PRELIMINARY VERSION

Geometry of Dirac Operators

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sometime around 1987

These notes are based on lectures I gave at the University of Chicago in the fall of 1987. The reader should be warned that these notes are still fairly rough in a few places (particularly in §6 and §7, which are incomplete). This is the preliminary version of a book, which I will probably not finish for at least another year. The book will go on to develop the theory of superconnections, determinant line bundles, and η-invariants associated to families of Dirac operators. (*That optimistic assessment was written a long time ago!*)

I have endeavored to keep the prerequisites to a minimum. A course in basic differential and Riemannian geometry as well as some background in Hilbert space theory should suffice. In particular, I use little or no topology (aside from de Rham cohomology) and develop the necessary analysis from scratch. There are many exercises scattered throughout these notes. Some of these do make more demands on the reader's background. While it is perhaps the author's laziness which breeds exercises, only the reader's laziness can defeat their intent.

There are two excellent books covering similar material. John Roe [R] treats the index theorem for a single operator (we have particularly profited from his work), and then goes on to discuss the Lefschetz theorem, Morse inequalities, and index theorem for covering spaces. The second book, by Berline, Getzler, and Vergne [BGV] takes a different approach to the analysis. They also include material on families of Dirac operators. We hope the student of the index theorem will profit by having available three complete accounts from varying viewpoints.

I have benefited greatly from my collaboration with Jean-Michel Bismut, as well as from discussions with Ezra Getzler, Richard Melrose, Duang Phong, and John Roe. I thank those who attended my lectures and offered valuable insights and comments. Of course, this material also reflects the influence of many teachers and colleagues.

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I welcome any comments, suggestions, corrections, criticisms, ...

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§1 Overview

The circle of ideas surrounding the Atiyah-Singer index theorem is so large that a comprehensive account could be the subject of a book in itself. The historical survey in this chapter is selective and incomplete. Our goal is to place the new developments in the theory, which are the major subject of these notes, into a larger context. We also take the opportunity to introduce some basic notions and results. Unfortunately, greater demands will be made on the reader in this chapter than in any subsequent chapter. Thus while here we treat characteristic classes abstractly, in later chapters they appear as differential forms via Chern-Weil theory. Some concepts that we mention here—for example, from algebraic geometry, topology, and probability theory—will never recur. Others will be explained at length later. With this in mind the reader may wish to skim this chapter now, referring back at his leisure.

The commentaries in [A6] (cf. [A6,Vol. 3,p.475]) provide a firsthand historical account of the development of the index theorem.

§1.1 The Riemann-Roch Theorem

Let X be a smooth connected projective curve over $\mathbb{C}$, i.e., a one dimensional compact connected complex submanifold of some complex projective space. A divisor D is a finite set of points on X, with an integer $\operatorname{ord}_x(D)$ attached to each point $x \in D$, and a divisor determines a holomorphic line bundle on X. Let $\mathcal{L}(D)$ denote the space of holomorphic sections of this bundle. We can describe $\mathcal{L}(D)$ as the space of meromorphic functions on X which have a pole of order $\leq \operatorname{ord}_x(D)$ at each $x \in X$. A basic problem in the theory of curves is: Compute the dimension of $\mathcal{L}(D)$. While this is quite difficult in general, there is a *topological* formula for $\dim \mathcal{L}(D) - \dim \mathcal{L}(K-D)$, where K is a canonical divisor of X. This is the classical Riemann-Roch[1] formula:

$$\dim \mathcal{L}(D) - \dim \mathcal{L}(K-D) = \deg(D) - g + 1. \tag{1.1.1}$$

Here g is the genus of the curve X, its basic topological invariant, which is defined to be $\frac{1}{2}\operatorname{rank} H^1(X)$. Also, $\deg(D) = \sum \operatorname{ord}_x(D)$ is the sum of the integers used in the definition of D. In the special case $\deg(D) > 2g-2$, it can be shown that $\mathcal{L}(K-D) = 0$, so that (1.1.1) provides a complete solution to the problem stated above.

Let us immediately note one consequence of the Riemann-Roch formula. Take $D = \mathcal{O}$ to be the trivial divisor consisting of no points. Then $\mathcal{L}(\mathcal{O})$ is the space of constant functions and $\mathcal{L}(K)$ is the space of holomorphic differentials. We deduce that the latter has dimension g. It follows that

[1] Roch was Riemann's student. Riemann [Ri] proved the inequality $\dim \mathcal{L}(D) \geq \deg(D) - g + 1$ and then Roch [Ro] proved the more precise (1.1.1).

g is an integer, a fact which is not at all apparent from the definition of g. Therefore, one-half the Euler characteristic of X is an integer, our first example of an *integrality theorem.*

More generally, let X be a nonsingular projective variety of complex dimension n and V a holomorphic vector bundle over X. Then the cohomology groups $H^i(X,V)$ can be defined by sheaf theory, or alternatively as the cohomology of the *Dolbeault complex*

$$\Omega^{0,0}(X,V) \xrightarrow{\bar\partial} \Omega^{0,1}(X,V) \xrightarrow{\bar\partial} \Omega^{0,2}(X,V) \to \cdots \to \Omega^{0,n}(X,V). \tag{1.1.2}$$

Here $\bar\partial$ is the Cauchy-Riemann operator. The cohomology groups are finite dimensional and the *Euler characteristic* or *index* of (1.1.2) is defined by

$$\chi(X,V) = \sum_{i=0}^{n} (-1)^i \dim H^i(X,V). \tag{1.1.3}$$

As before one wants to compute $\dim H^0(X,V)$, which in general depends on more than simply topological data. But the index $\chi(X,V)$ does have a topological formula in terms of the Chern classes $c_i(X)$ and $c_i(V)$. The case $\dim X = \operatorname{rank} V = 1$ is covered by (1.1.1). For X a projective algebraic surface ($n = 2$) and V the trivial bundle of rank one, the result is commonly known as Noether's formula:

$$\chi(X) = \frac{1}{12}\bigl(c_1^2(X) + c_2(X)\bigr)[X]. \tag{1.1.4}$$

In (1.1.4) the Chern classes are evaluated on the fundamental class of X given by the natural orientation. Notice that the denominator of $1/12$ gives an integrality theorem for the Chern numbers of a projective surface.

The formula in higher dimensions (and for arbitrary vector bundles) was first proved by Hirzebruch [H1] in 1954, though there are earlier partial results of Todd and others. Hirzebruch's formula is expressed in terms of the Todd polynomials and the Chern character. Suppose that the tangent bundle $TX = L_1 \oplus \cdots \oplus L_n$ splits as a sum of line bundles, and set $y_i = c_1(L_i) \in H^2(X)$. Then the *Todd class* is

$$\operatorname{Todd}(X) = \prod_{i=1}^{n} \frac{y_i}{1 - e^{-y_i}}. \tag{1.1.5}$$

This is a cohomology class of (mixed) even degree. Similarly, if $V = K_1 \oplus \cdots \oplus K_r$ is a sum of line bundles, with $x_i = c_1(K_i)$, then the *Chern character* is

$$\operatorname{ch}(V) = \sum_{i=1}^{n} e^{x_i}. \tag{1.1.6}$$

The theory of characteristic classes allows us to extend these definitions to arbitrary TX and V. Finally, Hirzebruch's formula states

$$\chi(X,V) = \mathrm{Todd}(X)\,\mathrm{ch}(V)[X]. \tag{1.1.7}$$

We emphasize that Hirzebruch proved (1.1.7) only for algebraic manifolds.

The first step in Hirzebruch's proof is the derivation of his *signature theorem*. Recall that on a compact oriented real differentiable manifold X of dimension $4k$ there is a nondegenerate symmetric bilinear pairing on the middle cohomology $H^{2k}(X;\mathbb{R})$ given by the cup product followed by evaluation on the fundamental class:

$$\begin{aligned} H^{2k}(X;\mathbb{R}) \otimes H^{2k}(X;\mathbb{R}) &\longrightarrow \mathbb{R} \\ \alpha \quad\otimes\quad \beta \quad &\longmapsto (\alpha \smile \beta)[X] \end{aligned} \tag{1.1.8}$$

The signature $\mathrm{Sign}(X)$ of this pairing is called the signature of X. Now the *L-class* is the polynomial in the Pontrjagin classes of X determined by the formal expression

$$L(X) = \prod_{i=1}^{2k} \frac{y_i}{\tanh y_i}, \tag{1.1.9}$$

where $y_i, -y_i$ are the Chern roots of the complexified tangent bundle. Then Hirzebruch proves

$$\mathrm{Sign}(X) = L(X)[X]. \tag{1.1.10}$$

His proof uses Thom's cobordism theory [T] in an essential way. Both sides of (1.1.10) are invariant under oriented bordism and are multiplicative. Therefore, it suffices to verify (1.1.10) on a set of generators of the (rational) oriented bordism ring. The even projective spaces $\mathbb{CP}^{2n}$ provide a convenient set of generators, and the proof concludes with the observation that the L-class is characterized as evaluating to 1 on these generators. The Todd class enters (1.1.7) in a similar manner—its value on all projective spaces $\mathbb{CP}^n$ is 1 and it is characterized by this property.

Exercise 1.1.11. Derive (1.1.1) and (1.1.4) from (1.1.7).

Exercise 1.1.12. In the context of the classical Riemann-Roch formula prove that if $\deg(D) > 2g-2$, then $\mathcal{L}(K-D) = 0$ (cf. (2.2.33) and EXERCISE ON KODAIRA VANISHING).

One consequence of the Riemann-Roch-Hirzebruch theorem is that the characteristic number on the right hand side of (1.1.7), which *a priori* is a rational number, is actually an integer. This integer is identified as a combination of dimensions of cohomology groups by the left hand side. On the other hand, the right hand side is defined for any almost complex manifold. Hirzebruch was led to ask (as early as 1953) whether the *Todd genus* $\mathrm{Todd}(X)[X]$ of an almost complex manifold (much less a non-algebraic complex manifold) is an integer [H2]. He also asked analogous questions for real manifolds. Define the *$\hat{A}$-class*[2] of a real manifold X^{4k} by the formal expression

$$\hat{A}(X) = \prod_{i=1}^{2k} \frac{y_i/2}{\sinh y_i/2}. \tag{1.1.13}$$

Then the Todd class of an almost complex manifold can be expressed as

$$\mathrm{Todd}(X) = e^{c_1(X)/2}\hat{A}(X). \tag{1.1.14}$$

In particular, $\mathrm{Todd}(X)$ depends only on the Pontrjagin classes and the first Chern class. Recalling that the mod 2 reduction of c_1 is the second Stiefel-Whitney class, it is reasonable to ask: If a real manifold X^{4k} has $w_2(X) = 0$, then is $\hat{A}(X)[X]$ an integer?[3] This was later proved true (initially up to a power of 2) by Borel and Hirzebruch [BH] in the late 50's using results of Milnor [Mi]. Still the question remained: What is the integer $\hat{A}(X)[X]$?

While these topological questions were being formulated and attacked, the Riemann-Roch-Hirzebruch theorem was extended in a new direction by Grothendieck [BS] in 1957. A decisive step was Grothendieck's introduction of K-theory. Let X be a smooth algebraic variety. Then $K(X)$ is the free abelian group generated by coherent algebraic sheaves on X modulo the equivalence $\mathcal{F} \sim \mathcal{F}' + \mathcal{F}''$ if there is a short exact sequence $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$. One can replace 'coherent algebraic sheaves' by 'holomorphic vector bundles' in this definition, and one fundamental result is that the group $K(X)$ is unchanged. Thus Chern classes and the Chern character are defined for elements of $K(X)$. (Grothendieck refines these to take values in the *Chow ring* of X.) If $f\colon X \longrightarrow X'$ is a morphism of varieties, and $\mathcal{F}$ a sheaf over X, then the direct image sheaf $R^i f_*(\mathcal{F})$ is the sheaf on X' associated to the presheaf $U' \longmapsto H^i(f^{-1}(U'), \mathcal{F})$. The map

$$f_!\colon \mathcal{F} \longmapsto \sum (-1)^i R^i f_*(\mathcal{F}) \in K(X') \tag{1.1.15}$$

extends to $f_!\colon K(X) \longrightarrow K(X')$, as can be seen from the long exact sequence in cohomology.

Now let $f\colon Z \longrightarrow Y$ be a *proper* morphism between nonsingular irreducible quasiprojective varieties. There is a pushforward f_* in cohomology (or on the Chow rings). Grothendieck's theorem states that for $z \in K(Z)$,

$$\mathrm{ch}\big(f_!(z)\big)\,\mathrm{Todd}(Y) = f_*\big(\mathrm{ch}(z)\,\mathrm{Todd}(Z)\big). \tag{1.1.16}$$

[2]Hirzebruch had previously defined an A-class which differs from the $\hat{A}$-class by a power of 2, whence the notation $\hat{A}$.

[3]Or, more generally, if a real manifold X admits an element $c \in H^2(X)$ whose reduction mod 2 is $w_2(X)$, and $\mathrm{Todd}(X)$ is defined by (1.1.14) (where c replaces $c_1(X)$), then is $\mathrm{Todd}(X)[X]$ an integer?

This reduces to Hirzebruch's theorem (1.1.7) upon taking Y to be a point and z the K-theory class of a holomorphic vector bundle.

The Todd class enters the proof via the special case where $f\colon Z \longrightarrow Y$ is the inclusion of a divisor and z is the class of the structure sheaf $\mathcal{O}_Z$. Now $R^i f_*(\mathcal{O}_Z) = 0$ for $i \geq 1$ and $R^0 f_*(\mathcal{O}_Z)$ is $\mathcal{O}_Z$ extended trivially to Y. Let L be the line bundle defined by the divisor Z. Then the exact sequence

$$0 \to L^{-1} \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0 \tag{1.1.17}$$

shows that

$$\mathcal{O}_Z = \mathcal{O}_Y - L^{-1} \tag{1.1.18}$$

in $K(Y)$. Notice here that $f^*(L)$ is the normal bundle to Z in Y. Set $x = c_1(L)$. Then from (1.1.18)

$$\mathrm{ch}\big(f_!(\mathcal{O}_Z)\big) = 1 - e^{-x}.$$

On the other hand

$$f_*\big(\mathrm{ch}(\mathcal{O}_Z)\big) = f_*(1) = x.$$

Thus $f_* \circ \mathrm{ch} = \mathrm{ch} \circ f_!$ up to the Todd class of L.

It is instructive at this stage to consider the inclusion of the zero section $f\colon Z \longrightarrow E$ in a rank k vector bundle $\pi\colon E \longrightarrow Z$. Then the sheaf $R^0 f_*\mathcal{O}_Z = \mathcal{O}_Z$ fits into the exact sequence

$$0 \to \pi^*{\textstyle\bigwedge}^k E^* \to \pi^*{\textstyle\bigwedge}^{k-1} E^* \to \cdots \to \pi^* E^* \to \mathcal{O}_E \to \mathcal{O}_Z \to 0 \tag{1.1.19}$$

of sheaves over E. (Compare (1.1.17)). Here E^* is the (sheaf of sections of the) dual bundle to $\pi\colon E \longrightarrow Z$, and the arrows in (1.1.19) at $e \in E$ are contraction by e. Thus in $K(E)$ we have

$$f_!(\mathcal{O}_Z) = {\textstyle\bigwedge}^*(E^*). \tag{1.1.20}$$

Note that $\pi\colon E \longrightarrow Z$ is the normal bundle to Z in E.

When Atiyah and Hirzebruch learned about Grothendieck's work, they immediately set out to investigate possible ramifications in topology. The first step was to define K-theory for arbitrary CW complexes X [AH1]. The definition is as for algebraic varieties, but with 'topological vector bundles' replacing 'coherent algebraic sheaves.' The basic building blocks of topology are the spheres, and the calculation of $K(S^n)$ quickly reduces to that of the stable homotopy groups of the unitary group. By a fortunate coincidence Bott had just computed (in 1957) these homotopy groups [B1], [B2]. His *periodicity theorem* became the cornerstone of the new topological K-theory. What results is a cohomology theory which satisfies all of the Eilenberg-MacLane axioms save one, the dimension axiom. Thus was born "extraordinary cohomology." Its efficacy as a tool for solving topology problems was immediate [A1], [Ad1], [Ad2] and lasting.

Returning to the Grothendieck program, Atiyah and Hirzebruch formulated a version of Riemann-Roch for smooth manifolds [AH2], [H3]. Let $f\colon Z \longrightarrow Y$ be a smooth map between differentiable manifolds, and suppose f is 'oriented' in the sense that there exists an element $c_1 \in H^2(Z)$ with

(1.1.21)
$$c_1 \equiv w_2(Z) - f^*w_2(Y) \pmod 2.$$

Recall that Grothendieck's theorem (1.1.16) is stated in terms of a map $f_!\colon K(Z) \to K(Y)$. In the topological category we cannot push forward vector bundles, as we could sheaves in the algebraic category, so we need a new construction.[4] Here we restrict our attention to immersions of complex manifolds to simplify the presentation.[5] Then (1.1.18) and (1.1.20) indicate the appropriate definition. Let $\pi\colon E \to Z$ be the normal bundle of Z in Y. By the tubular neighborhood theorem we can identify E with a neighborhood N of Z in Y. The *Thom complex* $\bigwedge^* E^*$ is defined on the total space of E by contraction (compare (1.1.19)):

(1.1.22)
$$0 \to \pi^*{\textstyle\bigwedge}^k E^* \xrightarrow{\iota(e)} \pi^*{\textstyle\bigwedge}^{k-1} E^* \to \cdots \to \pi^* E^* \xrightarrow{\iota(e)} E \times \mathbb{C} \to 0.$$

Notice that (1.1.22) is exact for $e \neq 0$, so the resulting K-theory element is supported on Z. By the tubular neighborhood theorem it is also defined on N, and extension by zero yields the desired element $f_!(1) \in K(Y)$. If $V \to Z$ is a vector bundle, then $f_!(V)$ is defined by tensoring (1.1.22) with π^*V.

EXERCISE 1.1.23. Compare the algebraic $f_!$ to the topological $f_!$. Can the topological definition be imitated in the algebraic setting? If not, where does the construction break down? What is the relationship between (1.1.2) and the dual complex to (1.1.22) (defined by exterior multiplication)?

The differentiable Riemann-Roch theorem states

(1.1.24)
$$\mathrm{ch}\bigl(f_!(z)\bigr)\,\mathrm{Todd}(Y) = f_*\bigl(\mathrm{ch}(z)\,\mathrm{Todd}(Z)\bigr), \qquad z \in K(Z).$$

[4] The definition of $f_!$ was not given in the original paper [AH2], but at least for the case of immersions all of the essential points appear in [AH2,§4].

[5] By embedding Z in a sphere, we can factor an arbitrary map $f\colon Z \to Y$ into an immersion followed by a projection: $Z \to S^N \times Y \to Y$. Bott Periodicity is used to calculate the "shriek map" $K(S^N \times Y) \to K(Y)$. For immersions of real manifolds (with an orientation of the normal bundle) Clifford multiplication on spinors replaces (1.1.22).

Given the definition of $f_!$, the proof is a routine exercise in topology, comparing the Thom isomorphisms in K-theory and cohomology. (See [AS1], [Ad3,§4–§5] for example.) Specialize now to the case where Y is a point, and suppose $w_2(Z) = 0$. Then we can choose the orientation class $c_1 \in H^2(Z)$ to be zero, so taking $z = 0$ in (1.1.24) we deduce, in view of (1.1.14), that $\hat{A}(Z)[Z] = f_!(1) \in K(\mathrm{pt}) = \mathbb{Z}$ is an integer. This argument provided a new proof of the integrality theorem discussed earlier, now with a topological interpretation of the integer $\hat{A}(Z)[Z]$.

One more point deserves mention here. We have used the Thom complex (1.1.22) and Todd class (1.1.5) to stress the analogy with Grothendieck's picture. But the spin representation of the orthogonal group and the $\hat{A}$-class will prove more fundamental to our considerations. In fact, the original version of the differentiable Riemann-Roch theorem [AH2] is stated in terms of the $\hat{A}$-class. A complete treatment of the spin group and Clifford algebras, and the relationship to periodicity and K-theory was given later by Atiyah, Bott, and Shapiro [ABS].

§**1.2 The Atiyah-Singer index**

The de Rham theorem is a precursor of the index theorem. Let X be an n-manifold, and consider the complex of differential forms

(1.2.1)
$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X),$$

where d is the exterior derivative of Elie Cartan. The de Rham cohomology vector spaces are defined by

(1.2.2)
$$H^p_{\mathrm{DR}}(X) = \frac{\ker[d\colon \Omega^p(X) \to \Omega^{p+1}(X)]}{\mathrm{im}[d\colon \Omega^{p-1}(X) \to \Omega^p(X)]}.$$

The theorem de Rham proved in his 1931 thesis [deR] states that there is a natural isomorphism $H^p_{\mathrm{DR}}(X) \cong H^p(X;\mathbb{R})$ of the de Rham cohomology with the usual real cohomology defined via singular cochains. (This is modern language; de Rham proved that there is a closed form with specified periods, unique modulo exact forms.) Notice that $H^p_{\mathrm{DR}}(X)$ relates to a differential operator and $H^p(X;\mathbb{R})$ comes from topology. Hodge,[6] motivated by questions in algebraic geometry, proved that on a closed *Riemannian* manifold there is a unique "best" form in each cohomology class. Namely, Hodge defined a duality operation $*\colon \Omega^p(X) \to \Omega^{n-p}(X)$, which depends on the Riemannian metric, and for compact manifolds asserted [Hod] that in each de Rham cohomology class there is a unique form ω satisfying

(1.2.3)
$$d\omega = 0, \qquad d(*\omega) = 0.$$

[6] For a review of Hodge's life and work, see [A5].

Such forms are termed *harmonic*.

Of course, analysts were developing techniques to handle more general differential equations than the system (1.2.3). By the late 50's there was much effort devoted to the study of elliptic boundary value problems in the plane. (For a historical review of these developments, see [Se2].) These workers derived formulæ for the *index* of such elliptic systems. (The first example of such a formula dates back to work of F. Noether in 1921.) The index is defined to be the dimension of the space of solutions to the equations minus the dimension of the space of solutions to the adjoint equations. Also at this time the nature of the *symbol* of a singular integral operator was clarified in work of Calderón-Zygmund, Horvath, Kohn, Mikhlin, and Seeley. These developments led I. M. Gel'fand [G] to conjecture (in 1960) that many important properties of the solutions are invariant under certain homotopies of the equations. In particular, he conjectured that the index is a topological invariant.

We set up the index problem on a closed manifold X. Let E^+, E^- be vector bundles over X, and suppose $P\colon C^\infty(E^+)\to C^\infty(E^-)$ is a differential operator of order k. In local coordinates we have

$$Pu = a^{i_1 i_2 \dots i_k}\frac{\partial^k u}{\partial x^{i_1}\partial x^{i_2}\dots\partial x^{i_k}} + \text{lower order terms,} \tag{1.2.4}$$

where $a^{i_1 i_2\dots i_k}$ is a bundle map $E^+\to E^-$ depending symmetrically on the i_j, and we sum over the indices i_j. This top order piece transforms as a symmetric tensor under coordinate changes, so defines the *symbol*

$$\sigma(P)\colon S^k(T^*X)\otimes E^+ \longrightarrow E^-. \tag{1.2.5}$$

View $\sigma(P)$ as a homogeneous polynomial of degree k in $T^*(X)$ with values in $\operatorname{Hom}(E^+,E^-)$. The differential operator P is *elliptic* if its symbol is invertible; that is, if for each nonzero $\theta\in T^*X$ the bundle map $\sigma(P)(\theta,\dots,\theta)$ is invertible. It follows from elliptic theory that P has finite dimensional kernel and cokernel. The index of P is

$$\operatorname{ind}P = \dim\ker P - \dim\operatorname{coker}P. \tag{1.2.6}$$

The primary geometric example is the Dirac operator on an even dimensional compact spin manifold; other geometric operators (the $\bar\partial$ operator on a Kähler manifold, the signature operator on an oriented Riemannian manifold) are simple modifications of this basic operator. Dirac is a first order operator acting on sections of the spinor bundles $S^\pm$, and its symbol

$$c\colon T^*X\otimes S^+ \longrightarrow S^- \tag{1.2.7}$$

is Clifford multiplication on spinors. The Dirac equation first entered physics to provide a relativistic equation for electrons [D]. In mathematics the Dirac operator has immediate topological significance—its symbol (1.2.7) defines an element of $K(T^*X)$ (which is roughly $S^+ - S^-$ over the zero section and trivial elsewhere). This is exactly the K-theory element Atiyah and Hirzebruch used in their Riemann-Roch theorem.

While the Dirac operator is natural from the viewpoint of K-theory, in view of Gel'fand's assertion that homotopies of elliptic operators should provide a key to the index problem, it was natural to consider more general operators. Their symbols (1.2.5) also determine elements of $K(T^*X)$ [A1]. In fact, to fully realize the "topology of elliptic operators" the newly developed *pseudodifferential operators* of Kohn, Nirenberg [KN] and Hörmander [Ho] were used.[7] The index (1.2.6) is then a homomorphism

$$\operatorname{ind}\colon K(T^*X) \longrightarrow \mathbb{Z}, \tag{1.2.8}$$

i.e., it depends only on the symbol.

On the topological side another homomorphism $K(T^*X) \to \mathbb{Z}$, the *topological index*, is easily defined. We already saw that for manifolds with an integral lift of w_2 (cf. (1.1.21)) there is a direct image map $f_!\colon K(X) \to \mathbb{Z}$. This is analogous to the integration map $H^*(X) \to \mathbb{Z}$ determined by an orientation class $[X] \in H_*(X)$ in ordinary homology. Now without an orientation on X there is still an integration $H^*(T^*X) \to \mathbb{Z}$, since the cotangent bundle carries a natural orientation.[8] Similarly, the cotangent bundle has a natural orientation in K-theory, and the topological index map

$$\text{t-ind}\colon K(T^*X) \longrightarrow \mathbb{Z} \tag{1.2.9}$$

is the resulting shriek map. The Riemann-Roch formula (1.1.24) translates (1.2.9) into cohomological terms.

We can now state the Atiyah-Singer index theorem.

Theorem 1.2.10 [AS2]**.** *The analytic index* (1.2.8) *equals the topological index* (1.2.9)*.*

This theorem realizes Gel'fand's goal as it expresses the index in terms of homotopy theoretic data. For the Dirac operator D the index formula reads

$$\operatorname{ind} D = \hat{A}(X)[X]. \tag{1.2.11}$$

[7] The algebra of pseudodifferential operators was developed from the *singular integral operators* of Calderón and Zygmund [CZ] and Mikhlin [M] (cf. [Se2]). We do not describe these in any detail here; indeed, one of the main points of these lectures is to illustrate that for Dirac operators the analysis is much simpler and more classical.

[8] We take cohomology with compact supports.

Finally, we have an analytic realization of the integer $\hat{A}(X)[X]$—it is the index of the Dirac operator. The index theorem (1.2.10) also proves the Riemann-Roch formula (1.1.7) for nonalgebraic complex manifolds. Many more applications of the index theorem can be found in [Pa] and [AS1].

The original proof (announced in [AS2] and carried out in detail in [Pa]) was based on Hirzebruch's bordism argument. There is a set of axioms which uniquely characterizes the index homomorphism, and both (1.2.8) and (1.2.9) were shown to verify the axioms. The trickiest part of the proof comes in the verification that the analytic index is a bordism invariant. As in Hirzebruch the actual index formula is essentially derived by checking enough examples.

A proof modeled on Grothendieck's work appeared in [AS3]. Thus the construction of the topological index—which is carried out by embedding the manifold X in a sphere, multiplying the symbol by the Thom class of the normal bundle, extending the result to the whole sphere, and applying Bott periodicity—is mimicked with pseudodifferential operators. The Thom class is represented by a standard operator (roughly half the de Rham complex), and the multiplicative and excisive properties of pseudodifferential operators give an operator on the sphere. In this way one is reduced to checking the index formula on spheres (it is enough to check S^2).

The advantage of this second proof is that it generalizes easily. In particular there is an index theorem for *fibrations* $\pi\colon Z \to Y$. The map π defines a family of manifolds, parametrized by Y, and a family of elliptic operators $P = \{P_y\}$ determines an element in $K(T(Z/Y))$, the K-theory of the tangent bundle along the fibers. The topological index

$$\text{t-ind}\colon K(T(Z/Y)) \longrightarrow K(Y) \tag{1.2.12}$$

is defined as before. Now the analytic index is

$$\operatorname{ind} P = \ker P_y - \operatorname{coker} P_y, \tag{1.2.13}$$

a formal difference of parametrized families of vector spaces. If $\dim\ker P_y$ is constant in y, then each term in (1.2.13) determines a vector bundle, and so $\operatorname{ind} P$ makes sense as an element of $K(Y)$. In general one makes sense of (1.2.13) as an element of K-theory, but this requires more argument [A2,Appendix].

Theorem 1.2.14 [AS4]**.** *The analytic index* (1.2.13) *equals the topological index* (1.2.12)*.*

The Riemann-Roch formula (1.1.24) then gives an explicit formula for the Chern character of the index in the rational cohomology of Y. For families of Dirac operators it asserts (compare (1.2.11))

$$\operatorname{ch}\operatorname{ind} P = \pi_*(\hat{A}(Z/Y)). \tag{1.2.15}$$

Exercise 1.2.16. Go through Hirzebruch's problem list [H2] to see which problems can be solved using the index theorem. What is the current status of his other problems?

§1.3 The heat equation method

As an offshoot of their work on the Lefschetz fixed point theorem, Atiyah and Bott [AB1], [A3] realized a new formula for the index of an elliptic operator $P\colon C^\infty(E^+) \to C^\infty(E^-)$. By elliptic theory the operators $\Delta^+ = P^*P$ and $\Delta^- = PP^*$ have discrete spectrum, with the eigenvalues tending to infinity at a controlled rate. Furthermore, the nonzero spectra of Δ^+ and Δ^- coincide. It follows that the heat operators $e^{-t\Delta^\pm}$ are defined for $t > 0$, are smoothing operators, and have finite traces

$$\operatorname{Tr} e^{-t\Delta^\pm} = \sum_{\lambda\in\operatorname{spec}(\Delta^\pm)} e^{-t\lambda}. \tag{1.3.1}$$

Since the nonzero spectra coincide we have the formula of Atiyah and Bott:[9]

$$\operatorname{ind} P = \operatorname{Tr} e^{-t\Delta^+} - \operatorname{Tr} e^{-t\Delta^-} \tag{1.3.2}$$

for any $t > 0$. As an aside, we note that as $t \to \infty$ the operators $e^{-t\Delta^\pm}$ tend to projection onto the kernel, whence the right hand side of (1.3.2) approaches the left hand side. On the other hand, differentiation in t shows that the right hand side of (1.3.2) is independent of t. Thus we obtain another proof of the Atiyah-Bott formula.

The heat operators $e^{-t\Delta^\pm}$ have Schwartz kernels $e^{-t\Delta^\pm}(x,y)$ which are maps $E^\pm_y \to E^\pm_x$, and the trace can be reexpressed as

$$\operatorname{Tr} e^{-t\Delta^\pm} = \int_X \operatorname{tr} e^{-t\Delta^\pm}(x,x)\,dx. \tag{1.3.3}$$

Here $e^{-t\Delta^\pm}(x,x)$ is an endomorphism of $E^\pm_x$ at each $x \in X$, and tr denotes its (finite dimensional) trace. For *differential* operators P, Seeley [Se1] (generalizing earlier work of Minakshisundarum and Pleijel [MP]) proved[10] that as $t \to 0$ there is an asymptotic expansion

$$\operatorname{tr} e^{-t\Delta^\pm}(x,x) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{\nu=0}^\infty a^\pm_\nu(x) t^\nu \quad \text{as } t \to \infty, \tag{1.3.4}$$

where the functions $a^\pm_\nu$ depend only on the coefficients of P at x together with their derivatives at x. (See also [Gre], [P1].) Combining (1.3.2)–(1.3.4) and taking note of the fact that the right hand side of (1.3.2) is independent of t, we let $t \to 0$ to deduce

$$\operatorname{ind} P = \int_X \bigl(a^+_{n/2}(x) - a^-_{n/2}(x)\bigr)\,dx. \tag{1.3.5}$$

[9] They first state an equivalent formula in terms of ζ-functions.

[10] Seeley works with the ζ-functions of $\Delta^\pm$, which are holomorphic functions of $s \in \mathbb{C}$ for $\operatorname{Re} s >> 0$. Seeley proved the existence of a meromorphic continuation of these ζ-functions to $s \in \mathbb{C}$, which is equivalent to (1.3.4).

This local formula for the index has immediate consequences. For example, it proves that the index is multiplicative in finite covers. But to recover the full index theorem it remains to determine $\bigl(a^+_{n/2}(x) - a^-_{n/2}(x)\bigr)\,dx$ explicitly.

McKean and Singer [MS] carried this out for the classical Gauss-Bonnet theorem on a 2-manifold. Then Patodi generalized their work to the de Rham complex in arbitrary dimensions [P1] and to the Dolbeault complex on Kähler manifolds [P2]. In these cases Patodi executed a virtuoso calculation to exhibit the cancelation (1.3.7) and prove the formulæ analogous to (1.3.8). (The $\hat{A}$ polynomial in the curvature is replaced by the Chern form for the Gauss-Bonnet-Chern theorem and the Todd form for the Dolbeault complex.) Notice that in Patodi's work the index polynomial comes by direct, albeit hard, computation. Gilkey [Gi1] studied local invariants of metrics and connections, and proved that those of sufficiently small order vanish, while the first nonvanishing ones are polynomials in Pontrjagin and Chern classes. Atiyah, Bott, and Patodi [ABP] then added a new ingredient—Weyl's theory of invariants of the orthogonal group—to simplify Gilkey's argument. The final results apply to any operator of Dirac type. We state the basic theorem for the Dirac operator on a spin manifold X.

Theorem 1.3.6. *Let D be the Dirac operator on X. Then in the notation of* (1.3.4) *we have*

$$a^+_\nu(x) - a^-_\nu(x) \equiv 0 \quad \text{for } \nu < n/2; \tag{1.3.7}$$

$$\bigl(a^+_{n/2}(x) - a^-_{n/2}(x)\bigr)\,dx = \sqrt{\det\left(\frac{\Omega/4\pi i}{\sinh \Omega/4\pi i}\right)}, \tag{1.3.8}$$

where Ω is the Riemannian curvature form on X.

The differential form in (1.3.8) is the Chern-Weil representative for $\hat{A}(X)$, so (1.3.8) and (1.3.5) combine to prove the index formula (1.2.11) for the Dirac operator. It is important to note that (1.3.7) implies that the difference of the *local* traces

$$\operatorname{tr} e^{-t\Delta^+}(x,x) - \operatorname{tr} e^{-t\Delta^-}(x,x) \tag{1.3.9}$$

converges as $t \to 0$. This local convergence is special to Dirac operators, and is crucial in what follows.

The Gilkey-Atiyah-Bott-Patodi proof of Theorem 1.3.6 is by invariance theory. They note that $\bigl(a^+_\nu(x) - a^-_\nu(x)\bigr)\,dx$ depends functorially on the metric, so is a combination of Pontrjagin forms. The vanishing in (1.3.7) is evident from the conformal weights under scaling. The actual formula in (1.3.8) comes by computing enough examples, as in Hirzebruch's work.

§1.4 New Techniques

Recently (since 1982) there have been many new developments related to the index theorem. They are the basis of this book. Although we pursue a purely mathematical approach, it is important to bear in mind that many of the ideas come directly from physics. We believe that the physical intuition to which these ideas owe their birth—i) supersymmetry and ii) the equivalence of the Hamiltonian and path integral representations of quantum theory—will continue to be an effective tool on a wide variety of mathematical fronts.

We now survey some new proofs of the index theorem for Dirac operators. These proofs start with the heat equation formula (1.3.2) for the index. As in Patodi's work, Theorem 1.3.6 is proved by direct analysis, and the $\hat{A}$-genus appears by explicit computation. But the new computations are simpler and more conceptual. Furthermore, the appearance of the $\hat{A}$-genus in index theory is directly related to its occurrence in other contexts.

In the physics literature it is the concept of supersymmetry [W1], [W2], applied to a particular quantum mechanical system, which leads to the index theorem [Ag], [Ge2], [FW]. We describe this briefly in a mathematical exposition of Witten's ideas due to Atiyah [A4]. The key mathematical idea in Atiyah's treatment is the localization theorem in equivariant cohomology [AB3], or more specifically a formula of Duistermaat and Heckman [DH]. Let M be a compact symplectic manifold with symplectic form ω, and suppose that the circle group acts on M preserving ω. Assume that the circle action is generated by a Hamiltonian function H. Then the formula of Duistermaat-Heckman is

$$\int_M e^{-tH}e^{\omega} = \int_{\operatorname{Fix} M} \frac{e^{-tH}e^{\omega}}{\prod_j (tm_j - iy_j)}. \tag{1.4.1}$$

Here e^{ω} is essentially the symplectic volume form; $\operatorname{Fix} M$ is the submanifold of M fixed by the circle action; the normal bundle to $\operatorname{Fix} M$ is $\oplus_j z^{m_j} L_j$, where L_j is a complex line bundle on which the circle acts by $z \mapsto z^{m_j}$, and $c_1(L_j) = y_j$; and t is an arbitrary complex number. This formula is called "exactness of stationary phase," since when $t = -i$ it evaluates the oscillatory integral on the left exactly by its stationary phase approximation on the right. The right hand side of (1.4.1) also has a natural representation in terms of differential forms [BV1], [Bi1].

The basic idea is to apply (1.4.1) to the free loop space M of a spin manifold X. There is a natural degenerate closed 2-form on M. In finite dimensions the degeneracy cycle represents the first Stiefel-Whitney class, and so is nonessential if the manifold is oriented. One can define the notion of orientability for loop space similarly, using the degeneracy cycle of its symplectic form. Then Witten shows that the loop space M is orientable if the manifold X is spin. Now the left hand side of (1.4.1) can be interpreted as the path integral representation of (1.3.2) via the Feynman-Kac formula. Here one must be careful to distinguish the symplectic volume from the Riemannian volume, the latter being what formally enters into the Feynman-Kac formula. The spin representation, and so the Dirac operator, appears in reconciling the two volumes. The right hand side is computed as follows. Fix M is the manifold of constant loops sitting inside the loop space, and the normal bundle decomposes as $\oplus_{n\ge 1} z^n(TX\otimes\mathbb{C})$. Thus if $\pm y_j$ are the Chern roots of $TX\otimes\mathbb{C}$, and $\dim X = 2m$, then the right hand side of (1.4.1) is

$$\begin{aligned}\int_X \frac{1}{\prod_j \prod_{n=1}^{\infty}(tn-iy_j)(tn+iy_j)} &= \int_X \Bigl(\prod_n n^2\Bigr)^{-m} \prod_j\prod_n \frac{1}{1+(y_j/n)^2} \\ &= \Bigl(\prod_n n^2\Bigr)^{-m} \int_X (2\pi)^m \prod_j \frac{2\pi y_j/2}{\sinh(2\pi y_y/2)}.\end{aligned} \tag{1.4.2}$$

CHECK WHY $t=1$ IN THIS FORMULA ON THE RHS. The infinite constant in front is thrown out, and the resulting integral is the $\hat{A}$-genus (1.1.13).

These arguments are interpreted by Bismut [Bi2], [Bi1] in terms of Wiener measure on loop space. In this way he deals with integrals over loop space rigorously, thus avoiding infinite constants. The heat kernel is represented in terms of Wiener measure with the aid of Lichnerowicz's formula, which expresses the Dirac Laplacian in terms of the covariant Laplacian. The localization to point loops as $t\to 0$ is natural in this picture. The variable t represents the total time during which a Brownian path exists, and as the time tends to zero, only constant loops have a significant probability of occurring. The evaluation of the integral over these point loops is accomplished using a formula of Paul Lévy [Lé]. He considers a Brownian curve in the plane which is conditioned to close after time 2π. Then the characteristic function (expectation value of e^{iz}) of the area enclosed by the random curve is $\pi z/\sinh \pi z$. This same calculation appears in Bismut's work, only there the curvature of X replaces z, and once again the $\hat{A}$-genus is obtained.

The $\hat{A}$-genus arises quite differently in a proof of the index theorem due to Berline and Vergne [BV2]. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then a standard formula asserts that the differential of the exponential map $\exp\colon \mathfrak{g}\to G$ at $a\in\mathfrak{g}$ is

$$d\exp_a = \frac{1-e^{-\operatorname{ad} a}}{\operatorname{ad} a} = \operatorname{Todd}^{-1}(\operatorname{ad} a). \tag{1.4.3}$$

It was always a mystery whether the occurrence of the Todd genus in (1.4.3) is related to the index theorem. Berline and Vergne noticed that if X is a Riemannian manifold, and $O(X)$ the principal bundle of orthonormal frames, then the differential of the Riemannian exponential map on $O(X)$ is given by a similar formula. Precisely, there is a natural isomorphism $T_pO(X)\cong\mathbb{E}^n\oplus\mathfrak{o}(n)$ via the Levi-Civita connection. The exponential map at p is $\exp\colon \mathbb{E}^n\oplus\mathfrak{o}(n)\to O(X)$, and its differential at $a\in\mathfrak{o}(n)$ is

$$\begin{aligned} d\exp_a\big|_{\mathbb{E}^n} &= \exp(-a)\operatorname{Todd}^{-1}\bigl((\tfrac12\Omega_p, a)\bigr). \\ d\exp_a\big|_{\mathfrak{o}(n)} &= \operatorname{Todd}^{-1}(\operatorname{ad} a);\end{aligned} \tag{1.4.4}$$

In this formula the Riemann curvature Ω, which takes values in $\mathfrak{o}(n)$, is contracted with a using the Killing form. The result is a 2-form, which can be identified as an element of $\mathfrak{o}(n)$. To prove the index theorem Berline and Vergne rewrite (1.3.2), (1.3.3) on the frame bundle. To compensate for the introduction of extra degrees of freedom in the fiber direction, they must study the behavior of the heat kernel along the fiber. It is at this stage, in the small time limit, where (1.4.4) appears. Ultimately, that is how the $\hat{A}$-class enters their proof.

