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On G-structures in gauge/string duality

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Summary

We study how G-structures can be used in the framework of gauge/string duality. The reason why such concept is so powerful is that G-structures are a very natural way of describing supersymmetry in a geometric setting. We investigate more specifically two different points.

First, we are interested in how G-structures can help with constructing and understanding supergravity backgrounds with sources that can be interpreted as flavours in the dual field theory. In particular, we show that the smearing procedure, for flavouring a background, is expressed more clearly when described in terms of G-structures. We discuss this problem in general terms, before applying the newly developed techniques to several examples, some already known and some new. We see that the way one adds and distributes branes in a supersymmetric background is strongly constrained by the preservation of supersymmetry.

We then also look at how one can develop solution-generating techniques from G-structures, that derive complex solutions out of simple ones, by turning on new fluxes. We specialise to the two cases of $SU(3)$ and G_2-structures. In both examples, we use the solution-generating methods on known solutions to create new, previously unknown solutions. We show that those techniques can be applied to flavoured as well as unflavoured backgrounds.

DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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STATEMENT 1

This thesis is the result of my own investigations, except where otherwise stated. Where correction services have been used, the extent and nature of the correction is clearly marked in a footnote.
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STATEMENT 2

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Chapter 1

Introduction

1.1 Motivations

First created as a possible theory for the strong interaction, string theory has become one of the most promising candidate for a theory of quantum gravity. Currently one of the biggest problems in theoretical physics, combining the theory of general relativity with the quantum world has proved very difficult. The main concept of string theory is that the fundamental objects of our universe are not particles but vibrating strings. This staggering assumption leads to surprising consequences, one of which being the existence of extra dimensions. If one also assumes the existence of supersymmetry, then string theory dictates that space-time has to be ten-dimensional (or eleven-dimensional for M-theory). But for the past fifteen years, string theory has been studied not only as a theory of quantum gravity, but also as a tool for understanding strongly coupled quantum field theories.

Indeed, in 1997, Maldacena proposed [1] that there exists a duality between a particular conformal field theory, namely four-dimensional $\mathcal{N} = 4$ super Yang-Mills, and Type IIB string theory on the ten-dimensional space $\text{AdS}_5 \times S^5$. Called the AdS/CFT correspondence, this idea has been further studied notably in [2] and [3] (see [4] for a review). The original idea is based on the existence of D-branes. D-branes are extended objects in string theory, on which lives a gauge theory. More precisely, on a stack of N_c Dp-branes lives a $(p + 1)$-dimensional $SU(N_c)$ gauge theory. The AdS/CFT correspondence relies on this double description of the branes. On the one hand, they are objects in string theory that live in a particular ten-dimensional space. On the other hand, they host a gauge theory.

Moreover, what is particularly interesting with this correspondence is that it is a strong/weak duality. This means that when one side of the correspondence is weakly coupled (that is we can apply perturbative techniques), the other side is strongly coupled (in which case it is very complicated to perform calculations). In the case of large N_c and large 't Hooft coupling, one can approximate the string-theory side by classical supergravity, and then one can study a strongly coupled gauge theory in the planar limit by studying a weakly coupled supergravity. Hence, this correspondence provides a tool for studying strongly coupled gauge theories.

In addition to the interest of studying $\mathcal{N} = 4$ super Yang-Mills, one may wonder if this correspondence can be adapted to study strongly coupled gauge theories that are more relevant physically, such as QCD. Indeed, QCD is strongly coupled at low energies, and the only way to study it was through numerical lattice simulations. Since the first paper on the AdS/CFT correspondence, a lot of work has been done to expand its scope, in many directions. This thesis is part of that project, to better understand how certain solutions in string theory can be dual to field theories, and what we can learn, through those dualities, about strongly coupled gauge theories. There are two main points on which $\mathcal{N} = 4$ super Yang-Mills is very different from realistic theories and in particular QCD. Firstly, $\mathcal{N} = 4$ super Yang-Mills in four dimensions is maximally supersymmetric, while QCD has no supersymmetry. This has been studied first in [5, 6]. Secondly, contrary to QCD that contains quarks, there is no matter in the fundamental representation of the gauge group in $\mathcal{N} = 4$ super Yang-Mills. Thus we have two possible directions of generalisation of the AdS/CFT correspondence to try to study other gauge theories, with less supersymmetry and a richer matter content. One additional parameter one can play with is the dimension of the field theory, and in some of the following chapters we also look at three-dimensional theories.

In the next two sections, we introduce ways to deal with the two issues mentioned above, namely how to reduce the amount of supersymmetry and how to introduce fundamental matter in the framework of the AdS/CFT correspondence. Since the new solutions we present do not have an AdS space in string theory, nor are they dual to conformal field theories, we refer to the generalisation of the AdS/CFT correspondence as gauge/string duality. In the following, we look at the reduction in the number of preserved supercharges: this issue is related to the geometry of the space considered in string theory, and one way to understand the link between geometry and supersymmetry is to use the concept of G-structures. But we first address the problem of adding fundamental matter which, by analogy

with the terminology used in QCD, we call "flavouring".

The rest of the thesis is then organised as follows.

In Chapter 2, based on [7], we show how one can understand the problem of flavouring in the language of G-structures. It allows us to make general statements on the type of backgrounds and flavours that are compatible with a given amount of supersymmetry in a fixed dimension. We then, as an example, apply the newly developed technique to the flavouring of a background dual to a three-dimensional $\mathcal{N} = 2$ gauge theory.

In Chapter 3, based on [8], we look at the flavouring of a particular Type IIA supergravity background with D6-branes. We then study how G-structures can help us to understand the lift to eleven dimensions of that particular Type IIA supergravity solution involving smeared D6-sources. We see that the sources correspond to geometric torsion in M-theory.

In Chapter 4, based on [9], we use the techniques of Chapter 2 in order to create a new background of Type IIB supergravity with flavours. The main specificity of this new solution is the use of Riemann surfaces with genus bigger than one as cycles for brane wrapping. This allows us to have an even richer matter content than previously. That is, in addition to the gauge bosons and fundamental matter, we have adjoint matter in the dual field theory. This means that this gauge theory is closely related to the ones created by Kutasov, and it indeed exhibits a Kutasov-like duality.

In Chapters 5 and 6, based on [10] and [11] respectively, we study how G-structures can be used to create solution-generating techniques. Chapter 5 is concerned with $SU(3)$-structure and Type IIB supergravity, while Chapter 6 deals with G_2-structure in Type IIA supergravity. Those techniques based on G-structures are compared with techniques based on chains of string dualities. It is shown that they are equivalent in the case where one does not consider the addition of flavours. Flavouring however is only compatible with G-structures, since the dualities are not well defined in that case.

In Chapter 7, based on [12], we investigate the addition of massive flavours to the Maldacena-Núñez background [13]. Using holomorphicity, we developed a new technique to deal with massive flavours on the gravity side, and in particular we find solutions that are regular everywhere.

Finally, we summarise the results presented in the different chapters.

1.2   Flavouring in gauge/string duality

As was mentioned before, if one wants to use gauge/string duality to study QCD-like theories, then it is imperative to consider the presence of matter transforming in the fundamental representation of the gauge group. As an abuse of language since we never deal exactly with QCD, we call such matter "quarks" or "flavours". Let us recall that the gauge/string duality is based on the fact that there is a stack of D-branes in string theory, and on those branes lives a gauge theory. Since those branes create the gauge group of the field theory, we call them colour branes, again by analogy to QCD. D-branes are extended objects where strings can end, and each end has an associated colour index, depending on which brane it belongs to. So strings which have both ends on branes of that particular stack have two colour indices, and hence represent gauge bosons. The fact that quarks are in the fundamental representation of the gauge group means that they have only one colour index. Thus only one end of the string representing a quark can be on a brane of the colour stack. But open strings cannot have their end-points floating around in space. They must belong to some brane. So in order to have strings that can represent flavours, we need to introduce other branes for the second end of the flavour string to be. We call those new branes "flavour branes".

As there is a gauge theory living on D-branes, adding new flavour branes in a background introduces *a priori* a new gauge sector. However, the purpose of those branes is to create flavours only. The way to get rid of this unwanted new gauge sector comes from the position and type of branes introduced. Indeed, flavour branes need to span, in the string-theory background, the directions where the field theory lives. But for flavour branes one takes higher-dimensional branes, so that there are extra dimensions to the ones of the field theory. If the setup is such that those extra dimensions are non-compact, then the gauge coupling of the new sector, as seen in the field theory, becomes zero.

In order to take into account the dynamics of the flavours, we need to consider the Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) actions for each flavour brane. If one considers those flavour branes as being placed in a fixed string-theory background, then one finds himself in the probe approximation. It has been first studied in [14, 15] (see also [16] for a review). That is, one does not take into account the backreaction of the flavour branes on the space-time solution. This is a good approximation as long as the number N_f of flavour branes added is very small compared to the number N_c (which is always taken to be large for the duality

to apply) of colour branes that created the background. This is related to what is called the quenched approximation in lattice field theory. Taking $N_f \ll N_c$ means that the flavours are non-dynamical, that they do not appear in loops. In field theory, this corresponds to the 't Hooft limit, that is

$$N_c \to \infty \,, \qquad \lambda = g_{YM}^2 N_c \text{ fixed}\,, \qquad N_f \text{ fixed}\,, \qquad \frac{N_f}{N_c} \to 0 \,, \qquad (1.2.1)$$

where λ is the 't Hooft coupling and g_{YM} the gauge coupling of the field theory. This is enough to study a number of properties of gauge theories with fundamental matter: chiral symmetry breaking or meson spectra for example. But making such approximation prevents us from seeing other very interesting phenomena. The fact that flavours do not run in loops means that we cannot study anything that comes from the quantum effect of quarks in the field theory, like Seiberg duality [17].

To address this issue, we need, at the level of the field theory, to take the Veneziano limit and not the 't Hooft limit anymore. It corresponds to having

$$N_c \to \infty \,, \qquad \lambda = g_{YM}^2 N_c \text{ fixed}\,, \qquad N_f \to \infty \,, \qquad \frac{N_f}{N_c} \text{ fixed}\,. \qquad (1.2.2)$$

In this limit, it is possible to have quarks running in loops, and so it allows phenomena based on the quantum effects of flavours. On the string side, taking N_f large and of the same order as N_c means that it is not possible to ignore the backreaction of the flavour branes anymore. Instead of solving for a background created by colour branes only, and studying the addition of flavour branes in this fixed space-time, one now needs to solve for the following action:

$$S = S_{IIA/B} + S_{\text{sources}} \,, \qquad (1.2.3)$$

where $S_{IIA/B}$ is the action of the supergravity we want to use (either Type IIA or IIB) and S_{sources} is the action for all the flavour branes. Now comes the question of how to arrange all those N_f branes. If one puts them as a stack somewhere in space, then one indeed gets $SU(N_f)$ as a global flavour symmetry. But then this localised stack breaks the symmetries of the background, leading to huge difficulties in solving the equations of motion (see however [18, 19]). In addition, if one considers a stack of N_f branes, then one cannot use for their action the DBI and WZ terms, since they do not take into account the non-abelian character of the problem. Another possibility, which is the one used all along in this thesis, is to separate the flavour branes and distribute them all around the space, in a

way that preserves the symmetries. This technique is called smearing. Since all the branes are now separated, one can this time use the DBI and WZ actions for each one, and S_{sources} is then simply the sum of the actions for each brane. In addition, by respecting the symmetries of the unflavoured background, it makes the problem of solving the equations of motion much simpler. One consequence however of separating the branes is that the flavour group as seen in the field theory is not $SU(N_f)$ anymore but is $U(1)^{N_f}$. This technique has first been used in [20] for non-critical strings and in [21] in the context of ten-dimensional string theory. See also [22] for a recent review, including some of the results presented in this thesis.

Smearing the flavour branes asks also the question of the stability of such a configuration. Indeed, if one tries to separate the branes, do they not attract each other, making the whole setup completely unstable? One possible solution to this problem lies within supersymmetry. Let us suppose that our unflavoured background preserves some supersymmetry. Then there are particular ways of putting branes in that space so that they still preserve the supersymmetry. And branes placed in this manner are stable. So, the way to ensure that the smeared setup is stable is to start from a supersymmetric background, and to arrange the flavour branes so that each one preserves the supersymmetry. This can be achieved practically by using G-structures. Indeed, as is explained in the next section, G-structures are a way to understand supersymmetry in a geometric context, which is exactly what is needed to deal with the smearing procedure.

1.3 G-structures

A well-known concept in differential geometry, G-structures provide a natural framework for talking about supersymmetry in geometric terms. It is very useful in the context of gauge/string duality, where one wants to build spaces in supergravity that preserve a certain amount of supercharges. In addition, it also provides us with the concept of calibration, which tells us where to put branes so that they also preserve supersymmetry. Once again, it is what we need to address the issue of flavouring, in particular for the smearing procedure. For a review of G-structures in a physics context, see for example [23] or [24] and references therein.

Let us first briefly describe what G-structures are in mathematical terms, before looking at how they relate to supersymmetry. First we need to know what the frame

bundle over a manifold $\mathcal{M}$ is. It is the fibre bundle for which the fibre in each point of $\mathcal{M}$ is the set of all the ordered bases of the tangent space at that point. For a point of $\mathcal{M}$ that belongs to two different patches of the manifold, we have two different bases given by the frame bundle. The way the two patches are related to each other defines transition functions that transform one base into the other. The set of all transition functions is a group called the structure group. Generically, for a d-dimensional manifold, the structure group is the group of d-dimensional real matrices $GL(d, \mathbb{R})$. However, if it is possible, by choosing appropriately the bases in different patches, to have a frame bundle with structure group G (where G is a subgroup of $GL(d, \mathbb{R})$), then one says that the manifold has a G-structure.

This mathematical definition is of little help to physicists. So let us now explain, with some examples, how this idea of G-structure can be used in supergravity. One consequence of the presence of a G-structure for a manifold is the existence of certain globally defined non-degenerate tensors that are invariant under G. The number and type of those depend on the particular structure one considers. Actually, in all this thesis, when we want to impose a particular structure on a manifold, we impose the existence of the appropriate globally defined invariant tensors. Indeed, if such objects exist, one can choose a frame bundle such that the form of all the invariant tensors is the same in all the patches. Then, only transition functions that do not modify that form are acceptable, which leads to a reduced structure group and thus to a G-structure. For example, working with a d-dimensional Riemannian manifold $\mathcal{M}$ means that there is a globally defined two-tensor on $\mathcal{M}$, the metric. So, for reasons we just mentioned, the structure group is reduced to $O(d)$, which is a subgroup of $GL(d, \mathbb{R})$. If, in addition, one defines a volume form on $\mathcal{M}$, then the structure group is further reduced to $SO(d)$. Since we always work with Riemannian manifolds equipped with a volume form, the most general structure we deal with in d dimensions is $SO(d)$-structure. That means that if we want to impose an additional structure, it has to be with a group G which is a subgroup of $SO(d)$.

