $\mathcal{C}$.

Verify that these choices give a model category structure on $\mathcal{C}^{\mathcal{D}}$.

1.25 Exercise. Suppose that $\mathcal{D}$ is the pullback category (two-sink category)

$$a \to b \leftarrow c\,.$$

Use the previous exercise + duality to obtain for free a model category structure on $\mathcal{C}^{\mathcal{D}}$.

1.26 Exercise. Formulate the notion of a category $\mathcal{D}$ in which the objects have nonnegative integer gradings and in which the grading of the source of a morphism is always strictly less than the grading of the target; call this, say, an *increasing category*, since the morphisms increase the nonnegative integer grading). Let $(\mathcal{C}, \mathcal{E})$ be a model category. Generalize the above to get model category structures on $\mathcal{C}^{\mathcal{D}}$ and on $\mathcal{C}^{\mathcal{D}^{\mathrm{op}}}$. (It would be tempting to call $\mathcal{D}^{\mathrm{op}}$ a *decreasing category*.)

As indicated on the slide, if $\mathcal{D}$ is any category there is a model category structure on $\mathbf{Top}^{\mathcal{D}}$ in which the fibrations are the maps of diagrams which give objectwise fibrations in $\mathbf{Top}$. The cofibrations are constructed as follows. Let D^n be the n-disk and $S^{n-1} = \partial D^n$ its boundary For each $x \in \mathcal{D}$, the n-disk D^n_x based at x is the functor $\mathcal{D} \to \mathbf{Top}$ given by

$$D^n_x(y) = \amalg_{\mathrm{Hom}_{\mathcal{D}}(x,y)} D^n = D^n \times \mathrm{Hom}_{\mathcal{D}}(x,y)\,.$$

The $n-1$ sphere $S^{n-1}_x = \partial D^n_x$ is defined similarly. A map $X \to Y$ in $\mathcal{C}^{\mathcal{D}}$ is a relative diagram cell inclusion if Y is obtained from X by iteratively (perhaps transfinitely) attaching cells of the form $(D^n_x, \partial D^n_x)$ for various n, x. The cofibrations in $\mathbf{Top}^{\mathcal{D}}$ are the retracts of relative cell inclusions.

1.27 Exercise. Produce a model category structure as above on $\mathbf{Sp}^{\mathcal{D}}$ (remember, $\mathbf{Sp}$ = simplicial sets). How about something similar for $(\mathbf{Ch}^+_R)^{\mathcal{D}}$? (These are called projective model category structures.)

1.28 Exercise. [49, VIII.2.4] Produce a model category structure on $\mathbf{Sp}^{\mathcal{D}}$ in which the cofibrations are the objectwise cofibrations. (This is much harder, and requires a willingness to let go of any desire to describe the fibrations explicitly.) This is called the injective model structure on $\mathbf{Sp}^{\mathcal{D}}$.

Remark. If $\mathcal{C}$ is a model category, it does not seem to be true in general that there is a model category structure on the diagram category $\mathcal{C}^{\mathcal{D}}$. Such a model category structure is known to exist only if $\mathcal{C}$ is nice (cofibrantly generated) [56, 11.6] or $\mathcal{D}$ is nice (above, see [56, Ch. 15] or [58, Ch. 5]). There are ways to work around this; one of them (roughly) [26] is to construct another category $\mathcal{D}'$ from $\mathcal{D}$ such that $\mathcal{D}'$ is nice (i.e. $\mathcal{C}^{\mathcal{D}'}$ has a model category structure) and the homotopy theory of $\mathcal{C}^{\mathcal{D}'}$ is equivalent to the homotopy theory of $\mathcal{C}^{\mathcal{D}}$.

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Dividends from a model category structure on $(\mathcal{C}, \mathcal{E})$

Calculate

- $\mathrm{Ho}(\mathcal{C})$ (or even $\mathbb{T}_{\mathrm{en}}(\mathcal{C}, \mathcal{E})$) from $(\mathcal{C}, \mathcal{E})$
- Diagrams: Theory of $(\mathcal{C}, \mathcal{E})^{(\mathcal{D}, \mathcal{F})} \quad \sim \quad \mathbb{T}(\mathcal{C}, \mathcal{E})^{\mathrm{h}\mathbb{T}(\mathcal{D}, \mathcal{F})}$

Construct

- Derived functors
- Homotopy limits & colimits

Identify

- Equivalences $\mathbb{T}(\mathcal{C}, \mathcal{E}) \sim \mathbb{T}(\mathcal{C}', \mathcal{E}')$

The exercises above suggest how to compute $\mathrm{Ho}(\mathcal{C}, \mathcal{E})$, or even the associated topologically enriched category, from a model category structure on $(\mathcal{C}, \mathcal{E})$. Equivalences between homotopy theories are discussed below, while homotopy limits/colimits will come up later. The diagram classification goes as follows. If $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{D}, \mathcal{F})$ are two relative categories, the relative category

$$\mathrm{Fun}((\mathcal{D}, \mathcal{F}), (\mathcal{C}, \mathcal{E})) = (\mathcal{C}, \mathcal{E})^{(\mathcal{D}, \mathcal{F})}$$

has as objects the functors $\mathcal{C} \to \mathcal{D}$ which take $\mathcal{E}$ to $\mathcal{F}$, has as morphisms the natural transformations, and has as equivalences (i.e. distinguished subcategory) those natural transformations which carry each object of $\mathcal{C}$ to an equivalence in $\mathcal{D}$. Then if $(\mathcal{C}, \mathcal{E})$ has a model category structure, the homotopy theory of this functor category is equivalent to the mapping object $\mathrm{Cat}^h(\mathbb{T}(\mathcal{D}, \mathcal{F}), \mathbb{T}(\mathcal{C}, \mathcal{E}))$. This has been worked out explicitly for $\mathcal{C} = \mathbf{Sp}$ but almost certainly holds in general.

☡ **1.29 Exercise.** It's possible to view this diagram classification claim as a generalization of bundle classification theory. How?

☡ **1.30 Exercise.** Let $\mathcal{C}$ be the category $\mathbf{Top}$ (or, with appropriate adjustments, the category $\mathbf{Sp}$, if this seems more convenient). Let $\mathcal{D}$ be the category $\mathbf{1} = 0 \to 1$ (treated as a homotopy theory with only the identity maps as equivalences). Consider as above the homotopy theory of functors $\mathcal{D} \to \mathcal{C}$. Show that if X and Y are CW-complexes, the set of equivalence classes of functors $F : \mathcal{D} \to \mathcal{C}$ with $F(0) \sim X$ and $F(1) \sim Y$ is in bijective correspondence with the set

$$\pi_0 \,\mathrm{Aut}^{\mathrm{h}}(X) \setminus [X, Y] \,/\, \pi_0 \,\mathrm{Aut}^{\mathrm{h}}(Y) \ .$$

Here $[X, Y]$ is the set of homotopy classes of maps from X to Y, $\mathrm{Aut}^{\mathrm{h}}(Z)$ is the group-like monoid of self-homotopy equivalences of Z, and the orbit sets are obtained from the composition action of self-equivalences on maps.

☡ ☡ **1.31 Exercise.** Looking at the previous exercise with a homotopy theoretic eye strongly suggests considering the double Borel construction

$$\mathrm{Aut}^{\mathrm{h}}(X) \backslash\backslash \, \mathrm{Map}(X, Y) // \, \mathrm{Aut}^{\mathrm{h}}(Y) \, .$$

What is the significance of this construction in terms of the functors $\mathcal{D} \to \mathcal{C}$? (Note that the above exercise amounts to the statement that π_0 of this construction classifies certain functors.) What happens if $\mathcal{D} = \mathbf{n}$, $n > 1$?

1.32 Exercise. In the above situation, compute homotopy classes of maps in $\mathcal{C}^{\mathcal{D}}$ from $X \to Y$ to $Z \to W$ in terms of homotopy constructions in $\mathcal{C}$. (You're computing the homotopy category of the homotopy theory $\mathrm{Cat}^h(\mathbb{T}_{\mathcal{D}}, \mathbb{T}(\mathcal{C}, \mathcal{E}))$.)

Slide 1-8

Slide 1-8

Equivalences between homotopy theories

Paradox

There is a homotopy theory of homotopy theories.

Equivalences vary with context

($\mathbb{T}_{\text{en}}$) $F\colon \mathcal{R} \to \mathcal{R}'$ is an equivalence if $\mathrm{Ho}(F)$ is an equivalence

of categories and $\mathrm{Hom}_{\mathcal{R}}(x, y) \sim \mathrm{Hom}_{\mathcal{R}'}(Fx, Fy)$

($\mathbb{T}_{\text{css}}$) $F\colon X_* \to Y_*$ is an equivalence if $X_n \sim Y_n, n \geq 0$

($\mathbb{T}_{\text{pair}}$) $F\colon (\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$ is an equivalence if (??)

Special $\mathbb{T}_{\text{pair}}$ case

Filling in (??) easier for model categories

There are set-theoretic problems with contemplating the homotopy theory of *all* homotopy theories, but these are easy to evade by sticking to homotopy theories whose objects and morphisms are sets in some chosen Grothendieck universe [44, §32]. The slide refers to the fact that determining whether or not a map between homotopy theories is an equivalence can be tricky, but there are some easy ways to check this in the model category case.

The simplest context in which to characterize equivalences between homotopy theories is $\mathbb{T}_{\text{en}}$: the category of topologically (respectively, simplicially) enriched categories. A functor $F : \mathcal{R} \to \mathcal{R}'$ between two of these objects is an equivalence if

- $\pi_0 F : \pi_0 \mathcal{R} \to \pi_0 \mathcal{R}'$ is an ordinary equivalence of categories (in other words F induces an equivalence of categories $\mathrm{Ho}(\mathcal{R}) \to \mathrm{Ho}(\mathcal{R}')$), and
- for any two objects x, y of $\mathcal{R}$, the map

$$\mathrm{Hom}_{\mathcal{R}}(x, y) \to \mathrm{Hom}_{\mathcal{R}'}(Fx, Fy)$$

  induced by F is an equivalence in $\mathbf{Top}$, i.e., a weak homotopy equivalence (resp. an equivalence in $\mathbf{Sp}$).

Slide 1-9

$\mathbb{T}(\mathcal{C},\mathcal{E}) \sim \mathbb{T}(\mathcal{C}',\mathcal{E}')$ for model categories

Conditions on adjoint functors $F\colon \mathcal{C} \leftrightarrow \mathcal{C}' : G$

1. $F(\hookrightarrow) = (\hookrightarrow)$ and $G(\twoheadrightarrow) = (\twoheadrightarrow)$
2. $f\colon A^{\mathrm{c}} \xrightarrow{\sim} G(B^{\mathrm{f}}) \quad \Leftrightarrow \quad f^{\flat}\colon F(A^{\mathrm{c}}) \xrightarrow{\sim} B^{\mathrm{f}}$

Definition 1. (1) $=$ *Quillen pair*, (1) $+$ (2) $=$ *Quillen equivalence*

Theorem 2. *A Quillen equivalence* (F,G) *induces* $\mathbb{T}(\mathcal{C},\mathcal{E}) \sim \mathbb{T}(\mathcal{C}',\mathcal{E}')$

In a Quillen pair, F preserves cofibrations and G preserves fibrations. The condition on a Quillen equivalence is that in addition, if A is a cofibrant object of $\mathcal{C}$ and B is a fibrant object of $\mathcal{C}'$, then a map $f\colon A \to G(B)$ is a equivalence in $\mathcal{C}$ if and only if the adjoint map $f^{\flat}\colon F(A) \to B$ is an equivalence in $\mathcal{C}'$.

1.33 Exercise. Show that if F and G form a Quillen pair, then F preserves acyclic cofibrations and G preserves acyclic fibrations. (Use the fact that these kinds of maps are characterized by lifting properties; see 1.19.)

It may be unclear how a Quillen pair (F,G) induces an equivalence of homotopy theories, since the functors in the pair do not necessarily preserve equivalences, and so do not directly induce morphisms of homotopy theories. The key observation is due to K. Brown [58, 1.1.12].

1.34 Exercise. Let F be a functor from a model category into some other category. If F takes all acyclic cofibrations to isomorphisms, then F takes all equivalences between cofibrant objects to isomorphisms.

There is also a dual form involving fibrations and fibrant objects. It's known that if $\mathcal{C}$ is a model category, then the morphisms of $\mathcal{C}$ which become isomorphisms in $\mathrm{Ho}(\mathcal{C})$ are exactly the equivalences (no additional morphisms are inverted). It follows that if (F,G) is a Quillen pair, then F preserves equivalences between cofibrant objects and so induces a map of pairs $(\mathcal{C}^{\mathrm{c}},\mathcal{E}^{\mathrm{c}}) \to (\mathcal{C}',\mathcal{E}')$, where $\mathcal{C}^{\mathrm{c}}$ is the full subcategory of cofibrant objects in $\mathcal{C}$ and $\mathcal{E}^{\mathrm{c}} = \mathcal{E} \cap \mathcal{C}^{\mathrm{c}}$. Now it is necessary to observe that the inclusion $(\mathcal{C}^{\mathrm{c}},\mathcal{E}^{\mathrm{c}}) \to (\mathcal{C},\mathcal{E})$ induces an equivalence of homotopy theories. The zigzag

$$\mathbb{T}(\mathcal{C},\mathcal{E}) \xleftarrow{\sim} \mathbb{T}(\mathcal{C}^{\mathrm{c}},\mathcal{E}^{\mathrm{c}}) \to \mathbb{T}(\mathcal{C}',\mathcal{E}')$$

is the map which if (F,G) is a Quillen equivalence induces the equivalence $\mathbb{T}(\mathcal{C},\mathcal{E}) \sim \mathbb{T}(\mathcal{C}',\mathcal{E}')$.

Slide 1-10

Slide 1-10

Examples of equivalences between homotopy theories

Examples

- $\mathbf{Top}$ and $\mathbf{Sp}$
- $\mathbf{Sp}_*$ and $\mathbf{sGrp}$
- Simplicial algebras and DG_+ algebras
- Chain complexes and $H\mathbb{Z}$-module spectra

Meta-examples

- $\mathbb{T}_{\text{en}}$, $\mathbb{T}_{\text{sc}}$, $\mathbb{T}_{\text{css}}$, and $\mathbb{T}_{\text{qc}}$ ($\mathbb{T}_{\text{pair}}$ belong here?)

Here $\mathbf{Sp}_*$ is the category of pointed simplicial sets; a map in this context is an equivalence if it induces an equivalence in $\mathbf{Sp}$ between the basepoint components.

Implicit above is the statement that $\mathbb{T}_{\text{sc}}$, $\mathbb{T}_{\text{css}}$, $\mathbb{T}_{\text{en}}$, etc. have model category structures. So it's important to be careful in thinking about "maps" between homotopy theories; such a map will necessarily be represented by an actual morphism only if the source homotopy theory is cofibrant in the appropriate sense and the target homotopy theory is fibrant.

Remark. It is *almost* true that a map $(\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$ of homotopy theories is an equivalence if and only if the induced map of diagram theories

$$\mathrm{Cat}^h(\mathbb{T}(\mathcal{C}', \mathcal{E}'), \mathbf{Sp}) \to \mathrm{Cat}^h(\mathbb{T}(\mathcal{C}, \mathcal{E}), \mathbf{Sp})$$

is an equivalence [40]. But not quite; the problem is an interesting one that arises even for discrete homotopy theories, i.e., ordinary categories.

1.35 Exercise. Give an example of a functor $F\colon \mathcal{C} \to \mathcal{C}'$ between (ordinary) categories such that (1) F induces an equivalence of categories $\mathbf{Set}^{\mathcal{C}'} \to \mathbf{Set}^{\mathcal{C}}$, but (2) F itself is *not* an equivalence of categories. (Hint: *retracts!*) Go on to characterize the functors $\mathcal{C} \to \mathcal{C}'$ which have property (1).

Lecture 2.

Homotopy limits and colimits

Slide 2-1

Slide 2-1

Colimits and related constructions

Colimit for $X\colon \mathcal{C} \to \mathcal{S}$

$$\operatorname{colim}\colon \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S} \ : \Delta$$

$$\operatorname{Hom}_{\mathcal{S}^{\mathcal{C}}}(X, \Delta(Y)) \cong \operatorname{Hom}_{\mathcal{S}}(\operatorname{colim} X, Y)$$

LKan_F **for** $X\colon \mathcal{C} \to \mathcal{S}$ **and** $F\colon \mathcal{C} \to \mathcal{D}$

$$\operatorname{LKan}_F\colon \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S}^{\mathcal{D}} \ : F^*$$

$$\operatorname{Hom}_{\mathcal{S}^{\mathcal{C}}}(X, F^*Y) \cong \operatorname{Hom}_{\mathcal{S}^{\mathcal{D}}}(\operatorname{LKan}_F X, Y)$$

Coend for $X\colon \mathcal{C}^{\mathbf{op}} \times \mathcal{C} \to \mathcal{S}$

$$A(\mathcal{C}) = \begin{tikzcd} a \arrow[r, "f"'] & b \arrow[dd] \\ & \\ a' \arrow[uu] \arrow[r] & b' \end{tikzcd} \quad \operatorname{coend} X \cong \operatorname{colim}_{A(\mathcal{C})}(f \mapsto X(a,b))$$

This slide describes colimits (left adjoints to diagonal functors), left Kan extensions (left adjoints to restriction functors) and coends (not described directly, but asserted to be given by some funny associated colimits).

The pictorial description of $A(\mathcal{C})$ signifies that an object of $A(\mathcal{C})$ is a solid arrow $f : a \to b$ in $\mathcal{C}$, while a morphism from the top solid arrow to the bottom one is the indicated peculiar type of commuting square. Note that taking left and right endpoints of the object arrows gives a functor $A(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. The category $A(\mathcal{C})$ is sometimes called the *arrow category* of $\mathcal{C}$, and sometimes the *twisted arrow category*. I prefer the second name because it's a reminder that one of the vertical arrows in a morphism is twisted backwards.

2.1 Exercise. Check that the colimit description of the coend [68, IX.6] is correct.

2.2 Exercise. If G is a (discrete) group, let $\mathcal{C}_G$ denote the category associated to G; this is the category with one object in which the maps from the object to itself are given by G. A functor $\mathcal{C}_G \to \mathcal{S}$ amounts to an object of $\mathcal{S}$ with an action of G. Let H be a subgroup of G, and $F\colon \mathcal{C}_H \to \mathcal{C}_G$ the natural functor. Compute the left Kan extension functor $\mathcal{S}^{\mathcal{C}_H} \to \mathcal{S}^{\mathcal{C}_G}$ if (a) $\mathcal{S}$ is the category of sets, or (b) $\mathcal{S}$ is the category of R-modules for a ring R. In both cases, compute the colimit functors $\mathcal{S}^{\mathcal{C}_G} \to \mathcal{S}$.

2.3 Exercise. It's clear that a colimit is an example of a Kan extension. It is also possible to compute Kan extensions in terms of colimits. If $d \in \mathcal{D}$, the *over category* (also called comma category) $F{\downarrow}d$ can be described pictorially as follows:

$$\begin{tikzcd}
c, F(c) \arrow[dd, dashed, "h"', "F(h)"] \arrow[dr] & \\
 & d \\
c', F(c') \arrow[ur] &
\end{tikzcd}$$

Show that $\mathrm{LKan}_F(X)$ is the functor which assigns to $d \in \mathcal{D}$ the colimit, over $F{\downarrow}d$, of the functor which assigns to $(c, F(c) \to d)$ the object $X(c)$ [56, 11.8]. Symbolically,

$$\mathrm{LKan}_F(X)(d) = \mathrm{colim}_{F\downarrow d}\,[(c, F(c) \to d) \mapsto X(c)]$$

This is sometimes expressed by saying that left Kan extensions can be computed pointwise.

2.4 Exercise. Show that if $F\colon \mathcal{C} \to \mathcal{D}$ is the inclusion of a full subcategory, then $\mathrm{LKan}_F(X)$ actually is an extension of X to $\mathcal{D}$ (in other words, the restriction of $\mathrm{LKan}_F(X)$ to $\mathcal{C}$ is isomorphic to X). Show by example that this is not necessarily the case in other situations.

Slide 2-2

Slide 2-2

Aside: notation for special coends

Monoidal category $\mathcal{S}$

- Bifunctor $\otimes\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$
- Usually associative, unital (commutative) up to . . .

$\otimes$ **over** $\mathcal{C}$ **of functors** $\mathcal{C} \to \mathcal{S}$

$$X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}, \quad Y : \mathcal{C} \to \mathcal{S}$$

$$X \otimes_{\mathcal{C}} Y := \text{coend of } (a, b) \mapsto X(a) \otimes Y(b)$$

Usually $\otimes = \times$

$$\mathcal{S} = \mathbf{Sp}, \quad \otimes = \times$$

$$\mathrm{Map}_{\mathbf{Sp}}(X \times_{\mathcal{C}} Y, Z) \cong \mathrm{Hom}_{\mathbf{Sp}^{\mathcal{C}}}(X, \mathrm{Map}(Y, Z))$$

This slide establishes a notation for "balanced products" of functors with values in a (symmetric) monoidal category in terms of an associated coend. The functors are both defined on some category $\mathcal{C}$, but one of them is required to be contravariant and one covariant. We'll be mostly in the case in which the monoidal structure is given by cartesian product in $\mathbf{Top}$ or $\mathbf{Sp}$.

Note that the last displayed isomorphism does in fact make sense: Y is a contravariant functor $\mathcal{C} \to \mathbf{Sp}$, and so the functor $\mathrm{Map}(Y, Z)$, sending a to $\mathrm{Map}(Y(a), Z)$ is a covariant functor.

2.5 Exercise. Convince yourself that the last displayed isomorphism on the slide is correct.

2.6 Exercise. Try to interpret the tensor product of a left module over a ring R with a right module as a coend in the above sense. It may be necessary to deal with additive categories ($=$ morphisms are abelian groups, composition is bilinear) and additive maps between them.

2.7 Exercise. Let G be a discrete group with associated category $\mathcal{C}_G$ (2.2). Let $X\colon \mathcal{C}_G \to \mathbf{Sp}$ be a contravariant functor (a right G-space) and $Y\colon \mathcal{C}_G \to \mathbf{Sp}$ a covariant functor (a left G-space). Show that $X \times_{\mathcal{C}_G} Y$ is the orbit space of action of G on $X \times Y$ obtained by converting the action on X to a left action using $g \mapsto g^{-1}$ and then taking the diagonal action on the product.

Slide 2-3 Slide 2-3

Homotopy colimits and related constructions

$\mathcal{C}$, $\mathcal{S}$ homotopy theories

Homotopy colimit for $X\colon \mathcal{C} \to \mathcal{S}$

$$\mathrm{hocolim}\colon \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S} \ : \Delta$$

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}}(X, \Delta(Y)) \sim \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}}(\mathrm{hocolim}\, X, Y)$$

$\mathrm{LKan}^{\mathbf{h}}_F$ **for** $X\colon \mathcal{C} \to \mathcal{S}$ **and** $F\colon \mathcal{C} \to \mathcal{D}$

$$\mathrm{LKan}^{\mathrm{h}}_F\colon \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S}^{\mathcal{D}} \ : F^*$$

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}}(X, F^*Y) \cong \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{D}}}(\mathrm{LKan}^{\mathrm{h}}_F\, X, Y)$$

Homotopy coend for $X\colon \mathcal{C}^{\mathbf{op}} \times \mathcal{C} \to \mathcal{S}$

$$\mathrm{hocoend}\, X = \mathrm{hocolim}_{A(\mathcal{C})} \text{ of } (a \to b) \mapsto X(a,b)$$

$$U \otimes^{\mathrm{h}}_{\mathcal{C}} V = \mathrm{hocolim}_{A(\mathcal{C})} \text{ of } (a \to b) \mapsto U(a) \otimes V(b)$$

Remark. Most of the time when homotopy colimits come up in these lectures, $\mathcal{C}$ is an ordinary discrete category, treated as a homotopy theory by taking the equivalences to be the isomorphisms. The objects $\mathcal{S}^{\mathcal{C}}$ and $\mathcal{S}^{\mathcal{D}}$ on this slide should really be $\mathcal{S}^{\mathrm{h}\mathcal{C}}$ and $\mathcal{S}^{\mathrm{h}\mathcal{D}}$, but it seemed too cluttered to have an “h” decorating both the function categories and the hom constructions.

This brings up the fact that the slide contains the first appearance of $\mathrm{Hom}^{\mathrm{h}}(U,V)$: this stands for the function space of maps from U to V, where U and V are objects in some homotopy theory $\mathbb{T}$. The homotopy type of $\mathrm{Hom}^{\mathrm{h}}(U,V)$ may be more or less deeply buried, depending on how the homotopy theory is presented. If the theory is given as a category $\mathcal{R}$ enriched over $\mathbf{Sp}$, then $\mathrm{Hom}^{\mathrm{h}}(U,V)$ is the space of maps $U \to V$ in $\mathcal{R}$. If $\mathbb{T}$ is given by a pair $(\mathcal{C},\mathcal{E})$, then in general $\mathrm{Hom}^{\mathrm{h}}(U,V)$ has to be computed as a space of zigzags [36], though if $(\mathcal{C},\mathcal{E})$ has a model structure there is a more direct approach (see 1.21, [58, 5.4], or [56, Ch. 17]). If $(\mathcal{C},\mathcal{E})$ has the structure of a simplicial model category [58, 4.2] [56, Ch. 9], then $\mathrm{Hom}^{\mathrm{h}}(U,V)$ is equivalent to the simplicial mapping space $\mathrm{Map}(U^{\mathrm{c}}, V^{\mathrm{f}})$, where U^{c} is a cofibrant replacement for U and V^{f} a fibrant replacement for V. In practice, $\mathrm{Hom}^{\mathrm{h}}(U, V)$ is usually given by any reasonable mapping space construction that respects equivalences in both variables.

The last displayed definition contains an underlying assumption that the pairing map $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is a map of homotopy theories (or, if $\mathcal{S}$ is a category with equivalences, can be suitably adjusted to become a map of homotopy theories). This won't be an issue for us, since in the monoidal category $(\mathbf{Sp}, \times)$ the monoidal operation $\times$ does preserve equivalences.

2.8 Exercise. Explain what $X \otimes^{\mathrm{h}}_{\mathcal{C}} Y$ should mean if X and Y are, respectively, contravariant and covariant functors from $\mathcal{C}$ to the category of chain complexes of modules over a commutative ring R.

Note. The adjunctions above are in the appropriate enriched category sense; for instance the second one asserts that the two functors

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}}(X, F^*Y) \text{ and } \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{D}}}(\mathrm{LKan}^{\mathrm{h}}_F X, Y)$$

are explicitly equivalent as functors

$$(\mathcal{S}^{\mathcal{C}})^{\mathrm{op}} \times \mathcal{S}^{\mathcal{D}} \to \mathbf{Sp}\,.$$

Depending on the model for homotopy theories that is currently on the workbench, this explicit equivalence may or may not be realized by a direct morphism in the functor category; it may well be specified as a zigzag of morphisms which are direct equivalences.

Slide 2-4

Slide 2-4

Model category dividend

Theorem
If $\mathcal{S}$ admits a model category structure:

Existence
$\mathrm{hocolim}$, $\mathrm{LKan}^{\mathrm{h}}_F$, $\mathrm{hocoend}$ exist for target $\mathcal{S}$.

Realizability

- $\mathrm{hocolim} \sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}$,
- $\mathrm{LKan}^{\mathrm{h}}_F \sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}^{\mathcal{D}}$, and
- $\mathrm{hocoend} \sim$ functor $\mathcal{S}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \to \mathcal{S}$.

Remark. This theorem may or may not have been proved in the form in which it's stated, but it's certainly true. The only issue is whether the various constructions available have been explicitly identified as homotopy colimits (etc.) in the sense of the previous slide.

Slide 2-5

Slide 2-5

Construction of $\operatorname{hocolim}$**, version I**

Assumptions

- $\mathcal{S}$ admits a model structure
- $\mathcal{S}^{\mathcal{C}}$ admits the projective model structure
  (fibrations are objectwise)

The assumptions hold if $\mathcal{S}$ is $\mathbf{Sp}$, or $\mathbf{Top}$.

Conclusion

$$\operatorname{hocolim} X \sim \operatorname{colim} X^{c}, \quad X \in \mathcal{S}^{\mathcal{C}}$$

The assumptions hold if $\mathcal{C}$ is an increasing category (1.26) or if $\mathcal{S}$ admits a cofibrantly generated model category structure. For instance, the above technique always works for computing homotopy pushouts (1.24; see [41, §10]). Every object in $\mathbf{Sp}$ is cofibrant, so a homotopy pushout in $\mathbf{Sp}$ can be calculated by converting the two maps involved into cofibrations (leaving the common domain unchanged) and then taking a pushout.

2.9 Exercise. Give an example to show that in $(\mathbf{Top}_{\text{he}})_*$ (the category of pointed topological spaces with pointed homotopy equivalences as the equivalences) the coproduct of two objects is not necessarily equivalent to the homotopy coproduct of the objects. (Even coproducts sometimes have to be derived.)

2.10 Exercise. Look ahead to slide 2–9. Can you think of an example in which the product of a collection of objects in a model category is not equivalent to their homotopy product (the product of their fibrant replacements)?

2.11 Exercise. How about an example different from the one you got by applying the "opposite category" trick to 2.9 ☺.

Slide 2-6

Slide 2-6

Construction of $\operatorname{hocolim}$**, version II**
$\mathbf{Top}, \mathbf{Sp}, \ldots$

(A) $|Y|$ **for** $Y : \Delta^{\mathbf{op}} \to \{\mathbf{Top}\text{ or }\mathbf{Sp}\}$

$$|Y| = \begin{cases} \Delta_* \times_{\Delta^{\mathrm{op}}} Y & \text{for } \mathbf{Top} \\ \Delta[*] \times_{\Delta^{\mathrm{op}}} Y & \text{for } \mathbf{Sp} \end{cases}$$

(B) $\operatorname{Repl}_*(X) : \Delta^{\mathbf{op}} \to \mathcal{S}$ **for** $X : \mathcal{C} \to \mathcal{S}$

$$\operatorname{Repl}_*(X)(\mathbf{n}) = \coprod_{f:\,\{0\to 1\to\cdots\to n\}\to\mathcal{C}} f(0)$$

$\operatorname{hocolim} =$ **(A)** $+$ **(B)**

$$\operatorname{hocolim} X \sim |\operatorname{Repl}_*(X)|$$

Note. The functor Δ_* is the functor $\mathbf{\Delta} \to \mathbf{Top}$ which sends the set $\mathbf{n}$ to the geometric n-simplex Δ_n (the space of convex linear combinations of points in $\mathbf{n}$. Similarly, $\Delta[*]$ is the functor $\mathbf{\Delta} \to \mathbf{Sp}$ which sends $\mathbf{n}$ to the simplicial set $\Delta[n]$ corresponding to the n-simplex, considered as an ordered simplicial complex. $\mathrm{Repl}_*(X)$ is called the *simplicial replacement* of the diagram X [18, XII].

2.12 Remark. This construction of the homotopy colimit works in many simplicial model categories. Sometimes it's a good idea to insist that the values of the functor X are cofibrant objects in the base category; this guarantees the coproducts which enter into the formation of $\mathrm{Repl}_*(X)$ have the correct equivalence type and that the gluing involved in the realization construction is well-behaved. It is something of a surprise that this cofibrancy condition is not necessary in $\mathbf{Top}$ [30]. It *is* necessary in $\mathbf{Top}_*$ (2.9).

Remark. In general, it is necessary to be careful in forming the realization of an arbitrary simplicial object in a simplicial model category. (If you *are* careful, the realization of the simplicial object should be equivalent to its homotopy colimit as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$.) Being careful in this case means checking to see that the simplicial object is Reedy cofibrant [58, 5.2] [49, VII]. Roughly, "Reedy cofibrant" means that combined images of the degeneracy maps at each level sit cofibrantly inside the object at that level, a request which is not unreasonable, since taking the realization involves collapsing out the images of the degeneracy maps. Every simplicial object in $\mathbf{Sp}$ is Reedy cofibrant, but the same is not true of every simplicial object in $\mathbf{Top}$; this leads to the tale of the thick realization of a simplicial topological space [91, Appendix].

2.13 Exercise. Check that the above simplicial construction does in fact give a homotopy pushout in $\mathbf{Top}$ which agrees up to equivalence with the sort of homotopy pushout from slide 2–5. (The outcome of the simplicial construction isn't nearly as elaborate as it might look. Realizing a simplicial object involves collapsing the images of degeneracy maps, and almost all of the pieces of the simplicial replacement for a pushout diagram are degenerate.)

2.14 Exercise. Verify that the simplicial set $\Delta[n]$ is the functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ which sends $\mathbf{m}$ to $\mathrm{Hom}_{\mathbf{\Delta}}(\mathbf{m}, \mathbf{n})$. Conclude that for any simplicial set X, $\mathrm{Hom}_{\mathbf{Sp}}(\Delta[n], X)$ is naturally isomorphic to X_n.