Our approach to the index theorem is based largely on a paper of Ezra Getzler [Ge1]. This paper reformulates Getzler's earlier treatment of the rescaling which used pseudodifferential symobl calculus [Ge2]. The main idea is to compute the coefficient (1.3.8) in the asymptotic expansion by localizing at a point. Thus, fix $x_0 \in X$ and choose normal coordinates x^k in which x_0 is the origin. The natural scaling around x_0,

$$\begin{aligned} x &\longrightarrow \epsilon x \\ t &\longrightarrow \epsilon^2 t, \end{aligned} \tag{1.4.5}$$

induces a family of Dirac Laplacians whose limit as $\epsilon \to 0$ is the standard constant coefficient Laplacian on $T_{x_0}X$. One crucial step is the introduction of a scaling on spinors, which already appears in [FW]. Then the limiting Laplacian can be written

$$-\sum_k \left(\frac{\partial}{\partial x^k} - \frac{1}{4} R_{k\ell} x^\ell \right)^2, \tag{1.4.6}$$

where $R_{k\ell}$ is the curvature of X at x_0. This operator has the general shape of the *harmonic oscillator*

$$P = -\frac{d^2}{dx^2} + a^2 x^2 \tag{1.4.7}$$

on $\mathbb{R}$, whose heat kernel is given by *Mehler's formula*

$$e^{-tP}(0,x) = \frac{1}{\sqrt{4\pi t}} \left(\frac{2at}{\sinh 2at} \right)^{1/2} \exp\left[\frac{-x^2}{4t} \left(\frac{2at}{\tanh 2at} \right) \right]. \tag{1.4.8}$$

We are only interested in the heat kernel along the diagonal (cf. (1.3.4)), which is

$$e^{-tP}(0,0) = \frac{1}{4\pi t} \left(\frac{2at}{\sinh 2at} \right)^{1/2} \tag{1.4.9}$$

for the harmonic oscillator. By general principles the heat kernel varies smoothly with the operator, so that (1.4.9) (with the curvature term in (1.4.6) replacing the parameter a) approximates the heat kernel of the Dirac Laplacian as $t \to 0$. The detailed calculation yields the $\hat{A}$-genus (1.3.8) directly from (1.4.9).

In Getzler's approach the index theorem for Dirac operators is quite elementary. Analytically, we need only the asymptotic expansion for the heat kernel together with its smooth dependence on parameters. Then some basic facts about Clifford algebras and the explicit formula (1.4.9) finish the proof. In its broad outline this is very similar to Patodi's method. But Patodi did not use the symmetry of the Clifford algebra to see the cancellation (1.3.7) and prove (1.3.8), so that his computations are more complicated than Getzler's.

So far we have only discussed new insights into old results. But these techniques also give new results for parametrized families of Dirac operators. Whereas the topological version of the index theorem starts with homotopy information (cf. (1.2.12)), the geometric version requires precise rigid geometric data. Thus let $\pi\colon Z \to Y$ be a smooth fibration of manifolds with a spin structure along the fibers. The extra geometric information we require is a metric along the fibers, i.e., a metric on $T(Z/Y)$, and a smoothly varying family of "horizontal subspaces" transverse to $\ker \pi_*$. Then the geometric family of Dirac operators is defined.[11] We then desire a differential geometric version of the analytic index (1.2.13).

The essential idea is due to Quillen [Q1], inspired by ideas from physics. Just as a connection on a smooth vector bundle is the differential geometric refinement of an equivalence class of topological vector bundles, so too is Quillen's *superconnection* the differential geometric refinement of an element of K-theory. Let $V = V^+ \oplus V^-$ be a smooth $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over a smooth manifold X. A superconnection on V consists of an ordinary connection $\nabla = \nabla^+ \oplus \nabla^-$ and a linear endomorphism $L\colon V \to V$ which anticommutes with the grading, i.e., $L(V^\pm) \subseteq V^\mp$. The superconnection is the operator $\nabla + L$ on the space $\Omega^*(X,V)$ of differential forms on X with values in V. The K-theory element corresponding to $\nabla + L$ is $V^+ \xrightarrow{L} V^-$ in $K(X)$.[12] Quillen defines the curvature $(\nabla + L)^2$ and the Chern character form $\operatorname{tr_s} \exp(\nabla + L)^2$, and proves that this differential form represents the topological Chern character of the associated K-theory element. The *supertrace* is the map

$$\begin{aligned} \operatorname{tr_s}\colon \Omega^*(X, \operatorname{End} V) &\longrightarrow \Omega^*(X) \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\longmapsto \operatorname{tr}\alpha - \operatorname{tr}\delta \end{aligned} \tag{1.4.10}$$

A small, but important generalization of Quillen's construction obtains by permitting L to be any odd element of the $\mathbb{Z}/2\mathbb{Z}$-graded algebra $\Omega^*(X, \operatorname{End} V)$.

Bismut [Bi3] applied Quillen's ideas to a geometric family of Dirac operators $\pi\colon Z \to Y$ with even dimensional fibers. Bismut constructs an *infinite dimensional* superconnection over Y. The

[11] The horizontal subspaces are not required to simply define Dirac; they enter in the Bismut superconnection below.

[12] The K-theory element is obtained using the "difference construction." It actually lives in $K(X, X_0)$, where $X_0 \subseteq X$ is the subspace on which L is invertible.

fiber of the vector bundle at $y \in Y$ is the space of spinor fields on Z_y, the fiber of π at y. It carries an L^2 inner product, coming from the metric on Z_y, and a unitary connection ∇ from the horizontal subspaces. The operator L is the Dirac operator D modified by a degree 2 term which is essentially the curvature T of the field of horizontal planes. Precisely, Bismut's superconnection with parameter t is

$$\nabla + \sqrt{t}D - \frac{c(T)}{4\sqrt{t}}. \tag{1.4.11}$$

The c denotes Clifford multiplication (1.2.7). We consider the superconnection (1.4.11) to represent the differential geometric shriek map $\pi_!$. Because D is Fredholm there is an element of $K(Y)$ associated to (1.4.11). Bismut proves that this element is precisely $\operatorname{ind} D$ as defined in (1.2.13). This assertion is a direct generalization of (1.3.2). Then Bismut considers the behavior as $t \to 0$, deriving a differential geometric Riemann-Roch formula.

Theorem 1.4.12 [Bi3]**.** *The Chern character of the Bismut superconnection approaches*

$$\int_{Z/Y} \hat{A}(\Omega^{(Z/Y)}) \tag{1.4.13}$$

as $t \to 0$.

Here $\Omega^{(Z/Y)}$ is the curvature of the connection on $T(Z/Y)$ determined by the geometric data. The differential form in (1.4.13), which of course represents in de Rham theory the cohomology class in (1.2.15), plays an important role in many applications. Bismut's approach to Theorem 1.4.12 was through the probabilistic representation of the heat kernel. Recently, Donnelly [Do] gave a proof following Getzler's ideas.

§1.5 Summary of Contents

Here follows a rough outline of Part I. (Part II will contain more about superconnections and will treat η-invariants and determinant line bundles, but it is not yet written!)

In Chapter 2 we introduce the basic material on Clifford algebras, spinors, and the Dirac operator. We more or less follow [ABS,Part I] in §2.1. The reader who does the exercises here will be rewarded in the later chapters. Our treatment of spinors on manifolds in §2.2 emphasizes the role of the frame bundle, as there the spin group is most clearly exhibited. We prove the important Weitzenböck formula. The generalization to vector bundle coefficients in §2.3 is straightforward. Here (in exercise form) is the spinor form of the classical geometric operators: the de Rham complex, the Dolbeault complex on a Kähler manifold, the signature operator, and the self-dual complex. It is important to realize that the results in Part I hold for all these operators of Dirac type.

Chapter 3 develops from scratch the basic results of elliptic theory for the Dirac operator. The treatment is more elementary than usual since we make good use of the Weitzenböck formula. The Sobolev spaces are introduced and their basic properties proved. Then we prove that the Dirac operator has discrete spectrum with smooth eigenfunctions, and that Dirac is Fredholm.

Our main analytic tool is the heat operator (Chapter 4). After writing the explicit solution on flat space in §4.1, we prove the basic existence of heat flow on compact manifolds. The basic estimate (4.2.19) is useful in estimating approximate solutions. Distributions are used in §4.3 to discuss the heat kernel. We also introduce the wave operator and prove that waves propagate with finite speed. Using this we can control the heat flow and prove, for example, that the amount of heat that flows a finite distance decreases exponentially in time for small time. In §4.4 we discuss more precise behavior of the heat kernel for small time. This is encoded in the asymptotic expansion (Theorem 4.4.1). Here one must come to grips with the fact that a small ball in a Riemannian manifold is not isometric to a ball in Euclidean space, so the heat kernel in Euclidean space is not an accurate approximation to the heat kernel in a manifold X. The usual technique (following Minakshisundaram and Pleijel) consists in altering the flat heat kernel to compensate for the curved geometry. We take the opposite tack, blowing up the geometry of the ball to approximate Euclidean space (cf. (1.4.5)). Then the asymptotic expansion (on the diagonal) amounts to the smooth dependence on parameters of the heat kernel. We need this in the limit where the ball in X expands to infinity, i.e., where X deforms to its tangent space at one point. As the tangent space is noncompact, some extra work is required to make the argument. This deformation argument is a differential geometric analog of the "deformation to the normal cone" construction in algebraic geometry. It is also a simpler version of the scaling used in Getzler's proof of the index theorem (§5.4).

We prove the Atiyah-Singer index theorem (for Dirac operators) in Chapter 5. In §5.1 we review the decomposition of the spin representation in even dimensions. The existence of different types of spinor fields gives the possibility of a nontrivial index problem—the index measures the difference in dimension of the two spaces of harmonic spinor fields. Intuitively, this is also the difference in dimension of the two spaces of *all* spinor fields. Of course, the space of spinor fields is infinite dimensional, so its dimension must be "renormalized." This is accomplished with the heat kernel, and the Atiyah-Bott formula is obtained (§5.2). In §5.3 we treat the harmonic oscillator in flat space and derive Mehler's formula for its heat kernel. The $\hat{A}$-class makes its appearance at this stage. The proof of the index theorem is now reduced to Getzler's scaling argument, which we present in §5.4.

The geometric treatment of a family of Dirac operators is based on Quillen's concept of a superconnection, which we introduce in Chapter 6. We begin with a review of connections on vector bundles and Chern-Weil theory, mostly in exercise form. This is presented as a warm-up to superconnections. It also allows us to give a brief introduction to characteristic classes, and so indicate the topological significance of the $\hat{A}$-class in the index formula. (Connections are treated in §2.2 from a different point of view.) Our exposition of superconnections in §6.2 and §6.3 is simply an expanded version of Quillen's original paper [Q1]. We give a novel viewpoint of the relationship between K-theory and superconnections in §6.4.

In Chapter 7 we return to Dirac operators, now in parametrized families. Our first task is to derive the basic equations of Riemannian geometry over a parameter space. This is carried out in §7.1. The Bismut superconnection (§7.2) is constructed out of this geometry.

An appendix contains some basic facts about exponential coordinates which we use in Chapter 5.

§2 The Dirac Operator

Consider the Laplace operator on $\mathbb{R}^4$:

$$\triangle = -\frac{\partial^2}{(\partial x^1)^2} - \frac{\partial^2}{(\partial x^2)^2} - \frac{\partial^2}{(\partial x^3)^2} - \frac{\partial^2}{(\partial x^4)^2}. \tag{2.1}$$

Our sign conventions render $\triangle$ a nonnegative operator. Motivated by the quantum mechanics of the electron, Dirac asked[13] whether there is a first order operator $\mathcal{D}$ such that $\mathcal{D}^2 = \triangle$. Suppose

$$\mathcal{D} = \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} + \gamma^4 \frac{\partial}{\partial x^4}. \tag{2.2}$$

Then the equation $\mathcal{D}^2 = \triangle$ is equivalent to

$$\begin{gathered} (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = (\gamma^4)^2 = -1 \\ \gamma^1\gamma^2 + \gamma^2\gamma^1 = \gamma^1\gamma^3 + \gamma^3\gamma^1 = \cdots = 0. \end{gathered} \tag{2.3}$$

It is easy to see that (2.3) has no scalar solutions. Dirac's great insight is that (2.3) has *matrix* solutions.

Exercise 2.4. Without reading further, construct 4×4 matrices γ^i satisfying (2.3). Try the lower dimensional cases first. What size matrices do you need? Do you need real or complex matrices? What about quaternionic solutions?

Atiyah and Singer rediscovered the *Dirac operator* (2.2) from the relationship of the $\hat{A}$-genus and K-theory to the representation theory of the spin group. We develop the algebra of equations (2.3) in §2.1, culminating in the spin representation. There is an abundance of material here, much of it in exercise form, and the reader is well-advised to absorb it in small doses. Further details can be found in [ABS], [Ch], [Gr], while a more sophisticated treatment of the spin representation is given in [PS,§12]. After reviewing some basic Riemannian geometry, we construct the Dirac operator in §2.2. There is a global topological obstruction to the existence of spinors on an oriented manifold X, the second Stiefel-Whitney class $w_2(X)$. When this obstruction vanishes the Dirac operator exists. Other geometric operators—the de Rham complex, signature operator, Dolbeault complex on a Kähler manifold—can be expressed in terms of the Dirac operator on spinors. Hence we term them *operators of Dirac type.* They sometimes exist even if $w_2(X) \neq 0$. (For example, the de Rham complex is defined on any manifold, and the signature operator on any oriented manifold.) All of our subsequent analysis applies to all operators of Dirac type.

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[13] Dirac worked in Minkowski space, which changes one of the minus signs in (2.1) to a plus sign.

§2.1 Clifford algebras and spinors

Let $\mathbb{E}^n$ denote Euclidean n-space with its usual inner product $(\cdot,\cdot)$. The linear isometries of $\mathbb{E}^n$ comprise the group $O(n)$ of orthogonal transformations. It is an old theorem of Cartan (later refined by Dieudonné) that every element of $O(n)$ can be written as a product of at most n reflections [J,p.352]. We encode this theorem into an algebra as follows. For any unit vector $v \in \mathbb{E}^n$ denote by $\overline{v}$ reflection in the hyperplane perpendicular to v. Notice that $\overline{v} = \overline{-v}$, so that the map from the unit sphere to reflections is 2:1. (Of course, the set of reflections coincides with the set of hyperplanes, the real projective $(n-1)$-space.)

We mean the following to be heuristic motivation for the introduction of the Clifford algebra. Namely, in order to multiply reflections we consider the algebra generated (over $\mathbb{R}$) by the $\overline{v}$. Identify $\overline{v} + \overline{w}$ with $\overline{v+w}$ and $a\overline{v}$ with $\overline{av}$ for any $a \in \mathbb{R}$. There are some relations in this algebra. First, since a reflection has order 2, for unit vectors v

$$\overline{v}^2 = \pm 1. \tag{2.1.1}$$

The ambiguity in the choice of sign arises from $\overline{v} = \overline{-v}$, but by continuity considerations we choose the same sign for all v. Since reflections in perpendicular planes commute, for orthogonal unit vectors v and w

$$\overline{v}\,\overline{w} = \pm \overline{w}\,\overline{v}. \tag{2.1.2}$$

Here, however, we *must* choose the minus sign. For $(v+w)/\sqrt{2}$ is a unit vector, and

$$\overline{\left(\frac{v+w}{\sqrt{2}}\right)}^2 = \frac{\overline{v}^2 + \overline{v}\,\overline{w} + \overline{w}\,\overline{v} + \overline{w}^2}{2},$$

so using (2.1.1) we obtain (2.1.2) with a minus sign. The algebra generated (with either sign in (2.1.1)) is called the *Clifford algebra*.

For reasons which will become clearer later we choose the minus sign in (2.1.1). Also, from now on we omit the bars from the notation, but use '$\cdot$' to denote multiplication in the Clifford algebra. Let v and w be arbitrary elements of $\mathbb{E}^n$. Then since the vector $v - \frac{(v,w)}{(w,w)}w$ is perpendicular to w, by (2.1.2)

$$\begin{gathered}\left(v - \frac{(v,w)}{(w,w)}w\right)\cdot w + w\cdot\left(v - \frac{(v,w)}{(w,w)}w\right) = 0,\\ v\cdot w + w\cdot v = 2(v,w)\frac{w\cdot w}{(w,w)},\end{gathered}$$

and from (2.1.1) we obtain

$$v \cdot w + w \cdot v = -2(v, w). \tag{2.1.3}$$

The single relation (2.1.3) describes the Clifford algebra completely. In terms of an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{E}^n$, we have

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}. \tag{2.1.4}$$

Here δ_{ij} has its usual meaning $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. We denote the Clifford algebra by $\mathrm{Cliff}(\mathbb{E}^n)$; it is the algebra generated by $e_1, \dots, e_n$ subject to the relation (2.1.4).

EXERCISE 2.1.5. Determine $\mathrm{Cliff}(\mathbb{E}^1)$, $\mathrm{Cliff}(\mathbb{E}^2)$, and $\mathrm{Cliff}(\mathbb{E}^3)$. What are their complexifications?

More formally, $\mathrm{Cliff}(\mathbb{E}^n)$ is the full tensor algebra of $\mathbb{R}^n$ modulo the ideal generated by the relation (2.1.3). Since (2.1.3) is homogeneous of degree 2, the $\mathbb{Z}$ grading of the tensor algebra passes to a $\mathbb{Z}/2\mathbb{Z}$ grading of the Clifford algebra. The elements of even degree form a subalgebra $\mathrm{Cliff}(\mathbb{E}^n)^+$. It is easy to see that the set $\{e_{i_1} \cdot e_{i_2} \cdots e_{i_k} : 1 \le k \le n,\ i_1 < i_2 < \cdots < i_k\}$ forms a basis for $\mathrm{Cliff}(\mathbb{E}^n)$, and the subset of elements with k even forms a basis for $\mathrm{Cliff}(\mathbb{E}^n)^+$. Hence $\dim \mathrm{Cliff}(\mathbb{E}^n) = 2^n$ and $\dim \mathrm{Cliff}(\mathbb{E}^n)^+ = 2^{n-1}$. Finally, we observe that $\mathbb{E}^n$ is embedded in the Clifford algebra in the obvious way.

EXERCISE 2.1.6. Define the Clifford algebra associated to vector space endowed with a bilinear form. (Hint: (2.1.3).) What is the Clifford algebra if this form is zero?

EXERCISE 2.1.7. Construct an isomorphism $\mathrm{Cliff}(\mathbb{E}^{n-1}) \cong \mathrm{Cliff}(\mathbb{E}^n)^+$.

EXERCISE 2.1.8. Show that there is a canonical isomorphism of *vector spaces* $\mathrm{Cliff}(\mathbb{E}^n) \cong \bigwedge(\mathbb{R}^n)$, where $\bigwedge(\mathbb{R}^n)$ is the full exterior algebra. This isomorphism does *not* preserve the algebra structure. In particular, $\mathrm{Cliff}(\mathbb{E}^n)$ has a canonical $\mathbb{Z}$ grading as a vector space. What is the grading in terms of a basis?

EXERCISE 2.1.9. Show that under the isomorphism $\mathrm{Cliff}(\mathbb{E}^n) \cong \bigwedge(\mathbb{R}^n)$ of Exercise 2.1.8, left Clifford multiplication $e\cdot$ by $e \in \mathbb{E}^n$ is

$$e\cdot = \epsilon(e) - \iota(e), \tag{2.1.10}$$

where $\epsilon(\cdot)$ is exterior multiplication and $\iota(\cdot)$ is interior multiplication (relative to the inner product in $\mathbb{E}^n$). What is right Clifford multiplication by e?

EXERCISE 2.1.11. The tensor algebra $\bigotimes \mathbb{R}^n$ has a filtration $F^0 \subset F^1 \subset \dots$ with $F^p = \bigoplus_{q \le p}(\bigotimes^q \mathbb{R}^n)$. Show that there is an induced filtration in the Clifford algebra. What is the associated graded algebra?

Recall that unit vectors in $\mathbb{E}^n$, viewed in the Clifford algebra, represent reflections. Thus to recover the action of reflections on $\mathbb{E}^n$ (with $\mathbb{E}^n$ still embedded in $\mathrm{Cliff}(\mathbb{E}^n)$) we expect to use conjugation. However, the ambiguity in sign necessitates a small computation:

$$\begin{aligned} e_i \cdot e_i \cdot e_i^{-1} &= e_i, \\ e_i \cdot e_j \cdot e_i^{-1} = -e_j \cdot e_i \cdot e_i^{-1} &= -e_j, \quad j \neq i. \end{aligned}$$

Thus we see that the action of a unit vector r by reflection on $v \in \mathbb{E}^n$ is represented in the Clifford algebra by $-r \cdot v \cdot r^{-1}$.

Exercise 2.1.12. Do explicit computations in $\mathrm{Cliff}(\mathbb{E}^2)$. In particular, write rotation through angle θ as the product of two reflections and compute the corresponding element in the Clifford algebra. Characterize the elements in $\mathrm{Cliff}(\mathbb{E}^2)$ which represent rotations. Prove that they form a group, and determine the homomorphism to the rotation group.

An arbitrary element $g \in O(n)$ is represented by a product of reflections, so inside $\mathrm{Cliff}(\mathbb{E}^n)$ we write

$$g = r_1 \cdots r_k \tag{2.1.13}$$

for some unit vectors r_i. (This representation is not unique.) Let us characterize such elements in the Clifford algebra. First we define the transpose antiautomorphism of $\mathrm{Cliff}(\mathbb{E}^n)$ by

$${}^t(v_1 \cdots v_k) = v_k \cdots v_1 \tag{2.1.14}$$

for $v_i \in \mathbb{E}^n$. To account for the grading we set

$$\beta(v_1 \cdots v_k) = (-1)^k v_k \cdots v_1 \tag{2.1.15}$$

Then it is easy to see that (2.1.13) satisfies

$$\begin{aligned} g \cdot \beta(g) &= 1, \\ g \cdot \mathbb{E}^n \cdot {}^t g &\subseteqq \mathbb{E}^n. \end{aligned} \tag{2.1.16}$$

Notice that the first equation implies that g is invertible; indeed its inverse is $\beta(g)$. Also, the action of g in the second equation generalizes the action of reflections on $\mathbb{E}^n$. Finally, the set of $g \in \mathrm{Cliff}(\mathbb{E}^n)$ which satisfy (2.1.16) forms a group G. Now there is a homomorphism $G \to O(n)$

(the action of $g \in G$ on $v \in \mathbb{E}^n$ is $g \cdot v \cdot {}^t g$), and by Cartan's theorem it is surjective. We proceed to determine the kernel.

Suppose $g \in G$ maps to the identity element in $O(n)$. Write $g = g_+ + g_-$ as a sum of even and odd elements. Then for any $v \in \mathbb{E}^n$ we have

$$g_+ \cdot v = v \cdot g_+ \qquad g_- \cdot v = -v \cdot g_-. \tag{2.1.17}$$

Choose $v = e_i$ a basis vector, and write $g_\pm = a_\pm + b_\pm \cdot e_i$ where $a_\pm, b_\pm$ do not involve e_i. Plugging into (2.1.17) we find

$$\begin{aligned} a_+ \cdot e_i - b_+ &= e_i \cdot a_+ + e_i \cdot b_+ \cdot e_i \qquad & a_- \cdot e_i - b_- &= -e_i \cdot a_- - e_i \cdot b_- \cdot e_i \\ &= a_+ \cdot e_i + b_+ & &= a_- \cdot e_i + b_- \end{aligned}$$

from which $b_+ = b_- = 0$. Repeating for all i we conclude that g_+, g_- are constants. Thus g_- is zero, since it is an odd element. From the first equation in (2.1.16) we have $g^2 = g_+^2 = 1$, and so $g = \pm 1$. Therefore, there is an exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G \longrightarrow O(n) \longrightarrow 1. \tag{2.1.18}$$

In fact, for $n \geq 2$ the group G is a nontrivial double cover of $O(n)$. (Consider the loop $\theta \mapsto \cos\theta + \sin\theta\, e_1 \cdot e_2$.) It is customary to refer to G as $\mathrm{Pin}(n)$. In our attempt to embed $O(n)$ inside an algebra, we succeeded in constructing a nontrivial double cover.

Endow $\mathbb{E}^n$ with its usual orientation, so that $\{e_1, \dots, e_n\}$ is an oriented basis. Then the identity component $SO(n)$ of $O(n)$ preserves the orientation. The identity component of the double cover is called $\mathrm{Spin}(n)$, and it sits inside $\mathrm{Cliff}(\mathbb{E}^n)$ as the elements g which satisfy (2.1.16) and

$$g \in \mathrm{Cliff}(\mathbb{E}^n)^+. \tag{2.1.19}$$

Finally, corresponding to (2.1.18) is the extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(n) \longrightarrow SO(n) \longrightarrow 1. \tag{2.1.20}$$

Since $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$, we see that $\mathrm{Spin}(n)$ is simply connected for $n \geq 3$, while $\mathrm{Spin}(2)$ is diffeomorphic to the circle.

Exercise 2.1.21. Construct the embedding of the Lie algebra $\mathfrak{o}(n)$ in $\mathrm{Cliff}(\mathbb{E}^n)$ induced by the inclusion $\mathrm{Spin}(n) \hookrightarrow \mathrm{Cliff}(\mathbb{E}^n)$. How does it act on $\mathbb{E}^n \subset \mathrm{Cliff}(\mathbb{E}^n)$? (Hint: Differentiate the $\mathrm{Spin}(n)$ action.)

Exercise 2.1.22. What is the group Spin(1)? What is its Lie algebra? What about Spin(2)?

Exercise 2.1.60. Repeat the preceding analysis for the Clifford algebra defined by using a plus sign on the right hand side of (2.1.4). Is the *Pin* group obtained different? What about the *Spin* group? At least check this for $n = 1$ and $n = 2$.

Exercise 2.1.23. Construct an isomorphism $\mathrm{Spin}(3) \cong SU(2)$. (Hint: The adjoint action of $SU(2)$ is a homomorphism $SU(2) \to SO(3)$ with kernel $\{\pm 1\}$.)

Exercise 2.1.24. Prove that $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ as follows. On $\bigwedge^2 \mathbb{E}^4$ the *Hodge $*$ operator* is characterized by

$$\alpha \wedge *\beta = (\alpha, \beta)\,\mathrm{vol}, \qquad \alpha, \beta \in \textstyle\bigwedge^2 \mathbb{E}^4,$$

where $(\cdot,\cdot)$ is the inner product on $\bigwedge^2 \mathbb{E}^4$ and $\mathrm{vol} \in \bigwedge^4 \mathbb{E}^4$ is the volume form. Then $*^2 = 1$, so $\bigwedge^2 \mathbb{E}^4 = \bigwedge^2_+ \oplus \bigwedge^2_-$ splits into the ± 1-eigenspaces of $*$. Clearly $\dim \bigwedge^2_\pm = 3$. The action of $SO(4)$ on $\bigwedge^2 \mathbb{E}^4$ preserves the splitting, so produces a homomorphism $SO(4) \to SO(3) \times SO(3)$. Show that this is surjective with kernel $\{\pm 1\}$. Now lift to the spin groups.

Let $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n) = \mathrm{Cliff}(\mathbb{E}^n) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the Clifford algebra. Denote the algebra of $k \times k$ matrices over $\mathbb{C}$ by $M_k(\mathbb{C})$.

Theorem 2.1.25. *There is a noncanonical isomorphism of algebras*

$$\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n) \cong \begin{cases} M_{2^{n/2}}(\mathbb{C}), & n \text{ even}, \\ M_{2^{(n-1)/2}}(\mathbb{C}) \oplus M_{2^{(n-1)/2}}(\mathbb{C}), & n \text{ odd}. \end{cases} \tag{2.1.26}$$

Two proofs are indicated in the following exercises.

Exercise 2.1.27. Prove Theorem 2.1.25 by induction. (Hint: Calculate the Clifford algebra explicitly for $n = 1, 2$. Then construct an isomorphism $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n) \cong \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^{n-2}) \otimes \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^2)$ by mapping the standard basis of $\mathbb{E}^n$ into the right hand side and checking the defining relation (2.1.4). Your formulas should involve $\sqrt{-1}$.)

Exercise 2.1.28. Define $\epsilon \in \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ by

$$\epsilon = i^{n(n+1)/2} e_1 \cdot e_2 \cdot \cdots \cdot e_n. \tag{2.1.29}$$

Show that $\epsilon^2 = 1$. If n is odd then ϵ is in the center of $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$, whereas if n is even ϵ commutes with $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^+$ and anticommutes with the elements $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^-$ of odd degree. Show that if n is even the endomorphism $x \mapsto \epsilon \cdot x \cdot \epsilon$ of $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ is $+1$ on $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^+$ and -1 on $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^-$.

Exercise 2.1.30 [Ch]. For n even prove that $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ is a simple algebra. (Hint: Suppose $\mathfrak{i}$ is a nontrivial two-sided ideal. Show how to multiply a nonzero element of $\mathfrak{i}$ on the left and the right by elements in the Clifford algebra to obtain an arbitrary element of $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$. Hence $\mathfrak{i} = \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$.) Conclude from the Wedderburn theorem that $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ is a matrix algebra. For n odd $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ is semisimple.

EXERCISE 2.1.31. Write down the isomorphism (2.1.26) explicitly for $n = 4$. What are the 4×4 matrices which represent the basis elements? (Hint: Try $n = 2$ first.)

EXERCISE 2.1.32. Determine the structure of the real Clifford algebras. (Hint: You may also want to consider indefinite inner products. Compute the low dimensional cases explicitly. Then try to find an induction argument as in Exercise 2.1.27. You will find a periodicity property, as in (2.1.26), but the period is not 2 as in the complex case. If you get stuck, consult [ABS], [Gr].)

Now Exercise 2.1.7 extends to the complexified Clifford algebras. Thus we have isomorphisms

$$\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n) \cong \begin{cases} \mathrm{End}(\mathbb{S}), & n \text{ even}, \\ \mathrm{End}(\mathbb{S}^+) \oplus \mathrm{End}(\mathbb{S}^-), & n \text{ odd}; \end{cases} \tag{2.1.33}$$

$$\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^+ \cong \begin{cases} \mathrm{End}(\mathbb{S}^+) \oplus \mathrm{End}(\mathbb{S}^-), & n \text{ even}, \\ \mathrm{End}(\mathbb{S}), & n \text{ odd}. \end{cases} \tag{2.1.34}$$

Here we have rewritten the matrix algebras in (2.1.26) as endomorphisms of a vector space. Although the isomorphisms (2.1.26) are not canonical,[14] we make a fixed choice of spin spaces in (2.1.33) and (2.1.34) once and for all. Note that in even dimensions the total *spin space* $\mathbb{S}$ splits into the sum of a "positive" piece $\mathbb{S}^+$ and a "negative" piece $\mathbb{S}^-$. Since the spin group $\mathrm{Spin}(n)$ sits inside the invertible elements of the even Clifford algebra, we obtain the *spin representation*

$$\gamma\colon\ \mathrm{Spin}(n) \longrightarrow \mathrm{Aut}(\mathbb{S}). \tag{2.1.35}$$

For n odd the spin representation is irreducible, while for n even it splits into the sum of the two *half-spin representations.* We denote the action of $x \in \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ on $\mathbb{S}$ by $c(x)$; thus γ is the restriction of c to $\mathrm{Spin}(n) \subset \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$.

EXERCISE 2.1.36. Show that if n is even then $c(\epsilon)$ (cf. (2.1.29)) distinguishes the two half-spin representations.

The following two exercises indicate other constructions of the spin representation.

EXERCISE 2.1.37. For $n = 2\ell$ construct the spin representation as follows. Choose an identification $\mathbb{R}^{2\ell} \cong \mathbb{C}^\ell$, for example so that $f_1 = e_1 + \sqrt{-1}\, e_2,\ f_2 = e_3 + \sqrt{-1}\, e_4,\ \dots$ forms a complex basis.

[14] But the projectivization $\mathbb{P}(\mathbb{S})$ is canonical.

Set $\mathbb{S} = \bigwedge \mathbb{C}^\ell$, and let ϵ denote exterior multiplication and ι interior multiplication (defined by the standard hermitian metric on $\mathbb{C}^\ell$). Define a map

$$\begin{aligned} \mathbb{R}^{2\ell} &\longrightarrow \operatorname{End}(\mathbb{S}) \\ v &\longmapsto \epsilon(v) - \iota(v) \end{aligned} \tag{2.1.38}$$

Show that this map extends to $\operatorname{Cliff}(\mathbb{E}^n)$, and gives another proof of the isomorphism (2.1.26) (for n even). Now restrict to $\operatorname{Spin}(n)$. How does the representation split into two irreducible pieces?

Exercise 2.1.39 [AB2,p.482]. Consider the complexified left regular representation of $\operatorname{Cliff}(\mathbb{E}^n)$, that is the action of $\operatorname{Cliff}(\mathbb{E}^n)$ on $\operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ by left Clifford multiplication. This representation is highly reducible. In fact, the transformations of $\operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ determined by right Clifford multiplication by $e_1 \cdot e_2$, $e_3 \cdot e_4$, $\dots$ have square -1 and mutually commute. So we can decompose $\operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ into the simultaneous eigenspaces of these transformations. Prove that the resulting representations of $\operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ are isomorphic. How can we decompose them into the half-spin representations if n is odd?

Exercise 2.1.40. In any dimension the isomorphism (2.1.33) determines a map

$$\operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n) \longrightarrow \operatorname{End}(\mathbb{S}) \cong \mathbb{S}^* \otimes \mathbb{S}. \tag{2.1.41}$$

This is an isomorphism for n even and for n odd an isomorphism onto the elements preserving the splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Show that for any $x \in \operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$, there are correspondences between the transformations

$$\begin{aligned} x\cdot &\longleftrightarrow c(x)^* \otimes 1 \\ \cdot x &\longleftrightarrow 1 \otimes c(x) \end{aligned} \tag{2.1.42}$$

on $\operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ and $\mathbb{S}^* \otimes \mathbb{S}$. Here $x\cdot$ (resp. $\cdot x$) denote left (resp. right) Clifford multiplication by x. (Hint: Use the fact that (2.1.41) is an algebra homomorphism and the composition law for matrix algebras.)

Since $\operatorname{Spin}(n)$ is a compact group, we can average an arbitrary inner product on $\mathbb{S}$ so that the spin representation (2.1.35) is unitary.

Exercise 2.1.43. Show that the natural inner products which arise in the constructions of Exercise 2.1.37 and Exercise 2.1.39 are preserved by the action of $\operatorname{Spin}(n)$.

Exercise 2.1.44. Prove that the spin representation is self-conjugate: $\mathbb{S} \cong \mathbb{S}^*$. One way to do this is with characters using Exercise 2.1.48. A more direct approach is the following. Extend β in (2.1.15) to a complex linear antiautomorphism of $\operatorname{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$. Define

$$\begin{aligned} \theta\colon \operatorname{End}(\mathbb{S}) &\longrightarrow \operatorname{End}(\mathbb{S}^*) \\ c(x) &\longmapsto c\bigl(\beta(x)\bigr)^* \end{aligned} \tag{2.1.45}$$

for $x \in \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$. Note that for n even this defines θ on all of $\mathrm{End}(\mathbb{S})$, whereas for n odd it defines θ only on the subalgebra of endomorphisms preserving the splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Show that θ is an isomorphism where it is defined. In general, if $\theta\colon \operatorname{End} V \to \operatorname{End} W$ is an isomorphism of endomorphisms of vector spaces, prove that θ determines an isomorphism $\hat{\theta}\colon V \to W$ (defined uniquely up to a scalar) which intertwines θ. Extend this statement to cover the case where n is odd. Now restrict (2.1.45) to $\mathrm{Spin}(n) \subset \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ to obtain the isomorphism of the spin representation with its contragradient. (Hint: (2.1.16).) How does the grading element ϵ (2.1.29) behave in (2.1.45)? Use this to show that the half-spin representations are self-conjugate in dimensions $n \equiv 0 \pmod 4$ and that $(\mathbb{S}^{\pm})^* \cong \mathbb{S}^{\mp}$ in dimensions $n \equiv 2 \pmod 4$.

The spin representation (2.1.35) extends to a unitary representation of the slightly larger group $\mathrm{Pin}(n)$. In particular, the transformation $\gamma_i = \gamma(e_i)$ is unitary. But since $\gamma_i^2 = -1$ we conclude that γ_i is also skew-Hermitian. The transformation γ_i represents Clifford multiplication by the element e_i. More generally, the restriction of (2.1.33) to $\mathbb{E}^n$ gives a map

$$c\colon \mathbb{E}^n \otimes \mathbb{S} \longrightarrow \mathbb{S} \tag{2.1.46}$$

which we call Clifford multiplication. When n is even it is easy to check that Clifford multiplication interchanges $\mathbb{S}^+$ and $\mathbb{S}^-$. Of great importance is the fact the (2.1.46) commutes with the $\mathrm{Spin}(n)$ action, i.e., is a map of $\mathrm{Spin}(n)$-modules. This is most easily checked using Exercise 2.1.39.

Exercise 2.1.47. Prove that the spin representation is irreducible if n is odd, and that the half-spin representations are irreducible if n is even. One way to do this is to restrict to the "extra special 2-group" generated by the elements $e_1 \cdot e_2, e_3 \cdot e_4, \dots$. Work out the representation theory of this finite group, and so give an alternative construction of the spin representation (cf. [R,§2]).

Exercise 2.1.48. Consider the matrix $\Omega \in \mathfrak{so}(n), n = 2\ell$, defined by

$$\Omega = \begin{pmatrix} 0 & -y_1 & & & \\ y_1 & 0 & & & \\ & & 0 & -y_2 & \\ & & y_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{2.1.49}$$

According to Exercise 2.1.21 this corresponds to an element in $\mathrm{Cliff}(\mathbb{E}^n)$. Show that this element is

$$x = \frac{1}{2}(y_1 e_1 \cdot e_2 + y_2 e_3 \cdot e_4 + \cdots + y_\ell e_{2\ell-1} \cdot e_{2\ell}). \tag{2.1.50}$$

Then $g = e^x$ is an element of $\mathrm{Spin}(n)$. Compute it explicitly. For any representation ρ of $\mathrm{Spin}(n)$, define the *character* $\chi_\rho(g) = \operatorname{Tr}\rho(g)$. Prove

$$\chi_{\mathbb{S}}(g) = \prod_{j=1}^{\ell} 2\cos y_j/2, \tag{2.1.51}$$

$$\chi_{\mathbb{S}^+ - \mathbb{S}^-}(g) = \prod_{j=1}^{\ell}(-2i\sin y_j/2). \tag{2.1.52}$$

(Hint: Try the case $n = 2$ first. For $\mathbb{S}^+ - \mathbb{S}^-$ you will need to use Exercise 2.1.36. If you get stuck, read §5.1.)

Exercise 2.1.53. The antiautomorphisms (2.1.14) and (2.1.15) extend to the complex Clifford algebra. (Be careful to conjugate the complex coefficient.) Now define groups $\mathrm{Pin}^c(n)$ and $\mathrm{Spin}^c(n)$ using (2.1.16) and (2.1.19) Define a map of $\mathrm{Spin}^c(n)$ to $SO(n)$ and determine the kernel. Determine a homomorphism $U(\ell) \to \mathrm{Spin}^c(2\ell)$ which lifts the natural homomorphism $U(\ell) \to SO(2\ell)$. Define the spin representation for $\mathrm{Spin}^c(n)$. What is $\mathrm{Spin}^c(3)$? $\mathrm{Spin}^c(4)$? (Hint: [ABS,§3].)

Exercise 2.1.54. Although Exercise 2.1.37 constructs the spin representation from the complex exterior algebra, it is a little misleading, as we explain in this exercise. Construct a nontrivial double cover $\tilde U(\ell)$ of the unitary group $U(\ell)$, and show that the natural homomorphism $i\colon U(\ell) \to SO(2\ell)$ lifts to a homomorphism $\tilde i\colon \tilde U(\ell) \to \mathrm{Spin}(2\ell)$. (Hint: Consider pairs $(A, u) \in U(\ell) \times U(1)$ such that $\det A = u^2$.) Construct a homomorphism $\det^{1/2}\colon \tilde U(\ell) \to U(1)$ whose square is the usual determinant $\det\colon U(\ell) \to U(1)$. Now show that the pullback of the spin representation γ to $\tilde U(\ell)$ is

$$i^*\gamma = \bigwedge \otimes \det{}^{-1/2}, \tag{2.1.55}$$

where $\bigwedge$ is the exterior algebra representation of $U(\ell)$. (Hint: Try $\ell = 1$ first. You may want to compute with characters using Exercise 2.1.48.) Thus we identify the underlying representation spaces

$$\mathbb{S} \cong \bigwedge\mathbb{C}^\ell \otimes (\det \mathbb{C}^\ell)^{-1/2}. \tag{2.1.56}$$

Also, we identify $\mathbb{E}^{2\ell} \cong \mathbb{C}^\ell$ via $e_{2k-1} + \sqrt{-1}\, e_{2k} = f_k$. Show that under these correspondences Clifford multiplication is

$$\begin{aligned} c(e_{2k-1}) &\longleftrightarrow [\epsilon(f_k) - \iota(f_k)] \otimes 1 \\ c(e_{2k}) &\longleftrightarrow \sqrt{-1}\,[\epsilon(f_k) + \iota(f_k)] \otimes 1, \end{aligned} \tag{2.1.57}$$

where the interior product is taken relative to the Hermitian form on $\mathbb{C}^\ell$. (Compare (2.1.38).) How is all of this related to Exercise 2.1.53?

The next exercises are crucial to our proof of the index theorem in Chapter 5.

EXERCISE 2.1.58. For $\epsilon \neq 0$ let $\mathbb{E}^n_\epsilon$ denote the inner product space $\langle \mathbb{R}^n, \epsilon^2(\cdot,\cdot)\rangle$, where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^n$. Thus $\epsilon^{-1}e_1,\dots,\epsilon^{-1}e_n$ is an orthonormal basis of $\mathbb{E}^n_\epsilon$. Construct a canonical algebra isomorphism $\mathrm{Cliff}(\mathbb{E}^n) \cong \mathrm{Cliff}(\mathbb{E}^n_\epsilon)$.

EXERCISE 2.1.59. Continuing the previous exercise, show that "$\lim\limits_{\epsilon\to 0} \mathrm{Cliff}(\mathbb{E}^n_\epsilon) = \bigwedge(\mathbb{R}^n)$" in the following sense. We change our point of view and identify the $\mathrm{Cliff}(\mathbb{E}^n_\epsilon)$ by vector space isomorphisms which are *not* algebra isomorphisms. These are specified by requiring them to be the identity map on the underlying vector space $\mathbb{R}^n$. Then we consider this fixed vector space to have a family of algebra structures parametrized by ϵ, the basis elements satisfying the relation

$$e_ie_j + e_je_i = -2\delta_{ij}\epsilon^2$$

in the ϵ algebra structure. Fix a basis element $e_I = e_{i_1}\cdot e_{i_2}\cdots e_{i_k}$, and set $|I| = k$. Let $c_\epsilon(e_I)$ denote left Clifford multiplication by e_I in the ϵ algebra structure. Then show

$$\lim_{\epsilon\to 0} c_\epsilon(e_I) = \epsilon(e_I)$$

under the canonical identification of vector spaces $\mathrm{Cliff}(\mathbb{E}^n) \cong \bigwedge(\mathbb{R}^n)$, where $\epsilon(\cdot)$ denotes exterior multiplication.[15] More formally, consider the algebra (over $\mathbb{R}$) generated by $e_1,\dots,e_n,\boldsymbol{\epsilon}$ subject to the relation

$$e_ie_j + e_je_i = -2\delta_{ij}\boldsymbol{\epsilon}^2.$$

Show that specializing $\boldsymbol{\epsilon} = \epsilon$ for $\epsilon \in \mathbb{R}$ yields $\mathrm{Cliff}(\mathbb{E}^n_\epsilon)$ for $\epsilon \neq 0$ and $\bigwedge(\mathbb{R}^n)$ for $\epsilon = 0$.