Let us now take the example of $SU(3)$-structure in six dimensions ($SU(3)$ is a subgroup of $SO(6)$) to look at how one can practically use G-structures, and in particular the invariant tensors that come with. To find out what the invariants are for a given G-structure, one can decompose tensor representations in terms of representations of the group G and look for singlets. For the case of $SU(3)$-structure, we can start with the one-form representation of $SO(6)$, denoted $\mathbf{6}$. Its decomposition in terms of representation of $SU(3)$ gives: $\mathbf{6} \rightarrow \mathbf{3} + \bar{\mathbf{3}}$, where $\mathbf{3}$ is the

one-form representation of $SU(3)$ and the bar stands for the complex conjugation. It means that any one-form on the six-dimensional manifold can be expressed as a sum of two one-forms, one in the **3** representation of $SU(3)$ and the other one in the $\bar{\mathbf{3}}$ representation. As there are no singlets appearing in this decomposition, it means that $SU(3)$-structure does not have an invariant one-form. If we now look at the decomposition of the two-form representation **15** and the three-form one **20**, we find

$$\mathbf{15} \to \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{8}\,,$$
$$\mathbf{20} \to \mathbf{1} + \bar{\mathbf{1}} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{6} + \bar{\mathbf{6}}\,. \tag{1.3.1}$$

In both cases, we notice that there is a singlet **1** in the decomposition. That means that there is an $SU(3)$-invariant two-form and an $SU(3)$-invariant three-form. The second singlet appearing in the decomposition of the three-form representation is the complex conjugate of the first one. So there is only one independent invariant three-form. If one looked at the decompositions of other representations of $SO(6)$, one would not find any more singlets. So the $SU(3)$-structure on a six-dimensional manifold is characterised by the existence of one globally defined $SU(3)$-invariant two-form (usually called J) and one such three-form (denoted as Ω). In addition to the structure, one can consider integrability conditions, which can lead to having a manifold of G-holonomy. For example, in an $SU(3)$-structure, adding the conditions

$$\mathrm{d}J = 0\,, \qquad \mathrm{d}\Omega = 0\,, \tag{1.3.2}$$

makes the manifold an $SU(3)$-holonomy manifold, also called a Calabi-Yau manifold. The way we use G-structures in this thesis is through those invariant forms and the integrability conditions (that are usually modified in various ways) they obey.

We now look at how G-structures relate to supersymmetry. The fact that a manifold preserves some supersymmetry means that there exist on that manifold some globally defined spinors. The number of those spinors is related to the number of unbroken supercharges. If we want to talk about spinors, we need to consider our manifold to be spin. Then we can perform the same type of decomposition of the spinor representation as we did for the tensor ones. Using again the example of $SU(3)$-structure on a six-dimensional manifold, we find that the spinor representation decomposes as follows:

$$\mathbf{4} \to \mathbf{1} + \mathbf{3}\,. \tag{1.3.3}$$

This indicates that having an $SU(3)$-structure imposes the existence of one invariant spinor. From this decomposition, one can also immediately see that an $SU(3)$-structure preserves one-quarter of the possible supercharges. Notice that this whole idea works in reverse as well. That is, if one imposes the existence on a manifold of unbroken supersymmetry, that is the existence of a number of globally defined spinors, then one is necessarily in presence of a G-structure, since the spinors constrain the way one can choose bases in the frame bundle. It is also interesting to notice that the invariant tensors of a given G-structure can be expressed as bilinears in its invariant spinors. Once again taking the example of an $SU(3)$-structure, we have one invariant spinor η (normalised such that $\eta^\dagger \eta = 1$), from which we can construct the invariant two-form and three-form as follows:

$$
\begin{aligned}
J_{ij} &= i \, \eta^\dagger \gamma_{ij} \, \eta \,, \\
\Omega_{ijk} &= \eta^T \gamma_{ijk} \, \eta \,,
\end{aligned}
\tag{1.3.4}
$$

where i, j, k run from 1 to 6 and γ_{ijk} denotes the antisymmetrised product of gamma matrices. This relationship between the invariant forms and spinors is very useful for the flavouring question, since it allows us to easily find the places where to put flavour branes such that they preserve supersymmetry. Indeed, as we discuss in more details in Chapter 2, one needs to wrap those branes on calibrated cycles, whose calibration form is related to the invariant forms. In $SU(3)$-structure for example, the invariant two-form J is the calibration form for two-cycles in the absence of $B_{(2)}$ field.

To summarise, G-structures provide a geometric framework for dealing with supersymmetry. As we realise all along this thesis, this way of presenting problems proves very useful in order to find solutions of supergravity, with or without flavours, that are interesting for gauge/strings duality, but also to improve our understanding of string theory in general.

Chapter 2

G-structures and backreacting flavours

2.1 Introduction

This chapter, based on work done in collaboration with Schmude [7], presents a first approach to the use of G-structures to address the problem of flavouring backgrounds in gauge/string duality. As was explained in the introduction, the study of some aspects of both QCD and supersymmetric gauge theories demands for a discussion of the Veneziano limit (1.2.2), where in particular the number of colours N_c and the number of flavours N_f are of the same order. So, on the supergravity side, it is imperative to go beyond the probe approximation and to take into account the backreaction of the flavour branes. When doing so, one should keep in mind a fundamental difference between the colour and flavour branes. While the former undergo a geometric transition and are replaced by fluxes, the latter are still present in the string dual – whether one includes their backreaction or treats them as probes – in order to realise the $SU(N_f)$ flavour symmetry (actually broken to $U(1)^{N_f}$ after smearing) in the bulk. Therefore one needs to consider the combined action

$$S = S_{IIB} + S_{\text{sources}} \tag{2.1.1}$$

where $S_{\text{sources}} = S_{DBI} + S_{WZ}$ is the brane action given by the DBI and WZ terms. This method of computing the backreaction was introduced in [25]. The localised branes become δ-function sources in the equations of motion, making the search for solutions of the above system highly non-trivial. See however [26, 27].

The most successful method for dealing with this issue was first developed in

[20]. See also [21, 28, 29]. By considering a continuous distribution of flavour branes over their transverse directions, one avoids the problem of the inclusion of localised sources, making the search for solutions much more feasible. An advantage of the smearing method lies in the fact that the inclusion of localised sources breaks local isometries on the string-theory side, leading to a violation of global symmetries in the full dual gauge theory – including the Kaluza-Klein (KK) modes. Such symmetries are restored after smearing. By now, many examples of string duals with both massless and massive flavours have been constructed in various dimensions ([30]-[41]).

The smearing of Dp-branes is usually determined by the use of a $(10 - p - 1)$-form Ξ, the smearing form, which is in general interpreted as a distribution density of the branes. One issue resides in finding a way to construct Ξ, such that it is possible to find a solution of the equations of motion. This problem is usually addressed on a case by case basis, and can be quite difficult to tackle when flavour embeddings cannot be identified with globally defined coordinates. If there is no obvious choice of Ξ, one uses the physical properties of the anticipated dual gauge theory, such as the mass of the fundamental fields or the unbroken symmetries, in order to impose constraints on its form.

To deal with this issue, we make use of some of the methods of modern string phenomenology. If one allows for the misnomer of thinking of string duals with flavours as string compactifications with non-compact internal manifolds, the two fields become virtually the same. So it is quite striking that, while the set-up of gauge/string duality with flavours is quite similar to that of string phenomenology, the methods used are quite different. In particular, the advanced mathematical methods of modern phenomenology such as the uses of generalised calibrated geometry, G-structures or generalised geometry, have been absent, before [7], from any discussion of flavours in gauge/string duality.

Geometric arguments have been used to tackle the question of supersymmetry conservation, but not to answer the issue of smeared flavours. However geometry is one of the main techniques used in the study of string compactifications, which is in many points similar to the search for backgrounds with gauge duals.

So this chapter is a first step towards bridging the gap between string phenomenology and gauge/string duality with flavours, by presenting a systematic method of finding backgrounds with smeared flavours, using tools from modern geometry. The main goal is to find the smearing form Ξ, and the strategy is to use generalised calibrated geometry [42]. The central concept of this field is that

of the calibration form $\mathcal{K}$, a $(p+1)$-form which can usually be constructed as a bilinear of the supersymmetry spinors of the background. It has the property that a brane is supersymmetric if and only if the pull-back $i^*(\mathcal{K})$ of the form onto the world-volume is equal to the induced volume form. It follows immediately that one can write the DBI action of any supersymmetric brane in terms of the pull-back of the calibration form. As all the backgrounds considered in this chapter are Type IIB backgrounds with only the dilaton Φ and one or two Ramond-Ramond (RR) fields excited, it is not necessary to make use of the full machinery of generalised calibrations or G-structures. A more complete treatment of some cases is done in following chapters of this thesis (see also [43, 44, 45] and references therein).

Let us turn to the central argument of this chapter. In the case of Type IIA/B backgrounds with Ramond-Ramond flux $F_{(p+2)}$, we can write the action of the smeared flavour branes in Einstein frame as (see [45])

$$S_{\text{sources}} = -T_p \int_{\mathcal{M}_{10}} \left(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)} \right) \wedge \Xi \,. \qquad (2.1.2)$$

As explained later, it is always possible to relate the smearing form to the calibration form using supersymmetry and the equations of motion as

$$\mathrm{d}\left[* e^{\frac{10-2p-4}{4}\Phi}\mathrm{d}\left(e^{\frac{p-3}{4}\Phi}\mathcal{K} \right) \right] = \pm 2\kappa_{10}^2 T_p \, \Xi \,, \qquad (2.1.3)$$

giving us a geometric constraint on the smearing form. In the following we shall study how equations (2.1.2) and (2.1.3) can be applied to address the problem of smeared flavours.

Proceeding rather pedagogically, Section 2.2.1 introduces the methods outlined above by studying three different, well-known examples. We show that our methods are not only capable of reproducing the known results, but they also provide some new, interesting ones. The examples studied are the $\mathcal{N}=1$ SQCD-like dual of [21, 28, 29], the three-dimensional $\mathcal{N}=1$ theory of [36] and the Klebanov-Witten theory [6] with massless [30] and massive [35] flavours. Following this we shall turn to the generic case (Section 2.2.2), showing how the action (2.1.2) can be constructed from purely geometric considerations and proving its equivalence with other actions used in the field of smeared flavours.

In Section 2.3, we shall finally apply our methods to the problem of flavouring a background dual to an $\mathcal{N}=2$ super Yang-Mills-like theory, first studied in [46, 47]. Interestingly, we are able to do so without an explicit knowledge of the

brane embeddings used. We find new analytic and asymptotic solutions of the flavoured and unflavoured equations of motion and discuss various properties of these backgrounds.

Following [48], we show, for the examples considered, how all the constraints imposed by supersymmetry on space-time can be understood and recovered from geometric grounds using methods such as G-structures.

2.2   The geometry of smeared branes

In the following, we shall now investigate what generalised calibrated geometry can teach us about string-theory duals with backreacting, smeared flavour branes. First we take a detailed look at three examples [21, 36, 30]. For each of these we briefly summarise the conventional approach to flavouring and then show explicitly that it can be nicely understood in terms of a suitable calibration form. In Section 2.2.2 we turn to the case of a generic supergravity dual.

2.2.1   Three examples

The string dual to an $\mathcal{N} = 1$ SQCD-like theory

Review of the $\mathcal{N} = 1$ SQCD-like string dual   As a first example we shall turn to the string dual to an $\mathcal{N} = 1$ SQCD-like theory [21, 28, 29]. It is based on the background of [13] which is given by the following solution of the Type IIB equations of motion[1]:

$$
\mathrm{d}s^2 = \alpha' g_s N_c \, e^{\frac{\Phi}{2}} \left[\frac{1}{\alpha' g_s N_c} \mathrm{d}x_{1,3}^2 + \mathrm{d}r^2 + e^{2h}(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) + \frac{1}{4}(\tilde{\omega}_i - A^i)^2 \right],
$$
$$
F_{(3)} = -\frac{1}{4} \bigwedge_a (\tilde{\omega}_a - A^a) + \frac{1}{4} \sum_a F^a \wedge (\tilde{\omega}_a - A^a),
$$

$$
(2.2.1)
$$

with

$$
\begin{aligned}
A^1 &= -a(r)\mathrm{d}\theta, & \tilde{\omega}_1 &= \cos\psi \, \mathrm{d}\tilde{\theta} + \sin\psi \sin\tilde{\theta} \, \mathrm{d}\tilde{\varphi}, \\
A^2 &= a(r)\sin\theta \, \mathrm{d}\varphi, & \tilde{\omega}_2 &= -\sin\psi \, \mathrm{d}\tilde{\theta} + \cos\psi \sin\tilde{\theta} \, \mathrm{d}\tilde{\varphi}, & (2.2.2) \\
A^3 &= -\cos\theta \, \mathrm{d}\varphi, & \tilde{\omega}_3 &= \mathrm{d}\psi + \cos\tilde{\theta} \, \mathrm{d}\tilde{\varphi}.
\end{aligned}
$$

The metric describes a space with topology $\mathbb{R}^{1,3} \times \mathbb{R} \times S^2 \times S^3$, where the three-sphere is parametrised by the Maurer-Cartan forms $\tilde{\omega}_i$ and the one-forms A^i de-

---

[1]Except where explicitly noted, we shall always use Einstein frame in this chapter.

scribe the fibration between the two spheres. It is interpreted as the near-horizon geometry of a stack of N_c D5-branes wrapping an S^2, thus describing the dynamics of a four-dimensional $\mathcal{N} = 1$, $SU(N_c)$ super Yang-Mills theory coupled to some extra matter. To keep the discussion as simple as possible, we shall focus on the so-called singular solution which is obtained from the assumption $a(r) = 0$.

The possibility of adding probe flavour branes to the above background (2.2.1) was studied in [49]. Using κ-symmetry, the authors found several classes of flavour D5-branes; the simplest of these is given by branes extending along (x^μ, r) and wrapping ψ. They are pointlike on the four-dimensional submanifold given by $(\theta, \varphi, \tilde{\theta}, \tilde{\varphi})$ and extend to $r = 0$, thus describing massless flavours. In what follows, the most important feature of this embedding is that we are able to identify world-volume coordinates ξ^α with space-time ones, (x^μ, r, ψ). So even at the level of the space-time coordinates X^M, there is a well-defined notion of coordinates tangential and transverse to the brane.

From the perspective of Type IIB string theory, it is clear that the addition of a large number of such branes to the system (2.2.1) deforms the geometry of the background. Given the form of the brane embeddings, it follows that a suitable ansatz for the deformed background should be of the form

$$ds^2 = e^{2f(r)}\left[dx_{1,3}^2 + dr^2 + e^{2h(r)}(d\theta^2 + \sin^2\theta\, d\varphi^2)\right.$$

$$\left. + \frac{e^{2g(r)}}{4}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2) + \frac{e^{2k(r)}}{4}(\tilde{\omega}_3 + \cos\theta\, d\varphi)^2\right] \tag{2.2.3}$$

$$F_{(3)} = -2N_c\, e^{-3f-2g-k} e^{123} + \frac{N_c}{2} e^{-3f-2h-k} e^{\theta\varphi 3}\,,$$

as the flavour branes are points on the four-dimensional transverse manifold while singling out the $U(1) \subset S^3$ parametrised by ψ. When writing (2.2.3) we introduced a vielbein

$$e^{x^i} = e^f dx^i\,, \quad e^1 = \frac{e^{f+g}}{2}\tilde{\omega}_1\,, \quad e^2 = \frac{e^{f+g}}{2}\tilde{\omega}_2\,, \qquad e^3 = \frac{e^{f+k}}{2}(\tilde{\omega}_3 + \cos\theta\, d\varphi)\,,$$

$$e^r = e^f dr\,, \quad e^\theta = e^{f+h} d\theta\,, \quad e^\varphi = e^{f+h}\sin\theta\, d\varphi\,,$$

$$\tag{2.2.4}$$

and made use of the convention $e^{A_1\cdots A_p} = e^{A_1} \wedge \cdots \wedge e^{A_p}$.