★ **2.15 Exercise.** The following is not hard to prove, but it is completely implausible at first sight. The coend formula for the realization of a simplicial object X_* in $\mathbf{Sp}$ exhibits $|X_*|$ as a quotient

$$|X_*| = \coprod_n (\Delta[n] \times X_n) \,/ \sim$$

where in this case $\sim$ is an equivalence relation generated by the morphisms in $\mathbf{\Delta}^{\mathrm{op}}$. But X_* is just a simplicial object in the category of simplicial sets, as such it amounts to a functor

$$X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}} \quad \text{or} \quad X \colon \;:\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$$

The second way of looking at X leads to the definition of the diagonal $\mathrm{diag}(X)$; this is the simplicial set

$$\mathrm{diag}(X) : \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{diag}} \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathbf{Set} \, .$$

Prove that for any simplicial object X in $\mathbf{Sp}$, $|X|$ is isomorphic to $\mathrm{diag}(X)$.

2.16 Exercise. In $\mathbf{Top}$, interpret the above simplicial homotopy colimit of $X \to Y$ as the mapping cylinder of the map. What's the (simplicial) homotopy colimit of $X \to Y \to Z$? Interpret a homotopy colimit in $\mathbf{Top}$ indexed by an arbitrary category in terms of building blocks of this kind [18, XII.2].

★ **2.17 Exercise.** Verify that the nerve (1.7) of a category $\mathcal{C}$ can be identified as the homotopy colimit of the constant functor which assigns to each object of $\mathcal{C}$ the one-point space. The constant functor is treated as taking values in $\mathbf{Sp}$, and the homotopy colimit is in $\mathbf{Sp}$.

2.18 Exercise. The above construction of the homotopy colimit works for the category $\mathbf{Sp}_*$ of pointed simplicial sets (and also in the category $\mathbf{Top}_*$ of pointed spaces, as long as the values of the functor X are assumed to be cofibrant, see 2.12). Verify that the homotopy colimit in the pointed category is obtained by forming the homotopy colimit in the unpointed category and collapsing out the nerve of the index category (i.e., the unpointed homotopy colimit of the one-point basepoint functor).

2.19 Exercise. Draw a picture of a homotopy coequalizer in $\mathbf{Top}$.

☡☡ **2.20 Exercise.** Filtering Δ_* or $\Delta[*]$ by skeleta gives an increasing filtration of $\mathrm{hocolim}\, X$, which leads to a homology spectral sequence for $h_*(\mathrm{hocolim}\, X)$ (here h_* is any homology theory). Identify the E_2 page of the spectral sequence as

$$E_2(i,j) = \mathrm{colim}_i \, h_j(X)$$

where colim_i is the i'th (classical) left derived functor of the colimit functor $\mathbf{Ab}^{\mathcal{C}} \to \mathbf{Ab}$ and $h_j(X)$ is the functor $\mathcal{C} \to \mathbf{Ab}$ obtained by applying h_j to X.

☡☡ **2.21 Exercise.** In the above situation, show that the E_2 page can be identified more explicitly as

$$E_2(i,j) = H_i N(h_j(\mathrm{Repl}_*(X))) \, .$$

Here $h_j(\mathrm{Repl}_*(X))$ is the simplicial abelian group obtained from $\mathrm{Repl}_*(X)$ by applying E_j, N is normalization as in (1.13), and H_i is the i'th homology group of a chain complex.

☡☡ **2.22 Exercise.** In an exercise that, in light of the above two, might seem obscurely confusing, show that if $A \in \mathbf{Ab}^{\mathbf{\Delta}^{\mathrm{op}}}$ is a simplicial abelian group, then

$$\mathrm{colim}_i \, A \cong H_i N(A) \, .$$

Resolve the confusion by observing that if $B : \mathcal{C} \to \mathbf{Ab}$ is any functor, $\mathrm{colim}_*(B)$ can be computed by a simplicial replacement formula.

Slide 2-7

Slide 2-7

Homotopy limits and related constructions

$\mathcal{C}$, $\mathcal{S}$ homotopy theories

Homotopy limit for $X\colon \mathcal{C}\to\mathcal{S}$

$$\Delta\colon \mathcal{S}\leftrightarrow \mathcal{S}^{\mathcal{C}}\ :\mathrm{holim}$$

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}}(\Delta(Y),X)\sim \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}}(Y,\mathrm{holim}\,X)$$

$\mathrm{RKan}^{\mathbf{h}}_{F}$ **for** $X\colon \mathcal{C}\to\mathcal{S}$ **and** $F\colon \mathcal{C}\to\mathcal{D}$

$$F^*\colon \mathcal{S}^{\mathcal{D}}\leftrightarrow \mathcal{S}^{\mathcal{C}}\ :\mathrm{RKan}^{\mathrm{h}}_{F}$$

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}}(F^*Y,X)\cong \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{D}}}(Y,\mathrm{RKan}^{\mathrm{h}}_{F}\,X)$$

Homotopy end for $X\colon \mathcal{C}^{\mathbf{op}}\times\mathcal{C}\to\mathcal{S}$

$$\mathrm{hoend}\,X=\mathrm{holim}_{A(\mathcal{C})}\ \text{of}\ (a\to b)\mapsto X(a,b)$$

Remark. The opposite of a homotopy theory is a homotopy theory, so a hapless lecturer who was pressed for time could point out that homotopy limits are just homotopy colimits in the opposite category, and leave it at that. . .

In this slide, as in 2–3, $\mathcal{S}^{\mathcal{C}}$ and $\mathcal{S}^{\mathcal{D}}$ should really be $\mathcal{S}^{\mathrm{h}\mathcal{C}}$ and $\mathcal{S}^{\mathrm{h}\mathcal{D}}$

2.23 Exercise. Formulate the notion of right Kan extension, and show that right Kan extensions can be computed pointwise (2.3). It will probably be necessary to consider the *under categories* $d{\downarrow}F$:

$$\begin{tikzcd} & F(c), \arrow[dd, dashed, "F(h)"'] & c \arrow[dd, dashed, "h"] \\ d \arrow[ur] \arrow[dr] & & \\ & F(c'), & c' \end{tikzcd}$$

Identify some interesting right Kan extension.

2.24 Exercise. If $X,Y:\mathcal{C}\to\mathcal{S}$ are two functors between ordinary categories, there is a functor $H_{X,Y}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathbf{Set}$ given by

$$H_{X,Y}(c,c')=\mathrm{Hom}_{\mathcal{S}}(X(c),Y(c'))\,.$$

Check that the end of this functor is isomorphic to $\mathrm{Hom}_{\mathcal{S}^{\mathcal{C}}}(X,Y)$.

Remark. Suppose that $\mathcal{C}$ and $\mathcal{S}$ are homotopy theories. The mapping theory $\mathcal{S}^{\mathrm{h}\mathcal{C}}$ is defined so that if $X,Y:\mathcal{C}\to\mathcal{S}$ are objects of the theory, then

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}(X,Y)}\sim \mathrm{hoend}_{\mathcal{C}}\left[(c,c')\mapsto \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}}(X(c),Y(c'))\right]\ .$$

Slide 2-8

Another model category dividend

Theorem
If $\mathcal{S}$ admits a model category structure:

Existence
holim, $\operatorname{RKan}^{\mathrm{h}}_{F}$, hoend exist for target $\mathcal{S}$.

Realizability

- $\operatorname{holim} \sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}$,
- $\operatorname{RKan}^{\mathrm{h}}_{F} \sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}^{\mathcal{D}}$, and
- $\operatorname{hoend} \sim$ functor $\mathcal{S}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \to \mathcal{S}$.

Remark. This really does follow from the corresponding remark about homotopy colimits, etc., because the notion of a model category is self-dual (1.18).

Slide 2-9

Construction of holim, version I

Assumptions

- $\mathcal{S}$ admits a model structure
- $\mathcal{S}^{\mathcal{C}}$ admits the injective model structure
  (cofibrations are objectwise)

The assumptions hold if $\mathcal{S}$ is $\mathbf{Sp}$ ($\mathbf{Top}$?)

Conclusion

$$\operatorname{holim} X \sim \lim X^{\mathrm{f}}, \quad X \in \mathcal{S}^{\mathcal{C}}$$

★ **2.25 Exercise.** Determine what this says about homotopy pullbacks in $\mathbf{Sp}$ or $\mathbf{Top}$. What do you need to do to compute the homotopy limit of a tower

$$\cdots \to X_n \to X_{n-1} \to \cdots X_1 \to X_0$$

in a model category.

Slide 2-10

Slide 2-10

Construction of holim**, Version II**
$\mathbf{Top}, \mathbf{Sp}, \ldots$

(A) $\operatorname{Tot} Y$ **for** $Y : \Delta \to \{\mathbf{Top} \text{ or } \mathbf{Sp}\}$

$$\operatorname{Tot} Y = \begin{cases} \operatorname{Map}_{\mathbf{Top}^{\Delta}}(\Delta_*, Y) & \text{for } \mathbf{Top} \\ \operatorname{Map}_{\mathbf{Sp}^{\Delta}}(\Delta[*], Y) & \text{for } \mathbf{Sp} \end{cases}$$

(B) $\operatorname{Repl}^*(X) : \Delta \to \mathcal{S}$ **for** $X : \mathcal{C} \to \mathcal{S}$

$$\operatorname{Repl}^*(X)(\mathbf{n}) = \prod_{f:\ \{0\to 1\to\cdots\to n\}\to\mathcal{C}} f(n)$$

$\operatorname{holim} =$ **(A)** $+$ **(B)**

$$\operatorname{holim} X \sim \operatorname{Tot}\operatorname{Repl}^*(c \mapsto X(c)^{\mathrm{f}})$$

Remark. The phrase $c \mapsto X(c)^{\mathrm{f}}$ here stands for the functor that results from the objectwise process of making every value of the functor X fibrant. In $\mathbf{Top}$, this doesn't require any action. The dual issue (replacing X by $c \mapsto X(c)^{\mathrm{c}}$) does in fact come up in forming homotopy colimits, but I didn't emphasize it because (1) every object of $\mathbf{Sp}$ is cofibrant, and (2) by accident or good luck, depending on your point of view, the issue is not important if the target category is $\mathbf{Top}$.

★ **2.26 Exercise.** Interpret this formula for holim more explicitly (at least in $\mathbf{Top}$). A point in the holim is

- For each $c \in \mathcal{C}$, a point in $x_c \in X(c)$.
- For each arrow $f : c \to c'$ in $\mathcal{C}$, a path in $X(c')$ between $f(x_c)$ and $x_{c'}$.
- For each composition $c \to c' \to c''$ in $\mathcal{C}$, a 2-simplex in $X(c'')$ with (?).
- (?)

★ **2.27 Exercise.** What construction does this give for the homotopy pullback in $\mathbf{Top}$?

2.28 Exercise. Let G be a discrete group, and X a G-space, treated as a functor $\mathcal{C}_G \to \mathbf{Sp}$ (or $\mathcal{C}_G \to \mathbf{Top}$) (2.2). Show that $\operatorname{hocolim} X$ is the Borel construction $EG \times_G X$; in other words, the bundle over BG associated to the action of G on X. Check that $\operatorname{holim} X$ is the space of sections of this bundle. There is special terminology and notation for these homotopy limits and colimits: $\operatorname{holim} X$ is called the homotopy fixed point set of the action of G on X and denoted $X^{\mathrm{h}G}$, while $\operatorname{hocolim} X$ is called the homotopy orbit space of the action of G on X and denoted $X_{\mathrm{h}G}$.

☡☡ **2.29 Exercise.** Filtering Δ_* or $\Delta[*]$ by skeleta gives in this case a second quadrant homotopy spectral sequence for $\pi_* \operatorname{holim} X$. (Some problems do come up. you need a basepoint to define homotopy groups, and there's no hope of choosing a basepoint if for instance $\operatorname{holim} X$ is empty. So assume that X is a diagram of pointed objects to guarantee a distinguished point in $\operatorname{holim} X$.) Compare this with 2.20. Show that the E^2 page of the spectral sequence is given by

$$E_2(-i, j) = \lim^i \pi_j X$$

where $\lim^i$ is the i'th right derived functor of $\lim\colon \mathbf{Ab}^{\mathcal{C}} \to \mathbf{Ab}$. Or maybe not; remember that π_0 is just a set, while π_1 is a possibly nonabelian group. This spectral sequence might occupy more second quadrant than you expect [14].

Slide 2-11

Slide 2-11

Properties of homotopy (co)limits

for functors $\mathcal{C} \to \mathcal{S}$

Universal maps

$$\begin{aligned} \operatorname{hocolim} X &\to \operatorname{colim} X \\ \lim X &\to \operatorname{holim} X \end{aligned}$$

Homotopy invariance

$$X \xrightarrow{\sim} Y \quad \Longrightarrow \begin{cases} \operatorname{hocolim} X \xrightarrow{\sim} \operatorname{hocolim} Y \\ \operatorname{holim} X \xrightarrow{\sim} \operatorname{holim} Y \end{cases}$$

Mapping adjointness

$$\begin{aligned} \operatorname{Hom}^{\mathsf{h}}_{\mathcal{S}}(\operatorname{hocolim} X, Y) &\sim \operatorname{holim}_{\mathcal{C}^{\mathrm{op}}} \text{ of } c \mapsto \operatorname{Hom}^{\mathsf{h}}(X(c), Y) \\ \operatorname{Hom}^{\mathsf{h}}_{\mathcal{S}}(Y, \operatorname{holim} X) &\sim \operatorname{holim}_{\mathcal{C}} \text{ of } c \mapsto \operatorname{Hom}^{\mathsf{h}}(Y, X(c)) \end{aligned}$$

Remark. The two homotopy limits on the right in the last display are homotopy limits of functors into $\mathbf{Sp}$ (or $\mathbf{Top}$).

Slide 2-12

Slide 2-12

Mysteries of (I) and (II) revealed

for $X\colon \mathcal{C} \to \mathbf{Sp}$

Homotopy colimit

Suppose that X takes on cofibrant values.

$$\begin{array}{rcccc} \operatorname{hocolim} X & \sim & * & \times^{\mathsf{h}}_{\mathcal{C}} & X \\ \text{(I)} & \sim & * & \times_{\mathcal{C}} & X^{\mathsf{c}}_{\text{proj}} \\ \text{(II)} & \sim & *^{\mathsf{c}}_{\text{proj}} & \times_{\mathcal{C}} & X \end{array}$$

Homotopy limit

Suppose that X takes on fibrant values.

$$\begin{array}{rcccc} \operatorname{holim} X & \sim & \operatorname{Hom}^{\mathsf{h}}(& *, & X) \\ \text{(I)} & \sim & \operatorname{Map}(& *, & X^{\mathsf{f}}_{\text{inj}}) \\ \text{(II)} & \sim & \operatorname{Map}(& *^{\mathsf{c}}_{\text{proj}}, & X) \end{array}$$

This is a lot like homological algebra: if you want to compute the derived tensor product of two chain complexes, you can make either one or the other projective, it doesn't matter which.

Remark. In the first box, $*$ is a contravariant constant functor, and in the second box it's a covariant one. The simplicial formula for $\operatorname{hocolim} X$ arises from taking the following cofibrant model for $*$ in the projective model category structure on $\mathbf{Sp}^{\mathcal{C}^{\mathrm{op}}}$:

$$(*^{\mathrm{c}}_{\mathrm{proj}})(x) = \mathrm{N}(x{\downarrow}\mathcal{C})\,.$$

See 2.23 for a description of the under category $c{\downarrow}\mathcal{C}$; for brevity, $\mathcal{C}$ here stands for the identify functor on $\mathcal{C}$.

2.30 Exercise. Check that this formula does provide a cofibrant model for $*$, and that $(*^{\mathrm{c}}_{\mathrm{proj}}) \times_{\mathcal{C}} X$ does in fact describe the simplicial model for $\operatorname{hocolim} X$.

In the second box, Map stands for the simplicial mapping complex and $*^{\mathrm{c}}_{\mathrm{proj}}$ is a cofibrant model for $*$ in the projective model category structure on $\mathbf{Sp}^{\mathcal{C}}$. The cosimplicial formula for $\operatorname{holim} X$ arises from taking the cofibrant model given by

$$(*^{\mathrm{c}}_{\mathrm{proj}})(x) = \mathrm{N}(\mathcal{C}{\downarrow}x)\,.$$

Lecture 3.
Spaces from categories

In this section we'll look at various constructions on categories which give interesting results when the nerve functor is applied. At then end there's a short discussion of terminal functors and initial functors (generalization of terminal objects and initial objects). The basic properties of these functors don't directly involve nerves of categories, but the functors are tied to nerves in a couple of different ways.

Slide 3-1

Slide 3-1

Categories vs. Spaces

Geometrization

$$\mathrm{N} : \mathbf{Cat} \to \mathbf{Sp}$$

Properties

- $F\colon \mathcal{C} \to \mathcal{D} \quad \mapsto \quad \mathrm{N}\,F\colon\ \mathrm{N}\,\mathcal{C} \to \mathrm{N}\,\mathcal{D}$
- $\tau\colon F \overset{\cdot}{\to} G \quad \mapsto \quad H\colon\ \mathrm{N}\,\mathcal{C} \times \Delta[1] \to \mathcal{D}$

Advantages

- Categories more visible than spaces.
- Natural transformations more accessible than homotopies.

Can homotopy colimits (coends) be built in?

$$F\colon \mathcal{C} \to \mathbf{Cat} \implies \operatorname{hocolim}_{\mathcal{C}} \mathrm{N}(F) \quad \sim \quad \mathrm{N}(?)$$

It is pretty clear that if $\mathcal{C}$ is a category then the nerve $\mathrm{N}\,\mathcal{C}$ of $\mathcal{C}$, considered up to equivalence in $\mathbf{Sp}$, does not capture a lot of the structure in $\mathcal{C}$.

3.1 Exercise. Give an example of two categories $\mathcal{C}$ and $\mathcal{D}$ such that $\mathrm{N}\,\mathcal{C}$ and $\mathrm{N}\,\mathcal{D}$ are equivalent (i.e. weakly homotopy equivalent) as simplicial sets, but such that $\mathcal{C}$ and $\mathcal{D}$ are not equivalent as categories.

But there is something to be said.

3.2 Exercise. Show that $\mathrm{N}\,\mathcal{C}$ and $\mathrm{N}\,\mathcal{D}$ are weakly equivalent as simplicial sets if and only if the category pairs $(\mathcal{C}, \mathcal{C})$ and $(\mathcal{D}, \mathcal{D})$ give equivalent homotopy theories. (In other words, the nerve $\mathrm{N}\,\mathcal{C}$, considered up to the usual notion of equivalence for simplicial sets, exactly captures the homotopy theory that results from inverting all of the arrows in $\mathcal{C}$.)

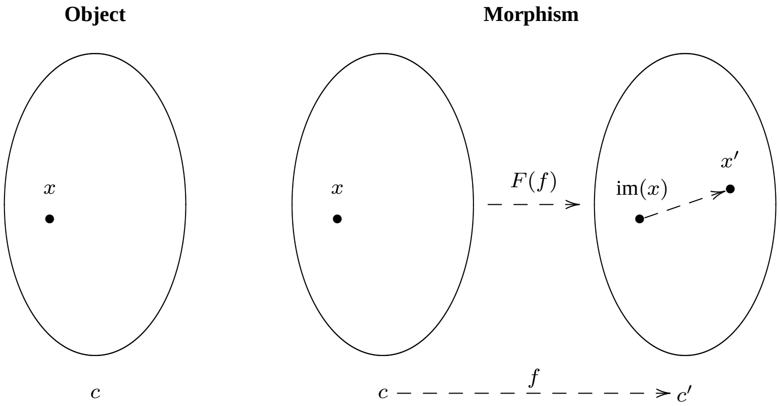
3.3 Exercise. Show that $\mathrm{N}\,\mathcal{C}$, considered up to isomorphism of simplicial sets, determines $\mathcal{C}$ up to isomorphism of categories. (Hint: look at the left adjoint to $\mathrm{N} : \mathbf{Cat} \to \mathbf{Sp}$).

Trying to interpolate between the previous two exercises might well lead to the notion of a quasicategory.

Slide 3-2

The Grothendieck Construction

$F : \mathcal{C} \to \mathbf{Cat} \quad \Longrightarrow \quad$ category $\mathcal{C} \ltimes F$



The Grothendieck construction $\mathcal{C} \ltimes F$ is also denoted $\mathcal{C} \wr F$, $\int_{\mathcal{C}} F$, or in general anything else that somehow connotes adding up the values of F over $\mathcal{C}$. The slide is supposed to indicate that an object of the category $\mathcal{C} \ltimes F$ is a pair (c, x), where $c \in \mathcal{C}$ and $x \in F(c)$; a morphism $(c, x) \to (c', x')$ is then a pair (f, g), where f is a morphism $c \to c'$ in $\mathcal{C}$ and g is a morphism $F(f)(x) \to x'$ in $F(c')$.

3.4 Exercise. Suppose that F is the constant functor $\mathcal{C} \to \mathbf{Cat}$ with value $\mathcal{D}$. Show that $\mathcal{C} \ltimes F$ is the product category $\mathcal{C} \times \mathcal{D}$.

3.5 Exercise. Suppose that the group G acts on the group H by automorphisms, and let $F : \mathcal{C}_G \to \mathbf{Grp}$ be the corresponding functor (2.2). Show that $\mathcal{C} \ltimes F$ is the category corresponding to the semidirect product group $G \ltimes H$. (The intent here is that the $\ltimes$ symbol is oriented so that the closed triangle points sideways towards the normal subgroup.)

Slide 3-3

Thomason's Theorem

$F : \mathcal{C} \to \mathbf{Cat}$

Theorem

$$\mathrm{N}(\mathcal{C} \ltimes F) \quad \sim \quad \mathrm{hocolim}_{\mathcal{C}}\, \mathrm{N}(F)$$

Example

$F : \mathcal{C} \to \mathbf{Set}$, $\mathcal{C} \ltimes F =$ Transport Category

- Object: $(c, x),\ c \in \mathcal{C},\ x \in F(c)$
- Morphism: $f : c \to c'$ with $f(x) = x'$

N(Transport Category) $\sim$ hocolim F

The reference for Thomason's theorem is [97].

3.6 Exercise. In the above situation, show that the nerve of the transport category is actually isomorphic to $\operatorname{hocolim} F$, where $\operatorname{hocolim} F$ is formed according to the simplicial formula.

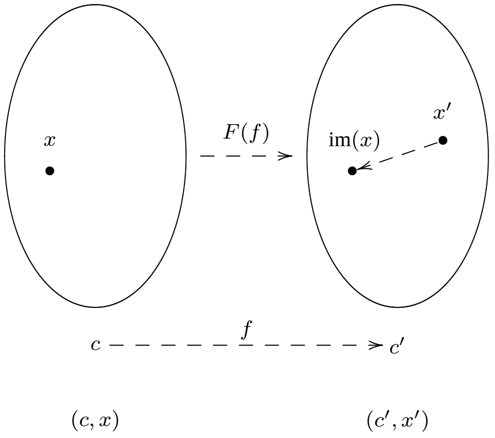
3.7 Exercise. Consider the possibility that $\mathcal{C}\ltimes F$ might be the homotopy colimit of F in the category of homotopy theories. Can you find any evidence for or against this suggestion? Is there any reason to believe that the nerve functor from categories to simplicial sets should commute with (homotopy) colimits?

Slide 3-4

Slide 3-4

Variations on $\mathcal{C}\ltimes F$ (all $\sim$ on N)
$F : \mathcal{C} \to \mathbf{Cat}$

$\mathcal{C}\ltimes(F^{\mathrm{op}})$



Also have $(\mathcal{C}\ltimes F)^{op}$ and $(\mathcal{C}\ltimes(F^{op}))^{op}$!

This slide refers implicitly to the fact that for any category $\mathcal{D}$, the nerve of $\mathcal{D}^{\mathrm{op}}$ is naturally equivalent in $\mathbf{Sp}$ to the nerve of $\mathcal{D}$. We'll see why in a minute. Meanwhile, we've come up with four different categories whose nerves represent $\operatorname{hocolim} \mathrm{N}(F)$.

3.8 Exercise. Draw $(\mathcal{C}\ltimes F)^{\mathrm{op}}$ and $(\mathcal{C}\ltimes(F^{\mathrm{op}}))^{\mathrm{op}}$.

3.9 Exercise. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor, and that $d \in \mathcal{D}$. Show that $\mathrm{N}(d{\downarrow}F)$ is the homotopy colimit over $\mathcal{C}$ of the functor $c \mapsto \operatorname{Hom}_{\mathcal{D}}(d, F(c))$. Show that $\mathrm{N}(F{\downarrow}d)$ is the homotopy colimit over $\mathcal{C}^{\mathrm{op}}$ of the functor $c \mapsto \operatorname{Hom}_{\mathcal{D}}(F(c), d)$.

Note that $\mathcal{C}\ltimes F^{\mathrm{op}}$ is quite weird, in that the two arrows that comprise a morphism in some sense go in different directions.

For a functor $F : \mathcal{C} \to \mathbf{Cat}$, call the categories $F(c)$, $c \in \mathcal{C}$, the *fibre categories* for the obvious projection functor $\mathcal{C}\ltimes F \to \mathcal{C}$.

3.10 Exercise. Show that if all of these fibre categories have contractible nerve, then the projection $\mathrm{N}(\mathcal{C}\ltimes F) \to \mathrm{N}(\mathcal{C})$ is an equivalence of simplicial sets. (Use the homotopy invariance of homotopy colimits, the fact that the nerve of $\mathcal{C}$ is $\mathrm{hocolim}_{\mathcal{C}} *$, and some sort of functoriality which I haven't stated but which must be built into Thomason's theorem.)

3.11 Exercise. Show that adjoint functors

$$F\colon \mathcal{C} \leftrightarrow \mathcal{D} : G$$

induce inverse equivalences (up to homotopy) between $\mathrm{N}\,\mathcal{C}$ and $\mathrm{N}\,\mathcal{D}$. Hint: natural transformations give homotopies (1.8). Deduce (again 1.9) that any category with an initial object or a terminal object has a contractible nerve.

Recall that if $\mathcal{C}$ is a category, $A(\mathcal{C})$ stands for the twisted arrow category of $\mathcal{C}$ (2.1). There are natural functors $A(\mathcal{C}) \to \mathcal{C}$ and $A(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$.

★ **3.12 Exercise.** Show that $A(\mathcal{C}) \to \mathcal{C}$ can be identified as $\mathcal{C}\ltimes F \to \mathcal{C}$ for some functor $\mathcal{C} \to \mathbf{Cat}$. Observe that all of the fibre categories have contractible nerves. Show that the same is true of $A(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$. Conclude that in the diagram

$$\mathcal{C} \leftarrow A(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$$

both of the functors induce equivalences on nerves [85, p. 94].

Slide 3-5

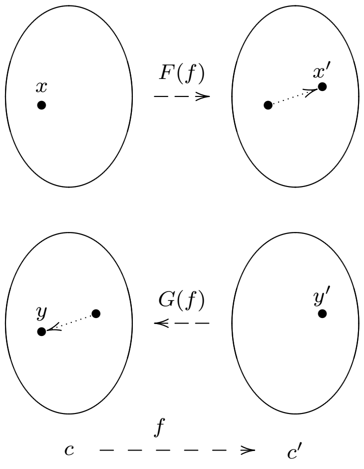
Slide 3-5

Extension to homotopy coends

$F\colon \mathcal{C} \to \mathbf{Cat},\ G\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat} \qquad \mathrm{N}(G\rtimes\mathcal{C}\ltimes F) \sim \mathrm{N}(G) \times^h_{\mathcal{C}} \mathrm{N}(F)$

Morphism

$G\rtimes\mathcal{C}\ltimes F = \{(c,x,y)\}$



There’s a discussion of the Grothendieck construction model for homotopy coends in [36, §9]. It’s terse, and I don’t thing the phrase “homotopy coend”’ appears anywhere, but the argument does produce an equivalence between the nerve of a two-sided Grothendieck construction and a simplicial model for the homotopy coend along the lines of slide 2–6.

The slide signifies that an object of $G{\rtimes}\mathcal{C}{\ltimes}F$ is a triple (c, x, y) where $c \in \mathcal{C}$, $x \in F(c)$, and $y \in G(c)$. A morphism $(c, x, y) \to (c', x', y')$ is itself a triple, consisting of a map $f : c \to c'$ in $\mathcal{C}$, a map $F(f)(x) \to x'$ in $F'(c)$ and a map $G(f)(y') \to y$ in $G(c)$. It should be pretty clear how to compose these triples.

The reader is left to ponder the multitudinous variants ($(G^{\text{op}}){\rtimes}\mathcal{C}{\ltimes}F$, $G{\rtimes}\mathcal{C}{\ltimes}(F^{\text{op}})$, . . .), all of which have equivalent nerves. One point that sometimes disorients me a bit is the fact that the opposite category construction $(-)^{\text{op}}$ is a *covariant* functor on $\mathbf{Cat}$.

Working with Grothendieck constructions can be unexpectedly tricky. Here’s an example (which when sorted out leads to yet larger collections of categories with equivalent nerves). Suppose that $\mathcal{C}$ is category with equivalences $\mathcal{E}$ (i.e. a homotopy theory) and that x and y are objects of $\mathcal{C}$. Consider the diagram category $\mathrm{ZiZaZig}(x, y)$ described by the following picture:

$$\begin{tikzcd}
& A \arrow[r] \arrow[dl, "\sim"] \arrow[dd, dashed, "\sim"] & B \arrow[dd, dashed, "\sim"] & \\
x & & & y \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
& A' \arrow[r] \arrow[ul, "\sim"] & B' &
\end{tikzcd}$$

The solid arrow zigzags are the objects of the category, and the pictured commutative diagram (which includes the dashed arrows) gives a morphism between the upper zigzag and the lower one. The problem is to understand the homotopy type of the nerve of this category. There are several ways to take this homotopy type apart; the most symmetrical one is to observe that an object consists of a triple (U, V, h), where $U : A \to x$ is an object of the over category $\mathcal{E}{\downarrow}x$, $V : y \to B$ is an object of the under category $y{\downarrow}\mathcal{E}$, and h is an element of $F(U, V) = \mathrm{Hom}_{\mathcal{C}}(A, B)$. The functor F is contravariant in U, covariant in V, and takes values in sets ($=$ discrete categories). A morphism $(U, V, h) \to (U', V', h')$ in $\mathrm{ZiZaZig}(x, y)$ consists of a morphism $u : U \to U'$ in $\mathcal{E}{\downarrow}x$ and a morphism $v : V \to V'$ in $y{\downarrow}\mathcal{E}$ such that $u^*(h') = v_*(h)$ (this is the commutativity of the central square).

A first guess based on looking at the variances might have

$$\mathrm{ZiZaZig}(x, y) \cong \mathcal{D}{\ltimes}F \quad \text{for} \quad \mathcal{D} = (\mathcal{E}{\downarrow}x)^{\text{op}} \times (y{\downarrow}\mathcal{E})$$

but this would be hasty.

3.13 Exercise. Show that $((\mathcal{E}{\downarrow}x)^{\mathrm{op}} \times (y{\downarrow}\mathcal{E}))\ltimes F$ is the diagram category

$$\begin{tikzcd}
& A \arrow[r] \arrow[dl, "\sim"'] & B \arrow[dd, dashed, "\sim"] & \\
x & & & y \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
& A' \arrow[r] \arrow[ul, "\sim"] \arrow[uu, dashed, "\sim"] & B' &
\end{tikzcd}$$

A better formula is this

$$\mathrm{ZiZaZig}(x, y) \cong (y{\downarrow}\mathcal{E}) \ltimes F \rtimes (\mathcal{E}{\downarrow}x)$$

3.14 Exercise. Make sense of the above formula. In particular, if G is a contravariant functor $\mathcal{C} \to \mathbf{Cat}$, define $G \rtimes \mathcal{C}$ and observe that its nerve is equivalent to $\mathrm{hocolim}_{\mathcal{C}^{\mathrm{op}}} \mathrm{N}(G)$. Observe nevertheless that $G \rtimes \mathcal{C}$ is not quite isomorphic to $(\mathcal{C}^{\mathrm{op}})\ltimes G$, or even to the opposite of this last category. Observe further that the notation $(y{\downarrow}\mathcal{E}) \ltimes F \rtimes (\mathcal{E}{\downarrow}x)$ above now makes sense in two ways, either as

$$[\,(y{\downarrow}\mathcal{E}) \ltimes F\,] \;\rtimes\; (\mathcal{E}{\downarrow}x)$$

or as

$$(y{\downarrow}\mathcal{E}) \;\ltimes\; [\,F \rtimes (\mathcal{E}{\downarrow}x)\,]$$

Show that these two interpretations give isomorphic categories (thank goodness) and that the nerve of this object (and hence the nerve of $\mathrm{ZiZaZig}(x, y)$) is in fact equivalent to $\mathrm{hocolim}_{(\mathcal{E}\downarrow x)^{\mathrm{op}}\times(y\downarrow\mathcal{E})} H$.