§2.2 Spinors on manifolds

First consider flat space $\mathbb{R}^n$. A differential form on $\mathbb{R}^n$ is a smooth map $\mathbb{R}^n \to \bigwedge\mathbb{R}^{n*}$ Here we make the standard identification of $T^*_x\mathbb{R}^n$ with $\mathbb{R}^{n*}$ at all $x \in \mathbb{R}^n$. In standard coordinates $x^1,\dots,x^n$ the differentials dx^k form a basis for the covectors at each point. The exterior derivative on differential forms is the operator

$$d = \epsilon(dx^k)\frac{\partial}{\partial x^k}, \tag{2.2.1}$$

[15] A rather unfortunate clash of notation!

where summation over k is implicit in the notation, and $\epsilon(\cdot)$ denotes exterior multiplication. The *Dirac operator* on Euclidean space $\mathbb{E}^n$ is defined similarly. (A metric is needed to define the Dirac operator, which explains why we use $\mathbb{E}^n$ instead of $\mathbb{R}^n$.) A *spinor field* on $\mathbb{E}^n$ is a smooth map $\mathbb{E}^n \to \mathbb{S}$, where $\mathbb{S}$ is the spin space on the dual $\mathbb{E}^{n*}$. Again we make the natural identification of $T_x^*\mathbb{E}^n$ with $\mathbb{E}^{n*}$. The Dirac operator

$$\mathcal{D} = c(dx^k)\frac{\partial}{\partial x^k} \tag{2.2.2}$$

is defined by replacing exterior multiplication in (2.2.1) with Clifford multiplication (2.1.46); it operates on spinor fields. In even dimensions the spin space decomposes as $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, and since $c(\cdot)$ interchanges $\mathbb{S}^+$ and $\mathbb{S}^-$ we see that $\mathcal{D}$ maps positive spinor fields into negative spinor fields and vise versa. Finally, we compute

$$\begin{aligned}\mathcal{D}^2 &= c(dx^k)\frac{\partial}{\partial x^k}c(dx^\ell)\frac{\partial}{\partial x^\ell}\\ &= \sum_k c(dx^k)^2\frac{\partial^2}{(\partial x^k)^2} + \sum_{k<\ell} c(dx^k)c(dx^\ell)\left[\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\right]\\ &= -\sum_k \frac{\partial^2}{(\partial x^k)^2}.\end{aligned} \tag{2.2.3}$$

The second term in the second line vanishes since mixed partial derivatives commute while $c(dx^k)$ and $c(dx^\ell)$ anticommute for $k \neq \ell$. Hence $\mathcal{D}^2$ is the (nonnegative) Laplace operator acting on spinor fields.

To define spinors and the Dirac operator on curved spaces we need to review some basic Riemannian geometry. We introduce the frame bundle so that we can write tensor fields as functions and differential operators as vector fields. It is also the natural framework to describe spin structures. A *Riemannian manifold* is a smooth manifold X together with a smoothly varying inner product on the tangent spaces. We will assume that X is oriented. Then at each $x \in X$ we consider the set $SO(X)_x$ of oriented orthonormal bases of the tangent space T_xX. These are orientation-preserving isometries $p\colon \mathbb{E}^n \to T_xX$; the basis corresponding to p is $p(e_1),\dots,p(e_n)$. The group $SO(n)$ acts on $SO(X)_x$ *on the right* by changing basis, i.e., by right composition of p with $g\colon \mathbb{E}^n \to \mathbb{E}^n$, and so the $SO(X)_x$ glue together to form a principal $SO(n)$ bundle $\pi\colon SO(X) \to X$. The fundamental theorem of Riemannian geometry asserts that this frame bundle carries a unique torsionfree connection, the Levi-Civita connection. It is a field H of n-planes on $SO(X)$, complementary to the vertical (kernel of π_*), and invariant under the action of $SO(n)$. We call the n-plane H_p at $p \in SO(X)$ the *horizontal space* at p. Now p is a basis of $T_{\pi(p)}X$, and H_p maps isomorphically to $T_{\pi(p)}X$ via π. Let $\partial_1(p),\dots,\partial_n(p)$ be the basis of H_p corresponding to p. The ∂_k are smooth vector fields on $SO(X)$. More invariantly, $\partial = e^k\partial_k$ is an $\mathbb{E}^{n*}$-valued vector field on $SO(X)$. Here $\{e^k\}$ is the standard basis of $\mathbb{E}^{n*}$ dual to the basis $\{e_k\}$ of $\mathbb{E}^n$.

EXERCISE 2.2.4. What is the action of $SO(n)$ on the ∂_k? (Hint: Consider ∂.)

Let $\xi \in T_xX$ be a tangent vector. Then given a basis $p \in SO(X)_x$ we can specify ξ by its n coordinates. Thus a tangent vector is a map $\xi\colon SO(X)_x \to \mathbb{E}^n$. Of course, the coordinates change in a specified fashion under $SO(n)$—for $g \in SO(n)$ we have

$$\xi(p \cdot g) = g^{-1} \cdot \xi(p), \tag{2.2.5}$$

where g^{-1} acts on $\mathbb{E}^n$ by the standard action. A vector field on X is then an $SO(n)$-equivariant function $\xi\colon SO(X) \to \mathbb{E}^n$.[16] We can differentiate $\xi = \xi^i e_i$ along the $\mathbb{E}^{n*}$-valued vector field $\partial = e^k \partial_k$ to obtain

$$\partial\xi = (\partial_k \xi^i)\, e^k \otimes e_i\colon SO(X) \longrightarrow \mathbb{E}^{n*} \otimes \mathbb{E}^n. \tag{2.2.6}$$

This is the *covariant derivative* of ξ.

EXERCISE 2.2.7. How does $\partial\xi$ transform under $SO(n)$? Show that $\partial\xi$ is a section of $T^*X \otimes TX$. Represent other tensor fields as functions on the frame bundle, and extend the covariant derivative to act on them.

We can write the exterior derivative in terms of ∂. A differential form is an $SO(n)$-equivariant map $SO(X) \to \bigwedge\mathbb{E}^{n*}$, and (compare (2.2.1))

$$d = \epsilon(\partial) = \epsilon(e^k)\partial_k. \tag{2.2.8}$$

EXERCISE 2.2.9. Verify $d^2 = 0$. You will need to use the fact that the Levi-Civita connection is torsionfree. Write d^* in this language. What about $\bar\partial$ on a complex manifold?

Now $[\partial_k, \partial_\ell]$ is a vertical vector field on $SO(X)$; this is equivalent to the fact that the Levi-Civita connection is torsionfree. Since $SO(n)$ acts simply transitively on the fibers of $\pi\colon SO(X) \to X$, we can identify the vertical tangent spaces with the Lie algebra $\mathfrak{o}(n)$. Let $\{E_i^j\}_{i<j}$ be the standard basis of $\mathfrak{o}(n)$. The skew-symmetric map E_i^j is defined on the basis elements by

$$\begin{aligned} E_i^j\colon e_j &\longmapsto e_i \\ e_i &\longmapsto -e_j \\ e_k &\longmapsto 0, \quad k \neq i, j. \end{aligned} \tag{2.2.10}$$

[16] The reader should recognize that this is essentially the classical point of view—a tensor field is a collection of functions transforming in a specified manner under change of basis. The classical writers (before Cartan) used coordinates instead of orthonormal bases.

Then we write

$$[e^k\partial_k, e^\ell\partial_\ell] = [\partial_k, \partial_\ell]e^k \otimes e^\ell = -R^i_{jk\ell}\, e^k \otimes e^\ell \otimes E^j_i. \tag{2.2.11}$$

$R^i_{jk\ell} = R_{ijk\ell}$ is the *Riemann curvature tensor*. It transforms as a 2-form on X with values in skew-symmetric endomorphisms of the tangent bundle. Hence it is skew in i, j and k, ℓ. We denote this 2-form by

$$\Omega^{(X)} = R^i_{jk\ell}\, e^k \wedge e^\ell \otimes E^j_i. \tag{2.2.12}$$

What is not immediately apparent is the Bianchi identity

$$R_{ijk\ell} + R_{i\ell jk} + R_{ik\ell j} = 0. \tag{2.2.13}$$

The contracted tensor

$$R_{j\ell} = \sum_k R_{kjk\ell} \tag{2.2.14}$$

is the *Ricci curvature*; it is symmetric in j, ℓ. After one more contraction we obtain the function

$$R = \sum_\ell R_{\ell\ell}. \tag{2.2.15}$$

It is appropriately termed the *scalar curvature.*

Next we introduce spinors on X. These are tensors which transform under $\mathrm{Spin}(n)$. Since the frame bundle $SO(X)$ only transforms under $SO(n)$, we must lift to a bundle on which $\mathrm{Spin}(n)$ acts.

Definition 2.2.16. *A spin structure on X is a principal $\mathrm{Spin}(n)$ bundle $\mathrm{Spin}(X)$ which double covers $SO(X)$, and for which the diagram*

$$\begin{tikzcd} \mathrm{Spin}(X)\times \mathrm{Spin}(n) \arrow[r] \arrow[d, "2:1"] & \mathrm{Spin}(X) \arrow[d, "2:1"] \\ SO(X)\times SO(n) \arrow[r] & SO(X) \end{tikzcd}$$

commutes.

The last statement simply asserts that the action of $\mathrm{Spin}(n)$ on $\mathrm{Spin}(X)$ is compatible with the action of $SO(n)$ on $SO(X)$. Spin structures may not exist, and even if they do exist there is not in general a unique isomorphism class. The obstruction to existence is the second Stiefel-Whitney class $w_2(X) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$, and if this obstruction vanishes then $H^1(X;\mathbb{Z}/2\mathbb{Z})$ acts simply transitively on the set of isomorphism classes of spin structures.

Exercise 2.2.17. Prove these topological assertions (cf. [AB2,p.480]).

Exercise 2.2.37. Suppose that X is an oriented Riemannian manifold with boundary which is endowed with a spin structure. Construct an induced spin structure on ∂X. Does the Cartesian product of two spin manifolds carry a natural spin structure? What about the quotient by a free group action? What about an oriented submanifold of a spin manifold?

Exercise 2.2.18. Describe explicitly the spin structures on S^1. Prove that if X is a spin manifold with boundary, then ∂X inherits a natural spin structure. Which spin structure on the circle is the boundary of the spin structure on the disk? Describe the spin structures on a Riemann surface X of genus g. There are 2^{2g} of them. Describe geometrically the action of $H^1(X;\mathbb{Z}/2\mathbb{Z})$ on the set of spin structures in this case. (Hint: Consider the restriction of a spin structure to circles in X.)

Exercise 2.2.19. Find an oriented closed 4-manifold which does not admit a spin structure. Are there any such examples in dimension 2 or 3?

Exercise 2.2.38. Investigate the notion of a *pin structure* on a Riemannian manifold (with no orientation or orientability hypothesis).

Let X be a spin manifold. For us this means that X is oriented and carries a Riemannian metric. A *spinor field* ψ is then a function $\psi\colon \operatorname{Spin}(X) \to \mathbb{S}$ which satisfies

$$\psi(p\cdot g) = \gamma(g^{-1})\psi(p), \qquad g \in \operatorname{Spin}(n),\ p \in \operatorname{Spin}(X). \tag{2.2.20}$$

Recall that γ denotes the spin representation (2.1.35) (associated to the dual space $\mathbb{E}^{n*}$). Alternatively, the spin representation determines a vector bundle $S \to X$ associated to $\operatorname{Spin}(X)$; a spinor field is a section of S. (An element of S_x is a map $\psi\colon \operatorname{Spin}(X)_x \to \mathbb{S}$ satisfying (2.2.20).) This spin bundle decomposes in even dimensions as $S = S^+ \oplus S^-$. The *Dirac operator* is (cf. (2.2.2))

$$\mathcal{D} = c(\partial) = c(e^k)\partial_k = \gamma^k\partial_k. \tag{2.2.21}$$

As on $\mathbb{R}^n$ it maps spinor fields to spinor fields, interchanging sections of S^+ and sections of S^- in even dimensions. Formally speaking, since $\gamma^k = c(e^k)$ and ∂_k are skew-adjoint and commute, the Dirac operator $\mathcal{D}$ is (formally) self-adjoint. The following two exercises make this idea rigorous.

Exercise 2.2.22. Let E, F be unitary representations of $\operatorname{Spin}(n)$ and $\sigma \in \operatorname{Hom}(\mathbb{E}^{n*}, \operatorname{Hom}(E,F))$ an invariant element. Show that $\sigma(\partial)$ defines a first order differential operator. What is σ for the

Dirac operator? For the exterior derivative d? For the adjoint d^* of the exterior derivative? For the covariant derivative?

Exercise 2.2.23. The formal adjoint D^* of a differential operator D is characterized by the equation

$$\langle D\psi, \chi\rangle = \langle \psi, D^*\chi\rangle, \tag{2.2.24}$$

where the L^2 inner product is defined using the metrics on E, F and integration against the volume form on X (cf. (3.1.1)). Since D^* is also a differential operator, it suffices to verify (2.2.24) for (local) compactly supported sections. For σ as in Exercise 2.2.22, define $\sigma^* \in \operatorname{Hom}(\mathbb{E}^{n*}, \operatorname{Hom}(F,E))$ by $\sigma^*(e) = -\sigma(e)^*$. (Note the minus sign!) Prove that $\sigma^*(\partial) = \sigma(\partial)^*$. Check this for the exterior derivative d and its adjoint d^*. Verify that the Dirac operator (2.2.21) is formally self-adjoint.

Exercise 2.2.25. What is the Dirac operator on S^1? Compute it for both spin structures. What is the Dirac operator on a Riemann surface? How is it related to the Cauchy-Riemann operator $\bar\partial$? (Hint: Try this on $\mathbb{R}^2 \cong \mathbb{C}$ first.)

Exercise 2.2.26. Let ψ be a spinor field and f a function. Use the Leibnitz rule to show

$$\mathcal{D}(f\psi) = c(df)\psi + f\mathcal{D}\psi. \tag{2.2.27}$$

What is the corresponding formula for $\mathcal{D}\big(c(\alpha)\psi\big)$, where α is a 1-form? (Your answer will involve Clifford multiplication of higher degree forms on spinor fields.)

Finally, as in (2.2.3), we compute

$$\begin{aligned}\mathcal{D}^2 &= \gamma^k\partial_k\,\gamma^\ell\partial_\ell\\ &= \sum_k(\gamma^k)^2\,{\partial_k}^2 + \sum_{k<\ell}\gamma^k\gamma^\ell\,[\partial_k,\partial_\ell]\\ &= -\sum_k {\partial_k}^2 + \sum_{\substack{i<j\\ k<\ell}} R^i_{jk\ell}\,\gamma^k\gamma^\ell\dot\gamma(E^j_i).\end{aligned} \tag{2.2.28}$$

In the last line we used the basic Clifford identity $(\gamma^k)^2 = -1$, the definition of curvature (2.2.11), and the fact that vertical vector fields act on associated bundles by *minus* the differential of the defining representation.[17] The skew matrix $E^j_i \in \mathfrak{o}(n)$ acts on spinors by the differential $\dot\gamma$ of the

[17] More explicitly, let $P \to X$ be an arbitrary principal G bundle, $Z \in \mathfrak{g}$ an element of the Lie algebra of G, $\tilde Z$ the vertical vector field on P determined by Z, γ a representation of G, and ψ a section of the vector bundle associated to ρ. Note that ψ satisfies equation (2.2.20). Thus if g_t is the 1-parameter group generated by Z, then

$$\tilde Z\psi(p) = \frac{d}{dt}\Big|_{t=0}\,\psi(p\cdot g_t) = \frac{d}{dt}\Big|_{t=0}\,\gamma(g_t^{-1})\psi(p) = -\dot\gamma(Z)\psi(p)$$

spin representation. This should have been computed in Exercise 2.1.21 as

$$\dot\gamma(E_i^j) = \frac{1}{2}\gamma^j\gamma^i. \tag{2.2.29}$$

(Try Exercise 2.1.25 first! Note here that we have taken the dual action on $\mathbb{E}^{n*}$, which is by *minus* the transpose.) Hence the last term in (2.2.28) is

$$\sum_{\substack{i<j\\ k<\ell}} R^i_{jk\ell}\,\gamma^k\gamma^\ell\dot\gamma(E_i^j) = \frac{1}{8}\sum_{\substack{i\neq j\\ k\neq\ell}} R_{ijk\ell}\,\gamma^k\gamma^\ell\gamma^j\gamma^i. \tag{2.2.30}$$

Now if $k\neq\ell\neq j$ then the summand vanishes, by the Bianchi identity (2.2.13). When $k=j$ we obtain

$$\frac{1}{8}\sum_k R_{ikk\ell}\gamma^\ell\gamma^i = -\frac{1}{8}R_{i\ell}\gamma^\ell\gamma^i = \frac{1}{8}\sum_\ell R_{\ell\ell} = \frac{1}{8}R. \tag{2.2.31}$$

Here we have used the definitions of the Ricci curvature (2.2.14) and the scalar curvature (2.2.15). Similarly, when $\ell=j$ in (2.2.30) we have

$$-\frac{1}{8}\sum_\ell R_{i\ell k\ell}\gamma^k\gamma^i = -\frac{1}{8}R_{ik}\gamma^k\gamma^i = \frac{1}{8}R. \tag{2.2.32}$$

Combining (2.2.28)–(2.2.31) we deduce the *Weitzenböck formula*

$$\mathcal{D}^2 = -\sum_k {\partial_k}^2 + \frac{R}{4}. \tag{2.2.33}$$

This formula is important in the sequel. The operator $-\sum_k {\partial_k}^2$ is the (flat, covariant) Laplacian on spinor fields. Since the formal adjoint (Exercise 2.2.23) of the covariant derivative $\partial = e^k\partial_k$ is

$$\partial^* = -\iota(e^k)\partial_k, \tag{2.2.34}$$

where $\iota(e^k)$ is the inner product with e^k, we can rewrite this operator as $\partial^*\partial$. It is a nonnegative (formally) self-adjoint operator.[18]

Exercise 2.2.35. Prove that if $R\geq 0$ on X, and $R>0$ at one point, then there are no harmonic spinors on X, i.e., the Dirac operator has no kernel. This is Lichnerowicz's vanishing theorem [L].

Exercise 2.2.36. Give an example of a nontrivial harmonic spinor.

Exercise 2.2.39. Suppose $\{\xi_k\}$ is a local oriented orthonormal frame, i.e., a local section of $SO(X)$. Write $\mathcal{D}$, $\mathcal{D}^2$, ∇^*, and $\nabla^*\nabla$ in terms of the covariant derivatives ∇_{ξ_k}.

[18] The covariant derivative is often denoted ∇, in which case the covariant Laplacian is $\nabla^*\nabla$.

§2.3 Generalized Dirac operators

Let $V \to X$ be a Hermitian vector bundle endowed with a unitary connection ∇. The connection is a first order differential operator $\nabla\colon C^\infty(V) \to C^\infty(T^*X \otimes V)$; it satisfies Leibnitz's rule and preserves the metric. (See Chapter 6 for more about connections on vector bundles.) If X is a spin manifold, and $\mathcal{D}$ its Dirac operator, we can construct the *coupled Dirac operator*

$$\mathcal{D}_V : C^\infty(S \otimes V) \longrightarrow C^\infty(S \otimes V) \tag{2.3.1}$$

as the composition

$$C^\infty(S \otimes V) \xrightarrow{\partial\otimes 1 + 1 \otimes \nabla} C^\infty(T^*X \otimes S \otimes V) \xrightarrow{c(\cdot)} C^\infty(S \otimes V).$$

The operator $\partial \otimes 1 + 1 \otimes \nabla$ is a connection on $S \otimes V$. If ψ is a spinor field, φ a section of V, and ξ_k a local orthonormal frame field, with θ^k the dual coframe field, then

$$\mathcal{D}_V(\psi \otimes \varphi) = c(\theta^k)\partial_{\xi_k}\psi \otimes \varphi + c(\theta^k)\psi \otimes \nabla_{\xi_k}\varphi. \tag{2.3.2}$$

Exercise 2.3.3. Prove that the operator $\mathcal{D}_V$ is formally self-adjoint (cf. Exercise 2.2.23).

We can also describe the generalized Dirac operator in terms of the frame bundle. Suppose $Q \to X$ is a principal bundle with connection to which (V, ∇) is associated. Form the product $SO(X) \times_X Q \to X$; it is a principal bundle over X with connection. The vector fields ∂_k lift to this bundle, and $\partial = e^k\partial_k$ is the connection on $P \times_X Q$ to which the connection $\partial \otimes 1 + 1 \otimes \nabla$ on $S \otimes V$ is associated. Now the coupled Dirac operator is $\mathcal{D}_V = \gamma^k\partial_k$ as in (2.2.21), only now the ∂_k live on $P \times_X Q$ and act on functions $P \times_X Q \to \mathbb{S} \otimes \mathbb{V}$ which transform appropriately.

Notice that we might have $Q = SO(X)$. In other words, the auxiliary vector bundle V may be *intrinsic* (associated to the tangent bundle). Then we can replace $SO(X) \times_X SO(X)$ by $SO(X)$. The Dirac operator is $\mathcal{D}_V = \gamma^k\partial_k$ as before, operating on spinor fields coupled to some tensor fields.

Important examples appear in the following exercises. The analysis that we carry out in the sequel applies to all of these generalized Dirac operators.

Exercise 2.3.41. As in Exercise 2.2.39 write the coupled Dirac operator in terms of a local orthonormal frame field.

Exercise 2.3.4. Set $\overline{\nabla} = \partial \otimes 1 + 1 \otimes \nabla$, the connection on $S \otimes V$. Deduce the generalized Weitzenböck formula

$$\mathcal{D}_V^2 = \overline{\nabla}^*\overline{\nabla} + \frac{R}{4} + c(\Omega^{(V)}), \tag{2.3.5}$$

where $\Omega^{(V)} \in \Omega^2(X, \operatorname{End} V)$ is the curvature of ∇, and Clifford multiplication by a 2-form is defined using Exercise 2.1.8.

In the next several exercises, we develop the important examples of generalized Dirac operators. This amounts to rewriting some of the usual operators on differential forms in terms of spinors and Clifford algebras. Notice that the coupling bundle V in these examples is *intrinsic*, i.e., associated to the tangent bundle.

EXERCISE 2.3.6. For $V = S$ show that in even dimensions $\mathcal{D}_V$ is the complexification of the operator

$$d + d^* \colon \Omega^*(X) \longrightarrow \Omega^*(X) \tag{2.3.7}$$

on differential forms. (Hint: Exercise 2.1.8, (2.1.42).) Show that the kernel consists of harmonic forms (cf. (1.2.3)). What happens in odd dimensions? Investigate carefully the situation in 3 dimensions. Notice that (2.3.7) is defined on any manifold X.

In even dimensions the *chiral Dirac operator* D is the restriction of $\mathcal{D}$ to S^+. Thus given V as above we form

$$D_V \colon C^\infty(S^+ \otimes V) \longrightarrow C^\infty(S^- \otimes V). \tag{2.3.8}$$

For the rest of this section we work in even dimensions.

EXERCISE 2.3.9. Suppose E_i are vector bundles over X, and

$$0 \longrightarrow C^\infty(E_0) \xrightarrow{d_0} C^\infty(E_1) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^\infty(E_n) \longrightarrow 0 \tag{2.3.10}$$

a sequence of first order differential operators with $d_i \circ d_{i-1} = 0$. The i^{th} cohomology of the complex (2.3.10) is then

$$H^i = \ker d_i / \operatorname{im} d_{i-1}.$$

Assume E_i are Hermitian and d_i^* is the adjoint of d_i. Make sense of

$$d + d^* \colon \bigoplus_i C^\infty(E_{2i}) \longrightarrow \bigoplus_i C^\infty(E_{2i+1}). \tag{2.3.11}$$

What is the kernel of (2.3.11)? What is the kernel of its adjoint? If the H^i are finite dimensional, prove that the index of (2.3.11) is finite and equal to $\sum_i (-1)^i \dim H^i$, the *Euler characteristic* of (2.3.10).

Exercise 2.3.12. Recall the grading operator ϵ introduced in Exercise 2.1.28. Show that $\pm c(\epsilon)\otimes c(\epsilon)$ induces the grading on $\mathbb{S}\otimes\mathbb{S}$ corresponding to the $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$, where we use the $+$ sign if n is divisible by 4 and the $-$ sign if $n\equiv 2 \pmod 4$. (Hint: Exercise 2.1.40, Exercise 2.1.44.) Hence

$$\begin{aligned}\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^{\pm} &\cong \mathbb{S}^+\otimes\mathbb{S}^+ \;\oplus\; \mathbb{S}^-\otimes\mathbb{S}^-,\\ \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^{\mp} &\cong \mathbb{S}^+\otimes\mathbb{S}^- \;\oplus\; \mathbb{S}^-\otimes\mathbb{S}^+.\end{aligned} \tag{2.3.13}$$

(Hint: Exercise 2.1.36.) Again the signs depend on the value of $n \pmod 4$. Also, we can identify $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^{\pm}\cong\bigwedge^{\pm}(\mathbb{R}^n)\otimes\mathbb{C}$, the even and odd parts of the exterior algebra. Under these isomorphisms the action of $c(\cdot)\otimes 1$ on $\mathbb{S}\otimes\mathbb{S}$ corresponds to the action of $\epsilon(\cdot)-\iota(\cdot)$ on $\bigwedge(\mathbb{R}^n)\otimes\mathbb{C}$. (Hint: (2.1.10), (2.1.42).) Conclude that for $V=\pm S^+\mp S^-$ we can identify D_V with the complexification of the real operator

$$d+d^*\colon \Omega^+(X)\longrightarrow\Omega^-(X). \tag{2.3.14}$$

(What do we mean by the difference of two bundles?) (2.3.14) is a collapsed version of the de Rham complex (1.2.1). Show that the index of (2.3.14) is $\chi(X)$, the Euler characteristic of X. (The index is the dimension of the kernel minus the dimension of the kernel of the adjoint.) Notice that (2.3.14) is defined on any manifold X, though the identification of the index with the Euler characteristic is valid only if X is compact.

Exercise 2.3.15. The Hodge $*$ operator is defined on $\bigwedge(\mathbb{E}^n)$ by the formula

$$\alpha\wedge *\beta = (\alpha,\beta)\,\mathrm{vol},\qquad \alpha,\beta\in\bigwedge(\mathbb{E}^n), \tag{2.3.16}$$

where $\mathrm{vol}\in\bigwedge^n\mathbb{E}^n$ is the volume form. Notice that we require an orientation on $\mathbb{E}^n$. Show that under the isomorphism $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)\cong\bigwedge(\mathbb{E}^n)\otimes\mathbb{C}$, left Clifford multiplication by ϵ (2.1.29) is

$$\epsilon\cdot = i^{p(p-1)+n/2}\,*\,. \tag{2.3.17}$$

What is right Clifford multiplication by ϵ? The involution

$$\tau(\alpha)=i^{p(p-1)+n/2}*\alpha \tag{2.3.18}$$

induces a decomposition $\bigwedge(\mathbb{E}^n)\otimes\mathbb{C}\cong\bigwedge_+(\mathbb{E}^n)\oplus\bigwedge_-(\mathbb{E}^n)$. Notice that if n is divisible by 4, then τ is defined on the *real* exterior algebra $\bigwedge(\mathbb{E}^n)$. Prove

$$\begin{aligned}\textstyle\bigwedge_+(\mathbb{E}^n)&\cong\mathbb{S}^+\otimes\mathbb{S},\\ \textstyle\bigwedge_-(\mathbb{E}^n)&\cong\mathbb{S}^-\otimes\mathbb{S}.\end{aligned} \tag{2.3.19}$$

(CHECK THE SIGNS CAREFULLY!)

Exercise 2.3.20. Continuing the previous exercise, on any oriented even dimensional Riemannian manifold X we decompose the complexified differential forms into $\Omega_+(X) \oplus \Omega_-(X)$, according to the involution (2.3.18). Prove that for $V = S$ we can identify D_V with

$$d + d^* \colon \Omega_+(X) \longrightarrow \Omega_-(X). \tag{2.3.21}$$

This is called the *signature operator*. If n is divisible by 4, then (2.3.21) is the complexification of a real operator. Identify the kernel (and the kernel of the adjoint) when X is compact.

Exercise 2.3.22. Let X^{4k} be a compact oriented manifold of dimension $4k$. Then there is a symmetric pairing on $\Omega^{2k}(X)$ defined by

$$\alpha \otimes \beta \longmapsto \int_X \alpha \wedge \beta. \tag{2.3.23}$$

By Stokes' theorem this passes to a symmetric pairing on the de Rham cohomology $H^{2k}(X)$, and by Poincaré duality it is nondegenerate. Now a nondegenerate symmetric bilinear form on a real vector space has a *signature*, which is the dimension of the maximal subspace on which the form is positive definite minus the dimension of the maximal subspace on which the form is negative definite. The signature of (2.3.23) is called the signature of X, denoted $\mathrm{Sign}(X)$. Compute the signature of some familiar manifolds ($\mathbb{CP}^n$, tori). How does the signature behave under products?

Exercise 2.3.24. Prove that the index of (2.3.21) on X^{4k} compact, oriented is $\mathrm{Sign}(X)$.

Exercise 2.3.25. Identify D_V in terms of differential forms for $V = S^+$ and $V = S^-$. Is D_V defined over the reals? On what sorts of manifolds is D_V defined? Express D_V as the collapsed version of an elliptic complex. It is called the *(anti-)self-dual complex.* On X^4 the $*$ operator is an involution on 2-forms, which leads to the decomposition $\Omega^2(X) \cong \Omega^2_+(X) \oplus \Omega^2_-(X)$. The self-dual-complex is

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d_+} \Omega^2_+(X) \to 0. \tag{2.3.26}$$

Assuming X compact, compute the index of (2.3.26) in terms of $\chi(X)$ and $\mathrm{Sign}(X)$. Define the anti-self-dual complex in 4 dimensions and compute its index. This complex is fundamental in the work of Simon Donaldson on the topology of 4-manifolds. (See [FU] for an introduction.)

Exercise 2.3.27. An *almost complex structure* on a differentiable manifold X of dimension $2m$ is a smoothly varying complex structure J_x on the tangent spaces T_xX, i.e., an endomorphism of T_xX

whose square is -1. If X also carries a Riemannian metric g, then we ask that J be skew-symmetric (and hence orthogonal). There is then an exterior 2-form

$$\omega(X,Y) = g(X,JY) \tag{2.3.28}$$

defined. (Note that $g+\sqrt{-1}\,\omega$ is a Hermitian metric.) We call X *Kähler* if ω is closed. Reconcile this definition with any other definition of 'Kähler' with which you are familiar. On any almost complex manifold X define a subbundle $U(X)$ of the orthogonal frame bundle $SO(X)$ which consists of isometries $p\colon \mathbb{E}^{2m}\to T_xX$ which carry the standard complex structure on $\mathbb{E}^{2m}$ to the complex structure J_x on T_xX. (Fix the standard complex structure on $\mathbb{E}^{2m}$ to be $J(e_{2k-1}) = e_{2k}, J(e_{2k}) = -e_{2k-1}$.) Then $U(X)$ is a principal $U(2m)$ bundle. Show that X is Kähler if and only if the vector fields ∂_k on $SO(X)$ are tangent to $U(X)$ along $U(X)$. This is a nice geometric interpretation of the Kähler condition.

Exercise 2.3.29. Recall that on any complex m-manifold X the complex differential forms decompose into $\Omega^*(X) = \oplus\Omega^{p,q}(X)$, and that the exterior differential also decomposes into $d=\partial+\bar\partial$, with $\partial^2=\bar\partial^2=0$. The *Dolbeault complex* is then

$$0\longrightarrow\Omega^{0,0}(X)\xrightarrow{\bar\partial}\Omega^{0,1}(X)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\Omega^{0,m}(X)\longrightarrow 0. \tag{2.3.30}$$

Now suppose that X is Kähler with unitary frame bundle $U(X)$ as in Exercise 2.3.27. Define complex vector fields

$$\bar\partial_k = \frac{1}{2}(\partial_{2k-1}+\sqrt{-1}\,\partial_{2k}),\quad k=1,\dots,m, \tag{2.3.31}$$

and a $\overline{\mathbb{C}^m}^*$-valued vector field

$$\bar\partial = \bar f^k\bar\partial_k, \tag{2.3.32}$$

where $f_1,\dots,f_m$ is the standard unitary basis of $\mathbb{C}^m$,

$$f_k = e_{2k-1}+\sqrt{-1}\,e_{2k}, \tag{2.3.33}$$

and $\bar f^1,\dots,\bar f^m$ the corresponding basis of $\overline{\mathbb{C}^m}^*$. Show that (2.3.32) represents the $\bar\partial$ operator in (2.3.30). (Hint: Act on $U(m)$ equivariant functions $U(X)\to\bigwedge\overline{\mathbb{C}^m}^*$. Try first to understand the equation $d=\partial+\bar\partial$ in this language, rewriting (2.2.8) in terms of f^k and $\bar f^k$ using (2.3.33).) Show that the adjoint is

$$\bar\partial^* = -\iota(\bar f^k)[\frac{1}{2}(\partial_{2k-1}-\sqrt{-1}\,\partial_{2k})]. \tag{2.3.34}$$

Exercise 2.3.35. Let X be a Kähler manifold with unitary frame bundle $U(X)$. Recall from Exercise 2.1.54 that its structure group $U(n)$ has a nontrivial double cover $\tilde{U}(n)$. Define a $\tilde{U}(n)$ structure by analogy with Definition 2.2.16. Show that X has a $\tilde{U}(n)$ structure if and only if $w_2(X) = 0$. (Recall that since X is complex, $w_2(X)$ is the mod 2 reduction of $c_1(X)$.) Describe a 1:1 correspondence between $\tilde{U}(n)$ structures and spin structures.

Exercise 2.3.36. Suppose X is a Kähler manifold with $\tilde{U}(n)$ structure $\tilde{U}(X)$, which is a subbundle of a spin structure $\mathrm{Spin}(X)$. Since the vector fields ∂_k are tangent to $\tilde{U}(X)$, we can restrict the Dirac operator (2.2.21) to $\tilde{U}(X)$. Now use Exercise 2.1.54 to rewrite the result in terms of differential forms. Use the Hermitian metric to identify $\mathbb{C}^m \cong \overline{\mathbb{C}^m}^*$. Also, the bundle associated to $\tilde{U}(X)$ via the representation $\det^{-1/2}$ of Exercise 2.1.54 is a square root $K^{1/2}$ of the *canonical bundle* $K = \det T^*X$. (The inverse is accounted for since we use the cotangent bundle.) Finally, then, using (2.3.32) and (2.3.34) you should find that the chiral Dirac operator restricts to

$$D = \frac{1}{2}(\bar\partial + \bar\partial^*)\colon \Omega^{0,\mathrm{even}}(X, K^{1/2}) \longrightarrow \Omega^{0,\mathrm{odd}}(X, K^{1/2}). \tag{2.3.37}$$

Here, for any vector bundle E we let $\Omega^{p,q}(X,E)$ denote the (p,q)-forms with values in E. Show that up to a factor of 2 the operator in (2.3.37) is the collapse of the complex

$$0 \to \Omega^{0,0}(X, K^{1/2}) \xrightarrow{\bar\partial} \Omega^{0,1}(X, K^{1/2}) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,m}(X, K^{1/2}) \to 0. \tag{2.3.38}$$

Note that $\bar\partial$ in (2.3.38) is defined purely from the holomorphic structure of $K^{1/2}$. Show that on a Riemann surface ($m = 1$) the chiral Dirac operator is complex skew-adjoint.

Exercise 2.3.39. Finally, if X is Kähler with a $\tilde{U}(n)$ structure (or equivalently a spin structure), identify the coupled chiral Dirac operator $D_{K^{-1/2}}$ with

$$D_{K^{-1/2}} = \frac{1}{2}(\bar\partial + \bar\partial^*)\colon \Omega^{0,\mathrm{even}}(X) \longrightarrow \Omega^{0,\mathrm{odd}}(X), \tag{2.3.40}$$

the collapse[19] of the Dolbeault complex (2.3.30). Hence this operator is of Dirac type. Its index is called the *arithmetic genus* of X. What is the arithmetic genus of a Riemann surface of genus g?

Exercise 2.3.42. Work out the Weitzenböck formula for the $\bar\partial$ complex (cf. Exercise 2.3.4). Use it to deduce a vanishing theorem (cf. Exercise 2.2.35).

[19] up to a factor of 2

§3 Ellipticity

The methods of functional analysis will aid our study of the Dirac operator, once we settle on appropriate function spaces. Consider the simplest Dirac operator $i\frac{d}{dx}$ on the real line $\mathbb{R}$. The smooth functions form a linear space $C^\infty(\mathbb{R})$ on which $i\frac{d}{dx}$ acts, but in the natural topology $C^\infty(\mathbb{R})$ is a Fréchet space. Analysis on Fréchet spaces is far too tricky for our needs, so we try instead the space $C^k(\mathbb{R})$ of functions with k bounded continuous derivatives. This is a Banach space, but still not best for our purposes. Hilbert spaces are the nicest among topological vector spaces, and the basic example here is the space $L^2(\mathbb{R})$ of square integrable functions.

Exercise 3.1. Show that $i\frac{d}{dx}$ defines an unbounded operator on $L^2(\mathbb{R})$. Prove that it is self-adjoint [St,§X.2]. Prove that the Dirac operator on any compact manifold X defines an unbounded self-adjoint operator on $L^2(X)$. (This will follow from our work later, but if you try it now you will appreciate the difficulties.)

Exercise 3.1 is a powerful result. For self-adjoint operators one has available the spectral theorem. However, we prefer another tack, restricting ourselves to bounded operators and simpler Hilbert space theory. The price we pay is small—we must define more complicated Hilbert spaces. As the Dirac operator involves differentiation, in §3.1 we introduce Hilbert spaces of functions with derivatives, the *Sobolev spaces*. Rather than develop the theory of general elliptic differential (or pseudodifferential) operators, in §3.2 we specialize to generalized Dirac operators. This simplifies the theory tremendously. The main result is the existence of a basis of smooth eigenfunctions.

§3.1 Sobolev Spaces

We begin with a somewhat general setting. Let X be a *compact* Riemannian manifold. The metric on X determines a volume form dx. Suppose $E \to X$ is a vector bundle with a smooth hermitian metric $(\cdot,\cdot)$ and a smooth unitary connection ∇. The metric is conjugate linear in the second variable. For $\varphi,\psi \in C^\infty(E)$ smooth sections of E, we define the L^2 inner product

$$\langle\varphi,\psi\rangle_{L^2} = \int_X \big(\varphi(x),\psi(x)\big)\,dx. \tag{3.1.1}$$

Then $L^2(E)$ is the completion

$$L^2(E) = \overline{\{\psi \in C^\infty(E) : \|\psi\|_{L^2} < \infty\}}. \tag{3.1.2}$$

We denote the L^2 inner product by $\langle\cdot,\cdot\rangle$ and the L^2 norm by $\|\cdot\|$. Similarly, for a nonnegative integer ℓ we define the Sobolev H_ℓ inner product

$$\langle\varphi,\psi\rangle_{H_\ell} = \int_X \big[(\varphi,\psi) + (\nabla\varphi,\nabla\psi) + \cdots + (\nabla^\ell\varphi,\nabla^\ell\psi)\big]\,dx. \tag{3.1.3}$$

In (3.1.3) the Riemannian metric and the inner product on E combine to produce an inner product on sections of $(T^*X)^k \otimes E, 0 \le k \le \ell$. The Sobolev space $H_\ell(E)$ is then the completion

$$H_\ell(E) = \overline{\{\psi \in C^\infty(E)\colon \|\psi\|_{H_\ell} < \infty\}}. \tag{3.1.4}$$

$H_\ell(E)$ is a Hilbert space. Notice that $H_0(E) = L^2(E)$; we use the notations interchangeably. The inequality

$$\|\psi\|_{H_\ell} \le \|\psi\|_{H_{\ell'}}, \quad \ell' > \ell \tag{3.1.5}$$

is trivial.

Exercise 3.1.6. Write smooth complex-valued functions on the circle as Fourier series $f = \sum a_n e^{in\theta}$, $a_n \in \mathbb{C}$. When is $f \in H_\ell(S^1)$? What is $\|f\|_{H_\ell}$? Extend to functions on the n-torus.

Exercise 3.1.7. Define $H_\ell(\mathbb{R})$ and $H_\ell(\mathbb{R}^n)$. Compare with the Sobolev spaces on compact manifolds. (Hint: Consider constant functions.) Guess the definition of $H_\ell^{\mathrm{loc}}(\mathbb{R}^n)$, the space of functions locally in $H_\ell()$.

Exercise 3.1.24. The space $H_k(E)$ is sometimes denoted '$L_k^2(E)$'. Guess the definition of the space $L_k^p(E)$ for $1 \le p \le \infty$. We do not need these L^p spaces for the linear analysis, but they do enter into *nonlinear* elliptic theory.

Exercise 3.1.8. Prove that $i\frac{d}{dx}\colon H_1(\mathbb{R}) \to H_0(\mathbb{R})$ is bounded. How does $H_1(\mathbb{R})$ enter into the considerations of Exercise 3.1?

The Sobolev norms depend on the choice of metric on X, metric on E, and connection on E. The next result asserts that different choices lead to equivalent norms.

Proposition 3.1.9. *If $\|\cdot\|'_{H_\ell}$ is the Sobolev norm for different choices of metric on X, metric on E, and connection on E, then $\|\cdot\|_{H_\ell}$ and $\|\cdot\|'_{H_\ell}$ are equivalent. In other words, there exist constants c, C (depending on ℓ) such that for all $\psi \in C^\infty(E)$*

$$c\|\psi\|_{H_\ell} \le \|\psi\|'_{H_\ell} \le C\,\|\psi\|_{H_\ell}. \tag{3.1.10}$$

Throughout we use symbols like c, C to denote generic constants. Their meaning may vary from line to line, as well as within the same equation.