One can also interpret the ansatz (2.2.3) from the gauge-theory point of view. The $U(1)$ describes the R-symmetry of the flavoured theory, which one demands not to be broken classically by the addition of massless flavours.

Studying the dilatino and gravitino variations of the deformed background, one

obtains the projections satisfied by the supersymmetric spinor ϵ

$$\Gamma_{r123}\epsilon = \epsilon, \quad \Gamma_{r\theta\varphi3}\epsilon = \epsilon, \quad \epsilon = \sigma_3\epsilon, \tag{2.2.5}$$

as well as the BPS equations

$$4f = \Phi,$$

$$h' = \frac{1}{4}N_c\, e^{-2h-k} + \frac{1}{4}e^{-2h+k} = \frac{1}{2}e^{3f}F_{\theta\varphi3} + \frac{1}{4}e^{-2h+k},$$

$$g' = -N_c\, e^{-2g-k} + e^{-2g+k} = \frac{1}{2}e^{3f}F_{123} + e^{-2g+k},$$

$$k' = \frac{1}{4}N_c\, e^{-2h-k} - N_c\, e^{-2g-k} - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k}$$

$$= \frac{1}{2}e^{3f}\big(F_{\theta\varphi3} + F_{123}\big) - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k},$$

$$\Phi' = -\frac{1}{4}N_c\, e^{-2h-k} + N_c\, e^{-2g-k} = -\frac{1}{2}e^{3f}\big(F_{\theta\varphi3} + F_{123}\big). \tag{2.2.6}$$

It is *a priori* not obvious that the flavour branes mentioned earlier are still super-symmetric brane embeddings for the deformed background for arbitrary functions g, h, k. One therefore has to check again that probes with world-volume directions as before, $\xi^\alpha = (x^\mu, r, \psi)$, still preserve all of the backgrounds supersymmetries.

Having deformed the original background one turns to the system given by the combined action (2.1.1). One can anticipate that the brane action contributes to the energy-momentum tensor in the Einstein equations, adds a source term for the three-form field strength and modifies the dilaton equation by a contribution related to the DBI action.

For the case of N_f flavour branes localised at $(\theta_0, \varphi_0, \tilde\theta_0, \tilde\varphi_0)$, the brane action is ($\imath^*$ denoting the pull-back onto the world-volume)

$$S_{\text{sources}} = T_5 \sum_{N_f} \left(-\int_{\mathcal{M}_6} d^6\xi\, e^{\frac{\Phi}{2}} \sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_6} \imath^* \left(C_{(6)} \right) \right)\bigg|_{(\theta_0, \varphi_0, \tilde\theta_0, \tilde\varphi_0)}. \tag{2.2.7}$$

As these branes are localised in the four transverse directions, the equations of motion contain δ-function sources, making the search for solutions a difficult endeavour. The idea is therefore to smoothly distribute the branes over the transverse directions. If one assumes a transverse brane distribution with density

$$\Xi = \frac{N_f}{(4\pi)^2} \sin\theta \sin\tilde\theta\, d\theta \wedge d\varphi \wedge d\tilde\theta \wedge d\tilde\varphi, \tag{2.2.8}$$

the action (2.2.7) may be generalised to

$$
\begin{aligned}
S_{\text{sources}} &= T_5 \left(-\frac{N_f}{(4\pi)^2} \int_{\mathcal{M}_{10}} \mathrm{d}^{10}x\, e^{\frac{\Phi}{2}} \sin\theta \sin\tilde{\theta} \sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Xi \right) \\
&= T_5 \left(-\int_{\mathcal{M}_{10}} \mathrm{d}^{10}x\, e^{\frac{\Phi}{2}} \sqrt{-g_{(10)}} |\Xi| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Xi \right) ,
\end{aligned}
\tag{2.2.9}
$$

where we have defined the modulus of a p-form Ξ as

$$
|\Xi| \equiv \sqrt{\frac{1}{p!} \Xi_{M_1 \dots M_p} \Xi^{M_1 \dots M_p}} ,
\tag{2.2.10}
$$

and have checked the equality of the first and second lines by explicit calculation.

Let us take a look at how the brane action modifies the second-order equations of motion, starting with the Ramond-Ramond field strength. Here the relevant part of the total action is

$$
S = \int_{\mathcal{M}_{10}} -\frac{1}{2\kappa_{10}^2} \frac{e^{-\Phi}}{2} (F_{(7)} \wedge *F_{(7)}) + T_5\, C_{(6)} \wedge \Xi .
\tag{2.2.11}
$$

If we vary the potential $C_{(6)}$,

$$
\begin{aligned}
\delta_C S &= \int_{\mathcal{M}_{10}} -\frac{1}{2\kappa_{10}^2} \frac{e^{-\Phi}}{2} \left(\mathrm{d}\delta C_{(6)} \wedge *F_{(7)} + F_{(7)} \wedge *\mathrm{d}\delta C_{(6)} \right) + T_5 \int \delta C_{(6)} \wedge \Xi \\
&= \int_{\mathcal{M}_{10}} \delta C_{(6)} \wedge \left(\frac{1}{2\kappa_{10}^2} \mathrm{d}\left(*e^{-\Phi} F_{(7)} \right) + T_5\, \Xi \right)
\end{aligned}
$$

$$
\Rightarrow \mathrm{d}F_{(3)} = 2\kappa_{10}^2\, T_5\, \Xi .
$$

$$
\tag{2.2.12}
$$

The change in the dilaton and Einstein equations does not take such a nice geometric form. Choosing $T_5 = \frac{1}{(2\pi)^5}$, $2\kappa_{10}^2 = (2\pi)^7$, the complete equations of motion are

$$
0 = \mathrm{d}F_{(3)} - (2\pi)^2 \Xi ,
$$

$$
0 = \frac{1}{\sqrt{-g_{(10)}}} \partial_\mu (g^{\mu\nu} \sqrt{-g_{(10)}} \partial_\nu \Phi) - \frac{1}{12} e^{\Phi} F_{(3)}^2 - \frac{N_f}{8} e^{\frac{\Phi}{2}} \frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}} \sin\theta \sin\tilde{\theta} ,
$$

$$
\begin{aligned}
0 = R_{\mu\nu} &- \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\lambda \Phi \partial^\lambda \Phi \right) \\
&- \frac{1}{12} e^{\Phi} \left(3 F_{\mu\kappa\lambda} F_\nu{}^{\kappa\lambda} - \frac{1}{2} g_{\mu\nu} F_{(3)}^2 \right) - T_{\mu\nu}^{\text{flvr}} ,
\end{aligned}
$$

$$T_{\mu\nu}^{\text{flvr}} = -\frac{N_f}{4}\sin\theta\sin\tilde{\theta}\frac{1}{2}e^{\frac{\Phi}{2}}g_{\mu\alpha}g_{\nu\beta}\hat{g}_{(6)}^{\alpha\beta}\frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}}\,.$$ (2.2.13)

The search for solutions of (2.2.13) is simplified considerably by a powerful result, found in [50, 51], which states that any solution of the system of BPS equations satisfying the modified Bianchi identity of (2.2.12) solves also the Einstein and dilaton equations and is therefore a solution of (2.2.13).

So we turn again to the issue of the BPS equations. As the brane embeddings are supersymmetric, the projections (2.2.5) imposed on the spinor ϵ remain the same. However, the three-form field strength $F_{(3)}$ is modified by the appearance of the source term in (2.2.13). To incorporate this, one makes a new ansatz for the field strength of (2.2.3)

$$F_{(3)} = -2N_c\, e^{-3f-2g-k}e^{123} - \frac{N_f - N_c}{2}e^{-3f-2h-k}e^{\theta\varphi3}\,.$$ (2.2.14)

It follows that the BPS equations (2.2.6) change to

$$4f = \Phi\,,$$

$$h' = \frac{1}{4}(N_c - N_f)e^{-2h-k} + \frac{1}{4}e^{-2h+k} = \frac{1}{2}e^{3f}F_{\theta\varphi3} + \frac{1}{4}e^{-2h+k}\,,$$

$$g' = -N_c\, e^{-2g-k} + e^{-2g+k} = \frac{1}{2}e^{3f}F_{123} + e^{-2g+k}\,,$$

$$k' = \frac{1}{4}(N_c - N_f)e^{-2h-k} - N_c\, e^{-2g-k} - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k}$$

$$= \frac{1}{2}e^{3f}(F_{\theta\varphi3} + F_{123}) - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k}\,,$$

$$\Phi' = -\frac{1}{4}(N_c - N_f)e^{-2h-k} + N_c\, e^{-2g-k} = -\frac{1}{2}e^{3f}(F_{\theta\varphi3} + F_{123})\,.$$ (2.2.15)

It is curious to note that, when written in terms of $F_{\theta\varphi3}$ and F_{123}, the BPS equations of the deformed and flavoured systems are the same – see (2.2.6) and (2.2.15). The change in the BPS equations stems solely from the modification of the field strength. This should not come as a surprise, since the brane embeddings are supersymmetric[2].

---

[2]Whether the BPS equations are modified by the flavouring procedure is – to some extent – a matter of taste. It depends on whether one makes a sufficiently generic ansatz for the three-form field strength to accommodate the source term. From the perspective of a physicist who is interested in the properties of the dual gauge theory, it is more appropriate to consider the BPS equations of the flavoured and unflavoured theories as different, as some phenomena, such

By construction $F_{(3)}$ satisfies the modified Bianchi identity. Thus any solution of (2.2.15) solves the flavouring problem for the Maldacena-Núñez (MN) background. For a discussion of these solutions and their physical interpretation see [21, 28, 29].

In the above background, the generalisation of the action (2.2.7) to (2.2.9) is fairly intuitive and simple, because there is only one stack of flavour branes with world-volume coordinates that can be globally identified with space-time coordinates. However we can already anticipate the shortcomings of this definition. On a technical level, the first line of (2.2.9) is inherently dependent on the coordinate split while the second is non-linear in the smearing form Ξ. From a more formal point of view, it is also unsatisfying that the formalism of those equations treats the DBI and WZ contributions to the brane action on an unequal footing. One should recall that, roughly speaking, the DBI action defines the tree level couplings of the brane to the Neveu-Schwarz (NS) sector of the background while the couplings to Ramond-Ramond fields are contained in the WZ term. A standard string-theory calculation shows the cancellation of the effects of closed strings from the two sectors on supersymmetric branes. So it would be desirable to see an explicit symmetry between the two terms even after smearing. Adopting once again a more physics centred perspective, we might also wonder if there are any constraints on the choice of the smearing form. *E.g.* one should note that the smearing form does not agree with the volume form induced on the four-cycle $(\theta, \varphi, \tilde{\theta}, \tilde{\varphi})$. At first glance it might appear that there are none. After all, the cancellations between parallel BPS branes allow us to place them at arbitrary separations. As we later explain, however, there are constraints on Ξ which can be traced back to the geometric structure of the background.

The perspective of generalised calibrated geometry The properties of generalised calibrations and their relation to supersymmetry are discussed in detail in Appendix 2.A. As the backgrounds considered in this chapter are not fully generic, yet only include dilaton and Ramond-Ramond fields in Type IIB supergravity, we do not make use of the most general concept of a generalised calibration. Again we refer to [43, 44]. For our purposes it is sufficient to think of calibrations as $(p + 1)$-forms $\mathcal{K}$, such that a Dp-brane with embedding $X^M(\xi)$ is supersymmetric

---

as Seiberg duality, become apparent at the level of the first-order BPS equations [21]. For a mathematician on the other hand, it might be more important to think about the close link between supersymmetry and geometry which is evident in this chapter – the fact that the flavour branes are supersymmetric is then reflected by the invariance of the BPS equations in terms of $\mathcal{F}_{ABC}$.

if and only if it satisfies

$$\imath^* \left(\mathcal{K}_{(p+1)} \right) = \sqrt{-\hat{g}_{(p+1)}} \, \mathrm{d}^{p+1}\xi \,. \tag{2.2.16}$$

As discussed in the appendix, this can be understood as a simple rephrasing of the κ-symmetry condition on the supersymmetric spinor ϵ,

$$\Gamma_\kappa \epsilon = \epsilon \,. \tag{2.2.17}$$

For us the most interesting feature of (2.2.16) is that when pulled back onto the world-volume of the brane, the calibration form is equivalent to the induced volume form, and one may write the DBI action as

$$S_{DBI} = -T_p \int_{\mathcal{M}_{p+1}} e^{\frac{p-3}{4}\Phi} \imath^*\left(\mathcal{K} \right) \,. \tag{2.2.18}$$

Furthermore, if the p-brane couples electrically to the flux given by $F_{(p+2)}$, supersymmetry in Einstein frame requires [42]

$$\mathrm{d}\left(e^{\frac{p-3}{4}\Phi} \mathcal{K} \right) = F_{(p+2)} \,. \tag{2.2.19}$$

In the case at hand, the calibration six-form is given by

$$\mathcal{K} = \frac{1}{6!} \left(\epsilon^\dagger \sigma_3 \otimes \Gamma_{a_0 \dots a_5} \epsilon \right) e^{a_0 \dots a_5} \,. \tag{2.2.20}$$

The evaluation of the calibration form requires only the chirality of the Type IIB spinors, $\boldsymbol{\epsilon} = \Gamma^{11} \boldsymbol{\epsilon}$ and knowledge of the projections imposed on the supersymmetric spinors (2.2.5). From the last of these it follows that one of the Majorana-Weyl spinors of Type IIB is fixed to zero, $\boldsymbol{\epsilon} = \left(\begin{smallmatrix} \epsilon \\ 0 \end{smallmatrix} \right)$. Thus there is only one calibration six-form and we may use ϵ instead of $\boldsymbol{\epsilon}$. In Section 2.3 we encounter an example with two calibration forms. Combining the supersymmetric projections (2.2.5) with the definition (2.2.20) yields

$$\mathcal{K}_{x^0 x^1 x^2 x^3 \theta \varphi} = \epsilon^\dagger \Gamma_{x^0 x^1 x^2 x^3 \theta \varphi} \epsilon = -\epsilon^\dagger \Gamma_{r123} \epsilon = -1 \,. \tag{2.2.21}$$

The second equality makes use of chirality, the third of the supersymmetric projections and the normalisation $\epsilon^\dagger \epsilon = 1$. Finally we are left with

$$\mathcal{K} = e^{x^0 x^1 x^2 x^3} \wedge \left(e^{r3} - e^{\theta \varphi} - e^{12} \right) \,. \tag{2.2.22}$$