Slide 3-6

Slide 3-6

The parallel universe of Hom and $\otimes$

Setup

$R \to S$ aug. k-algebras	$F\colon \mathcal{C} \to \mathcal{D}$ categories
${}_R M,\ {}_S N$	${}_{\mathcal{C}}X,\ {}_{\mathcal{D}}Y$

Dictionary

$k \otimes_R M$	$\mathrm{hocolim}_{\mathcal{C}}\, X \sim * \times^{\mathrm{h}}_{\mathcal{C}} X$
$\mathrm{Hom}_R(k, M)$	$\mathrm{holim}_{\mathcal{C}}\, X \sim \mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(*, X)$
$S \otimes_R M$	$\mathrm{LKan}^{\mathrm{h}}_F(X) \sim \mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} X$
$\mathrm{Hom}_R(S, M)$	$\mathrm{RKan}^{\mathrm{h}}_F(X) \sim \mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(\mathcal{D}, X)$

At this point we're beginning to work our way towards some preliminary applications of the Grothendieck construction; more applications will come up later on.

The notation ${}_{\mathcal{C}}X$ on this slide denotes for short that X is a covariant functor from $\mathcal{C}$ to, say, $\mathbf{Sp}$. In the notation $\mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} X$, $\mathcal{D}$ is shorthand for the contravariant functor $\mathcal{C} \to \mathbf{Set}^{\mathcal{D}}$ given by

$$c \mapsto \mathrm{Hom}_{\mathcal{D}}(F(c), -)\,.$$

(Note that the object on the right actually is a functor from $\mathcal{D}$ to $\mathbf{Set}$.) The indicated homotopy coend can be computed objectwise in $\mathcal{D}$; in symbols

$$(\mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} X)(d) = \mathrm{Hom}_{\mathcal{D}}(F(-), d) \times^{\mathrm{h}}_{\mathcal{C}} X\,.$$

The fact that $\mathcal{D}$ stands for a $\mathcal{D}$ Hom-functor which is contravariant on $\mathcal{C}$ is silently implied by the fact that $\mathcal{D}$ appears on the left side of the homotopy coend.

Similarly, $\mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(\mathcal{D}, X)$ stands for the functor $\mathcal{D} \to \mathbf{Sp}$ which sends d to

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(\mathrm{Hom}_{\mathcal{D}}(d, F(-)), X)\,.$$

The fact that $\mathcal{D}$ here represents a covariant Hom-functor on $\mathcal{C}$ is necessitated by the fact that $\mathcal{D}$ appears inside of a $\mathrm{Hom}^{\mathrm{h}}$ construction in which the second component, namely X, is covariant.

The parallel universes here involve two constructions (restriction of a module along a ring homomorphism, pullback of a diagram along a functor), each of which has both left and right adjoints. In both cases there's an underlying homotopy theory (at least if you replace modules over the rings by chain complexes of modules) and both the left adjoint and the right adjoint can be usefully derived (although the slide doesn't refer to the possibility of deriving the algebraic constructions). The main point of the slide is psychological rather than mathematical: if you use notation for left and right homotopy Kan extensions which is similar to familiar algebraic notation, you find yourself led to correct conclusions.

3.15 Exercise. Find some reason to believe that the indicated formulas for the right and left homotopy Kan extensions are correct.

For the material on the next few slides, see [57] or [39, §9].

Slide 3-7

Slide 3-7

Properties of Kan extensions

Transitivity (pushing forward over functors)

$$\begin{tikzcd} M, R \arrow[r] \arrow[rr, bend right] & S \arrow[r] & T & X, \mathcal{C} \arrow[r] \arrow[rr, bend right] & \mathcal{D} \arrow[r] & \mathcal{S} \end{tikzcd}$$

On the left

Algebra $T \otimes_R M \cong T \otimes_S S \otimes_R M$

Topology $\mathcal{S} \times^{\mathrm{h}}_{\mathcal{C}} X \sim \mathcal{S} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} X$

On the right

Algebra $\mathrm{Hom}_R(T, M) \cong \mathrm{Hom}_S(T, \mathrm{Hom}_R(S, M))$

Topology $\mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(\mathcal{S}, X) \sim \mathrm{Hom}^{\mathrm{h}}_{\mathcal{D}}(\mathcal{S}, \mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(\mathcal{D}, X))$

3.16 Exercise. What does this transitivity say about homotopy colimits?

3.17 Exercise. Check the transitivity properties by using adjointness properties of the homotopy Kan extensions.

Slide 3-8

Slide 3-8

Does pulling back preserve $\mathrm{hocolim}$?

Cofinality (pulling back)

$R \to S, \quad {}_S N$ $\qquad\qquad$ $\mathcal{C} \to \mathcal{D}, \quad {}_{\mathcal{D}} Y$

On the left

Algebra $\forall N : \quad k \otimes_R N \quad \cong \quad k \otimes_S N$?

Topology $\forall Y : \quad \mathrm{hocolim}\, F^*(Y) \sim \quad \mathrm{hocolim}\, Y$?

Topology $\forall Y : \quad * \times^{\mathrm{h}}_{\mathcal{C}} Y \quad \sim \quad * \times^{\mathrm{h}}_{\mathcal{D}} Y$?

Solution

Algebra $k \otimes_R N \cong (k \otimes_R S) \otimes_S N,$ want $\boxed{k \otimes_R S \cong k}$

Topology $* \times^{\mathrm{h}}_{\mathcal{C}} Y \sim (* \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{D}) \times^{\mathrm{h}}_{\mathcal{D}} Y$ want $\boxed{* \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{D} \sim *}$

In this slide, $* \times^{\mathrm{h}}_{\mathcal{C}} Y$ stands for $* \times^{\mathrm{h}}_{\mathcal{C}} F^*Y$, where F denotes the functor $\mathcal{C} \to \mathcal{D}$. This notation makes the analogy with the ring context clearer. Given $F\colon \mathcal{C} \to \mathcal{D}$ the slide looks at the question of whether for every $Y : \mathcal{D} \to \mathbf{Sp}$, the natural map $\mathrm{hocolim}\, F^*Y \to \mathrm{hocolim}\, Y$ is an equivalence.

3.18 Exercise. (for fun) Can you think of any augmented k-algebra maps $R \to S$ for which $k \otimes_R S \cong k$? What about maps for which the analogous statement is true for derived tensor products?

Terminal functors

Definition 3. $F : \mathcal{C} \to \mathcal{D}$ *terminal* if $* \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{D} \sim *$.

Theorem
$F : \mathcal{C} \to \mathcal{D}$ terminal, ${}_{\mathcal{D}}Y \implies \operatorname{hocolim}_{\mathcal{C}} Y \sim \operatorname{hocolim}_{\mathcal{D}} Y$

Interpretation (Grothendieck construction!)

$$(* \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{D})(d) = \mathrm{N}(d{\downarrow}F) \qquad \begin{tikzcd} & F(c),\ c \arrow[dd, dashed, "F(h)"', "h"] \\ d \arrow[ur] \arrow[dr] & \\ & F(c'),\ c' \end{tikzcd}$$

This theorem guarantees that if for each $d \in \mathcal{D}$ the under category $d{\downarrow}F$ has a contractible nerve, then $F : \mathcal{C} \to \mathcal{D}$ preserves homotopy colimits (in other words, for each $Y : \mathcal{D} \to \mathbf{Sp}$, $\operatorname{hocolim} F^*Y \sim \operatorname{hocolim} Y$.

3.19 Exercise. Prove that this is an if and only if condition. Hint: consider the functors on $\mathcal{D}$ given by $\mathrm{Hom}_{\mathcal{D}}(d, -)$ for various d. Show that each one has a contractible homotopy colimit (use the Grothendieck construction to interpret this in category theoretical terms). Investigate what it means for $F^* \mathrm{Hom}_{\mathcal{D}}(d, -)$ to have a contractible homotopy colimit.

3.20 Exercise. (Sanity check.) Show that if τ is a terminal object of a category $\mathcal{C}$, then the inclusion $\{\tau\} \to \mathcal{C}$ is a terminal functor.

3.21 Exercise. Given $F : \mathcal{C} \to \mathcal{D}$, prove that if for each $d \in \mathcal{D}$ the nerve of $d{\downarrow}F$ is *connected*, then F preserves arbitrary colimits. Is this an if and only if statement?

3.22 Exercise. Prove Quillen's Theorem A, which states that if $F : \mathcal{C} \to \mathcal{D}$ is a functor with the property that for each $d \in \mathcal{D}$ the under category $d{\downarrow}F$ has nerve equivalent to a point, then F induces an equivalence on nerves.

Somewhat trickier is Quillen's Theorem B, which states the following. Suppose that the nerve of $\mathcal{D}$ is connected (to make the statement simpler) and that for each map $f : d \to d'$ in $\mathcal{D}$ the (obvious) induced map $\mathrm{N}(d'{\downarrow}F) \to \mathrm{N}(d{\downarrow}F)$ is an equivalence. Then the homotopy fibre of the map $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ is equivalent to $\mathrm{N}(d{\downarrow}F)$ for any $d \in \mathcal{D}$. Assume the following statement:

Theorem 3.23. *Suppose that* $\mathrm{N}(\mathcal{D})$ *is connected, and that* $F : \mathcal{D} \to \mathbf{Sp}$ *is a functor which takes each morphism of* $\mathcal{D}$ *to an equivalence in* $\mathbf{Sp}$. *Then the homotopy fibre of the natural map* $\operatorname{hocolim}_{\mathcal{D}} F \to \operatorname{hocolim}_{\mathcal{D}} * = \mathrm{N}(\mathcal{D})$ *is equivalent in a natural way to* $F(d)$ *for any* $d \in \mathcal{D}$.

3.24 Exercise. Given Theorem 3.23, prove Quillen's Theorem B. Do this by taking the Grothendieck construction of the functor $d \mapsto (d{\downarrow}F)$ on $\mathcal{D}^{\mathrm{op}}$ and arguing that for categorical reasons (natural transformations, adjoint functors, etc.) the nerve of this category is equivalent to $\mathrm{N}(\mathcal{C})$.

3.25 Exercise. Prove another version of Quillen's Theorem B in which the under categories are replaced by over categories. (Note that $d \mapsto (F{\downarrow}d)$ gives a covariant functor $\mathcal{D} \to \mathbf{Cat}$.

3.26 Exercise. This represents an attempt to calculate the homotopy fibre of $\mathrm{N}(F)\colon \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ when neither version of Quillen's Theorem B applies. The idea is that two directions competing with one another are bound to be better than a single direction (and if something doesn't work the first time, try it again). For each $d \in \mathcal{D}$, let $d \updownarrow F$ be the category

$$\begin{tikzcd}
& d_0 \arrow[r] \arrow[ld] \arrow[dd, dashed] & F(c), \arrow[dd, dashed, "F(f)"'] & c \arrow[dd, dashed, "f"] \\
d & & & \\
& d_0' \arrow[r] \arrow[lu] & F(c'), & c'
\end{tikzcd}$$

The formula $d \mapsto (d \updownarrow F)$ gives a functor $A_2\colon \mathcal{D} \to \mathbf{Cat}$. Use the argument in 3.24 to calculate the homotopy fibre of $\mathrm{N}(F)$ under the assumption that A_2 sends each morphism of $\mathcal{D}$ to a functor which induces an equivalence on nerves. Now prove the same thing with A_2 replaced by A_{2n}, where $A_{2n}(d)$ is the category $d \updownarrow^{2n} F$:

$$\begin{tikzcd}
& d_0 \arrow[r] \arrow[ld] \arrow[dd, dashed] & d_1 \arrow[dd, dashed] & d_2 \arrow[l] \arrow[r] \arrow[dd, dashed] & {} & & {} & d_{2n-2} \arrow[l] \arrow[r] \arrow[dd, dashed] & F(c), \arrow[dd, dashed, "F(f)"'] & c \arrow[dd, dashed, "f"] \\
d & & & & & \cdots & & & & \\
& d_0' \arrow[r] \arrow[lu] & d_1' & d_2' \arrow[l] \arrow[r] & {} & & {} & d_{2n-2}' \arrow[l] \arrow[r] & F(c'), & c'
\end{tikzcd}$$

Can you see the homotopy fibre taking shape? What happens in the colimit? (Another question: is there any advantage to using patterns with two adjacent arrows pointing in the same direction?)

Slide 3-10

Slide 3-10

Does pulling back preserve holim?

Cofinality (pulling back)
$R \to S, \quad {}_S N$ $\qquad$ $\mathcal{C} \to \mathcal{D}, \quad {}_{\mathcal{D}}Y$

On the right
Algebra $\forall N : \quad \operatorname{Hom}_R(k, N) \quad \cong \quad \operatorname{Hom}_S(k, N)$?
Topology $\forall Y : \quad \operatorname{holim} F^*(Y) \sim \quad \operatorname{holim} Y$?
Topology $\forall Y : \quad \operatorname{Hom}^{\mathsf{h}}_{\mathcal{S}^{\mathcal{C}}}(*, Y) \quad \sim \quad \operatorname{Hom}^{\mathsf{h}}_{\mathcal{S}^{\mathcal{D}}}(*, Y)$?

Solution
Algebra $\operatorname{Hom}_R(k, N) \cong \operatorname{Hom}_S(S \otimes_R k, N)$ $\boxed{S \otimes_R k \cong k}$
Topology $\operatorname{Hom}^{\mathsf{h}}_{\mathcal{S}^{\mathcal{C}}}(*.Y) \cong \operatorname{Hom}^{\mathsf{h}}_{\mathcal{S}^{\mathcal{D}}}(\mathcal{D} \times^{\mathsf{h}}_{\mathcal{C}} *, Y)$ $\boxed{\mathcal{D} \times^{\mathsf{h}}_{\mathcal{C}} * \sim *}$

Here, given $F\colon \mathcal{C} \to \mathcal{D}$, the question is whether for every functor $Y : \mathcal{D} \to \mathbf{Sp}$, the natural map $\operatorname{holim} Y \to \operatorname{holim} F^*Y$ is an equivalence.

Slide 3-11

Slide 3-11

Initial functors

Definition 4. $F : \mathcal{C} \to \mathcal{D}$ *initial* if $\mathcal{D} \times^{\mathsf{h}}_{\mathcal{C}} * \sim *$.

Theorem
$F : \mathcal{C} \to \mathcal{D}$ initial, ${}_{\mathcal{D}}Y \implies \operatorname{holim}_{\mathcal{C}} Y \sim \operatorname{holim}_{\mathcal{D}} Y$

Interpretation (Grothendieck construction again!)

$$\begin{tikzcd}
 & c, F(c) \arrow[dd, dashed, shift right, "h"'] \arrow[dd, dashed, shift left, "F(h)"] \arrow[dr] & \\
(\mathcal{D} \times^{\mathsf{h}}_{\mathcal{C}} *)(d) = \mathrm{N}(F{\downarrow}d) & & d \\
 & c', F(c') \arrow[ur] &
\end{tikzcd}$$

3.27 Exercise. (Another sanity check.) Verify that if ι is an initial object of $\mathcal{D}$, then the inclusion $\{\iota\} \to \mathcal{D}$ is an initial functor.

3.28 Exercise. Does the converse of the theorem on the slide hold? (In other words, if a functor F preserves all homotopy limits, is F initial?)

The theorem on the slide is Bousfield and Kan's cofinality theorem for homotopy limits [18, XI.9.2], which they use [18, XI.10.6] to identify the R-completion as what in our terms would be called a homotopy right Kan extension (slide 5–9).

Lecture 4.
Homology decompositions

In this lecture, we'll use the machinery of homotopy colimits and Grothendieck constructions to construct homology approximations for classifying spaces of finite groups.

Slide 4-1

Slide 4-1

Approximation data for BG

Approximating BG by B(subgroups)

- $F : \mathcal{D} \to \mathbf{Sp}$
- $\forall d,\, F(d) \sim BH_d, \quad H_d \subset G$
- Approximation: $\operatorname{hocolim} F \to BG$

$$\begin{tikzcd}
BH_1 \arrow[loop above] \arrow[dr, Rightarrow] \arrow[drrr, dotted] & & & \\
& BH_2 \arrow[loop above] \arrow[rr, dotted] & & BG \\
BH_3 \arrow[loop above] \arrow[ur, Rightarrow] \arrow[urrr, dotted] & & &
\end{tikzcd}$$

Homology decomposition $\quad\Leftrightarrow\quad \operatorname{hocolim} F \sim_p BG$

The slide is a bit vague. The idea is that there is a natural transformation from F to the constant functor with value BG which induces a map $\operatorname{hocolim} F \to BG$. For each $d \in \mathcal{D}$ the map $F(d) \to BG$ is supposed to be equivalent to $BH_d \to BG$ for some subgroup $H_d \subset G$. The symbol $\sim_p$ denotes a $\mathbb{Z}/p$-homology isomorphism.

Slide 4-2

Slide 4-2

Approximation data from G-orbits

G-orbit $\Leftrightarrow$ B(subgroup)

$(X \in \mathcal{O}_G) \quad\Leftrightarrow\quad (X_{\mathrm{h}G} \sim BG_x,\, x \in X)$

$S : \mathcal{D} \to \mathcal{O}_G \quad\Longrightarrow\quad S_{\mathrm{h}G}$ is approximation data

$$\begin{tikzcd}
X_1 \arrow[loop above] \arrow[dr, Rightarrow] \arrow[drrr, dotted] & & & \\
& X_2 \arrow[loop above] \arrow[rr, dotted] & & * \\
X_3 \arrow[loop above] \arrow[ur, Rightarrow] \arrow[urrr, dotted] & & &
\end{tikzcd}$$

Here $\mathcal{O}_G$ is the category of G-orbits, i.e., transitive G-sets. The slide points out that if you have a diagram of G-orbits, it's easy to take homotopy orbit spaces and construct a diagram of classifying spaces. Since the orbits all map to the trivial orbit, their homotopy orbit spaces all map to BG. (The slide does not ask the question of whether this last map is a mod p homology isomorphism.)

4.1 Exercise. Given a transitive G-set X, considered as a functor $\mathcal{C}_G \to \mathbf{Set}$, use the Grothendieck construction (in this case the transport category construction) to construct a category with nerve $X_{\mathrm{h}G}$. Why is this nerve equivalent to BG_x for $x \in X$?

4.2 Exercise. [28] Let $(\mathcal{C}, \mathcal{E})$ be the homotopy theory of G-spaces, where $\mathcal{E}$ is the class of G-maps which are ordinary weak homotopy equivalences. Let $(\mathcal{D}, \mathcal{F})$ be the homotopy theory given by the category $\mathbf{Sp}{\downarrow}BG$ of spaces over BG with equivalences $\mathcal{F}$ the maps of spaces over BG which are ordinary weak homotopy equivalences of spaces. Show that these two homotopy theories are equivalent. Hint: $(\mathcal{C}, \mathcal{E})$ has a projective model category structure, while $(\mathcal{D}, \mathcal{F})$ has a model category structure inherited from $\mathbf{Sp}$ (same fibrations and cofibrations).

4.3 Exercise. What happens with the above exercise if you replace $\mathbf{Sp}{\downarrow}BG$ by $\mathbf{Sp}{\downarrow}B$ for an arbitrary space B? Don't ignore the possibility that B might not be connected.

Slide 4-3

Slide 4-3

Obtaining alternative approximation data

Collection C: set of subgroups of G closed under conjugation

Subgroup diagram of C **(**$H_i \in C$**)**

$$\mathcal{I}_C \quad = \quad \begin{tikzcd} G/H_1 \arrow[d, dotted] \\ G/H_2 \end{tikzcd}$$

$I(G/H) = G/H$

$I_{\mathrm{h}G}(G/H) \sim BH$

Centralizer diagram of C **(**$\mathrm{im}(H_i) \in C$**)**

$$\mathcal{J}_C \quad = \quad \begin{tikzcd} H_1 \arrow[dr, hook, squiggly, "\Sigma_1"] \arrow[dd, dotted] & \\ & G \\ H_2 \arrow[ur, hook, squiggly, "\Sigma_2"'] & \end{tikzcd}$$

$J(H, \Sigma) = \Sigma$

$J_{\mathrm{h}G}(H, \Sigma) \sim BZ_G(\mathrm{im}\, H)$

Given a collection of subgroups of G closed under conjugation, this slide describes two ways of obtaining an associated diagram of transitive G-sets.

In the subgroup diagram for C, the category $\mathcal{I}_C$ is the category of G-orbits G/H for $H \in C$. The approximation functor S assigns to G/H the transitive G-set G/H itself.

In the centralizer diagram for C, the category $\mathcal{J}_C$ has as objects the pairs (H, Σ), where H is a group and Σ is a conjugacy class of monomorphisms $H \to G$ with image an element of C. (Σ is a orbit for the action of G by conjugation on the set of group monomorphisms $H \to G$.) A morphism $(H_1, \Sigma_1) \to (H_2, \Sigma_2)$ is a group homomorphism $H_1 \to H_2$ such that the obvious diagram commutes up to G-conjugacy. The decomposition functor S assigns to (H, Σ) the G-orbit Σ. On the slide, $Z_G(\operatorname{im} H)$ denotes the centralizer in G of the image of H under any of the homomorphisms contained in Σ

4.4 Exercise. Suppose that C consists of only a single conjugacy class of subgroup, so that up to equivalence of categories both $\mathcal{I}_C$ and $\mathcal{J}_C$ have only a single object. Describe the approximation diagrams $J_{\mathrm{h}G} \colon \mathcal{J}_C \to \mathbf{Sp}$ and $I_{\mathrm{h}G} \colon \mathcal{I}_C \to \mathbf{Sp}$ and calculate their homotopy colimits directly.

Slide 4-4

Slide 4-4

Six $\mathbb{Z}/p$-homology decompositions

Collections **($p \mid \#(G)$)**

- $C_1 = \{$non-trivial p-subgroups$\}$
- $C_2 = \{$non-trivial elementary abelian p-subgroups$\}$
- $C_3 = \{V \in C_2 \mid V = {}_pZ(Z_G(V))\}$

$\mathbb{Z}/p$-homology decompositions?

C	**Subgroup decomposition**	**Centralizer decomposition**
C_1	Yes	Yes
C_2	Yes	Yes
C_3	Yes	Yes

Finally, here's a claim that specific nontrivial homology decompositions actually exist, six of them in fact. For each of the three specified collections of subgroups, there are two associated decompositions.

Some references for this material and what follows later on in this lecture are [33], [34] and, holding the world record for the number of decompositions and the detail in which they're studied, [50]. Many of the ideas go back to Webb (e.g. [99]) and Jackowski-McClure [59], or even further.

4.5 Exercise. What happens with the above six decompositions if p doesn't divide the order of G?

4.6 Exercise. [33, 1.21] Let C_4 be the collection consisting only of the trivial subgroup, and C_5 the collection consisting only of G itself. Show that these two collections also give homology decompositions. Are they interesting?

4.7 Exercise. If the subgroup decomposition for C gives a $\mathbb{Z}/p$-homology decomposition for G, does the same hold true for any collection C' strictly containing C?

Slide 4-5

How to obtain the six decompositions

$\mathcal{K}_C = \{$poset C under inclusion$\}$, $K_C = \mathrm{N}(\mathcal{K}_C)$

Identify hocolims
$I\colon\ \mathcal{I}_C \to \mathbf{Sp}$ hocolim $I_{\mathrm{h}G} \sim (K_C)_{\mathrm{h}G}$ (subgroup diagram)
$J\colon\ \mathcal{J}_C \to \mathbf{Sp}$ hocolim $J_{\mathrm{h}G} \sim (K_C)_{\mathrm{h}G}$ (cent'lizer diagram)

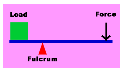
Relate posets

$$K_{C_1} \sim K_{C_2} \sim K_{C_3}$$

(via G-maps)

Start here

$$(K_{C_1})_{\mathrm{h}G} \sim_p BG$$



The aim of this slide is to describe an economical way to study various decompositions. Given a collection C of subgroups of G, the subgroup and centralizer decompositions associated to C provide two approximations to BG. It turns out that these approximations are essentially the same, and can be identified with the natural map $(K_C)_{\mathrm{h}G} \to BG$, where K_C is the nerve of the poset (under inclusion) of the given by the subgroups contained in C. (G acts on this poset by conjugation.) Thus one of these approximations gives a homology decomposition if and only if the other one does.

We're thrown back on studying K_{C_1}, K_{C_2} and K_{C_3}, but it turns out that they are all equivalent to one another, by maps which respect the actions of G. So we either have six homology decompositions to look at (two for each collection) or none. Ken Brown breaks the suspense [1, V.3.1] with the equivalence at the bottom of the slide.

Slide 4-6

Identify hocolim for subgroup diagram

$$I_{\mathrm{h}G}\colon \mathcal{I}_C \to \mathbf{Sp} \qquad I(G/H) = G/H$$

Reduction to hocolim I

Want:	hocolim$(I_{\mathrm{h}G})$	$\sim$	$(K_C)_{\mathrm{h}G}$
Have:	hocolim$(I_{\mathrm{h}G})$	$\sim$	(hocolim $I)_{\mathrm{h}G}$
Need:	hocolim I	$\sim$	K_C

Grothendieck construction

$$\begin{tikzcd}
 & & (x_1 \in G/H_1) \arrow[r, mapsto] \arrow[dd, dotted] & G_{x_1} \arrow[dd, dotted] & & \\
\mathcal{I}_C \ltimes I & = & & & = & \mathcal{K}_C \\
 & & (x_2 \in G/H_2) \arrow[r, mapsto] & G_{x_2} & &
\end{tikzcd}$$

Equivalence of categories, G-equivariant

The problem is to show that the homotopy colimit of the subgroup diagram associated to a collection C, is equivalent to the Borel construction of the action of G on the nerve of the poset given by C. The proof technique is to express both the homotopy colimit and the Borel construction in categorical terms using Grothendieck Constructions and then observe that the two categories which come out of the process are equivalent.

4.8 Exercise. Actually, this isn't quite the argument which the slide pretends to describe. Can you carry out the argument in the above deceptive paragraph?

Slide 4-7

Identify hocolim for centralizer diagram

$$J_{\mathrm{h}G} \colon \mathcal{J}_C \to \mathbf{Sp} \qquad J(H,\Sigma) = \Sigma$$

Reduction to $\mathrm{hocolim}\, J$

Want: $\mathrm{hocolim}(J_{\mathrm{h}G}) \sim (K_C)_{\mathrm{h}G}$

Have: $\mathrm{hocolim}(J_{\mathrm{h}G}) \sim (\mathrm{hocolim}\, J)_{\mathrm{h}G}$

Need: $\mathrm{hocolim}\, J \sim K_C$

Grothendieck construction

$$\begin{tikzcd}
 & H_1 \arrow[dr, hook] \arrow[dd, dotted] & & {} \arrow[r, mapsto] & \mathrm{im}(H_1) \arrow[dd, dotted] & \\
\mathcal{J}_C \ltimes J \quad = & & G & & & = \quad \mathcal{K}_C \\
 & H_2 \arrow[ur, hook] & & {} \arrow[r, mapsto] & \mathrm{im}(H_2) &
\end{tikzcd}$$

Equivalence of categories, G-equivariant

An object of the Grothendieck construction $\mathcal{J}_C \ltimes J$ consists of a conjugacy class Σ of monomorphisms $H \hookrightarrow G$ with image in C together with an element of Σ; this amounts to a monomorphism $H \hookrightarrow G$ with image in C. The morphisms in the Grothendieck construction are commutative diagrams (in $\mathbf{Grp}$) of the indicated type.

Slide 4-8

Relate posets $\quad K_{C_2} \sim K_{C_1}$

$\mathcal{K}_{C_1}$ = {nontrivial p-subgroups} $\mathcal{K}_{C_2}$ = {nontrivial elementary abelian p-subgroups}

An initial functor $\iota \colon \mathcal{K}_{C_2} \to \mathcal{K}_{C_1}$

$$\begin{tikzcd}
 & V \arrow[d, dotted] \arrow[dr] & & \text{Identity functor} \arrow[d] \\
(\iota{\downarrow}P) \sim * & V \cdot {}_pZ(P) \arrow[r] & P & \text{Middle functor} \\
 & {}_pZ(P) \arrow[u, dotted] \arrow[ur] & & \text{Constant functor} \arrow[u]
\end{tikzcd}$$

$$\mathcal{K}_{C_2} \to \mathcal{K}_{C_1} \text{ initial} \implies K_{C_2} \sim K_{C_1}$$

Slide 4-9

Slide 4-9

Relate posets $K_{C_3} \sim K_{C_2}$

$\mathcal{K}_{C_2} = \{$nontrivial elementary abelian p-subgroups$\}$ $\mathcal{K}_{C_3} = \{V \in C_2 \mid V = {}_pZ(Z_G(V))\}$

Adjoint functors $\beta : \mathcal{K}_{C_2} \leftrightarrow \mathcal{K}_{C_3} : \iota$
$(\beta\iota \to \text{id}, \quad \text{id} \to \iota\beta)$

$$\begin{tikzcd}
W & W \arrow[l, mapsto, "\iota"'] \\
V \arrow[r, mapsto, "\beta"] & {}_pZ(Z_G(V))
\end{tikzcd}$$

β and ι give inverse $\sim$ on nerves.

Lecture 5.
Localizations

This lecture discusses localizations of homotopy theories, more particularly homology localizations of the homotopy theory of spaces, and even more particularly $\mathbb{Z}/p$-homology localization L_p. It concludes with a discussion of the Bousfield-Kan p-completion functor C_p, which is a type of approximation to L_p. The functor C_p comes up in an essential way in the study of the cohomology of function spaces.

First, the general setup.

Slide 5-1

Slide 5-1

Left Bousfield localization

Setup
$(\mathcal{C}, \mathcal{E})$ a homotopy theory, $\mathcal{E} \subset \mathcal{F} \subset \mathcal{C}$ $\quad (\mathcal{E} = \sim,\ \mathcal{F} = \approx)$

Definition 5. $I\colon (\mathcal{C}, \mathcal{E}) \to (\mathcal{C}, \mathcal{F})$ a *left Bousfield localization* if $\exists J$

$$I\colon (\mathcal{C}, \mathcal{E}) \leftrightarrow (\mathcal{C}, \mathcal{F}) \ : J \quad J \text{ full \& faithful}$$

Properties

- $L = JI$ (localization functor), X *local* $\Leftrightarrow X \in \operatorname{im}(L)$
- $L^2 \sim L$
- $X \overset{\approx}{\to} L(X)$ terminal among $X \overset{\approx}{\to} Y$
- $X \overset{\approx}{\to} L(X)$ initial among $X \to$ (local)

Remark. A functor $F\colon \mathcal{U} \to \mathcal{V}$ between homotopy theories is "full and faithful" if for every pair (x, y) of objects of $\mathcal{U}$ the natural map $\operatorname{Hom}^{\mathrm{h}}_{\mathcal{U}}(x, y) \to \operatorname{Hom}^{\mathrm{h}}_{\mathcal{V}}(Fx, Fy)$ is an equivalence in $\mathbf{Sp}$. In the homotopical situation this is an indivisible concept; as far as I can see, there's no good way to break "full" and "faithful" apart.

The process of localizing a homotopy theory involves adding new equivalences to the ones that were there before (just as localizing a ring involves adding new units). From the $\mathbb{T}_{\text{pair}}$ point of view, this is passing from a pair $(\mathcal{C}, \mathcal{E})$ to $(\mathcal{C}, \mathcal{F})$ where $\mathcal{E} \subset \mathcal{F}$. There is always a map : $(\mathcal{C}, \mathcal{E}) \to (\mathcal{C}, \mathcal{F})$, and I is called a left Bousfield localization if I is a left homotopy adjoint, i.e., I has a right homotopy adjoint J such that J is full and faithful [16], [56, 3.3]. This means that the homotopy theory $(\mathcal{C}, \mathcal{F})$ is essentially embedded in $(\mathcal{C}, \mathcal{E})$ as, well, the image of J.

Recall that the notion of homotopy adjoint is easiest to express from the $\mathbb{T}_{\text{en}}$ point of view (2–3).

5.1 Exercise. Show that L converts a morphism in $\mathcal{F}$ to a morphism in $\mathcal{E}$.

5.2 Exercise. What exactly do the terms *terminal* and *initial* mean as they are used on the slide?

Slide 5-2

Slide 5-2

Example with discrete categories

Context

- $\mathcal{C} = \mathbf{Ab}$
- $R = \mathbb{Z}[1/p]$
- $\mathcal{E} =$ (isos), $\mathcal{F} = \{f \mid R \otimes f$ an iso$\}$
- $(\mathcal{C}, \mathcal{E}) \sim \mathbf{Mod}_{\mathbb{Z}}$, $(\mathcal{C}, \mathcal{F}) \sim \mathbf{Mod}_R$

Properties

- $L = R \otimes (-)$, X *local* $\Leftrightarrow X$ an R-module
- $L^2 \sim L$
- $X \xrightarrow{\approx} L(X)$ terminal among $X \xrightarrow{\approx} Y$
- $X \xrightarrow{\approx} L(X)$ initial among $X \to$ (local)

This is in fact a legitimate example: the homotopy theory $(\mathbf{Mod}_{\mathbb{Z}}, \mathcal{F})$ is equivalent to the discrete category $\mathbf{Mod}_R$. Usually the process of inverting morphisms creates higher homotopy groups in the morphism spaces $\mathrm{Hom}^{\mathrm{h}}_{\mathcal{C},\mathcal{F}}(-,-)$, but not in this case [36, 7.3].

