Dirac Charge Quantization, K-Theory, and Orientifolds

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November 4, 2009

Ongoing joint work with Jacques Distler and Greg Moore

Summary of results in `arXiv:0906.0795`

TWO FEATURES OF TOPOLOGY

- Scale independence Let Σ be a compact Riemannian 2-manifold with Gauss curvature K. Then

$$\text{Euler}(\Sigma) = \frac{1}{2\pi}\int_{\Sigma} K\, d\mu_{\Sigma}.$$

Small distance scale on RHS (Riemannian metric). Large distance scale on LHS (e.g. triangulation). Physics picture is analogous.

- Torsion and Integrality The de Rham cohomology groups $H^q_{\mathrm{dR}}(M)$ of a smooth manifold M are real vector spaces. The integral cohomology $H^q(M;\mathbb{Z})$ is an abelian group which maps to $H^q_{\mathrm{dR}}(M)$ with image a full lattice and kernel the torsion subgroup of $H^q(M;\mathbb{Z})$.

Note: We can substitute a *generalized* cohomology theory for integer cohomology.

DIRAC QUANTIZATION OF CHARGE

Classical Maxwell:

	M^4	Minkowski spacetime
	$F \in \Omega^2(M)$	electromagnetic field
	$j_B \in \Omega^3(M)_{cs}$	magnetic current
	$j_E \in \Omega^3(M)_{cs}$	electric current

The currents have compact *spatial* support and $dj_E = dj_B = 0$. On any spacelike hypersurface $N = \mathbb{E}^3$ their "homotopy classes" are

$$Q_B = [j_B] \in H^3_{\mathrm{dR}}(N)_c \cong \mathbb{R} \qquad \text{magnetic charge}$$

$$Q_E = [j_E] \in H^3_{\mathrm{dR}}(N)_c \cong \mathbb{R} \qquad \text{electric charge}$$

Maxwell's equations assert

$$dF = j_B$$
$$d * F = j_E$$

Notice the electromagnetic duality symmetry $F \leftrightarrow *F$, $j_B \leftrightarrow j_E$.

In the *quantum* theory charge is constrained to be an integer. This can be viewed as $H^3(N;\mathbb{Z})_c \subset H^3_{\mathrm{dR}}(N)_c$ (which is $\mathbb{Z} \subset \mathbb{R}$). This is encoded by introducing a gauge field with *topology*: the electromagnetic field F is the normalized curvature of a connection on a circle bundle.

Analogs of F, j_E, j_B appear as differential forms of arbitrary degrees in many theories. In an theory with n spacetime dimensions

$$\deg j_B + \deg j_E = n + 2.$$

In the Euclidean formulation of QFT we work on a Riemannian manifold X, assumed compact for convenience. Then charges lie in generalized cohomology groups of X.

- The choice of cohomology theory for each gauge field is based on physical considerations, including matching features of short distance and long distance theories (scale invariance).
- There may be torsion charges, and they obey a Heisenberg uncertainty relation (F.-Moore-Segal, `hep-th/0605198, 0605200`).

ORIENTIFOLDS IN STRING THEORY

String theory: X 10-dimensional spacetime (assumed compact)
Σ 2-dimensional worldsheet (also compact)

2d short distance theory on Σ: maps $\Sigma \to X$, spinor fields on Σ, ...

10d long distance approximation on X: metric and other fields on X ...

The fields on X are background data for the 2d theory, so play a dual role. Their topological features have manifestations in *both* theories, and can serve as a guide to construct the long distance approximation. Example in 4-dimensional gauge theories: 't Hooft anomaly matching.

The orientifold construction plays an important role in string phonomenology, the landscape, etc. The Type I superstring is a special case. Our formulation includes the orbifold construction of string theory.