As e^3 is the only part of the vielbein containing $\mathrm{d}\psi$, it is obvious that equation (2.2.16) is satisfied and we recover the result of [49] that the embedding in question is supersymmetric. Noting that

$$\sqrt{-\hat{g}_{(6)}}\,\mathrm{d}^6\xi = e^{x^0 x^1 x^2 x^3 r 3}\,,$$

$$\Xi = 4N_f e^{-4f-2g-2h} e^{\theta\varphi 12}\,,$$

(2.2.23)

it is easy to see that we may write the smeared brane action (2.2.9) as

$$S_{\text{sources}} = T_5 \int_{\mathcal{M}_{10}} \left(-e^{\frac{\Phi}{2}}\mathcal{K} + C_{(6)}\right) \wedge \Xi\,.$$

(2.2.24)

Contrary to (2.2.9), this is independent of coordinates, linear in the smearing form, and it treats the DBI and WZ contributions to the brane action on an equal footing. Concerning the supersymmetry condition (2.2.19), we find

$$\mathrm{d}\left(e^{\frac{\Phi}{2}}\mathcal{K}\right) = e^{-f+\frac{\Phi}{2}} e^{x^0 x^1 x^2 x^3} \wedge \left[e^{-2g}\left(2e^k - 6e^{2g}f' - 2e^{2g}g' - e^{2g}\Phi'\right) e^{r12} \right.$$

$$\left. + e^{-2h}\left(\frac{1}{2}e^k - 6e^{2h}f' - 4e^{2h}h' - e^{2h}\Phi'\right) e^{r\theta\varphi} \right].$$

(2.2.25)

Using the BPS equations (2.2.6) or (2.2.15), one may verify for the three-form field strength with (2.2.14) and without sources (2.2.3) that $\mathrm{d}(e^{\frac{\Phi}{2}}\mathcal{K}) = F_{(7)}$ is satisfied. We can exploit the calibration form even further. From $e^{-\Phi} * F_{(7)} = -F_{(3)}$ and $\mathrm{d}F_{(3)} = (2\pi)^2\Xi$, it follows that

$$e^{-\Phi} * \mathrm{d}\left(e^{\frac{\Phi}{2}}\mathcal{K}\right) = -F_{(3)}\,,$$

$$\mathrm{d}\left[e^{-\Phi} * \mathrm{d}\left(e^{\frac{\Phi}{2}}\mathcal{K}\right)\right] = -(2\pi)^2\,\Xi\,.$$

(2.2.26)

Again note that these equations hold with or without the backreaction of the source terms – in the latter case with $\Xi = 0$. One should think of them rather as a characteristic of the supersymmetries preserved by the background than a property of the branes.

When we first introduced the smearing form in (2.2.8), it appeared that its choice was rather arbitrary. After all supersymmetry allows us to place branes at arbitrary separations. However, (2.2.26) is not a result of supersymmetry alone yet

rather an interplay of supersymmetry and the Einstein equations, as the following illustrates.

$$\mathrm{d}\left(e^{\frac{\Phi}{2}}\mathcal{K}\right) \stackrel{\text{SUSY}}{=} F_{(7)}, \qquad *e^{-\Phi}F_{(7)} \equiv -F_{(3)}, \qquad \mathrm{d}F_{(3)} \stackrel{\text{EOM}}{=} (2\pi)^2 \,\Xi. \qquad (2.2.27)$$

BPS equations and G-structures We showed before that the requirement of supersymmetry is related to geometry, notably with the calibration form. As supersymmetry gives us the BPS equations of the system, it is logical to think that one can retrieve those equations through geometric considerations, namely G-structures. When looking at the supersymmetric gravitino equation, we can identify $F_{(3)}$ with a torsion (straightforward in string frame), defining a new co-variant derivative $\widetilde{\nabla}_\mu$ such that

$$\widetilde{\nabla}_\mu \epsilon = 0. \qquad (2.2.28)$$

This means that we have a covariantly constant spinor satisfying certain projections (2.2.5). $\epsilon = \sigma_3 \epsilon$ states that there is only one structure. The other two tell us that, in the six-dimensional internal manifold, there is a covariantly constant complex chiral spinor η obeying

$$\gamma_{r123}\eta = \eta, \qquad \gamma_{r\theta\varphi3}\eta = \eta, \qquad (2.2.29)$$

where γ_i are the gamma matrices of the six-dimensional internal manifold. We can choose the chirality of η to be

$$i\,\gamma_{r123\theta\varphi}\eta = -\eta. \qquad (2.2.30)$$

Then we recognise that the six-dimensional manifold has an $SU(3)$-structure. It has a fundamental two-form J and a holomorphic three-form Ω defined as

$$J_{mn} = i\,\eta^\dagger \gamma_{mn}\eta, \qquad (2.2.31)$$

$$\Omega_{mnp} = \eta^T \gamma_{mnp}\eta. \qquad (2.2.32)$$

Supersymmetry imposes the following conditions on the forms (see [48]):

$$\mathrm{d}\left(e^{\Phi} *_6 J\right) = 0, \qquad (2.2.33)$$

$$\mathrm{d}\left(e^{\frac{5}{4}\Phi}\Omega\right) = 0. \qquad (2.2.34)$$

From those equations, plus the generalised calibration condition (2.2.26), we can retrieve the BPS equations of the system, imposing $4f = \Phi$.

An $\mathcal{N} = 1$, three-dimensional example

We turn now to the string dual of a three-dimensional $\mathcal{N} = 1$ theory that was discussed in [36]. We leave the discussion rather brief, only exhibiting the equivalence of the actions (2.2.9) and (2.2.24) for this example. In comparison to the $\mathcal{N} = 1$ SQCD-like dual of the previous section, the situation is complicated by the fact that there are three stacks of branes. While it is possible to find coordinates such that the world-volume of one of these stacks may be identified with space-time coordinates, it is not possible to do so for all three stacks simultaneously. The system has the topology $\mathbb{R}^{1,2} \times \mathbb{R} \times S^3 \times S^3$. As in Section 2.2.1, we shall work with a simplification, the truncated system, for which the background is given by

$$e^{x^i} = e^f \mathrm{d}x^i, \quad e^r = e^f \mathrm{d}r, \quad e^i = \frac{e^{f+h}}{2}\sigma^i, \quad e^{\hat{i}} = \frac{e^{f+g}}{2}\left(\omega^i - \frac{1}{2}\sigma^i\right),$$
$$(2.2.35)$$
$$F_{(3)} = -2N_c\, e^{-3g-3f} e^{\hat{1}\hat{2}\hat{3}} + \frac{1}{2}N_c\, e^{-g-2h-3f}\left(e^{13\hat{2}} - e^{12\hat{3}} - e^{23\hat{1}}\right).$$

σ^i and ω^i are sets of Maurer-Cartan forms parametrising the two three-spheres. The projections satisfied by the supersymmetric spinor η are

$$\Gamma_{1\hat{1}2\hat{2}}\eta = -\eta, \quad \Gamma_{1\hat{1}3\hat{3}}\eta = -\eta, \quad \Gamma_{2\hat{2}3\hat{3}}\eta = -\eta, \quad \Gamma_{r\hat{1}\hat{2}\hat{3}}\eta = \eta, \quad \eta = \sigma_3\eta.$$
$$(2.2.36)$$

And the BPS equations take the form

$$\Phi' = N_c\, e^{-3g} - \frac{3}{4}N_c\, e^{-g-2h},$$

$$h' = \frac{1}{2}e^{g-2h} + \frac{1}{2}N_c\, e^{-g-2h},$$
$$(2.2.37)$$

$$g' = e^{-g} - \frac{1}{4}e^{g-2h} + \frac{N_c}{4}e^{-g-2h} - N_c\, e^{-3g},$$

$$\Phi = 4f.$$

Once more, it follows from $\eta = \sigma_3\eta = \left(\begin{smallmatrix}\eta\\0\end{smallmatrix}\right)$ that there is only one calibration six-form which is given by (assuming $\Gamma^{11}\eta = -\eta$)

$$\mathcal{K} = e^{012} \wedge \left(e^{r1\hat{1}} + e^{r2\hat{2}} + e^{r3\hat{3}} - e^{123} + e^{3\hat{1}\hat{2}} - e^{2\hat{1}\hat{3}} + e^{1\hat{2}\hat{3}}\right).$$
$$(2.2.38)$$

From the calibration condition for supersymmetric branes, $\imath^*(\mathcal{K}) = \mathrm{d}\xi^6\sqrt{-\hat{g}_{(6)}}$, one can see immediately that there are supersymmetric D5-brane embeddings

with tangent vectors[3] $(\partial_{x^0}, \partial_{x^1}, \partial_{x^2}, E_r, E_i, E_{\hat{i}})$, $i \in \{1, 2, 3\}$. We also learn from (2.2.38) that these embeddings are absolutely equivalent. They were originally derived in [36] using κ-symmetry. There the authors introduced a standard set of Maurer-Cartan forms ω, σ to parametrise the two three-spheres, and then found a coordinate representation of the $(\partial_\mu, \partial_r, E_3, E_{\hat{3}})$ branes given by $(x^\mu, r, \psi_1, \psi_2)$. Subsequently they argued from the symmetries of the space that there are also $1\hat{1}$ and $2\hat{2}$ embeddings, whose coordinate representation would become apparent upon using different Maurer-Cartan forms. As we mentioned earlier, it does not seem possible to find global coordinates for this system in which all three flavour-brane embeddings have good coordinate representations – thus this is an ideal setting for using the calibration form (2.2.38).

Our analysis here shall start with the $3\hat{3}$ embeddings. In [36], the smeared action was given by

$$S_{D5} = T_5 \left(-\int \mathrm{d}^{10}x\, e^{\frac{\Phi}{2}} \sqrt{-G_{10}}\, |\Xi^{(1)}| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Xi^{(1)} \right) ,$$

$$\Xi^{(1)} = -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{12\hat{1}\hat{2}} ,$$

$$|\Xi^{(1)}| = \frac{N_f}{\pi^2} e^{-4f-2h-2g} , \qquad\qquad (2.2.39)$$

$$\sqrt{-G_{10}} = \frac{1}{64} e^{10f+3g+3h} \sin\theta \sin\tilde{\theta} ,$$

$$\sqrt{-\hat{G}_6} = \frac{1}{4} e^{6f+g+h} .$$

Now

$$\mathcal{K} \wedge \Xi^{(1)} = -\frac{N_f}{\pi^2} e^{-4f-2h-2g} \sqrt{-G_{10}}\, \mathrm{d}^{10}x = \mathrm{d}^{10}x \sqrt{-G_{10}}\, |\Xi^{(1)}| . \qquad (2.2.40)$$

---

[3] When labelling brane embeddings in terms of their tangent vectors, one should think of the brane being along the submanifold spanned by the integral curves of the tangent vector fields. That is, if one were to find coordinates y^M such that

$$\partial_{x^0} = \partial_{y^0} , \quad \partial_{x^1} = \partial_{y^1} , \quad \partial_{x^2} = \partial_{y^2} , \quad E_r = \partial_{y^3} , \quad E_1 = \partial_{y^4} , \quad E_{\hat{1}} = \partial_{y^5} ,$$

the corresponding $1\hat{1}$ brane embedding would be given by

$$Y^\alpha(\xi) = \xi^\alpha \qquad Y^a = \text{const.} \quad \alpha \in \{0, \ldots, 5\} \quad a \in \{6, \ldots, 9\} .$$

One should note, however, that it is necessary to verify that the distribution given by the tangent vectors is integrable, *i.e.* to verify that the coordinates y^M exist. One can do so using Frobenius theorem, which states that a distribution given by vectors T_a is integrable if and only if it is in involution, that is $[T_a, T_b] = f_{abc} T_c$.

Thus again, we may write the action of one stack of $(3\hat{3})$ branes as

$$S_{D5} = T_5 \int_{\mathcal{M}_{10}} \left(-e^{\frac{\Phi}{2}} \mathcal{K} + C_{(6)} \right) \wedge \Xi^{(1)} \,. \tag{2.2.41}$$

The above may be easily generalised to the case of three stacks of D5-branes as the expression is linear in Ξ:

$$
\begin{aligned}
S_{D5} &= T_5 \int_{\mathcal{M}_{10}} \left(-e^{\frac{\Phi}{2}} \mathcal{K} + C_{(6)} \right) \wedge \Xi \,, \\
\Xi &= \Xi^{(1)} + \Xi^{(2)} + \Xi^{(3)} \,, \\
\Xi^{(2)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{1 3 \hat{1} \hat{3}} \,, \\
\Xi^{(3)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{2 3 \hat{2} \hat{3}} \,,
\end{aligned}
\tag{2.2.42}
$$

where $\Xi^{(2)}$ is the smearing form for branes extending along $2\hat{2}$ and $\Xi^{(3)}$ smears the $1\hat{1}$ embedding. The linearity of the above expression gives a good motivation for the use of $\sum_i |\Xi^{(i)}|$ instead of $|\Xi|$ in the original action of [36]

$$S_{D5} = T_5 \left(-\int d^{10}x \, e^{\frac{\Phi}{2}} \sqrt{-G_{10}} \sum_{i=1}^{3} |\Xi^{(i)}| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Xi \right) \,. \tag{2.2.43}$$

Independently of whether one uses the action (2.2.42) or (2.2.43), the Bianchi identity is modified to $dF_{(3)} = 2\kappa_{10}^2 T_5 \,\Xi$. Accordingly, one changes the ansatz for the field strength by adding a term $f_{(3)}$ which is not closed,

$$
\begin{aligned}
F_{(3)} &\mapsto F_{(3)} + f_{(3)} \,, \\
f_{(3)} &= 2 N_f \, e^{-g-2h-3f} \left(e^{1 2 \hat{3}} + e^{2 3 \hat{1}} - e^{1 3 \hat{2}} \right) \,.
\end{aligned}
\tag{2.2.44}
$$

The BPS equations (2.2.37) change to

$$
\begin{aligned}
\Phi' &= N_c e^{-3g} - \frac{3}{4}(N_c - N_f)e^{-g-2h} \,, \\
h' &= \frac{e^{g-2h}}{2} + \frac{N_c - 4N_f}{2} e^{-g-2h} \,, \\
g' &= e^{-g} - \frac{1}{4}e^{g-2h} - N_c e^{-3g} + \frac{N_c - 4N_f}{4} e^{-g-2h} \,, \\
\Phi &= 4f \,.
\end{aligned}
\tag{2.2.45}
$$

Let us now turn to the supersymmetric condition (2.2.19). A straightforward yet

tedious calculation yields

$$\mathrm{d}\left(e^{\frac{\Phi}{2}}\mathcal{K}\right) = e^{\frac{\Phi}{2}-f}e^{012} \wedge \left\{ \left(2e^{-g} - 6f' - 2g' - h' - \Phi'\right)\left(e^{r1\hat{2}\hat{3}} - e^{r2\hat{1}\hat{3}} + e^{r3\hat{1}\hat{2}}\right) \right.$$
$$\left. + \frac{e^{-2h}}{2}\left(-3e^g + 12e^{2h}f' + 6e^{2h}h' + e^{2h}\Phi'\right)e^{r123} \right\}.$$
$$(2.2.46)$$

Using the BPS equations (2.2.37) or (2.2.45) respectively, one can verify that $e^{-\Phi} * \mathrm{d}\left(e^{\frac{\Phi}{2}}\mathcal{K}\right) = -F_{(3)}$ is satisfied in both the deformed and flavoured cases. Furthermore we know that $\mathrm{d}F_{(3)} = (2\pi)^2 \,\Xi$, thus we are again able to obtain a constraint on the smearing form as

$$\mathrm{d}\left[e^{-\Phi} * \mathrm{d}\left(e^{\frac{\Phi}{2}}\mathcal{K}\right)\right] = -(2\pi)^2 \,\Xi.$$
$$(2.2.47)$$

We immediately see why there has to be three stacks of flavour D5-branes in the backreacted solution – the calibration form respects the symmetries of the two three-spheres and from (2.2.47) it follows that the same holds true for the smearing form. It would therefore not be possible to obtain a smeared system with only one or two of the three stacks.