Slide 5-3

Slide 5-3

Another model category dividend

$\mathcal{E} \subset \mathcal{F} \subset \mathcal{C}$

Definition (f a morphism of $\mathcal{C}$, X an object)

- $f \perp X$ if $\mathrm{Hom}^{\mathrm{h}}_{(\mathcal{C},\mathcal{E})}(f, X)$ is an equivalence in $\mathbf{Sp}$.
- $f^{<} = \{g : f \perp X \implies g \perp X\}$.

Assume

- $(\mathcal{C}, \mathcal{E})$ is a model category^{++}
- there is a map f such that $\mathcal{F} = f^{<}$

Theorem

1. $(\mathcal{C}, \mathcal{E}) \to (\mathcal{C}, \mathcal{F})$ is a left Bousfield localization.
2. $(\mathcal{C}, \mathcal{F})$ is a model category with $\mathcal{C}^{\mathrm{c}}_{\mathcal{F}} = \mathcal{C}^{\mathrm{c}}_{\mathcal{E}}$.
3. L is the fibrant replacement functor in $(\mathcal{C}, \mathcal{F})$.

This slide describes a common situation in which it's possible to construct and identify a left Bousfield localization. (Maybe [24] it's the *only* situation!) Say that a map f looks invisible to an object X in a model category $(\mathcal{C}, \mathcal{E})$ (or X can't see f) if $\mathrm{Hom}^{\mathrm{h}}(f, X)$ is an equivalence. The idea is to start with a map f and let $\mathcal{F} = f^{<}$ be the class of all maps which look look invisible to all spaces which can't see f.

Then, under relatively mild conditions (the $^{++}$), $(\mathcal{C}, \mathcal{E}) \to (\mathcal{C}, \mathcal{F})$ is a left Bousfield localization [56, ch.4] [48].

This is called *localization with respect to the map* f. Under the relatively mild conditions mentioned above, it turns out that the class of newly inverted maps $f^{<}$ is (up to $\mathcal{E}$) the class of maps which can be built from f by homotopy colimit constructions. (This is not quite true: one direction is OK [56, 4.2.9] but the other one is bogus [56, 2.1.6]. But it's not far off.)

5.3 Exercise. Show that the localization on slide 5–2 is localization with respect to the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by p.

5.4 Exercise. Show that in the above situation an object X is local (i.e. $L(X) \sim X$) if and only if $f \perp X$. (Recall that if $f \perp X$ then $g \perp X$ for all $g \in \mathcal{F}$.) Conclude that the class of local objects is closed under homotopy limits.

Slide 5-4

Examples of model category localizations (I)

Localization with respect to a map
Pick your favorite map f and let $\mathcal{F} = f^{<}$

$\mathcal{C} = \mathbf{Sp}$, $f = (S^{n+1} \to *)$

- $f^{<} = \{g : \pi_i(g) \text{ iso for } i \le n\}$
- $L = P_n$ (n'th Postnikov stage)
- (local) $=$ (π_i vanishes for $i > n$)

$\mathcal{C} = \mathbf{Ch}_{\mathbb{Z}}$, $f = \oplus_i(\Sigma^i \mathbb{Z}[1/p] \to 0)$

- $f^{<} = \{g : \mathbb{Z}/p \otimes^{\mathrm{h}} g \text{ is an equivalence}\}$
- $L =$ derived p-completion
- (local) $=$ homology groups are Ext-p-complete

Remark. An abelian group A is *Ext-p-complete* [18, VI] [51] if the natural map $A \to \mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$ is an isomorphism. Here "the natural map" is the map induced by the connecting homomorphism in the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}/p^\infty \to 0\,.$$

5.5 Exercise. Show that if A is a finitely generated abelian group, then $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$ is isomorphic to the p-completion $A\hat{}_p = \lim A/p^n A$, and that $A\hat{}_p$ is Ext-p-complete.

☡☡ **5.6 Exercise.** Show that $A \mapsto \mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$ is the zero'th left derived functor (in the classical sense of Cartan and Eilenberg) of the functor $A \mapsto A\hat{}_p$ on the category of abelian groups [18, VI.2]. This is an unusual situation in which it turns out to be interesting to take the left derived functors of a functor which itself is not left exact, so the zero'th left derived functor does not coincide with the original functor. What's the first left derived functor in this case?

Slide 5-5

Slide 5-5

Examples of model category localizations (II)

Localization with respect to a homology theory E_*
Pick your favorite E_*, let $\mathcal{F} = \{E_*\text{-isos}\}$, and follow Bousfield.

Theorem (Bousfield)
There exists a magic morphism f with $f^< = \mathcal{F}$.

Widely applicable

- Spaces
- Spectra
- $\mathbf{Ch}_R$ (e.g, $E_*(X) = H_*(A \otimes^{\mathrm{h}}_R X)$)
- simplicial universal algebras

The original references for localization with respect to homology are [12] (for the space case) and [13] (for the spectrum case). It's worth thinking about the construction of the magic morphism f and about why the construction works; the lemma to look for is [12, 11.2] or [49, X.2.8]. Don't make the mistake of thinking that the map f is necessarily subject to human comprehension. For instance, if $E_* = H_*(;\mathbb{Z}/p)$, the f can be taken to be the coproduct of all monomorphic $H\mathbb{Z}/p_*$-equivalences between countable simplicial sets (actually, for set theoretic reasons, in forming the coproduct that gives f you would only take one representative from each *isomorphism class* of monomorphic $H\mathbb{Z}/p_*$-equivalence).

5.7 Exercise. Show that a space X is local with respect to E_* (i.e. $L(X) \sim X$) if and only if any E_*-isomorphism $A \to B$ induces an equivalence $\mathrm{Hom}^{\mathrm{h}}(B, X) \to \mathrm{Hom}^{\mathrm{h}}(A, X)$ (cf. 5.4).

Slide 5-6

Slide 5-6

Localization with respect to R-homology, $R \subset \mathbb{Q}$

Properties
$L_R =$ localization in $\mathbf{Sp}$ with respect to $H_*(-; R)$

- L_R preserves components, connectivity, nilpotency.
- X 1-connected $\implies \pi_i L_R(X) \cong R \otimes \pi_i(X)$
- X 1-connected $\implies H_i(L_R X; \mathbb{Z}) \cong R \otimes H_i(X; \mathbb{Z})$
- L_R preserves fibrations of connected nilpotent spaces.

Algebraization if $R = \mathbb{Q}$
$\mathbf{Sp}_1 =$ 1-connected spaces, $\mathcal{F} = \{H_*(-;\mathbb{Q})$ isos$\}$
$\mathbf{Lie}_0 = \{$0-connected DG Lie alg$/\mathbb{Q}\}$, $\mathcal{H} = \{H_*$ isos$\}$.

$$\boxed{(\mathbf{Sp}_1, \mathcal{F}) \quad \sim \quad (\mathbf{Lie}_0, \mathcal{H})}$$

If $R \subset \mathbb{Q}$, localization with respect to R-homology is relatively simple, at least for 1-connected spaces. The statement that the homotopy theory of simply-connected rational spaces is equivalent to the homotopy theory of connected differential graded Lie algebras over $\mathbb{Q}$ is due to Quillen [83]; this was one of the first applications of model category machinery. Sullivan came up with a dual approach [96] that works in the finite-type case and includes the theory of minimal models (see also [17]). See [55] for an up-to-date survey of rational homotopy theory.

5.8 Exercise. Prove that any simply-connected space X is $H\mathbb{Z}$-local, i.e., $L_{\mathbb{Z}}X \sim X$.

☡ **5.9 Exercise.** As remarked above, homology localization can also be done in the category of spectra [13]. Give an example of a nontrivial spectrum X such that $L_{\mathbb{Z}}X \sim *$.

A connected space X is *nilpotent* if $\pi_1(X)$ is a nilpotent group and, for each $i \geq 2$ the natural action of $\pi_1(X)$ on $\pi_i(X)$ is nilpotent (= trivial up to a finite filtration or in loose terms "upper triangular").

★ **5.10 Exercise.** Show that a connected space X is nilpotent if and only if $\pi_1(X)$ is a nilpotent group and for each $i \geq 2$ the natural action of $\pi_1(X)$ on $H_i(\tilde{X};\mathbb{Z})$ is nilpotent. (Here $\tilde{X}$ is the universal cover of X, and the action is induced by covering transformations.)

★ **5.11 Exercise.** Show that a connected space X is nilpotent if and only if each Postnikov stage of X can be built from Eilenberg-Mac Lane spaces by a finite number of principal fibrations with connected fibre.

☡ **5.12 Exercise.** Is "connected" necessary in the previous exercise?

Slide 5-7

Slide 5-7

Localization with respect to $\mathbb{Z}/p$-homology

Properties (Completion?)
L_p = localization in $\mathbf{Sp}$ with respect to $H_*(-;\mathbb{Z}/p)$

- L_p preserves components, connectivity, nilpotency.
- X 1-connected, fin. type $\implies \pi_i L_p(X) \cong \mathbb{Z}_p \otimes \pi_i(X)$
- X 1-connected $\implies H_i(L_pX;\mathbb{Z}) \cong$??
- L_p preserves fibrations of connected nilpotent spaces.

General formula for homotopy groups of L_pX
X 1-connected $\implies$

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{i-1}X) \to \pi_i L_pX \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_iX) \to 0$$

The slide points out that from the point of view of the homotopy groups, localizing a 1-connected space with respect to $H\mathbb{Z}/p$ looks like a type of completion.

★ **5.13 Exercise.** Verify that the description on the slide of the $H\mathbb{Z}/p$-localization of a 1-connected space X of finite type is correct. One approach is to calculate the localization of an Eilenberg-MacLane space directly, and then use the fact that the class of $H\mathbb{Z}/p$-local spaces is closed under homotopy limits, together with various homology arguments, to climb up the Postnikov tower of X. (Note that in order to compute the localization of a space Y, it's enough to produce a local space Y' and an $H\mathbb{Z}/p$-equivalence $Y \to Y'$.) Start by showing that $K(\mathbb{Z}/p, n)$ is local with respect to $H\mathbb{Z}/p$ (5.4). . Extend this to p-GEMs (i.e., products of $K(\mathbb{Z}/p.n)$'s for various n, also known as mod p Generalized Eilenberg-Mac Lane spaces) Using homotopy limit arguments (pullback, towers) show that (products of) $K(\mathbb{Z}/p^j, n)$ and $K(\mathbb{Z}_p, n)$ (various j, n) are also $H\mathbb{Z}/p$-local. Show that $K(\mathbb{Z}, n) \to K(\mathbb{Z}_p, n)$ gives an isomorphism on $H\mathbb{Z}/p$, and conclude that this is a localization map. Now go Postnikov.

5.14 Exercise. Show that if X is one-connected, then $L_p(L_\mathbb{Q}(X)) \sim *$. What about $L_\mathbb{Q}(L_p(X))$?

★ **5.15 Exercise.** The short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}/p^\infty \to 0$$

leads to a fibration sequence

$$B\mathbb{Z}/p^\infty \to K(\mathbb{Z}, 2) \to K(\mathbb{Z}[1/p], 2)$$

Use this or any other technique to show that the map $B\mathbb{Z}/p^\infty \to K(\mathbb{Z}, 2)$ is an $H\mathbb{Z}/p$-equivalence, and so induces an equivalence $L_p(B\mathbb{Z}/p^\infty) \to K(\mathbb{Z}_p, 2) \sim L_p(BS^1)$.

☡ **5.16 Exercise.** Compute $H^*(B\mathbb{Z}/p^\infty; \mathbb{Z})$ and $H^*(B\mathbb{Z}/p^\infty; \mathbb{Q})$. How can this be? It's enough to make a person reread [93, 5.4] with a big cup of coffee.

☡ ☡ **5.17 Exercise.** If A is any direct sum of copies of $\mathbb{Z}$, show that $K(A_{\hat{p}}, n)$ is $H\mathbb{Z}/p$-local (5.5). Be careful. Calculate that $K(A, n) \to K(A_{\hat{p}}, n)$ is an $H\mathbb{Z}/p$-isomorphism, and conclude that this map is an $H\mathbb{Z}/p$-localization map. Using various Postnikov and fibration arguments in conjunction with 5.6, derive the short exact sequence at the bottom of the last slide.

Slide 5-8

Slide 5-8

Mixing $L_\mathbb{Q}(X)$ with the $L_p(X)$'s to recover X

The Arithmetic Square (Sullivan)
X nilpotent $\Longrightarrow$ homotopy fibre square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p L_p(X) \arrow[d] \\ L_\mathbb{Q}X \arrow[r] & L_\mathbb{Q}(\prod_p L_p(X)) \end{tikzcd}$$

★ **5.18 Exercise.** If X is 1-connected, use a direct homology calculation to show that the arithmetic square is a homotopy fibre square. The right vertical map is an $H\mathbb{Q}$-equivalence, and the lower horizontal map is an $H\mathbb{Z}/p$-equivalence (5.14). Let P be the homotopy pullback of the square. Argue that P maps to $L_{\mathbb{Q}}(X)$ by an $H\mathbb{Q}$-equivalence, and that for any prime q the map $P \to \prod_p L_p(X)$ is an $H\mathbb{Z}/q$-equivalence. Observe that for any q the projection $\prod_p L_p(X) \to L_q(X)$ is an $H\mathbb{Z}/q$-equivalence (tricky!). Conclude that $X \to P$ induces an isomorphism on integral homology, and finish the argument by using the fact that both spaces are $H\mathbb{Z}$-local (5.8).

☡☡ **5.19 Exercise.** The trickiness mentioned above is relatively minor and has to do with avoiding the temptation to invoke an imaginary infinite Künneth formula. Find an example of a sequence of spaces X_n such that each X_n has, say, the rational cohomology of a point, but $\prod_n X_n$ does not. How about an example with integral homology?

5.20 Exercise. Let A and B be sets of primes, and write L_A, for instance, for $L_{\mathbb{Z}[p^{-1}:p\in A]}$. Show that if X is 1-connected there is a homotopy fibre square

$$\begin{tikzcd}
L_{A\cap B}(X) \arrow[r] \arrow[d] & L_A(X) \arrow[d] \\
L_B(X) \arrow[r] & L_{A\cup B}(X)
\end{tikzcd}$$

Many people have studied a number of problems of the following type (under the general heading of genus problems): given a space X, how many spaces Y are there (up to equivalence) with the property that $L_{\mathbb{Q}}Y \sim L_{\mathbb{Q}}X$ and $L_p(Y) \sim L_p(X)$ for all p. If such a Y exists, call it a *remixing* of X, since it's obtained by taking X apart into p-complete and rational pieces, and then mixing them together in some way.

5.21 Exercise. Let X be a sphere. Produce a remixing of X not equivalent to X.

(Cheap trick?) Most of the time, if X has finitely generated integral homology groups the interesting remixings of X are those which also have finitely generated integral homology. Call these the *nice* remixings.

5.22 Exercise. Let X be a sphere. Show that any nice remixing of X is equivalent to X.

☡ **5.23 Exercise.** Can you produce a 2-cell complex with a nontrivial nice remixing?

☡☡ **5.24 Exercise.** In theory, all remixings of X can be constructed by taking the maps $\prod_p L_p(X) \to L_{\mathbb{Q}}(\prod_p L_p(X))$ and $L_{\mathbb{Q}}(X) \to L_{\mathbb{Q}}(\prod_p L_p(X))$ in the arithmetic square for X and adjusting them. What adjustments are possible? Which adjustments give nice remixings? See [101] for a proof that a finite 1-connected complex has only a finite number of homotopically distinct nice remixings.

Slide 5-9

An approximation to L_p

$H\mathbb{Z}/p$-localization functor L_p

$$\begin{tikzcd}
\text{$H\mathbb{Z}/p$-local spaces} \arrow[r, "I"] \arrow[d, "J"'] & \text{$H\mathbb{Z}/p$-local spaces} \\
\text{spaces} \arrow[ur, "\mathrm{RKan}^{\mathrm{h}}_J(I) \sim L_p"'] &
\end{tikzcd}$$

Bousfield-Kan p-completion functor C_p ($L_p \to C_p$)

$$\begin{tikzcd}
\text{p-GEMs} \arrow[r, "i"] \arrow[d, "j"'] & \text{$H\mathbb{Z}/p$-local spaces} \\
\text{spaces} \arrow[ur, "\mathrm{RKan}^{\mathrm{h}}_j(i) \sim C_p"'] &
\end{tikzcd}$$

Recall that a p-GEM is a product of copies of $K(\mathbb{Z}/p, n)$'s for various n. The category of p-GEMs is to be taken as a full subcategory of spaces.

The fact that $\mathrm{RKan}^{\mathrm{h}}_J(I) \sim L_p$ follows from the fact that homotopy Kan extensions can be computed pointwise (cf. 2.3 and 2.23) and the fact that for any space X, $X{\downarrow}J$ has $X \to L_p(X)$ has a (homotopically) initial object (slide 5–1). The functor C_p, denoted $(\mathbb{Z}/p)_\infty$ in [18], is defined by an explicit cosimplicial construction [18, I.4]; the expression as a homotopy right Kan extension is from [18, XI.10.6].

5.25 Exercise. (This is probably very hard, or impossible.) For functors $F : \mathcal{C} \to \mathcal{D}$ and $X : \mathcal{C} \to \mathbf{Sp}$, observe that the right homotopy Kan extension $\mathrm{RKan}^{\mathrm{h}}_F(X)$ can be computed by the formula

$$\mathrm{RKan}^{\mathrm{h}}_F(X)(d) \sim \mathrm{Hom}^{\mathrm{h}}_{\mathbf{Sp}^{\mathcal{C}}}(c \mapsto \mathrm{Hom}^{\mathrm{h}}_{\mathcal{D}}(d, c), X)\,.$$

(This is the formula for the right homotopy Kan extension from slide 3–7.) Now for any space X, let $K(X)$ be the functor on p-GEMs given by $A \mapsto \mathrm{Hom}^{\mathrm{h}}(X, A)$. This is a kind of cochain functor; the homotopy groups of $\mathrm{Hom}^{\mathrm{h}}(X, A)$ are various cohomology groups of X with coefficients in the homotopy groups of A (in other words, various products of mod p cohomology groups of X). Applied to the case of C_p, the above formula states that $C_p(X)$ is the space of natural maps, as A ranges through p-GEMs, of $K(X, A)$ to A. In other words, $C_p(X)$ is some sort of double-dual of X over the category of p-GEMs. The problem is to find some interesting relationship between this remark and Mandell's theorem [69] that (under certain finiteness and nilpotency assumptions) $C_p(X)$ is the space of E_∞ $\mathbb{F}_p$-algebra maps from $C^*(X, \mathbb{F}_p)$ to $\bar{\mathbb{F}}_p$.

Slide 5-10

Slide 5-10

The Bousfield-Kan p-completion C_p

Properties

- C_p preserves disjoint unions, connectivity
- C_p preserves $H\mathbb{Z}/p_*$-nilpotent fibrations
- $C_p(\sim_p) = \sim$
- X nilpotent $\implies L_p(X) \sim C_p(X)$.

Computability
Unstable Adams spectral sequence $\Rightarrow \pi_* C_p(X)$

Building C_pX if H_*X has finite type
$X \to \{X_s\}$, X_s p-finite, $H^*X \cong \operatorname{colim} H^*X_s$:

$$C_p(X) \sim \operatorname{holim} X_s$$

The properties of C_p are mostly worked out in [18]; for instance, preservation of disjoint unions is I.7.1 and preservation of connectivity is I.6.1. The *fibre lemma* [18, II.5.1] states that C_p preserves fibrations in which the monodromy action of $\pi_1(\text{base})$ on each homology group $H_i(\text{fibre}; \mathbb{Z}/p)$ is nilpotent. The fact that C_p takes $H\mathbb{Z}/p$-equivalences to equivalences follows from the previous slide, or from [18, I.6.2]. The statement that $L_p(X) \sim \mathbb{C}_p(X)$ if X is nilpotent is from [12, §4].

The following piece of information will prove absolutely indispensable later on.

★ **5.26 Exercise.** Show that if Q is a finite p-group, then *any* action of Q on a vector space over $\mathbb{F}_p$ is nilpotent.

The last box on the slide refers to a tower of spaces $\{X_s\}$ under X, spaces which are *p-finite* in the sense that each space has a finite number of components, each homotopy group of each component is a finite p-group, and each component has only a finite number of non-trivial homotopy groups. The last condition on the tower is that the natural map

$$\operatorname{colim}_s H^*(X_s; \mathbb{Z}/p) \to H^*(X; \mathbb{Z}/p)$$

is an isomorphism. The statement is that if $H_*(X; \mathbb{Z}/p)$ is of finite type then such a tower exists and, given any such tower, $C_p(X)$ is equivalent to its homotopy limit. This is a combination of [18, III.6.4] and 6.28.

Remark. There's a very interesting alternative approach to the construction of $C_p(X)$ in [18, IV]. Suppose without loss that X is pointed and connected, and let GX be Kan's group model [70] [49, V.5] for ΩX. Then C_pX is equivalent to $B(GX)^{\wedge}_p$, where $(GX)^{\wedge}_p = \lim GX/\Gamma_i^{(p)}GX$ is the (dimensionwise) p-lower-central-series completion of GX.

Slide 5-11

Slide 5-11

The Bousfield-Kan p-completion C_p: good & bad

Definition

- $L_pX \sim C_pX = X$ p-good
- $L_pX \not\sim C_pX = X$ p-bad

p**-good examples**

- X nilpotent (e.g. $\pi_1 X$ trivial)
- $\pi_1 X$ finite

p**-bad examples**

- $S^1 \vee S^1$
- $S^1 \vee S^n$

News flash

$C_p(p\text{-bad } X)$ sighted in the wild!

The space $C_p(X)$ is always $H\mathbb{Z}/p$-local (it's a homotopy limit of p-GEMs, cf. 5–9), and so $C_pX \sim L_pX$ if and only if $X \to C_pX$ is an $H\mathbb{Z}/p$-equivalence. This last is the definition of "$\mathbb{Z}/p$-good" in [18, I.5]. The p-good examples are from [18]. Bousfield [15] provides the indicated p-bad examples.

The last statement promises that in the near future we will encounter the p-completion of a p-bad space in a relatively simple geometric setting that doesn't seem to deserve such a visitation.

Lecture 6.

Cohomology of function spaces

In this section we'll look at how algebra can be combined with homotopy theory to compute the cohomology of mapping spaces with domain $B\mathbb{Z}/p$. The machinery only works sometimes (we'll try to figure out why this is true) but it works amazingly often.

Slide 6-1

Slide 6-1

The Steenrod algebra A_p

$p =$ fixed prime, $H^* = H^*(-;\mathbb{Z}/p)$

p **odd**	**Structure of A_p**	$p = 2$
• $\{\beta, \mathcal{P}^i, i \geq 0\}$		$\{\mathrm{Sq}^i, i \geq 0\}$
• $\mathcal{P}^0 = 1$		$\mathrm{Sq}^0 = 1, \mathrm{Sq}^1 = \beta$
• Adem relations $\mathcal{P}^i\mathcal{P}^j = \cdots$		$\mathrm{Sq}^i\,\mathrm{Sq}^j = \cdots$
• $A_p \to A_p \otimes A_p$		$\mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$
• $\lvert\beta\rvert = 1, \lvert\mathcal{P}^i\rvert = 2i(p-1)$		$\lvert\mathrm{Sq}^i\rvert = i$

All in all

$A_p =$ cocommutative Hopf algebra over $\mathbb{F}_p$

The traditional place to learn about the Steenrod operations is in Steenrod and Epstein [94]; the best place to learn about the Steeenrod algebra is in Milnor [72], and the most enigmatic source for the Adem relations is Bullett-Macdonald [23]. Here's a question that Quillen tossed out as a throwaway during a lecture I was at years ago, and that many other people have probably thought of since. The answer says everything there is to say about the Steenrod algebra (short of computing Ext!) in *very* few words.

6.1 Exercise. Milnor identifies the dual of the mod 2 Steenrod algebra as a commutative Hopf algebra over $\mathbb{F}_2$ with antipode. A Hopf algebra like this is exactly a cogroup object in the category of commutative $\mathbb{F}_2$-algebras, so $\mathrm{Hom}(A_2, -)$ gives a functor which assigns a group to every commutative $\mathbb{F}_2$-algebra. Giving A_2 as a Hopf algebra is exactly the same as specifying this functor. What's the functor?

6.2 Exercise. What's the above functor in the odd primary case?

Slide 6-2

Slide 6-2

Modules and algebras over A_p

Modules over A_p

- *Graded* modules $\mathrm{Sq}^i : M^k \to M^{k+i}$
- $\mathcal{U} =$ unstable modules $\mathrm{Sq}^i\, x = 0,\ i > |x|$
- M, N (unstable) modules $\implies$ $M \otimes N$ (unstable) module

Algebra over A_p

- Graded algebras, $\mu\colon M \otimes M \to M$ respects A_p
- $\mathcal{K} =$ unstable algebras $\mathrm{Sq}^{|x|}\, x = x^2$
- R, S (unstable) algebras $\implies$ $R \otimes S$ (unstable) algebra

Geometry

- H^*(suspension spectrum) $\in \mathcal{U}$
- H^*(space) $\in \mathcal{K}$

The slide glosses over the fact that $\mathcal{K}$ is the category of unstable *graded-commutative* algebras over A_p.

6.3 Exercise. What kind of geometric object would give rise to an algebra over A_p which is *not* unstable?

6.4 Exercise. Give an example of an object of $\mathcal{K}$ which is not isomorphic to H^*X for any space X.

6.5 Exercise. Why are objects in $\mathcal{K}$ and $\mathcal{U}$ necessarily concentrated in non-negative degrees?

Slide 6-3

Slide 6-3

The functor T

$V = V_1 = \mathbb{Z}/p, \mathbb{H} = H^*BV$

Left adjoint(s) to $\mathbb{H} \otimes -$

$T_{\mathcal{U}} \colon \mathcal{U} \leftrightarrow \mathcal{U} :\ (\mathbb{H} \otimes -)$ $\boxed{T_{\mathcal{U}} = T_{\mathcal{K}}\ (= T)}$ $T_{\mathcal{K}} \colon \mathcal{K} \leftrightarrow \mathcal{K} :\ (\mathbb{H} \otimes -)$

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "T_{\mathcal{K}}"'] \arrow[d, "\text{forget}"'] & \mathcal{K} \arrow[d, "\text{forget}"] \\
\mathcal{U} \arrow[r, "T_{\mathcal{U}}"] & \mathcal{U}
\end{tikzcd}$$

Crucial properties

- T is exact
- T preserves $\otimes$
- Same properties with $V_1 \leftrightarrow V_n = (\mathbb{Z}/p)^n$

The functor T is *left* adjoint to tensoring with $\mathbb{H}$, and, incredibly enough, you get the same result (to the extent that this makes any sense) whether you compute T in the category $\mathcal{U}$ of unstable modules or in the category $\mathcal{K}$ of unstable algebras. The universal reference for all things T is [64].

6.6 Exercise. Let $\mathbb{F}_p$ be the trivial object of $\mathcal{K}$ concentrated in degree 0. Show that there are natural maps

$$T(R) \to \mathbb{H} \otimes R \to R$$

(the second one induced by the unique $\mathcal{K}$-map $\mathbb{H} \to \mathbb{F}_p$).

6.7 Exercise. Calculate directly that $T(\mathbb{F}_p) \to \mathbb{F}_p$ is an isomorphism. Now try $T(\Sigma\mathbb{F}_p) \cong \Sigma\mathbb{F}_p$. (This probably requires looking a little more carefully at the Steenrod algebra action on $\mathbb{H}$.) Use this last calculation and the fact that T preserves $\otimes$ to conclude that T commutes with suspension. Go on to show that if M is any finite object of $\mathcal{U}$ or $\mathcal{K}$ (i.e. an object of finite type which is concentrated in a finite number of degrees) then the natural map $T(M) \to M$ is an isomorphism.

6.8 Exercise. If R be an object of $\mathcal{K}$, we'll use the phrase *module over* R to denote a graded module M over R (in the usual sense) with the property that the multiplication map $R \otimes M \to M$ is a morphism in $\mathcal{U}$. If M and N are modules over R, observe that $\operatorname{Tor}_i^R(M,N)$ is naturally an object of $\mathcal{U}$ (and also a module over R). Prove that there are natural isomorphisms

$$T(\operatorname{Tor}_i^R(M.N)) \cong \operatorname{Tor}_i^{TR}(TM,TN)\,.$$

Slide 6-4

Slide 6-4

The functor $T \leftrightarrow$ function spaces out of BV

$X \in \mathbf{Sp}$, want to understand $\operatorname{Hom}^{\mathrm{h}}(BV,X)$.

Consequence of adjointness

$$X \leftarrow BV \times \operatorname{Hom}^{\mathrm{h}}(BV,X)$$
$$H^*X \to \mathbb{H} \otimes H^* \operatorname{Hom}^{\mathrm{h}}(BV,X)$$
$$\lambda_X : T(H^*X) \to H^* \operatorname{Hom}^{\mathrm{h}}(BV,X)$$

Question

How often is λ_X an isomorphism?

Assume: H^*X of finite type, $T(H^*X)$ of finite type.

Whenever we work with TH^*X, we will always assume that H^*X is of finite type and that TH^*X is of finite type.

Remark. Morel [79] works out the entire theory of T without any of these finite type conditions, at the price of replacing the category of spaces by the category of pro-p-finite spaces. There's a philosophical point here. The mod p homology of $X \in \mathbf{Sp}$ is naturally expressed as a filtered colimit $\operatorname{colim}_\alpha H_*(X_\alpha;\mathbb{Z}/p)$, where X_α ranges over the finite subcomplexes of X. The dual cohomology H^*X is then $\lim_\alpha H^*X_\alpha$, so the cohomology has a natural profinite structure. The functor T doesn't see this structure, and so is vulnerable to becoming geometrically irrelevant in some situations. There are three ways to evade this problem:

1. modify T (which might damage it beyond repair),
2. stay within the finite type world, so the profinite structure on H^* is trivial
3. modify the notion of "space" in order to get rid of the profinite topology on H^*.

We choose alternative (2), while Morel works out (3)..., conceptually more satisfying but a little more abstract. (Note that the cohomology of a pro-p-finite space does *not* have a profinite structure, although its homology does.)

6.9 Exercise. What's the geometric significance of the natural map $T(H^*X) \to H^*X$ (see 6.6). In other words, what's the relationship between this map and the above natural map $T(H^*X) \to H^* \operatorname{Hom}^{\mathrm{h}}(BV, X)$? What is suggested by the fact that the map of 6.6 is an equivalence if H^*X is finite (6.7)?

Slide 6-5

Slide 6-5

Lannes works his magic

Tweak of the question

$$\lambda^{\mathrm{c}}_X : T(H^*X) \to H^* \operatorname{Hom}^{\mathrm{h}}(BV, C_pX)$$

N-conditions on $(T(H^*X), \operatorname{Hom}^{\mathbf{h}}(BV, C_pX))$

1. $T(H^*X)^1 = 0$
2. $\operatorname{Hom}^{\mathrm{h}}(BV, C_pX)$ is an H-space, is nilpotent, is p-complete
3. $\beta : T(H^*X)^1 \to T(H^*X)^2$ is injective
4. $\pi_1 \operatorname{Hom}^{\mathrm{h}}(BV, C_pX)$ is finite

Lannes Theorem

$$T(H^*X)^0 \cong H^0 \operatorname{Hom}^{\mathrm{h}}(BV, C_pX).$$

λ^{c}_X is an isomorphism if any **N**-condition holds.

The next few slides will sketch the proof of the theorem. This might make the **N**-conditions a little easier to understand.

At this point even the map λ^{c} is a little mysterious. Somewhere in here is a claim that if H^*X has finite type (and TH^*X has finite type?) there exists a dashed lift in the diagram

$$\begin{tikzcd}
H^*(C_pX) \arrow[r, "\lambda_{C_pX}"] \arrow[d] & H^* \operatorname{Hom}^{\mathrm{h}}(BV, C_pX) \arrow[d] \\
T(H^*X) \arrow[r, "\lambda_X"'] \arrow[ur, dashed, "\lambda^{\mathrm{c}}_X"] & H^* \operatorname{Hom}^{\mathrm{h}}(BV, X)
\end{tikzcd}$$

It's probably unclear why anything like C_pX should show up here. One useful remark is that H^*X only depends on the $\mathbb{Z}/p$-homology type of X, and so anything that's constructed out of H^*X, such as TH^*X, can at the very best only give information about L_pX. This would explain an appearance of L_pX in the theorem, but there you are, it's C_pX, not L_pX. We'll see what's going on in a minute.