Let Y be a smooth compact 10-manifold with involution $\sigma\colon Y \to Y$. The spacetime X is the quotient $Y /\!/ (\mathbb{Z}/2\mathbb{Z})$ in the sense of *orbifolds* in differential geometry. Two analogs of Maxwell fields:

- The Neveu-Schwarz B-field whose field strength is a 3-form $H \in \Omega^3(Y)$ with $\sigma^* H = -H$.
- The Ramond-Ramond field whose currents are differential forms on Y of various degrees, invariant or anti-invariant under σ.

The geometry imposes a background RR current, which was computed by Morales-Scrucca-Serrone (1999) in the 2d worldsheet theory. Let $i\colon F \hookrightarrow Y$ be the fixed point set of σ. Their RR charge formula is

$$-i_*\left\{2^{5-r}\sqrt{\frac{L'(F)}{L'(\nu)}}\right\} \qquad L' = \prod \frac{x/4}{\tanh x/4}\,,$$

where $r\colon F \to \mathbb{Z}$ is the codimension, $\nu \to F$ is the normal bundle, and L' is a *modified* Hirzebruch L-genus—Hirzebruch has 2 in place of 4.

RR CHARGE QUANTIZATION

$$-i_* \left\{ 2^{5-r} \sqrt{\frac{L'(F)}{L'(\nu)}} \right\}$$

The formula may be interpreted as a de Rham *current* supported on the fixed point set F. Or it may be interpreted as a real cohomology class, which is the RR *charge*.

Dirac charge quantization implies that there is a refinement to an integral generalized cohomology group.

Plan: Give geometric models for the B-field and RR field which encode charge quantization. Interpret and compute the background RR charge in the abelian group and recover above formula in the vector space.

We go further and define precisely all fields and the action for both the 2d and 10d theories for general orientifolds.

SPECIAL CASE: TYPE I SUPERSTRING

Y compact spin 10-manifold $\qquad H = 0$
σ the trivial involution ($F = Y$) $\qquad$ RR currents of degrees 4,8.

Quantization of RR charge: In the mid '90s it was realized (Minasian-Moore, Witten, ...) that Ramond-Ramond charge is quantized by K-theory. For Type I the appropriate flavor is KO-theory. The RR charge lies in $KO^0(Y)$.

Recover differential forms by tensoring with $\mathbb{R}$. Recall

$$KO(\mathrm{pt}; \mathbb{R}) \cong \mathbb{R}[u_c^{\pm 2}],$$

where u_c^2 has degree 4. (We recall u_c later.) The Chern character

$$KO^0(Y) \xrightarrow{\mathrm{ch}} H\big(Y; \mathbb{R}[u_c^{\pm 2}]\big)^0 \cong H^0(Y;\mathbb{R}) \oplus H^4(Y;\mathbb{R})u_c^{-2} \oplus H^8(Y;\mathbb{R})u_c^{-4}$$

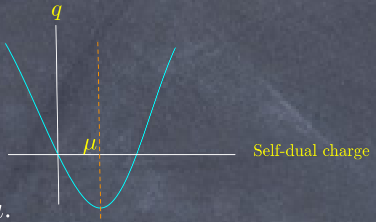
is an iso after tensoring with $\mathbb{R}$. Currents in $\Omega^{\{0,4,8\}}(Y)\, u_c^{\{0,-2,-4\}}$.

Problem: Too many charges. Throw out degree 0 (no field strength in degree -1). But *both* magnetic and electric currents in degrees 4 and 8.

SELF-DUAL FIELDS

The electromagnetic duality of the classical Maxwell equations persists in the (free) quantum theory and allows us to define self-dual fields.

- Quantization of charge by a Pontrjagin self-dual cohomology theory.
- There is a specified isomorphism from the magnetic charge group to the electric charge group. This single abelian group is the self-dual charge group.
- For non-self-dual abelian gauge fields there is a bilinear pairing between magnetic and electric charges. For self-dual fields a specified quadratic form q on the self-dual charge group refines this bilinear form. It has center of symmetry μ determined by q.



- The background self-dual charge is $-\mu$.