We can again use G-structures to derive the BPS equations for the system. In this case the internal manifold is seven-dimensional, with a covariantly constant spinor which satisfies

$$\gamma_{1\hat{1}2\hat{2}}\eta = -\eta, \qquad \gamma_{1\hat{1}3\hat{3}}\eta = -\eta, \qquad \gamma_{r\hat{1}\hat{2}\hat{3}}\eta = \eta.$$
$$(2.2.48)$$

We recognise here a manifold with G_2-holonomy. Its associative three-form ϕ is defined as

$$\phi_{mnp} = -i\,\eta^T \gamma_{mnp}\,\eta.$$
$$(2.2.49)$$

The condition imposed by supersymmetry is

$$\mathrm{d}\left(e^{\Phi} *_7 \phi\right) = 0.$$
$$(2.2.50)$$

Together with the generalised calibration condition, and assuming $\Phi = 4f$, this condition provides us with a method to rederive the BPS equations (2.2.37), (2.2.45).

The Klebanov-Witten model

Finally we take a look at the Klebanov-Witten model for the cases of massless [30] and massive flavours [35]. The Klebanov-Witten model [6] is based on D3-branes at the tip of the conifold and is dual to a certain $\mathcal{N} = 1$ gauge theory. So apart from the dilaton and the metric, there is self-dual $F_{(5)}$ flux due to the D3-branes. In contrast to the previous two examples, one uses D7-branes to introduce flavour degrees of freedom into the system. These source $F_{(1)}$, so the suitable ansatz for the relevant deformed, flavoured background is

$$
\mathrm{d}s^2 = h^{-\frac{1}{2}}\mathrm{d}x_{1,3}^2
$$

$$
+ h^{\frac{1}{2}}\left[e^{2f}\mathrm{d}\rho^2 + \frac{e^{2g}}{6}\sum_{i=1,2}(\mathrm{d}\theta_i^2 + \sin^2\theta_i\,\mathrm{d}\varphi_i^2) + \frac{e^{2f}}{9}(\mathrm{d}\psi + \sum_{i=1,2}\cos\theta_i\,\mathrm{d}\varphi_i)^2\right],
$$

$$
F_{(5)} = 27\pi N_c\, e^{-4g-f}h^{-5/4}\left(e^{x^0x^1x^2x^3\rho} - e^{\theta_1\varphi_1\theta_2\varphi_2\psi}\right),
$$

$$
F_{(1)} = \frac{N_f(\rho)}{4\pi}\left(\mathrm{d}\psi + \cos\theta_1\,\mathrm{d}\varphi_1 + \cos\theta_2\,\mathrm{d}\varphi_2\right),
$$

$$
(2.2.51)
$$

with $\psi \in [0, 4\pi[$, $\theta_i \in [0, \pi]$, $\varphi_i \in [0, 2\pi[$ and $\rho \in \mathbb{R}$. There is an obvious choice of vielbein

$$
e^{x^i} = h^{-1/4}\mathrm{d}x^i, \qquad e^\rho = h^{1/4}e^f\mathrm{d}\rho,
$$

$$
e^{\theta_i} = \frac{1}{\sqrt{6}}h^{1/4}e^g\mathrm{d}\theta_i, \qquad e^{\varphi_i} = \frac{1}{\sqrt{6}}h^{1/4}e^g\sin\theta_i\,\mathrm{d}\varphi_i, \qquad (2.2.52)
$$

$$
e^\psi = \frac{1}{3}h^{1/4}e^f(\mathrm{d}\psi + \cos\theta_1\,\mathrm{d}\varphi_1 + \cos\theta_2\,\mathrm{d}\varphi_2).
$$

The flavour branes behave differently in the massless or massive case. In the former, the authors of [30] used two stacks of branes whose world-volume coordinates may once more be identified with space-time ones:

$$
\begin{aligned}
\xi_1^\alpha &= (x^\mu, \rho, \theta_2, \varphi_2, \psi) \quad \theta_1 = \text{const.} \quad \varphi_1 = \text{const.}\\
\xi_2^\alpha &= (x^\mu, \rho, \theta_1, \varphi_1, \psi) \quad \theta_2 = \text{const.} \quad \varphi_2 = \text{const.}
\end{aligned}
\qquad (2.2.53)
$$

So prior to smearing, the system has a global $U(N_f) \times U(N_f)$ flavour symmetry – one group for each set of D7-branes. This is obviously a four-parameter family of embeddings, which can be smeared over the transverse (θ_i, φ_i) directions. In the massive case, the embeddings are more complicated. In the field theory, the mass term breaks the global symmetry to the diagonal $U(N_f) \times U(N_f) \mapsto U(N_f)$, which corresponds to the two stacks joining into one on the string-theory side. There is again a four-parameter family of brane embeddings, yet as the generic embedding

is much more complicated than those of (2.2.53), we shall only look at one representative, trusting that the calibration form ensures that we make use of the whole family of branes. Choosing world-volume coordinates $\xi = (x^\mu, \theta_1, \varphi_1, \theta_2, \varphi_2)$, this is given by

$$X^M(\xi) = \left(x^\mu, \ \rho_q - \frac{2}{3}\log\sin\frac{\theta_1}{2} - \frac{2}{3}\log\sin\frac{\theta_2}{2}, \ \theta_1, \ \varphi_1, \theta_2, \ \varphi_2, \ \varphi_1 + \varphi_2 + 2\beta \right)$$

$$\rho_q, \beta = \text{const.}$$

$$(2.2.54)$$

The constant ρ_q denotes the minimal radius reached by the brane and may therefore be identified as the mass.

The branes have an eight-dimensional world-volume and we therefore need to construct the calibration eight-form. In the case at hand, this requires the knowledge of the supersymmetric spinors on the conifold. These were discussed in [52]. Our conventions however are those of [30]. The supersymmetric spinor ϵ is related to a constant spinor η as $\epsilon = h^{-1/8} e^{-\frac{i}{2}\psi}\eta$. Both satisfy the projections

$$i\,\sigma_2 \otimes \Gamma_{x^0 x^1 x^2 x^3}\eta = \eta\,, \qquad \Gamma_{r\psi} = i\,\sigma_2\eta\,,$$

$$\Gamma_{\theta_1\varphi_1} = -i\,\sigma_2\eta\,, \qquad\qquad \Gamma_{\theta_2\varphi_2} = -i\,\sigma_2\eta\,.$$

$$(2.2.55)$$

From equation (2.A.5), it follows that the calibration form for D7-branes is given by

$$\mathcal{K} = \frac{1}{8!}\left(\eta^\dagger i\,\sigma_2 \otimes \Gamma_{a_0\ldots a_7}\eta\right)e^{a_0\ldots a_7}\,,$$

$$(2.2.56)$$

which we may evaluate using (2.2.55) to be

$$\mathcal{K} = e^{x^0 x^1 x^2 x^3} \wedge \left(e^{\rho\theta_1\varphi_1\psi} + e^{\rho\theta_2\varphi_2\psi} - e^{\theta_1\varphi_1\theta_2\varphi_2}\right)\,.$$

$$(2.2.57)$$

At this point we may calculate the pull-backs $\imath^*\left(\mathcal{K}\right)$ for both embeddings (2.2.53) and (2.2.54). Finding $\imath^*\left(\mathcal{K}\right) = \sqrt{-\hat{g}_{(8)}}\,\mathrm{d}^8\xi$, we do thus verify that the brane embeddings are indeed supersymmetric.

In Einstein frame, the integrand of the DBI action is $e^\Phi\sqrt{-\hat{g}_{(8)}}\,\mathrm{d}^8\xi = e^\Phi\imath^*\left(\mathcal{K}\right)$. As before, supersymmetry requires this to satisfy $\mathrm{d}\left(e^\Phi\mathcal{K}\right) = F_{(9)}$. Making use of the definition $F_{(1)} = e^{-2\Phi} * F_{(9)}$ and the modified Bianchi identity $\mathrm{d}F_{(1)} = -\Xi$, we arrive at the following:

$$F_{(9)} = \mathrm{d}(e^\Phi\mathcal{K}) = 3h^{-\frac{1}{4}}e^{-f}\frac{N_f(\rho)}{4\pi}e^{x^0 x^1 x^2 x^3 \rho\theta_1\varphi_1\theta_2\varphi_2}\,,$$

$$F_{(1)} = e^{-2\Phi} * F_{(9)} = -3h^{-\frac{1}{4}}e^{-f}\frac{N_f(\rho)}{4\pi}e^{\psi} ,$$

$$\Xi = -dF_{(1)} = \frac{N_f(\rho)}{4\pi}\Big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\Big) \qquad (2.2.58)$$

$$+ \frac{N_f'(\rho)}{4\pi}d\rho \wedge \Big(d\psi + \cos\theta_1\, d\varphi_1 + \cos\theta_2\, d\varphi_2\Big) ,$$

$$N_f(\rho) = \frac{4\pi}{3}e^{-2g-\Phi}\Big(4e^{2g}g' + e^{2g}\Phi' - 4e^{2f}\Big) .$$

The name for the function $N_f(\rho)$ has been chosen in anticipation of what is to come – it denotes the effective number of flavours at a given energy scale. It should not be confused with N_f, the number of flavour branes.

One should notice that the only assumptions made in deriving (2.2.58) are the form of $F_{(5)}$ and the vielbein describing the deformed background (2.2.52). That is, the above relations hold for all types of D7-branes one might want to smear, massless or massive. They allow us to write down the BPS equations of the system, which can be derived from the supersymmetric variations [35] or using geometric methods:

$$g' = e^{2f-2g} , \qquad f' = 3 - 2e^{2f-2g} - \frac{3N_f(\rho)}{8\pi}e^{\Phi} ,$$

$$\Phi' = \frac{3N_f(\rho)}{4\pi}e^{\Phi} , \qquad h' = -27\pi N_c\, e^{-4g} . \qquad (2.2.59)$$

Note that there are four first-order equations for the five functions Φ, f, g, h, N_f. Furthermore, the smearing procedure always uses the same action,

$$S_{\text{sources}} = T_7 \int_{\mathcal{M}_{10}} \Big(-e^{\Phi}\mathcal{K} + C_{(8)}\Big) \wedge \Xi . \qquad (2.2.60)$$

The authors of [30, 35] used an action of the type encountered in (2.2.9) and (2.2.43), yet once more the equivalence with (2.2.60) may be shown explicitly – we also present a general proof of the validity of (2.2.60) in Section 2.2.2.

Given that the discussion up to this point is completely independent of the type of brane one wants to smear, one might ask how to distinguish between the different classes of potential flavour branes. The answer to that question lies in the choice of the function $N_f(\rho)$.

However, even before looking at specific choices of $N_f(\rho)$, the generic form of Ξ in (2.2.58) tells us quite a bit about possible smeared-brane configurations. For once, it is not possible to break the $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry of the background, as this is the inherent symmetry of Ξ (The $\mathbb{Z}_2$ describes the exchange of the two spheres). So, for massless branes, we are only able to smear both stacks

simultaneously.

The massless branes may be identified with the coordinates given by (2.2.53). Thus they are smeared by the terms proportional to $d\theta_i \wedge d\varphi_i$. As the smearing form is symmetric under the exchange $(\theta_1, \varphi_1) \leftrightarrow (\theta_2, \varphi_2)$, it is clear that we have to smear both stacks of branes. *I.e.* one cannot assume $\Xi_{\theta_1 \varphi_1}$ to vanish without $\Xi_{\theta_2 \varphi_2}$ vanishing as well. The term involving $d\rho$ on the other hand is not transverse to the world-volume defined by (2.2.53). In order to smear only massless branes, one needs this term to vanish. *I.e.* massless branes require

$$N'_f(\rho) = 0 \,. \tag{2.2.61}$$

Using this constraint the system (2.2.59) is fully determined and can be solved. In that case, we can see from (2.2.60) that the last term in (2.2.57) – which does not contain e^ρ – does not contribute. Interpreting the smearing form as a brane density, we may identify the overall factor with the number of flavours,

$$N_f = 4\pi N_f(\rho) \,. \tag{2.2.62}$$

That is, our decision to smear N_f massless branes with a constant number of flavours imposes two constraints on the system, namely (2.2.61) and (2.2.62).

Our choice for $N_f(\rho)$ may also be interpreted using the local geometry of the brane embeddings instead of their global coordinates. The vectors

$$\left(\partial_{x^\mu}, \partial_\rho, \partial_\psi \right) \tag{2.2.63}$$

are tangent to either stack of branes. As the smearing form should – locally – define a volume orthogonal to these vectors, we demand[4]

$$\imath_{\partial_{x^\mu}} \Xi = \imath_{\partial_\rho} \Xi = \imath_{\partial_\psi} \Xi = 0 \,. \tag{2.2.64}$$

It follows that $4\pi N_f(\rho) = \text{const.} = N_f$.

Turning to the massive case, the authors of [35] used

$$N'_f(\rho) = 3N_f \, e^{3\rho_q - 3\rho}(3\rho - 3\rho_q) \,. \tag{2.2.65}$$

---

[4]Interior multiplication of forms with vectors is defined as

$$(\imath_X \omega)_{N_1 \ldots N_{p-1}} = X^M \omega_{M N_1 \ldots N_{p-1}}$$

In principle, one would expect that one can combine the knowledge of the embedding (2.2.54) together with the general form for Ξ in order to derive this form for $N_f(\rho)$, as we did for massless branes, yet we were unable to do so. The reason might be that the authors of [35] considered not just the single representative of the family of massive embeddings, yet a complete distribution. Our analysis contributes to the construction of $N_f(\rho)$ in so far, however, as the derivation in [35] requires the assumption that the $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry cannot be broken, while we have shown that this is not an assumption, but an innate property of the background. More on that topic is discussed in Chapter 7.

Once more, one invokes [51] and needs only to study the BPS equations (2.2.59) together with the modified Bianchi identity to find solutions of the second-order equations. We refer to the original papers for a discussion of the solutions.