★ **6.10 Exercise.** Show that if X is one-connected, the natural map $\kappa : \mathrm{Hom}^{\mathrm{h}}(BV, X) \to \mathrm{Hom}^{\mathrm{h}}(BV, C_pX)$ induces is a $\mathbb{Z}/p$-homology equivalence. (Show that the homotopy fibre F of $X \to C_pX$ has uniquely p-divisible homotopy groups. Argue that the homotopy fibre of κ over a map $f : BV \to C_pX$ is the space of sections of a bundle over BV with fibre F induced from f. If it helps, interpret this space of sections as a homotopy fixed point set $F^{\mathrm{h}V}$ for an action of V on F depending on f. Show that the homotopy groups of such a space of sections (=homotopy fixed point set) are uniquely p-divisible, and thus (careful with π_1!) that the space is $\mathbb{Z}/p$-acyclic.)

☡ **6.11 Exercise.** How far beyond 1-connected spaces does the above result extend? What happens if π_1X is finite? What if X is a connected loop space (careful: the homotopy fibre F might not be connected)?

6.12 Exercise. The slide is a bit cavalier; produce an example of an H-space which is not nilpotent. (Presumably, the author of the slide did *not* mean to allow this as a possibility.)

6.13 Exercise. Show that the statement about $T(H^*X)^0$ is equivalent to the statement that there is an isomorphism $[BV, C_pX] \cong \mathrm{Hom}_{\mathcal{K}}(H^*X, \mathbb{H})$.

Slide 6-6

Slide 6-6

Outlining the proof

X p-finite $\leftrightarrow$

- π_0X finite
- π_iX finite p-group
- $\pi_iX = 0$ for $i \gg 0$.

One step at a time

1. λ_X an iso if $X = K(\mathbb{Z}/p, n)$
2. λ_X an iso if X is p-finite
3. A tour of towers
4. λ^{c}_X exists, iso if any **N**-condition holds

We'll approach the proof in three steps, with a digression on towers of spaces between the second and the third. The line of argument is from [29], augmented by [92].

In the definition of p-finite, the conditions on π_iX are supposed to hold for all possible choices of a basepoint in X. A space X is p-finite if and only if X has a finite number of components and each component is p-finite.

6.14 Exercise. Given a fibration $F \to E \to B$ (with connected base B), show that if B and F are p-finite, so is E. Show that if E and B are p-finite, so is F. On the other hand, give an example in which F and E are p-finite, but B isn't.

6.15 Exercise. Let C be the smallest class of spaces with the following properties:

1. C is closed under equivalences,
2. C contains $K(\mathbb{Z}/p, n)$ for all $n \geq 1$, and
3. if X in C and $f : X \to K(\mathbb{Z}/p, n)$ is a map for some $n \geq 1$, then the homotopy fibre of f belongs to C.

Show that a space X is p-finite if and only $\pi_0 X$ is finite and each component of X belongs to C.

Slide 6-7

Slide 6-7

The case $X = K = K(\mathbb{Z}/p, n)$

This slide: $Y^* = H^*Y, \quad Y^*$ of finite type

T_Y = left adjoint to $(Y^* \otimes -)$ on $\mathcal{K}$

$M \; = \mathrm{Hom}^{\mathrm{h}}(Y, K) \sim \prod K(\mathbb{Z}/p, m_i)$

Theorem

$$T_Y(K^*) \cong H^* \, \mathrm{Hom}^{\mathrm{h}}(Y, K)$$

Proof (Yoneda, $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{K}}$**)**

(1) $$\mathrm{Hom}(T_Y(K^*), Z^*) \cong \mathrm{Hom}(K^*, Y^* \otimes Z^*)$$

(2) $$\mathrm{Hom}(M^*, Z^*) \cong [Z, M] \cong [Y \times Z, K] \cong \mathrm{Hom}(K^*, Y^* \otimes Z^*)$$

$$T(H^*K) \cong H^* \, \mathrm{Hom}^{\mathrm{h}}(BV, K)$$

This proof relies on the fact that $H^*K(\mathbb{Z}/p, n)$ is a free object of $\mathcal{K}$ on a generator of degree n; choosing a generator of $H^nK(\mathbb{Z}/p, n)$ leads to a natural isomorphism

$$\mathrm{Hom}_{\mathcal{K}}(H^*K(\mathbb{Z}/p, n), R) \cong R^n \, .$$

Since such a choice also leads to a natural isomorphism

$$[Z, K(\mathbb{Z}/p, n)] \cong H^n Z$$

it follows that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{K}}(H^*K(\mathbb{Z}/p, n), H^*Z) \cong [Z, K(\mathbb{Z}/p, n)]$$

This isomorphism is independent of any choice of generator. A similar natural isomorphism arises if $K(\mathbb{Z}/p, n)$ is replaced by a product of mod p Eilenberg-MacLane spaces (such as $\mathrm{Hom}^{\mathrm{h}}(Y, K(\mathbb{Z}/p, n))$).

6.16 Exercise. Prove that if Y is any space, then $\operatorname{Hom}^{\mathrm{h}}(Y, K(\mathbb{Z}/p, n))$ is equivalent to a product of mod p Eilenberg-MacLane spaces. (This is probably a lot easier in the simplicial world.)

6.17 Exercise. The application of Yoneda's lemma above seems to be questionable, because the argument shows only that the two objects in question ($T_Y(K^*)$ and M^*) represent the same functor on the subcategory of $\mathcal{K}$ given by objects Z^* which can be realized as the cohomology of a space. What saves the day?

6.18 Exercise. An argument identical to the one on the slide shows that the Lannes map

$$\lambda_X : T(H^*X) \to H^* \operatorname{Hom}^{\mathrm{h}}(BV, X)$$

is an equivalence if X is a finite product of mod p Eilenberg-MacLane spaces. What happens if X is an infinite product of such Eilenberg-MacLane spaces? What about if X is just tall ($X = \prod_{i=1}^{\infty} \mathcal{K}(\mathbb{Z}/p, i)$) or just wide ($X = \prod_{i=1}^{\infty} K(\mathbb{Z}/p, n)$) ?

Slide 6-8

Slide 6-8

The case in which X is p-finite.

$$X \to Y \to K = K(\mathbb{Z}/p, n)$$

$$H^*X \Leftarrow \operatorname{Tor}^{H^*K}(H^*Y, \mathbb{F}_p)$$ EMSS

λ_Y an iso (Ex. 6.15)

Mapping side **(EMSS + $\operatorname{Hom}^{\mathrm{h}}$ preserves fibrations)**

$$H^* \operatorname{Hom}^{\mathrm{h}}(BV, X) \Leftarrow \operatorname{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p)$$

Space side **(T is nice)**

$$TH^*X \Leftarrow \operatorname{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p)$$

$$T(H^*X) \cong H^* \operatorname{Hom}^{\mathrm{h}}(BV, X)$$

The argument with the Eilenberg-Moore spectral sequence is from [29].

The point here is to work by induction on the complexity of X. By Exercise 6.15 we can get any connected p-finite X by starting from a point and repeatedly taking fibres of maps into mod p Eilenberg-MacLane spaces.

6.19 Exercise. Why is it clear that if λ_Z is an isomorphism for each component Z of X, then λ_X is an isomorphism?

6.20 Exercise. Suppose that $F \to E \to B$ is a fibration and that both H^*B and H^*E are of finite type. The mod p cohomology Eilenberg-Moore spectral sequence for this fibration converges strongly (i.e. with finite filtrations in each degree) if and only if the monodromy action of $\pi_1 B$ on each one of the cohomology groups H^iF is nilpotent (trivial up to a finite filtration). So why does the mapping side spectral sequence on the slide converge?

6.21 Exercise. The "T is nice" tag is a little bit too glib. The nice properties of T do lead to a formula

$$\mathrm{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p) \cong T\left\{\mathrm{Tor}^{H^*K}(H^*Y, \mathbb{F}_p)\right\}$$

but this a formula in the category of graded objects in $\mathcal{U}$ (one Tor_i for each i). Following the spectral sequence through is no problem, since T is exact and commutes with colimits (why?), but there's some geometric work to be done in the limit in order to stitch the graded pieces together. Figure out what this work is, and explain how to do it. It may help to look at Rector's paper [87].

Slide 6-9

Slide 6-9

Tower $\{U_s\} = \{\cdots \to U_n \to U_{n-1} \to \cdots \to U_0\}$

$$\mathrm{colim}\, H^*U_s \overset{?}{=} H^*\, \mathrm{holim}\, U_s$$

Assume: each U_n p-finite, $\mathrm{colim}\, H^*U_s$ of finite type

N-conditions on $(H^*_\infty = \mathrm{colim}\, H^*U_s,\ U_\infty = \mathrm{holim}\, U_s$**)**

1. $H^1_\infty = 0$
2. U_∞ is an H-space, is nilpotent, is p-complete
3. $\beta : H^1_\infty \to H^2_\infty$ is injective
4. $\pi_1 U_\infty$ is finite

Tower Theorem

$H^0_\infty \cong H^0 U_\infty$, always.

$H^*_\infty \cong H^* U_\infty$ if any **N**-condition holds.

See [92] for general background about towers.

The exercises below are meant to set up some of the machinery that's used in working with towers. They more or less lead to a proof of the above theorem, at least in some cases. **N**-condition (3) is a list of three successively more general alternatives: either U_∞ is an H-space, or more generally U_∞ is nilpotent, or even more generally U_∞ is p-complete (here and later on, a space X is p-complete if $X \sim C_pX$). Shipley [92] proves the tower theorem under the assumption that U_∞ is p-complete.

6.22 Exercise. Prove that if each component of U_∞ has a finite fundamental group, or if $H^1_\infty = 0$, or if the stated Bockstein injectivity holds, then U_∞ is p-complete. (Hint: Show that if a component of U_∞ has a finite fundamental group, then this group is a finite p-group; show that if the Bockstein injectivity holds then each component of U_∞ has a finite mod p vector space as fundamental group; and show that if $H^1_\infty = 0$ then each component of U_∞ is simply connected.)

First, there is the general problem of understanding homotopy limits of towers and their homotopy groups. Note that it is possible to talk about towers in any category $\mathcal{C}$; these are just functors from the poset of nonpositive integers into $\mathcal{C}$. A map between towers is a natural transformation of functors.

6.23 Exercise. (Homotopy limit of a tower) Let $\{U_s\}$ be a tower in $\mathbf{Sp}$. Show that $\operatorname{holim} U_s$ can be computed in either of two ways:

- replace $\{U_s\}$ up to equivalence by a tower of fibrations, and take the limit of this replacement tower, or
- take the space of tuples $(u_0, \omega_0, u_1, \omega_1, \ldots)$, where $u_i \in U_i$ and ω_i is a path in U_i between u_i and the image of u_{i+1}.

(It's worth it to try to make simplicial sense of the second description.) Let $\sigma \colon \prod U_s \to \prod U_s$ be the shift map, which sends $(u_0, u_1, \ldots)$ to $(u'_0, u'_1, \ldots)$, where u'_i is the image of u_{i+1} in U_i. Conclude that $\operatorname{holim} U_s$ can be interpreted as the homotopy pullback

$$\begin{tikzcd}
\operatorname{holim} U_s \arrow[r, dashed] \arrow[d, dashed] & \prod U_s \arrow[d, "{(\mathrm{id},\sigma)}"] \\
\prod U_s \arrow[r, "\mathrm{diag}"] & (\prod U_s) \times (\prod U_s)
\end{tikzcd}$$

or as the homotopy equalizer

$$\begin{tikzcd}
\operatorname{holim} U_s \arrow[r, dashed] & \prod U_s \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\sigma"'] & \prod U_s
\end{tikzcd}$$

or even as the homotopy fixed point set of σ acting on $\prod U_s$ (i.e., the homotopy fixed point set of the action on $\prod U_s$ of the free monoid on one generator, where the generator acts by σ).

If $\{A_s\}$ is a tower of abelian group, the shift map $\sigma : \prod A_s \to \prod A_s$ is defined in the same way as above. The kernel of $(1 - \sigma) : \prod A_s \to \prod A_s$ is the limit $\lim A_s$ (check this); the cokernel is denoted $\lim^1 A_s$.

6.24 Exercise. (Milnor sequences) [49, VI.2.15] Show that if $\{U_s\}$ is a tower of pointed one-connected spaces, there are short exact (Milnor) sequences

$$0 \to \lim^1 \pi_{i+1} U_s \to \pi_i \operatorname{holim} U_s \to \lim \pi_i U_s \to 0$$

Define $\lim^1$ for a tower of nonabelian groups in such a way that these sequences can be extended to handle π_1.

6.25 Exercise. Show that a short exact sequence

$$0 \to \{A_s\} \to \{B_s\} \to \{C_s\} \to 0$$

of towers of (abelian) groups leads to a six-term exact sequence

$$0 \to \lim A_s \to \lim B_s \to \lim C_s \to \lim^1 A_s \to \lim^1 B_s \to \lim^1 C_s \to 0\,.$$

6.26 Exercise. (Pro-isomorphisms) A tower $\{A_s\}$ of abelian groups (groups, pointed sets) is *pro-trivial* if for any $s \geq 0$ there is a $t > s$ such that the tower map $A_t \to A_s$ is trivial. Show that if $\{A_s\}$ is pro-trivial then $\lim A_s$ is trivial and $\lim^1 A_s$ (if applicable) is trivial. A map $f\colon \{A_s\} \to \{B_s\}$ of towers is a *pro-isomorphism* if $\ker(f)$ and $\operatorname{coker}(f)$ are protrivial. Show that if f is a pro-isomorphism then f induces isomorphisms on $\lim$ and on $\lim^1$ (if applicable).

6.27 Exercise. Show that if $\{A_s\}$ is a tower of finite groups, then $\lim^1 A_s$ is trivial (possibly tricky). Conclude that if $\{U_s\}$ is a tower of pointed connected spaces with finite homotopy groups then there are isomorphisms

$$\pi_i \operatorname{holim} U_s \cong \lim \pi_i U_s \,.$$

A tower is *pro-constant* if it is pro-isomorphic to a constant tower (one in which all of the bonding maps are isomorphisms).

6.28 Exercise. (Finite type demystified) Suppose that $\{V_s\}$ is a tower of finite-dimensional vector spaces over $\mathbb{F}_p$. Let $V_s^{\#}$ be the $\mathbb{F}_p$-dual of V_s. Show that $\operatorname{colim} V_s^{\#}$ is finite-dimensional over $\mathbb{F}_p$ if and only if $\{V_s\}$ is pro-constant.

Now we can go on to the real problem: computing the cohomology of the homotopy limit of a tower.

6.29 Exercise. (The simplest case) Consider the tower $\{U_s\}$, where $U_s \cong K(\mathbb{Z}/p^s, n)$, ($n \geq 1$) a fixed integer, and the bonding maps are the the obvious ones $K(\mathbb{Z}/p^s, n) \to K(\mathbb{Z}/p_{s-1}, n{-}1)$. (There's a whole story about whether merely specifying the bonding maps up to homotopy in this way actually determines the tower up to equivalence–the answer is no [100] – but the issues don't come up in our particular situation). Observe that $\operatorname{holim} U_s \sim K(\mathbb{Z}_p, n)$ and calculate that in this case the natural map

$$\operatorname{colim} H^* U_s \to H^* \operatorname{holim} U_s$$

is an isomorphism. (It's probably a good idea to prove along the way that $H^*K(\mathbb{Z}_p, n)$ is isomorphic to $H^*K(\mathbb{Z}, n)$; this is implicit in the discussion of $\mathbb{Z}/p$-homology localization above.) Short of explicit calculation for each n, it looks to me as if it might work to prove this for $n = 1$ by calculation and then use an induction based on path-loop fibrations and (colimits of) cohomology Serre spectral sequences.

6.30 Exercise. Suppose that $\{A_s\}$ is a tower of finite abelian p-groups, and consider the tower $\{A_s/pA_s\}$ obtained by reducing mod p. Show that if $\{A_s/pA_s\}$ is pro-trivial then the original tower is pro-trivial too. Now argue (tricky, but true, I think) that if $\{A_s/pA_s\}$ is pro-constant, then $\{A_s\}$ is pro-isomorphic to a finite direct sum of towers of two types; the first type is a constant tower $\{Z/p^n\}$ and the second type is the "p-adic integer" tower $\{\mathbb{Z}/p^s\}$ from the previous exercise. Conclude that in this case too there is an isomorphism

$$\operatorname{colim} H^*K(A_s, n) \cong H^* \operatorname{holim} K(A_s, n) \qquad \text{any } n \geq 1.$$

6.31 Exercise. (The one-connected case) Suppose that $\{U_s\}$ is a tower of one-connected pointed p-finite spaces such that $\operatorname{colim} H^*U_s$ is of finite type. We would like to show that the map

$$\operatorname{colim} H^*U_s \to H^* \operatorname{holim} U_s$$

is an isomorphism. Proceed (for instance) by showing, using induction on n, that

1. $\operatorname{colim} H^*P_nU_s$ is of finite type, and
2. $\operatorname{colim} H^*P_nU_s \to H^* \operatorname{holim} P_nU_s$ is an isomorphism.

(Here P_n denotes the n'th Postnikov stage.) For the first case ($n = 2$) observe using the above exercises that the assumption

$$\operatorname{colim} H^2U_s \text{ is finite dimensional over } \mathbb{F}_p$$

implies that

$$\{\pi_2(U_s) \otimes \mathbb{Z}/p\} \text{ is pro-constant}$$

and hence that the tower $\{P_2U_s\}$ is well-behaved. Proceed from one n to the next by using (colimits of) Serre spectral sequences.

6.32 Exercise. Explain how the **N**-conditions work, and invent some new ones of your own.

Slide 6-10

Slide 6-10

T and maps from BV

H^*X, $T(H^*X)$ of finite type

BK: $C_pX \sim \operatorname{holim}\{X_s\}$, X_s p-finite, $\operatorname{colim} H^*X_s \cong H^*X$

Proof of Lannes Theorem
$U_s = \operatorname{Hom}^{\mathrm{h}}(BV, X_s)$

1. $U_\infty \sim \operatorname{Hom}^{\mathrm{h}}(BV, C_pX)$
2. $H^*U_s \cong T(H^*X_s)$
3. $\operatorname{colim} T(H^*X_s) = \operatorname{colim} T(H^*X_s) \cong TH^*X$
4. Tower Theorem

$$T(H^*X)^0 \cong H^0U_\infty$$
$T(H^*X) \cong H^*U_\infty$ if any **N**-condition holds.

6.33 Exercise. Suppose that X is a one-connected finite complex. Recall from 6.7 that $T(H^*X) \to H^*X$ is an isomorphism. Conclude that that evaluation at the basepoint gives an equivalence $\operatorname{Hom}^{\mathrm{h}}(BV, C_pX) \to C_pX$. Use the arithmetic square or other arguments (6.10) to deduce that basepoint evaluation $\operatorname{Hom}^{\mathrm{h}}(BV, X) \to X$ is an equivalence. This is the Miller's Theorem (formerly known as the Sullivan Conjecture) in the one-connected case.

★ **6.34 Exercise.** Extend the previous exercise to the case in which BV is replaced by BG for a finite p-group G. Use induction on the size of G, the fact that any nontrivial G contains a central element of order p, and the fact that if K is a normal subgroup of G there is an equivalence $X^{\mathrm{h}G} \sim (X^{\mathrm{h}K})^{\mathrm{h}(G/K)}$. There's a more general principle here: if X is a space and $F \to E \to B$ is a fibration sequence over a connected base B, then $\mathrm{Hom}^{\mathrm{h}}(E, X)$ can be identified as the space of sections of an associated fibration over B with $\mathrm{Hom}^{\mathrm{h}}(F, X)$ as the fibre. (With some acrobatics, this could be derived from the transitivity of homotopy right Kan extensions, but it's easier to draw a picture.) If the inclusion $X \to \mathrm{Hom}^{\mathrm{h}}(F, X)$ of the constant maps is an equivalence, it follows that $\mathrm{Hom}^{\mathrm{h}}(B, X) \to \mathrm{Hom}^{\mathrm{h}}(E, X)$ is also an equivalence. This is another way to look at the inductive step above.

☡ **6.35 Exercise.** What are the obstacles to extending the previous exercise to the case in which G is an arbitrary finite group? Do homology decompositions help?

Slide 6-11

Slide 6-11

In the presence of a volunteer...

Tower version $\{U_s\}$ **as before**
V-condition: $\exists Y \to \{U_x\}$ inducing $H^*Y \cong \operatorname{colim} H^*U_s$.

Theorem: $\operatorname{holim} U_s \sim C_pY$

T-**version** $H^*X, T(H^*X)$ **as before**
V-condition: $\exists BV \times Y \to X$ inducing $H^*Y \cong T(H^*X)$.

Theorem: $\mathrm{Hom}^{\mathrm{h}}(BV, C_pX) \sim C_pY$

Y is a (cohomological) *volunteer*.

This slide describes an assumption very different from the **N**-conditions under which it is possible to use T to compute something about a function space. Suppose there is a homological "volunteer" for the function complex, in the sense there is a space Y together with a map

$$BV \times Y \to X \quad \text{(i.e., } Y \to \mathrm{Hom}^{\mathrm{h}}(BV, X))$$

which induces an isomorphism

$$H^*Y \xleftarrow{\cong} T(H^*X)$$

(We're still assuming that H^*X is of finite type and that TH^*X is of finite type.) Then the discussion after slide 5–10, in conjunction with the tower machinery above, produces an equivalence $C_pY \sim \mathrm{Hom}^{\mathrm{h}}(BV, C_pX)$. Note that in this case there is *no* claim that $T(H^*X) \cong H^* \mathrm{Hom}^{\mathrm{h}}(BV, C_pX)$; whether or not this is true depends entirely on whether or not Y is p-good (5–11).

Lecture 7.
Maps between classifying spaces

In this lecture, we'll look at maps from classifying spaces of finite p-groups to various other kinds of classifying spaces. The first step is to decipher what the functor T has to say about the cohomology of homotopy fixed point sets. The lecture ends with a question about whether the machinery can be used to give a simple combinatorial description of the homotopy type of C_pBG for G a finite group.

Slide 7-1

Cohomology of homotopy fixed point sets

$V = \mathbb{Z}/p$, acts on X

$X^{\mathbf{h}V}$ **and a close relative**

$$\begin{tikzcd} & X_{\mathrm{h}V} \arrow[d] \\ BV \arrow[ur, dashed] \arrow[r] & BV \end{tikzcd} \qquad \boxed{X^{\mathrm{h}V}} \qquad \begin{tikzcd} X^{\mathrm{h}V} \arrow[r] \arrow[d] & \mathrm{Hom}^{\mathrm{h}}(BV, X_{\mathrm{h}V}) \arrow[d] \\ * \arrow[r, "\mathrm{id}"] & \mathrm{Hom}^{\mathrm{h}}(BV, BV) \end{tikzcd}$$

$$\begin{tikzcd} & X_{\mathrm{h}V} \arrow[d] \\ BV \arrow[ur, dashed] \arrow[r, squiggly] & BV \end{tikzcd} \qquad \boxed{X^{\mathcal{H}V}} \qquad \begin{tikzcd} X^{\mathcal{H}V} \arrow[r] \arrow[d] & \mathrm{Hom}^{\mathrm{h}}(BV, X_{\mathrm{h}V}) \arrow[d] \\ \mathrm{Hom}^{\mathrm{h}}(BV, BV)_1 \arrow[r, "\mathrm{id}"] & \mathrm{Hom}^{\mathrm{h}}(BV, BV) \end{tikzcd}$$

Definition & question **($\mathbb{H} \cong$ identity piece of $T\mathbb{H}$)**

$$\mathcal{F}\mathrm{ix}(X, V) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(X_{\mathrm{h}V}) \;\to\; H^*X^{\mathcal{H}V} \qquad \text{(iso?)}$$

This slide defines a variant of the homotopy fixed point set, and poses the question of how to calculate its cohomology. We've seen before (2.28 that $X^{\mathrm{h}V}$ is the space of maps $BV \to X_{\mathrm{h}V}$ which cover the identity map of BV (in other words, the space of sections of $X_{\mathrm{h}V} \to BV$). The fattened up homotopy fixed point set $X^{\mathcal{H}V}$ is defined to be the space of maps $BV \to X_{\mathrm{h}V}$ which cover the identity map of BV up to homotopy.

As we've seen before, $T\mathbb{H}$ is naturally isomorphic to $H^* \operatorname{Hom}^{\mathrm{h}}(BV, BV)$.

7.1 Exercise. Observe that $\operatorname{Hom}^{\mathrm{h}}(BV, BV)$ is equivalent to a disjoint union of copies of BV, one for each homomorphism $V \to V$.

The heading of the definition block identifies $\mathbb{H}$ as the summand of $T\mathbb{H}$ corresponding to the component of $\operatorname{Hom}^{\mathrm{h}}(BV, BV)$ containing the identity map. In the definition of $\mathcal{F}\mathrm{ix}(X, V)$, $\mathbb{H}$ is supposed to be treated as a module over $T\mathbb{H}$ via the restriction map $T\mathbb{H} \to \mathbb{H}$.

7.2 Exercise. Show that construction $M \mapsto \mathbb{H} \otimes_{T\mathbb{H}} T(M)$ is exact on the category of object in $\mathcal{U}$ which are (compatibly) modules over $\mathbb{H}$. (Hint: $\mathbb{H}$ is a summand of $T\mathbb{H}$.)

The question on the slide should be easy to answer, since it just asks whether a particular summand of $T(H^* X_{\mathrm{h}V})$ accurately calculates the cohomology of a component of the mapping space $\operatorname{Hom}^{\mathrm{h}}(BV, X_{\mathrm{h}V})$.

7.3 Exercise. Show that for any V-space X, $X^{\mathcal{H}V}$ is equivalent in a natural way to $BV \times X^{\mathrm{h}V}$. What's special about this situation? (Why isn't the corresponding statement true when $X_{\mathrm{h}V} \to BV$ is replaced by some arbitrary fibration $E \to B$?)

Slide 7-2

Slide 7-2

X a finite complex: setting the scene

$V = \mathbb{Z}/p$, acts on X

$$\mathcal{F}\mathrm{ix}(X,V) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(H^*X_{\mathrm{h}V})$$

M an $\mathbb{H}$-module: $\mathcal{F}\mathrm{ix}(M) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(M)$

$\mathcal{F}$**ix**$(M) \cong 0$ **if** $M = H^*S^n$

$$T\mathbb{H} \to TM = H^*(\ \mathrm{Hom}^{\mathrm{h}}(BV,BV) \leftarrow \mathrm{Hom}^{\mathrm{h}}(BV,S^n)\ )$$

$\mathcal{F}$**ix**$(M) \cong 0$ **if** M **is finite**

Exactness.

$\mathcal{F}$**ix**$(X,V) \cong \mathbb{H} \otimes H^*(X^V)$ **(Smith Theory)**

$$\text{(finite)} \leftarrow H^*(BV \times X^V) \leftarrow H^*(X_{\mathrm{h}V}) \leftarrow \text{(finite)}$$

In the tensor products on the slide, $\mathbb{H}$ is made into a module over $T(\mathbb{H})$ by the restriction $T(\mathbb{H}) \to \mathbb{H}$ to the cohomology of the identity component of $\mathrm{Hom}^{\mathrm{h}}(BV,BV)$. The last slide constructed an algebraic object $\mathcal{F}\mathrm{ix}(X,V)$ which is an approximation to $H^*X^{\mathcal{H}V}$ in the same way as TH^*X is an approximation to $H^*\,\mathrm{Hom}^{\mathrm{h}}(BV,X)$. This slide computes $\mathcal{F}\mathrm{ix}(X,V)$ if X is a finite complex with a celluar V-action. There are several steps:

1. Extend the construction of $\mathcal{F}\mathrm{ix}(X,V)$ to give $\mathcal{F}\mathrm{ix}(M)$, where M is an unstable algebra(module) over A_p which is a module over $\mathbb{H}$. The connection is that $\mathcal{F}\mathrm{ix}(X,V) \cong \mathcal{F}\mathrm{ix}(H^*X_{\mathrm{h}V})$.
2. $M \mapsto \mathcal{F}\mathrm{ix}(M)$ is exact (7.2).
3. $\mathcal{F}\mathrm{ix}(\text{finite module}) = 0$.
4. By Smith theory to see that there is a map

   $$H^*(BV \times X^V) \leftarrow H^*X_{\mathrm{h}V}$$

   which is an isomorphism modulo finite modules.

The conclusion is that there is an isomorphism

$$\mathcal{F}\mathrm{ix}(X,V) \cong \mathcal{F}\mathrm{ix}(X^V,V) \cong \mathbb{H} \otimes H^*(X^V)\,,$$

where the last isomorphism comes from the fact that T preserves $\otimes$ and takes finite modules to themselves.

7.4 Exercise. In computing $\mathcal{F}\mathrm{ix}(H^*S^n)$, H^*S^n is treated as a trivial module over $\mathbb{H}$, in other words, as a module in which the action factors through the unique ring homomorphism $\mathbb{H} \to \mathbb{F}_p$. Verify that $\mathcal{F}\mathrm{ix}(H^*S^n)$ is zero, and show carefully how this implies that $\mathcal{F}\mathrm{ix}(M) = 0$ for any finite module M. (You may need to use a little bit of interplay between T on the category $\mathcal{K}$ of unstable algebras and T on the category $\mathcal{U}$ of unstable modules).

Slide 7-3

Slide 7-3

X a finite complex (II)

$V = \mathbb{Z}/p$, acts on X

$BV \times X^V$ is a volunteer for $(C_pX)^{\mathcal{H}V}$!

Theorem

$$\begin{aligned}(C_pX)^{\mathcal{H}V} &\sim BV \times C_p(X^V)\\ (C_pX)^{\mathrm{h}V} &\sim C_p(X^V)\end{aligned}$$

Corollary (T-free, V-free)
Q a finite p-group, X a finite Q-complex $\implies$

$$(C_pX)^{\mathrm{h}Q} \sim C_p(X^Q)$$

7.5 Exercise. There is a little sleight of hand hidden behind the innocent slide. The fact that $(C_pX)^{\mathcal{H}V}$ appears depends on the equivalence

$$C_p(X_{\mathrm{h}V}) \sim (C_pX)_{\mathrm{h}V}$$

Show that this follows from the fibre lemma (slide 5–10).

7.6 Exercise. Prove the corollary on the slide, using induction on the size of Q.

Slide 7-4

Slide 7-4

Aside: sighting of C_p(p-bad space)

- $Y = S^1 \vee S^1$
- $X = Y * S^1 * S^1 * \cdots * S^1$
- $Q = \mathbb{Z}/p$
- Q-actions: trivial on Y, rotation on S^1, diagonal on X

Y has problems ($C_pY \neq L_pY$, $C_pC_pY \neq C_pY \ldots$)

p-bad news

$$(C_pX)^{\mathrm{h}Q} \sim_p C_pY$$

You can run but you can't hide.

7.7 Exercise. Prove that the news really is bad, by showing that

1. $X^Q \cong Y$, and
2. $X^{\mathrm{h}Q} \sim_p C_pY$

In particular, the homology of the relatively pathological space C_pY comes up even if you're just interested in homotopy fixed point sets of finite groups acting on finite simply-connected complexes. For the second item, it's probably easiest to show that if X is one-connected and Q is a finite p-group acting on X, then $X^{\mathrm{h}Q} \sim_p (C_pX)^{\mathrm{h}Q}$ (compare with 6.10).

Slide 7-5

Slide 7-5

Maps $BQ \to BG$: individual components

- G a connected compact Lie group,
- Q a finite p-group, $\rho : Q \to G$
- $Z(B\rho) = \mathrm{Hom}^{\mathrm{h}}(BQ, BG)_{B\rho}$

Adjoint action and the free loop space

$$\begin{tikzcd}
 & EG \times_G (G^{\mathrm{ad}}) \arrow[r, "\sim"] \arrow[d] & BG^{S^1} \arrow[d] \\
BQ \arrow[ur, dashed, "\lambda_2"] \arrow[urr, dashed, "\lambda_1"'] \arrow[r, "B\rho"'] & BG \arrow[r, "="'] & BG
\end{tikzcd}$$

$$\Omega Z(B\rho) \sim \{\lambda_1\text{'s}\} \sim \{\lambda_2\text{'s}\} \sim G^{\mathrm{h}Q}$$

Theorem (C_p-free)

$$Z(B\rho) \ \sim_p \ B(Z_G(\rho Q))$$

The results on this slide and the next one appear in a different form in [32]; there's a significant generalization (in which finite p-groups are replaced by p-toral groups) in [80].

In this slide, the action of Q on G is by conjugation via the homomorphism ρ.

7.8 Exercise. Show that if G is a compact Lie group (or any simplicial group or reasonable topological group) the free loop space fibration $BG^{S^1} \to BG$ over BG is equivalent in the category of objects over BG to the Borel construction associated to the conjugation (adjoint) action of G on itself.

