The $(\infty,1)$-category of Types

Geometry, Topology and Physics Seminar

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<u>Problem</u>:

- Type theory does not seem to allow us to describe infinitely coherent algebraic structures.

- One way to describe the situation is that type theory gives us a theory of <u>types</u> but not <u>presentations</u> <u>of</u> <u>types</u>.

- In set-based mathematics, a presentation of an algebraic object is additional structure, distinct from the object itself.
- In higher/proof relevant mathematics, we must remember all higher relations of algebraic structures.
- Thus the difference between a presentation of a structure and the structure itself is blurred.

Higher Structure = Presentation + Property

- In light of these facts, it seems reasonable to suppose that a foundational theory for higher mathematics should come equipped with a theory of presentations.

- This has some precedence in the type theory literature: "Levitation"

- There is some flexibility in the types of presentations we allow. I suggest that the Opetopes are a natural choice.

The Plan

- Directly axiomatize opetopic types in type theory to serve as our theory of presentations

- Opetopic Types will be defined in terms of ordinary types.

- We will consider that the equations which make opetopic types well-defined belong to the meta-theory.

What is an Opetopic Type?

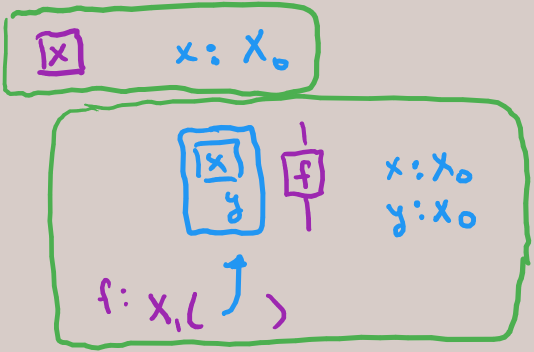
- An infinite sequence of type families:

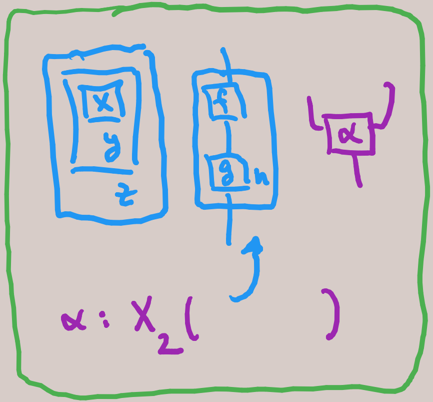
$$
\begin{aligned}
X_0 &: \mathrm{Type} \\
X_1 &: \mathrm{Frm}(X_0) \to \mathrm{Type} \\
X_2 &: \mathrm{Frm}(X_0, X_1) \to \mathrm{Type} \\
X_3 &: \mathrm{Frm}(X_0, X_1, X_2) \to \mathrm{Type} \\
&\vdots
\end{aligned}
$$

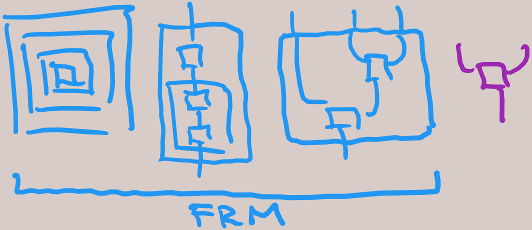
- Frames are the opetopic notion of "boundary"

$$\mathbb{O}\mathrm{Type}\,(n+1) = \sum_{X : \mathbb{O}\mathrm{Type}\, n} \mathrm{Frm}(X) \to \mathrm{Type}$$

Opetopic Types Visualized







Opetopes and Monads

- If $X : \mathbb{O}\mathrm{Type}\ n$, it determines a type $\mathrm{Frm}(X)$ of valid boundary configurations.
- There is now a monad

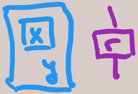
  $$\mathrm{Pd} : (\mathrm{Frm}(X) \to \mathrm{Type}) \to (\mathrm{Frm}(X) \to \mathrm{Type})$$

- This monad associates to any choice of fillers, the type of well-formed pasting diagrams built from these new cells.

Example

$X : \mathrm{Type}$

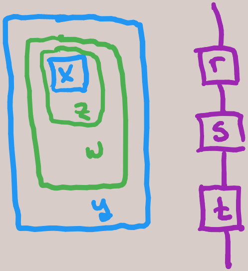
$R : X \times X \to \mathrm{Type}$



$\mathrm{Frm}(X)$ $\quad R$

$r : R(x,y)$

$x \xrightarrow{r} y$



$\mathrm{Frm}(X)$

$\rho : \mathrm{Pd}(R)$

$r : R(x,z)$

$s : R(z,w)$

$t : R(y,w)$

$\rho : \mathrm{Pd}(R)(x,y)$

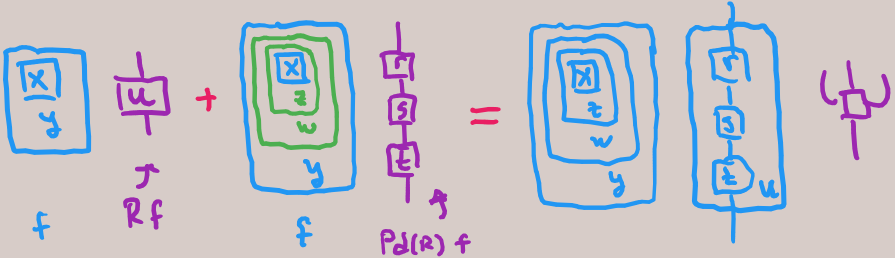
$x \xrightarrow{r} z \xrightarrow{s} w \xrightarrow{t} y$