RR FIELD IN TYPE I

The quadratic form q is most easily defined on an auxiliary compact, spin 12-manifold M where it is integer-valued:

$$\begin{aligned} q\colon KO^0(M) &\longrightarrow KO^{-12}(\mathrm{pt}) \cong \mathbb{Z} \\ x &\longmapsto \pi^M_*\bigl(\lambda^2(x)\bigr) \end{aligned}$$

$x \mapsto \lambda^2(x)$ is the second exterior square, a quadratic function.
$\pi^M\colon M \to \mathrm{pt}$ and π^M_* is the induced pushforward (integration) on KO.

Define a KO-theoretic Wu class $\Xi(M) \in KO^0(M)$ by

$$\pi^M_*\bigl(\psi_2(x)\bigr) = \pi^M_*\bigl(\Xi(M)\,x\bigr), \qquad x \in KO^0(M).$$

$x \mapsto \psi_2(x)$ is the Adams operation, a homomorphism.

Theorem (F.-Hopkins, 2000): The center μ of q satisfies

$$2\mu = \Xi(M).$$

This theorem determines the background RR charge $-\mu$ up to torsion of order 2. A standard computation in KO-theory gives the formula in rational cohomology

$$-\sqrt{\hat{A}(Y)}\,\mathrm{ch}\,\mu = -2^5\sqrt{L'(Y)}, \qquad L' = \prod \frac{x/4u_c}{\tanh x/4u_c} \qquad (*)$$

We include a normalizing factor $\sqrt{\hat{A}} = \prod \dfrac{x/2u_c}{\sinh x/2u_c}$. Only even powers of u_c occur in $\hat{A}$ and L'.

The theorem refines the Green-Schwarz anomaly cancellation mechanism to the integers; their formulas are recovered from $(*)$ which gives the rational characteristic classes of the real "gauge bundle" on Y. For example, it has rank $2^5 = 32$. See `hep-th/0011220` for details.

As a preliminary to general orientifolds we describe a model for the B-field and RR charges.

TWISTINGS OF KR-THEORY

There are many approaches to twistings of K-theory: Donovan-Karoubi, Rosenberg, Atiyah-Segal, Bouwknegt-Carey-Mathai-Murray-Stevenson, etc. We adapt F.-Hopkins-Teleman (arXiv:0711.1906) to KR-theory and include degree shifts as twistings.

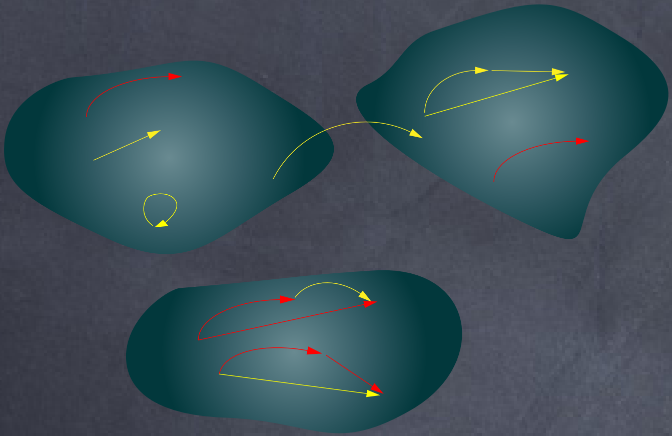
Let X be a local quotient groupoid in the sense that locally it is isomorphic to $S/\!/G$ for S a nice space (e.g. manifold) and G a compact Lie group. We write

$$X : \quad X_0 \underset{p_0}{\overset{p_1}{\leftleftarrows}} X_1$$

Specify a double cover $\pi\colon X_w \to X$ by a homomorphism $\phi\colon X_1 \to \mathbb{Z}/2\mathbb{Z}$. Then X_w is represented by the groupoid

$$X_w : \quad X_0 \underset{p_0}{\overset{p_1}{\leftleftarrows}} X_1'$$

where $X_1' = \{(a \xrightarrow{f} b) \in X_1 : \phi(f) = 0\}$ is the kernel of ϕ. It is classified by $w \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ (cohomology of geometric realization).