7.9 Exercise. Show that if L is a loop space then $C_pL \sim_p L$. Generalize this to show that if Q is a finite p-group acting on L by loop maps, then $C_p(L^{\mathrm{h}Q}) \sim_p (C_pL)^{\mathrm{h}Q}$. You might want to assume without loss of generality that L is connected (why is this OK?) and pass to the classifying spaces using Bousfield and Kan's fibre lemma ($\Omega C_p \sim \mathbb{C}_p\Omega$ if the input space is 1-connected) and something along the lines of $\Omega((-)^{\mathrm{h}Q}) \sim (\Omega(-))^{\mathrm{h}Q}$. (Compare with 6.10, where the action is trivial but the space involved is not a loop space.)

7.10 Exercise. Prove the theorem. This involves using the previous exercise to fool around with $\sim_p$ and C_p, and illustrates how technicalities can sometimes disappear from some final statement. The simple but notationally toxic argument which I barely restrained myself from putting on the slide reads

$$\text{Pf: } G^{\mathrm{h}Q} \sim_p (C_pG)^{\mathrm{h}Q} \sim C_p(G^Q) = C_pZ_G(\rho Q) \sim_p Z_G(\rho Q)$$

Slide 7-6

Slide 7-6

Maps $BQ \to BG$: how many components?
G, Q as before

New ingredient ***(Q acts on X)***
$H^*X < \infty$ & $\chi(X) \neq 0 \mod p$ $\Longrightarrow$ $(C_pX)^{\mathrm{h}Q} \neq \emptyset$

Theorem: $[BQ, BG] \cong \mathrm{Hom}(Q, G)/G$ **(Sketch)**

- OK if G is p-toral
- $\exists K \subset G$ with K p-toral, $\chi(G/K) \neq 0 \mod p$

$$(G/K)^Q \cong \begin{tikzcd} & K \arrow[d] \\ Q \arrow[ur, squiggly] \arrow[r] & G \end{tikzcd} \qquad (G/K)^{\mathrm{h}Q} \sim \begin{tikzcd} & BK \arrow[d] \\ BQ \arrow[ur, dashed] \arrow[r] & BG \end{tikzcd}$$

$$\coprod_{\langle\rho\rangle} BZ_G(\rho) \sim_p \mathrm{Hom}^{\mathrm{h}}(BQ, BG) \qquad \Leftrightarrow \qquad \mathcal{C}_G^{\mathcal{C}_Q} \sim_p \mathcal{C}_G^{\mathrm{h}\mathcal{C}_Q}$$

The "new ingredient" comes from an algebraic study of $\mathcal{F}\mathrm{ix}(–)$, which we'll talk about later on in more detail on slide 9–3. It's clear that the non-emptiness statement is true if X is a genuine finite complex with a cellular action of Q, since counting cells shows that that $\chi(X^Q) = \chi(X) \mod p$, and so $\chi(X) \neq 0 \mod p$ implies that $(C_pX)^{\mathrm{h}Q} \sim C_p(X^Q)$ is nonempty. The problem is that here we have to handle a case in which the "action" of Q on X $(= G/K)$ arises from a fibration over BQ with X as fibre, and so corresponds to a geometric action of Q on an infinite complex equivalent to X. The Smith theory arguments above depend on actual finiteness of the action, and so the arguments need to be replaced.

In the statement of the Theorem, $\mathrm{Hom}(Q, G)/G$ stands for the set of G-conjugacy classes of group homomorphisms $Q \to G$. The p-toral compact Lie groups which appear in the proof are groups which fit into an exact sequence

$$1 \to T \to K \to P \to 1$$

in which T is a torus (a finite product of copies of the circle group) and P is a finite p-group. The squiggly arrow in the left-hand commutative diagram is supposed to convey that the diagram is supposed to commute only up to conjugacy in G

7.11 Exercise. Prove that if G p-toral and Q is a finite p-group, then homotopy classes of maps $BQ \to BG$ correspond bijectively to conjugacy classes of maps $Q \to G$.

One way to do this is by inspecting the extension in which G by definition sits (the statement is pretty clear if G is a torus or a finite p-group). Here's another approach. Show that it is possible to find a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & (\mathbb{Z}/p^\infty)^r \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (S^1)^r \arrow[r] & G \arrow[r] & P \arrow[r] & 1
\end{tikzcd}$$

in which the left vertical arrow is the inclusion of the p-primary torsion subgroup of $(S^1)^r$. (Don't make the mistake of taking G' to be the set of p-primary torsion elements in G; this isn't necessarily a subgroup.) Now show that there is a fibration sequence

$$(B\mathbb{Z}[1/p])^r \longrightarrow BG' \longrightarrow BG$$

and use this to compare $\mathrm{Hom}^{\mathrm{h}}(BQ, BG')$ to $\mathrm{Hom}^{\mathrm{h}}(BQ, BG)$, as well as to compare $\mathrm{Hom}(Q, G')/G'$ to $\mathrm{Hom}(Q, G)/G$.

The subgroup K of G which appears on the slide is the "p-normalizer" N_pT of a maximal torus T in G. This is the inverse image in the normalizer NT of T of a Sylow p-subgroup of the Weyl group $W = (NT)/T$. Note that the Euler characteristic $\chi(G/NT)$ is 1.

7.12 Exercise. Try to make some sense of the "proof" which is sketched for the theorem. The argument has something to do with the fact that in light of the previous discussion there is a connection (involving occasional insertion and/or deletion of C_p) between $(G/K)^Q$ and $(G/K)^{\mathrm{h}Q}$. But the space $(G/K)^Q$ only makes sense if there is a homomorphism $Q \to G$ to look at. How does the "new ingredient" contribute to showing that any map $BQ \to BG$ is obtained from a homomorphism $Q \to G$?

The last statement on the slide is an interpretation of the calculation we've made on this slide and the previous one; the interpretation is in the language of homotopy theories. The statement is that if Q and G are interpreted as categories enriched over topological spaces, i.e., homotopy theories, then the category of functors from Q to G is $\mathbb{Z}/p$-equivalent, as an enriched category, to the derived or homotopy-invariant category of functors from Q to G.

7.13 Exercise. (see 1.2) A functor $\mathcal{C}_Q \to \mathcal{C}_G$ is just a group homomorphism $\rho : Q \to G$. A morphism between two such functors is an element $g \in G$ with $g\rho g^{-1} = \rho'$. What does this have to do with $\coprod_{\langle\rho\rangle} BZ_G(\rho)$?

The following exercise shows that maps from the classifying space of a finite group into the classifying space of a compact connected Lie group are not directly related to group homomorphisms if more than one prime divides the order of the finite group.

7.14 Exercise. Produce a map $B\Sigma_3 \to BS^3$ which is not realized by any homomorphism $\Sigma_3 \to S^3$. Here Σ_3 is the symmetric group on 3 letters (or equivalently the unique nontrivial semidirect product $\mathbb{Z}/2 \ltimes \mathbb{Z}/3$) and S^3 is the multiplicative group of unit quaternions (or equivalently the special unitary group $SU(2)$). It might help to observe that BS^3 is $H\mathbb{Z}$-local (since it is simply connected) and that there is an integral homology equivalence

$$B\Sigma_3 \sim_{\mathbb{Z}} L_2(B\Sigma_3) \vee L_3(B\Sigma_3) \sim \mathbb{R}P^\infty \vee L_3(B\Sigma_3)\,.$$

In constructing the map to BS^3, it's good to notice that $L_3(B\Sigma_3)$ is equivalent to $L_3(B(\mathbb{Z}/4 \ltimes \mathbb{Z}/3))$.

Slide 7-7

Slide 7-7

Maps into C_pBG for G finite

Q a finite p-group, G compact Lie

G connected $\Longrightarrow$ $\mathrm{Hom}^{\mathrm{h}}(BQ, BG)$ $\sim_p$ $\mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BG))$

Theorem

G finite $\Longrightarrow$ $\mathrm{Hom}^{\mathrm{h}}(BQ, BG)$ $\sim_p$ $\mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BG))$

1. find a faithful $\rho\colon G \to U(n) = K$
2. $$\begin{tikzcd} K/G \arrow[r] \arrow[d] & BG \arrow[r] \arrow[d] & BK \arrow[d] \\ C_p(K/G) \arrow[r] & C_p(BG) \arrow[r] & C_p(BK) \end{tikzcd}$$
3. Apply $\mathrm{Hom}^{\mathrm{h}}(BQ, -)$, compare bases and fibres.

On this slide, Q is a finite p-group and G is a compact Lie group. The first statement, involving connected G, is low-tech, in the sense that it does not depend on anything beyond obstruction theory; the statement has already come up a couple of times in the exercises for this lecture. On the other hand, the statement doesn't have that much force, since neither $\mathrm{Hom}^{\mathrm{h}}(BQ, BG)$ nor $\mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BG))$ is easy to compute. The statement in the theorem looks the similar, but it's really worlds different. If G is finite, $\mathrm{Hom}^{\mathrm{h}}(BQ, BG)$ is very easy to compute in terms of homomorphisms and centralizers (1.2, 7.13), but $\mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BG))$ is *a priori* inaccessible. The first results like this were due to Mislin [74].

The theorem actually doesn't require G to be finite (it works for a general compact Lie G); it's just that we will mostly be interested in the finite case. There are several technicalities to deal with in the proof of the theorem, and these are either irritating or fun, depending on your feelings about C_p! The comments from this point on give advice on how to wrestle with the technicalities, and can safely be ignored by the weary, the bored, and the otherwise occupied. But look at the starred remark.

7.15 Exercise. Why is the sequence on the slide involving the completed classifying spaces a fibration sequence?

7.16 Exercise. (Compare bases.) Once more, convince yourself that

$$\mathrm{Hom}^{\mathrm{h}}(BQ, BK) \to \mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BK))$$

is a $\mathbb{Z}/p$-homology equivalence. In particular, the map is a bijection on components. Don't forget about this map; we'll have to come back to it.

7.17 Exercise. (Compare fibres.) Any map $BQ \to BK$ is represented by a homomorphism $\rho : Q \to K$ (as above). The homotopy fibre of $\mathrm{Hom}^{\mathrm{h}}(BQ, BG) \to \mathrm{Hom}^{\mathrm{h}}(BQ, BK)$ over $B\rho$ is $(K/G)^{\mathrm{h}Q}$, while the homotopy fibre over the image point in $\mathrm{Hom}^{\mathrm{h}}(BQ, B(C_pBK)$ is $(C_p(K/G))^{\mathrm{h}Q}$. (In each case Q is acting from the left on K/G via the homomorphism ρ.) We would like to argue that the map

$$(K/G)^{\mathrm{h}Q} \to (C_p(K/G))^{\mathrm{h}Q}$$

is a $\mathbb{Z}/p$-homology equivalence. This seems tricky; the target is $\mathbb{Z}/p$-equivalent to $C_p((K/G)^Q)$, but the source is hard to identify. Solve the problem by taking $(K/G)^{\mathrm{h}Q}$ seriously as a homotopy fibre of mapping spaces and (using the previous slides) interpreting it as a space p-equivalent to

$$\coprod_{\sigma} Z_K(Q)/Z_G(Q)$$

(where σ ranges over G-conjugacy classes of homomorphisms $Q \to G$ which up to K-conjugacy lift ρ). Observe that this coproduct is exactly $(K/G)^Q$. Now prove that $(K/G)^Q$ is p-good.

7.18 Exercise. (Add base to fibre, carefully.) Observe that, given a map

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d] \\
F' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

in which the outer vertical arrows are $\mathbb{Z}/p$-homology equivalences, the center vertical arrow is *not* necessarily a $\mathbb{Z}/p$-homology equivalence. Explain why such weirdness will not take place if $\pi_q(B')$ is a finite p-group. Show that $\pi_1(\mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BK)_{B\rho}$ is a finite p-group. One way is to argue as follows:

1. $\Omega\,\mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BK)_{B\rho} \sim_p Z_K(\rho(Q))$. This implies that π_0 of this loop space (or equivalently π_1 of the mapping space) is a finite group.
2. $\mathrm{Hom}^{\mathrm{h}}(BQ, C_p(BK)_{B\rho}$ is $H\mathbb{Z}/p$-local. (Why?)
3. if X is an $H\mathbb{Z}/p$-local space with a finite fundamental group, then the fundamental group is a finite p-group. (Why?)

★ **Remark.** Something really interesting has happened here: you've just proved that if Q is a finite p-subgroup of the connected compact Lie group K, then the group of components of the centralizer in K of Q is also a finite p-group.

7.19 Exercise. Whew! Assemble the above pieces into a proof of the theorem.

Slide 7-8

Slide 7-8

Fusion functors

G a finite group

$\mathcal{P} = \{\text{finite } p\text{-groups}\}$
$\bar{\mathcal{P}} = \{\text{finite } p\text{-groups}\} + \{monomorphisms\}$ $\qquad I\colon \bar{\mathcal{P}} \to \mathcal{P}$

Fusion functor **(X a space)**

$$\Phi_X : \mathcal{P}^{\mathrm{op}} \to \mathbf{Set}, \qquad \Phi_X(Q) = [BQ, X]$$

$$\Phi_{BG} \sim \Phi_{C_p\,BG}$$

Does Φ_{BG} determine $C_p\,BG$?

Frugal fusion functor

$$\Psi_G : \bar{\mathcal{P}}^{\mathrm{op}} \to \mathbf{Set}, \quad \Psi_G(Q) = \{Q \hookrightarrow\rightsquigarrow G\}$$

$$\Phi_{BG} \cong \mathrm{LKan}_{I^{\mathrm{op}}}\,\Psi_G$$

Does Ψ_G determine C_pBG ?

The basic question here is what sort of algebraic data determines the homotopy type of C_pBG; this is interesting because of the idea that C_pBG captures "the finite group G at the prime p". Of course G itself (which is certainly algebraic!) determines the homotopy type of C_pBG, but G contains too much information. The group G can not usually be reconstructed from C_pBG, and in fact there are many examples of finite groups G, G' such that G is not isomorphic to G' but $C_pG \sim C_pG'$.

★ **7.20 Exercise.** Find examples of such pairs, the more complicated the better.

☡ **7.21 Exercise.** Are there examples of finite groups G, G' such that $C_pBG \sim C_pBG'$ for every prime p, but G is not isomorphic to G'? Can there be such a pair in which the equivalences are induced by map $G \to G'$?

The funny notation for $\Psi_G(Q)$ is meant to denote that $\Psi_G(Q)$ is the set of conjugacy classes of monomorphisms $Q \to G$.

7.22 Exercise. Recall that $\Phi_{BG}(Q)$ can be indentified with the set of conjugacy classes of homomorphisms $Q \to G$. Verify that Φ_{BG} is isomorphic to the left Kan extension of Ψ_G along $I^{\mathrm{op}}\colon \bar{\mathcal{P}} \to \mathcal{P}$.

☡ **7.23 Exercise.** The fusion functors are defined in terms of conjugacy classes of maps $Q \to G$, or in other words homotopy classes $[BQ, BG]$ of unpointed maps $BQ \to BG$; previous slides have shown that $[BQ, BG]$ is isomorphic to $[BQ, C_pBG]$. What happens if you decide that you don't like working up to conjugacy, and you would like to look at genuine homomorphisms $Q \to G$, or in other words homotopy classes of pointed maps $BQ \to BG$. What's the relationship between such homotopy classes and the homotopy classes of pointed maps $BQ \to C_pBG$?

Slide 7-9

Slide 7-9

Fusion systems

G a finite group, $\bar{\mathcal{P}} = p$-groups + monos

$$\begin{tikzcd}
 & & Q \arrow[rd, hook, dotted, "\langle f\rangle"] \arrow[dd, dashed] & & & Q \arrow[rd, hook, "f"] \arrow[dd, dashed] \arrow[ddr, phantom, "\copyright"] & \\
\Psi_G \rtimes \bar{\mathcal{P}} \ltimes * & = & & G & \sim & & G \\
 & & Q' \arrow[ru, hook, dotted, "\langle f'\rangle"'] & & & Q' \arrow[ru, hook, "f'"'] &
\end{tikzcd}$$

Fusion system $\mathcal{F} = \mathcal{F}_p(G)$

- $P \subset G$ a Sylow p-sugroup
- $\mathcal{F} = \{Q \subset P\}$
- $Q \to_{\mathcal{F}} Q' \ = \ f \colon Q \to Q$ s.t. $\exists g \colon f = g(-)g^{-1}$

$$\mathcal{F} \quad \sim \quad \Psi_G \rtimes \bar{\mathcal{P}} \ltimes *$$

Does $\mathcal{F}$ determine C_pBG?

The diagrammatic description of the transport category (Grothendieck construction) $\Psi_G \rtimes \bar{\mathcal{P}} \ltimes *$ is meant to indicate that this category consists of pairs $(Q, \langle f\rangle)$ where Q is a finite p-group and $\langle f\rangle$ is a conjugacy class of monomorphisms $Q \to G$. A morphism $(Q, \langle f\rangle) \to (Q', \langle f'\rangle)$ is a map $Q \to Q'$ such that the indicated diagram commutes in the only way it can. This category is equivalent to the category whose objects consist of pairs (Q, f), where Q is a finite p-group and f is a monomorphism $Q \to G$; a morphism $(Q, f) \to (Q', f')$ is a homomorphism $Q \to Q'$ such that the indicated diagram commutes *up to conjugacy* (this is what the little circled c means).

★ **7.24 Exercise.** Show that these two categories at the top of the slide are in fact equivalent.

The objects of $\mathcal{F}$ are the subgroups Q of P; a morphism $Q \to Q'$ is a homomorphism of groups which is realized by conjugation with some element $g \in G$. (The morphism data does not include the choice of a particular element g realizing $Q \to Q'$). Sometimes $\mathcal{F}$ is taken to have as its objects all of the p-subgroups of G; this gives an equivalent category.

★ **7.25 Exercise.** Verify that $\mathcal{F}$ is equivalent as a category to $\Psi_G \rtimes \bar{\mathcal{P}} \ltimes *$.

★ **7.26 Exercise.** The category $\mathcal{F}$ comes supplied with a faithful functor $\mathcal{F} \to \bar{\mathcal{P}}$ (where $\bar{\mathcal{P}}$ as above is the category of finite p-groups and monomorphisms). Find an explicit way to reconstruct the fusion functor Ψ_G from $\mathcal{F}$ and $\mathcal{F} \to \bar{\mathcal{P}}$.

The question of whether the fusion data determines the homotopy type of $C_p(BG)$ is sometimes called the "Martino-Priddy" problem. It's a question of whether the converse to one of the main results in [74] is true.

Lecture 8.

Linking systems and p-local classifying spaces

Slide 8-1

Slide 8-1

Does $\mathcal{F}$ determine C_pBG?

G a finite group, $\mathcal{F} = \mathcal{F}_p(G)$

$$\mathcal{F} \sim \{\, Q \overset{\langle f \rangle}{\hookrightarrow} G \,\}$$

Centralizer diagram $\implies$ something missing ***($C = \{Q \subset G\}$)***

$$C_pBG \sim_p \operatorname{hocolim}_{\mathcal{F}} J \qquad J\colon \langle f \rangle \mapsto BZ_G(fQ)$$

Troubling questions

1. Need $\mathcal{F}$ **and** J (better, C_pJ) to get C_pBG?
2. J too fancy? $C_pBZ_G(fQ)$ not “algebraic”
3. Circular scam? $J(1 \to G) = BG \sim_p C_pBG$

Solution to (2) and (3): **find better** C

Let C be the collection of all p-subgroups of BG (4–3). The category $\mathcal{F}$ is equivalent to the category which parametrizes the centralizer decomposition for BG which is associated to the collection C. We continue to look at the question of whether $\mathcal{F}$ determines the homotopy type of C_pBG.

8.1 Exercise. Verify that the collection C of all nontrivial p-subgroups of G satisfies the condition from 4–5, i.e.

$$(K_C)_{\mathrm{h}G} \sim_p BG\,.$$

(In fact, show that K_C is contractible.) This implies that the homotopy colimit of the centralizer diagram associated to C is $\mathbb{Z}/p$-equivalent to BG.

The slide points out that we should not expect $\mathcal{F}$ itself to give C_pBG, since (according to the lore of the centralizer decomposition) the $\mathbb{Z}/p$-homology type of BG is given by the homotopy colimit of a highly nontrivial functor J on $\mathcal{F}$. So what should we do? Accepting the functor J or even C_pJ as part of the structure is not an attractive option, since the values of $\mathbb{C}_pJ$ are complicated, and in fact one of these values is the very object, C_pBG, which we would like to understand. Remember, the goal here is to find a some sort of data which

1. is reasonably simple and combinatorial,
2. determines the homotopy type of C_pBG, and
3. can be derived from the homotopy type of C_pBG.

The functor J satisfies (1) and (2), but to an embarrassing degree, since J determines G itself ($J(\{1\}) \sim BG$). The functor C_pJ satisfies (2) and (3), but again in a pointless circular way ($C_pJ(\{1\}) \sim C_pBG$). So we need a new idea.

Slide 8-2

Slide 8-2

Switch to the p-centric collection

A more economical approach

Definition: $Q \subset G$ p-**centric** **(Q a p-group)**

$Z(Q) =$ Sylow p-subgroup in $Z_G(Q)$

Theorem $C = \{p$-**centric** $Q \subset G\}$

$$\begin{aligned} BG &\sim_p (K_C)_{\mathrm{h}G} & & \\ &\sim_p \operatorname{hocolim} I & & I(-) \sim BQ \\ &\sim_p \operatorname{hocolim} J & & J(-) \sim BZ_G(Q) \end{aligned}$$

Fewer subgroups, smaller(?) centralizers, rarely $G = Z_G(Q)\ldots$

The way in which the centralizers in the p-centric centralizer decomposition are effectively *very* much smaller than centralizers in general will become clear on the next slide. The theorem on the slide appears in [33].

8.2 Exercise. Check that the theorem is true (and pretty tame) if p does not divide the order of G.

8.3 Exercise. If p divides the order of G, show that any Sylow p-subgroup of G is p-centric.

★ **8.4 Exercise.** The theorem implies something particularly interesting if a Sylow p-subgroup of G is abelian. What is this?

★ **8.5 Exercise.** Show that a p-subgroup Q of G is p-centric if and only if the centralizer $Z_G(Q)$ is isomorphic to the product of the center $Z(Q)$ of Q with a group of order prime to p.

8.6 Exercise. Use the previous exercise to give necessary and sufficient conditions for G itself to appear as the centralizer of a p-centric subgroup. What can you say about C_pBG in this case?

8.7 Exercise. Show that if Q, Q' are p-centric subgroups of G with $Q \subset Q'$, then Q is also a p-centric subgroup of Q'.

A map $f\colon Q \to G$ is p-centric if f is a monomorphism and $f(Q)$ is a p-centric subgroup of G.

☡ **8.8 Exercise.** Given a map $f\colon Q \to G$, consider the induced map $f'\colon BQ \to C_pBG$. Give a homotopical condition on f' which is equivalent to the statement that the map f is p-centric.

Slide 8-3

Slide 8-3

The p-centric centralizer diagram

$C = \{p\text{-centric } Q \subset G\}$

$$\mathcal{F}_{\mathrm{c}} \sim \{\, Q \xhookrightarrow{\langle f \rangle} G\,,\ f(Q) \subset G \ p\text{-centric}\}$$

Centralizer decomposition

$$C_pBG \sim_p \operatorname{hocolim} J \qquad J : \langle f \rangle \mapsto BZ_G(fQ)$$

Centralizers: simplify at p to centers

$$Q \ p\text{-centric} \implies Z_G(Q) \cong Z(Q) \times \ p'\text{-group}$$

$$\implies C_pBZ_G(Q) \sim BZ(Q)$$

$$C_pBG \ \sim \ \operatorname{hocolim} BZ(fQ)$$

Centers: algebraic, and easily extracted from C_pBG

$$BZ(fQ) \ \sim_p \ \mathrm{Map}(BQ, C_pBG)_f$$

8.9 Exercise. There's something that has been taken for granted for a while. Show that if $\mathcal{D}$ is a category and $F \to G$ is a natural transformation between two functors $\mathcal{D} \to \mathbf{Sp}$ which induces a $\mathbb{Z}/p$-equivalence $F(d) \sim_p G(d)$ for each $d \in \mathcal{D}$, then F induces $\operatorname{hocolim} F \sim_p \operatorname{hocolim} d$. (cf. 2.20)

8.10 Exercise. Give an example to show that the above conclusion no longer holds if $\operatorname{hocolim}$ is replaced by holim. Does it help to assume that the values of F and G are highly connected? Find some cases in which you can say something positive.

★ **8.11 Exercise.** Suppose that Q and Q' are p-centric subgroups of G. Show that if $gGg^{-1} \subset Q'$ then conjugation with g^{-1} induces a map $Z(Q') \to Z(Q)$

Slide 8-4

Slide 8-4

The categorical p-centric centralizer model
$\mathcal{F}_{\mathrm{c}} = \{\langle f\rangle : f\colon Q \hookrightarrow G \ p\text{-centric}\ \}$

$$\text{Functor } \mathcal{Z}\colon \mathcal{F}_{\mathrm{c}} \to \mathbf{Grpd}, \qquad \mathcal{Z}(\langle f\rangle) \sim Z(Q)$$

Grothendieck construction

$$\begin{aligned} C_pBG &\sim_p \operatorname{hocolim} B\mathcal{Z} \\ &\sim_p \mathrm{N}(\mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}}) \end{aligned}$$

$\mathcal{F}$ vs. $\mathcal{Z} \rtimes \mathcal{F}_{\mathbf{c}}$

$$\mathcal{F} \sim \begin{tikzcd} Q \arrow[r, hook] \arrow[d, dashed] & G \arrow[d, dashed, "gZ_G(Q)"] \\ Q' \arrow[r, hook] & G \end{tikzcd} \qquad \mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}} \sim \begin{tikzcd} Q \arrow[r, hook, "p.\text{-c}"] \arrow[d, dashed] & G \arrow[d, dashed, "gZ'_G(Q)"] \\ Q' \arrow[r, hook, "p.\text{-c}"] & G \end{tikzcd}$$

$$Z_G(Q) \cong Z(Q) \times Z'_G(Q)$$

The slide claims that $\mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}}$ is equivalent to the category whose objects consist of pairs (Q, f), where f is a monomorphism $Q \to G$ with p-centric image. A homomorphism $(Q, f) \to (Q', f')$ is a homomorphism $Q \to Q'$ together with a coset $gZ'_G(Q)$ such that the map $Q \to Q'$ is realized inside of G by $g(-)g^{-1}$.

8.12 Exercise. Verify that these morphisms really can be composed.

8.13 Exercise. Describe $\mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}}$ up to equivalence as a category whose objects are the subgroups of a given Sylow p-subgroup of G,

Probably the easiest way to represent the functor $\mathcal{Z}$ is by the following formula

$$\mathcal{Z}(\langle f\rangle) = \Pi_1^{(p)} \operatorname{Hom}^{\mathrm{h}}(BQ, BG)_{\langle f\rangle}\,.$$

The right-had side denotes the p-primary part of the fundamental groupoid of the component of the space of maps $BQ \to BG$ corresponding to the conjugacy class $\langle f\rangle$ of homomorphisms $Q \to G$. This is also the p-primary part of the component of the category of functors $\mathcal{C}_Q \to \mathcal{C}_G$ corresponding to f.

8.14 Exercise. Suppose that $\mathcal{H}$ is a connected groupoid with the property that each vertex group is the product of a finite abelian p-group and a finite group of order prime to p. Convince yourself that the “p-primary part” of this groupoid makes functorial sense (as a groupoid with the same objects a $\mathcal{H}$). If in doubt, think about $\Pi_1 C_p\,\mathrm{N}(\mathcal{H})$.

★ **8.15 Exercise.** Check that the assertions on the slide are correct.

1. The functor $\mathcal{Z}$ makes sense,
2. there is a natural transformation from the centralizer decomposition functor J for the p-centric collection (slide 8–1) into the functor $B\mathcal{Z}$ which gives a $\mathbb{Z}/p$-homology equivalence $J \to B\mathcal{Z}$, and

3. the Grothendieck construction $\mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}}$ is (at least up to equivalence) the category described by the picture on the slide.

8.16 Exercise. Suppose that $\mathcal{A}$ is a connected abelian groupoid (i.e., each vertex group is abelian), that G is a group, and that $I\colon \mathcal{C}_G \to \mathbf{Grpd}$ is a functor whose unique value $I(*)$ is $\mathcal{A}$. Consider the functor $J\colon \mathcal{C}_G \to \mathbf{Grp}$ obtained by composing F with the functor from groupoids to abelian groups which assigns to $\mathcal{A}$ the abelian group $H_1\,\mathrm{N}(\mathcal{A})$. Is there a morphism $J \to I$ (in the category of functors $\mathcal{C}_G \to \mathbf{Grpd}$) which gives a categorical equivalence $J(*) \to I(*)$? How about a zigzag of such morphisms? It is more or less the same thing to ask whether $\mathrm{N}(I)$ is equivalent to $\mathrm{N}(J)$ in the homotopy theory of functors $\mathcal{C}_G \to \mathbf{Sp}$. Either prove that the answer is yes or give a counterexample.

Slide 8-5

Slide 8-5

The linking system

Linking system $\mathcal{L}_{\mathrm{c}} := \mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}}$

Recovering $\mathcal{L}_{\mathrm{c}}$ from C_pBG $X = C_pBG$

$$\begin{tikzcd} & Q, \arrow[dd, "h"'] & BQ \arrow[dd, "Bh"'] \arrow[dr, dotted, "\langle f\rangle : p\text{.c.}"] & \\ \mathcal{F}_{\mathrm{c}} \quad \sim & & & X \\ & Q', & BQ' \arrow[ur, dotted, "\langle f'\rangle : p\text{.c.}"'] & \end{tikzcd}$$

$$\mathcal{Z}\langle f\rangle = \Pi_1\,\mathrm{Hom}^{\mathrm{h}}(BQ, X)_f$$

Now just form $\mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}}$

Theorem

$$C_pBG \sim C_pBG' \iff \mathcal{L}_{\mathrm{c}}(G) \sim \mathcal{L}_{\mathrm{c}}(G')$$

In the diagram formula for $\mathcal{F}_{\mathrm{c}}$ above, the objects of the category are the pairs $(Q, \langle f\rangle)$, where Q is a finite p-group and $\langle f\rangle$ is a homotopy class of p-centric (8.8) maps $BQ \to C_pBG$; a homomorphism $(Q, \langle f\rangle) \to (Q', \langle f'\rangle)$ is a homomorphism $Q \to Q'$ which makes the obvious diagram commute (in the homotopy category). The functor $\mathcal{Z}$ assigns to $(Q, \langle f\rangle)$ the fundamental groupoid of the indicated mapping space component.

8.17 Exercise. Check that this construction for $\mathcal{L}_{\mathrm{c}} = \mathcal{Z} \rtimes \mathcal{F}_{\mathrm{c}}$, which takes C_pBG as input, gives a category equivalent to the one from slide 8–4.

In the statement of the theorem, the $\sim$ on the left is equivalence between spaces and the $\sim$ on the right is equivalence of categories.

8.18 Exercise. Let $\mathcal{A}$ be the full subcategory of the category of (unpointed) spaces consisting of spaces equivalent to BQ, Q a finite p-group. Let $\mathcal{B}$ be the category with the single object C_pBG and only the identity map. Construct a category $\mathcal{C}$ by taking the disjoint union of $\mathcal{A}$ and $\mathcal{B}$. There are no maps in $\mathcal{C}$ from the object in $\mathcal{B}$ to objects in $\mathcal{A}$, but if $BQ \in \mathcal{A}$ the maps $BQ \to C_pBG$ are the ordinary p-centric maps of spaces $BQ \to C_pBG$ (8.8). Calculate the homotopy left Kan extension over $\mathcal{C}$ of the inclusion map $\mathcal{A} \to \mathbf{Sp}$.