Proof. We only treat a change of connection and $\ell = 1$, the general case being similar. Hence suppose $\nabla' = \nabla + A$ for some matrix valued 1-form A. Since X is compact we have the bound $|A| \le C$, where $|A|$ is the pointwise operator norm. From the Cauchy-Schwarz inequality we conclude

$$\begin{aligned}\langle \nabla'\psi, \nabla'\psi\rangle &= \langle \nabla\psi, \nabla\psi\rangle + 2\,\mathrm{Re}\langle \nabla\psi, A\psi\rangle + \langle A\psi, A\psi\rangle \\ &\le \|\nabla\psi\|^2 + C\,\|\nabla\psi\|\,\|\psi\| + C\,\|\psi\|^2.\end{aligned} \tag{3.1.11}$$

Taking note of (3.1.5) we obtain

$$\begin{aligned}\|\psi\|_{H_1}^{\prime 2} &= \|\psi\|^2 + \|\nabla'\psi\|^2 \\ &\le \|\psi\|^2 + \|\nabla\psi\|^2 + C\,\|\psi\|_{H_1}^2 + C\,\|\psi\|^2 \\ &\le C\,\|\psi\|_{H_1}^2.\end{aligned}$$

Take the square root to obtain the desired inequality; the reverse inequality is obtained by switching ∇ and ∇'.

Exercise 3.1.12. Write out the proof of Proposition 3.1.9 for the general case.

Some elementary properties of the Sobolev spaces are summarized in

Proposition 3.1.13.

(1) *There is a bounded inclusion* $H_{\ell'}(E) \hookrightarrow H_\ell(E)$ *for* $\ell' > \ell$.
(2) *The covariant derivative is a bounded map* $\nabla\colon H_\ell(E) \to H_{\ell-1}(E)$.
(3) *A vector bundle map* $L\colon E \to F$ *extends to a bounded map* $H_\ell(E) \to H_\ell(F)$ *for all* ℓ.
(4) *Any* k^{th} *order differential operator* $P\colon C^\infty(E) \to C^\infty(F)$ *extends to a bounded map* $H_\ell(E) \to H_{\ell-k}(F)$ *for all* ℓ.

Proof. Assertion (1) is the inequality (3.1.5), and (2) is trivial from the definitions. To prove (3) we show that for any $\psi \in C^\infty(E)$ we have

$$\|L\psi\|_{H_\ell} \le C\,\|\psi\|_{H_\ell}. \tag{3.1.14}$$

Our proof is by induction on ℓ, and at the same time we show that for $\ell \ge 1$

$$\|[L, \nabla^\ell]\psi\| \le C\,\|\psi\|_{H_{\ell-1}}. \tag{3.1.15}$$

Now (3.1.14) is easy for $\ell = 0$. Similarly, $[L, \nabla] = -\nabla L$ is a bundle map, so that (3.1.15) follows easily for $\ell = 1$. Assuming (3.1.14) and (3.1.15) for $k < \ell$, we write

$$[L, \nabla^\ell] = [L, \nabla]\nabla^{\ell-1} + \nabla[L, \nabla]\nabla^{\ell-2} + \nabla^2[L, \nabla]\nabla^{\ell-3} + \cdots + \nabla^{\ell-1}[L, \nabla]. \tag{3.1.16}$$

Since $[L, \nabla]$ is a bundle map, our induction hypothesis implies that $[L, \nabla]\colon H_k(E) \to H_k(F)$ is bounded for $k \le \ell - 1$. Then (2) and (3.1.16) show that $[L, \nabla^\ell]\colon H_{\ell-1}(E) \to H_0(F)$ is bounded, which is (3.1.15). Thus

$$\begin{aligned}\|\nabla^\ell L\psi\| &\le \|L\nabla^\ell\psi\| + \|[L, \nabla^\ell]\psi\| \\ &\le C\left(\|\nabla^\ell\psi\| + \|\psi\|_{H_{\ell-1}}\right) \\ &\le C\,\|\psi\|_{H_\ell},\end{aligned} \tag{3.1.17}$$

and now by the induction hypothesis,

$$\begin{aligned}\|L\psi\|_{H_\ell} &\le C\left(\|\nabla^\ell L\psi\| + \|L\psi\|_{H_{\ell-1}}\right)\\ &\le C\left(\|\psi\|_{H_\ell} + \|\psi\|_{H_{\ell-1}}\right)\\ &\le C\,\|\psi\|_{H_\ell},\end{aligned}$$

which completes the proof of (3). Assertion (4) follows from (2) and (3), since a general k^{th} order differential operator has the form

$$L_k \circ \nabla^k + L_{k-1} \circ \nabla^{k-1} + \cdots + L_0 \tag{3.1.18}$$

for some bundle maps L_j.

There are two basic lemmas about Sobolev spaces which we need in the next section.

Lemma 3.1.19 (Rellich). *The inclusion $H_1(E) \hookrightarrow H_0(E)$ is compact.*

Proof. For any sequence $\{\varphi_i\} \subset H_1(E)$ with $\|\varphi_i\|_{H_1} \le 1$, we must find a subsequence which converges in $H_0(E)$. Cover X by a finite number of coordinate charts U_α with trivializations of E over U_α, and let ρ_α be a partition of unity subordinate to $\{U_\alpha\}$. Then it suffices to prove that a subsequence of vector-valued functions $\{\rho_\alpha\varphi_i\}$ converges. Take the coordinate charts to live on the n-torus $\mathbb{T}^n$. We are thus reduced to proving that $\iota\colon H_1(\mathbb{T}^n) \hookrightarrow H_0(\mathbb{T}^n)$ is compact. Since $H_0(\mathbb{T}^n)$ is complete, it suffices to show that given $\epsilon > 0$ we can find cover the image $\iota(B) \subset H_0(\mathbb{T}^n)$ of the closed unit ball $B \subset H_1(\mathbb{T}^n)$ by finitely many balls of radius ϵ.

For any function f on $\mathbb{T}^n$ we write its Fourier expansion $f(x) = \sum\limits_\nu \hat f_\nu e^{i\nu\cdot x}$ as a sum over multiindices $\nu = \langle \nu_1, \dots, \nu_n\rangle \in \mathbb{Z}^n$. Then $\|f\|^2_{H_1} = \sum\limits_\nu (1+|\nu|^2)|\hat f_\nu|^2$, where $|\nu|^2 = \nu_1^2 + \cdots + \nu_n^2$. Set

$$Z_N = \{f \in H_1(\mathbb{T}^n)\colon \hat f_\nu = 0 \text{ for } |\nu| \le N\}.$$

Then for N sufficiently large any $f \in B \cap Z_N$ satisfies $\|f\|_{H_0} < \epsilon/\sqrt{2}$, since

$$\sum_{|\nu|>N} |\hat f_\nu|^2 \le \sum_{|\nu|>N} \frac{1+|\nu|^2}{1+N^2}\,|\hat f_\nu|^2 \le \frac{\|f\|^2_{H_1}}{1+N^2} \le \frac{1}{1+N^2},$$

which tends to zero as $N \to \infty$. The orthogonal complement $Z_N^\perp$ consists of smooth functions whose Fourier coefficients $\hat f_\nu$ vanish for $|\nu| > N$; it is finite dimensional and orthogonal to Z_N in both the H_1 and H_0 inner products. The intersection $B \cap Z_N^\perp$ is compact in both the H_1 and H_0 topologies, so can be covered by a finite set of balls of H_0-radius $\epsilon/\sqrt{2}$ with centers $f_1, \dots, f_M$ in $Z_N^\perp$. Then the balls of radius ϵ with those centers cover $\iota(B) \subset H_0$.

Exercise 3.1.20. Prove that the inclusion $H_{\ell'}(E) \hookrightarrow H_\ell(E)$ is compact for $\ell' > \ell$.

The *Sobolev embedding theorem* bounds pointwise norms by integral norms.

Lemma 3.1.21 (Sobolev). *If $\ell - \frac{n}{2} > k$ then $H_\ell(E) \hookrightarrow C^k(E)$.*

Proof. We give a proof due to Louis Nirenberg [N]. Consider first the case $k = 0$. We must estimate the sup norm of a smooth section ψ in terms of the $H_\ell(E)$ norm. Since this is a local estimate, we may as well work with functions on $\mathbb{R}^n$ with support contained in the ball of radius R about the origin. Then repeated integration by parts in a fixed radial direction gives

$$\psi(0) = -\int_0^R \frac{\partial\psi}{\partial r}\,dr = \cdots = C\int_0^R r^{\ell-1}\frac{\partial^\ell\psi}{\partial r^\ell}. \tag{3.1.22}$$

The volume form in polar coordinates is vol $= r^{n-1}dr\,d\theta$. Integrate (3.1.22) over the unit sphere to obtain

$$\begin{aligned} |\psi(0)| &= C\left|\int_{B_R(0)} \frac{\partial^\ell\psi}{\partial r^\ell} r^{\ell-n}\,\mathrm{vol}\right| \\ &\le C\left(\int_{B_R(0)}\left|\frac{\partial^\ell\psi}{\partial r^\ell}\right|^2\mathrm{vol}\right)^{1/2}\left(\int_{B_R(0)} r^{2(\ell-n)}\,\mathrm{vol}\right)^{1/2}. \end{aligned} \tag{3.1.23}$$

The first factor is bounded by the H_ℓ norm of ψ, and the second factor is finite for $\ell > n/2$. This is the desired statement for $k = 0$. For $k > 0$ apply (3.1.23) to the derivatives of ψ.

§3.2 Elliptic Theory for Dirac Operators

Let X be a compact n dimensional spin manifold, with spin bundle S, and $V \to X$ a Hermitian bundle with connection. Then we have a coupled Dirac operator $\mathcal{D}$ acting on sections of $S \otimes V$ (cf. (2.3.1)). It is formally self-adjoint (Exercise 2.3.3). For convenience we denote the Sobolev space $H_\ell(S \otimes V)$ by H_ℓ. Then(3.1.13(4)) implies that

$$\begin{gathered} \mathcal{D}\colon H_\ell \longrightarrow H_{\ell-1} \\ \mathcal{D}^2\colon H_\ell \longrightarrow H_{\ell-2} \end{gathered}$$

are bounded operators. We determine their structure.

Our treatment requires only a simple estimate and elementary Hilbert space theory. To simplify matters we introduce the antidual space H_{-1} to H_1. For a smooth function f define the H_{-1} norm of f to be the least constant C such that

$$\langle f, \psi\rangle \le C\,\|\psi\|_{H_1} \qquad \text{for } \psi \in H_1. \tag{3.2.1}$$

Let H_{-1} be the completion of C^∞ with respect to the norm (3.2.1). By definition there is a pairing $H_{-1} \otimes \overline{H}_1 \to \mathbb{C}$, and both H_1 and H_{-1} are complete. This pairing is also nondegenerate (Exercise 3.2.2). It follows that H_{-1} is the antidual space to H_1. Notice that elements of H_{-1} are distributions which are not generally functions.

EXERCISE 3.2.2. Prove that (3.2.1) extends to a *nondegenerate* pairing $H_{-1} \otimes \overline{H}_1 \to \mathbb{C}$. (Hint: C^∞ functions are dense in both H_1 and H_{-1}.)

EXERCISE 3.2.3. Show that $L^2 \hookrightarrow H_{-1}$ is bounded. Prove that $\mathcal{D}^2\colon H_1 \to H_{-1}$ is bounded.

Our basic estimate is known as *Gårding's inequality.*

Proposition 3.2.4. *There exists $\kappa > 0$ such that for any $\psi \in H_1$,*

$$\langle \mathcal{D}^2\psi, \psi\rangle + \kappa\langle \psi, \psi\rangle \ge \|\psi\|^2_{H_1}. \tag{3.2.5}$$

Proof. It suffices to verify (3.2.5) for smooth ψ. Recall the generalized Weitzenböck formula (2.3.5), which implies

$$\psi + \nabla^*\nabla\psi = \mathcal{D}^2\psi - \left(\frac{R}{4} + \Omega^{(V)} - 1\right)\psi.$$

Then since the curvatures are bounded on X,

$$\langle \psi, \psi\rangle + \langle \nabla\psi, \nabla\psi\rangle \le \langle \mathcal{D}^2\psi, \psi\rangle + \kappa\langle \psi, \psi\rangle,$$

which is (3.2.5).

Corollary 3.2.6. *The inner product $\langle\langle \psi, \varphi\rangle\rangle = \langle \mathcal{D}^2\psi, \varphi\rangle + \kappa\langle \psi, \varphi\rangle$ is equivalent to the H_1 inner product.*

Theorem 3.2.7. *$\mathcal{D}^2 + \kappa\colon H_1 \to H_{-1}$ is an isomorphism.*

Proof. Using (3.2.5) and the definition of the H_{-1} norm we easily deduce

$$\|(\mathcal{D}^2 + \kappa)\psi\|_{H_{-1}} \ge \|\psi\|_{H_1}. \tag{3.2.8}$$

This implies that $\mathcal{D}^2 + \kappa$ is injective and has closed range. We must prove that it is also onto. For any $f \in H_{-1}$ consider the antilinear functional $\varphi \mapsto \langle f, \varphi\rangle$ on H_1. This is bounded, by the definition of H_{-1}. Corollary 3.2.6 implies that any bounded functional on H_1 is represented by the inner product $\langle\langle \cdot, \cdot\rangle\rangle$, whence we can find $\psi \in H_1$ such that

$$\langle f, \varphi\rangle = \langle (\mathcal{D}^2 + \kappa)\psi, \varphi\rangle$$

for all $\varphi \in H_1$. This implies $(\mathcal{D}^2 + \kappa)\psi = f$. So $\mathcal{D}^2 + \kappa$ is onto. Now (3.2.8) shows that the inverse is bounded.

Let T denote the composition

$$T\colon H_0 \hookrightarrow H_{-1} \xrightarrow{(\mathcal{D}^2+\kappa)^{-1}} H_1 \hookrightarrow H_0. \tag{3.2.9}$$

By Theorem 3.2.7 and Lemma 3.1.19 we see that T is compact. Since for $\psi, \varphi \in H_1$ we have

$$\langle (\mathcal{D}^2 + \kappa)\psi, \varphi\rangle = \langle \psi, (\mathcal{D}^2 + \kappa)\varphi\rangle, \tag{3.2.10}$$

it follows easily that T is self-adjoint. It is clear that T is a positive operator. We now apply a result from Hilbert space theory, the spectral theorem for positive self-adjoint compact operators. It asserts the existence of a complete orthonormal basis $\{\psi_n\}$ of H_0 and positive numbers $\mu_1 \geq \mu_2 \geq \dots$ such that

$$\begin{aligned} &(i)\ T\psi_n = \mu_n \psi_n; \\ &(ii)\ \text{For any } c > 0 \text{ there is a finite number of } \mu_n > c; \\ &(iii)\ \lim_{n\to\infty} \mu_n = 0. \end{aligned} \tag{3.2.11}$$

This is quite an elementary theorem in Hilbert space theory, as we remind the reader in the following exercise.

EXERCISE 3.2.12. Prove the previous assertion as follows. Consider the quadratic form $\psi \mapsto \langle T\psi, \psi\rangle$ on the unit ball in H_0. Since T is compact it achieves its maximum, say at ψ_1. Show that T preserves the orthogonal complement to $\mathbb{C}\cdot\psi_1$, and iterate the argument.

EXERCISE 3.2.13. Make these conclusions explicit for the circle. What is $\mathcal{D}^2$? What are the ψ_n? The μ_n? Have you previously seen a proof of this theorem (on the circle)? If so, compare to our approach.

Our goal is to prove that the "eigenspinor fields" ψ_n are in H_ℓ for all ℓ, and hence by the Sobolev Lemma 3.1.21 all ψ_n are smooth. This result is called *elliptic regularity*, as it asserts that solutions to an elliptic equation with smooth coefficients ($(\mathcal{D}^2 - \lambda_n)\psi = 0$) are smooth. As a first step, notice that the definition of T and equation (3.2.11(i)) imply $\psi_n \in H_1$. Then setting $\lambda_n = 1/\mu_n - \kappa$ we conclude

$$\begin{aligned} &(i)\ \mathcal{D}^2\psi_n = \lambda_n \psi_n; \\ &(ii)\ \text{For any } a > 0 \text{ there is a finite number of } \lambda_n < a; \\ &(iii)\ \lim_{n\to\infty} \lambda_n = \infty. \end{aligned} \tag{3.2.14}$$

Also, since $\lambda_n = \lambda_n\langle\psi_n,\psi_n\rangle = \langle\mathcal{D}^2\psi_n,\psi_n\rangle = \langle\mathcal{D}\psi_n,\mathcal{D}\psi_n\rangle$ we have

$$(iv)\ \lambda_n \geq 0.$$

The main step in the regularity argument is the following basic elliptic estimate.

Proposition 3.2.15. *For any $\psi\in H_{\ell+1}$, $\ell\geq 0$,*

$$\|\psi\|_{H_{\ell+1}} \leq C\left(\|\mathcal{D}\psi\|_{H_\ell} + \|\psi\|_{H_\ell}\right). \tag{3.2.16}$$

Proof. The case $\ell=0$ follows immediately from Gårding's inequality (3.2.5). Assume that (3.2.16) holds for smaller values of ℓ. As usual, it suffices to consider smooth ψ. We claim that $[\mathcal{D},\nabla]$ is a tensor (zero$^{\text{th}}$ order differential operator). To see this we compute on the frame bundle of $S\otimes V$ (cf. the discussion following (2.3.2)). Thus we write $\nabla = t(e^k)\partial_k$ and $\mathcal{D} = c(e^k)\partial_k$, where $t(e^k)$ denotes tensor product by e^k and $c(e^k)$ denotes Clifford multiplication. Then since $c(e^k)$ and $t(e^\ell)$ commute,

$$\begin{aligned}[\mathcal{D},\nabla] &= c(e^k)t(e^\ell)\partial_k\partial_\ell - t(e^\ell)c(e^k)\partial_\ell\partial_k\\ &= \sum_{k<\ell} t(e^\ell)c(e^k)[\partial_k,\partial_\ell].\end{aligned}$$

Since $-[\partial_k,\partial_\ell] = \dot\gamma(R_{k\ell})\otimes 1\,+\,1\otimes\Omega^{(V)}_{k\ell}$ acting on sections of $S\otimes V$ is the curvature tensor, the claim is proved. Hence the induction hypothesis yields (for $\psi\in H_{\ell+1}$)

$$\begin{aligned}\|\nabla\psi\|_{H_\ell} &\leq C\left(\|\mathcal{D}\nabla\psi\|_{H_{\ell-1}} + \|\nabla\psi\|_{H_{\ell-1}}\right)\\ &\leq C\left(\|\nabla\mathcal{D}\psi\|_{H_{\ell-1}} + \|[\mathcal{D},\nabla]\psi\|_{H_{\ell-1}} + \|\nabla\psi\|_{H_{\ell-1}}\right)\\ &\leq C\left(\|\mathcal{D}\psi\|_{H_\ell} + \|\psi\|_{H_{\ell-1}} + \|\psi\|_{H_\ell}\right)\\ &\leq C\left(\|\mathcal{D}\psi\|_{H_\ell} + \|\psi\|_{H_\ell}\right).\end{aligned} \tag{3.2.17}$$

The desired inequality (3.2.16) follows by adding $\|\psi\|_{H_\ell}$ to both sides of (3.2.17).

Corollary 3.2.18. *For any $\psi\in H_{\ell+2}$, $\ell\geq 0$,*

$$\|\psi\|_{H_{\ell+2}} \leq C\left(\|\mathcal{D}^2\psi\|_{H_\ell} + \|\psi\|_{H_\ell}\right). \tag{3.2.19}$$

Proof. Applying (3.2.16) four times, and using $\|\mathcal{D}\psi\|_{H_{\ell-1}} \leq \|\psi\|_{H_\ell}$, we have

$$\begin{aligned}\|\psi\|_{H_{\ell+2}} &\leq C\left(\|\mathcal{D}\psi\|_{H_{\ell+1}} + \|\psi\|_{H_{\ell+1}}\right)\\ &\leq C\left(\|\mathcal{D}^2\psi\|_{H_\ell} + \|\mathcal{D}\psi\|_{H_\ell} + \|\psi\|_{H_\ell}\right)\\ &\leq C\left(\|\mathcal{D}^2\psi\|_{H_\ell} + \|\mathcal{D}^2\psi\|_{H_{\ell-1}} + \|\mathcal{D}\psi\|_{H_{\ell-1}} + \|\psi\|_{H_\ell}\right)\\ &\leq C\left(\|\mathcal{D}^2\psi\|_{H_\ell} + \|\psi\|_{H_\ell}\right).\end{aligned}$$

Exercise 3.2.20. Prove that for any $k \geq 0, \psi \in H_{\ell+k}$,

$$\|\psi\|_{H_{\ell+k}} \leq C\left(\|\mathcal{D}^k\psi\|_{H_\ell} + \|\psi\|_{H_\ell}\right). \tag{3.2.21}$$

We would like to apply Corollary 3.2.18 directly to conclude that if $\psi, \mathcal{D}^2\psi \in H_\ell$, then $\psi \in H_{\ell+2}$. While this assertion is true (Corollary 3.2.28), to apply (3.2.19) directly we need to approximate ψ by smooth functions f_n so that $f_n \to \psi$ and $\mathcal{D}^2 f_n \to \mathcal{D}^2\psi$ in the H_ℓ norm. One approach [R,§5] is via *Friedrich's mollifiers*. We opt for a different strategy, employing *difference quotients* as in [N].

We work now in coordinates, which we take to lie on the torus $\mathbb{T}^n$. Then for any (vector-valued) function $f \in L^2(\mathbb{T}^n)$ and any nonzero $h \in \mathbb{T}^n$, define the difference quotient

$$f^h(x) = \frac{f(x+h) - f(x)}{|h|}. \tag{3.2.22}$$

We need some elementary properties of difference quotients.

Lemma 3.2.23. *For any $\ell \geq 0$,*

1. *If $f(\cdot) \in H_\ell$, then $f(\,\cdot + h) \in H_\ell$ and $\|f(\,\cdot + h)\|_{H_\ell} = \|f(\cdot)\|_{H_\ell}$.*
2. *If $f \in H_{\ell+1}$, then $f^h \in H_\ell$ and $\|f^h\|_{H_\ell} \leq C\|f\|_{H_{\ell+1}}$.*
3. *If $f \in H_\ell$ and $\|f^h\|_{H_\ell} \leq C$ for all sufficiently small $|h|$, then $f \in H_{\ell+1}$.*

Proof. (1) and (2) are immediate. For (3) we use the Fourier transform $f(x) = \sum_\nu \hat{f}_\nu e^{i\nu\cdot x}$, as in the proof of Lemma 3.1.19. Then

$$\|f\|^2_{H_\ell} = \sum_\nu (1 + |\nu|^2 + |\nu|^4 + \cdots + |\nu|^{2\ell})|\hat{f}_\nu|^2. \tag{3.2.24}$$

Now

$$|\widehat{f^h}_\nu|^2 = \left|\frac{e^{ih\cdot\nu} - 1}{|h|}\right|^2 |\hat{f}_\nu|^2,$$

so setting $h_i = \langle 0, \dots, 1, \dots, 0\rangle$ the standard basis vector, we obtain

$$\lim_{\epsilon\to 0}\sum_i |\widehat{f^{\epsilon h_i}}_\nu|^2 = |\nu|^2|\hat{f}_\nu|^2.$$

Hence for any N,

$$\begin{aligned}\sum_{|\nu|<N}(|\nu|^2 + |\nu|^4 + \cdots + |\nu|^{2(\ell+1)})|\hat{f}_\nu|^2 &= \lim_{\epsilon\to 0}\sum_i\sum_{|\nu|<N}(1 + |\nu|^2 + \cdots + |\nu|^{2\ell})|\widehat{f^{\epsilon h_i}}_\nu|^2\\ &\leq \lim_{\epsilon\to 0}\sum_i \|f^{\epsilon h_i}\|^2_{H_\ell},\\ &\leq C,\end{aligned}$$

from which $f \in H_{\ell+1}$.

The main step in the proof of the regularity theorem is the following result.

Proposition 3.2.25. *For any $\ell \geq 1$, if $\psi \in H_\ell$ and $\mathcal{D}^2\psi \in H_{\ell-1}$, then $\psi \in H_{\ell+1}$.*

Proof. As in the proof of Lemma 3.1.19, let ρ_α be a finite partition of unity over whose supports E is trivial. Then $\psi = \sum \rho_\alpha\psi$ with $\rho_\alpha\psi \in H_\ell$ and $\mathcal{D}^2(\rho_\alpha\psi) \in H_{\ell-1}$. It suffices to prove that $\rho_\alpha\psi \in H_{\ell+1}$. Thus we reduce to the case where ψ is a vector-valued function on the torus with an arbitrary Riemannian metric defining the Dirac operator. In this coordinate system we have (cf. (3.1.18))

$$\mathcal{D}^2 = L_2 \circ \nabla^2 + L_1 \circ \nabla + L_0$$

for some matrix functions L_i. Now the formation of difference quotients commutes with ∇ but not with multiplication by $L_i(x)$. In fact,

$$\mathcal{D}^2\psi^h = (\mathcal{D}^2\psi)^h - \big((\mathcal{D}^2)^h\psi\big)(x+h), \tag{3.2.26}$$

where $(\mathcal{D}^2)^h$ is the second order differential operator

$$(\mathcal{D}^2)^h = L_2^h \circ \nabla^2 + L_1^h \circ \nabla + L_0^h. \tag{3.2.27}$$

Hence by (3.2.19), (3.2.26), Lemma 3.2.23(2), and Proposition 3.1.13(4),

$$\begin{aligned}\|\psi^h\|_{H_\ell} &\leq C(\|\mathcal{D}^2\psi^h\|_{H_{\ell-2}} + \|\psi^h\|_{H_{\ell-2}})\\ &\leq C(\|(\mathcal{D}^2\psi)^h\|_{H_{\ell-2}} + \|(\mathcal{D}^2)^h\psi\|_{H_{\ell-2}} + \|\psi\|_{H_{\ell-2}})\\ &\leq C(\|\mathcal{D}^2\psi\|_{H_{\ell-1}} + \|\psi\|_{H_\ell})\\ &\leq C.\end{aligned}$$

Now Lemma 3.2.23(3) implies that $\psi \in H_{\ell+1}$.

Corollary 3.2.28. *For any $\ell \geq 1$, if $\psi \in H_\ell$ and $\mathcal{D}^2\psi \in H_\ell$, then $\psi \in H_{\ell+2}$.*

Proof. Apply Proposition 3.2.25 twice.

The regularity is now an easy induction argument.

Corollary 3.2.29. *The eigenspinor fields $\psi_n \in H_\ell$ for all ℓ, and $\mathcal{D}^2\psi_n = \lambda_n\psi_n$. It follows that ψ_n are smooth.*

Proof. We already know $\psi_n \in H_1$ and $\mathcal{D}^2\psi_n \in H_1$. So Corollary 3.2.28 implies $\psi_n \in H_3$. Now since $\mathcal{D}^2\psi_n = \lambda_n\psi_n \in H_3$, another application of Corollary 3.2.28 yields $\psi_n \in H_5$. Continue by induction. The smoothness follows from the Sobolev Lemma 3.1.21.

A useful consequence of this discussion is

Proposition 3.2.30. *For any $\psi \in H_1$, if $(\mathcal{D}^2)^k\psi \in H_0$ for all k, then ψ is smooth.*

In particular, the hypothesis is satisfied if $\mathcal{D}^2\psi$ is smooth. We leave the proof to the reader. Further consequences appear in the exercises below.

Exercise 3.2.31. Show that $\mathcal{D}^2 \colon H_\ell \to H_{\ell-2}$ is a *Fredholm map*. This means that its range is closed, and its kernel and cokernel are finite dimensional.

Exercise 3.2.32. Prove that $\mathcal{D}^2 \colon L^2 \to L^2$ is an unbounded self-adjoint operator.

Exercise 3.2.33. What can you deduce about the spectrum of $\mathcal{D}$? Prove that $\mathcal{D}$ is Fredholm.

Exercise 3.2.34. Show that all of our results are valid if we replace $\mathcal{D}^2$ by the Laplace operator acting on functions. (Hint: You can see this as a special case by a judicious choice of V. However, it is useful to work through it directly.)

§4 The Heat Equation

Our main tool for studying analytic invariants of Dirac operators is the heat equation. Classically, the heat operator is $e^{-t\triangle}$ for $\triangle$ the Laplace operator on functions. If $f(x)$ is an initial temperature distribution, then $(e^{-t\triangle}f)(x)$ is the temperature distribution after time t. We study "spinor-valued heat," which flows according to the operator $e^{-t\mathcal{D}^2}$. As $t \to \infty$ the heat spreads all over and so reflects global properties of the underlying manifold. In particular, $e^{-t\mathcal{D}^2}$ approaches projection onto the kernel of $\mathcal{D}$. On the other hand, if the support of the initial data f lies in a compact set K, then for small t the solution $e^{-t\mathcal{D}^2}f$ is exponentially small away from K. Thus the heat operator connects global ($t \to \infty$) and local ($t \to 0$). In this chapter we derive the basic properties of heat flow on compact manifolds. In this (preliminary) version of these notes our treatment of the asymptotic expansion of the heat kernel is incomplete. We offer a few excuses. First, the standard argument (essentially due to Minakshisundaram and Pleijel) is nicely presented in [R]. Secondly, we were unable to obtain a satisfactory account at one point (Assertion 4.4.8). Our current understanding involves slightly more advanced concepts than we would like, and we still hope to circumvent some of these in the future.

§4.1 Solution on $\mathbb{E}^n$

Consider first the real line $\mathbb{R}$ with standard coordinate x. The heat equation is a partial differential equation for a real-valued function $u(t,x) = u_t(x)$ of two variables, defined for $t > 0$ and $x \in \mathbb{R}$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{4.1.1}$$

The solution depends on a choice of initial data. Let $\hat{u}_t(\xi)$ be the Fourier transform of u_t:

$$\hat{u}_t(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, u(x)e^{-ix\xi}. \tag{4.1.2}$$

Then (4.1.1) is equivalent to

$$\frac{\partial \hat{u}}{\partial t} + \xi^2 \hat{u} = 0, \tag{4.1.3}$$

and the general solution is

$$\hat{u}_t(\xi) = C(\xi)e^{-t\xi^2}, \tag{4.1.4}$$

where $C(\xi)$ is an arbitrary function. Note

$$C(\xi) = \hat{u}_0(\xi). \tag{4.1.5}$$

Let us assume that $\hat{u}_0(\xi) \equiv 1$. Then by the Fourier inversion formula,

$$u_t(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi \, e^{ix\xi} e^{-t\xi^2} = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}. \tag{4.1.6}$$

This formula is the foundation for all our work. As $t \to 0$ the solution $u_t(x)$ approaches a δ-function at the origin. More precisely, for any smooth function h,

$$\lim_{t\to 0} \int_{\mathbb{R}} u_t(x) h(x) = h(0). \tag{4.1.7}$$

Of course, this follows immediately from $\hat{u}_0(\xi) \equiv 1$.

Exercise 4.1.8. Verify directly that $u_t(x)$ in (4.1.6) solves (4.1.1). What is the solution for arbitrary initial data $f(x)$? In other words, find the solution $\tilde{u}_t(x)$ of (4.1.1) with $\tilde{u}_0(x) = f(x)$.

The extension to Euclidean space $\mathbb{E}^n$ is straightforward. Let $\{x^k\}$ be standard coordinates. Now the Laplacian

$$\triangle = -\sum \frac{\partial^2}{(\partial x^k)^2} \tag{4.1.9}$$

replaces $-\partial^2/\partial x^2$ in (4.1.1). Then

$$u_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \tag{4.1.10}$$

solves the scalar heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \triangle u &= 0 \\ u_0 &= \delta, \end{aligned} \tag{4.1.11}$$

where δ is the δ-function at the origin.

Exercise 4.1.12. Check that (4.1.10) solves (4.1.11). Write down the solution for initial condition δ_y, the δ-function at y.

Exercise 4.1.13. Let $\mathbb{R}^n$ be endowed with an arbitrary inner product and denote the associated norm by $|\cdot|$. Prove that (4.1.10) is a solution to the heat equation (4.1.11), where now the Laplacian $\triangle = -g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$ is defined with respect to the given metric g_{ij}.

Exercise 4.1.14. Consider now the Dirac operator (2.2.2) on Euclidean space. Write down the solution to the heat equation $\partial u/\partial t + \mathcal{D}^2 u = 0$ with u_0 a δ-function at the origin with value $\sigma \in \mathbb{S}$. (Now u is spinor-valued.)

§4.2 Heat Flow on Compact Manifolds

Let X be a compact Riemannian spin n-manifold, $V \to X$ a Hermitian vector bundle with unitary connection. We denote $E = S \otimes V$, where S is the spin bundle over X. The heat equation on X is a partial differential equation for a time-dependent section ψ_t of E:

$$H\psi = \frac{\partial \psi}{\partial t} + \mathcal{D}^2\psi = 0. \tag{4.2.1}$$

As before, we need to specify initial data. Then the solution exists and is unique.

Proposition 4.2.2. *Fix $f \in H_\ell(E) = H_\ell$ for $\ell \geq 0$. Then there is a unique solution ψ_t, $t > 0$, of* (4.2.1) *with $\lim\limits_{t\to 0} \psi_t = f$ in H_ℓ. Furthermore, ψ_t is smooth for $t > 0$, and $\lim\limits_{t\to\infty} \psi_t = P_{\ker \mathcal{D}}(f)$, where $P_{\ker \mathcal{D}}$ is orthogonal projection onto* $\ker \mathcal{D} \subset H_0$*. Finally,*

$$\|\psi_t\| \leq \|f\|. \tag{4.2.3}$$

We will extend Proposition 4.2.2 to distributional initial data (e.g. δ-functions) in the next section.

Proof. Consider first $\ell = 0$. Recall the spectral decomposition $\mathcal{D}^2\psi_n = \lambda_n\psi_n$ from (3.2.14). Define

$$\begin{aligned} e^{-t\mathcal{D}^2} : H_0 &\longrightarrow H_0 \\ \psi_n &\longmapsto e^{-t\lambda_n}\psi_n. \end{aligned} \tag{4.2.4}$$

Since $\{\psi_n\}$ is a complete orthonormal basis of H_0, it is clear that $e^{-t\mathcal{D}^2}$ is a bounded operator of norm ≤ 1. Now set

$$\psi_t = e^{-t\mathcal{D}^2} f. \tag{4.2.5}$$

Then

$$\frac{\partial \psi}{\partial t} = -\mathcal{D}^2 e^{-t\mathcal{D}^2} f = -\mathcal{D}^2\psi, \tag{4.2.6}$$

so that ψ does indeed solve (4.2.1). Furthermore, if $f = \sum\limits_n f_n\psi_n$, with $\sum\limits_n |f_n|^2 < \infty$, then

$$\lim_{t\to 0} \psi_t = \lim_{t\to 0} \sum_n f_n e^{-t\lambda_n}\psi_n = \sum_n f_n\psi_n = f \tag{4.2.7}$$

in H_0. Also,

$$\lim_{t\to\infty}\psi_t = \lim_{t\to\infty}\sum_n f_n e^{-t\lambda_n}\psi_n = \sum_{\lambda_n=0} f_n\psi_n = P_{\ker\mathcal{D}}(f). \tag{4.2.8}$$

As for the inequality,

$$\|\psi_t\|^2 = \sum_n e^{-2t\lambda_n}|f_n|^2 \le \sum_n |f_n|^2 \le \|f\|^2. \tag{4.2.9}$$

The inequality proves uniqueness, since if ψ' is another solution to (4.2.1), then so is $\psi-\psi'$, but with initial condition identically zero. Then (4.2.3) shows $\|\psi_t-\psi'_t\|=0$.

It remains to show that ψ_t is smooth for $t>0$. By Proposition 3.2.30 it suffices to bound $\|(\mathcal{D}^2)^k\psi\|$ for all k. But

$$\|(\mathcal{D}^2)^k\psi\|^2 = \|(\mathcal{D}^2)^k e^{-t\mathcal{D}^2}f\|^2 = \sum(\lambda_n)^{2k}e^{-2t\lambda_n}|f_n|^2. \tag{4.2.10}$$

Since $\lambda\mapsto\lambda^{2k}e^{-2t\lambda}$ is uniformly bounded in λ for any fixed $t>0$ ("exponentials beat polynomials"), we conclude

$$\|(\mathcal{D}^2)^k\psi\| \le C_k\|f\|^2, \tag{4.2.11}$$

as desired. Notice that this proves

$$e^{-t\mathcal{D}^2}\colon H_0 \longrightarrow H_\ell \tag{4.2.12}$$

is a bounded linear map for $t>0$ and all ℓ.

For initial data $f\in H_\ell$, $\ell>0$, we have only to show $\lim_{t\to 0}\psi_t = f$ in H_ℓ. But by (4.2.12) we know $e^{-t\mathcal{D}^2}\colon H_\ell\to H_\ell$ is bounded for $t\ge 0$, whence as $t\to 0$ we have $e^{-t\mathcal{D}^2}f\to f$ in H_ℓ (cf. Exercise 4.2.15 and Exercise 4.2.23).

Several remarks are in order. First, the Sobolev Lemma 3.1.21 (Sobolev) implies that

$$e^{-t\mathcal{D}^2}\colon H_0 \longrightarrow C^\infty, \quad t>0 \tag{4.2.13}$$

is bounded. Thus we call $e^{-t\mathcal{D}^2}$ a *smoothing operator*. Next, we have $\lim_{t\to\infty}e^{-t\mathcal{D}^2} = P_{\ker\mathcal{D}}$ in the operator norm topology on H_0. But $\lim_{t\to 0}e^{-t\mathcal{D}^2} = \mathrm{id}$ only holds in the strong topology. For smooth initial data this is true in all H_ℓ, $\ell\ge 0$, so by Sobolev we obtain

Corollary 4.2.14. *If f is smooth, then $\lim_{t\to 0}\psi_t = f$ in the C^∞ topology.*

EXERCISE 4.2.15. Justify the interchange of lim and $\sum$ in (4.2.7) and (4.2.8). Prove that $\lim_{t\to\infty} e^{-t\mathcal{D}^2}$ exists in the norm topology viewing $e^{-t\mathcal{D}^2}$ as an operator $H_0 \to H_0$. What about $e^{-t\mathcal{D}^2}\colon H_0 \to H_\ell$? Why doesn't $\lim_{t\to 0} e^{-t\mathcal{D}^2}$ exist in the norm topology? Prove in detail that $\lim_{t\to 0} e^{-t\mathcal{D}^2} = \mathrm{id}$ in the strong operator topology (for any ℓ).

EXERCISE 4.2.16. Let $F\colon [0,\infty) \to \mathbb{R}$ and define $F(\mathcal{D})$ by $F(\mathcal{D})\psi_n = F(\lambda_n)\psi_n$. Prove that $F(\mathcal{D})$ is smoothing if F is a *Schwartz function*, i.e., a smooth function which decreases more rapidly than the inverse of any polynomial.

EXERCISE 4.2.17. Suppose that y is a parameter on which the metric on X, metric on V, and connection on V depend smoothly. Then the Dirac operator $\mathcal{D}_y$ also varies smoothly. Show that if the initial condition f_y varies smoothly (in some H_ℓ), then so too does the solution ψ_t^y to the heat equation vary smoothly (in C^∞). Is this statement uniform in t as $t \to 0$?

Next we derive a basic estimate for solutions to the heat equation. We will use it to estimate approximate solutions.

Proposition 4.2.18. *Suppose ψ_t is any smooth spinor field depending smoothly on t for $t \ge 0$. Let $H = \frac{\partial}{\partial t} + \mathcal{D}^2$ be the heat operator. Then there exist constants C_ℓ (independent of ψ) such that for all $t \ge 0$,*

$$\|\psi_t\|_{H_\ell} \le C_\ell\left(\int_0^t ds\,\|(H\psi)_s\|_{H_\ell} \;+\; \|\psi_0\|_{H_\ell}\right). \tag{4.2.19}$$

Here $H\psi$ is a time-varying spinor field whose value at time s is $(H\psi)_s$. Notice that (4.2.3) is a special case of (4.2.19).

Proof. We claim that

$$\psi_t = \int_0^t ds\, e^{-(s-t)\mathcal{D}^2}(H\psi)_s \;+\; e^{-t\mathcal{D}^2}\psi_0. \tag{4.2.20}$$

To see this denote the right hand side of (4.2.20) by φ_t. Then

$$\frac{\partial\varphi}{\partial t} = -\mathcal{D}^2\varphi + H\psi. \tag{4.2.21}$$

Hence $\psi - \varphi$ is annihilated by H and vanishes at $t = 0$. Now the uniqueness statement in Proposition 4.2.2 implies (4.2.20).

The estimate (4.2.19) follows once we bound the operator norm of $e^{-t\mathcal{D}^2}\colon H_\ell \to H_\ell$ uniformly in t. For $\ell = 0$ we have $\|e^{-t\mathcal{D}^2}\| \le 1$, since the eigenvalues relative to an orthonormal basis are ≤ 1. For higher ℓ we employ the elliptic estimate (3.2.21) using the fact that $\mathcal{D}$ and $e^{-t\mathcal{D}^2}$ commute:

$$\begin{aligned} \|e^{-t\mathcal{D}^2} f\|_{H_\ell} &\le C(\|\mathcal{D}^\ell e^{-t\mathcal{D}^2} f\| + \|f\|) \\ &\le C(\|e^{-t\mathcal{D}^2}\mathcal{D}^\ell f\| + \|f\|) \\ &\le C(\|\mathcal{D}^\ell f\| + \|f\|) \\ &\le C\|f\|_{H_\ell}. \end{aligned} \tag{4.2.22}$$

Exercise 4.2.23. Prove that the map $t \mapsto e^{-t\mathcal{D}^2}$ is continuous in the operator norm topology on H_ℓ. (Hint: Derive the identity

$$e^{-P_2} - e^{-P_1} = -\int_0^1 ds\, e^{-sP_1}(P_2 - P_1)e^{-(1-s)P_2}, \tag{4.2.24}$$

and apply it to $P_1 = t\mathcal{D}^2$, $P_2 = t'\mathcal{D}^2$.) Observe that $\|e^{-t\mathcal{D}^2}\|$ is finite at $t = 0$ and $t \to \infty$. Conclude that $\|e^{-t\mathcal{D}^2}\|$ is uniformly bounded in t. This is an alternative proof of (4.2.22).