Pictured is the groupoid X. Yellow arrows f satisfy $\phi(f) = 0$; red arrows f satisfy $\phi(f) = 1$. The groupoid X_w has only the yellow arrows.

Extend the groupoid to a simplicial space with X_n the space of compositions of n arrows.

$$X: \quad X_0 \leftleftarrows X_1 \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} X_2 \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}} X_3 \cdots$$

For V is a complex vector space, $\phi \in \mathbb{Z}/2\mathbb{Z}$, set

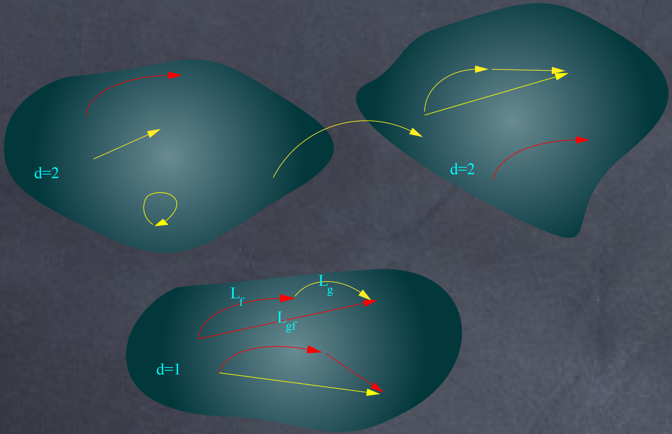
$$^{\phi}V = \begin{cases} V, & \phi = 0; \\ \overline{V}, & \phi = 1. \end{cases}$$

Definition: A *twisting* of $KR(X_w)$ is a triple $\beta = (d, L, \theta)$ consisting of a locally constant function $d\colon X_0 \to \mathbb{Z}$, a $\mathbb{Z}/2\mathbb{Z}$-graded complex line bundle $L \to X_1$, and for $(a \xrightarrow{f} b \xrightarrow{g} c) \in X_2$ an isomorphism

$$\theta\colon {}^{\phi(f)}L_g \otimes L_f \xrightarrow{\cong} L_{gf}.$$

There are consistency conditions for d on X_1 and for θ on X_3.

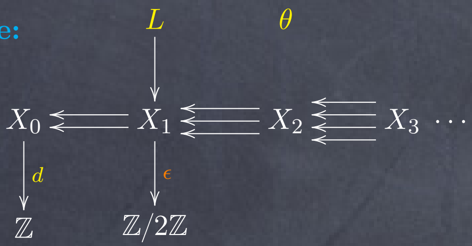
Warning: In general, we replace X by a locally equivalent groupoid.



Consistency conditions:

- The degree d is equal on components of X_0 connected by an arrow.
- There is an isomorphism $\theta\colon \overline{L_g} \otimes L_f \to L_{gf}$ for the labeled arrows.

Another picture:



We define a 2-groupoid of twistings with a commutative composition law. Isomorphism classes of twistings of $KR(X_w)$ are classified by

$$H^0(X;\mathbb{Z}) \times H^1(X;\mathbb{Z}/2\mathbb{Z}) \times H^{w+3}(X;\mathbb{Z}),$$

$$d \qquad\qquad \epsilon \qquad\qquad (L,\theta)$$

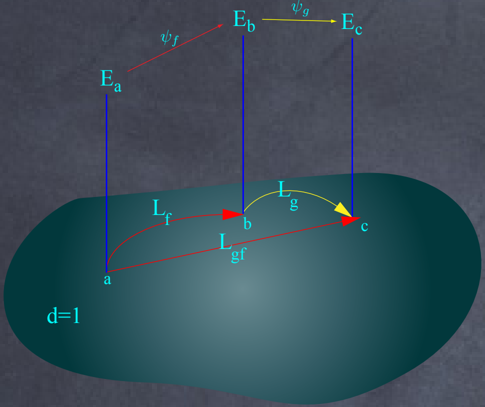
where the last factor is cohomology in the local system defined by $X_w \to X$. This is an isomorphism of sets but *not* of abelian groups.