Anticipating the possibility of using the formalism presented up to this point in order to smear branes whose coordinate representation is unknown, we shall now discuss the problem of correctly interpreting the smearing form Ξ. Using the vielbein it takes the form

$$\Xi = \frac{6N_f(\rho)}{\sqrt{h}}e^{-2g}(e^{\theta_1\varphi_1} + e^{\theta_2\varphi_2}) + \frac{6N_f'(\rho)}{\sqrt{h}}e^{-2f}e^{\rho\psi}. \qquad (2.2.66)$$

In the case of massless embeddings (2.2.53), the second term disappeared and it is straightforward to interpret the first as a distribution on the space transverse to the two stacks of D7-branes. If we did not know about the massive embeddings (2.2.54), it would be tempting to interpret the term including N_f' as the distribution of a third stack of branes extending along x^μ, wrapping $(\theta_1, \varphi_1, \theta_2, \varphi_2)$ and positioned at fixed (ρ, ψ). That is, we would think of this term as a contribution of compact, smeared D7-branes. The presence of such branes is potentially disastrous as the gauge theory in their world-volume could remain dynamical from a four-dimensional point of view. In the case at hand, the eight-dimensional gauge coupling behaves as $g_{YM} \sim g_s \alpha'^2$, which vanishes for $\alpha' \to 0$, the decoupling limit of the D3-branes. When using D5-branes on the other hand, this does not have to happen. For the massive Klebanov-Witten model, we know that our interpretation in terms of compact D7-branes is wrong as we are smearing a single stack of massive ones. Keeping this in mind, we conclude that it is not straightforward to know which branes have been smeared by simply investigating Ξ.

Again we would like to carry on the procedure we used previously to find the BPS equations (2.2.59) through geometric properties. However, in the previous

examples, the starting point was to identify $F_{(3)}$ with a torsion. In the Klebanov-Witten model, there is no $F_{(3)}$ but instead $F_{(1)}$ and $F_{(5)}$. As a consequence, it is not straightforward to transform the supersymmetric gravitino variation into a covariant derivative. In this case as in the other ones, supersymmetry should nevertheless impose conditions on the geometry of the internal manifold. This information is encoded in the formalism of G-structures, which we use more formally in following chapters.

2.2.2   The generic case

The three examples of the previous section provide us with all the intuition needed to understand the relation between generalised calibrated geometry and supergravity duals with backreacted, smeared flavours. For a Type IIA/B background with Ramond-Ramond flux $F_{(p+2)}$ and arbitrary dilaton, we expect that we should always be able to write the action in terms of the calibration and smearing form as

$$S_{\text{sources}} = -T_p \int_{\mathcal{M}_{10}} (e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)}) \wedge \Xi .\qquad (2.2.67)$$

Now, as we discuss in Appendix 2.A, supersymmetry imposes

$$\mathrm{d}\left(e^{\frac{p-3}{4}\Phi}\mathcal{K}\right) = F_{(p+2)} .\qquad (2.2.68)$$

Combining this with the modified Bianchi identity $\mathrm{d}F_{(10-p-2)} = \pm 2\kappa_{10}^2 T_p \Xi$, as derived in (2.2.12), we may link the calibration and the smearing form

$$\mathrm{d}\left[*e^{\frac{10-2p-4}{4}\Phi}\mathrm{d}\left(e^{\frac{p-3}{4}\Phi}\mathcal{K}\right)\right] = \pm 2\kappa_{10}^2 T_p \Xi .\qquad (2.2.69)$$

The overall sign depends on the dimension $p+1$ of the branes used for the flavouring. In what follows, we shall give a more formal argument why the action (2.2.67) is appropriate to describe smeared branes, show that it is equivalent to the actions previously used in the literature and finally examine some of the consequences of the above relations.

The smeared brane action

The problem of smearing a generic DBI+WZ system takes a rather simple form from a mathematical point of view. Here we are dealing with two spaces, the world-volume $\mathcal{M}_{p+1}$ and space-time $\mathcal{M}_{10}$, which are related by the embedding

map

$$X : \mathcal{M}_{p+1} \to \mathcal{M}_{10}$$
$$\xi^\alpha \mapsto X^M(\xi) \,.$$

(2.2.70)

As integrals of scalars are ill-defined on manifolds, it is mandatory for this discussion to think of the brane action as an integral of differential forms. For the WZ term, the integrand is the pull-back of the relevant electrically coupled gauge potential onto the world-volume,

$$\int_{\mathcal{M}_{p+1}} \imath^* \left(C_{(p+1)} \right) \,.$$

(2.2.71)

Whereas we integrate over the induced volume form and the dilaton in the case of the DBI action[5],

$$\int_{\mathcal{M}_{p+1}} \mathrm{d}^{p+1}\xi \, e^{\frac{p-3}{4}\Phi} \sqrt{-\hat{g}_{(p+1)}} \,.$$

(2.2.72)

The crucial point is that there is no way to *a priori* identify the DBI integrand with a $(p+1)$-form in space-time, as the induced volume form is usually not thought of as the pull-back of a differential form. Indeed, we were rather careless in Section 2.2.1 as we did not discriminate between the set of form-fields in the world-volume of the brane, $\Omega(\mathcal{M}_{p+1})$, and that defined on all of space-time, $\Omega(\mathcal{M}_{10})$.

One might argue that we should be able to somehow push the induced volume form forward onto space-time. This is certainly the case if we are able to identify world-volume with space-time coordinates. In the case of the string dual of the $\mathcal{N} = 1$ SQCD-like theory, this was strikingly obvious. As a matter of fact, the action written in the first line of (2.2.9) is exactly of the form (2.2.67). In a generic situation however, we cannot expect to be able to find such a set of global coordinates. Moreover the natural operations induced by maps between manifolds are push-forwards of vectors and pull-backs of forms. And as they connect spaces of different dimensions, they cannot be assumed to be invertible.

This is where calibrated geometry comes in. As we have seen before, supersymmetric branes satisfy $\imath^* (\mathcal{K}) = \sqrt{-\hat{g}_{(p+1)}} \, \mathrm{d}^{p+1}\xi$. Making use of this fact allows us to treat the DBI and WZ terms on a democratic footing, as both integrands can now be written as pull-backs of $(p+1)$-forms defined on space-time.

We shall now show that the action (2.2.67) can always be written in the form used in [30, 36]. Essentially the whole discussion boils down to the fact that we

---

[5]The discussion in this section considers branes without world-volume gauge fields or the NS potential $B_{(2)}$. The presence of a $B_{(2)}$ field is discussed in Chapter 5. See also [34, 43, 44].

may locally choose nice coordinates. Let us assume that we have a single stack of supersymmetric p-branes. Locally, we may choose coordinates $x^M = (z^\mu, y^m)$ such that the branes extend along the z^μ; that is for worldsheet coordinates ξ^μ and embeddings $X^M(\xi)$ we have

$$\partial_\nu X^M = \begin{cases} \delta_\nu^M & M \in \{0, \dots, p\} \\ 0 & M \notin \{0, \dots, p\} \end{cases} \tag{2.2.73}$$

The vectors ∂_μ are tangent to the brane. They span a subset of $T\mathcal{M}_{10}$ which may be thought of as the embedding of the tangent space $T\mathcal{M}_{p+1}$ of the brane into that of space-time. Orthonormalising the ∂_μ, we obtain a new basis of $T\mathcal{M}_{p+1}$ given by some E_α. *I.e. span*$(E_\alpha) = T\mathcal{M}_{p+1} \subset T\mathcal{M}_{10}$. It follows from the construction that the E_α are closed under the Lie bracket, *i.e.* $[E_\alpha, E_\beta] \in span(E_\gamma)$. Therefore $E_\alpha^m = 0$ and the matrix E_α^μ is invertible. We may complete the set E_α to a basis of the whole tangent space, $E_A = (E_\alpha, E_a)$. Naturally, there is a dual basis of covectors, $e^A = (e^\alpha, e^a)$ which we may use as a vielbein.

Having constructed a vielbein suitable for our purposes, we shall now express the DBI action in terms of that vielbein. As the two bases are dual we have

$$0 = E_\alpha e^b = E_\alpha^M e_M^b \tag{2.2.74}$$

Contracting with $(E_\alpha^\mu)^{-1} = e_\mu^\alpha$, we obtain

$$e_\mu^b = 0 \tag{2.2.75}$$

This is quite important. It means that the components e^a of the vielbein are not pulled back onto the brane world-volume whereas all the e^α are. After all, the pull-back acts as $\imath^* (\omega_M \mathrm{d}x^M) = \omega_\mu \mathrm{d}\xi^\mu$. It follows that the volume form induced onto the brane world-volume is given by the pull-back of the forms e^α

$$\sqrt{-\hat{g}_{(p+1)}} \, \mathrm{d}^{p+1}\xi = \bigwedge_\alpha \imath^* (e^\alpha) \, . \tag{2.2.76}$$

The DBI action in this frame is therefore given by

$$S_{DBI} = -T_p \int_{\mathcal{M}_{p+1}} e^{\frac{p-3}{4}\Phi} \bigwedge_\alpha \imath^* (e^\alpha) \, . \tag{2.2.77}$$

In the final part of our discussion, we impose some constraints on the calibra-

tion and smearing forms, and show that an action of the form (2.1.2) can always be rewritten in the form (2.2.9). For the calibration form to satisfy $\imath^* \left(\mathcal{K} \right) = \sqrt{-\hat{g}_{(p+1)}} \, \mathrm{d}^{p+1}\xi$, it has to include $\bigwedge_\alpha e^\alpha$. So we may assume it to be of the form $\mathcal{K} = \bigwedge_\alpha e^\alpha + \tilde{\mathcal{K}}$, where $\tilde{\mathcal{K}}$ is a $(p+1)$-form which does not depend on all the indices α simultaneously and therefore includes some of the e^a. It follows that $\imath^* \left(\tilde{\mathcal{K}} \right) = 0$. The smearing form is defined on the space transverse to the branes. This space has a one-form basis given by $\mathrm{d}y^m$. As we saw above, $e^a_\mu = 0$ and it follows that we may write the smearing form in this basis as

$$
\begin{aligned}
\Xi &= \frac{1}{(10-p-1)!} \Xi_{m_1 \dots m_{10-p-1}} \mathrm{d}y^{m_1} \wedge \dots \wedge \mathrm{d}y^{m_{10-p-1}} \\
&= \frac{1}{(10-p-1)!} \Xi_{a_1 \dots a_{10-p-1}} e^{a_1 \dots a_{10-p-1}} = \Xi_{(p+2)\dots 9} e^{(p+2)\dots 9} \, .
\end{aligned}
\tag{2.2.78}
$$

That is, locally the smearing form is defined by a single scalar function $\Xi_{(p+2)\dots 9}$ and includes the wedge product over all the transverse components of the vielbein, $\bigwedge_a e^a$. We see immediately that $\tilde{\mathcal{K}} \wedge \Xi = 0$. Moreover

$$
\mathcal{K} \wedge \Xi = e^{0\dots 9} \, \Xi_{(p+2)\dots 9} \, .
\tag{2.2.79}
$$

The trick is now to associate the indices of the function $\Xi_{(p+2)\dots 9}$ with something other than those of the relevant components of the vielbein, as we need those for the overall volume form $e^{0\dots 9} = \sqrt{-g_{(10)}} \, \mathrm{d}^{10}x$. As the form reduces to a function and we are working in flat indices, we may resolve this as follows:

$$
\begin{aligned}
\mathcal{K} \wedge \Xi &= e^{0\dots 9} \, \Xi_{(p+2)\dots 9} = e^{0\dots 9} \sqrt{\Xi_{(p+2)\dots 9} \Xi^{(p+2)\dots 9}} \\
&= \sqrt{-g_{(10)}} \, \mathrm{d}^{10}x |\Xi| \, ,
\end{aligned}
\tag{2.2.80}
$$

with the modulus of the smearing form defined as in (2.2.10). As the wedge product is linear, one may immediately generalise our argument here for multiple stacks of branes, thus proving our initial assertion.

As an immediate application of the results of this section, we shall take a brief look at central extensions of supersymmetric algebras. From the equations of motion (2.2.13), it follows that the smearing form is exact, $\mathrm{d}F_{(10-p-2)} = \pm 2\kappa_{10}^2 \, T_p \, \Xi$. In addition, supersymmetry requires $(e^{\frac{p-3}{4}\Phi} \mathcal{K} - C_{(p+1)})$ to be closed. It follows that we may write the smeared brane action (2.2.67) as a surface integral at infinity,

$$
S_{\text{sources}} = -\frac{1}{2\kappa_{10}^2} \int_{S^9_\infty} (e^{\frac{p-3}{4}\Phi} \mathcal{K} - C_{(p+1)}) \wedge F_{(10-p-2)} \, .
\tag{2.2.81}
$$

This takes the form of a charge. From the original discussion of generalised calibrated geometry in [42], we recall the fact that probe-brane actions relate to central charges in supersymmetry algebras – as one would expect for BPS objects. We conjecture that the charge defined by (2.2.81) has the same interpretation.

2.3 $\mathcal{N} = 2$ gauge/string duality in three dimensions

Let us now apply the methods described in the previous section to the flavouring of an $\mathcal{N} = 2$ super Yang-Mills-like dual in three dimensions. A string dual can be found in the unflavoured case by constructing a domain-wall solution in seven-dimensional gauged supergravity and then lifting it to ten dimensions. It then describes a stack of NS5-branes wrapping a three-sphere. Details and physical interpretation of this solution can be found in [46] and [47]. We first describe the unflavoured solution using notations from [47] before studying the addition of flavours.