§4.3 The Heat Kernel

As a preliminary to considering distributional initial data, we define Sobolev spaces H_ℓ for negative values of ℓ. For $\ell \ge 0$ let $H_{-\ell}$ be the antidual space to H_ℓ relative to the H_0 pairing $\langle\cdot,\cdot\rangle$. More explicitly, the $H_{-\ell}$ norm of a smooth function f is the best constant C in the inequality $\langle f, \psi\rangle \le C\|\psi\|_{H_\ell}$. Then $H_{-\ell}$ is the completion of C^∞ relative to this norm (cf. Exercise 3.2.2).

Exercise 4.3.1. Define the Hilbert space structure on $H_{-\ell}$.

Exercise 4.3.2. Show that $\psi = \sum a_n\psi_n \in H_\ell$ (for any $\ell \in \mathbb{Z}$) if and only if $\sum(1+\lambda_n^2)^\ell |a_n|^2 < \infty$.

Exercise 4.3.3. Prove that $\bigcup_{\ell\in\mathbb{Z}} H_\ell$ is the (anti-) dual to C^∞, the space of distributions on X (with values in E).

Exercise 4.3.4. Prove that $\mathcal{D}^2$ extends to a bounded operator $H_\ell \to H_{\ell-2}$ for any $\ell \in \mathbb{Z}$.

The δ-function lives in H_ℓ for ℓ sufficiently small.

Proposition 4.3.5. *Fix $y \in X$ and $\sigma \in E_y$, and define a functional*

$$\delta_\sigma(\psi) = \big(\sigma, \psi(y)\big). \tag{4.3.6}$$

Then $\delta_\sigma \in H_{-\ell}$ *for* $\ell > n/2$*, where* $n = \dim X$*.*

Proof. $|\delta_\sigma(\psi)| \le |\sigma|\,|\psi(y)| \le C\|\psi\|_{H_\ell}$ for $\ell > n/2$, by the Sobolev embedding theorem.

By duality we extend any operator $A_\ell\colon H_\ell \to H_\ell$, $\ell \ge 0$, to $A_{-\ell}\colon H_{-\ell} \to H_{-\ell}$; simply define $A_{-\ell}$ to be the dual of A_ℓ^* relative to the H_0 pairing. Thus $\langle A_{-\ell}\theta, \psi\rangle = \langle \theta, A_\ell^*\psi\rangle$ for $\theta \in H_{-\ell}$, $\psi \in H_\ell$. In particular, we extend the heat operator to $e^{-t\mathcal{D}^2}\colon H_{-\ell} \to H_{-\ell}$.

Proposition 4.3.7. *For* $t > 0$ *the heat operator* $e^{-t\mathcal{D}^2}\colon H_{-\ell} \to C^\infty$ *is smoothing.*

Proof. As in Proposition 4.2.2 and (4.2.13).

Now we construct the *heat kernel* as the heat flow with initial data a δ-function.

Proposition 4.3.8. *For* $\sigma \in E_y$ *let* $k_t^{(\sigma)} = e^{-t\mathcal{D}^2}(\delta_\sigma)$*. Then* $\sigma \mapsto k_t^{(\sigma)}(x)$ *determines a linear map*

$$p_t(x,y)\colon E_y \longrightarrow E_x \tag{4.3.9}$$

which is smooth in x*,* y*, and* $t > 0$*. Also,*

$$p_t(y,x) = p_t(x,y)^*, \tag{4.3.10}$$

and

$$\lim_{t\to 0} p_t(x,y) = \delta_{x=y}\cdot \mathrm{id}_{E_x} \tag{4.3.11}$$

is the $\mathrm{Hom}(E_y,E_x)$*-valued distribution supported on the diagonal* $x = y$ *as indicated. Here the limit is taken in the* $H_{-\ell}$ *topology for* $\ell > n/2$ *and* y *fixed. Finally, for any smooth spinor field* f*,*

$$\psi_t(x) = \int_X dy\, p_t(x,y)f(y) \tag{4.3.12}$$

is the unique solution to the heat equation with initial condition f*.*

In (4.3.12) the integrand is a map $X \to E_x$, so the integration is well-defined.

Proof. By Proposition 4.3.7, $k_t^\sigma(x) = p_t(x,y)\sigma$ is smooth in x. To see the smoothness in $t > 0$, differentiate the heat equation: $\dfrac{\partial^i k_t^\sigma(x)}{\partial t^i} = (-1)^i(\mathcal{D}^2)^i k_t^\sigma(x)$. Since $e^{-t\mathcal{D}^2}$ is formally self-adjoint, for any $\sigma \in E_y$ and $\tau \in E_x$

$$\begin{aligned} \langle e^{-t\mathcal{D}^2}\delta_\sigma, \delta_\tau\rangle &= \langle \delta_\sigma, e^{-t\mathcal{D}^2}\delta_\tau\rangle, \\ (p_t(x,y)\sigma, \tau) &= (\sigma, p_t(x,y)\tau), \end{aligned} \tag{4.3.13}$$

which is (4.3.10). The smoothness of $p_t(x,y)$ in y follows immediately. Next, any smooth spinor field f can be written as the distribution

$$f = \int_X dy\, \delta_{f(y)},$$

since then

$$\langle f, \psi\rangle = \int_X dy\, \delta_{f(y)}(\psi) = \int_X dy\, \big(f(y), \psi(y)\big).$$

Therefore,

$$e^{-t\mathcal{D}^2} f = \int_X dy\, e^{-t\mathcal{D}^2}\delta_{f(y)} = \int_X dy\, p_t(x,y) f(y),$$

proving (4.3.12). Equation (4.3.11) follows from the fact that $\lim\limits_{t\to 0} e^{-t\mathcal{D}^2}\delta_\sigma = \delta_\sigma$ in $H_{-\ell}$, $\quad \ell > n/2$.

Exercise 4.3.14. Extend Exercise 4.2.17 to show that the heat kernel varies smoothly in a parameter.

In the remainder of this chapter we study the behavior of $p_t(x,y)$ as $t \to 0$ in the C^∞ topology. For $x \neq y$ the heat kernel approaches zero exponentially fast. We show this next, following the presentation of [R,§5]. The main idea is to express the heat kernel in terms of the fundamental solution of the wave equation.

The *wave equation* for spinor fields is

$$\frac{\partial \varphi}{\partial t} = i\mathcal{D}\varphi. \tag{4.3.15}$$

As for the heat equation we need to specify initial conditions

$$\varphi_0 = f. \tag{4.3.16}$$

Then formally the solution to the wave equation is $\varphi_t = e^{it\mathcal{D}}f$. The wave exists for all time, and the heat operator can be expressed in terms of the wave operator.

Proposition 4.3.17. *There is a bounded operator $e^{it\mathcal{D}}\colon H_\ell \to H_\ell$ for all t, ℓ. Hence* (4.3.15) *has a unique solution $\varphi_t = e^{it\mathcal{D}}f \in H_\ell$ for initial data $f \in H_\ell$. It satisfies*

$$\|\varphi_t\| = \|f\| \tag{4.3.18}$$

if $f \in L^2$. The solution φ_t is smooth if f is. Finally,

$$e^{-t\mathcal{D}^2} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} du\, e^{-u^2/4t} e^{iu\mathcal{D}}. \tag{4.3.19}$$

Exercise 4.3.20. Prove Proposition 4.3.17. The proof is very similar to the proof of Proposition 4.2.2. (Hint: Verify (4.3.19) on each eigenspace separately. For which topologies on the space of operators is (4.3.19) valid?)

The wave equation has the crucial property that its solutions propagate with unit speed.

Proposition 4.3.21. *Let $K \subset X$ be compact and suppose* $\operatorname{supp} f \subset K$*. Let φ_t be the unique solution to the wave equation* (4.3.15) *with initial condition f. Then for sufficiently small*[20] *t we have* $\operatorname{supp}\varphi_t \subseteq B_{|t|}(K) = \{x \in X\colon\ \operatorname{dist}(x,K) \le |t|\}$.

Proof. Consider first an arbitrary *smooth* solution φ_t to (4.3.15) (with arbitrary smooth initial data). We estimate the energy $\int |\varphi_t|^2$ in a ball. Fix any $x \in X$ and sufficiently small $R > 0$. Let ω be the 1-form $\omega(\xi) = (\xi \cdot \varphi, \varphi)$. Then

$$\tag{4.3.22} \begin{aligned} \frac{d}{dt}\int_{B_R(x)} dy\,|\varphi(y)|^2 &= i\int_{B_R(x)} dy\,(\mathcal{D}\varphi(y),\varphi(y)) - (\varphi(y),\mathcal{D}\varphi(y)) \\ &= -i\int_{B_R(x)} dy\,d^*\omega \\ &= i\int_{S_R(x)} dy\,\omega(\nu) \\ &= i\int_{S_R(x)} dy\,(\nu\cdot\varphi(y),\varphi(y)), \end{aligned}$$

where $S_R(x) = \partial B_R(x)$ is the sphere of radius R with unit normal ν, and we have used the identity (4.3.26) below (see Exercise 4.3.25) in the second inequality. It follows that

$$\tag{4.3.23} \left|\frac{d}{dt}\int_{B_{R-|t|}(x)} dy\,|\varphi(y)|^2\right| = \left|\int_{S_{R-|t|}(x)} dy\,(i\nu\cdot\varphi(y),\varphi(y)) - (\varphi(y),\varphi(y))\right| \le 0$$

by Cauchy-Schwartz. This is a general estimate for solutions to the wave equation.

Returning to the hypotheses of the proposition, assume first that the initial spinor field f is smooth. Fix $x \notin K$ and choose $R < \operatorname{dist}(x,K)$. Then $\int_{B_R(x)} |\varphi_0|^2 = 0$ by hypothesis. Hence (4.3.23) implies $\int_{B_{R-|t|}(x)} |\varphi_t|^2 = 0$ for $|t| < R$, which proves the proposition for smooth f. Initial data $f \in H_\ell$ can be approximated by smooth data, and the proposition follows from the smooth case.

Exercise 4.3.24. Write out the details of this approximation argument.

Exercise 4.3.25. Let φ, ψ be smooth spinor fields. Define a 1-form $\omega(\xi) = (\xi\cdot\varphi,\psi)$, where $\xi\cdot\varphi$ is the Clifford product of the vector ξ and spinor φ (using the identification of vectors and covectors

[20] Here 'sufficiently small' depends on the injectivity radius of X.

via the metric.) Prove

$$d^*\omega = (\varphi, \mathcal{D}\psi) - (\mathcal{D}\varphi, \psi). \tag{4.3.26}$$

(Hint: Compute on the frame bundle, writing $\omega = \sum_k (\gamma^k \psi, \varphi) e^k$ and $d^* = -\iota(e^k)\partial_k$.)

Now we show that $p_t(x,y) \to 0$ as $t \to 0$ away from the diagonal (where x and y coincide).

Proposition 4.3.27. *Fix $d > 0$ (but d less than the injectivity radius of X). Then there exist constants C, c such that*

$$\|p_t(x,y)\|_{C^k} \le Ce^{-c/t} \tag{4.3.28}$$

for $\operatorname{dist}(x,y) > d$. *In fact, any $c < d^2/4$ works, and then C depends on c and k.*

Proof. Fix $\sigma \in E_y$ of unit norm, and set $k_t(x) = p_t(x,y)\sigma$. By (4.3.19) and Proposition 4.3.21 we can write

$$k_t(x) = \frac{1}{\sqrt{4\pi t}} \int_{|u|>d} du\, e^{-u^2/4t} (e^{iu\mathcal{D}} \delta_\sigma)(x). \tag{4.3.29}$$

Fix $c < d^2/4$. Then

$$k_t(x) = e^{-c/t} \int_{|u|>d} du\, \frac{1}{\sqrt{4\pi t}} e^{-(u^2/4-c)/t} (e^{iu\mathcal{D}} \delta_\sigma)(x). \tag{4.3.30}$$

We need to bound the C^k norm of the integral (uniformly in t); by Sobolev it suffices to bound the H_ℓ norms, and by the elliptic theory these are bounded by the H_0 norms of $\mathcal{D}^k$ applied to the integral. Rather than estimate the operator applied to δ_σ, we estimate the H_0 operator norm of

$$\psi \longmapsto \frac{1}{\sqrt{4\pi t}} \int_{|u|>d} du\, e^{-(u^2/4-c)/t} \mathcal{D}^k e^{iu\mathcal{D}} \psi. \tag{4.3.31}$$

Since there is a basis of eigenfunctions we replace $\mathcal{D}$ by an eigenvalue μ, and then

$$\begin{aligned} \frac{1}{\sqrt{4\pi t}} \int_{|u|>d} du\, e^{-(u^2/4-c)/t} \mu^k e^{iu\mu} &\le \frac{Ce^{-\epsilon/t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} du\, e^{-u^2/4t} \mu^k e^{iu\mu} \\ &= Ce^{-\epsilon/t} e^{-t\mu^2} \mu^k \\ &\le C \end{aligned} \tag{4.3.32}$$

uniformly in $t > 0$ and $\mu \ge 0$. This is the required estimate.

§4.4 The Asymptotic Expansion

The small time behavior of the heat kernel $p_t(x,y)$ for x near y depends on the local geometry of X near y. This is made precise by an *asymptotic expansion* for $p_t(x,y)$. In this section we indicate a derivation of this asymptotic expansion along the diagonal $x=y$. As noted above, our account in these preliminary notes omits a crucial step in the proof. Still, we think the geometric ideas sufficiently clear to warrant presentation at this time. An exposition of the Minakshisundaram-Pleijel approach is given in [R,§5].

The result we are after is the following.

Theorem 4.4.1. *There exists an asymptotic expansion*

$$p_t(y,y) \sim (4\pi t)^{-n/2}\sum_{j=0}^{\infty} A_j(y)t^j \quad \textit{as } t\to 0, \tag{4.4.2}$$

where A_j are smooth sections of $\operatorname{End}E$ *whose values at y depend only on the infinite jet of the geometry (metrics and connections) at y. Furthermore, $A_0\equiv 1$ and the expansion is uniform in y.*

More precisely, for any N

$$\left| p_t(y,y) - (4\pi t)^{-n/2}\sum_{j=0}^{N} A_j(y)t^j\right| \le C_N t^{N-n/2+1/2} \tag{4.4.3}$$

for small t. Our method will also give asymptotic expansions for all derivatives $\partial_x^\alpha p_t(x,y)\big|_{x=y}$ restricted to the diagonal, but we will not need this. (The standard result of Minakshisundaram and Pleijel is stronger; it is an asymptotic expansion for $p_t(x,y)$ in a full neighborhood of the diagonal.)

Fix $y\in X$. We construct a deformation of X with a single real parameter ϵ, call it $\mathbf{X}\to\mathbb{R}$, such that the fiber $\mathbf{X}_\epsilon$ is diffeomorphic to X for $\epsilon\ne 0$ and $\mathbf{X}_0$ is identified with T_yX. Intuitively, we blow up the geometry near y as $\epsilon\to 0$. This is very much like the "deformation to the normal cone" construction used in the modern algebro-geometric proof of the Grothendieck-Riemann-Roch Theorem [F,§5,§15].[21]

Let

$$\exp\colon B_r\to U\subset X \tag{4.4.4}$$

be the restriction of the exponential map at y to the ball of radius r in T_yX. We choose r so that (4.4.4) is a diffeomorphism. Denote by Σ_y denote the one point compactification of T_yX. Then

[21] I owe this observation to Bill Fulton and Spencer Bloch.

$\exp^{-1}$ extends to a (degree one) map $F\colon X \to \Sigma_y$. We assume that the extension maps $X \setminus U$ into $\Sigma_y \setminus B_r$. Think of Σ_y as $T_yX \cup \{\infty\}$ so that scalar multiplication makes sense as a map $\Sigma_y \to \Sigma_y$, though as usual we leave $0 \cdot \infty$ undefined. Let

$$\mathbf{X} \subset (\mathbb{R} \times \Sigma_y \times X) \setminus (\{0\} \times \{\infty\} \times X) = \{\langle \epsilon, a, x\rangle \neq \langle 0, \infty, x\rangle\}$$

be the solution set of the equation

$$F(x) = \epsilon a. \tag{4.4.5}$$

It is easy to verify that $\mathbf{X}$ is a smooth manifold of dimension $n+1$ and that the projection $\mathbf{X} \to \mathbb{R}$ onto the first factor is a submersion. Also, the fibers $\mathbf{X}_\epsilon$ of this projection are diffeomorphic to X for $\epsilon \neq 0$ and $\mathbf{X}_0 = \{\langle 0, a, y\rangle \colon a \neq \infty\}$. The identification $\mathbf{X}_0 \cong T_yX$ is quite natural. For if $a \in T_yX$ then (4.4.5) determines a curve $x_\epsilon = F^{-1}(\epsilon a) = \exp(\epsilon a)$ for small ϵ. Since $d\exp_0$ is the identity map (A.2), the tangent to this curve at 0 is equal to a. It is also quite natural to identify $\mathbf{X}_1$ with X.

Covariant geometric data on X lifts to the deformation $\mathbf{X}$ as follows. Let $i\colon \mathbf{X} \hookrightarrow \mathbb{R} \times \Sigma_y \times X$ denote the inclusion and $\pi_3\colon \mathbb{R} \times \Sigma_y \times X \to X$ projection onto the third factor. Then $i^*\pi_3^*$ lifts data from X to $\mathbf{X}$. For example, the vector bundle $i^*\pi_3^*(TX)$ is the tangent bundle along the fibers of $\mathbf{X} \to \mathbb{R}$. In particular, it carries a Riemannian metric. This gives a Riemannian structure on $\mathbf{X}_\epsilon$. The vector bundle $\mathbf{V} = i^*\pi_3^*(V)$ is a deformation of $V \to X$ to the trivial bundle $T_yX \times V_y \to T_yX$. (Here V_y is the fiber of V at y.) Also, $\mathbf{V}$ inherits a metric and connection from V. It is clear from the construction that all of the lifted geometric data on $\mathbf{X}$ is smooth. Recall that $E = S \otimes V$ is the bundle of V-valued spinors. Denote $\mathbf{E} = i^*\pi_3^*(E)$.

We next explain the sense in which the deformed geometry encodes scaling near y. For $\epsilon \neq 0$ let

$$\begin{aligned} T_\epsilon\colon T_yX &\longrightarrow T_yX \\ a &\longmapsto \epsilon a \end{aligned} \tag{4.4.6}$$

be the scaling map. Then according to (4.4.4) the map

$$\exp \circ T_\epsilon\colon B_{r/|\epsilon|} \longrightarrow U \tag{4.4.7}$$

is a diffeomorphism of the ball of radius $r/|\epsilon|$ in T_yX onto the neighborhood U of y in X. By (4.4.5) we can identify either "side" of (4.4.7) with a neighborhood of $\langle \epsilon, 0, y\rangle$ in $\mathbf{X}_\epsilon$. As described in the preceding paragraph, geometric data on X restricts to U and then pulls back via (4.4.7) to give geometric data on $B_{r/|\epsilon|}$. At $\epsilon = 1$ we have the original data on U in exponential coordinates. For

$\epsilon \neq 0$ the data on $B_{r/|\epsilon|}$ is the pullback of this original data on B_r by the scaling operator T_ϵ^*. Finally, at $\epsilon = 0$ we obtain the original (constant) data on T_yX. This applies to the bundle V together with its metric and connection, which lift via exponential coordinates to a bundle over B_r. Applying T_ϵ^* we obtain a bundle (with metric and connection) over $B_{r/|\epsilon|}$, and in the limit $\epsilon \to 0$ this becomes the trivial bundle $T_yX \times V_y \to T_yX$ with constant metric and trivial connection. We must be careful with the intrinsic Riemannian metric. For if $g = g_{k\ell}(a)da^k da^\ell$ is the original metric in exponential coordinates, then the metric at ϵ is $g_{k\ell}(\epsilon a)da^k da^\ell$, which is $\epsilon^2 T_\epsilon^*(g)$. We do not transform the da^k via T_ϵ^*.

Now we have a smooth family of manifolds (but *not* a fiber bundle!) with smoothly varying geometric data. Hence the Dirac operators on $\mathbf{X}_\epsilon$ have smoothly varying coefficients. Furthermore, the heat kernel exists on the compact manifold $\mathbf{X}_\epsilon$ ($\epsilon \neq 0$) by Proposition 4.3.8 and on $\mathbf{X}_0 \cong T_yX$ by the explicit formula (4.1.10) (cf. Exercise 4.1.14). We specify initial data as follows. The definition of $\mathbf{E}$ implies that $\mathbf{E}_{\langle \epsilon,0,y\rangle} \cong E_y$ for any ϵ. Fix $\sigma \in E_y$ and let $k_t^\epsilon(\mathbf{x})$ be the solution to the heat equation in $\mathbf{X}_\epsilon$ with initial data δ_σ. Thus $k_t^\epsilon(\mathbf{x}) \in \mathbf{E}_{\langle \epsilon,\mathbf{x}\rangle}$ and $\mathbf{x} = \langle a, x\rangle$ satisfies $F(x) = \epsilon a$. By our previous work we know that $k_t^\epsilon(\mathbf{x})$ is smooth in $\mathbf{x}$ and t for fixed ϵ.

Assertion 4.4.8. *$k_t^\epsilon(\mathbf{x})$ is uniformly smooth in $\epsilon, \mathbf{x}$ for t bounded away from 0.*

This is the statement that the solution to the heat equation depends smoothly on parameters. We leave Assertion 4.4.8 unproved in this preliminary version.[22] For a family of compact manifolds (so for $\epsilon \neq 0$) the smooth dependence on parameters was stated in Exercise 4.2.17, and is fairly easy to prove. The difficulty here is the noncompactness of $\mathbf{X}_0$.

Accepting Assertion 4.4.8 we prove the asymptotic expansion. As a preliminary we give

EXERCISE 4.4.9. Suppose ψ is a section of E and ρ a function. Prove

$$\mathcal{D}^2(\rho\psi) = c(\nabla^2\rho)\psi - 2(d\rho, \nabla\psi) + \rho\mathcal{D}^2\psi. \tag{4.4.10}$$

(Hint: Compute on the frame bundle, as in Exercise 2.2.26.)

The diffeomorphism (4.4.7) identifies a neighborhood of $\langle \epsilon, 0, y\rangle$ in $\mathbf{X}_\epsilon$ with $B_{r/|\epsilon|} \subset T_yX$. Parallel transport in V along radial geodesics lifts (4.4.7) to an identification of $B_{r/|\epsilon|} \times E_y$ with $E\big|_U$. In these exponential coordinates the heat flow in $\mathbf{X}_\epsilon$ appears as $k_t^\epsilon(a) \in E_y$ for $a \in B_{r/|\epsilon|}$. The following lemma relates the heat kernel in U to a scaled heat kernel for the scaled geometry at ϵ.

Lemma 4.4.11. *Fix $r' < r$. Then there exist constants C, c independent of $\epsilon \neq 0$ such that for $|a| < r'$*

$$\Big| k_t^1(a) - |\epsilon|^{-n} k_{t/\epsilon^2}^\epsilon(a/\epsilon) \Big| \le C|\epsilon|^{-n} e^{-c/t}. \tag{4.4.12}$$

[22] One approach, kindly suggested by Richard Melrose, is to express the heat kernel in terms of the wave kernel and prove smooth dependence on parameters for the wave kernel instead. The advantage of the wave equation is that its fundamental solution propagates with finite speed (Proposition 4.3.21), so is easier to control. We haven't yet worked out a satisfactory account.

In fact, we can choose $c = {r'}^2/4$.

The estimate (4.4.12) extends to all derivatives of k.

Proof. Set $\delta = (r - r')/2$. Fix a C^∞ cutoff function $\pi \colon B_r \to [0,1]$ satisfying $\rho(a) = 1$ for $|a| \le r'$ and $\rho(a) = 0$ for $|a| \ge r - \delta$. Let

$$\psi_t(x) = \begin{cases} k^1_t(a) - \rho(a)|\epsilon|^{-n} k^\epsilon_{t/\epsilon^2}(a/\epsilon), & a \in B_r \ \ (x \in U), \\ k^1_t(x), & x \in X \setminus U \end{cases} \tag{4.4.13}$$

define a smooth time-varying section of E over X. We will employ Proposition 4.2.18 and the Sobolev Lemma 3.1.21 to estimate $|\psi|$. As usual, $H = \frac{\partial}{\partial t} + \mathcal{D}^2$ denotes the heat operator on X. First we estimate $H\psi$. Now $H(k^1) \equiv 0$ by definition. Since the geometry in $B_{r/|\epsilon|}$ is T^*_ϵ applied to the geometry at $\epsilon = 1$, except for the Riemannian metric, which is the pullback metric times ϵ^2, the Dirac operator at ϵ is $|\epsilon| T^*_\epsilon \mathcal{D} (T^*_\epsilon)^{-1}$. Hence the heat operator at ϵ is $H_\epsilon = \frac{\partial}{\partial t} + \epsilon^2 T^*_\epsilon \mathcal{D}^2 (T^*_\epsilon)^{-1}$. Now

$$\begin{aligned} 0 &= H_\epsilon(k^\epsilon_t) \\ &= \frac{\partial k^\epsilon_t}{\partial t} + \epsilon^2 T^*_\epsilon \mathcal{D}^2 (T^*_{1/\epsilon} k^\epsilon_t) \\ &= \frac{1}{\epsilon^2} \frac{\partial (T^*_{1/\epsilon} k^\epsilon_t)}{\partial t} + \mathcal{D}^2 (T^*_{1/\epsilon} k^\epsilon_t) \\ &= \frac{\partial (T^*_{1/\epsilon} k^\epsilon_{t/\epsilon^2})}{\partial t} + \mathcal{D}^2 (T^*_{1/\epsilon} k^\epsilon_{t/\epsilon^2}), \end{aligned} \tag{4.4.14}$$

i.e., $k^\epsilon_{t/\epsilon^2}(a/\epsilon)$ is annihilated by H. Thus the only contribution to $H\psi$ comes from $H(\rho(a)|\epsilon|^{-n} k^\epsilon_{t/\epsilon^2}(a/\epsilon))$ for $r - \delta < |a| < r$. Computing with equation (4.4.10) we derive

$$H\psi = |\epsilon|^{-n} \Big\{ B\big(\nabla^2 \rho, k^\epsilon_{t/\epsilon^2}(a/\epsilon)\big) + B\big(\nabla\rho, \nabla k^\epsilon_{t/\epsilon^2}(a/\epsilon)\big) \Big\}, \tag{4.4.15}$$

where $B(\cdot,\cdot)$ are certain bilinear expressions. The Sobolev norms of $\nabla\rho$ and $\nabla^2\rho$ are constants, and the norms of k and its derivatives decay exponentially by Proposition 4.3.27:

$$\begin{aligned} \|k^\epsilon_{t/\epsilon^2}(a/\epsilon)\|_{C^k} &\le C e^{-({r'}^2/4\epsilon^2)\big/(t/\epsilon^2)}, \\ &= C e^{-{r'}^2/4t} \end{aligned} \tag{4.4.16}$$

for $|a| > r - \delta = r' + \delta$. Combining (4.4.15) and (4.4.16),

$$\|(H\psi)_t\|_{H_\ell} \le C|\epsilon|^{-n} e^{-{r'}^2/4t}. \tag{4.4.17}$$

Next we show that $\lim_{t\to 0}\psi_t = 0$ in C^∞. First, by definition $\lim_{t\to 0} k^\epsilon_t(a) = \delta_\sigma$ in $H_{-\ell}$, $\ell > n/2$. This is equivalent to

$$\lim_{t\to 0}\int \bigl(k^\epsilon_t(a),\psi(a)\bigr)\,da = \bigl(\sigma,\psi(0)\bigr).$$

The change of variables $a \to a/\epsilon$ shows that $\lim_{t\to 0}|\epsilon|^{-n}k^\epsilon_t(a/\epsilon) = \delta_\sigma$ in $H_{-\ell}$, whence $\lim_{t\to 0}\psi_t = 0$ in $H_{-\ell}$. Now set (cf. (4.2.20))

$$\varphi_t = -\int_0^t ds\, e^{-(s-t)\mathcal{D}^2}(H\psi)_s.$$

Then φ_t is smooth, $\lim_{t\to 0}\varphi_t \equiv 0$ in C^∞, and $H(\psi+\varphi) = 0$. Since $\lim_{t\to 0}(\psi_t+\varphi_t) = 0$ in $H_{-\ell}$, and solutions to the heat equation are unique, we conclude that $\varphi = \psi$. This proves that $\lim_{t\to 0}\psi_t \equiv 0$ in C^∞.

Finally, from (4.4.17) and the basic estimate (4.2.19) we obtain

$$\|\psi_t\|_{H_\ell} \le C|\epsilon|^{-n}e^{-r'^2/4t},$$

which implies the desired result by the Sobolev embedding theorem.

By the smooth dependence on ϵ near $\epsilon = 0$ for $t = 1$ and $a = 0$ (Assertion 4.4.8) there is a Taylor series[23]

$$k^\epsilon_1(0) = \sum_{i=0}^M b_i\epsilon^i + O(|\epsilon|^{M+1}), \tag{4.4.18}$$

with $b_i \in E_y$. Also, $b_0 = (4\pi)^{-n/2}$ by (4.1.10). Two applications of (4.4.12) show that for $\epsilon > 0$

$$|k^\epsilon_1(0) - k^{-\epsilon}_1(0)| \le Ce^{-c/\epsilon^2}. \tag{4.4.19}$$

So up to an exponentially small error, $k^\epsilon_1(0)$ is an even function of ϵ. Therefore, $b_{2i+1} = 0$ in the Taylor series (4.4.18). Again by (4.4.12) we have

$$k^1_t(0) = |\epsilon|^{-n}k^\epsilon_{t/\epsilon^2}(0) + O(|\epsilon|^{-n}e^{-c/t})$$

[23]Here we could differentiate before setting $a = 0$, and so obtain asymptotic expansions for derivatives of the heat kernel along the diagonal.

for any ϵ. Choose $\epsilon = t^{1/2}$. Then using (4.4.18)

$$
\begin{aligned}
k_t^1(0) &= t^{-n/2}\sum_{j=0}^{N} b_{2j}t^j + O(t^{N-n/2+1/2}) + O(t^{-n/2}e^{-c/t}) \\
&= (4\pi t)^{-n/2}\sum_{j=0}^{N} a_j t^j + O(t^{N-n/2+1/2})
\end{aligned} \tag{4.4.20}
$$

with $a_0 = 1$. Define $A_j \in \operatorname{End} E_y$ by $A_j(\sigma) = a_j$. Then (4.4.20) determines the asymptotic expansion (4.4.2).

It should be clear from the argument that the A_j depend only on the infinite jet of the geometry at y.

Exercise 4.4.21. Write out a detailed proof of this last assertion. In other words, show that if X' is another compact manifold endowed with a connected Hermitian bundle $V' \to X'$; $y \in X'$; $\phi\colon U \to U'$ an isometry from a neighborhood of y to a neighborhood of y'; and $\tilde{\phi}\colon V\big|_U \to V'\big|_{U'}$ a lift which preserves the metric and connection, then the asymptotic expansion for the heat kernel of X at y agrees with the asymptotic expansion for the heat kernel of X' at y'.

To see the smooth dependence on y, introduce y as a variable in the problem. Thus instead of blowing up a neighborhood of y in X, blow up a neighborhood of the diagonal Δ in $X\times X$. Denote a typical point of $X\times X$ by $\langle y, x\rangle$; in other words, the first factor is the parameter space and the second factor the "physical space." Let $\Sigma \to X$ be the bundle whose fiber at $y \in X$ is the compactified tangent space Σ_y. We identify Σ with $j^*\pi_1^*(\Sigma) \to \Delta$, where $\pi_1\colon X\times X \to X$ is projection onto the first factor and $j\colon \Delta \hookrightarrow X\times X$ the inclusion of the diagonal. Then the exponential map restricts on a neighborhood of the zero section of Σ to a diffeomorphism onto a neighborhood of Δ in $X\times X$. Extend its inverse to a degree one map $F\colon X\times X \to \Sigma$ (which preserves fibers). Now construct the deformation space $\mathcal{X} \subset \mathbb{R}\times\Sigma\times X$ as the solution set of the equation $F(x) = \epsilon a$, where now $a \in \Sigma$. (Again we exclude $0\cdot\infty$.) The composite projection $\mathcal{X} \to \mathbb{R}\times\Sigma \to \mathbb{R}\times X$ is a submersion, and the fiber $\mathcal{X}_{\langle\epsilon,y\rangle}$ is the fiber $\mathbf{X}_\epsilon$ above. Lift the geometric data as before. Clearly everything varies smoothly with y, hence so do the Taylor series coefficients b_i (4.4.18), whence also the asymptotic expansion coefficients a_j (4.4.20).

Exercise 4.4.22. Suppose P is a second order elliptic differential operator with scalar symbol defined on a ball B in $\mathbb{R}^n$. Glue B into any compact manifold X and extend P to a second order elliptic differential operator on X. Prove that at any $y \in B$ there is an asymptotic expansion for the heat kernel of P (along the diagonal), and that the expansion is independent of the extension. (Hint: Construct the deformation $\mathbf{X}$ and lift P to a family of operators on $\mathbf{X}_\epsilon$. What is the operator on $\mathbf{X}_0$?)

§5 The Index Theorem

On an even dimensional spin manifold the spinor fields decompose into two types—'positive' and 'negative'—and the Dirac operator $\mathcal{D}$ exchanges positive and negative spinor fields. Let D denote the restriction of the Dirac operator to positive spinor fields, which is the *chiral Dirac operator* introduced in §2.3. The adjoint D^* is the restriction of $\mathcal{D}$ to the negative spinor fields. The *index* of D is

$$\operatorname{ind} D = \dim\ker D - \dim\ker D^*. \tag{5.1}$$

In this chapter we prove the Atiyah-Singer index theorem [AS2], which gives a topological formula for $\operatorname{ind} D$.[24] We actually prove a stronger result (Theorem 5.4.1), the so-called *local* Atiyah-Singer index theorem [ABP,§7]. Theorem 5.2 is the global version, the original Atiyah-Singer result.

Theorem 5.2. *Let X be a Riemannian spin manifold with curvature $\Omega^{(X)}$, and $V \to X$ a Hermitian vector bundle endowed with a unitary connection whose curvature is $\Omega^{(V)}$. Then the index of the chiral Dirac operator D acting on V-valued spinor fields is*

$$\operatorname{ind} D = \int_X \hat{A}(\Omega^{(X)})\,\operatorname{ch}(\Omega^{(V)}), \tag{5.3}$$

where for any skew-symmetric matrix Ω

$$\hat{A}(\Omega) = \sqrt{\det\left(\frac{\Omega/4\pi i}{\sinh\,\Omega/4\pi i}\right)}, \tag{5.4}$$

and for any skew-Hermitian matrix Ω

$$\operatorname{ch}(\Omega) = \operatorname{Tr}\, e^{i\Omega/2\pi}. \tag{5.5}$$

The index appears as the integral of a differential form, and the identification with characteristic classes is through Chern-Weil theory (cf. Chapter 6). What is crucial is that the integrand, and not just the integral, has a geometric interpretation. The first step in the proof of Theorem 5.2 is an analytic expression for the index in terms of the heat equation, due to Atiyah and Bott [AB1]. The integrand in (5.3) is the small time limit of a local trace of the heat kernel. This result, Theorem 5.4.1, is in many ways more fundamental to our approach than the index theorem itself,

[24] In fact, the theorem of Atiyah and Singer applies to arbitrary elliptic pseudodifferential operators. For a derivation of the full Atiyah-Singer theorem from the index theorem for Dirac operators, see [ABP].

which is a corollary. We prove Theorem 5.4.1 by a scaling argument. As in Chapter 4 this scaling is essentially a deformation from the manifold to a given tangent space. Unfortunately, the deformation used here does not seem to have a good *intrinsic* globalization. Since we only use local information in any case, it suffices to consider the asymptotic expansion. The scaling of Chapter 4 is modified, and there is a new limiting heat operator on the tangent space—a modified harmonic oscillator. *Mehler's formula* for the heat kernel of the ordinary harmonic oscillator on flat space yields an explicit formula for the heat kernel of this limiting operator. The $\hat{A}$ genus comes from the explicit form of Mehler's formula (5.3.10). Although we work with the asymptotic expansions in this chapter, the reader should keep in mind the global deformation constructed in Chapter 4. The analogy, then, between this proof of the index theorem and the modern proof of the Grothendieck-Riemann-Roch theorem [F,§15] is quite strong. Our approach in this chapter is largely based on Getzler [Ge1], as presented in [R], though as we discussed in Chapter 1 many of these ideas also appear in the physics literature. In particular, Friedan and Windy [FW] use the scaling (5.4.12) in their approach to the index theorem.

§**5.1 Chirality**

In Chapter 2 (especially Theorem 2.1.25, (2.1.34), Exercise 2.1.37, Exercise 2.1.39, Exercise 2.1.36) we observed that the spin representation $\mathbb{S}$ of $\mathrm{Spin}(n)$ splits into $\mathbb{S}^+ \oplus \mathbb{S}^-$ for n even. Recall that the entire complexified Clifford algebra $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ acts on $\mathbb{S}$. Let $e_1, \dots, e_n$ be the standard oriented orthonormal basis of $\mathbb{E}^n$, and consider the element $\epsilon = i^{n/2} e_1 \cdot e_2 \cdot \cdots \cdot e_n$ (cf. Exercise 2.1.28). Since n is even ϵ anticommutes with e_i, so commutes with $e_i \cdot e_j$. Therefore ϵ commutes with $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^+$ and so with $\mathrm{Spin}(n) \subset \mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)^+$. On the other hand, a simple computation shows that $\epsilon^2 = 1$. Hence the spin representation $\mathbb{S}$ breaks up according to the eigenspaces of ϵ (acting on $\mathbb{S}$). The $+1$ eigenspace is denoted $\mathbb{S}^+$ and the -1 eigenspace $\mathbb{S}^-$. The group $\mathrm{Spin}(n)$ preserves each of these eigenspaces. These two irreducible representations of $\mathrm{Spin}(n)$ are called the *half-spin representations.*

Let X be a spin manifold with even dimension n. Then the spin bundle $S \to X$ splits into $S = S^+ \oplus S^-$; simply apply the previous construction pointwise. Sections of S^+ are 'positive spinor fields' and sections of S^- are 'negative spinor fields.' Relative to this decomposition the Dirac operator takes the form

(5.1.1) $$\mathcal{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix},$$

where

(5.1.2) $$D\colon C^\infty(S^+) \longrightarrow C^\infty(S^-)$$

is the *chiral Dirac operator*.

The vector space $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ is an example of a $\mathbb{Z}/2\mathbb{Z}$-*graded vector space*.[25] A $\mathbb{Z}/2\mathbb{Z}$-graded vector space is simply a vector space $\mathbb{F}$ together with a decomposition $\mathbb{F} = \mathbb{F}^+ \oplus \mathbb{F}^-$. The *superdimension* of $\mathbb{F}$ is defined by

$$\dim_s \mathbb{F} = \dim \mathbb{F}^+ - \dim \mathbb{F}^-. \tag{5.1.3}$$

The grading operator ϵ is the element of $\operatorname{End}\mathbb{F}$ which is $+1$ on $\mathbb{F}^+$ and -1 on $\mathbb{F}^-$. Notice that $\dim_s(\mathbb{F}) = \operatorname{tr}(\epsilon)$. Consider the action of ϵ on $\operatorname{End}(\mathbb{F})$ by conjugation. As the square of this action is 1, we can split $\operatorname{End}(\mathbb{F}) = \operatorname{End}(\mathbb{F})^+ \oplus \operatorname{End}(\mathbb{F})^-$ according to the $+1$ and -1 eigenspaces. Elements of $\operatorname{End}(\mathbb{F})^+$ preserve $\mathbb{F}^+$ and $\mathbb{F}^-$, while elements of $\operatorname{End}(\mathbb{F})^-$ exchange them. Said differently, elements of $\operatorname{End}(\mathbb{F})^+$ are diagonal 2×2 matrices (relative to the decomposition $\mathbb{F} = \mathbb{F}^+ \oplus \mathbb{F}^-$), and elements of $\operatorname{End}(\mathbb{F})^-$ are off-diagonal matrices. The *supertrace* of $A \in \operatorname{End}(\mathbb{F})$ is

$$\operatorname{tr}_s(A) = \operatorname{tr}(\epsilon A). \tag{5.1.4}$$

In terms of matrices,

$$\operatorname{tr}_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{tr}(a) - \operatorname{tr}(d). \tag{5.1.5}$$

Notice that $\dim_s(\mathbb{F}) = \operatorname{tr}_s(1)$. Suppose that A, B are homogeneous elements of $\operatorname{End}(\mathbb{F})$. Let $|A| = 0$ if $A \in \operatorname{End}(\mathbb{F})^+$ and $|A| = 1$ if $A \in \operatorname{End}(\mathbb{F})^-$. The *supercommutator* $[A, B]_s$ is defined by

$$[A, B]_s = AB - (-1)^{|A|\,|B|} BA. \tag{5.1.6}$$

Lemma 5.1.7. *The supertrace vanishes on supercommutators:* $\operatorname{tr}_s[A, B]_s = 0$.

Proof. Using the fact that the ordinary trace vanishes on ordinary commutators, for homogeneous elements we have

$$\begin{aligned}
\operatorname{tr}_s[A, B]_s &= \operatorname{tr}\left(\epsilon AB - (-1)^{|A|\,|B|}\epsilon BA\right) \\
&= \operatorname{tr}\left(\epsilon AB - (-1)^{|A|\,|B|}A\epsilon B\right) \\
&= \operatorname{tr}\left(\epsilon AB - (-1)^{|A|\,|B|}(-1)^{|A|}\epsilon AB\right) \\
&= \left(1 - (-1)^{|A|(|B|+1)}\right)\operatorname{tr}(\epsilon AB).
\end{aligned}$$

[25] This is sometimes termed a *super vector space*. We give a more complete treatment in Chapter 6.

The factor in front vanishes unless A is odd and B is even, in which case ϵAB is off-diagonal and $\mathrm{tr}(\epsilon AB)$ vanishes.

We need a formula for the supertrace on the spin space $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Recall (2.1.26) that the complexified Clifford algebra $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$ is $\mathrm{End}(\mathbb{S})^+$ in even dimensions. We use the standard basis $\{e_I = e_{i_1} \cdot \dots \cdot e_{i_k}\}$ of the Clifford algebra.

Lemma 5.1.8. *Let $c = c^I e_I$ (summed on I) be an arbitrary element of* $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{E}^n)$*. Then*

$$\mathrm{tr}_{\mathrm{s}}(c) = (-2i)^{n/2} c^{12\cdots n}. \tag{5.1.9}$$

Here $c^{12\cdots n}$ is the coefficient of the "volume form" $e_1 \cdot \dots \cdot e_n$.

EXERCISE 5.1.10. Characterize the volume form using the inner product and orientation. Hence verify that the right hand side of (5.1.9) is independent of the choice of oriented orthonormal basis.