Key point: We can realize twistings as objects in a cohomology theory. Special case: involution on X_w acts trivially—so twistings of $KO(X)$—twistings classified by Postnikov truncation $ko < 0 \cdots 2 >$ of connective ko with homotopy groups $\pi_{\{0,1,2\}} = \{\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\}$.

An object in twisted $KR^q(X_w)$ may be represented by a pair (E, ψ), with $E \to X_0$ a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford$_q$-module, for $(a \xrightarrow{f} b) \in X_1$

$$\psi : {}^{\phi(f)}(L_f \otimes E_a) \xrightarrow{\cong} E_b$$

Consistency condition on X_2:



Warning: In general we need to use a more sophisticated model in which E has infinite rank and an odd skew-adjoint Fredholm operator.

TWISTED $KR^{\bullet}(\mathrm{pt})$

$X = \mathrm{pt} /\!/ (\mathbb{Z}/2\mathbb{Z})$ $\qquad$ $d = 0$

$E = E^0 \oplus E^1$ complex $\mathbb{Z}/2\mathbb{Z}$-graded $\qquad$ $\sigma \colon E \to E$ antilinear

4 twistings: $\quad \sigma$ even/odd $\quad$ and $\quad \sigma^2 = \pm 1$.

A cyclic group of order 4 with generator β_1: σ odd, $\sigma^2 = -1$. So the group of isomorphism classes of twistings of $KR(\mathrm{pt})$ is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$. Denote $\beta_\ell = \ell\beta_1$, $\ell \in \mathbb{Z}$.

Bott class: $u \in KR^{\beta_1+2}(\mathrm{pt})$ modeled as $\mathbb{C}^{1|1} = \mathbb{C} \oplus \mathbb{C}$ with

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma(z^0, z^1) = (-\overline{z^1}, \overline{z^0}).$$

u is invertible: multiplication by u is real Bott periodicity in KR. Drop σ to obtain the complex Bott element $u_c \in K^2(\mathrm{pt})$.

After inverting 2 we have, as a $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z})$-graded ring,

$$KR[1/2]^{\bullet}(\mathrm{pt}) \cong \mathbb{Z}[1/2][u^{\pm 1}, u_c^{\pm 2}]/(u^4 - u_c^4).$$

DIFFERENTIAL OBJECTS

Definition: A *differential twisting* of $KR(X_w)$ is a quintet $\check{\beta} = (d, L, \theta, \nabla, B)$ where $\beta = (d, L, \theta)$ is a twisting, ∇ is a covariant derivative on L, and $B \in \Omega^2(X_0)$ satisfies

$$(-1)^{\phi} p_1^* B - p_0^* B = \frac{i}{2\pi} \operatorname{curv}(\nabla) \qquad \text{on } X_1.$$

The 3-form $H = dB$ is a global *twisted* form: $(-1)^{\phi} p_1^* H = p_0^* H$. It is the *curvature* of $\check{\beta}$. (Ungraded version in Schreiber-Schweigert-Waldorf).

- $\exists$ *finite dim* model of twisted differential KR-theory $\widecheck{KR}^{\check{\beta}}(X_w)$.
- Cohomological interpretation $\Longrightarrow$ topological models for differential objects (Hopkins-Singer). Products, pushforwards, . . .
- Other models for differential objects in H and K-theory. (Deligne, Simons-Sullivan, Bunke-Kreck-Schick-Schroeder-Wiethaup, . . .)
- No general *equivariant* differential theory. Ordinary cohomology (Gomi). Finite group actions in K-theory (Szabo-Valentino, Ortiz).