Slide 8-6

Slide 8-6

$\mathcal{L}_{\mathrm{c}}$ vs. $\mathcal{F}_{\mathrm{c}}$ – the orbit picture

$Z_G(Q) \cong Z(Q) \times \mathbb{Z}'_G(Q)$

Three categories $X = BG, C_pBG$

Category	Object	Morphism
$\mathcal{L}_{\mathrm{c}}$	$Q + f_{p.\mathrm{c.}} \colon BQ \to X$	$h \colon Q \to Q'_* + \omega$
$\mathcal{F}_{\mathrm{c}}$	$Q + \langle f_{p.\mathrm{c.}} \rangle \in [BQ, X]$	$h \colon Q \to Q'$
$\bar{\mathcal{O}}_{\mathrm{c}}$	$Q + \langle f_{p.\mathrm{c.}} \rangle \in [BQ, X]$	$\langle h \rangle \in [BQ, BQ']$

Three categories

$$\begin{tikzcd}
Q \arrow[r, "f\,:p.\mathrm{c.}"] \arrow[d, squiggly] & G \arrow[d, dashed] & \square \\
Q' \arrow[r, "f'\,:p.\mathrm{c.}"'] & G &
\end{tikzcd}$$

$\mathcal{L}_{\mathrm{c}}$: $\boxed{gZ'_G(Q)}$

$\mathcal{F}_{\mathrm{c}}$: $\boxed{gZ_G(Q)}$

$\bar{\mathcal{O}}_{\mathrm{c}}$: $\boxed{Q'gZ_G(Q)}$

$$\begin{tikzcd}
\mathcal{L}_{\mathrm{c}} \arrow[r] \arrow[d] & \{Q\} \arrow[d] \\
\mathcal{F}_{\mathrm{c}} \arrow[r, "\alpha"] \arrow[d] & \{Q\} \arrow[d] \\
\bar{\mathcal{O}}_{\mathrm{c}} \arrow[r, "\beta"] \arrow[uu, mapsto, bend left=60, "- \ltimes \mathcal{Q}"] & \mathrm{Ho}\{BQ\}
\end{tikzcd}$$

Conclusions

$$\mathcal{Q} = \begin{tikzcd}
 & \{BQ\} \arrow[d] \\
\bar{\mathcal{O}}_{\mathrm{c}} \arrow[r, "\beta"'] \arrow[ur, dashed, "\lambda"] & \mathrm{Ho}\{BQ\}
\end{tikzcd}$$

$\mathcal{F}_{\mathrm{c}} + \alpha = \bar{\mathcal{O}}_{\mathrm{c}} + \beta$

$\mathcal{L}_{\mathrm{c}} = \mathcal{F}_{\mathrm{c}} + u + \lambda$

This is quite a complicated slide. It gives descriptions for each of three categories; two of these categories we have already run into. The first description is geometric, the second is algebraic; the two descriptions give equivalent categories, not necessarily isomorphic ones. In part, this systematizes what has appeared before. The symbol X stands for either BG or C_pBG.

The (p-centric) linking category $\mathcal{L}_{\mathrm{c}}$. From the geometric point of view, an object of $\mathcal{L}_{\mathrm{c}}$ is a pair (Q, f), where Q is a finite p-group and f is a p-centric map $BQ \to X$. A map $(Q, f) \to (Q', f')$ is a homomorphism $h : Q \to Q'$ together with an equivalence class of paths ω in $\mathrm{Hom}^{\mathrm{h}}(BQ, X)$ connecting $f' \cdot Bh$ to f. If $X = C_pBG$ ths is simply a homotopy class of paths.

8.19 Exercise. How can you describe the equivalence relation on the class of paths if $X = BG$? (Remember that $\pi_1 \mathrm{Hom}^{\mathrm{h}}(BQ, BG)_f \sim BZ_G(fQ)$.)

From the algebraic point of view, an object in $\mathcal{L}_{\mathrm{c}}$ is a pair (Q, f), where Q is a finite p-group and $f : Q \to G$ is a homomorphism. A map $(Q, f) \to (Q', f')$ is a homomorphism $h : Q \to Q'$ together with a coset $gZ'_G(fQ)$ of $Z'_G(fQ)$ in G such that $gfg^{-1} = f'h$.

8.20 Exercise. Convince yourself that these two descriptions give the same category.

The (p-centric) fusion system $\mathcal{F}_{\mathrm{c}}$. From the geometric point of view, an object of $\mathcal{F}_{\mathrm{c}}$ consists of a pair $(Q, \langle f \rangle)$, where Q is a finite p-group and $\langle f \rangle$ is a homotopy class of p-centric maps $BQ \to X$. A map $(Q, \langle f \rangle) \to (Q', \langle f' \rangle)$ is a homomorphism

$h : Q \to Q'$ such that $f' \cdot Bh$ is homotopic to f. Algebraically, an object consists of a pair (Q, f) where Q is a finite p-group and $f : Q \to G$ is a p-centric homomorphism. A map $(Q, f) \to (Q', f')$ is a coset $gZ_G(fQ)$ of $Z_G(fQ)$ in G such that $gfg^{-1} = f'h$.

8.21 Exercise. Convince yourself (if you haven't before, see the remarks after slide 7–9) that the two descriptions give equivalent categories.

The (reduced, p-centric) orbit category $\bar{\mathcal{O}}_{\mathrm{c}}$. In the geometric interpretation, the objects in $\bar{\mathcal{O}}_{\mathrm{c}}$ are pairs $(Q, \langle f \rangle)$, where Q is a finite p-group and $\langle f \rangle$ is a homotopy class of p-centric maps $BQ \to X$. A map $(Q, \langle f \rangle) \to (Q', \langle f' \rangle)$ is a homotopy class $\langle h \rangle$ of maps $BQ \to BQ'$ such that $\langle f' \rangle \cdot \langle h \rangle = \langle f \rangle$. (Note that h amounts to a Q'-conjugacy class of maps $Q \to Q'$.) Algebraically, an object consists of a pair (Q, f) where Q is a finite p-group and $f : Q \to G$ is a p-centric map. A map $(Q, f) \to (Q', f')$ is a Q'-conjugacy class of maps $\langle h \rangle \colon Q \to Q'$ together with a $(Q', Z_G(Q)$ double coset $Q'gZ_G(Q)$ in G such that $f' \cdot \langle h \rangle$ contains gfg^{-1}.

8.22 Exercise. Again, check that these two descriptions give equivalent categories.

8.23 Exercise. Show that the p-centric orbit category of G (suitably defined) can be represented by the picture

$$\begin{tikzcd}
Q \arrow[r, "f:p.\mathrm{c}."] \arrow[d, squiggly] & G \arrow[d, dashed, "\boxed{Q'g}"] \\
Q' \arrow[r, "f':p.\mathrm{c}."'] & G
\end{tikzcd}$$

All three categories come with reference maps, the first two into the category of finite p-groups (and p-centric maps), the third into the category of finite p-groups and conjugacy classes of (p-centric) maps between them. (This is denoted on the slide as homotopy classes of maps between their classifying spaces.) See 8.7 for the fact that the reference functors take morphisms in the domain categories to appropriately p-centric maps in the target categories.

The maps $\mathcal{L}_{\mathrm{c}} \to \mathcal{F}_{\mathrm{c}} \to \bar{\mathcal{O}}_{\mathrm{c}}$ are pretty clear. There is a functor $\mathcal{Q} \colon \bar{\mathcal{O}}_{\mathrm{c}} \to \mathbf{Grpd}$ which sends an object $m = \ (f \colon Q \to G)$ of $\bar{\mathcal{O}}_{\mathrm{c}}$ to the groupoid consisting of all objects of $\mathcal{L}_{\mathrm{c}}$ which project to m and all morphisms between these objects which project to the identity map of m.

8.24 Exercise. Check that $\mathcal{Q}$ can be made into a functor in a natural way (why this should be true will become clear in the next exercise).

8.25 Exercise. Show that $\mathcal{L}_{\mathrm{c}}$ is equivalent to the Grothendieck construction $\bar{\mathcal{O}}_{\mathrm{c}} \ltimes \mathcal{Q}$.

8.26 Exercise. Show that $\mathcal{Q}(Q \to G)$ is naturally isomorphic to Q in the category of groupoids and natural isomorphism classes of functors between them. (This is isomorphic to the category of groupoids and homotopy classes of maps between their nerves.)

8.27 Exercise. Conclude that giving the category $\mathcal{L}_{\mathrm{c}}$ provides a lift λ of the functor β, a lift which takes values in the category $\mathbf{Sp}$ rather than (as β does) in the category $\mathrm{Ho}(\mathbf{Sp})$.

8.28 Exercise. Conversely, given a lift λ of β to $\mathbf{Sp}$, show show that some sort of a plausible candidate $\mathcal{L}'_{\mathrm{c}}$ for $\mathcal{L}_{\mathrm{c}}$ can be constructed by the formula $\mathcal{L}'_{\mathrm{c}} = \bar{\mathcal{O}}_{\mathrm{c}} \ltimes \Pi_1\lambda$.

8.29 Exercise. Explain how giving the pair $(\mathcal{F}_{\mathrm{c}}, \alpha)$ is equivalent to giving the pair $(\bar{\mathcal{O}}_{\mathrm{c}}, \beta)$.

At last!!

8.30 Exercise. Prove that if G and G’ are two groups with the property that $\mathcal{F}_{\mathrm{c}}(G) \sim \mathcal{F}_{\mathrm{c}}(G')$ but $C_p(B) \not\sim C_p(G')$, then the functor $\beta : \bar{\mathcal{O}}_{\mathrm{c}}(G) \to \mathrm{Ho}(\mathbf{Sp})$ has two inequivalent lifts $\lambda, \lambda' : \bar{\mathcal{O}}_{\mathrm{c}}(G) \to \mathbf{Sp}$.

Slide 8-7

Slide 8-7

Lifting from $\mathrm{Ho}(\mathbf{Sp})$ to $\mathbf{Sp}$

An example **(K a group)**

$$\begin{tikzcd}
 & \{BQ\} \arrow[d] & & & B\,\mathrm{Aut}^{\mathrm{h}}(BQ) \arrow[d] \\
\mathcal{C}_K \arrow[r] \arrow[ur, dashed, "\lambda"] & \mathrm{Ho}\{BQ\} & = & BK \arrow[r] \arrow[ur, dashed, "\lambda"] & B\,\mathrm{Out}(Q)
\end{tikzcd}$$

Obstruction $\in H^3(BK; Z(Q))$, $\langle\lambda\rangle \leftrightarrow H^2(BK, Z(Q))$

A generalization

$$\begin{tikzcd}
 & \{BQ\} \arrow[d] \\
\beta\colon \mathcal{D} \arrow[r] \arrow[ur, dashed, "\lambda"] & \mathrm{Ho}\{BQ\}_{p\text{-c}}
\end{tikzcd}$$

Obstruction $\in \lim^3_{\mathcal{D}^{\mathrm{op}}} \mathcal{Z}$, $\langle\lambda\rangle \leftrightarrow \lim^2_{\mathcal{D}^{\mathrm{op}}} \mathcal{Z}$, $\mathcal{Z}(d) = Z(\pi_1\beta(d))$

This slide looks at the question of taking a diagram in the homotopy category of spaces and rigidifying it to a diagram in the category of spaces itself (lifting a diagram from $\mathrm{Ho}(\mathbf{Sp})$ to $\mathbf{Sp}$). The focus is on a particular special case, but it might be worth pointing out why this case is special and particularly tractable. The general problem is to construct a functor $\lambda : \mathcal{D} \to \mathbf{Sp}$ which covers a given functor $\beta\colon \mathcal{D} \to \pi_0\mathbf{Sp} = \mathrm{Ho}(\mathbf{Sp})$. The global approach is to construct the space of such functors; the components of this space give equivalence classes of the sought-for lifts, and each component is the classifying space of the space of self-equivalences of its personal lift (where self-equivalences are to be taken over the base functor $\mathcal{D} \to \mathrm{Ho}(\mathbf{Sp})$). Of course the space of such functors might be empty, which would signify that no lift exists.

In general the lifting space can be written as a massive homotopy limit; this is the homotopy limit, over a category of simplices in $\mathrm{N}\,\mathcal{D}$, of a functor which assigns to any simplex

$$\sigma = d_0 \to d_1 \to \cdots d_n$$

the space of lifts of the restriction of β to σ. This can be calculated without too much trouble as an iterated homotopy coend (iterated Borel construction)

$$* \times^{\mathrm{h}}_{A_0} M_0 \times^{\mathrm{h}}_{A_1} \cdots \times^{\mathrm{h}}_{A_{n-1}} M_{n-1} \times^{\mathrm{h}}_{A_n} *$$

where $X_i = \beta(d_i)$, $A_i = \mathrm{Aut}^{\mathrm{h}}_1(X_i)$ is the identity component of the space of self-equivalences of X_i, and $M_i = \mathrm{Hom}^{\mathrm{h}}(X_i, X_{i+1})_{f_i}$ is the component of the space of maps $X_i \to X_{i+1}$ corresponding to $f_i = \beta(d_i \to d_{i+1})$.

Now for the punch line.

★ **8.31 Exercise.** Show that if each X_i above is the classifying space of a finite p-group Q_i, and each map f_i is a p-centric map, then the above iterated homotopy coend collapses into

$$B^2 ZQ_n) = K(Z(Q_n), 2)$$

It follows from this and a cofinality argument that the desired space of lifts can be written as the homotopy limit over $\mathcal{D}$ itself of a functor on $\mathcal{D}^{\mathrm{op}}$ which assigns to d a space equivalent to $B^2\mathcal{Z}(d)$, where $\mathcal{Z}(d) = Z(\pi_1\beta(d))$. According to general principles, there is an obstruction in $\lim^3 \mathcal{Z}$ which is zero if and only if the space of lifts is non empty; if this obstruction vanishes, the set of components corresponds bijectively to $\lim^2 \mathcal{Z}$.

☡☡ **8.32 Exercise.** Work out these general principles. Here's one statement. Suppose that $\mathcal{D}$ is a category, A is a functor from $\mathcal{D}$ to abelian groups, and $n \geq 1$ is an integer. Give a classification up to equivalence all functors $F : \mathcal{D} \to \mathbf{Sp}$ such that $\pi_i F \cong *$, $i \neq n$, and $\pi_n F \cong A$; the functors should be in bijective correspondence with elements of $\lim^{n+1} A$. The classification should be along the lines of the theory of k-invariants. For instance, the functor has a section, in the sense that $\mathrm{holim}\, F$ is nonempty, if and only if it corresponds to the zero element of $\lim^{n+1} A$. If the functor *does* have a section, then the components of the space of sections (i.e., the holim) correspond bijectively to $\lim^n A$. (Note that F is not assumed to be a functor into pointed spaces, so applying $\pi_n F$ is probably best interpreted as $H_n(F; \mathbb{Z})$.) If $\mathcal{D}$ is the category of a group G, what have you recovered?

This explains the bottom half of the slide. The top half describes a simpler problem which is meant to put above machinery in a more familiar context. If the category $\mathcal{D}$ is the category of the group K, the problem is one of taking an action of K on BQ up to homotopy and rigidifying it to an ordinary action, i.e., lifting a map $BK \to P_1 B\,\mathrm{Aut}^{\mathrm{h}}(BQ)$ to a map $BK \to B\,\mathrm{Aut}^{\mathrm{h}}(BQ)$ or, equivalently, turning an abstract kernel into a group extension. The neatest way to solve this is to analyze the homotopy groups of $B\,\mathrm{Aut}^{\mathrm{h}}(BQ)$ and use obstruction theory.

★ **8.33 Exercise.** Review the obstruction theory. It has already come up, at least implicitly, that the only nonvanishing homotopy groups of $B\,\mathrm{Aut}^{\mathrm{h}}(BQ)$ (for Q any discrete group) are

$$\pi_i = \begin{cases} \mathrm{Out}(Q) & i = 1 \\ Z(Q) & i = 2 \end{cases}$$

Slide 8-8

Slide 8-8

Fusion relations suffice!

$\mathcal{F}_c$ makes a spectacular comeback.

$G =$ finite group

$\mathcal{Z} : \bar{\mathcal{O}}_c^{\mathrm{op}} \to \mathrm{Ho}(\mathbf{Sp})$, $\mathcal{Z}(Q \to G) = Z(Q)$

Oliver's Theorem

$$\lim\nolimits^2 \mathcal{Z} = 0$$

Consequences

There is only one way to enrich $\mathcal{F}_c$ to $\mathcal{L}_c$.

$$\mathcal{F}_c(G) \ \sim_{\{Q\}}\ \mathcal{F}_c(G') \quad \Leftrightarrow \quad C_p(G) \ \sim\ C_p(G')$$

Oliver proves this by a miraculous calculation, in some sense case-by-case, that depends on the classification of finite simple groups. The odd primary case is in [81], the two-primary case in [82]

The notation $\mathcal{F}_c(G) \ \sim_{\{Q\}}\ \mathcal{F}_c(G')$ signifies that these two categories are equivalent in a way which respects the reference maps to the category of finite p-groups.

Slide 8-9

Slide 8-9

p-local finite group X

I. Fusion data (many versions) **(+ Axioms. . .)**

- (Frugal) fusion functor (Ψ) Φ, fusion category $\bar{\mathcal{P}} \ltimes \Psi$
- Fusion system $\mathcal{F}$ based on "Sylow" group P

II. p-centric fusion data (two versions)

- p-centric subcategory $\mathcal{F}_c \subset \mathcal{F}$
- p-centric orbit category $\bar{\mathcal{O}}_c$, $\quad \beta : \bar{\mathcal{O}}_c \to \mathrm{Ho}\{BQ\}$

III. Linking system

- $\mathcal{L}_c = \Pi_1\lambda \rtimes \bar{\mathcal{O}}_c$, $\lambda\colon \bar{\mathcal{O}}_c \to \{BQ\}$ a lift of β

*C_pBG **sans** G!*

$$BX \sim C_p\,\mathrm{N}(\mathcal{L}_c) \sim C_p(\mathrm{hocolim}\,\lambda \qquad\qquad)$$

This slide describes various ways of presenting a p-local finite group; the idea is to take exactly the same kind of data which determines C_pBG for G a finite group, without insisting that the data come from some G. Some references are [20], [21], and [22].

The *fusion data* can be presented in various ways, either as a fusion functor Φ, Ψ, (slide 7–8) or as a fusion system $\mathcal{F}$ (slide 7–9) based on a finite p-group P. (P serves as a

Sylow “p-subgroup” for the p-local finite groups. There are various axioms that imply in particular that it is possible to go back and forth among these presentations.

The *p-centric fusion data* sits inside the fusion data. For instance, if $\mathcal{F}$ is described as a category whose objects are the subgroups of P and whose morphisms are certain maps between them, then $\mathcal{F}_{\mathrm{c}}$ is the subcategory consisting of subgroups Q such that every object in $\mathcal{F}$ isomorphic to Q is a p-centric subgroup of P.

The *linking system* is specified in the same way as when a group G is present. One way to do this (not mentioned on the slide) is to describe $\mathcal{L}_{\mathrm{c}}$ directly as a suitable extension of $\mathcal{F}_{\mathrm{c}}$. Another way is to construct the “orbit category” $\bar{\mathcal{O}}_{\mathrm{c}}$ as a quotient of $\mathcal{F}_{\mathrm{c}}$, notice that the axioms give a functor β from $\bar{\mathcal{O}}_{\mathrm{c}}$ to the homotopy category of spaces (actually the homotopy category of classifying spaces of finite p-groups and p-centric maps between these classifying spaces) and then describe $\mathcal{L}_{\mathrm{c}}$ in terms of a lift $\lambda\colon \bar{\mathcal{O}}_{\mathrm{c}} \to \mathbf{Sp}$ of β. (Of course, λ might not exist if a certain characteristic element in $\lim^3_{\bar{\mathcal{O}}_{\mathrm{c}}} \mathcal{Z}$ refuses to vanish!)

Lecture 9.

p-compact groups

A p-compact group is meant to be homotopical a analogues of a compact Lie group. In this lecture I'll define p-compact groups and describe how to prove at least some things about them. This is just an introduction to the theory, which has recently produced a complete classification of the objects [4] [77] [78] [3]. For basic information about p-compact groups see [42], [65], or [76].

Slide 9-1

Definition of a p-compact group

Not quite like p-local finite groups

Dictionary

Compact Lie group	p-compact group
group G	*loop space* X
compact, smooth	$H^*X < \infty, \quad BX \sim C_pBX$
$\rho\colon G \to H$	$B\rho : BX \to BY$
$\ker \rho = \{1\}$	$H^*\text{fibre}(BX \to BY) < \infty$
H/G	$Y/X = \text{fibre } BX \to BY$
$Z_H(\rho G)$	$Z_Y(\rho X) = \Omega \operatorname{Hom}^{\mathrm{h}}(BX, BY)_{B\rho}$
abelian	$Z_X(X) \sim X$
torus $T = (S^1)^r$	p-complete torus $\hat{T} = C_pT$
...	...

For p-local finite groups, part of the problem is coming up with the definition; you analyze the p-completion C_pBG for G a finite group and abstract what seem to be its essential features. The upshot is that p-local finite groups come with an explicit way of building them up out of finite p-groups. The case of p-compact groups is different. The definition is simple: X is a p-compact group if X is a loop space, H^*X is finite (written on the slide as $H^*X < \infty$), and BX is p-complete. This is a straightforward stab at capturing something that looks like a finite loop space $(H^*X < \infty)$ but whose homotopy theory is concentrated at the prime p $(BX \sim C_pBX)$. The trick is to extract any structure at all from these objects. The approach is to reinterpret as many group-theoretic constructions as possible in homotopy language and then follow your nose.

9.1 Exercise. Show that if G is a compact Lie group and π_0G is a p-group, then $\Omega C_pBG \sim C_pG$ is a p-compact group.

Remark. If G is a finite group, then C_pBG gives a p-local finite group. If G is a compact Lie group with π_0G a p-group, then C_pBG gives a p-compact group. There's clearly [22] some kind of common generalization here!

9.2 Exercise. Show that if $\rho\colon G \to H$ is a homomorphism of finite p groups, then ρ is a monomorphism if and only if the homotopy fibre of $B\rho$ is $H\mathbb{Z}/p$-finite.

9.3 Exercise. Exhibit a homomorphism $\rho\colon G \to H$ of discrete groups which is not a monomorphism but such that the homotopy fibre of $B\rho$ is $H\mathbb{Z}/p$-finite for every prime p. (This suggests that the above dictionary has limits to its applicability.)

At this point, it's natural to ask "why the p in p-compact?" It seems much more appealing to study finite loop spaces, in other words, loop spaces X such that H_*X is (totally) finitely generated (so that X looks homologically like a finite complex). You might even be willing to assume that X is connected.

Why concentrate at a prime p?

The answer is that the integral theory is more complicated than the one-prime-at-a-time theories.

Consider, for instance, the humble case of the 3-sphere S^3; this sphere can be identified with the group of unit quaternions and so has a finite loop space structure. Rector proved the following disturbing theorem.

Theorem. [88] There are uncountably many homotopically distinct spaces Z with $\Omega Z \sim S^3$.

The proof consists of taking BS^3 apart into pieces via the arithmetic square, and then reattaching the pieces to one another by adjusting the maps in the square. All of Rector's exotic Z's are obtained in this way. But in fact, it turns out that this is the *only* way to obtain such Z's. First, an exercise.

9.4 Exercise. Show that up to homotopy there is a unique space Z such that $\Omega Z \sim L_{\mathbb{Q}}S^3$.

The following also turns out to be true, for any prime p. This lecture is just a bit too short to get to a proof.

Theorem [45] Up to homotopy there is a unique space Z with $\Omega Z \sim L_p(S^3)\,(\sim C_pS^3)$.

This implies that the pathology Rector discovered disappears if you work one prime at a time; it arises only because there are many possible choices involved in gluing up the primary information with the arithmetic square.

So why not leave the integers for later[1] and work one prime at a time? This is the philosophy of p-compact groups.

☡☡ **9.5 Exercise.** Rector's exotic Z's can actually be distinguished from one another by using the action of the Steenrod algebra. Explain how this can possibly be true, since each Z has $C_p(Z) \sim C_p(BS^3) \sim C_pHP^\infty$.

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[1] But AD 2008 is already later [5] [75].

Slide 9-2

Slide 9-2

Some basic properties of p-compact groups

X a p-compact group, $V = \mathbb{Z}/p$, $\rho\colon V \to X$

$$Z_X(\rho V) \to X \qquad \text{(eval at } *)$$

Theorem: centralizer is a p-compact group.
$Z_X(\rho V)$ is a p-compact group.

Theorem: centralizer is a "subgroup" of X.
$Z_X(\rho V) \to X$ is a monomorphism.

Theorem: there exist nontrivial ρ's.
$X \not\sim * \implies \exists$ nontrivial $\rho\colon V \to X$

This slide gives three statements about p-compact groups which we'll try to prove. The statements have to be interpreted according to the dictionary on the previous slide; here are the interpretations.

- $\rho\colon V \to X$ is really $B\rho\colon BV \to BX$.
- $Z_X(\rho V)$ is really the loop space $\Omega\,\mathrm{Hom}^{\mathrm{h}}(BV, BX)_{B\rho}$.
- Evaluating at the basepoint of BQ gives a map $\mathrm{Hom}^{\mathrm{h}}(BV, BX) \to BX$, i.e. a homomorphism $Z_X(\rho V) \to X$.
- The map $Z_X(\rho V) \to X$ is a monomorphism if $X/Z_X(\rho V)$ is $H\mathbb{Z}/p$-finite, i.e., if the homotopy fibre of $\mathrm{Hom}^{\mathrm{h}}(BV, BX) \to BX$ is $H\mathbb{Z}/p$-finite.
- A nontrivial homomorphism $\rho : V \to X$ is really a *non-null* map $BV \to BX$.

Slide 9-3

Slide 9-3

More cohomology of homotopy fixed point sets

$V = \mathbb{Z}/p$, acts on X, $H^*X < \infty$, $X \sim C_pX$

$$H^*X_{\mathrm{h}V} \cong \text{finitely generated module over } \mathbb{H} = H^*BV$$

Algebraic Smith theory (no need for X^V)

- (finite) $\leftarrow \mathbb{H} \otimes$ (finite) $\leftarrow H^*X_{\mathrm{h}V} \leftarrow$ (finite)
- $\mathcal{F}$ix$(X, V) \cong \mathbb{H} \otimes$ (finite)

Consequences
If any **N**-condition holds

- $H^*X^{\mathrm{h}V} < \infty$
- $H^*(X, X^{\mathrm{h}V})_{\mathrm{h}V} < \infty$
- $\chi(X^{\mathrm{h}V}) = \chi(X) \mod p$

This slide deals with the situation in which X is a p-complete space ($X \sim C_pX$), H^*X is finite (denoted $H^*X < \infty$), and $V \cong \mathbb{Z}/p$ acts on X. The main references are [46] [47] and [64]. The first remark is that the cohomology of the homotopy orbit space $X_{\mathrm{h}V}$ is in this case a finitely generated module over $\mathbb{H}$.

★ **9.6 Exercise.** Prove this (probably by using the Serre spectral sequence and the fact that $\mathbb{H}$ is a noetherian ring).

Of course, $H^*X_{\mathrm{h}V}$ is also a module over the Steenrod algebra in a way that's compatible with the $\mathbb{H}$-module structure. Algebraic Smith theory says that in this situation there is always a finite algebra R over A_p, and a map

$$H^*X_{\mathrm{h}V} \to \mathbb{H} \otimes R$$

with finite kernel and cokernel. As in slides 7–2 and 7–3, this implies that if some **N**-condition holds, $H^*X^{\mathrm{h}V} \cong R$. By naturality the map $H^*X_{\mathrm{h}V} \to H^*(X^{\mathrm{h}V})_{\mathrm{h}V}$ is then the above map $H^*X_{\mathrm{h}V} \to \mathbb{H} \otimes R$, and so the relative homology $H^*(X, X^{\mathrm{h}V})_{\mathrm{h}V}$ is finite.

Remark. Algebraic Smith theory is actually a little more explicit. Let $S \subset \mathbb{H}$ be the multiplicative subset generated by a nonzero element in degree 2 (if $p = 2$, let S be the multiplicative subset generated by the unique non-zero element of degree 1). Then $S^{-1}H^*X_{\mathrm{h}V}$ admits a natural action of A_p which is not necessarily unstable. Let $\mathrm{Un}(S^{-1}H^*X_{\mathrm{h}V})$ be the maximal unstable submodule. Algebraic Smith theory guarantees that this object splits as a tensor product

$$\mathrm{Un}(S^{-1}H^*X_{\mathrm{h}V}) \cong \mathbb{H} \otimes R$$

for some *finite* A_p-algebra R, and the natural map

$$H^*X_{\mathrm{h}V} \to \mathbb{H} \otimes R$$

becomes an isomorphism when $S^{-1}(-)$ is applied. If X is a genuine finite complex with a cellular action of V, then $R \cong H^*X^V$. So in geometric situations R is the cohomology of the fixed point set, and in homotopy theoretic situations a candidate for the cohomology of the homotopy fixed point set. In the geometric setting X^V is a volunteer for $X^{\mathrm{h}V}$ (6–11), hence giving result at the bottom of 7–3. In the purely homotopy theoretic setting of the current slide there are usually no volunteers, and we have to rely on **N**-conditions to extract information from T.

Suppose that (X, A) is a pair of spaces, that $f : Y \to X$ is a principal V-covering, and that $B = f^{-1}A$. Suppose in addition that $H^*(X, A)$ and $H^*(Y, B)$ are finite.

★ **9.7 Exercise.** Prove that $\chi(Y, B) = p\chi(X, A)$. Here $\chi(X, A)$ for instance is the alternating sum of the ranks of the mod p homology groups of (X, A). (Use the fact that if I is the augmentation ideal of the group ring $\mathbb{F}_p[V]$, then $I^p = 0$ and $I^k/I^{k+1} \cong \mathbb{Z}/p$ for $k < p$. Stitch together many long exact twisted cohomology sequences.) Deduce the final congruence on the slide.

Slide 9-4

Slide 9-4

The centralizer of V in X is a p-compact group.
$V = \mathbb{Z}/p$, $\rho : V \to X$

Formula for $Z_X(\rho V)$ $\Lambda BX = \mathrm{Hom}^{\mathbf{h}}(S^1, BX)$

$$\begin{tikzcd}
 & & \Lambda BX \arrow[dd] & & B\rho^*\Lambda BX \arrow[dd] & & \\
Z_X(\rho V) & \sim & & \sim & & \sim & X^{\mathrm{h}V} \\
 & BV \arrow[r, "B\rho"'] \arrow[uur, dashed] & BX & & BV \arrow[uu, dashed, bend right] & &
\end{tikzcd}$$

Algebraic Smith theory **N-condition: $X^{\mathbf{h}V}$ is an H-space**
$H^*X^{\mathrm{h}V} < \infty$

End game $Z = Z_X(\rho V)$
BX p-complete $\implies$ BX p-local $\implies$ BZ p-local
$+\ \pi_0 Z$ finite $\implies$ $\pi_1 BZ$ a finite p-group
$\implies$ BZ p-good, p-complete

The slide begins by describing $Z_X(\rho V) = \Omega\,\mathrm{Hom}^{\mathrm{h}}(BV, BZ)_{B\rho}$ as the space of lifts of $B\rho$ into the free loop fibration over BX or equivalently the space of sections of a pullback fibration over BV with X. This in turn can be interpreted the homotopy fixed point set of a suitable action of V on X. As in 7.8, this is a conjugation action.

It follows from algebraic Smith theory that $H^*X^{\mathrm{h}V} \cong H^*Z_X(\rho V)$ is finite. One of the **N**-conditions is automatically satisfied, since $X^{\mathrm{h}V}$ is an H-space, in fact a loop space. In proving that $Z_X(\rho V)$ is a p-compact group, the end game is showing that $BZ_X(\rho V)$ is p-complete.

9.8 Exercise. Work out the "end game" reasoning sketched on the slide.

Slide 9-5

Slide 9-5

The centralizer of V in X is a subgroup
$V = \mathbb{Z}/p$, $\rho : V \to X$, $Z = Z_X(\rho V)$

Construction of X/Z: V acts on BX^p by permutation.

$$\begin{tikzcd}
X^p/X \arrow[r] \arrow[d, mapsto, "(-)^{\mathrm{h}V}"] & BX \arrow[r, "\mathrm{diag}"] \arrow[d, mapsto, "(-)^{\mathrm{h}V}"] & BX^p \arrow[d, mapsto, "(-)^{\mathrm{h}V}"] \\
\coprod_\rho X/Z\rho \sim (X^p/X)^{\mathrm{h}V} \arrow[r] & \coprod_\rho BZ\rho \arrow[r] & BX
\end{tikzcd}$$

$H^*(\coprod_\rho X/Z\rho) < \infty$ **by algebraic Smith theory**
N-condition:

$$\begin{tikzcd}
X/Z\rho \arrow[r] & BZ\rho \arrow[r] & BX
\end{tikzcd} \implies \coprod_\rho X/Z\rho \text{ is } p\text{-complete}$$

In order to show that the centralizer of V in X is a "subgroup" of X, it's necessary to show that the homotopy fibre of the map $e : BZ = \mathrm{Map}(BV, BX)_{B\rho} \to BX$ has finite mod p cohomology. (Here e stands for evaluation at the basepoint.) The idea of the argument is to express e in terms of homotopy fixed point sets of V-actions, and then look at the information about the homotopy fibre of e that this expression reveals.

The space BZ is one component of $\mathrm{Hom}^{\mathrm{h}}(BV, BX)$; in using the argument on the slide, it's handy to treat all of the components at once. In the coproducts $\coprod_\rho$ on the slide, ρ runs over all conjugacy classes of homomorphisms $V \to X$ (homotopy classes of maps $BV \to BX$), and $Z\rho$ stands for $Z_X(\rho V)$.

9.9 Exercise. Show that $\mathrm{Hom}^{\mathrm{h}}(BV, BX)$ is the homotopy fixed point set of the trivial action of V on X.

The space BX is slighly trickier. But BX is the homotopy fixed point set of the trivial action of $\{1\}$ on X, and so, by transitivity of homotopy right Kan extensions, BX is also the homotopy fixed point set of the V-space given by the homotopy right Kan extension of X from $\mathcal{C}_{\{1\}}$ to $\mathcal{C}_V$. This is a complicated way of explaining something that's usually just called Shapiro's lemma.