Proof of Lemma 5.1.8. Fix a multiindex I and assume $i \notin I$. (We can always find such an i if $I \neq 12\cdots n$.) Then it is easy to verify that $e_I = -\frac{1}{2}[e_i, e_i e_I]_{\mathrm{s}}$. Hence $\mathrm{tr}_{\mathrm{s}}(e_I) = 0$ by Lemma 5.1.7. Finally, recalling that $\epsilon = i^{n/2} e_{12\cdots n}$ and that $\dim \mathbb{S} = 2^{n/2}$, we have

$$\mathrm{tr}_{\mathrm{s}}(e_{12\cdots n}) = \mathrm{tr}(\epsilon e_{12\cdots n}) = \mathrm{tr}\big((-i)^{n/2}\big) = (-2i)^{n/2}.$$

§5.2 The Atiyah-Bott Formula

Let X be a compact spin manifold with spin bundle $S \to X$, and suppose $V \to X$ is a Hermitian bundle with unitary connection. Let $E = S \otimes V$. Now suppose that $n = \dim X$ is even. Then $S = S^+ \oplus S^-$ splits, and therefore we can write $E = E^+ \oplus E^-$, where $E^+ = S^+ \otimes V$ and $E^- = S^- \otimes V$. The Sobolev spaces of sections of E are also $\mathbb{Z}/2\mathbb{Z}$-graded, and the elliptic theory of Chapter 3 and Chapter 4 applies to each separately. Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be the space of L^2 sections of E. Now the Dirac operator squared (cf. (5.1.1)) is

$$\mathcal{D}^2 = \begin{pmatrix} D^*D & 0 \\ 0 & D\,D^* \end{pmatrix}. \tag{5.2.1}$$

Each of D^*D and $D\,D^*$ is elliptic. It follows that each of $\mathcal{H}^+$ and $\mathcal{H}^-$ has an orthonormal basis of smooth eigenspinor fields. Suppose that $\psi \in \mathcal{H}^+$ is an eigenspinor field for D^*D with *nonzero* eigenvalue λ. Then $D\,D^*D\psi = \lambda D\psi$, so that $D\psi \in \mathcal{H}^-$ is an eigenspinor field for $D\,D^*$ with eigenvalue λ. It follows easily that for nonzero λ the Dirac operator D maps the λ-eigenspace

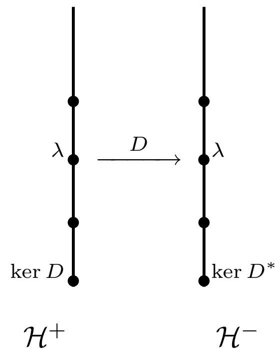

Figure 1

of D^*D isomorphically onto the λ-eigenspace for $D\,D^*$. Hence the nonzero spectra of D^*D and $D\,D^*$ coincide. No claim is made about the zero eigenvalue. Indeed, the index of D measures the discrepancy between the dimension of the kernels. We represent the situation schematically in Figure 1.

With these preliminaries we can give a formula for the index $\operatorname{ind} D = \dim\ker D - \dim\ker D^*$. In a formal sense the preceding argument demonstrates that $\operatorname{ind} D = \dim\mathcal{H}^+ - \dim\mathcal{H}^-$. Of course, this makes no sense as each of $\dim\mathcal{H}^+$, $\dim\mathcal{H}^-$ is infinite. Write (formally) $\dim\mathcal{H}^+ = \sum_{\lambda\in\operatorname{spec}(D^*D)} 1$ and $\dim\mathcal{H}^- = \sum_{\lambda\in\operatorname{spec}(D\,D^*)} 1$. We will obtain a well-defined expression by replacing the constant function 1 with a function $F(\lambda)$ which renders the sums absolutely convergent. Therefore, we have the following

Lemma 5.2.2. *Let $F(\lambda)\colon \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying $F(0) = 1$ and $\sum_{\lambda\in\operatorname{spec}(D^*D)} |F(\lambda)| < \infty$. Then*

$$\begin{aligned} \operatorname{ind} D &= \sum_{\lambda\in\operatorname{spec}(D^*D)} F(\lambda) - \sum_{\lambda\in\operatorname{spec}(D\,D^*)} F(\lambda) \\ &= \operatorname{Tr} F(D^*D) - \operatorname{Tr} F(D\,D^*) \\ &= \operatorname{Tr}_s F(\mathcal{D}^2). \end{aligned} \tag{5.2.3}$$

Exercise 5.2.4. Write out a careful proof of Lemma 5.2.2. Notice that the supertrace is only defined if each of the traces converges absolutely.

Our study of the heat equation in Chapter 4 was motivated by the fact that the hypotheses of Lemma 5.2.2 are satisfied by $F(\lambda) = e^{-t\lambda}$ for any $t > 0$.

Proposition 5.2.5. *The trace of the heat operator*

$$\operatorname{Tr} e^{-tD^*D} = \sum_{\lambda\in\operatorname{spec}(D^*D)} e^{-t\lambda} \tag{5.2.6}$$

converges. In fact, denoting by $p_t(x,y)^+$ the heat kernel of D^*D, we have

$$\operatorname{Tr} e^{-tD^*D} = \int_X dy\ \operatorname{tr} p_t(y,y)^+. \tag{5.2.7}$$

Recall that the heat kernel $p_t(x,y)$ of $\mathcal{D}^2$ is a map $E_y \to E_x$. Since $\mathcal{D}^2$ preserves chirality, the heat kernel does also. Hence $p_t(x,y)^+$ is the restriction of $p_t(x,y)$ to E_y^+.

Proof. The essential point is that the heat kernel is smooth. Let ψ_n^+ be an orthonormal basis of eigenspinor fields for $\mathcal{H}^+$, with $D^*D\psi_n^+ = \lambda_n \psi_n^+$. Recall that for $\sigma \in E_y^+$ we define a distribution $\delta_\sigma(\psi) = (\sigma, \psi(y))$ (cf. (4.3.6)). Thus

$$\begin{aligned} p_t(x,y)^+\sigma &= e^{-tD^*D}(\delta_\sigma)(x) \\ &= e^{-tD^*D}\left(\sum_n \langle \delta_\sigma, \psi_n^+\rangle \psi_n^+\right)(x) \\ &= \sum_n e^{-t\lambda_n}\psi_n^+(x)(\sigma, \psi_n^+(y)). \end{aligned}$$

Therefore, setting $x = y$ and tracing over σ we obtain

$$\operatorname{tr} p_t(y,y)^+ = \sum_n e^{-t\lambda_n}|\psi_n^+(y)|^2. \tag{5.2.8}$$

Now $p_t(y,y)^+$ is a smooth function of y, by Proposition 4.3.8, so the integral of (5.2.8) over X is finite. This justifies the interchange of $\sum_n$ and $\int_X$ in the computation:

$$\begin{aligned} \int_X dy\ \operatorname{tr} p_t(y,y)^+ &= \int_X dy \sum_n e^{-t\lambda_n}|\psi_n^+(y)|^2 \\ &= \sum_n e^{-t\lambda_n}\int_X dy\,|\psi_n^+(y)|^2 \\ &= \sum_n e^{-t\lambda_n} \\ &= \operatorname{Tr} e^{-tD^*D} \end{aligned} \tag{5.2.9}$$

Exercise 5.2.10. Justify (5.2.9) in detail. (Hint: Use the monotone convergence theorem.)

Corollary 5.2.11. *The index of the Dirac operator is*

$$\operatorname{ind} D = \operatorname{Tr}_s e^{-t\mathcal{D}^2} = \int_X dy \ \operatorname{tr}_s p_t(y,y) \tag{5.2.12}$$

for any t.

In (5.2.12) the integrand is the finite dimensional (local) supertrace of $p_t(y,y)\colon E_y \to E_y$.

Proof. Proposition 5.2.5 is true for $D\,D^*$ replacing D^*D and $-$ replacing $+$. Then

$$\text{Corollary } 5.2.11 = \text{Proposition } 5.2.5^+ - \text{Proposition } 5.2.5^-,$$

using Lemma 5.2.2.

§5.3 Mehler's Formula

Our final step to the proof of the index theorem is an explicit expression for the heat kernel of the harmonic oscillator in flat space. Formulas (4.1.6), (4.1.10) give the heat kernel of the scalar Laplace operator on $\mathbb{E}^n$. Now we want to solve the equation (on $\mathbb{R}$)

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + a^2x^2v = 0 \tag{5.3.1}$$

for $v_t(x)$ a real valued function of $t > 0, x \in \mathbb{R}$, with

$$\lim_{t\to 0} v_t = \delta_0 \tag{5.3.2}$$

the δ-function at 0. Equation (5.3.1) is related to the harmonic oscillator, as we explain in the exercises below. Encouraged by the form of (4.1.6) we guess a solution to (5.3.1) of the form

$$v_t(x) = \alpha(t)e^{-x^2\beta(t)/2}. \tag{5.3.3}$$

Substituting into (5.3.1) we obtain the pair of equations

$$(\log\alpha)^{\cdot} = -\beta \tag{5.3.4}$$

$$\dot\beta = 2(a^2 - \beta^2), \tag{5.3.5}$$

where we adopt Newton's notation for time derivatives. A rather inspired substitution into (5.3.5) is $\beta(t) = a\coth\gamma(t)$, from which we conclude $\dot\gamma = 2a$, or $\gamma = 2at + C$ for some constant C. Hence

$$\begin{aligned}\beta(t) &= a\coth(2at+C)\\ &= \frac{1}{2}(\log\sinh(2at+C))^{\cdot}.\end{aligned} \tag{5.3.6}$$

Comparing with (5.3.4) we conclude

$$\alpha(t) = C'[\sinh(2at+C)]^{-1/2} \tag{5.3.7}$$

for some constant C'. Equations (5.3.3), (5.3.6), and (5.3.7) determine a solution to (5.3.1). To satisfy the initial condition (5.3.2) we find $C = 0$ and $C' = \sqrt{a/2\pi}$. Therefore,

Proposition 5.3.8. *The function*

$$v_t(x) = \frac{1}{\sqrt{4\pi t}}\left(\frac{2at}{\sinh 2at}\right)^{1/2}\exp\left[\frac{-x^2}{4t}\left(\frac{2at}{\tanh 2at}\right)\right] \tag{5.3.9}$$

solves (5.3.1) *with initial condition* (5.3.2)*. In particular, the solution at the origin is*

$$v_t(0) = \frac{1}{\sqrt{4\pi t}}\left(\frac{2at}{\sinh 2at}\right)^{1/2}. \tag{5.3.10}$$

§5.4 The scaling argument

The main result we are after is

Theorem 5.4.1. *Let X be a Riemannian spin manifold with curvature $\Omega^{(X)}$, and $V \to X$ a Hermitian vector bundle endowed with a unitary connection whose curvature is $\Omega^{(V)}$. Denote the heat kernel of the full Dirac operator on V-valued spinor fields by $p_t(x,y)$. Then for each $y \in X$ the limit $\lim_{t\to 0}\operatorname{tr_s} p_t(y,y)\,dy$ exists and is computed by*

$$\lim_{t\to 0}\operatorname{tr_s} p_t(y,y)\,dy = \left[\hat A(\Omega^{(X)})\operatorname{ch}(\Omega^{(V)})\right]_{(n)}(y), \tag{5.4.2}$$

where dy is the volume form of X, and $\hat A, \operatorname{ch}$ are defined in (5.4), (5.5)*. Here $[\omega]_{(n)}$ denotes the n-form component of a differential form ω.*

There are cancellations implicit in (5.4.2), since neither $\lim_{t\to 0} p_t(y,y)$ nor $\lim_{t\to 0} \operatorname{tr} p_t(y,y)^{\pm}$ is finite, as is clear from the asymptotic expansion (4.4.2). These cancellations are exhibited explicitly in Proposition 5.4.16. The index theorem (5.2) follows immediately from Theorem 5.4.1 by taking $t\to 0$ in (5.2.12).

Fix $y\in X$. As in (4.4.4) we use exponential coordinates to identify $B_r\subset T_yX$ with a neighborhood of y in X. Parallel transport in V along radial geodesics identifies $S_{\exp a}\cong S_y$ and $V_{\exp a}\cong V_y$ for $a\in B_r$. In these coordinates the heat kernel is

$$\begin{aligned} q_t(a) = p_t(\exp a, y) &\in \operatorname{Hom}(S_y, S_{\exp a})\otimes \operatorname{Hom}(V_y, V_{\exp a}) \\ &\cong \operatorname{End}(S_y)\otimes\operatorname{End}(V_y) \\ &\cong \operatorname{Cliff}(T_y^*X)^+\otimes \operatorname{End}(V_y). \end{aligned} \tag{5.4.3}$$

(Recall that the spin bundle is built from the Clifford algebra of the cotangent space, since the Dirac operator (2.2.21) is defined using Clifford multiplication by cotangent vectors.) Now according to Lemma 5.1.8 the supertrace $\operatorname{tr_s} p_t(y,y)$ only depends on the coefficient of the volume form in $\operatorname{Cliff}(T_y^*X)^+$. Hence in addition to the scaling (4.4.6), which was used to obtain the asymptotic expansion, we introduce a scaling on the Clifford algebra to study this coefficient.

The relevant algebraic facts were set out in Exercise 2.1.58 and Exercise 2.1.59, which we review here. For $\epsilon\neq 0$ let $\mathbb{E}^n_\epsilon$ denote $\mathbb{R}^n$ with the inner product which renders the standard basis elements $e_1,\dots,e_n$ mutually orthogonal and of square length ϵ^2. The Clifford algebra $\operatorname{Cliff}(\mathbb{E}^n_\epsilon)$ is generated by the e_i subject to the relation

$$e_ie_j + e_je_i = -2\delta_{ij}\epsilon^2.$$

For $\epsilon = 1$ we recover the usual Clifford algebra $\operatorname{Cliff}(\mathbb{E}^n)$. Let $I = i_1\dots i_k$ be a multiindex with $1\le i_1<\dots<i_k\le n$, and $|I| = k$. Denote $e_I = e_{i_1}\cdot\dots\cdot e_{i_k}$; the e_I form a basis of $\operatorname{Cliff}(\mathbb{E}^n_\epsilon)$. There is a canonical isomorphism of algebras

$$\begin{aligned} \operatorname{Cliff}(\mathbb{E}^n) &\longrightarrow \operatorname{Cliff}(\mathbb{E}^n_\epsilon) \\ e_I &\longmapsto \epsilon^{-|I|}e_I. \end{aligned} \tag{5.4.4}$$

On the other hand, denoting the usual basis of the exterior algebra by $\hat e_I = e_{i_1}\wedge\dots\wedge e_{i_k}$, the map

$$\begin{aligned} {\textstyle\bigwedge}(\mathbb{R}^n) &\longrightarrow \operatorname{Cliff}(\mathbb{E}^n_\epsilon) \\ \hat e_I &\longmapsto e_I \end{aligned} \tag{5.4.5}$$

is an isomorphism of *vector spaces*, and so induces an algebra structure on the vector space $\bigwedge(\mathbb{R}^n)$ for each ϵ. Let $c_\epsilon(\hat e)$ denote left multiplication by $\hat e\in\bigwedge(\mathbb{R}^n)$ in the algebra structure induced by (5.4.5). Then

$$\lim_{\epsilon\to 0} c_\epsilon(\hat e) = \epsilon(\hat e), \tag{5.4.6}$$

where $\epsilon(\cdot)$ is exterior multiplication. Finally, the composite of (5.4.4) and $(5.4.5)^{-1}$ is the map

$$\begin{aligned} U_\epsilon\colon\ \mathrm{Cliff}(\mathbb{E}^n) &\longrightarrow \textstyle\bigwedge(\mathbb{R}^n) \\ e_I &\longmapsto \epsilon^{-|I|}\hat{e}, \end{aligned} \tag{5.4.7}$$

and from (5.4.6) we deduce

$$\lim_{\epsilon\to 0} \epsilon^{|I|} c_\epsilon(U_\epsilon e_I) = \epsilon(\hat{e}_I). \tag{5.4.8}$$

Notice that (5.4.4)–(5.4.8) transform properly under orthogonal transformations, so make sense on any inner product space.

Exercise 5.4.9. Write a formula for c_ϵ in terms of exterior multiplication and interior multiplication (relative to the standard inner product). Use it to prove (5.4.6).

Recall the scaling map $T_\epsilon(a) = \epsilon a$ in (4.4.6). Define[26] the scaled heat kernel

$$q_t^\epsilon = \epsilon^n U_\epsilon(T_\epsilon^* q_{\epsilon^2 t}). \tag{5.4.10}$$

It is a function of $a \in B_{r/|\epsilon|}$ with values in $\bigwedge(T_y^*X)\otimes \mathrm{End}(V_y)$. (Here $U_\epsilon\colon\ \mathrm{Cliff}(T_y^*X) \to \bigwedge(T_y^*X)$.) Note that $q_t^1 = q_t$. To be more explicit we fix an orthonormal basis $e^1,\dots,e^n$ of T_y^*X, and write

$$q_t(a) = \sum_I q_t(a)_I\, e^I,$$

where $e^I \in \mathrm{Cliff}(T_y^*X)$ and $q_t(a)_I \in \mathrm{End}(V_y)$. Then

$$q_t^\epsilon(a) = \epsilon^n \sum_I \epsilon^{-|I|} q_{\epsilon^2 t}(\epsilon a)_I\, \hat{e}^I; \tag{5.4.11}$$

in short, the scaling is the substitution

$$\begin{aligned} a &\longmapsto \epsilon a \\ t &\longmapsto \epsilon^2 t \\ e^i &\longmapsto \epsilon^{-1} e^i \end{aligned} \tag{5.4.12}$$

[26] We can essentially identify q_t^ϵ with $U_\epsilon(k^\epsilon_{\epsilon^2 t})$, where $k^\epsilon_{\epsilon^2 t}$ is as in Assertion 4.4.8. Since the supertrace only depends on the coefficient of the volume form, which scales by ϵ^{-n} under U_ϵ, it follows from Lemma 4.4.11 that $\mathrm{tr_s}\, q_1^\epsilon(0)$ is approximately $\mathrm{tr_s}\, q_{\epsilon^2}(0)$. This motivates the scaling U_ϵ: we have $\lim\limits_{t\to 0} \mathrm{tr_s}\, q_t(0) = \lim\limits_{\epsilon\to 0} \mathrm{tr_s}\, q_1^\epsilon(0)$ (provided the latter exists, which we prove below). Notice that only the continuity of q_1^ϵ at $\epsilon = 0$ is needed, whereas the asymptotic expansion requires the smoothness at $\epsilon = 0$.

together with an overall scaling by ϵ^n.

Next, we study the effect of the scaling on the asymptotic expansion (4.4.2)

$$q_t(0) \sim (4\pi t)^{-n/2} \sum_{j,I} A_{j,I}\, t^j e^I, \quad A_{j,I} \in \operatorname{End}(V_y), \tag{5.4.13}$$

valid as $t \to 0$. Namely, substituting (5.4.12) into (5.4.13) yields

$$\begin{aligned} q_t^\epsilon(0) &\sim (4\pi t)^{-n/2} \sum_{j,I} \epsilon^{2j-|I|} A_{j,I}\, t^j \hat{e}^I \\ &= (4\pi)^{-n/2} \sum_{j,I} \epsilon^{2j-|I|} A_{j,I}\, t^{j-n/2} \hat{e}^I. \end{aligned} \tag{5.4.14}$$

By (5.1.9) the supertrace is given as $(-2i)^{n/2}$ times the coefficient of the volume form $\hat{e}^{12\dots n}$:

$$\operatorname{tr_s} q_t^\epsilon(0) \sim (2\pi i)^{-n/2} \sum_j \epsilon^{2j-n} \operatorname{tr} A_{j,12\dots n} t^{j-n/2}. \tag{5.4.15}$$

Theorem 5.4.1 follows from a more precise result about the asymptotic expansion.

Proposition 5.4.16. *We have*

(1) $A_{j,I} = 0 \quad$ *if* $|I| > 2j$*;*
(2) $\operatorname{tr} A_{n/2,12\dots n}\, dy = (2\pi i)^{n/2} \left[\hat{A}(\Omega^{(X)}) \operatorname{ch}(\Omega^{(V)})\right]_{(n)} (y)$.

Granting Proposition 5.4.16, and setting $\epsilon = 1$ in (5.4.15) we conclude

$$\operatorname{tr_s} q_t(0) \sim (2\pi i)^{-n/2} \sum_{j \ge n/2} \operatorname{tr} A_{j,12\dots n} t^{j-n/2},$$

from which

$$\lim_{t \to 0} \operatorname{tr_s} q_t(0)\, dy = (2\pi i)^{-n/2} \operatorname{tr} A_{n/2,12\dots n}\, dy = \left[\hat{A}(\Omega^{(X)}) \operatorname{ch}(\Omega^{(V)})\right]_{(n)} (y).$$

This is (5.4.2).

Proof of Proposition 5.4.16. The logic of the proof is slightly tricky. First we show that (5.4.14) is the asymptotic expansion[27] (at $a = 0$) associated to a certain heat operator $\partial/\partial t + P_\epsilon$ on $B_{r/|\epsilon|}$. It is crucial to remember that the asymptotic expansion is determined by the operator P_ϵ near $a = 0$.

[27] We rely on Exercise 4.4.22 here.

Next we show that $\lim_{\epsilon\to 0} P_\epsilon = P_0$ exists and is a second order elliptic operator on T_yX. We compute its heat kernel explicitly, from which we can easily derive the asymptotic expansion at $a = 0$. No theory of the operator P_0 on T_yX is used here; in particular, we make no claim about the uniqueness of our solution (though it is unique in appropriate function spaces). Nevertheless, the asymptotic expansion is unique and is the same for any solution, since it is determined by P_0 near $a = 0$. Finally, the coefficients in the asymptotic expansion depend smoothly on the operator, which completes the proof. Notice that (1) and the existence of a limit in (2) follow merely from the existence of a good limiting operator P_0.

The heat kernel $q_t(a)$ satisfies the heat equation

$$\left(\frac{\partial}{\partial t} + \mathcal{D}^2\right) q = 0, \tag{5.4.17}$$

where the Dirac operator $\mathcal{D}$ acts on the Clifford algebra valued function q by left Clifford multiplication. Furthermore, $\lim_{t\to 0} q_t$ is the δ-function at $a = 0$. Let $S_\epsilon = U_\epsilon T_\epsilon^*$ be the scaling operator in (5.4.10), but without the scaling in t, and apply S_ϵ to both sides of (5.4.17) to conclude (compare (4.4.14))

$$\left(\frac{\partial}{\partial t} + \epsilon^2 S_\epsilon \mathcal{D}^2 S_\epsilon^{-1}\right) q^\epsilon = 0. \tag{5.4.18}$$

Also, $\lim_{t\to 0} q_t^\epsilon$ is the δ-function at $a = 0$. Therefore, q^ϵ is the heat kernel for the operator

$$P_\epsilon = \epsilon^2 S_\epsilon \mathcal{D}^2 S_\epsilon^{-1} \tag{5.4.19}$$

on $B_{r/|\epsilon|}$. To compute $\lim_{\epsilon\to 0} P_\epsilon$ we pass to exponential coordinates (cf. Appendix).

The Weitzenböck formula (2.3.5) asserts

$$P_1 = \mathcal{D}^2 = \nabla^*\nabla + \frac{R}{4} + c(\Omega^{(V)}). \tag{5.4.20}$$

Recall that in exponential coordinates (cf. Appendix A) we use Γ_k to denote the Levi-Civita connection form and A_k to denote the connection form on V. Also, the Γ_k act by left Clifford multiplication on Clifford algebra valued functions. Finally, $\begin{bmatrix} m \\ k\ell \end{bmatrix}$ are the classical Levi-Civita symbols in coordinates (cf. (A.16)). Thus from (A.17), (5.4.20), and (A.8) we derive

$$\begin{aligned} P_1 = &-g^{k\ell}(a)\left(\frac{\partial}{\partial a^k} + \frac{1}{2}c\bigl(\Gamma_k(a)\bigr) + A_k(a)\right)\left(\frac{\partial}{\partial a^\ell} + \frac{1}{2}c\bigl(\Gamma_\ell(a)\bigr) + A_\ell(a)\right) \\ &+ g^{k\ell}(a)\begin{bmatrix} m \\ k\ell \end{bmatrix}(a)\left(\frac{\partial}{\partial a^m} + \frac{1}{2}c\bigl(\Gamma_m(a)\bigr) + A_m(a)\right) + \frac{R(a)}{4} + c\bigl(\Omega^{(V)}(a)\bigr). \end{aligned} \tag{5.4.21}$$

The extra factor of 1/2 comes from identifying the skew-symmetric matrix Γ_k with an element of the Clifford algebra (cf. (2.2.29)). Conjugation by S_ϵ amounts to the substitutions $a \to \epsilon a$ and $c(e) \to c_\epsilon(U_\epsilon e)$. Thus

$$(5.4.22)\quad \begin{aligned} P_\epsilon = -g^{k\ell}(\epsilon a)\left(\frac{\partial}{\partial a^k} + \frac{1}{2}\epsilon c_\epsilon\big(U_\epsilon\Gamma_k(\epsilon a)\big) + \epsilon A_k(\epsilon a)\right)\left(\frac{\partial}{\partial a^\ell} + \frac{1}{2}\epsilon c_\epsilon\big(U_\epsilon\Gamma_\ell(\epsilon a)\big) + \epsilon A_\ell(\epsilon a)\right) \\ + \epsilon g^{k\ell}(\epsilon a)\begin{bmatrix} m \\ k\ell \end{bmatrix}(\epsilon a)\left(\frac{\partial}{\partial a^m} + \frac{1}{2}\epsilon c_\epsilon\big(U_\epsilon\Gamma_m(\epsilon a)\big) + \epsilon A_m(\epsilon a)\right) + \epsilon^2\frac{R(\epsilon a)}{4} + \epsilon^2 c_\epsilon\big(U_\epsilon\Omega^{(V)}(\epsilon a)\big). \end{aligned}$$

Now using (A.5), (A.6), (A.14), (A.15), and (5.4.8) we compute

$$(5.4.23)\qquad P_0 = \lim_{\epsilon\to 0} P_\epsilon = -\sum_k\left(\frac{\partial}{\partial a^k} - \frac{1}{4}\epsilon\big(\Omega^{(X)}_{k\ell}(0)\big)a^\ell\right)^2 + \epsilon\big(\Omega^{(V)}(0)\big).$$

P_0 is a generalized harmonic oscillator.

For simplicity write

$$(5.4.24)\qquad \Omega = \Omega^{(X)}(0), \qquad F = \Omega^{(V)}(0),$$

and set

$$(5.4.25)\qquad Q = -\sum_k\left(\frac{\partial}{\partial a^k} - \frac{1}{4}\epsilon(\Omega_{k\ell})a^\ell\right)^2.$$

It remains to solve for the heat kernel. First, note that the two terms in (5.4.23) commute, so that

$$(5.4.26)\qquad e^{-tP_0} = e^{-tQ}e^{-tF}.$$

Now Ω is a skew-symmetric matrix whose entries are 2-forms. We are free to choose the coordinates a^k so that Ω is block diagonal:

$$(5.4.27)\qquad (\Omega_{k\ell}) = \begin{pmatrix} 0 & -\omega_1 & & & \\ \omega_1 & 0 & & & \\ & & 0 & -\omega_2 & \\ & & \omega_2 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Then Q decouples into a sum of operators of the form

$$(5.4.28)\qquad -\left(\frac{\partial}{\partial x} + \frac{1}{4}\omega y\right)^2 - \left(\frac{\partial}{\partial y} - \frac{1}{4}\omega x\right)^2 = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{16}\omega^2(x^2+y^2) + \frac{1}{2}\omega\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Now the first two terms on the right hand side are invariant under rotations, and the last term annihilates rotationally invariant functions. Hence we can drop the last term—the heat operator for the sum of the first two terms equals the heat operator for the full operator Q. According to (5.3.10) the heat kernel (at the origin) for $-\partial^2/\partial x^2 - \omega^2 x^2/16$ is

$$\frac{1}{\sqrt{4\pi t}}\left(\frac{it\omega/2}{\sinh it\omega/2}\right)^{1/2}. \tag{5.4.29}$$

The heat kernel for the y variable contributes the same factor, so the heat kernel (at the origin) for Q is

$$(4\pi t)^{-n/2}\sqrt{\det\left(\frac{t\Omega/2}{\sinh t\Omega/2}\right)}. \tag{5.4.30}$$

Combining (5.4.26) and (5.4.30) we conclude that the heat kernel of P_0 at the origin is

$$(4\pi t)^{-n/2}\sqrt{\det\left(\frac{t\Omega/2}{\sinh t\Omega/2}\right)}e^{-tF}. \tag{5.4.31}$$

The asymptotic expansion as $t \to 0$ of (5.4.31) is computed by forming the Taylor series of the last two factors. This gives an expansion of the form

$$(4\pi)^{-n/2}\sum_j P_j(\frac{\Omega}{2}, -F)t^{j-n/2}, \tag{5.4.32}$$

where P_j is a homogeneous polynomial of degree j. Hence the coefficient of $t^{j-n/2}$ is a $2j$-form. But this coefficient is the limit as $\epsilon \to 0$ of the coefficient of $t^{j-n/2}$ in (5.4.14):

$$\lim_{\epsilon\to 0}\sum_I \epsilon^{2j-|I|}A_{j,I}\hat{e}^I = P_j(\frac{\Omega}{2}, -F), \tag{5.4.33}$$

which implies statement (1) in Proposition 5.4.16, and also gives an explicit formula for $\sum\limits_{|I|=2j} A_{j,I}\hat{e}^I$. Finally,

$$P_{n/2}(\frac{\Omega}{2}, -F) = (2\pi i)^{n/2}P_{n/2}(\frac{\Omega}{4\pi i}, \frac{iF}{2\pi}), \tag{5.4.34}$$

and assertion (2) in Proposition 5.4.16 follows by combining (5.4), (5.5), (5.4.14), (5.4.31), (5.4.32), and (5.4.34). This completes the proof of Proposition 5.4.16 and so of the Atiyah-Singer index theorem.

§5.5 The Index Theorem for Generalized Dirac Operators

This section is a guided set of exercises working out the index formula (5.3) for the generalized Dirac operators of §2.3. We begin with the relationship between the Chern character and representation theory, which is a special case of the general picture explained in [BH]. In our discussion X is a closed Riemannian spin manifold of dimension $n = 2\ell$.

EXERCISE 5.5.1. Let Ω_0 be the skew-symmetric matrix

$$\Omega_0 = \begin{pmatrix} 0 & 2\pi x_1 & & & \\ -2\pi x_1 & 0 & & & \\ & & 0 & 2\pi x_2 & \\ & & -2\pi x_2 & 0 & \\ & & & & \ddots \end{pmatrix}, \tag{5.5.2}$$

Show that

$$\hat{A}(\Omega_0) = \prod_{j=1}^{\ell} \frac{x_j/2}{\sinh x_j/2}. \tag{5.5.3}$$

Note that we can always express the curvature $\Omega^{(X)}$ in the form (5.5.2) by a judicious choice of basis. Then the x_j are real 2-forms.

EXERCISE 5.5.4. Suppose $\rho\colon \mathrm{Spin}(n) \to \mathrm{Aut}(\mathbb{V})$ is a representation of $\mathrm{Spin}(n)$, and $V_\rho \to X$ the associated bundle $V_\rho = \mathrm{Spin}(X) \times_\rho \mathbb{V}$. Verify that V_ρ inherits a connection whose curvature is

$$\Omega^{(V_\rho)} = \dot{\rho}(\Omega^{(X)}), \tag{5.5.5}$$

where $\dot{\rho}\colon \mathfrak{so}(n) \to \mathrm{End}(\mathbb{V})$ is the induced homomorphism of Lie algebras. Verify that for Ω_0 as in (5.5.2),

$$\operatorname{Tr} e^{\frac{i}{2\pi}\dot{\rho}(\Omega_0)} = \chi_\rho(x_1, \dots, x_\ell) \tag{5.5.6}$$

is a symmetric function of the x_j. It is called the *character* of ρ. Hence the *index polynomial* associated to the coupled chiral Dirac operator D_{V_ρ} is

$$P_\rho(x_1, \dots, x_\ell) = \left(\prod_{j=1}^{\ell} \frac{x_j/2}{\sinh x_j/2} \right) \chi_\rho(x_1, \dots, x_\ell). \tag{5.5.7}$$

Next we calculate the index polynomial P_ρ for the operators introduced in §2.3.

EXERCISE 5.5.8. Use Exercise 2.1.48 to compute χ_ρ for the spin representation and the difference of spin representations:

$$\chi_{\mathbb{S}^+-\mathbb{S}^-}(x_1,\dots,x_\ell)=(-1)^\ell\prod_{j=1}^{\ell}2\sinh x_j/2,$$
$$\chi_{\mathbb{S}}(x_1,\dots,x_\ell)=\prod_{j=1}^{\ell}2\cosh x_j/2.$$

Conclude that

$$P_{\mathbb{S}^+-\mathbb{S}^-}(x_1,\dots,x_\ell)=(-1)^\ell\prod_{j=1}^{\ell}x_j, \tag{5.5.9}$$
$$P_{\mathbb{S}}(x_1,\dots,x_\ell)=\prod_{j=1}^{\ell}\frac{x_j}{\tanh x_j/2}. \tag{5.5.10}$$

Show that the term of degree ℓ in (5.5.10) is the same as the term of degree ℓ in

$$\prod_{j=1}^{\ell}\frac{x_j}{\tanh x_j} \tag{5.5.11}$$

if ℓ is even. Finally, write (5.5.9), (5.5.10) and (5.5.11) as invariant polynomials in the matrix Ω_0.

EXERCISE 5.5.12. Combine Exercise 2.3.12 and Exercise 5.5.8 to conclude

$$\chi(X)=\int_X \operatorname{Pfaff}(\Omega^{(X)}), \tag{5.5.13}$$

where for any skew-symmetric matrix Ω,

$$\operatorname{Pfaff}(\Omega)=\sqrt{\det(\Omega/2\pi i)}, \tag{5.5.14}$$

(Recall that when ℓ is odd the coupling bundle is $S^- - S^+$, which takes care of the sign in (5.5.9).) Formula (5.5.14) is called the *generalized Gauss-Bonnet theorem*. This was first proved by Allendoerfer and Weil [AW], and shortly thereafter Chern [C] gave an intrinsic proof. Treat the special case $n = 2$ to recover the classical Gauss-Bonnet theorem.

EXERCISE 5.5.15. Combine Exercise 2.3.24 and Exercise 5.5.8 to conclude

$$\operatorname{Sign}(X) = \int_X L(\Omega^{(X)}), \tag{5.5.16}$$

where for any skew-symmetric matrix Ω,

$$L(\Omega) = \sqrt{\det\left(\frac{\Omega/2\pi i}{\tanh \Omega/2\pi i}\right)}. \tag{5.5.17}$$

This is Hirzebruch's signature theorem (1.1.10).

EXERCISE 5.5.18. Compute the index polynomials for the self-dual complex and the anti-self-dual complex (Exercise 2.3.25).

EXERCISE 5.5.19. Recall from Exercise 2.3.39 that on a Kähler manifold X the operator $\frac{1}{2}(\bar\partial + \bar\partial^*)$ can be identified with the chiral Dirac operator coupled to $K^{-1/2}$, where K is the canonical bundle. To calculate the index, which is the arithmetic genus of X, we must compute $\operatorname{ch} K^{-1/2}$. This is slightly different than the previous examples, since $K^{-1/2}$ is not associated to a representation of $\operatorname{Spin}(2\ell)$, but rather to a representation of $\tilde U(\ell)$ (cf. Exercise 2.1.54). Now on a Kähler manifold with spin structure, the bundle of spin frames $\operatorname{Spin}(X)$ has a reduction $\tilde U(X)$ with structure group $\tilde U(\ell)$. Restricted to this bundle the curvature is a skew-Hermitian matrix (whose entries are 2-forms). The matrix

$$\begin{pmatrix} -2\pi i x_1 & & \\ & -2\pi i x_2 & \\ & & \ddots \end{pmatrix} \tag{5.5.20}$$

maps to Ω_0 under the natural map $\mathfrak{u}(\ell) \to \mathfrak{so}(2\ell)$. Compute that the character of $\det^{1/2}$ (the representation which defines $K^{-1/2}$) is

$$\chi_{\det^{1/2}}(x_1, \dots, x_\ell) = \prod_{j=1}^{\ell} e^{x_j/2}. \tag{5.5.21}$$

Conclude that the index polynomial is

$$P_{\det^{1/2}}(x_1, \dots, x_\ell) = \prod_{j=1}^{\ell} \frac{x_j}{1 - e^{-x_j}}. \tag{5.5.22}$$

Finally,

$$\mathrm{index}(\bar\partial + \bar\partial^*) = \int_X \mathrm{Todd}(\Omega^{(X)}), \tag{5.5.23}$$

where

$$\mathrm{Todd}(\Omega) = \sqrt{\det\left(\frac{\Omega/2\pi i}{1 - e^{-\Omega/2\pi i}}\right)}. \tag{5.5.24}$$

Equation (5.5.23) is a generalization of the classical Riemann-Roch theorem, first proved by Hirzebruch [H1] for smooth projective varieties (cf. Chapter 1).

§6 Superconnections

The basic invariant of a Dirac operator is its index, which is an integer. For a family of Dirac operators there are more sophisticated invariants. In the original treatment of Atiyah and Singer [AS4] these invariants take values in cohomology or K-theory. We start with more precise geometric data and so find more refined invariants. Thus instead of cohomology classes we obtain differential forms (whose de Rham cohomology classes are the old invariants). An element of K-theory is an equivalence class of differences of vector bundles. Differential geometers study equivalence classes of ordinary vector bundles via connections. Quillen [Q1], [Q2] introduced *superconnections* as the differential geometric representative of an element of K-theory. After we review the theory of connections and Chern-Weil theory (mostly in exercise form), we develop the formalism of superconnections in §6.2, following [Q1]. The main construction is the super Chern character, which we develop in §6.3. Our treatment of K-theory in §6.4 is incomplete in these preliminary notes.

§6.1 Review of Connections

Suppose $F \to Y$ is a smooth finite dimensional vector bundle over a smooth manifold Y. A connection ∇ on F is a first order differential operator on sections of F. Thus if s is a section of F and ξ a vector field on Y, then $\nabla_\xi s$ is a new section of F. The connection satisfies the Leibnitz rule

$$\nabla_\xi(fs) = (\xi f)s + f\nabla_\xi s. \tag{6.1.1}$$

Let $\Omega^* = \Omega^*_Y$ denote the algebra of differential forms on Y, and $\Omega^0(F)$ the space of smooth sections of F. Then $\Omega^*(F) = \Omega^0(F) \otimes_{\Omega^0} \Omega^*$ is the space of differential form valued sections of F. It is a module over Ω^*, and we let Ω^* act by *right* multiplication. A typical element in this module is $s\theta$, where $s \in \Omega^0(F)$, $\theta \in \Omega^*$. Since $\nabla_\xi s$ is tensorial in ξ, it follows that ∇s is a 1-form valued section of F. Thus

$$\nabla \colon \Omega^0(F) \longrightarrow \Omega^1(F). \tag{6.1.2}$$

We can extend to an operator

$$\nabla \colon \Omega^*(F) \longrightarrow \Omega^*(F) \tag{6.1.3}$$

by setting (for $\sigma \in \Omega^0(F)$, $\theta \in \Omega^*$)

$$\nabla(\sigma\theta) = (\nabla\sigma)\theta + \sigma\, d\theta. \tag{6.1.4}$$

Note that ∇ raises the degree of the differential form by one. An easy computation shows that ∇^2 is linear over Ω^*:

$$\begin{aligned}\nabla^2(\sigma\theta) &= \nabla\big((\nabla\sigma)\theta + \sigma\, d\theta\big)\\ &= (\nabla^2\sigma)\theta - (\nabla\sigma)d\theta + (\nabla\sigma)d\theta + \sigma\, d^2\theta \\ &= (\nabla^2\sigma)\theta.\end{aligned} \tag{6.1.5}$$

The minus sign in the second line comes from moving d past a 1-form. Let

$$\Omega^*(\operatorname{End} F) = \Omega^* \otimes_{\Omega^0} \Omega^0(\operatorname{End} F) \tag{6.1.6}$$

be the algebra of differential forms with values in $\operatorname{End} F$. Locally an element of $\Omega^*(\operatorname{End} F)$ is a matrix of differential forms. A typical element has the form θA for $\theta \in \Omega^*$, $A \in \Omega^0(\operatorname{End} F)$, and the multiplication is $\theta A \cdot \theta' A' = (\theta \wedge \theta')A\,A'$.

Proposition 6.1.7. *$\Omega^*(\operatorname{End} F)$ is the algebra of endomorphisms of $\Omega^*(F)$ which are linear over Ω^*.*

Proof. Any $\alpha \in \Omega^*(\operatorname{End} F)$ acts as an endomorphism of $\Omega^*(F)$ by left multiplication. Write $\alpha = \theta A$ and choose $\sigma = s\theta' \in \Omega^*(F)$. Then $\alpha(\sigma) = (As)(\theta\wedge\theta')$, and α is clearly linear over Ω^*. Conversely, if α is an endomorphism of $\Omega^*(F)$, linear over Ω^*, then the restriction of α to $\Omega^0(F)$ determines the desired element of $\Omega^*(\operatorname{End} F)$.

It follows from (6.1.5) and Proposition 6.1.7 that $\nabla^2 \in \Omega^2(\operatorname{End} F)$. (Since ∇ raises degree by 1, it is clear that ∇^2 is a 2-form.) We call ∇^2 the *curvature* of the connection ∇.[28]

These algebras and modules admit a natural $\mathbb{Z}$-grading. Thus the differential forms Ω^* form a $\mathbb{Z}$-graded algebra according to the decomposition $\Omega^* = \oplus_{i=0}^{\infty}\Omega^i$ into homogeneous forms. Exterior multiplication satisfies $\Omega^i \wedge \Omega^j \subset \Omega^{i+j}$. Furthermore, $\Omega^*(F)$ is a $\mathbb{Z}$-graded module over Ω^*, and $\Omega^*(\operatorname{End} F)$ is also a $\mathbb{Z}$-graded algebra (whose homogeneous elements are graded endomorphisms of $\Omega^*(F)$). The following definition is simply a restatement of (6.1.4).

Definition 6.1.8. *A connection is a derivation of $\Omega^*(F)$ over Ω^* of degree $+1$*

We develop some properties of connections in the exercises. Many of these will be repeated in the context of superconnections.

Exercise 6.1.9. Show that the difference of two connections is an element of $\Omega^1(\operatorname{End} F)$. If ∇ is a connection and $\omega \in \Omega^1(\operatorname{End} F)$, what is the curvature of $\nabla + \omega$?