The foregoing provides an explicit model of the B-field on an orientifold. We formulate everything in a *model-independent* manner.

NSNS SUPERSTRING BACKGROUND

The (Neveu-Schwarz)^2 =NSNS fields are relevant for both the worldsheet (2d) and spacetime (10d) theories.

Definition (Distler-F.-Moore): An *NSNS superstring background* consists of:

(i) a 10-dimensional smooth orbifold X together with Riemannian metric and real-valued scalar (dilaton) field;

(ii) a double cover $\pi\colon X_w \to X$ (orientifold double cover);

(iii) a differential twisting $\check{\beta}$ of $KR(X_w)$ (B-field);

(iv) and a *twisted spin structure* $\kappa\colon \Re(\beta) \to \tau^{KO}(TX - 10)$.

- An orbifold (in the sense of Satake) is presented by a local quotient groupoid which is locally $S/\!/\Gamma$ with S a manifold, Γ *finite*.
- κ is an isomorphism of twistings of $KO(X)$ whose existence relates $w_1(X), w_2(X)$ to the isomorphism class of β and w.
- Only the underlying topological twisting β enters today.

RAMOND-RAMOND CURRENT

Definition (con't): An *RR current* is an object $\check{j}$ in $\widecheck{KR}^{\check{\beta}}(X_w)$ with quadratic form q.

Recall $KO^0_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{pt}) \cong RO(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$, $\epsilon =$sign representation.
M a 12-manifold with double-cover M_w and twisted spin structure κ.
Quadratic form q:

$$\begin{array}{ccc} & & KR^{\beta}(M_w) \\ & & \downarrow \\ KO^{\Re(\beta)}_{\mathbb{Z}/2\mathbb{Z}}(M_w) & \xrightarrow[\cong]{\kappa} & KO^{\tau^{KO}(TM-12)}_{\mathbb{Z}/2\mathbb{Z}}(M_w) \\ & & \downarrow \pi_*^{M_w} \\ & & KO^{-12}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{pt}) \cong \mathbb{Z} \times \mathbb{Z}\epsilon \\ & & \downarrow \\ & & \mathbb{Z} \end{array}$$

$$\begin{array}{c} j \\ \downarrow \\ \kappa\bar{j}j \\ \downarrow \\ \pi_*^{M_w}(\kappa\bar{j}j) \\ \downarrow \\ \epsilon\text{-component } \pi_*^{M_w}(\kappa\bar{j}j) \end{array}$$

BACKGROUND RR CHARGE

Assume $X = Y /\!/ (\mathbb{Z}/2\mathbb{Z})$ is the global quotient of a compact 10-manifold by an involution σ, fixed point set $i\colon F \hookrightarrow Y$, normal bundle $\nu \to F$.

Theorem (Distler-F.-Moore), in progress: The center $\mu \in KR^{\beta}(Y)$ of q localizes to F after inverting 2. The background charge is

$$-\sqrt{\hat{A}(Y)}\ \mathrm{ch}\,\mu = -i_*\left\{(-1)^{(d+r-2\ell)/4}\,2^{5-r}\,u_c^{(d+r-2\ell)/2}\,u^{\ell-r}\sqrt{\frac{L'(F)}{L'(\nu)}}\right\}$$

$$\left.\begin{array}{l} d\colon F \to \mathbb{Z} \\ \ell\colon F \to \mathbb{Z}/4\mathbb{Z} \end{array}\right\} \text{ determined by } \beta\big|_y = \beta_\ell + d \text{ for } y \in F$$

$r\colon F \to \mathbb{Z}$ codimension of fixed point set F

$2\ell \equiv d + r \pmod 4$

This 10d derivation generalizes the Morales-Scrucca-Seronne 2d formula to arbitrary orientifolds and refines the background RR charge to $\mathbb{Z}$.

A TIGHT FITTING SYSTEM

Cyan is data. Yellow is 10d theory. Green is 2d theory.