2.3.1   The unflavoured solution

In the unflavoured case, we consider only NS5-branes wrapping a three-sphere. So the non-zero fields in Type IIB supergravity are the metric $g_{\mu\nu}$, the dilaton Φ and the NS three-form field strength $H_{(3)}$. The solution found in [47] is, in string frame,

$$ds^2 = d\xi_{1,2}^2 + \frac{2z}{g^2}d\Omega_3^2 + \frac{e^{2x}}{g^2}(dz^2 + d\psi^2) + \frac{1}{g^2\Omega}\sin^2\psi(E_1^2 + E_2^2),$$

$$e^{2\Phi} = \left(\frac{2z}{g^2}\right)^{3/2}\frac{e^{-2A+x}}{\Omega},$$

$$H_{(3)} = \frac{ge^{-2x}}{2z\Omega^{1/2}}\left[\cos\psi(e^{124} - e^{236} - e^{135}) - e^{2x}\sin\psi e^{127}\right]$$

$$- \frac{ge^{-2x}\sin\psi}{\Omega^{3/2}}\left[e^{6x}\sin^2\psi + e^{2x}(4\cos^2\psi + 1) - 3e^{-2x}\cos^2\psi - \frac{\cos^2\psi}{z}\right]e^{567}$$

$$- \frac{ge^{-2x}\cos\psi}{\Omega^{3/2}}\left[e^{4x}\sin^2\psi - 3 + e^{-4x}\cos^2\psi - \frac{e^{2x}\sin^2\psi}{z}\right]e^{456}.$$

$$(2.3.1)$$

A and x are functions of z defined as

$$e^{-2x} = \frac{I_{3/4}(z) - cK_{3/4}(z)}{I_{-1/4}(z) + cK_{1/4}(z)},$$

$$e^{A+3x/2} = z\left(I_{-1/4}(z) + cK_{1/4}(z)\right),$$
(2.3.2)

where I_α and K_α are the modified Bessel functions and c is an integration constant. In the previous equations, we used the vielbein

$$e^a = \frac{\sqrt{2z}}{g}S^a \quad a = 1,2,3\,, \qquad e^7 = \frac{1}{g\Omega^{1/2}}(\cos\psi\,\mathrm{d}z - e^{2x}\sin\psi\,\mathrm{d}\psi)\,,$$

$$e^4 = \frac{1}{g\Omega^{1/2}}(e^{2x}\sin\psi\,\mathrm{d}z + \cos\psi\,\mathrm{d}\psi)\,, \qquad e^8 = \mathrm{d}\xi^1\,,$$

$$e^5 = \frac{1}{g\Omega^{1/2}}\sin\psi E_1\,, \qquad\qquad e^9 = \mathrm{d}\xi^2\,,$$

$$e^6 = \frac{1}{g\Omega^{1/2}}\sin\psi E_2\,, \qquad\qquad e^0 = \mathrm{d}\xi^0\,,$$
(2.3.3)

with

$$\sigma^1 = \cos\tilde{\beta}\,\mathrm{d}\tilde{\theta} + \sin\tilde{\beta}\sin\tilde{\theta}\,\mathrm{d}\tilde{\varphi}\,,$$

$$\sigma^2 = \sin\tilde{\beta}\,\mathrm{d}\tilde{\theta} - \cos\tilde{\beta}\sin\tilde{\theta}\,\mathrm{d}\tilde{\varphi}\,,$$

$$\sigma^3 = \mathrm{d}\tilde{\beta} + \cos\tilde{\theta}\,\mathrm{d}\tilde{\varphi}\,,$$

$$S^1 = \cos\varphi\frac{\sigma^1}{2} - \sin\varphi\frac{\sigma^2}{2}\,,$$

$$S^2 = \sin\theta\frac{\sigma^3}{2} - \cos\theta\left(\sin\varphi\frac{\sigma^1}{2} + \cos\varphi\frac{\sigma^2}{2}\right)\,,$$

$$S^3 = -\cos\theta\frac{\sigma^3}{2} - \sin\theta\left(\sin\varphi\frac{\sigma^1}{2} + \cos\varphi\frac{\sigma^2}{2}\right)\,,$$
(2.3.4)

$$E_1 = \mathrm{d}\theta + \cos\varphi\frac{\sigma^1}{2} - \sin\varphi\frac{\sigma^2}{2}\,,$$

$$E_2 = \sin\theta\left(\mathrm{d}\varphi + \frac{\sigma^3}{2}\right) - \cos\theta\left(\sin\varphi\frac{\sigma^1}{2} + \cos\varphi\frac{\sigma^2}{2}\right)\,,$$

$$\Omega = e^{2x}\sin^2\psi + e^{-2x}\cos^2\psi\,,$$

$$\theta, \tilde{\theta}, \psi \in [0,\pi]\,, \qquad \varphi, \tilde{\varphi} \in [0,2\pi[\,, \qquad \tilde{\beta} \in]0,4\pi]\,,$$

and $\mathrm{d}\Omega_3^2 = \sigma^i\sigma^i$. We know that Type IIB supergravity contains thirty-two supercharges that can be described by an $SO(2)$ doublet of chiral spinors $\epsilon = (\epsilon^-, \epsilon^+)$. Their chirality is expressed as

$$\Gamma_{11}\epsilon = \Gamma_{1234567890}\epsilon = -\epsilon\,.$$
(2.3.5)

This background preserves four supercharges, corresponding to $\mathcal{N} = 2$ in three dimensions. This means that ϵ has to verify the projections

$$\Gamma^{1256}\epsilon = \epsilon\,, \qquad \Gamma^{1346}\epsilon = \epsilon\,, \qquad \Gamma^{4567}\epsilon = \sigma_3\epsilon\,, \tag{2.3.6}$$

where σ_3 is the third Pauli matrix.

2.3.2 Deformation of the solution

We now work again in Einstein frame. We first notice that, in the solution of the previous section, e^4 and e^7 mix the z and ψ coordinates. In order to simplify this, we make a common change of coordinates, first proposed in [53]:

$$\rho = \sin\psi\,\frac{e^{A-x/2}}{(2zg^2)^{1/4}}\,,$$
$$\sigma = \sqrt{g}\,\frac{\cos\psi}{(2z)^{3/4}}e^{A+3x/2}\,. \tag{2.3.7}$$

We then get that $e^4 = h_1(\rho,\sigma)\mathrm{d}\rho$ and $e^7 = h_2(\rho,\sigma)\mathrm{d}\sigma$. Let us now deform the metric by modifying the vielbein in (2.3.3)

$$
\begin{aligned}
e^a &= e^{-f/2}\sqrt{j(\rho,\sigma)}\,S^a \quad a = 1,2,3\,, &\quad e^7 &= e^{-f/2}\sqrt{h_2(\rho,\sigma)}\,\mathrm{d}\sigma\,, \\
e^4 &= e^{-f/2}\sqrt{h_1(\rho,\sigma)}\,\mathrm{d}\rho\,, &\quad e^8 &= e^{-f/2}\mathrm{d}\xi^1\,, \\
e^5 &= e^{-f/2}\sqrt{h_1(\rho,\sigma)k(\rho,\sigma)}\,E_1\,, &\quad e^9 &= e^{-f/2}\mathrm{d}\xi^2\,, \\
e^6 &= e^{-f/2}\sqrt{h_1(\rho,\sigma)k(\rho,\sigma)}\,E_2\,, &\quad e^0 &= e^{-f/2}\mathrm{d}\xi^0\,.
\end{aligned} \tag{2.3.8}
$$

It gives us the following ansatz for the metric:

$$\mathrm{d}s^2 = e^{-f(\rho,\sigma)}\Big(\mathrm{d}\xi_{1,2}^2 + j(\rho,\sigma)\mathrm{d}\Omega_3^2 + h_1(\rho,\sigma)\Big[\mathrm{d}\rho^2 + k(\rho,\sigma)(E_1^2 + E_2^2)\Big] + h_2(\rho,\sigma)\mathrm{d}\sigma^2\Big)\,. \tag{2.3.9}$$

It is straightforward to see that this ansatz leaves the topology of the previous solution invariant.

2.3.3 Calibration, smearing and G-structures

We are now interested in adding flavour D5-branes to the background. Following the usual method, we first deform the unflavoured solution for D5-branes. Then we find calibrated cycles where we can put supersymmetric D5-branes. We finally smear them and find a solution that includes their backreaction.

The solution in the previous section describes NS5-branes. As we are interested in the infrared (IR) behaviour of the gauge dual, we want to consider D5-branes. So we first perform an S-duality on the solution. It gives a new solution of Type IIB supergravity describing D5-branes, for which the non-zero fields are the metric, the dilaton and the Ramond-Ramond three-form such that

$$g_{\mu\nu}^{NS5} \to g_{\mu\nu}^{D5}, \qquad \Phi^{NS5} \to -\Phi^{D5}, \qquad H_{(3)}^{NS5} \to F_{(3)}^{D5}, \qquad \sigma_3 \to \sigma_1. \qquad (2.3.10)$$

As we want to keep the same number of supercharges, and just deform the previous solution, we impose the same projections on the supersymmetric spinors as (2.3.6). We then define a new $SO(2)$ doublet

$$\boldsymbol{\eta} = \begin{pmatrix} \eta^- \\ \eta^+ \end{pmatrix} = \begin{pmatrix} \epsilon^- + \epsilon^+ \\ \epsilon^- - \epsilon^+ \end{pmatrix}, \qquad (2.3.11)$$

such that (2.3.6) becomes

$$\Gamma^{1256} \boldsymbol{\eta} = \boldsymbol{\eta},$$
$$\Gamma^{1346} \boldsymbol{\eta} = \boldsymbol{\eta}, \qquad (2.3.12)$$
$$\Gamma^{4567} \boldsymbol{\eta} = \sigma_3 \boldsymbol{\eta}.$$

Notice that $\boldsymbol{\eta}$ is still a doublet of chiral spinors that satisfies

$$\Gamma_{11} \boldsymbol{\eta} = -\boldsymbol{\eta}. \qquad (2.3.13)$$

From the third projection, we see that η^- and η^+ are both non-zero, but behave differently under the action of gamma matrices. So for each spinor we can construct a six-dimensional generalised calibration form

$$\mathcal{K}^- = \eta^{-T} \Gamma_{089abc} \, \eta^- \, e^{089abc},$$
$$\mathcal{K}^+ = \eta^{+T} \Gamma_{089abc} \, \eta^+ \, e^{089abc}. \qquad (2.3.14)$$

Those forms can be written as

$$\mathcal{K}^- = e^{089} \wedge \phi^-,$$
$$\mathcal{K}^+ = e^{089} \wedge \phi^+, \qquad (2.3.15)$$

where ϕ^+ and ϕ^- are three-forms. Using the supersymmetric variations of the gravitino and the dilatino and identifying $F_{(3)}$ with a torsion term, it is possible to

define two covariant derivatives $\widetilde{\nabla}^+$ and $\widetilde{\nabla}^-$ such that

$$\widetilde{\nabla}^+ \eta^+ = 0\,,$$

$$\widetilde{\nabla}^- \eta^- = 0\,. \tag{2.3.16}$$

So the existence of $\eta^\pm$ imposes that the internal manifold admits a G-structure. With both spinors satisfying the projections (2.3.12), it is possible to define two different G_2-structures in the seven-dimensional space with tangent directions $\{1, 2, 3, 4, 5, 6, 7\}$. The corresponding associative three-forms are ϕ^+ and ϕ^-. We want the flavour branes we add to preserve the same supercharges as in the unflavoured solution. From [48], we know that there is in fact an $SU(3)$-structure in that space, for which the three-dimensional calibration form is

$$\Psi = \frac{1}{2}(\phi^- - \phi^+)\,. \tag{2.3.17}$$

So the calibration form for D5-branes in this geometry is

$$\mathcal{K} = e^{089} \wedge \Psi\,. \tag{2.3.18}$$

We have

$$\phi^- = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}\,,$$

$$\phi^+ = -e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}\,. \tag{2.3.19}$$

So,

$$\mathcal{K} = e^{089} \wedge \left(e^{123} + e^{145} + e^{246} - e^{356}\right)\,. \tag{2.3.20}$$

In order to find solutions for the deformed background, we first need to provide an ansatz for the Ramond-Ramond form $F_{(3)}$:

$$F_{(3)} = e^{-3\Phi/4}\Big(F_{124}(\rho, \sigma)e^{124} + F_{135}(\rho, \sigma)e^{135} + F_{236}(\rho, \sigma)e^{236} + F_{127}(\rho, \sigma)e^{127}$$

$$+ F_{456}(\rho, \sigma)e^{456} + F_{567}(\rho, \sigma)e^{567}\Big)\,, \tag{2.3.21}$$

and we assume the dilaton depends only on ρ and σ. As mentioned previously, we know from [51] that the conservation of supersymmetry gives us first-order differential equations that, in addition to imposing the Bianchi identity for $F_{(3)}$, solve the equations of motion. One way to find those equations is to study the

Type IIB supersymmetry transformations of the dilatino and the gravitino

$$\delta\lambda = \frac{1}{2}\Gamma^\mu \partial_\mu \Phi \eta + \frac{1}{24}e^{\Phi/2}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\sigma_3\eta = 0\,,$$

$$\delta\psi_\mu = \nabla_\mu \eta + \frac{1}{96}e^{\Phi/2}F_{\nu\rho\sigma}(\Gamma_\mu{}^{\nu\rho\sigma} - 9\delta_\mu^\nu\Gamma^{\rho\sigma})\sigma_3\eta = 0\,.$$

$$(2.3.22)$$

Another way is to use the geometric properties of the space, using G-structures and generalised calibration conditions. We need to assume that $\Phi = 2f$. Otherwise, we can look at the dilatino variation to get an additional condition. From it we get

$$\partial_\rho \Phi = \frac{e^{(2f-\Phi)/4}\sqrt{h_1}}{2}(F_{127} - F_{567})\,,$$

$$\partial_\sigma \Phi = \frac{e^{(2f-\Phi)/4}\sqrt{h_2}}{2}(F_{135} + F_{236} + F_{456} - F_{124})\,.$$

$$(2.3.23)$$

Then we remember that $\mathcal{K}^-$ is a generalised calibration and ϕ^- defines a G_2-structure. So we get two conditions on those forms

$$\mathrm{d}\left(e^{\Phi/2}\mathcal{K}^-\right) = -e^\Phi *_{10} F_{(3)}\,,$$

$$\mathrm{d}\left(e^\Phi *_7 \phi^-\right) = \mathrm{d}\left(e^\Phi *_{10} \mathcal{K}^-\right) = 0\,.$$

$$(2.3.24)$$

Using the conditions on the dilaton, those two equations give us

$$f = \frac{\Phi}{2}\,, \qquad\qquad\qquad F_{127} = -\frac{\sqrt{h_1 k}}{j}\,, \qquad (2.3.25)$$

$$F_{135} = -F_{124}\,, \qquad\qquad\qquad F_{236} = -F_{124}\,. \qquad (2.3.26)$$

$$\partial_\rho \Phi = -\frac{j\sqrt{h_1}F_{567} + h_1\sqrt{k}}{2j}\,, \qquad \partial_\sigma \Phi = \frac{\sqrt{h_2}(F_{456} - 3F_{124})}{2}\,, \quad (2.3.27)$$

$$\partial_\rho j = 2h_1\sqrt{k}\,, \qquad\qquad\qquad \partial_\sigma j = 2j\sqrt{h_2}F_{124}\,, \qquad (2.3.28)$$

$$\partial_\rho k = 2\sqrt{k} - \frac{h_1 k^{3/2}}{j} + k\frac{h_1^{3/2}F_{567} - \partial_\rho h_1}{h_1}\,, \quad \partial_\sigma k = 0\,, \qquad (2.3.29)$$

$$\partial_\rho h_2 = h_2\frac{j\sqrt{h_1}F_{567} + h_1\sqrt{k}}{j}\,, \qquad \partial_\sigma h_1 = h_1\sqrt{h_2}(F_{124} - F_{456})\,,$$

$$(2.3.30)$$

Moreover, we must have

$$\partial_\rho \partial_\sigma \Phi = \partial_\sigma \partial_\rho \Phi\,,$$

$$\partial_\rho \partial_\sigma j = \partial_\sigma \partial_\rho j\,.$$

$$(2.3.31)$$

So we get

$$\partial_\rho F_{124} = -\frac{j\sqrt{h_1}F_{124}F_{567} + h_1\sqrt{k}(3F_{124} + 2F_{456})}{2j},$$

$$\frac{\partial_\rho F_{456}}{\sqrt{h_1}} = -\frac{\partial_\sigma F_{567}}{\sqrt{h_2}} - \frac{\sqrt{h_1 k}(4F_{124} + 5F_{456}) + jF_{124}F_{567}}{2j}.$$

$$(2.3.32)$$

Let us now eliminate the components of $F_{(3)}$ in (2.3.27) to (2.3.30) and try to solve those equations. We get

$$h_1 = \frac{e^{-2\Phi}}{j}e^{a(\rho)},$$

$$h_2 = e^{-2\Phi}e^{b(\sigma)},$$

$$e^{2\Phi} = \frac{2\sqrt{k}}{j\partial_\rho j}e^a,$$

$$F_{124} = \frac{e^{(a-b)/2}k^{1/4}\partial_\sigma j}{\sqrt{2\partial_\rho j}\,j^{3/2}},$$

$$(2.3.33)$$

$$F_{456} = \frac{e^{(a-b)/2}k^{1/4}(\partial_\sigma j\partial_\rho j - 2j\partial_\sigma\partial_\rho j)}{\sqrt{2}(j\partial_\rho j)^{3/2}},$$

$$F_{567} = \frac{\sqrt{k}(\partial_\rho j)^2 - j((2 + \sqrt{k}a')\partial_\rho j - 2\sqrt{k}\partial_\rho^2 j)}{\sqrt{2}k^{1/4}j(\partial_\rho j)^{3/2}},$$

$$\partial_\rho k = 2\sqrt{k} - ka'.$$

We notice that $b(\sigma)$ is arbitrary, which corresponds to the fact that it is always possible to redefine the σ coordinate. To simplify the problem, we are taking $b = 0$ in the following sections.