Exercise 6.1.10. Reconcile our present view of connections with the treatment in Chapter 2 of the Riemannian connection and curvature.

[28] In previous chapters we denoted this curvature by $\Omega^{(F)}$.

Exercise 6.1.11. Let $\iota\colon F' \hookrightarrow F$ be a subbundle, and $Q\colon F \to F'$ a projection. Fix a connection ∇ on F. Define the projected connection $Q\nabla\iota$ on F' and compute its curvature.

Exercise 6.1.12. For a vector space $\mathbb{F}$ we let $\mathrm{Gr}_k(\mathbb{F})$ be the space of k-dimensional subspaces of $\mathbb{F}$. Endow $\mathbb{F}$ with an inner product. Than an element of $\mathrm{Gr}_k(\mathbb{F})$ is described by a rank k orthogonal projection $e \in \mathrm{End}(\mathbb{F})$. The trivial bundle $\mathrm{Gr}_k(\mathbb{F}) \times \mathbb{F}$ has a canonical subbundle $S \to \mathrm{Gr}_k(\mathbb{F})$ whose fiber over $e \in \mathrm{Gr}_k(\mathbb{F})$ is the image of e. Define a natural connection on S and compute its curvature.

Exercise 6.1.13. The *trace* is a linear map $\mathrm{tr}\colon \Omega^0(\mathrm{End}\,F) \to \Omega^0$. Note that $\mathrm{tr}[A,B] = 0$. Extend to a map $\mathrm{tr}\colon \Omega^*(\mathrm{End}\,F) \to \Omega^*$, and show that the trace of a *supercommutator* is zero: $\mathrm{tr}[\alpha,\beta]_s = 0$ for $\alpha,\beta \in \Omega^*(\mathrm{End}\,F)$. (If α,β are homogeneous, then $[\alpha,\beta]_s = \alpha\beta + (-1)^{|\alpha||\beta|}\beta\alpha$.) If $\alpha \in \Omega^*(\mathrm{End}\,F)$ and ∇ is any connection, prove

$$d\ \mathrm{tr}\,\alpha = \mathrm{tr}(\nabla\alpha). \tag{6.1.14}$$

Here $\nabla\alpha$ is defined using the connection on $\mathrm{End}\,F$ induced from the connection on F. As an operator on $\Omega^*(F)$ it acts by the supercommutator $[\nabla,\alpha]_s$.

Exercise 6.1.15. Let ∇ be a connection on F. Prove that $\mathrm{tr}\,\nabla^2$ is closed for all $k \geq 0$. Conclude that $\mathrm{tr}\,e^{-\nabla^2}$ is a closed form.

Exercise 6.1.16. Let ∇_u be a 1-parameter family of connections on F. Show that $\dot\nabla_u = \frac{d\nabla_u}{du}$ is an element of $\Omega^1(\mathrm{End}\,F)$. Prove that

$$\frac{d\ \mathrm{tr}\,e^{-\nabla_u^2}}{du} = -d\{\mathrm{tr}(e^{-\nabla_u^2}\dot\nabla_u)\}. \tag{6.1.17}$$

Conclude that if ∇,∇' are two connections on F, then $\mathrm{tr}\,e^{-\nabla^2} - \mathrm{tr}\,e^{-\nabla'^2}$ is exact.

Let ∇ be a connection on F. The *Chern character* of ∇ is the differential form

$$\mathrm{ch}(\nabla) = \mathrm{tr}\,e^{-\nabla^2}. \tag{6.1.18}$$

It is an inhomogeneous differential form on Y with components in degrees 0, 2, 4, … . Our normalization is not the usual one.[29] (Usually one takes $\mathrm{ch}(\nabla) = \mathrm{tr}\exp(i\nabla^2/2\pi)$.) By Exercise 6.1.16 the de Rham cohomology class of $\mathrm{ch}(\nabla)$ is independent of the connection ∇, and so is a topological invariant of F. In fact, the de Rham cohomology class of $(\frac{-i}{2\pi})^k$ times the k-form component of $\mathrm{ch}(\nabla)$ is the k^{th} *Chern character class* of F. We develop this *Chern-Weil Theory* and the relationship to characteristic classes in the exercises below.

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[29] and we are not at all happy about it. However, when we come to superconnections there seems to be no nice way to normalize so that the differential forms represent the usual Chern character classes. Hence we choose (6.1.18) to be consistent with our choice for superconnections below.

§6.2 Superconnections in finite dimensions

Suppose now that $F = F^+ \oplus F^-$ is $\mathbb{Z}/2\mathbb{Z}$-graded. We replace the $\mathbb{Z}$-algebras and modules of the previous section with $\mathbb{Z}/2\mathbb{Z}$-graded algebras and modules. Thus we forget the $\mathbb{Z}$-grading on Ω^* and view $\Omega^* = \Omega^+ \oplus \Omega^-$ as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra: A homogeneous differential form is even if its degree is even and odd if its degree is odd. The space of sections $\Omega^0(F) = \Omega^0(F^+) \oplus \Omega^0(F^-)$ is $\mathbb{Z}/2\mathbb{Z}$-graded, and so $\Omega^*(F)$ is a graded[30] module over Ω^*. The endomorphisms $\Omega^0(\operatorname{End} F)$ are also graded: An element A is even if it preserves the homogeneous components of $\Omega^0(F)$ (i.e., is a diagonal matrix) and odd if it permutes the homogeneous components of $\Omega^0(F)$ (i.e., is an off-diagonal matrix). As in (6.1.6) we set

$$\Omega^*(\operatorname{End} F) = \Omega^* \hat{\otimes}_{\Omega^0} \Omega^0(\operatorname{End} F), \tag{6.2.1}$$

but now the tensor product is taken in the graded sense.

EXERCISE 6.2.2. Suppose $\mathcal{A}, \mathcal{B}$ are two graded algebras. Their graded tensor product $\mathcal{A}\hat{\otimes}\mathcal{B}$ is generated by elements $a \otimes b$,
$a \in \mathcal{A}, b \in \mathcal{B}$ with multiplication on homogeneous elements defined by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'. \tag{6.2.3}$$

Here $|a|$ is the degree of a, which is 0 if a is even and 1 if a is odd. Verify that $\mathcal{A}\hat{\otimes}\mathcal{B}$ is a graded algebra.

EXERCISE 6.2.4. A graded algebra $\mathcal{A}$ is commutative if

$$aa' = (-1)^{|a||a'|} a'a \tag{6.2.5}$$

for all homogeneous a, a'. Give an example of a graded commutative algebra. Give an example of a graded algebra which is not commutative. Show that if $\mathcal{A}, \mathcal{B}$ are commutative then so is $\mathcal{A}\hat{\otimes}\mathcal{B}$.

EXERCISE 6.2.6. The *graded commutator* (or *supercommutator*) of homogeneous elements $a, a' \in \mathcal{A}$ is (cf. (5.1.6))

$$[a, a']_s = aa' - (-1)^{|a||a'|} a'a. \tag{6.2.7}$$

Write the appropriate Jacobi identity for $[\cdot,\cdot]_s$. If $\mathcal{A}, \mathcal{B}$ are graded algebras, compute $[a \otimes b, a' \otimes b']_s$.

[30] We use 'graded' for '$\mathbb{Z}/2\mathbb{Z}$-graded.' Often the word 'super' is substituted for '$\mathbb{Z}/2\mathbb{Z}$-graded,' whence the terminology 'superalgebra,' 'superconnection,' etc.

Exercise 6.2.8. $\Omega^*(F)$ can be viewed as either a left or a right Ω^*-module. If $\sigma \in \Omega^*(F), \theta \in \Omega^*$ are homogeneous, then

$$\sigma\theta = (-1)^{|\theta||\sigma|}\theta\sigma. \tag{6.2.9}$$

Proposition 6.1.7 still holds—$\Omega^*(\operatorname{End} F)$ is the graded algebra of endomorphisms of $\Omega^*(F)$ as a graded Ω^*-module. An even element of $\Omega^*(\operatorname{End} F)$ is a matrix with even forms on the diagonal and odd forms on the off-diagonal; an odd element of $\Omega^*(\operatorname{End} F)$ is a matrix with odd forms on the diagonal and even forms on the off-diagonal.

Definition 6.2.10. *A superconnection $\boldsymbol{\nabla}$ on F is a graded derivation of $\Omega^*(F)$ over Ω^* of odd degree. Thus for $\sigma \in \Omega^*(F), \theta \in \Omega^*$ homogeneous,*

$$\boldsymbol{\nabla}(\sigma\theta) = (\boldsymbol{\nabla}\sigma)\theta + (-1)^{|\sigma|}\sigma\, d\theta. \tag{6.2.11}$$

This should be compared with Definition 6.1.8. Note that we could equally write (6.2.11) as

$$\boldsymbol{\nabla}(\theta\sigma) = d\theta\,\sigma + (-1)^{|\theta|}\theta\,\boldsymbol{\nabla}\sigma. \tag{6.2.12}$$

Let $\nabla = \nabla^+ \oplus \nabla^-$ be an ordinary connection (in the sense of (6.1.2)) on $F = F^+ \oplus F^-$ which preserves the grading. Thus ∇^+ is a connection on F^+ and ∇^- is a connection on F^-. We can extend ∇ to a graded derivation of $\Omega^*(F)$ satisfying (6.2.11), and so extend ∇ to a superconnection. Note that the extension is different than that of an ordinary connection. Thus if $s^- \in \Omega^0(F^-)$ and $\theta \in \Omega^*$, then for the superconnection extension $\boldsymbol{\nabla}$ we have

$$\boldsymbol{\nabla}(s^-\theta) = (\nabla^- s^-)\theta - s^- d\theta. \tag{6.2.13}$$

The difference of two superconnections is an odd element of $\Omega^*(\operatorname{End} F)$. Hence the general superconnection can be written

$$\boldsymbol{\nabla} = \nabla + \omega = \begin{pmatrix} \nabla^+ + \omega^{++} & \omega^{+-} \\ \omega^{-+} & \nabla^- + \omega^{--} \end{pmatrix}, \tag{6.2.14}$$

where $\omega \in \Omega^*(\operatorname{End} F)^-$, $\omega^{++} \in \Omega^{\text{odd}}(\operatorname{End} F^+)$, $\omega^{+-} \in \Omega^{\text{even}}\big(\operatorname{Hom}(F^-, F^+)\big)$, $\omega^{-+} \in \Omega^{\text{even}}\big(\operatorname{Hom}(F^+, F^-)\big)$, and $\omega^{--} \in \Omega^{\text{odd}}(\operatorname{End} F^-)$.

The *curvature* of $\boldsymbol{\nabla}$ is the operator $\boldsymbol{\nabla}^2$. As for ordinary connections (see (6.1.5)) it is linear over Ω^*:

$$\begin{aligned}\boldsymbol{\nabla}^2(\sigma\theta) &= \boldsymbol{\nabla}\big((\boldsymbol{\nabla}\sigma)\theta + (-1)^{|\sigma|}\sigma\, d\theta\big) \\ &= (\boldsymbol{\nabla}^2\sigma)\theta + (-1)^{|\sigma|+1}(\boldsymbol{\nabla}\sigma)d\theta + (-1)^{|\sigma|}\boldsymbol{\nabla}\sigma\, d\theta + s\, d^2\theta \\ &= (\boldsymbol{\nabla}^2\sigma)\theta.\end{aligned} \tag{6.2.15}$$

So $\boldsymbol{\nabla}^2$ is an even element of $\Omega^*(\operatorname{End} F)$—it has even forms on the diagonal and odd forms on the off-diagonal. If $\boldsymbol{\nabla} = \nabla + \omega$,

$$\boldsymbol{\nabla}^2 = \nabla^2 + \nabla\omega + \omega^2. \tag{6.2.16}$$

Suppose F^+, F^- are Hermitian, ∇^+, ∇^- are unitary connections, and $L\colon F^+ \to F^-$ is a linear map. Then

$$\mathcal{L} = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}$$

is an odd element of $\Omega^*(\operatorname{End} F)$. We define a family of superconnections

$$\boldsymbol{\nabla}_t = \nabla + \sqrt{t}\mathcal{L} = \begin{pmatrix} \nabla^+ & \sqrt{t}L^* \\ \sqrt{t}L & \nabla^- \end{pmatrix}. \tag{6.2.17}$$

The curvature of $\boldsymbol{\nabla}$ is

$$\boldsymbol{\nabla}_t^2 = \nabla^2 + \sqrt{t}\,\nabla\mathcal{L} + t\mathcal{L}^2 = \begin{pmatrix} {\nabla^+}^2 + tL^*L & \sqrt{t}\nabla L^* \\ \sqrt{t}\nabla L & {\nabla^-}^2 + tL\,L^* \end{pmatrix}. \tag{6.2.18}$$

Notice that the diagonal entries are sums of a 0-form and a 2-form, whereas the off-diagonal entries are 1-forms. Also, ∇L denotes the transformation $\nabla^- L - L\nabla^+$ and ∇L^* denotes $\nabla^+ L^* - L^*\nabla^-$. We study the family of superconnections (6.2.17) in detail in the next section.

Exercise 6.2.19. Let $\mathbb{F} = \mathbb{F}^+ \oplus \mathbb{F}^-$ be a graded vector space. Set $Y = \operatorname{End}(\mathbb{F}^+, \mathbb{F}^-)$, and F^+, F^- the obvious trivial bundles over Y. There is a canonical vector bundle map $L\colon F^+ \to F^-$. Define the superconnection (6.2.17) and compute its curvature.

§6.3 The super Chern character

Recall that the supertrace is defined on endomorphisms of a graded vector space $\mathbb{F} = \mathbb{F}^+ \oplus \mathbb{F}^-$. Let ϵ be the involution which defines the grading; it is $+1$ on $\mathbb{F}^+$ and -1 on $\mathbb{F}^-$. Then for $A \in \operatorname{End} F$ we have (cf. (5.1.4))

$$\operatorname{tr}_s A = \operatorname{tr}(\epsilon A). \tag{6.3.1}$$

Thus if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathrm{tr_s}\, A = \mathrm{tr}\, a - \mathrm{tr}\, d$. By Lemma 5.1.7 the supertrace vanishes on supercommutators: $\mathrm{tr_s}[A,B]_s = 0$.

Now let $F = F^+ \oplus F^-$ be a graded bundle over Y, as before. Then $\mathrm{tr_s}\colon \Omega^0(\mathrm{End}\, F) \to \Omega^0$ extends to an operator

$$\mathrm{tr_s}\colon \Omega^*(\mathrm{End}\, F) \longrightarrow \Omega^* \tag{6.3.2}$$

in the obvious way: $\mathrm{tr_s}(\theta A) = (\mathrm{tr_s}\, A)\theta$. This extension satisfies

$$\mathrm{tr_s}[\alpha,\beta]_s = 0 \tag{6.3.3}$$

for $\alpha,\beta \in \Omega^*(\mathrm{End}\, F)$, where the supercommutator is taken in the superalgebra $\Omega^*(\mathrm{End}\, F)$.

Exercise 6.3.4. Prove (6.3.3). (Compare Exercise 6.1.13.)

If $\boldsymbol{\nabla}$ is any superconnection on F, then for $\alpha \in \Omega^*(\mathrm{End}\, F)$ we have

$$d\ \mathrm{tr_s}\, \alpha = \mathrm{tr_s}(\boldsymbol{\nabla}\alpha), \tag{6.3.5}$$

where $\boldsymbol{\nabla}\alpha$ denotes the endomorphism $[\boldsymbol{\nabla},\alpha]_s$ of $\Omega^*(F)$.

Exercise 6.3.6. Prove (6.3.5). (Compare (6.1.14). Hint: Compute locally, writing $\boldsymbol{\nabla} = d + \omega$ relative to some trivialization of F.)

Now we define the *super Chern character*[31] by analogy with (6.1.18). Thus let $\boldsymbol{\nabla}$ be a superconnection on F. Set

$$\mathrm{ch_s}(\boldsymbol{\nabla}) = \mathrm{tr_s}\, e^{-\boldsymbol{\nabla}^2}. \tag{6.3.7}$$

It is an even differential form, with components in degrees 0, 2, 4, ...

Proposition 6.3.8. *The differential form* $\mathrm{ch_s}(\boldsymbol{\nabla})$ *is closed. If* $\boldsymbol{\nabla}_u$ *is a family of superconnections parametrized by a real parameter* u*, then*

$$\frac{d\ \mathrm{tr_s}\, e^{-\boldsymbol{\nabla}_u^2}}{du} = -d\left\{\mathrm{tr_s}(e^{-\boldsymbol{\nabla}_u^2}\,\dot{\boldsymbol{\nabla}}_u)\right\}. \tag{6.3.9}$$

In particular, if $\boldsymbol{\nabla}$ *and* $\boldsymbol{\nabla}'$ *are two superconnections on* F*, then* $\mathrm{ch_s}(\boldsymbol{\nabla}) - \mathrm{ch_s}(\boldsymbol{\nabla}')$ *is exact.*

[31] In the literature this is simply called the 'Chern character.' We introduce the 'super' for greater clarity.

Note that $\dot{\boldsymbol{\nabla}} = d\boldsymbol{\nabla}_u/du$ in (6.3.9) is an odd element of $\Omega^*(\operatorname{End} F)$. Equation (6.3.9) is sometimes called the *transgression formula*.

Proof. Using (6.3.5) we calculate

$$\begin{aligned} d\, \mathrm{ch_s}(\boldsymbol{\nabla}) &= d\, \mathrm{tr_s}\, e^{-\boldsymbol{\nabla}^2} \\ &= \mathrm{tr_s}[\boldsymbol{\nabla}, e^{-\boldsymbol{\nabla}^2}]_s \\ &= 0, \end{aligned} \tag{6.3.10}$$

since $\boldsymbol{\nabla}$ and $e^{-\boldsymbol{\nabla}^2}$ commute. To verify the *transgression formula* (6.3.9), we compute (omitting 'u' from the notation)

$$\frac{d}{du}\mathrm{tr_s}\, e^{-\boldsymbol{\nabla}_u^2} = -\,\mathrm{tr_s}(e^{-\boldsymbol{\nabla}^2}\boldsymbol{\nabla}\dot{\boldsymbol{\nabla}} + e^{-\boldsymbol{\nabla}^2}\dot{\boldsymbol{\nabla}}\boldsymbol{\nabla}), \tag{6.3.11}$$

and

$$\begin{aligned} -d(\mathrm{tr_s}\, e^{-\boldsymbol{\nabla}^2}\dot{\boldsymbol{\nabla}}) &= -\,\mathrm{tr_s}\big([\boldsymbol{\nabla}, e^{-\boldsymbol{\nabla}^2}\dot{\boldsymbol{\nabla}}]_s\big) \\ &= -\,\mathrm{tr_s}(\boldsymbol{\nabla} e^{-\boldsymbol{\nabla}^2}\dot{\boldsymbol{\nabla}} + e^{-\boldsymbol{\nabla}^2}\dot{\boldsymbol{\nabla}}\boldsymbol{\nabla}). \end{aligned} \tag{6.3.12}$$

Since $\boldsymbol{\nabla}$ and $e^{-\boldsymbol{\nabla}^2}$ commute we conclude that (6.3.11) and (6.3.12) are equal, as desired. The last statement in the proposition follows by setting

$$\boldsymbol{\nabla}_u = \boldsymbol{\nabla} + u(\boldsymbol{\nabla}' - \boldsymbol{\nabla})$$

and integrating (6.3.9) from $u = 0$ to $u = 1$.

Corollary 6.3.13. *Let $\nabla = \nabla^+ \oplus \nabla^-$ be a connection on the (finite dimensional) bundle $F = F^+ \oplus F^-$, and $\boldsymbol{\nabla} = \nabla + \omega$ any superconnection on F. Then $\mathrm{ch_s}(\boldsymbol{\nabla})$ and $\mathrm{ch}(\nabla^+) - \mathrm{ch}(\nabla^-)$ are cohomologous in de Rham theory.*

Here $\mathrm{ch}(\nabla^\pm)$ is the Chern character of the ordinary connection on $F^\pm$, as defined in (6.1.18).

Proof. Set $\boldsymbol{\nabla}_u = \nabla + u\omega$ and apply (6.3.9). Note that $\mathrm{ch_s}(\boldsymbol{\nabla}_1) = \mathrm{ch_s}(\boldsymbol{\nabla})$ and $\mathrm{ch_s}(\boldsymbol{\nabla}_0) = \mathrm{ch}(\nabla^+) - \mathrm{ch}(\nabla^-)$.

In the rest of this section we study in detail the Chern character form γ_t of the superconnection (6.2.17). Thus

$$\gamma_t = \mathrm{ch_s}(\boldsymbol{\nabla}_t) = \mathrm{tr_s}\, e^{-(t\mathcal{L}^2 + \sqrt{t}\,\nabla\mathcal{L} + \nabla^2)}. \tag{6.3.14}$$

Observe first that

$$\gamma_0 = \mathrm{ch_s}(\nabla) = \mathrm{ch}(\nabla^+) - \mathrm{ch}(\nabla^-). \tag{6.3.15}$$

Now consider $t \to \infty$. If L is invertible then we expect the first term $t\mathcal{L}^2$ in the exponential to dominate (6.3.14), and so expect $\gamma_t \to 0$. This is correct, but to prove it requires a few preliminary formulæ which are important in their own right.

The main result already appeared in (4.2.24); it is sometimes called *Duhamel's formula.* Although we apply it here to finite dimensional matrices, we state it in greater generality. Recall that a *Banach algebra* is a Banach space with an algebra structure whose multiplication satisfies $\|AB\| \le \|A\|\,\|B\|$.

Proposition 6.3.16. *Let $\mathcal{A}$ be a Banach algebra. Then for $P_1, P_2 \in \mathcal{A}$ we have*

$$e^{-P_2} - e^{-P_1} = -\int_0^1 ds\, e^{-sP_1}(P_2 - P_1)e^{-(1-s)P_2}. \tag{6.3.17}$$

Proof. Observe

$$\frac{d}{ds}\left(e^{-sP_1}e^{-(1-s)P_2}\right) = e^{-sP_1}(P_2 - P_1)e^{-(1-s)P_2}, \tag{6.3.18}$$

and integrate from $s = 0$ to $s = 1$.

We rewrite (6.3.17) in the form

$$e^{-P_2} = e^{-P_1} - e^{-P_1}(P_2 - P_1)\#e^{-P_2}, \tag{6.3.19}$$

the '#' being shorthand for the convolution in (6.3.17). Then iterating (6.3.19) we obtain

(6.3.20)
$$\begin{aligned} e^{-P_2} &= e^{-P_1} - e^{-P_1}(P_2 - P_1)\#e^{-P_1} + e^{-P_1}(P_2 - P_1)\#e^{-P_1}(P_2 - P_1)\#e^{-P_1} - \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k [e^{-P_1}(P_2 - P_1)]^{\#k}\#e^{-P_1}. \end{aligned}$$

This should be viewed as the noncommutative analog of a Taylor series. Indeed, if P_1 and $P_2 - P_1$ commute, then we can write

$$e^{-P_2} = e^{-(P_2 - P_1)}e^{-P_1} = \sum_{k=0}^{\infty} (-1)^k \frac{(P_2 - P_1)^k}{k!} e^{-P_1}.$$

It is important to realize that the k^{th} term in (6.3.20) is an integral over a k-simplex of volume $1/k!$. For example, the second term is

$$\int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{-s_2 P_1}(P_2 - P_1)e^{-(s_1-s_2)P_1}(P_2 - P_1)e^{-(1-s_1)P_1}. \tag{6.3.21}$$

The following exercises show how Duhamel's formula lends insight into the structure of γ_t.

Exercise 6.3.22. Apply Duhamel's formula to (6.3.14), choosing $P_2 = \boldsymbol{\nabla}_t^2 = t\mathcal{L}^2 + \sqrt{t}\,\nabla\mathcal{L} + \nabla^2$ and $P_1 = t\mathcal{L}^2$. By writing out (6.3.17) deduce that the 0-form component of γ_t is $\operatorname{tr}_s e^{-t\mathcal{L}^2}$. Write out the next term in (6.3.20) to see explicitly that the 1-form component of γ_t vanishes. What is the formula for the 2-form component?

Exercise 6.3.23. Write an explicit formula for the 1-form component of $\operatorname{tr}_s(e^{-\boldsymbol{\nabla}_t^2}\dot{\boldsymbol{\nabla}}_t)$. This term appears in the transgression formula (6.3.9).

Next we apply Duhamel's formula to derive a standard estimate on exponentials.

Proposition 6.3.24. *Let $\mathcal{A}$ be a Banach algebra, and fix $A, B \in \mathcal{A}$. Suppose $\|e^{sA}\| \le Me^{sa}$ for some constants $M, a > 0$ and all $0 \le s \le 1$. Then $\|e^{A+B}\| \le Me^{a+M\|B\|}$.*

Proof. By (6.3.20) we write

$$e^{A+B} = \sum_{k=0}^{\infty} (e^A B)^{\#k} \# e^A. \tag{6.3.25}$$

The k^{th} term is an integral over the k-simplex involving $k+1$ terms of the form $e^{(s_i - s_{i+1})A}$ and k factors of B. (See (6.3.21) for the case $k = 2$.) Since the volume of the k-simplex is $1/k!$, it follows from our hypothesis that the norm of the k^{th} term is bounded by $\dfrac{M^{k+1}e^a\|B\|^k}{k!}$, whence

$$\|e^{A+B}\| \le \sum_{k=0}^{\infty} \frac{M^{k+1}e^a\|B\|^k}{k!} = Me^{a+M\|B\|}.$$

Exercise 6.3.26. These ideas occur in the theory of ordinary differential equations. Suppose $A(t)$ is a curve of $N \times N$ matrices, say real matrices, and $u(t)$ is a variable curve in $\mathbb{R}^N$. Write an explicit solution to the equation

$$\frac{du}{dt} + Au = 0 \tag{6.3.27}$$

with initial condition u_0. Vary the initial conditions to see the solution as a curve of invertible matrices. Can you see this in terms of integration of a time-varying vector field on some manifold?

Show that if $A(t)$ is constrained to lie in the Lie algebra of a Lie group G, then the solution to (6.3.27) lies in G. Derive the equation for parallel transport on a principal bundle (or vector bundle) with connection.

With these preliminaries in hand, we return to our study of the superconnection $\boldsymbol{\nabla}_t = \nabla + \sqrt{t}\mathcal{L}$. Recall that γ_t denotes the super Chern character form of $\boldsymbol{\nabla}_t$.

Proposition 6.3.28. *Suppose $L\colon F^+ \to F^-$ is invertible. Then $\lim_{t\to\infty} \gamma_t = 0$ and*

$$\gamma_t = -d\left(\frac{1}{2\sqrt{t}}\int_t^\infty \operatorname{tr_s}\left[e^{-(t\mathcal{L}^2+\sqrt{t}\,\nabla\mathcal{L}+\nabla^2)}\mathcal{L}\right]\right). \tag{6.3.29}$$

Proof. Since $\mathcal{L}^2$ is invertible, $\|\mathcal{L}^2\| \geq a$ for some $a > 0$, so that $\|e^{-st\mathcal{L}^2}\| \leq e^{-sta}$ for all $s, t \geq 0$. Now Proposition 6.3.24 implies

$$\|e^{-(t\mathcal{L}^2+\sqrt{t}\,\nabla\mathcal{L}+\nabla^2)}\| \leq e^{-ta+C}$$

for some constant C This approaches zero as $t \to \infty$, and since $\operatorname{tr_s}$ is a continuous map, it follows that $\lim_{t\to\infty} \gamma_t = 0$. Equation (6.3.29) now follows from the transgression formula (6.3.9), integrated from $u = t$ to $u = \infty$.

If L is not invertible, then the same argument works away from the kernel of L. It is useful to recall our favorite picture (Figure 1) at this juncture. The operator $\mathcal{L} = \begin{pmatrix} L^*L & 0 \\ 0 & L\,L^* \end{pmatrix}$ is nonnegative self-adjoint, so for fixed y decomposes $F_y^+ \oplus F_y^-$ into a finite sum of eigenspaces. The eigenspaces of $L^*L, L\,L^*$ for nonzero eigenvalues are isomorphic via the map L. The situation is illustrated schematically in Figure 2. As y varies the eigenvalues vary continuously, the multiplicities of the eigenvalues can change, etc. In particular, there is no guarantee that $\ker L$ and $\ker L^*$ have constant rank (cf. Exercise 6.2.19).

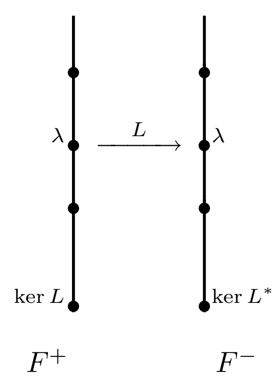

Figure 2

For the next result, we make an explicit assumption about the kernels.

Proposition 6.3.30. *Suppose* $\ker\mathcal{L} = \ker L \oplus \ker L^*$ *has locally constant rank. Let* $P_{\ker\mathcal{L}}$ *denote the orthogonal projection of* F *onto* $\ker\mathcal{L}$*, and* $P_{\ker\mathcal{L}}\nabla$ *the projected connection on* $\ker\mathcal{L}$ *(cf. Exercise 6.1.11). Then*

$$\lim_{t\to\infty}\gamma_t = \operatorname{ch}_s(P_{\ker\mathcal{L}}\nabla) = \operatorname{ch}(P_{\ker L}\nabla^+) - \operatorname{ch}(P_{\ker L^*}\nabla^-).$$

Proof. We can split the supertrace in the definition of γ_t into the sum of a supertrace over $\ker\mathcal{L}$ and a supertrace over the orthogonal complement. The argument of Proposition 6.3.28 applies to show that the supertrace over the complement converges to zero as $t\to\infty$. Hence

$$\begin{aligned}\lim_{t\to\infty}\gamma_t &= \lim_{t\to\infty}\operatorname{tr}_s\left(P_{\ker\mathcal{L}}e^{-(t\mathcal{L}^2+\sqrt{t}\,\nabla\mathcal{L}+\nabla^2)}P_{\ker\mathcal{L}}\right)\\ &= \lim_{t\to\infty}\operatorname{tr}_s e^{-P_{\ker\mathcal{L}}(t\mathcal{L}^2+\sqrt{t}\,\nabla\mathcal{L}+\nabla^2)P_{\ker\mathcal{L}}}.\end{aligned}\tag{6.3.31}$$

Now $P_{\ker\mathcal{L}}(t\mathcal{L}^2)P_{\ker\mathcal{L}} = 0$ is obvious. In fact, the restriction of $\nabla\mathcal{L}$ to $\ker\mathcal{L}$ is also zero. For example, the restriction of ∇L to $\ker L$, projected to $\ker L^*$, is

$$P_{\ker L^*}(\nabla^- L - L\nabla^+)P_{\ker L} = P_{\ker L^*}\nabla^- LP_{\ker L} - P_{\ker L^*}L\nabla^+P_{\ker L},$$

and $LP_{\ker L} = P_{\ker L^*}L = 0$. The argument for the adjoint similar. Hence

$$\lim_{t\to\infty}\gamma_t = \operatorname{tr}_s e^{-P_{\ker\mathcal{L}}\nabla^2P_{\ker\mathcal{L}}}.\tag{6.3.32}$$

But $P_{\ker\mathcal{L}}\nabla^2P_{\ker\mathcal{L}}$ is the curvature of the projected connection $P_{\ker\mathcal{L}}\nabla$ (cf. Exercise 6.1.11), whence the proposition.

When L has variable rank the same statement is morally true, only now $\ker\mathcal{L}$ is a sheaf, rather than a vector bundle. We will see how to handle this in the next section.

§6.4 K-Theory

Suppose Y is a compact space. The set of complex vector bundles on Y is a semigroup; that is, given two vector bundles F, F' over Y we can form their direct sum $F\oplus F'$. The idea of *K-theory* is to add inverses to this situation. Thus we define a *K-bundle*[32] to be a formal difference of vector

[32] Standard language distinguishes easily between a vector bundle and its equivalence class under isomorphisms of vector bundles. By contrast, an element of K-theory is understood to be an equivalence class, and there is no standard word to denote a member of the equivalence class. Hence we coin the phrase *K-bundle* .

bundles $F^+ - F^-$ over Y. It is clear how to add and subtract elements: $(F^+ - F^-) + ({F'}^+ - {F'}^-) = F^+ \oplus {F'}^+ - F^- \oplus {F'}^-$, and $-(F^+ - F^-) = F^- - F^+$. There is a natural equivalence relation: $F^+ - F^- \sim {F'}^+ - {F'}^-$ if there is an isomorphism $F^+ \oplus {F'}^- \cong {F'}^+ \oplus F^-$. The set of equivalence classes form the K-theory group $K(Y)$.

We will need to patch K-bundles which are only defined locally. This is analogous to the patching description of a vector bundle. The K-bundles we are interested in will come in this form. Also, in this way we extend the definition of K-theory to noncompact manifolds (including infinite dimensional manifolds). Thus let $\{U_\alpha\}$ be a cover of a manifold Y, and suppose $F_\alpha^+ - F_\alpha^- \to U_\alpha$ is a K-bundle over U_α. The patching isomorphisms are maps

$$g_{\alpha\beta}\colon F_\alpha^+ \oplus F_\beta^- \big|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} F_\beta^+ \oplus F_\alpha^- \big|_{U_\alpha \cap U_\beta}$$

which satisfy the cocycle condition

$$g_{\alpha\beta} \oplus g_{\beta\gamma} = g_{\alpha\gamma} \oplus \mathrm{id}_{F_\beta^+ \oplus F_\beta^-} .$$

We define the collection $\mathbf{F} = \{F_\alpha^+ - F_\alpha^-, g_{\alpha\beta}\}$ to be a K-bundle over Y. Notice that we do not require that each $F_\alpha^+ - F_\alpha^-$ be a trivial bundle, though this can be arranged by choosing the covering $\{U_\alpha\}$ fine enough. If $\mathbf{F}' = \{{F'}_\alpha^+ - {F'}_\alpha^-, g'_{\alpha\beta}\}$ is another K-bundle , then an isomorphism $\mathbf{F} \cong \mathbf{F}'$ is specified by a collection of maps $f_\alpha\colon F_\alpha^+ \oplus {F'}_\alpha^- \xrightarrow{\sim} {F'}_\alpha^+ \oplus F_\alpha^-$ which intertwine appropriately with the $g_{\alpha\beta}, g'_{\alpha\beta}$.

Exercise 6.4.1. Write down the precise definition. Verify that it defines an equivalence relation.

Exercise 6.4.2. Suppose $L : F^+ \to F^-$ is an isomorphism. Prove that the K-bundle $F^+ - F^-$ is equivalent to the trivial bundle (of rank 0).

We denote the set of equivalence classes of K-bundles over Y by $K(Y)$—this is our extended definition of K-theory. Notice that $K(Y)$ is an abelian group.

Exercise 6.4.3. Show that if Y is compact, then any K-bundle in this new sense is equivalent to a global K-bundle $F^+ - F^- \to Y$ (for globally defined F^+, F^-). Is this also true for locally compact spaces?

Exercise 6.4.4. Some topologists are inclined to define $K(Y)$ as the set of homotopy classes of maps $Y \to \mathbb{Z} \times BU$, where BU is the classifying space of the stable unitary group. How does our definition compare with the topologists'? You can make this comparison for any CW-complex Y; our definition doesn't require that Y be a manifold.

Exercise 6.4.5. Give a definition of $K^1(Y)$ in the spirit of our treatment. What about the relative groups $K(Y, A)$? Can you describe some of the boundary maps in the standard exact sequences (e.g. $K^1(A) \to K(Y, A)$)?

Now consider a map $F^+ \xrightarrow{L} F^-$ of vector bundles over Y. We want to construct an element of $K(Y)$ in this situation. The obvious candidate is the global K-bundle $F^+ - F^-$, but this does not take into account the map L. Rather, we want to use L to "cancel" the parts of the bundles where L is an isomorphism. One way of expressing this is through the exact sequence

$$0 \to \ker L \to F^+ \xrightarrow{L} F^- \to \operatorname{coker} L \to 0. \tag{6.4.6}$$

Since every sequence of vector bundles is split, there is an isomorphism $F^+ \oplus \operatorname{coker} L \cong F^- \oplus \ker L$, and so $F^+ - F^- \sim \ker L - \operatorname{coker} L$ represent the same element of $K(Y)$. However, as we pointed out at the end of the last section, in general $\ker L$, $\operatorname{coker} L$ are not of constant rank, so do not form vector bundles. Hence this naive construction is insufficient. Assume now that F^+, F^- are endowed with Hermitian metrics. Then we can decompose F^+, F^- according to $L^*L, L\,L^*$, as in Figure 2. Now for each $a > 0$ let $U_a \subset Y$ be the set of points where a is not in the spectrum of L^*L (hence not in the spectrum of $L\,L^*$). Let $F_a^{\pm} \subset F$ be the sum of the eigenspaces for eigenvalues less than a. Since the eigenvalues vary continuously, $F_a^{\pm}$ have constant rank over connected components of U_a, and so form vector bundles.

EXERCISE 6.4.7. Prove this last assertion rigorously.

Therefore, the K-bundle $F_a^+ - F_a^- \to U_a$ is well-defined. To patch, fix $b > a$ and consider a point in $U_a \cap U_b$. Let $F_{a,b}^{\pm} \subset F^{\pm}$ be the sum of the eigenspaces for eigenvalues between a and b. Then $F_b^{\pm} \cong F_a^{\pm} \oplus F_{a,b}^{\pm}$, and the map $L\colon F_{a,b}^+ \to F_{a,b}^-$ is an isomorphism. Hence the patching map $g_{ab}\colon F_a^+ \oplus F_b^- \to F_b^+ \oplus F_a^-$ is canonically defined (from L). This constructs a globally defined K-bundle .

The reader will quickly realize that the preceding paragraph is silly. The K-bundle we have just constructed is naturally equivalent to $F^+ - F^-$. Our motivation here is the infinite dimensional case, which we take up in the next chapter. There $F^{\pm}$ are bundles of spinor fields and L is the Dirac operator. Because these bundles are infinite dimensional, the patching construction is necessary to define an element of K-theory.

EXERCISE 6.4.8. Compare our construction here to the *difference bundle* construction in K-theory [ABS,Part 2]. Extend our construction to complexes of Hermitian bundles.

§7 Families of Dirac Operators

We now take up the index problem for families of Dirac operators. In their original treatment [AS4], Atiyah and Singer work in a purely topological framework. Their starting point is a homotopy class of a family of symbols parametrized by a topological space Y. The index of the family is an element in the K-theory of Y, and its Chern character is a real cohomology class on Y. We start with differential geometric data—a family of Riemannian manifolds (possibly endowed with an external vector bundle with metric and connection) parametrized by a *manifold* Y. Hence the index invariants come in precise differential geometric form—an infinite dimensional superconnection on Y whose associated Chern character is a differential form on Y. Of course, these more precise indices represent the K-theory index and its Chern character. After dispensing with the basic geometry of a family of Riemannian manifolds in §7.1, we discuss the Bismut superconnection and its associated K-theory element. The Riemann-Roch formula in this context computes the limiting Chern character as the parameter t in the superconnection tends to zero; it is a generalization of Theorem 5.4.1. In this preliminary version we omit the proof of Theorem 7.2.21. The results in this chapter are originally due to Bismut [Bi3].

§7.1 Families of Riemannian Manifolds

By a family of smooth manifolds we mean a fiber bundle $\pi\colon Z \to Y$, where Z and Y are smooth manifolds. Thus π is a submersion, and Z is locally (in Y) diffeomorphic a product $Y \times X$ for some manifold X. We allow Y and Z to be infinite dimensional manifolds, though we can carry out all of our arguments by working over finite dimensional submanifolds of Y. The fiber X in our examples will always be a smooth finite dimensional closed manifold. In terms of Steenrod's definition [Ste] π is a fiber bundle with structure group $\mathrm{Diff}(X)$ and typical fiber X. We denote the fiber $\pi^{-1}(y)$ by Z_y. In the trivial case where Y is a point, the family of manifolds reduces to a single manifold. The tangent bundle TZ has a canonical subbundle $\ker \pi$, which is the tangent bundle along the fibers. We denote it by $T(Z/Y)$. Notice that there is no God-given complement.

Definition 7.1.1. *A Riemannian structure on the family of manifolds $\pi\colon Z \to Y$ consists of a metric $g^{(Z/Y)}$ on $T(Z/Y)$ and a projection $P\colon TZ \to T(Z/Y)$.*

The kernel of the projection P is a complement to the vertical inside TZ. We term elements of $\ker P$ 'horizontal.' Definition 7.1.1 generalizes the notion of a Riemannian structure on a single manifold. The justification for including the projection P as part of the definition is Lemma 7.1.5 below.

Exercise 7.1.2. Given a fibration $\pi\colon Z \to Y$, show that there exist Riemannian structures. Prove that the space of Riemannian structures on a fixed fibration is contractible.

Exercise 7.1.3. Associate to $\pi\colon Z \to Y$ a principal $\mathrm{Diff}(X)$ bundle. Show that P determines a connection on this bundle, and vice versa. What is its curvature?

EXERCISE 7.1.4. Let X be a smooth manifold and $Y = \mathrm{Met}(X)$ the space of Riemannian metrics on X. In what sense is Y a manifold? Describe the natural Riemannian structure on the product fibering $Y \times X \to Y$. The group of diffeomorphisms $\mathrm{Diff}(X)$ acts smoothly on Y and on $Y \times X$. Construct a Riemannian structure on the quotient. Is it still a product? How can we correct for the fact that the action of $\mathrm{Diff}(X)$ is not free?

For a single Riemannian manifold the Levi-Civita theorem states that there is a unique torsionfree metric connection. The analog for a Riemannian family is a distinguished metric connection $\nabla^{(Z/Y)}$ on $T(Z/Y)$. We define $\nabla^{(Z/Y)}$ in terms of an auxiliary metric.

Lemma 7.1.5. *Choose any metric $g^{(\ker P)}$ on the horizontal spaces, and combine it with the Riemannian structure of π to produce a metric $g^{(Z)}$ on Z (relative to which $T(Z/Y)$ and $\ker P$ are orthogonal). Denote the Levi-Civita connection by $\nabla^{(Z)}$. Fix a horizontal vector field $\tilde{\xi}$ and vertical vector fields α, β. Then the second fundamental form $II_{\tilde{\xi}}(\alpha,\beta)$ and the component $(\nabla^{(Z)}_{\tilde{\xi}}\alpha,\beta)$ of the projected Levi-Civita connection are independent of $g^{(\ker P)}$. In fact, we have the formulas*

$$II_{\tilde{\xi}}(\alpha,\beta) = -\frac{1}{2}(\mathcal{L}_{\tilde{\xi}}g^{(Z/Y)})(\alpha,\beta) \tag{7.1.6}$$

$$(\nabla^{(Z)}_{\tilde{\xi}}\alpha,\beta) = ([\tilde{\xi},\alpha],\beta) - II_{\tilde{\xi}}(\alpha,\beta). \tag{7.1.7}$$

In (7.1.6) we use $\mathcal{L}_{\tilde{\xi}}g^{(Z/Y)}$ to denote the Lie derivative. Implicit in this equation is the assertion that this Lie derivative is tensorial in $\tilde{\xi}$.