Definition of Frames

- Recall $\mathrm{CDType}(n+1) := \sum_{x:\,\mathrm{CDType}\,n} \mathrm{Frm}(x) \to \mathrm{Type}$

- With the monad Pd we can say what Frames are

$$\mathrm{Frm}(x, R) = \sum_{f:\mathrm{Frm}(x)} R\,f \times \mathrm{Pd}(R)\,f$$



Coherence Issues

- The central issue is defining the monad structure on Pd

$$\nu, \eta, \mu$$

(map, return, bind)

- We take these operators to be primitives and prescribe their computational behavior.

More Structure

- Having defined opetopic types, we have given ourselves the type of presentations.
- This is not enough to make much progress.
- We need the type theory of presentations.

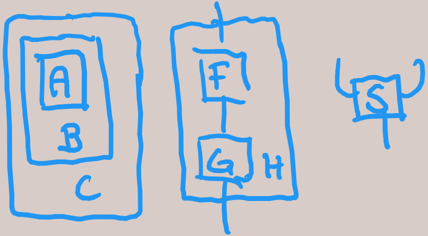
Context	$\rightsquigarrow$	$\mathbb{O}\mathrm{Type}$
Substitutions	$\rightsquigarrow$	$X \Rightarrow Y$
Types	$\not\rightsquigarrow$	$\mathbb{O}_u : \mathbb{O}\mathrm{Type}$
Terms	$\not\rightsquigarrow$	$\mathbb{O}_v : \mathbb{O}\mathrm{Type}$

The Universe of Opetopic Types

$X_0 := \mathrm{Type}$

$X_1 := \mathrm{Type} \times \mathrm{Type} \to \mathrm{Type}$

$(A, B) \mapsto A \times B \to \mathrm{Type}$



$A, B, C : \mathrm{Type}$

$F : A \times B \to \mathrm{Type}$

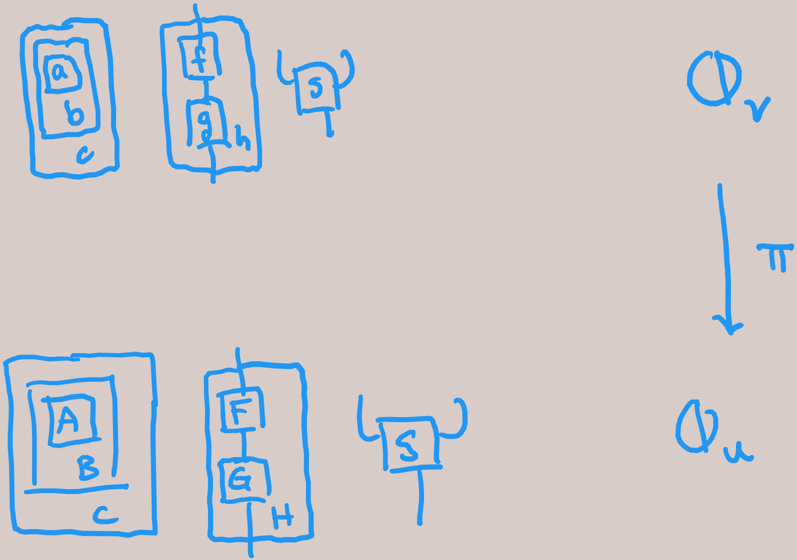
$G : B \times C \to \mathrm{Type}$

$H : A \times C \to \mathrm{Type}$

$S : (a : A)(b : B)(c : C)$

$\to F(a, b) \to G(b, c) \to H(a, c) \to \mathrm{Type}$

The Universal Fibration



What can we do with this?

- Definitions

  ∞-groupoid, $(\infty,1)$-category

  ∞-planar operad

  A_∞-monoid/group

  (∞,n)-category

- Constructions

  $\Sigma_0, \Pi_0, \mathrm{Grp}(X)$

  $X * Y, \ |X|$

  (∞-limits, colimits)

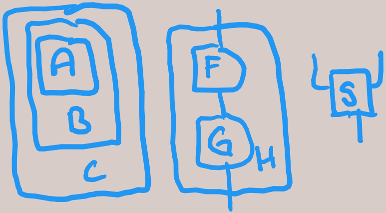
- Theorems

  Type $\simeq$ ∞-groupoid

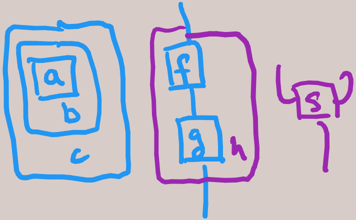
  1-category $\simeq$ truncated $(\infty,1)$-cat

And ....

Fibrant Relations



- We say S is fibrant if



$$\mathrm{isContr}\left(\sum_{h:H(a,c)} S\,a\,b\,c\,f\,g\,h\right)$$

The $(\infty,1)$-Category of Types

<u>Def</u> $\mathcal{S} :=$ the subobject of Φ_u
consisting of fibrant relations

<u>Thm</u> $\mathcal{S}$ is an $(\infty,1)$-category

Thanks!