9.10 Exercise. Show that the above homotopy right Kan extension is $BX^p = \prod_V BX$, with the obvious permutation action of V. Conclude that $(BX^p)^{\mathrm{h}V} \sim BX$.

9.11 Exercise. Show that the map $e\colon BZ \to BX$ is obtained up to homotopy by taking a component of the map $(BX)^{\mathrm{h}V} \to (BX^p)^{\mathrm{h}V}$ induced by the diagonal map $BX \to BX^p$.

But taking homotopy fixed point sets, or any homotopy limit construction, preserves fibration sequences (since homotopy limits commute with one another)

9.12 Exercise. The above statement is true, but is it actually relevant to this slide? Given a fibration $E \to B$ of V-spaces, how can you describe the fibre(s) of the induced map $E^{\mathrm{h}V} \to B^{\mathrm{h}V}$? Note that "fibres" might be appropriate, since $B^{\mathrm{h}V}$ might not be connected, and there's no obvious reason that the fibres over different components of $B^{\mathrm{h}V}$ should be equivalent.

9.13 Exercise. Verify the **N**-condition on the slide by showing that $\coprod_\rho Z/Z\rho$ is p-complete.

9.14 Exercise. Show that the set of conjugacy classes of homomorphisms $V \to X$ is finite.

★ **9.15 Exercise.** Suppose that G is a discrete group, and let G^p/G denote the quotient of G^p by the diagonal action of G on the right. The permutation action of V on G^p induces an action of V on G^p/G. Show that the fixed point set $(G^p/G)^V$ corresponds bijectively to the set of homomorphisms $\rho\colon V \to G$, and that under this correspondence the left action of G on $(G^p/G)^V$ (induced ultimately by the diagonal left action of G on G^p) gives the conjugation action of G on the set of homomorphisms $V \to G$. It's interesting even to look at $V = \mathbb{Z}/2$.

☡ **9.16 Exercise.** What adjustments can be made to the above arguments if V is replaced by an arbitrary finite p-group Q?

Slide 9-6

Slide 9-6

$\exists$ **non-trivial** $V \to X$
$V = \mathbb{Z}/p$, $X \not\sim *$

$$\textstyle\coprod_\rho X/Z\rho \sim (X^p/X)^{\mathrm{h}V}$$

Existence of a nontrivial ρ ($\rho \neq 0$)

Statement	Reason
$\chi(X) = 0 \mod p$	Milnor-Moore
$\chi(X^p/X) = 0 \mod p$	$X^p/X \sim X^{p-1}$
$\chi(\coprod_\rho X/Z\rho) = 0 \mod p$	algebraic Smith theory
$\chi(X/Z_0) = 1$	Miller's theorem

$\therefore \exists\rho \neq 0$ such that $\chi(X/Z\rho) \neq 0 \mod p$

The slide begins by recalling the formula from the previous slide for the "space of homomorphisms $\rho\colon V \to X$", i.e. the space

$$\coprod_{\langle\rho\rangle} X/Z_X(\rho V)$$

indexed by the set of conjugacy classes of homomorphisms $\rho\colon V \to X$, or equivalently the space $\mathrm{Hom}^{\mathrm{h}}_*(BV, BX)$ of pointed maps $BV \to BX$. The space has a trivial component, corresponding to the trivial homomorphism $V \to X$.

9.17 Exercise. Show that the centralizer Z_0 of the trivial homomorphism $V \to X$ is equivalent to X. Conclude that the component X/Z_0 of $\coprod_{\langle\rho\rangle} X/Z_X(\rho V)$ corresponding to $\rho = 0$ is contractible (cf. 6.33).

The problem here is to show that if X is not contractible itself the above space has a least one additional component; the proof actually shows a bit more. The first step is a theorem of Milnor-Moore on the structure of Hopf algebras [73], which implies that a nontrivial finite-dimensional connected cocommutative associative Hopf algebra over $\mathbb{F}_p$ with antipode has an Euler characteristic which is divisible by p. Let X_1 be the identity component of X. Since X is a loop space all of its components are equivalent to one another, and so

$$\chi(X) = \chi(X_1)(\#\pi_0 X)$$

By Milnor-Moore, $\chi(X_1)$ is divisible by p if X_1 is not contractible; by the definition of p-compact group, the order of $\pi_0 X$ is a power of p. The conclusion is that $\chi(X)$ is divisible by p unless X_1 is contractible and $\pi_0 X = *$, i.e., unless X is contractible.

☡☡ **9.18 Exercise.** Prove that if X is a connected p-compact group which is not contractible then $\chi(X)$ is actually zero on the nose, not just divisible by p. One idea that would work if X were a connected compact Lie group would be to observe that the Euler characteristic can be computed with rational coefficients.

Now X is compact, parallelizable, not contractible, and hence not rationally acyclic (it has a fundamental class). It follows that $H^*(X;\mathbb{Q})$ is a nontrivial finite-dimensional connected Hopf algebra over the rationals and thus, as a tensor product of exterior algebras on odd-dimensional generators, has Euler characteristic zero. There are obstacles in reasoning like this with a connected p-compact group X, starting with the fact that the actual integral or rational homology of the p-complete space X is going to be much too big to work with as it stands. Overcome the obstacles.

The next step on the slide is to observe that the $\chi(X^p/X)$ is divisible by p, which is immediate, since X^p/X is equivalent to X^{p-1}. Algebraic Smith theory, as described on slide 9–3 and applied on 9–5, guarantees that $\chi(X^p/X)^{\mathrm{h}V}$ is divisible by p. Since the component X/X_0 has Euler characteristic 1, there must other components, and in fact at least one other component with Euler characteristic not divisible by p.

★ **9.19 Exercise.** Show that the argument on the slide can be used with essentially no change to prove that if p divides the order of the finite group G, then there is an element of order p in G whose centralizer has index prime to p.

Slide 9-7

Slide 9-7

Existence of a Sylow p-subgroup for finite G
G finite, $V = \mathbb{Z}/p$

Seek $P \subset G$ such that $p \nmid \#(G/P)$

Induction on $\#(G)$ **(**$Z = Z_G(V)$**)**
Find $V \subset G$ with $p \nmid \#(G/Z)$.

$$\begin{tikzcd}
V \arrow[r, hook] & P \arrow[r, hook] \arrow[d, two heads] & Z \arrow[r, hook, "\text{prime to } p"] \arrow[d, two heads] & G \\
 & Q \arrow[r, hook, "\text{prime to } p", "\text{(induction)}"'] & Z/V &
\end{tikzcd}$$

This slide depicts a quick inductive proof of the existence of a Sylow p-subgroup in a finite group G (a Sylow p-subgroup is defined as a p-subgroup P such that p does not divide the order of G/P).

If $p \nmid \#(G)$, the trivial subgroup works. Otherwise, use the argument on slide 9–6 to show that there exists a subgroup $\mathbb{Z}/p \subset G$ such that the index in G of the centralizer $Z = Z_G(\mathbb{Z}/p)$ is prime to p. By induction, the quotient group Z/V possesses a Sylow p-subgroup Q. The inverse image P of Q in Z is then a Sylow p-subgroup of G.

Slide 9-8

Existence of a maximal torus in X

Seek $\hat{T} \subset X$ such that $\chi(X/\hat{T}) \neq 0$

Induction on size of X **($Z = Z_X(V)$)**

Find $V \subset X$ with $\chi(X/Z) \neq 0$.

$$\begin{tikzcd}
V \arrow[r, hook] & \hat{T} \arrow[r, hook] \arrow[d, two heads] & Z \arrow[r, hook, "\chi\neq 0"] \arrow[d, two heads] & X \\
& \hat{S} \arrow[r, hook, "\chi\neq 0", "\text{(induction)}"'] & Z/V &
\end{tikzcd}$$

More or less OK if Z/V "smaller" than X

This slide describes how to prove the existence of a maximal torus in a p-compact group X; a maximal torus is defined to be a p-complete toral subgroup $\hat{G}$ (slide 9–1) such that $\chi(X/\hat{G}) \neq 0$. The idea is to use slide 9–6 to find a subgroup $V \subset X$ such that $\chi(X/Z) \neq 0$, where $Z = Z_X(V)$. Now construct the quotient p-compact group Z/V and find a maximal torus $\hat{S}$ in Z/V; the inverse image T of $\hat{S}$ in Z should then be a maximal torus in X.

There are several problems with this argument, some of which can be overcome directly.

9.20 Exercise. Why does the homomorphism $V \to X$ lift to a homomorphism $V \to Z$? Prove that this lift is a monomorphism.

9.21 Exercise. Figure out how to define the quotient p-compact group Z/V, and prove that it is a p-compact group with the property that there is a fibration sequence

$$BV \to BZ\ toBZ/V\ .$$

(Hint. Since V is presumably central in Z, it should be the case that the centralizer of V in Z is Z itself. Prove this. This implies that $\mathrm{Hom}^{\mathrm{h}}(BV, BZ)_{Bi} \sim BZ$, where $i\colon V \to Z$ is the inclusion. Now note that the identity component $\mathrm{Hom}^{\mathrm{h}}(BV, BV)_1 \sim BV$ acts on $\mathrm{Hom}^{\mathrm{h}}(BV, BZ)_{Bi} \sim BZ$ by composition, and define $B(Z/V)$ by taking the Borel construction of this action.)

9.22 Exercise. Observe that the p-compact group $\hat{G}$ on the slide is not necessarily a torus. Deduce from the fact that $B\hat{G}$ lies in a fibration sequence

$$BV \to B\hat{G} \to B\hat{S}$$

that $\hat{T}$ is either a torus itself or a product $V \times \hat{T}'$, where $\hat{T}'$ is a torus. Explain why this second possibility really isn't much of a problem.

The real problem here is the induction: it seems hard to guarantee that Z/V is smaller than V in any sense. In fact:

9.23 Exercise. Suppose that X is a p-complete torus. Show that $Z \sim X$ and (even worse) that $Z/V \sim X$.

This problem seems like a secondary issue. . . after all, a maximal torus does exist if X itself is a torus. Here's one way to proceed. Assume without loss that X is connected.

☡☡ **9.24 Exercise.** Use Smith theory arguments (as on slide 9–6) to show that any homomorphism $\mathbb{Z}/p \to X$ extends to a homomorphism $\mathbb{Z}/p^\infty \to X$. Prove that if $\mathbb{Z}/p \to X$ is a monomorphism, then so is $\mathbb{Z}/p^\infty \to X$ (in the sense that $H^* X/\mathbb{Z}/p^\infty$ is finite).

Note that $\mathbb{Z}/p^\infty$ is *not* a p-compact group.

☡ **9.25 Exercise.** Suppose that $i\colon \mathbb{Z}/p^\infty \to X$ is a homomorphism. Observe that $Bi\colon B\mathbb{Z}/p^\infty \to X$ extends over $C_p(B\mathbb{Z}/p^\infty) \to X$ and that $C_p(B\mathbb{Z}/p^\infty) \sim K(\mathbb{Z}_p, 2) \sim BC_p(S^1)$. Prove that if $\mathbb{Z}/p^\infty \to X$ is a monomorphism in the sense of the previous exercise, then the induced map $C_p(S^1) \to X$ is a monomorphism of p-compact groups.

Let $S = C_p(S^1)$. We can now consider a chain of monomorphisms

$$V \to S \to Z_X(S) \to Z_X(V) \to X\,.$$

where $V \to X$ has been chosen so that $\chi(X/Z_X(V)) \neq 0$.

☡☡ **9.26 Exercise.** What is $Z_X(S)$, why is it a p-compact group, and why is $Z_X(S) \to X$ a monomorphism?

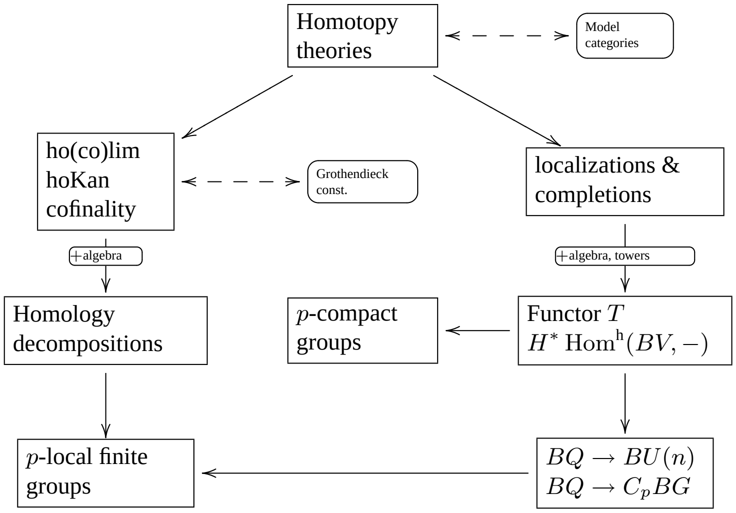
☡☡ **9.27 Exercise.** Show that $\dim(Z_X(S)/S) < \dim X$, where $\dim$ is mod p cohomological dimension. Proceed using an inductive argument to show that X possesses a maximal torus.

Lecture 10.
Wrapping up

In this lecture I'll give an overview of the first nine talks, and talk a little about some related topics that might be interesting. The remarks in this lecture are sketchy and incomplete. First, a short review of the material so far.

Slide 10-1

Another commutative diagram?



Homotopy theories

Model categories

ho(co)lim
hoKan
cofinality

Grothendieck const.

localizations & completions

+algebra

+algebra, towers

Homology decompositions

p-compact groups

Functor T
$H^* \operatorname{Hom}^{\mathrm{h}}(BV, -)$

p-local finite groups

$BQ \to BU(n)$
$BQ \to C_p BG$

This is a large-scale outline of the lecture series. Homotopy theories are the basic objects of study; these can be thought of as categories enriched over topological spaces or over simplicial sets (and in other ways, too). Homotopy (co)limits, (co)ends, and homotopy Kan extensions, as well as localizations, are defined and characterized in terms of homotopy theories. Model categories are handy tools for showing that these constructions exist and for figuring out what they give in particular cases. The Grothendieck construction is a clever device for building categorical models for homotopy colimits and coends.

The algebra of the T functor combines with tower technology to give a strong hold on the $\mathbb{Z}/p$-cohomology of function spaces with domain $B\mathbb{Z}/p$. With algebraic input of another sort (slide 4–5), the machinery of homotopy colimits, as supplemented by the Grothendieck construction, gives many homology decomposition formulas for classifying spaces of finite groups.

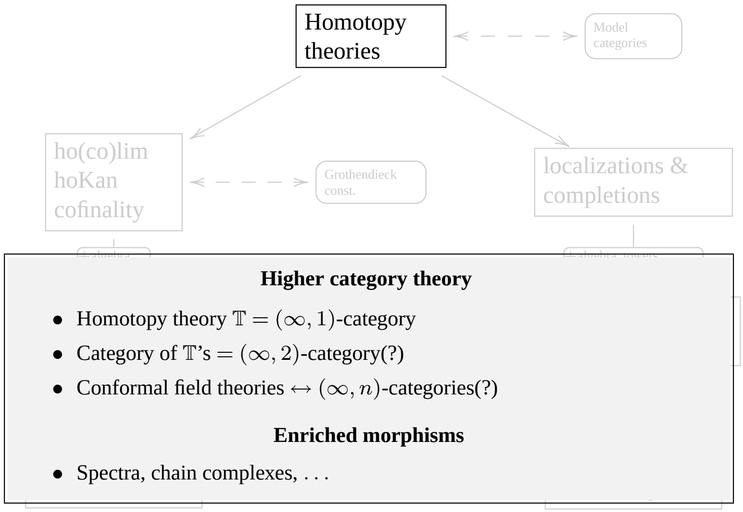
Finally, study of maps $BQ \to C_pBG$ (Q a finite p-group) reveals that these maps depend only on the p-subgroup structure of G. This is especially interesting in light of certain homology decompsitions of BG involving p-centric subgroups. Contemplating these structures leads leads to a relatively simple algebraic model for C_pBG. Other spaces which can be modeled in the same way are called p-local finite groups, and thought of as finite-group-like objects whose structure is concentrated at p.

Note: $BU(n)$ is used on the slide as shorthand for the classifying space of a generic connected compact Lie group.

Slide 10-2

Slide 10-2

Homotopy theories



Homotopy theories

Model categories

ho(co)lim
hoKan
cofinality

Grothendieck const.

localizations & completions

Higher category theory

- Homotopy theory $\mathbb{T} = (\infty, 1)$-category
- Category of $\mathbb{T}$'s $= (\infty, 2)$-category(?)
- Conformal field theories $\leftrightarrow$ (∞, n)-categories(?)

Enriched morphisms

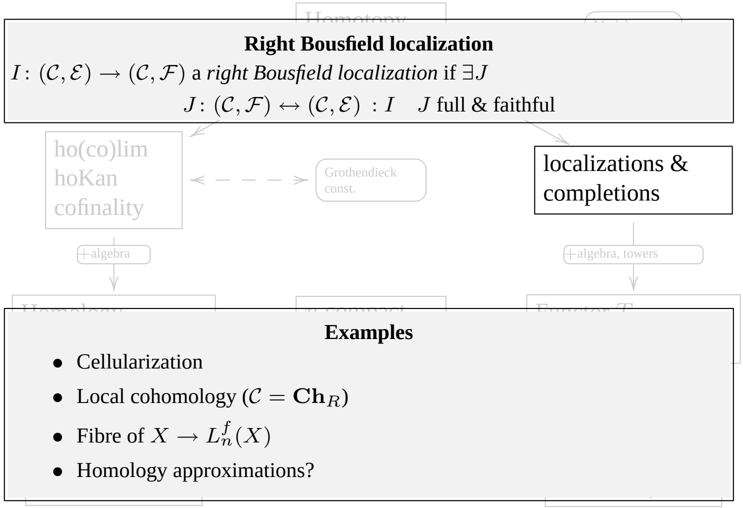
- Spectra, chain complexes, . . .

Accepting the notion of homotopy theory is just the first step on an long and apparently still treacherous journey. A homotopy theory is an $(\infty, 1)$ category, a category-like object in which the morphism between two objects form a space or (if you loop down) a topological (simplicial) groupoid. The "1" in $(\infty, 1)$ refers to this groupoid quality: all of the 1-morphisms are invertible. If $\mathcal{A}$ and $\mathcal{B}$ are two homotopy categories, the morphism object $\mathcal{B}^{\mathrm{h}\mathcal{A}}$ can be constructed as a homotopy category in its own right. There's no need for all of the maps in $\mathcal{B}^{\mathrm{h}\mathcal{A}}$ to be invertible, but each component of a mapping complex in $\mathcal{B}^{\mathrm{h}\mathcal{A}}$ is an ordinary space, which via looping corresponds to groupoid objects. To my mind (admittedly uninformed!) this signifies that the category of homotopy theories is an $(\infty, 2)$ category. Clearly it's possible to continue this progression. It turns out that there is even a reason to continue: for instance, (∞, n) categories are related to higher conformal field theories (think manifolds, bordisms, bordisms between bordisms, . . .) John Baez and Aaron Lauda have written a terrific article about things like this [8].

Another direction to take with homotopy theories is to introduce richer morphism objects, e.g. spectra instead of spaces. For instance, Dugger and Shipley have studied enriched model categories [31].

Slide 10-3

Localizations & completions



Right Bousfield localization

$I\colon (\mathcal{C},\mathcal{E}) \to (\mathcal{C},\mathcal{F})$ a *right Bousfield localization* if $\exists J$

$J\colon (\mathcal{C},\mathcal{F}) \leftrightarrow (\mathcal{C},\mathcal{E})\ :I \quad J$ full & faithful

Examples

- Cellularization
- Local cohomology ($\mathcal{C} = \mathbf{Ch}_R$)
- Fibre of $X \to L_n^f(X)$
- Homology approximations?

Here, as in slide 5–1 $(\mathcal{C},\mathcal{E})$ is a homotopy theory in the guise of a categorical pair, and $\mathcal{E} \subset \mathcal{F}$. The localization map $I\colon (\mathcal{C},\mathcal{E}) \to (\mathcal{C},\mathcal{F})$ is a right Bousfield localization if it is a right adjoint and its left adjoint is full and faithful. This comes up in a particular model category situation. Let A be an object in a model category $(\mathcal{C},\mathcal{E})$. Call a map f in $\mathcal{C}$ an A-equivalence if $\mathrm{Hom}^{\mathrm{h}}(f)$ is an equivalence in $\mathbf{Sp}$, and let $\mathcal{F}$ be the class of A-equivalences. Then, under mild conditions on $(\mathcal{C},\mathcal{E})$ the map $I\colon (\mathcal{C},\mathcal{E}) \to (\mathcal{C},\mathcal{F})$ is a right Bousfield localization; see [56, §5]. The functor JI is sometimes called a colocalization functor or cellularization functor, and $JI(X)$, written $\mathrm{Cell}_A(X)$ or $CW_A(X)$, is called the cellularization of X with respect to A. The natural map $\mathrm{Cell}_A(X) \to X$ is called the A-cellular approximation to X, and it's characterized by two properties:

1. $\mathrm{Cell}_A(X)$ is built from A by iterated homotopy colimits, and
2. $\mathrm{Cell}_A(X) \to X$ induces an equivalence $\mathrm{Hom}^{\mathrm{h}}(A, \mathrm{Cell}_A(X)) \to \mathrm{Hom}^{\mathrm{h}}(A, X)$.

In the topological setting this is studied by Farjoun [48]. If R is a commutative ring and $I \subset R$ is a finitely generated ideal, then local cohomology with respect to I is cellularization in the category $\mathbf{Ch}_R$ of chain complexes over R with respect to $\oplus_k \Sigma^k R/I$ [35];

10.1 Exercise. Suppose that X is a spectrum. Show that the homotopy fibre of $X \to L_n^f X$ [71] is the cellularization of X with respect to $\vee_k \Sigma^k A$, where A is any stable finite complex of type with $K(i)_* X = 0$ for $i \leq n$ and $K(n)_* X \neq 0$. Show that $X \to L_n^f X$ is localization with respect to the map $A \to *$.

10.2 Exercise. Suppose that G is a finite group and that C is a collection of subgroups of G. Show that the cellularization of the trivial G-space $*$ with respect to $A = \coprod_{H \in C} G/H$ is equivalent as G-space to the universal space EC (and nonequivariantly equivalent to the nerve of the poset represented by C). More generally, show that if X is a G-space, then $\mathrm{Cell}_A(X)$ is equivalent to the homotopy coend

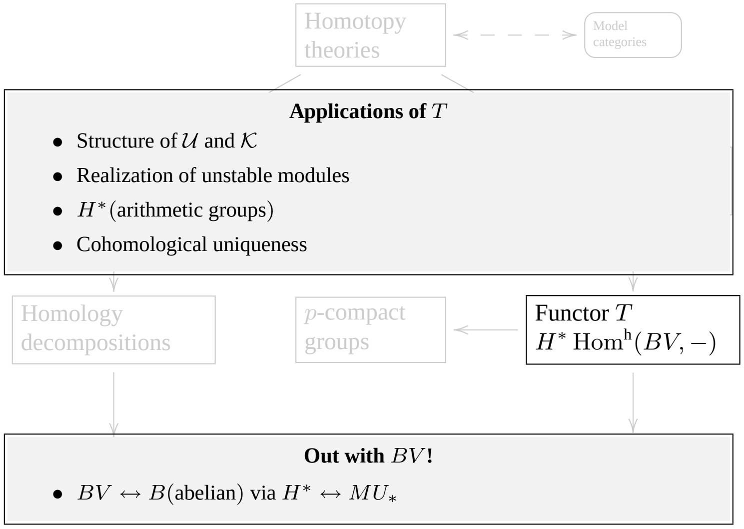
$$(G/H \mapsto X^H) \times^{\mathrm{h}}_{\mathcal{O}_C} (G/H \mapsto G/H)$$

over the category $\mathcal{O}_C = \{G/H : H \in C\}$ of C-orbits.

Slide 10-4

Slide 10-4

The functor T and $H^* \operatorname{Hom}^{\mathbf{h}}(BV, -)$



Homotopy theories

Model categories

Applications of T

- Structure of $\mathcal{U}$ and $\mathcal{K}$
- Realization of unstable modules
- H^*(arithmetic groups)
- Cohomological uniqueness

Homology decompositions

p-compact groups

Functor T
$H^* \operatorname{Hom}^{\mathrm{h}}(BV, -)$

Out with BV!

- $BV \leftrightarrow B$(abelian) via $H^* \leftrightarrow MU_*$

Henn, Lannes and Schwartz [53] have used the functor T to get some very detailed information about the category of unstable modules over A_p. For instance, it turns out that if G is a compact Lie group then H^*BG is determined *functorially* by the structure of H^*BG in a possibly large but finite range of dimensions. Schwartz [90] proved a conjecture of Kuhn about the nonrealizability of certain unstable modules M over A_p by a beautiful inductive reduction argument using T (and the Eilenberg-Moore spectral sequence). Henn [52] has figured out how to apply T to get information about the cohomology of some infinite discrete groups, specifically arithmetic groups.

There are many papers that use T to prove that various spaces, usually classifying spaces, are uniquely determined by their cohomology (see below, in the discussion of homology decompositions). But beware! Not all classifying spaces are determined by their cohomology algebras: compare $B\mathbb{Z}/4$ and $B\mathbb{Z}/8$. Even connectivity won't make the problem better.

10.3 Exercise. Find two connected compact Lie groups G, G' such that H^*BG is isomorphic to H^*BG' as an algebra over A_p but C_pBG is not equivalent to C_pBG'. (Hint: Try taking quotients of $SU(p^n) \times S^1$ by various finite central subgroups.)

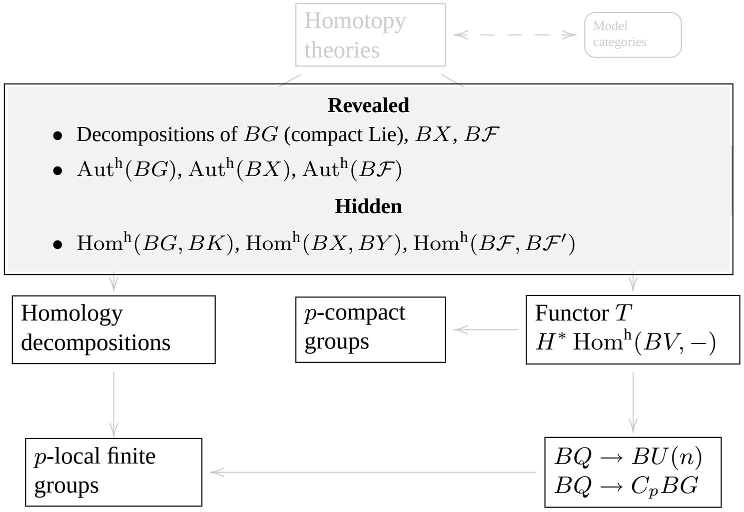
The two papers [2] [66] are especially fun because they look at examples in which more than one space (up to homotopy) realizes a given cohomology ring, and all such realizing spaces can be enumerated.

Suppose that A is an abelian compact Lie group. Lannes and Dehon [27] have obtained T-type information about $\mathrm{Hom}^{\mathrm{h}}(BA, C_pX)$ by using unitary bordism MU_* and periodic complex K-theory K_*.

Slide 10-5

Slide 10-5

Homology decompositions and maps to BG



Revealed

- Decompositions of BG (compact Lie), BX, $B\mathcal{F}$
- $\mathrm{Aut}^{\mathrm{h}}(BG)$, $\mathrm{Aut}^{\mathrm{h}}(BX)$, $\mathrm{Aut}^{\mathrm{h}}(B\mathcal{F})$

Hidden

- $\mathrm{Hom}^{\mathrm{h}}(BG, BK)$, $\mathrm{Hom}^{\mathrm{h}}(BX, BY)$, $\mathrm{Hom}^{\mathrm{h}}(B\mathcal{F}, B\mathcal{F}')$

Homology decompositions can be constructed for compact Lie groups, p-compact groups, and p-local finite groups, sometimes by using the functor T and sometimes with other techniques. Examples are in [59], [95], [25], [67]. In some sense, p-local finite groups are *defined* by the existence of a certain kind of homology decomposition for them.

These decompositions can sometimes be used to prove that a classifying space is uniquely determined by, say, its $\mathbb{Z}/p$-cohomology ring, taken as an algebra over $\mathbb{F}_p$. Many papers deal with examples of this; the simplest nontrivial case is BS^3 [45], but a case like BF_4, for instance [98], places much heavier demands on the machinery (and the machinists). All of the papers classifying p-compact groups that are mentioned before slide 9–1 depend in one way or another on decomposition ideas.

Decompositions can also be exploited to compute maps between classifying spaces and related objects. The most spectacular example of this is Jackowski-McClure-Oliver's calculation of the space of self maps of BG for G a connected compact Lie group [60]. This is not explicitly based on homology decomposition machinery, but there are homology decompositions lurking in the G-spaces (just as homology decompositions are derived from G-spaces on slide 4–5). It's possible to make similar self-equivalence calculations for p-compact groups [6] and for fusion systems [19].

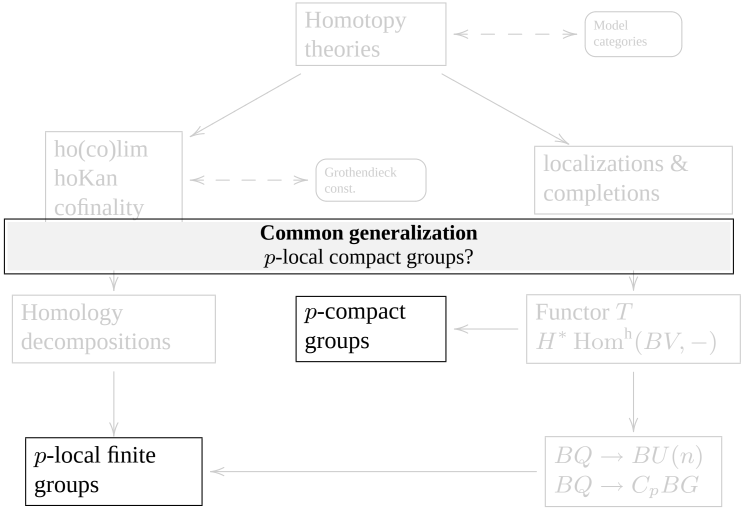
However, maps $BG \to BK$ remain a real mystery in general; representation theory isn't much of a guide unless G is a finite p-group or more generally a p-toral group. Jackowski, McClure, and Oliver have worked out some examples that show how complicated the situation can be [61].

☡☡ **10.4 Exercise.** Use T_V for various elementary abelian subgroups V (not just $V = \mathbb{Z}/p$) to prove that if G is a finite group and $p \mid \#(G)$ then the centralizer diagram for the collection of nontrivial elementary abelian subgroups of G does in fact give a homology decomposition of BG. Working backwards, use this to obtain the algebraic formula at the bottom of slide 4–5. Hint: Show that there is no problem with the cohomology decomposition if G is a finite p-group, because G has a central element of order p and so BG itself appears in the decomposition diagram. Observe that if P is a Sylow p-subgroup of G, then the transfer expresses H^*BG as a retract of H^*BP. Since the cohomology of the centralizer of an elementary abelian subgroup V of G is obtained by applying T_V to H^*BG, it should follow that in some sense yet to be determined the cohomology of the centralizer diagram for BG should be a retract of the cohomology of the centralizer diagram for BP and so the putative cohomological decomposition arrow for BG should be a retract of the arrow for BP. But the arrow for BP is an isomorphism, and a retract of an isomorphism is an isomorphism. Ken Brown's algebra is easier! But this kind of argument does lead to homology decompositions for compact Lie groups, p-compact groups, and p-local finite groups.

Slide 10-6

Slide 10-6

p-compact groups and p-local finite groups



Homotopy theories

Model categories

ho(co)lim
hoKan
cofinality

Grothendieck const.

localizations & completions

Common generalization
p-local compact groups?

Homology decompositions

p-compact groups

Functor T
$H^* \operatorname{Hom}^{\mathrm{h}}(BV, -)$

p-local finite groups

$BQ \to BU(n)$
$BQ \to C_p BG$

Recently, Broto, Levi and Oliver have taken the first steps towards merging the theory of p-compact groups and the theory of p-local finite groups [22]. They consider fusion systems which are based not on a finite p-group (thought of as a Sylow p-subgroup of the object under consideration) but on a discrete model for a p-toral group (thought of as the p-normalizer of the maximal torus in the object under consideration).

Slide 10-7

Slide 10-7

Conclusion

Many other directions

- Realization of polynomial algebras (Steenrod's problem)
- Finite loop spaces
- etc. etc., . . .

- `ArXiv`
- `hopf.math.purdue.edu`
- MathSciNet (review and reference crosslinks)

Happy Surfing!

Hmmm, are they still *other* directions if I point to references? See [7] for Andersen & Grodal's solution of the Steenrod problem. Andersen, et. al. [5] produce a long-sought-after example of a finite loop space which does not have the rational type of a compact Lie group. (The example has dimension 1254!)

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