Definition 7.1.8. *Let $\pi\colon Z \to Y$ be a family of Riemannian manifolds. The associated Riemannian connection $\nabla^{(Z/Y)}$ on $T(Z/Y)$ is the vertical projection of any Levi-Civita connection on Z. By Lemma 7.1.5 it is independent of the choice of metric on* $\ker P$.

Of course, the Riemannian connection $\nabla^{(Z/Y)}$ restricts to the Levi-Civita connection in each fiber. The point is that there is a unique torsionfree extension compatible with P, though it is awkward to state directly what 'torsionfree' and 'compatible with P' mean in this context.

EXERCISE 7.1.9. State directly what 'torsionfree' and 'compatible with P' mean in this context.

Proof of Lemma 7.1.5. Since the right hand sides of (7.1.6) and (7.1.7) are expressed purely in terms of $g^{(Z/Y)}$ and P, we have only to prove these formulas to conclude the independence from $g^{(\ker P)}$. For simplicity let g denote the metric $g^{(Z)}$ on Z and ∇ its associated Levi-Civita connection $\nabla^{(Z)}$. Now expand $\tilde{\xi}\cdot(\alpha,\beta)$ in two different ways:

$$\begin{aligned}\tilde{\xi}\cdot(\alpha,\beta) &= (\nabla_{\tilde{\xi}}\alpha,\beta) + (\alpha,\nabla_{\tilde{\xi}}\beta)\\ &= (\mathcal{L}_{\tilde{\xi}}g)(\alpha,\beta) + ([\tilde{\xi},\alpha],\beta) + (\alpha,[\tilde{\xi},\beta]).\end{aligned} \tag{7.1.10}$$

But since the connection is torsionfree,

$$(7.1.11) \qquad (\nabla_{\tilde{\xi}}\alpha, \beta) = ([\tilde{\xi}, \alpha], \beta) + (\nabla_\alpha \tilde{\xi}, \beta),$$

with a similar equation for α and β exchanged. Thus

$$(7.1.12) \qquad (\mathcal{L}_{\tilde{\xi}} g)(\alpha, \beta) = (\nabla_\alpha \tilde{\xi}, \beta) + (\alpha, \nabla_\beta \tilde{\xi}).$$

A standard argument shows that the two terms on the right hand side of (7.1.12) are equal:

$$(\nabla_\alpha \tilde{\xi}, \beta) - (\alpha, \nabla_\beta \tilde{\xi}) = -(\tilde{\xi}, \nabla_\alpha \beta) + (\nabla_\beta \alpha, \tilde{\xi}) = (\tilde{\xi}, [\beta, \alpha]) = 0.$$

Here we use the fact that $[\beta, \alpha]$ is a vertical vector field. Hence

$$(\mathcal{L}_{\tilde{\xi}} g)(\alpha, \beta) = 2(\nabla_\alpha \tilde{\xi}, \beta) = -2II_{\tilde{\xi}}(\alpha, \beta),$$

which is (7.1.6). Now (7.1.7) follows by combining (7.1.10), (7.1.11), (7.1.12), and (7.1.6).

It is interesting to notice that the preceding proof is local, so works for any foliation on Z; the foliation need not be a fibration.

There are two tensors associated to the Riemannian structure which we need in the next section. The first is the curvature of the horizontal distribution $\ker P$. It is a 2-form T on Y whose values are vector fields along the fibers of π. Let ξ_1, ξ_2 be vector fields on Y, denote by $\tilde{\xi}_1, \tilde{\xi}_2$ their horizontal lifts to Z, and set

$$(7.1.13) \qquad T(\xi_1, \xi_2) = [\tilde{\xi}_1, \tilde{\xi}_2] - \widetilde{[\xi_1, \xi_2]}.$$

The usual argument shows that T is tensorial in ξ_1, ξ_2.

The other tensor δ is the section of $(\ker P)^* \to Z$ which measures the local change in volume between fibers. Let **vol** denote the volume form of the vertical metric $g^{(Z/Y)}$; it is a section of $\bigwedge^n T(Z/Y)^* \to Z$. Suppose $\tilde{\xi}$ is a horizontal vector field. Then define $\delta(\tilde{\xi})$ by the equation

$$(7.1.14) \qquad \mathcal{L}_{\tilde{\xi}} \,\mathbf{vol} = \delta(\tilde{\xi})\, \mathbf{vol}\,.$$

We check that δ is tensorial as follows. Extend **vol** to a vertical n-form on Z using the splitting $TZ \cong T(Z/Y) \oplus \ker P$. Then

$$\mathcal{L}_{\tilde{\xi}}\, \mathbf{vol} = \iota_{\tilde{\xi}} d\, \mathbf{vol} + d\iota_{\tilde{\xi}}\, \mathbf{vol} = \iota_{\tilde{\xi}} d\, \mathbf{vol},$$

so that δ extends to the horizontal 1-form defined by

$$d\,\mathbf{vol} = \delta \wedge \mathbf{vol}\,. \tag{7.1.15}$$

Spinors and the Dirac operator are defined as for a single manifold. Set $n = \dim Z/Y = \dim X$. Let $SO(Z/Y) \to Z$ be the principal $SO(n)$ bundle of frames along the fiber. A point of $SO(Z/Y)$ over $z \in Z$ is an orientation-preserving isometry $p\colon \mathbb{E}^n \to T_z(Z/Y)$. The connection $\nabla^{(Z/Y)}$ determines a connection on this frame bundle. The vector fields $\partial_1, \dots, \partial_n$ are defined as the horizontal lifts of basis vectors. Notice that ∂_k is vertical (relative to π). As in Definition 2.2.16 a spin structure is a double cover of the frame bundle.

Definition 7.1.16. *A spin structure on $\pi\colon Z \to Y$ is a principal* $\mathrm{Spin}(n)$ *bundle* $\mathrm{Spin}(Z/Y)$ *which double covers $SO(Z/Y)$, and for which the diagram*

$$\begin{array}{ccc}
\mathrm{Spin}(Z/Y) \times \mathrm{Spin}(n) & \longrightarrow & \mathrm{Spin}(Z/Y) \\
\big\downarrow{\scriptstyle 2:1} \qquad \big\downarrow{\scriptstyle 2:1} & & \big\downarrow{\scriptstyle 2:1} \\
SO(Z/Y) \times SO(n) & \longrightarrow & SO(Z/Y)
\end{array}$$

commutes.

The topological obstruction to the existence of a spin structure is the second Stiefel-Whitney class $w_2(T(Z/Y)) \in H^2(Z; \mathbb{Z}/2\mathbb{Z})$.

Suppose that $\pi\colon Z \to Y$ is endowed with a spin structure. The associated spin bundle $S \to Z$ is defined using the spin representation γ, as in (2.2.20). If n is even there is a splitting $S = S^+ \oplus S^-$. The connection $\nabla^{(Z/Y)}$ induces a connection on S, which we denote $\dot\gamma\nabla^{(Z/Y)}$. (Recall that $\dot\gamma$ is the infinitesimal spin representation.) Now as in (2.2.21) the Dirac operator is

$$\mathcal{D} = c(\partial) = c(e^k)\partial_k = \gamma^k \partial_k. \tag{7.1.17}$$

It acts on spinor fields. In even dimensions we write $\mathcal{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$, with $D\colon C^\infty(S^+) \to C^\infty(S^-)$.

The Dirac operator restricted to the fiber Z_y is exactly the Dirac operator for the single Riemannian manifold Z_y. In particular, the projection P does not enter its definition. Notice, though, that the spin structures on the fibers Z_y must fit together to form a global spin structure on $Z \to Y$. The Weitzenböck formula (2.2.33) holds as before, where now R denotes the scalar curvature of the fiber.

Generalized Dirac operators are defined in this context by starting with a vector bundle $V \to Z$ with metric $g^{(V)}$ and unitary connection $\nabla^{(V)}$. (To define the Dirac operator we need only a connection in vertical directions, just as we only used $\nabla^{(Z/Y)}$ in vertical directions. Derivatives in horizontal directions enter in the Bismut superconnection below.) The generalized Dirac operator is the composition

$$C^\infty(S\otimes V) \xrightarrow{\dot\gamma\nabla^{(Z/Y)}\otimes 1+1\otimes\nabla^{(V)}} C^\infty(T^*X\otimes S\otimes V) \xrightarrow{c(\cdot)\otimes 1} C^\infty(S\otimes V). \tag{7.1.18}$$

As in §2 this could alternatively be described in terms of principal bundles.

§7.2 The Bismut Superconnection

We recapitulate the data which defines a geometric family of Dirac operators.

Definition 7.2.1. *Suppose $\pi\colon Z\to Y$ is a smooth family of manifolds endowed with a Riemannian structure (Definition 7.1.1) and a spin structure (Definition 7.1.16). We assume that the fibers of π are compact. Let $V\to Z$ be a complex vector bundle with Hermitian metric $g^{(V)}$ and unitary connection $\nabla^{(V)}$. Then this data determines a geometric family of Dirac operators, as described in the previous section.*

As for a single operator, the spin structure may be irrelevant for other operators of Dirac type (e.g. $\bar\partial$ operator on Kähler manifolds, signature operator).

Exercise 7.2.2. Let $E\to X$ be a Hermitian vector bundle over a Riemannian spin manifold X. Let Y be the space of unitary connections on E. Construct a natural geometric family of Dirac operators parametrized by Y. A *gauge transformation* is an automorphism of $E\to X$ which induces the identity map on X. The group $\mathcal{G}$ of gauge transformations acts on Y, though not quite freely. Construct a natural geometric family of Dirac operators parametrized by $Y/\mathcal{G}$. How do you handle the fact that $\mathcal{G}$ does not act freely?

Exercise 7.2.3. Let Y parametrize a geometric family of Dirac operators, and suppose $Y'\to Y$ is a smooth map. Construct a pullback family parametrized by Y'.

Exercise 7.2.4. Let $E\to M$ be a fixed Hermitian bundle with unitary connection, and X a compact Riemannian spin manifold. Construct a geometric family of Dirac operators parametrized by $\mathrm{Map}(X,M)$. How is this family related to Exercise 7.2.2? What are the relevant symmetries in this example? (Hint: Be careful about the spin structure.)

Exercise 7.2.5. Let X be a compact oriented 2-manifold of genus g. There are 2^{2g} spin structures on X (Exercise 2.2.18). Fix one. Let $\mathcal{M}$ be the set of metrics on X with constant Gauss curvature, and $\mathcal{D}$ the subgroup of $\mathrm{Diff}(X)$ consisting of diffeomorphisms which fix the given spin structure. Construct a geometric family of Dirac operators parametrized by $\mathcal{M}/\mathcal{D}$. (Hint: To construct a spin structure on the resulting family of manifolds you will need to consider the action of a double cover of $\mathcal{D}$.) Discuss the case $g=1$ in detail, identifying $\mathcal{M}$ and $\mathcal{D}$ explicitly.

Fix a geometric family of Dirac operators as in Definition 7.2.1. Assume that the fibers of $\pi\colon Z \to Y$ are even dimensional. Then for each $y \in Y$ the fiber Z_y is a Riemannian spin manifold with half-spin bundles $S_y^{\pm} \to Z_y$ and a Hermitian vector bundle $V_y \to Z_y$ with unitary connection. Let $\mathcal{H}_y^{\pm} = L^2(S_y^{\pm} \otimes V_y)$ be the Hilbert space of V_y-valued spinor fields on Z_y. The Hilbert spaces $\mathcal{H}_y^{\pm}$ patch to form *continuous* vector bundles over Y—the transition functions of $\pi\colon Z \to Y$, which take values in $\mathrm{Diff}(X)$, act continuously on $\mathcal{H}^{\pm}$ by pullback. However, the composition $L^2 \times C^{\infty} \to L^2$ is *not* differentiable, so the $\mathcal{H}^{\pm}$ are not smooth bundles. The subbundles of C^{∞} spinor fields are better, since composition $C^{\infty} \times C^{\infty} \to C^{\infty}$ is smooth, but the fibers are Fréchet spaces, not Hilbert spaces. We work with finite dimensional subbundles of $\mathcal{H}_y^{\pm}$ spanned by eigenspinor fields. By the elliptic theory §3 the eigenspinor fields are smooth, whence these subbundles are smooth finite dimensional vector bundles over Y.

Let us construct these subbundles now. (Compare with the discussion following (6.4.6).) We have our favorite picture at each $y \in Y$, as displayed in Figure 3. Let $U_a \subset Y$ be the set of points where a is not in the spectrum of $D_y^*D_y$. Let $\mathcal{H}_a^{\pm} \subset \mathcal{H}^{\pm}$ be the sum of eigenspaces for eigenvalues less than a.

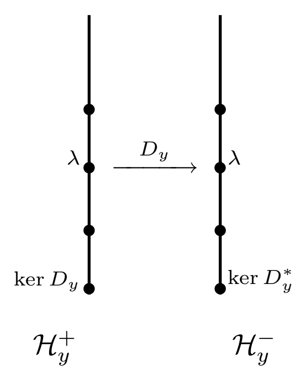

Figure 3

Lemma 7.2.6. *$\mathcal{H}_a^{\pm}$ has locally constant rank on U_a.*

Proof. Fix a compact subset of U_a and choose $\epsilon > 0$ such that the spectrum of $D_y^*D_y$ does not contain any point of $(a-\epsilon, a+\epsilon)$ for y in the compact set. Now fix a smooth function $F\colon [0,\infty) \to [0,1]$ which vanishes on $[a+\epsilon,\infty)$ and is identically 1 on $[0, a-\epsilon]$. Then the operator $P_y = F(D_y^*D_y)$ is smoothing (cf. Exercise 4.2.16) and varies smoothly in y. It is clear that P_y is projection onto $\mathcal{H}_a^+(y)$. Thus $\operatorname{rank} \mathcal{H}_a^+(y) = \operatorname{Tr} P_y$ is integral valued and smoothly varying. Hence it is locally constant.

So $\mathcal{H}_a^{\pm} \to U_a$ are vector bundles. Consider the K-bundle $\mathcal{H}_a^+ - \mathcal{H}_a^- \to U_a$. We patch these local K-bundles into an element of $K(Y)$, as in §6.4. Fix $b > a$ and let $\mathcal{H}_{a,b}^{\pm}$ be the sum of eigenspaces for eigenvalues between a and b. These are vector bundles over $U_a \cap U_b$. The patching map

$$g_{ab}\colon \mathcal{H}_a^+ \oplus \mathcal{H}_b^- \longrightarrow \mathcal{H}_b^+ \oplus \mathcal{H}_a^-$$

over $U_a \cap U_b$ are canonically defined using $\mathcal{H}_b^\pm \cong \mathcal{H}_a^\pm \oplus \mathcal{H}_{a,b}^\pm$ and the isomorphism $D\colon \mathcal{H}_{a,b}^+ \xrightarrow{\sim} \mathcal{H}_{a,b}^-$.

Definition 7.2.7. *The index bundle* $\operatorname{ind} D$ *of the family of Dirac operators is* $\{\mathcal{H}_a^+ - \mathcal{H}_a^-, g_{ab}\} \in K(Y)$.

This is the 'analytic index' of Atiyah and Singer [AS4].

We proceed to construct a family of superconnections on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Recall that $\mathcal{H}$ is only a continuous bundle, so differentiation of arbitrary sections is not well-defined. Therefore, the connection $\nabla^{(\mathcal{H})}$ that we construct below, and the family of superconnections $\boldsymbol{\nabla}_t$, only operates on sections of the dense subbundle of smooth spinor fields. The super Chern character $e^{-\boldsymbol{\nabla}_t^2}$, however, is a bounded (trace class) operator on all of L^2.

Let us first construct a connection $\nabla^{(\mathcal{H})}$ on $\mathcal{H} \to Y$. Suppose ψ is a smooth section of $\mathcal{H}$. We represent ψ as a section of the bundle $S \otimes V \to Z$, which we are assuming to be smooth over Z. (An arbitrary section of $\mathcal{H}$ is only L^2 in the fiber directions.) Fix $\xi \in T_yY$ and let $\tilde{\xi}$ be its horizontal lift to a vector field along the fiber Z_y. Then $\nabla_{\tilde{\xi}}^{(Z/Y)}\psi$ is a new section of $E \to Z_y$, and so defines an element in the fiber of $\mathcal{H}$ over y. (The connection $\nabla^{(Z/Y)}$ on $T(Z/Y)$ acts on the associated spin bundle S via the infinitesimal spin representation $\dot{\gamma}$, which we omit from the notation for convenience.) We call this the pointwise covariant derivative (on sections of $\mathcal{H}$). However, it does not preserve the L^2 inner product on $\mathcal{H}$, since it does not take into account the variation in the volume form across fibers. To correct we use the divergence form δ (7.1.15). Set

$$\nabla_\xi^{(\mathcal{H})} = \nabla_{\tilde{\xi}}^{(Z/Y)} + \frac{1}{2}\delta(\tilde{\xi}), \tag{7.2.8}$$

where the right hand side acts pointwise. We claim that $\nabla^{(\mathcal{H})}$ is unitary for the L^2 metric $\langle \cdot, \cdot \rangle$. For if φ, ψ are smooth sections of $\mathcal{H}$, and $\xi \in T_yY$, then

$$\begin{aligned}\xi \cdot \langle \varphi, \psi \rangle &= \xi \cdot \int_{Z/Y} (\varphi, \psi)\,\mathbf{vol} \\ &= \int_{Z/Y} \Bigl\{\tilde{\xi} \cdot (\varphi, \psi)\,\mathbf{vol} + (\varphi, \psi)\iota_{\tilde{\xi}} d\,\mathbf{vol}\Bigr\} \\ &= \int_{Z/Y} \Bigl\{(\nabla_{\tilde{\xi}}^{(Z/Y)}\varphi, \psi) + (\varphi, \nabla_{\tilde{\xi}}^{(Z/Y)}\psi) + (\varphi, \psi)\delta(\tilde{\xi})\Bigr\}\,\mathbf{vol} \\ &= \langle \nabla_\xi^{(\mathcal{H})}\varphi, \psi \rangle + \langle \varphi, \nabla_\xi^{(\mathcal{H})}\psi \rangle.\end{aligned} \tag{7.2.9}$$

Here $\int_{Z/Y}$ is integration over the fiber, which commutes with d. Notice that $\nabla^{(\mathcal{H})}$ preserves the splitting $\mathcal{H} \cong \mathcal{H}^+ \oplus \mathcal{H}^-$, since $\nabla^{(Z/Y)}$ and δ do.

It is convenient to observe that

$$\hat{\nabla}^{(E)} = \nabla^{(Z/Y)} + \frac{1}{2}\delta \tag{7.2.10}$$

defines a new connection on $T(Z/Y)$ (which does not preserve the metric). If $\Omega^{(Z/Y)}$ denotes the curvature of $\nabla^{(Z/Y)}$, then the curvature of $\hat{\nabla}^{(E)}$ is (cf. Exercise 6.1.9)

$$\hat{\Omega}^{(E)} = \Omega^{(Z/Y)} + \frac{1}{2}\nabla^{(Z/Y)}\delta. \tag{7.2.11}$$

Next we compute the curvature of $\mathcal{H}$. Recall that $T(\xi_1, \xi_2)$ is a vertical vector field, essentially the curvature of the horizontal distribution $\ker P$ (7.1.13).

Proposition 7.2.12. *The curvature of $\nabla^{(\mathcal{H})}$ is the first order differential operator (along the fibers)*

$$\Omega^{(\mathcal{H})}(\xi_1, \xi_2) = \hat{\nabla}^{(E)}_{T(\xi_1,\xi_2)} + \hat{\Omega}^{(E)}(\tilde{\xi}_1, \tilde{\xi}_2), \tag{7.2.13}$$

where both terms act by Clifford multiplication on spinors.

We will not need the actual formula (7.2.13), but only the fact that $\Omega^{(\mathcal{H})}$ is a first order differential operator. As a curvature it is a 2-form which acts as an endomorphism of the fiber. Here the fiber is the Hilbert space of spinor fields, and the content of the proposition is that the endomorphism is a first order differential operator.

Proof. We write $\nabla^{(\mathcal{H})}_{\xi} = \hat{\nabla}^{(E)}_{\tilde{\xi}}$ acting pointwise, whence

$$\begin{aligned}\Omega^{(\mathcal{H})}(\xi_1, \xi_2) &= [\nabla^{(\mathcal{H})}_{\xi_1}, \nabla^{(\mathcal{H})}_{\xi_2}] - \nabla^{(\mathcal{H})}_{[\xi_1,\xi_2]} \\ &= [\hat{\nabla}^{(E)}_{\tilde{\xi}_1}, \hat{\nabla}^{(E)}_{\tilde{\xi}_2}] - \hat{\nabla}^{(E)}_{[\tilde{\xi}_1,\tilde{\xi}_2]} + \hat{\nabla}^{(E)}_{T(\xi_1,\xi_2)} \\ &= \hat{\Omega}^{(E)}(\tilde{\xi}_1, \tilde{\xi}_2) + \hat{\nabla}^{(E)}_{T(\xi_1,\xi_2)}.\end{aligned}$$

We are now in a position to define a family of superconnections on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. A natural is $\nabla^{(\mathcal{H})} + \sqrt{t}\mathcal{D}$, a direct imitation of (6.2.17). One of Bismut's key insights is the addition of an extra term. We explain this term later (MAKE SURE!), noting now that it diverges as $t \to 0$.

Definition 7.2.14. *The Bismut superconnection with parameter $t > 0$ is the operator*

$$\boldsymbol{\nabla}_t = \nabla^{(\mathcal{H})} + \sqrt{t}\mathcal{D} - \frac{c(T)}{4\sqrt{t}}. \tag{7.2.15}$$

Here T is the curvature tensor (7.1.13), a 2-form whose values are vertical vector fields. Using the metric we can convert the vector field to a 1-form, and then it acts on spinors via Clifford multiplication c. Hence this additional term is an off-diagonal matrix of 2-forms (relative to the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$), and so is an odd element of $\Omega^*(\operatorname{End}\mathcal{H})$. The term $\sqrt{t}\mathcal{D}$ is a 0-form which is off-diagonal. Hence $\boldsymbol{\nabla}_t$ does define a superconnection (cf. (6.2.14)). In the trivial case where Y is a point, the superconnection reduces to the scaled Dirac operator $\sqrt{t}\mathcal{D}$.

Proposition 7.2.16. *The curvature* $\boldsymbol{\nabla}_t^2$ *at* $y \in Y$ *is a second order differential operator on* Z_y *with coefficients in the exterior algebra* $\bigwedge T_y^*Y$*. Therefore,* $e^{-\boldsymbol{\nabla}_t^2}$ *is a bounded trace class operator on* $\mathcal{H}$*, and the super Chern form*

$$\gamma_t = \operatorname{tr_s} e^{-\boldsymbol{\nabla}_t^2} \tag{7.2.17}$$

is a well-defined element of Ω^**, i.e., a smooth differential form on* Y*.*

Proof. We can compute $\boldsymbol{\nabla}_t^2$ directly (cf. (6.2.16)):

$$\boldsymbol{\nabla}_t^2 = t\mathcal{D}^2 + \Omega^{(\mathcal{H})} + \frac{c(T)^2}{16t} + \sqrt{t}[\nabla^{(\mathcal{H})}, \mathcal{D}] - \frac{1}{4\sqrt{t}}[\nabla^{(\mathcal{H})}, c(T)] - \frac{1}{4}(\mathcal{D}c(T) + c(T)\mathcal{D}). \tag{7.2.18}$$

The first term $t\mathcal{D}^2$ is the Dirac Laplacian. It is a second order differential operator with scalar symbol. By Proposition 7.2.12 the curvature $\Omega^{(\mathcal{H})}$ is a first order operator, and the remaining terms are evidently differential operators of order at most one. So in a local coordinate system x^k along the fiber Z_y, the operator $\boldsymbol{\nabla}_t^2$ takes the form

$$-g^{k\ell}(x)\frac{\partial^2}{\partial x^k \partial x^\ell} + b^k(x)\frac{\partial}{\partial x^k} + c(x), \tag{7.2.19}$$

where $g^{k\ell}(x)$ is the inverse metric, and b^k, c are endomorphisms of $S \otimes V$ with coefficients in $\bigwedge T_y^*Y$. (The term $c(T)^2/16t$ is a 4-form which acts as a multiplication operator, $\sqrt{t}[\nabla^{(\mathcal{H})}, \mathcal{D}]$ is a 1-form which acts as a first order differential operator, etc.) Notice that b^k, c consist of forms of positive degree. We view $\boldsymbol{\nabla}_t^2$ as a second order elliptic operator on Z_y, acting on sections of $S \otimes V \otimes \bigwedge T_y^*Y$. Here $\bigwedge T_y^*Y$ is to be interpreted as a trivial vector bundle on Z_y, and the differential form coefficients b^k, c act by left exterior multiplication.[33] Since the operator has scalar symbol, the elliptic theory of Chapter 3 and Chapter 4 applies to give the desired conclusion. Notice that the supertrace in (7.2.17) is taken over the $S \otimes V$ variables, but *not* over the exterior algebra variables. Also, we have implicitly used the smooth dependence of the heat kernel on parameters (Exercise 4.3.14), as the data varies smoothly in y. The differential form γ_t is smooth on Y (and smooth in t).

Exercise 7.2.20. As an alternative approach to analyzing (7.2.18), and so making sense of the super Chern character forms (7.2.17), apply Duhamel's formula (6.3.20) as in Exercise 6.3.22.

Bismut's theorem can now be stated.

[33] If Y is infinite dimensional, restrict to a finite dimensional submanifold. In any case, if we study the component of γ_t of degree $2k$, we can restrict to $2k$ dimensional subspaces of $\bigwedge T_y^*Y$.

Theorem 7.2.21 (Bismut [Bi3]**).** *The super Chern character form γ_t is a de Rham representative of the super Chern character $\mathrm{ch_s}(\mathrm{ind}\, D)$ of the index bundle for any $t > 0$. As $t \to 0$ there is a limit given by the local formula*

$$\lim_{t\to 0} \tilde{\gamma}_t = \int_{Z/Y} \hat{A}(\Omega^{(Z/Y)})\, \mathrm{ch}(\Omega^{(V)}), \tag{7.2.22}$$

where the homogeneous components of $\tilde{\gamma}_t$ are related to the homogeneous components of γ_t by scaling:

$$[\tilde{\gamma}_t]_{(2k)} = \left(\frac{-i}{2\pi}\right)^k [\gamma_t]_{(2k)}\,. \tag{7.2.23}$$

Notice that if Y is a point, then (7.2.22) reduces to the local index theorem Theorem 5.2. Turning this around, the local index theorem proves (7.2.22) for the component of degree zero. Also, taking cohomology classes in (7.2.22) yields the topological formula for $\mathrm{ch_s}(\mathrm{ind}\, D)$ given in [AS4]. The main import of Theorem 7.2.21 is an analytic representation of this topological result. We remark that the diverging term in (7.2.15) is there to ensure the existence of the limit in (7.2.22). It is rather strange that a diverging term is needed to accomplish this!

Appendix: Exponential Coordinates

Let X be a Riemannian manifold and fix $y \in X$. For r sufficiently small the exponential map (at y)

$$\exp\colon B_r \longrightarrow U \tag{A.1}$$

is a diffeomorphism from the ball of radius r in T_yX onto a neighborhood of y in X. By definition the line segment $\{ta\colon 0 \le t \le 1, a \in B_r\}$ is mapped onto the geodesic emanating from y in the direction a. Since the tangent to this geodesic at y is a, the differential of exp at 0 is the identity map:

$$d\exp_0 = \mathrm{id}\,. \tag{A.2}$$

We fix an orthonormal basis of T_yX relative to which we write $a = \langle a^1, \dots, a^n\rangle \in B_r$. The vector fields $\partial/\partial a^k$ are not in general orthonormal relative to the induced metric on B_r. Set

$$g_{k\ell}(a) = \left(\frac{\partial}{\partial a^k}, \frac{\partial}{\partial a^\ell}\right)(a). \tag{A.3}$$

It follows from (A.2) that

$$g_{k\ell}(0) = \delta_{k\ell}, \tag{A.4}$$

so that

$$g_{k\ell}(a) = \delta_{k\ell} + O(|a|). \tag{A.5}$$

Finally, define $g^{k\ell}$ as the inverse matrix

$$\left(g^{k\ell}\right) = \left(g_{k\ell}\right)^{-1}. \tag{A.6}$$

Now suppose $V \to X$ is a (real or complex) vector bundle with connection. We introduce "exponential coordinates," or more properly a *synchronous framing*, of V as follows. Fix an orthonormal (or unitary) frame $s_1, \dots, s_r$ of the fiber V_y. Then use parallel transport along radial geodesics in U to produce frames in V_x, $x \in U$. Thus a section of V over U, relative to this framing, is specified by a function $f\colon B_r \to \mathbb{R}^m$ (or $\mathbb{C}^m$)—the section is $f \cdot s = \sum_\alpha f^\alpha s_\alpha$. Define the connection forms A_k on B_r, with values in $m \times m$ matrices, by

$$\nabla_{\partial/\partial a^k}(f \cdot s) = \left(\frac{\partial f}{\partial a^k} + A_k f\right) \cdot s, \tag{A.7}$$

or more simply

$$\nabla_k = \nabla_{\partial/\partial a^k} = \frac{\partial}{\partial a^k} + A_k. \tag{A.8}$$

Now since the s_i are parallel on radial geodesics, for f^k *constant* we have

$$f^k \nabla_k s(tf^1, \dots, tf^k) = 0 \tag{A.9}$$

for all $t \geq 0$. In particular, this is true at the origin ($t = 0$), and for all f^k, from which

$$A_k(0) = 0. \tag{A.10}$$

Next, for $k \neq \ell$ take $f^k = f^\ell = 1$ and all other $f^i = 0$. Differentiating at $t = 0$ we find

$$\left(\frac{\partial}{\partial a^k} + \frac{\partial}{\partial a^\ell}\right)(A_k + A_\ell)(0) = 0,$$

But also $\dfrac{\partial A_k}{\partial a^k}(0) = \dfrac{\partial A_\ell}{\partial a^\ell}(0) = 0$, by the same argument with only one nonzero f^i, whence

$$\frac{\partial A_k}{\partial a^\ell}(0) + \frac{\partial A_\ell}{\partial a^k}(0) = 0. \tag{A.11}$$

Now the curvature $\Omega^{(V)}$ is a 2-form which is given in coordinates as

$$\begin{aligned} \Omega^{(V)} &= \frac{1}{2}\Omega^{(V)}_{k\ell}\, da^k \wedge da^\ell, \\ \Omega^{(V)}_{k\ell} &= [\nabla_k, \nabla_\ell] \\ &= \frac{\partial A_\ell}{\partial a^k} - \frac{\partial A_k}{\partial a^\ell} + [A_k, A_\ell]. \end{aligned} \tag{A.12}$$

Combining (A.10)–(A.12) we conclude

$$\Omega^{(V)}_{k\ell}(0) = -2\frac{\partial A_k}{\partial a^\ell}(0). \tag{A.13}$$

Finally, from (A.9) and (A.13) we see that the Taylor series for A_k at $a = 0$ is

$$A_k(a) = -\frac{1}{2}\Omega^{(V)}_{k\ell}(0)a^\ell + O(|a|^2). \tag{A.14}$$

These formulæ apply to the tangent bundle TX. Notice that the synchronous framing is different from the coordinate framing $\partial/\partial a^k$ (unless X is flat). Now we denote the connection forms in the *synchronous framing* by Γ_k and (A.14) is

$$\Gamma_k(a) = -\frac{1}{2}\Omega^{(X)}_{k\ell}(0)a^\ell + O(|a|^2). \tag{A.15}$$

We will have use for the classical Levi-Civita symbols $\begin{bmatrix} m \\ k\ell \end{bmatrix}$ for the connection relative to the *coordinate framing*:

$$\nabla_{\partial/\partial a^k}\frac{\partial}{\partial a^\ell} = \begin{bmatrix} m \\ k\ell \end{bmatrix}\frac{\partial}{\partial a^m}. \tag{A.16}$$

Suppose V is a metrized bundle with a metric connection ∇. Then the covariant Laplacian (cf. (2.2.33)) in these coordinates is

$$\nabla^*\nabla = -g^{k\ell}(\nabla_k\nabla_\ell - \begin{bmatrix} m \\ k\ell \end{bmatrix}\nabla_m). \tag{A.17}$$

For since $\nabla^* = -\iota(dx^k)\nabla_k$ (cf. (2.2.34)) and $\nabla_k dx^m = \begin{bmatrix} m \\ k\ell \end{bmatrix} dx^\ell$ (by duality from (A.16)), we compute

$$\begin{aligned}\nabla^*\nabla s &= -\iota(dx^k)\nabla_k(dx^m \otimes \nabla_m s)\\ &= -\iota(dx^k)\left\{-\begin{bmatrix} m \\ k\ell \end{bmatrix} dx^\ell \otimes \nabla_m s + dx^m \otimes \nabla_k\nabla_m s\right\},\end{aligned}$$

which is (A.17).

References

[Ad1] J. F. Adams, *Vector-fields on spheres*, Bull. Amer. Math. Soc **68** (1962), 39–41.

[Ad2] J. F. Adams, *Applications of the Grothendieck-Atiyah-Hirzebruch functor $K(X)$*, Proc. Int. Congr. Math. (1962), 435–441.

[Ad3] J. F. Adams, *On $J(X)$-II*, Topology **3** (1965), 137–71.

[AW] C. B. Allendoerfer, A. Weil, *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc. **53** (1943), 101–29.

[Ag] L. Alvarez-Gaumé, *Supersymmetry and the Atiyah-Singer index theorem*, Commun. Math. Phys. **90** (1983), 161–173.

[A1] M. F. Atiyah, *The Grothendieck ring in geometry and topology*, Proc. Int. Congr. Math. (1962), 442–446.

[A2] M. F. Atiyah, *K-Theory*, Benjamin, New York, 1967.

[A3] M. F. Atiyah, *Global aspects of the theory of elliptic differential operators*, Proc. Int. Congr. Math. (Moscow) (1966), 57–64.

[A4] M. F. Atiyah, *Circular symmetry and stationary-phase approximation*, in "Colloque en L'Honneur de Laurent Schwartz," Volume 1, Astérisque **131** (1985), 43–59.

[A5] M. F. Atiyah, *William Vallance Douglas Hodge*, Bull. London Math. Soc. **9** (1977), 99–118.

[A6] M. F. Atiyah, *Collected Works*, 5 Volumes, Oxford University Press, Oxford, England, 1987.

[AB1] M. F. Atiyah, R. Bott, *The Lefschetz fixed point theorem for elliptic complexes I*, Ann. of Math. **88** (1947), 374–407.

[AB2] M. F. Atiyah, R. Bott, *The Lefschetz fixed point theorem for elliptic complexes II*, Ann. of Math. **88** (1968), 451–491.

[AB3] M. F. Atiyah, R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[ABP] M. F. Atiyah, R. Bott, V. K. Patodi,, *On the heat equation and the index theorem*, Invent. math. **19** (1973), 279–330.

[ABS] M. F. Atiyah, R. Bott, A. A. Shapiro, *Clifford modules*, Topology **3** (1964), 3–38.

[AH1] M. F. Atiyah, F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Symp. Pure Math. **3** (1961), 7–38.

[AH2] M. F. Atiyah, F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281.

[AS1] M. F. Atiyah, I. M. Singer, *The index of elliptic operators III*, Ann. of Math. **87** (1968), 546–604.

[AS2] M. F. Atiyah, I. M. Singer, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. **69** (1963), 422–433.

[AS3] M. F. Atiyah, I. M. Singer, *The index of elliptic operators. I*, Ann. Math. **87** (1968), 484–530.

[AS4] M. F. Atiyah, I. M. Singer, *The index of elliptic operators. IV*, Ann. Math. **93** (1971), 119–138.

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators.*, Springer-Verlag, Berlin, 1992.

[BV1] N. Berline, M. Vergne, *Zéros d'un champ de vecteurs et classes caractéristiques équivariantes*, Duke Math J. **50** (1983), 539–549.

[BV2] N. Berline, M. Vergne, *A computation of the equivariant index of the Dirac operator*, Bull. Soc. math. Fr. **113** (1985), 305–345.

[Bi1] J.-M. Bismut, *Localization formulas, superconnections, and the index theorem for families*, Commun. Math. Phys. **103** (1986), 127–166.

[Bi2] J. M. Bismut, *The Atiyah-Singer theorems. A probalistic approach. I. The index theorem*, J. Funct. Anal. **57** (1984), 56–99.

[Bi3] J. M. Bismut, *The Atiyah-Singer Index Theorem for families of Dirac operators: two heat equation proofs*, Invent. math. **83** (1986), 91–151.

[BH] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces. I*, Amer. J. Math. **80** (1958), 458–538; *II*, Amer. J. Math. **81** (1959), 315–382; *III*, Amer. J. Math. **82** (1960), 491–504.

[BS] A. Borel, J.-P. Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136.

[B1] R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. **70** (1959), 313–337.

[B2] R. Bott, *The periodicity theorem for the classical groups and some of its applications*, Adv. Math. **4** (1970), 353–411.

[CZ] A. P. Calderón and A. Zygmund, *Singular integral operators and differential equations*, Amer. J. Math. **79** (1957), 901–21.

[C] S. S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. **45** (1944), 747–52.

[Ch] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[deR] G. de Rham, *Sur l'Analysis situs des variétés à n dimensions*, Journal de mathématiques pures at appliquées **10** (1931), 115–200.

[D] P. A. M. Dirac, Proc. Roy. Soc. A **117** (1928), 610.

[Do] H. Donnelly, *Local index theorem for families*, Michigan Math. J. **35** (1988), 11–20.

[DH] J. J. Duistermaat, G. J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase-space*, Invent. Math. **69** (1982), 259–269.

[FU] D. S. Freed, K. K. Uhlenbeck, *Instantons and Four-Manifolds*, MSRI Publications, Volume 1, Springer-Verlag, New York, 1984.

[FW] D. Friedan, P. Windey, *Supersymmetric derivation of the Atiyah-Singer index and the chiral anomaly*, Nucl. Phys. B **235** (1984), 395–416.

[F] W. Fulton, *Intersection Theory*, Springer-Verlag, N.Y., 1984.

[G] I. M. Gel'fand, *On elliptic equations*, Russian Math. Surveys **15** (1960), no. 3, 113–123.

[Ge1] E. Getzler, *A short proof of the local Atiyah-Singer index theorem*, Topology **25** (1986), 111-117.

[Ge2] E. Getzler, *Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem*, Commun. Math. Phys. **92** (1983), 167–178.

[Gi1] P. B. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Adv. Math. **10** (1973), 344–382.

[Gre] P. Greiner, *An asymptotic expansion for the heat equation*, Arch Ration. Mech. Anal. **41** (1971), 163–218.

[Gr] W. H. Greub, *Multilinear Algebra, 2nd edition*, Springer-Verlag, New York, 1978.

[H1] F. Hirzebruch, *Topological Methods in Algebraic Geometry, 3rd ed.*, (1st German ed. 1956), Springer-Verlag, Berlin, 1966.

[H2] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. of Math. **60** (1954), 213–236.

[H3] F. Hirzebruch, *A Riemann-Roch Theorem for differentiable manifolds*, Séminaire BOURBAKI (1959), no. 177.

[Hod] W. V. D. Hodge, *Harmonic functionals in a Riemannian space*, Proc. Lond. Math. Soc. **38** (1933), 72–95.

[Ho] L. Hörmander, *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517.

[J] N. Jacobson, *Basic Algebra I*, W. H. Freeman and Company, San Francisco, 1974.

[KN] J. J. Kohn, L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305.

[Lé] P. Lévy, *Wiener's random function, and other Laplacian random functions*, Second symposium of Berkeley Probability and Statistics, 1950, pp. 171–187.

[L] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris Sér. A–B **257** (1963), 7–9.

[MS] H. P. McKean, I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geo. **1** (1967), 43–69.

[M] S. G. Mikhlin, *Singular integral equations*, Amer. Math. Soc. translation Vol. 24, 1950, Uspehi Mat. Nauk. **3** (1948), no. 25, 29–112.

[Mi] J. Milnor, *On the cobordism ring Ω^*, and a complex analogue*, Amer. J. Math. **82** (1960), 505–21.

[MP] S. Minakshisundaram, A. Pleijel, *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds*, Canadian J. Math. **1** (1949), 242–256.

[N] L. Nirenberg, *On elliptic partial differential equations*, ??? **???**, 115–162.

[Pa] R. S. Palais, ed., *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies 57, Princeton University Press, Princeton, 1965.

[P1] V. K. Patodi, *Curvature and the eigenforms of the Laplace operator*, J. Diff. Geo. **5** (1971), 233–249.

[P2] V. K. Patodi, *An analytic proof of Riemann-Roch-Hirzebruch theorem for Kaehler manifolds*, J. Diff. Geo. **5** (1971), 251–283.

[PS] A. Pressley, G. Segal, *Loop Groups*, Oxford University Press, Oxford, 1986.

[Q1] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985), 89–95.

[Q2] D. Quillen, *Superconnections and the Laplace Transform*, Topology.

[Ri] B. Riemann, *Theorie der Abelschen functionen*, Journal für die reine und angewandte Mathematik **54** (1857), 115–155.

[Ro] G. Roch, *Ueber die anzahl der willkürlichen constanten in algebraischen functionen*, Journal für die reine und angewandte Mathematik **64** (1865), 372–376.

[R] J. Roe, *Elliptic operators, topology and asymptotic methods*, Pitman Research Notes in Mathematics Series 179, Longman Scientific & Technical, Essex, England, 1988.

[Se1] R. T. Seeley, *Complex powers of an elliptic operator*, Proc. Symp. Pure Math. **10** (1967), 288–307.

[Se2] R. T. Seeley, *Elliptic singular integral equations*, Proc. Symp. Pure Math. **10** (1967), 308–315.

[St] M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, AMS Colloquium Publications, Vol XV, American Mathematical Society, New York, 1932.

[Ste] Steenrod.

[T] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

[W1] E. Witten, *Constraints on supersymmetry breaking*, Nucl. Phys. B **202** (1982), 253–316.

[W2] E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom. **17** (1982), 661–692.

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