2.3.4   Addition and smearing of flavour branes

In order to add and smear flavour branes, one needs to find the smearing form Ξ. Following the prescription presented in the first part of this chapter, we know that this form is related to the calibration form of our background $\mathcal{K}$ (see (2.3.20)) through (we reabsorb factors of 2π in the definition of Ξ)

$$\Xi = dF_{(3)} = -d\left(e^{-\Phi} * d\left(e^{\Phi/2}\mathcal{K}\right)\right).$$

$$(2.3.34)$$

Using this, the ansatz for the metric and for $F_{(3)}$ and the equations found in the previous section ((2.3.27) to (2.3.32)), we can deduce that the most general form

of Ξ is

$$\Xi = e^{\Phi}\Big(N_{f1}(\rho,\sigma)[e^{2367} + e^{1357} - e^{1247}] + N_{f2}(\rho,\sigma)e^{4567}\Big)\,, \qquad (2.3.35)$$

with

$$\partial_{\sigma}F_{124} = \sqrt{h_2}\frac{j(F_{124}F_{456} - 5F_{124}^2 + 2N_{f1}e^{2\Phi}) - 2 - 2\sqrt{h_1k}F_{567}}{2j}\,, \qquad (2.3.36)$$

$$\frac{\partial_{\sigma}F_{456}}{\sqrt{h_2}} = \frac{\partial_{\rho}F_{567}}{\sqrt{h_1}} + \frac{3F_{567}^2}{2} + \frac{F_{567}(4j - h_1k)}{2j\sqrt{h_1k}} + \frac{3F_{456}(F_{456} - F_{124})}{2} - e^{2\Phi}N_{f2}\,.$$

$$(2.3.37)$$

Consistency between those equations and (2.3.33) imposes that

$$N_{f2} = N_{f1} + \frac{j}{h_1\sqrt{k}}\partial_{\rho}N_{f1}\,, \qquad (2.3.38)$$

$$0 = 2j^2\partial_{\rho}^2 j + 2e^a j\partial_{\sigma}^2 j + j(\partial_{\rho}j)^2 - e^a(\partial_{\sigma}j)^2 - j^2(a'\partial_{\rho}j + 4e^a N_{f1})\,. \qquad (2.3.39)$$

We now see that the only unknown we have is N_{f1}. Any function of ρ and σ is possible and gives first-order differential equations that solve the modified equations of motion for Type IIB supergravity plus flavour embeddings. Finding a solution then consists only on solving the second-order differential equation (2.3.39). However, while the choice of the function N_{f1} determines which branes are smeared, we are unable to derive the embedding of the supersymmetric branes that have been smeared. One might want to recall the discussion at the end of Section 2.2.1.

Different possibilities for the smearing form

As it was stated before, the starting point of adding smeared flavours is to choose a smearing form, which, in the case at hand, corresponds to choosing a function $N_{f1}(\rho,\sigma)$. A first possibility would be to take N_{f1} independent of ρ. It follows from (2.3.38) that

$$N_{f1} = N_{f2} = N_f(\sigma)\,. \qquad (2.3.40)$$

Then we can try to solve (2.3.39) by making the following ansatz for j:

$$j(\rho,\sigma) = G(\rho)^{2/3}H(\sigma)^2\,. \qquad (2.3.41)$$

We obtain

$$G' = c_1 e^{a/2}\,, \qquad\qquad \frac{H''}{H} = N_f\,, \qquad (2.3.42)$$

where c_1 is a constant. In the case where $a = 0$ and N_f is a constant, we can solve this and find

$$k = (\rho + \rho_0)^2 , \tag{2.3.43}$$

and

$$j = (c_1\rho + c_2)^{2/3} \cos(\sqrt{-N_f}\,\sigma + c_3)^2 \qquad \text{if } N_f < 0 , \tag{2.3.44}$$

$$j = (c_1\rho + c_2)^{2/3} \cosh(\sqrt{N_f}\,\sigma + c_3)^2 \qquad \text{if } N_f \geq 0 , \tag{2.3.45}$$

where c_2 and c_3 are integration constants. These provide analytic solutions of the equations of motion of Type IIB supergravity with modified Bianchi identity. When looking at the dilaton behaviour, we find

$$e^{2\Phi} = \frac{3(\rho + \rho_0)}{c_1(c_2 + c_1\rho)^{1/3} \cos(c_3 + \sqrt{-N_f}\,\sigma)^4} \qquad \text{if } N_f < 0 , \tag{2.3.46}$$

$$e^{2\Phi} = \frac{3(\rho + \rho_0)}{c_1(c_2 + c_1\rho)^{1/3} \cosh(c_3 + \sqrt{N_f}\,\sigma)^4} \qquad \text{if } N_f \geq 0 . \tag{2.3.47}$$

When $N_f < 0$, in (2.3.46), it is remarkable that there are singularities for $c_3 + \sqrt{-N_f}\,\sigma = \frac{\pi}{2} \bmod (2\pi)$. Those singularities may be a sign of the presence of the smeared flavour branes.

Another possibility would be to try to have a smearing form independent of one of the radial coordinates, instead of just the function N_{f1} as in the previous paragraph. For Ξ to be independent of σ, we have to take

$$N_{f1} = \frac{N(\rho)}{\sqrt{j}} . \tag{2.3.48}$$

Then (2.3.39) becomes

$$0 = 2j^2 \partial_\rho^2 j + 2e^a j \partial_\sigma^2 j + j(\partial_\rho j)^2 - e^a(\partial_\sigma j)^2 - j^2 a' \partial_\rho j - 4e^a N(\rho) j^{3/2} . \tag{2.3.49}$$

Taking here $N(\rho)$ to be constant, we get $N_{f2} = 0$ which suppresses one of the terms in the smearing form. Nevertheless, it is not obvious how to find a solution of the equation for j.

For Ξ to be independent of ρ, one needs to impose k to be a constant. Then

$$a(\rho) = 2a_1\rho ,$$
$$N_{f1} = \frac{e^{-a_1\rho}}{\sqrt{j}} N(\sigma) , \tag{2.3.50}$$

where a_1 is a strictly positive constant. We now have to solve:

$$0 = 2j^2 \partial_\rho^2 j + 2e^{2a_1\rho} j \partial_\sigma^2 j + j(\partial_\rho j)^2 - e^{2a_1\rho}(\partial_\sigma j)^2 - 2j^2 a_1 \partial_\rho j - 4e^{a_1\rho} N(\sigma) j^{3/2} . \quad (2.3.51)$$

In the case where $N(\sigma) = N_f$ is a constant, the smearing form is independent of any radial dependence. Then, we can find asymptotic solutions, considering ρ as the energy scale. One interesting fact is that it does not seem possible to ignore the term involving N_f in the IR, that is when ρ goes to zero. In the IR ($\rho \to 0$), we find that

$$j = e^{2a_1\rho/3} \left(\frac{3N_f}{a_1^2 - a_1} + c_1 e^{(1-a_1)\rho} \right)^{2/3} \qquad \text{if } a_1 \neq 1 ,$$

$$j = e^{2a_1\rho/3} \left(3N_f \rho + c_2 e^{-\rho} \right)^{2/3} \qquad \text{if } a_1 = 1 . \qquad (2.3.52)$$

In the ultraviolet (UV), we have two possibilities: we can decide that the term in N_f is suppressed or plays a role. The two cases give

$$j = c_3 e^{2a_1\rho/3} \sigma^2 \qquad \text{if we neglect the term in } N_f ,$$

$$j = e^{-2a_1\rho} \frac{N_f^2 \sigma^4}{4} . \qquad (2.3.53)$$

Comments on the solution

Firstly one can notice that none of the solutions presented in the previous section goes to the solution found in [47] in the limit where N_{f1} and N_{f2} go to zero, as expected from the dual gauge theory point of view.

We try to find a solution that describes a stack of N_c colour branes plus one or several stacks of smeared flavour branes. The number of colour branes is related to the Ramond-Ramond field $F_{(3)}$ through

$$\int_{S^3} F_{(3)} = 2\kappa_{10}^2 T_5 N_c , \qquad (2.3.54)$$

where S^3 is a three-sphere around the point where the colour branes are placed in the four-dimensional space transverse to their world-volume. We were not able to find a constant when calculating that integral for the solutions of the previous section. It means that either we did not find the right transverse four-dimensional space, or these results cannot have the usual interpretation of stacks of branes.

This relates to the most prominent problem of the method presented in this

section. As we mentioned in Footnote 3, it is necessary to verify the existence of a cycle wrapped by the branes. As we explicitly avoided the issue of considering the embedding smeared, one cannot be certain that the above solutions do describe smeared branes. In simple cases, when the smearing form does not have a term along the radial direction of the space, each component in the vielbein basis can usually be interpreted as the volume form of the space orthogonal to the brane smeared. In the case studied above, Ξ has to have a term in $d\rho$. So by analogy with the Klebanov-Witten case, it seems that we smear massive flavour branes. But we were not able to determine their embedding. However, the form of Ξ tells us it is not possible to smear massless flavour branes in this background. Moreover, knowing the explicit embedding of the flavour branes is not necessary to look at some properties of the gauge theory dual.

2.4 Conclusion

In this chapter, we applied generalised calibrations and G-structures to address the problem of adding smeared flavour branes to a supergravity background. In doing so, we made a first step towards a systematic study of backgrounds with a large number of smeared flavour branes. In Section 2.2, we showed that the smeared brane action of [45] is equivalent to those used previously in the literature on smeared flavour branes. This makes the symmetry between the DBi and the WZ terms apparent and the linearity in the smearing form Ξ manifest. Furthermore we were able to link the complete brane action to a conserved charge and to impose strong constraints on Ξ by relating it to the calibration form. While the explicit form of Ξ depends on the embedding smeared, this allowed us to explain various features of the examples in Section 2.2.1; in particular why the smearing has to preserve certain symmetries, which again implies that it is often only possible to smear several stacks of branes at once.

We exhibited the potential of our methods not only by studying known examples, yet by also flavouring a background dual to a three-dimensional $\mathcal{N} = 2$ super Yang-Mills-like theory (See Section 2.3). Here we found several solutions and some interesting features, notably the fact that it is not possible to smear massless flavours – a property which it would be nice to understand from the point of view of the dual gauge theory.

The formalism introduced in this chapter unifies the treatment of different possible embeddings for any single background, enabling for a general study of the

smearing procedure in a given background, instead of the case by case methods previously used. Even if it remains necessary to verify the existence of the cycles wrapped by the branes, their knowledge is not necessary for the actual calculation. However, as we have seen in the case of the three-dimensional $\mathcal{N} = 2$ duality, backgrounds constructed without any knowledge of the embeddings might be very difficult to interpret.

In the following chapters, we build on the results presented above, trying to improve our understanding of the flavouring procedure in the language of G-structures. In the next chapter, we apply our technique to the flavouring of a Type IIA supergravity background with an $SU(3)$-structure. We also ask the question of the lift to eleven dimensions of such a solution including sources, and see how G-structures can help in the understanding of such an issue.

2.A   A review of generalised calibrated geometry

We shall give here a short introduction to generalised calibrated geometry [42] in relation to supersymmetric brane embeddings. The discussion given ignores the case of world-volume fluxes and follows that of the review [54].

Calibration forms and supersymmetric brane embeddings   The standard method used when studying supersymmetric brane embeddings is κ-symmetry [55]. A brane embedding $X^M(\xi)$ is supersymmetric if it satisfies the equation

$$\Gamma_\kappa \epsilon = \epsilon \,, \tag{2.A.1}$$

where ϵ is a supersymmetric spinor of the background while Γ_κ is a linear map depending on the form of the embedding. For a D-brane of Type II string theory with world-volume gauge fields such that $\mathcal{F} = 2\pi\alpha'F - B_{(2)} = 0$, it reduces to

$$\Gamma_\kappa = \frac{1}{(1+p)!\sqrt{-\hat{g}_{(p+1)}}}\epsilon^{\alpha_0...\alpha_p} \begin{cases} (\Gamma^{11})^{\frac{p-2}{2}}\gamma_{\alpha_0...\alpha_p} & \text{(IIA)} \\[2ex] \sigma_3^{\frac{p-3}{2}} \, i\,\sigma_2 \otimes \gamma_{\alpha_0...\alpha_p} & \text{(IIB)} \end{cases} \tag{2.A.2}$$

which is invariant under Weyl transformations and therefore valid in string and Einstein frame. The definition uses the pull-back of the space-time gamma matrices onto the brane world-volume, $\gamma_\alpha = \partial_\alpha X^M \Gamma_M$.

Γ_κ is hermitian and squares to one. It follows that

$$\epsilon^\dagger \frac{1-\Gamma_\kappa}{2}\epsilon = \epsilon^\dagger \frac{1-\Gamma_\kappa}{2}\frac{1-\Gamma_\kappa}{2}\epsilon = \left\| \frac{1-\Gamma_\kappa}{2}\epsilon \right\|^2 \geq 0\,, \tag{2.A.3}$$

which implies that $\epsilon^\dagger\epsilon \geq \epsilon^\dagger\Gamma_\kappa\epsilon$, with equality if and only if the embedding is supersymmetric. Normalising the spinor such that $\epsilon^\dagger\epsilon = 1$ and using (2.A.2), we may rephrase this as

$$\sqrt{-\hat{g}_{(p+1)}} \geq \frac{1}{(p+1)!}\epsilon^{\alpha_0...\alpha_p} \begin{cases} \epsilon^\dagger(\Gamma^{11})^{\frac{p-2}{2}}\gamma_{\alpha_0...\alpha_p}\epsilon & \text{(IIA)} \\[2ex] \epsilon^\dagger\sigma_3^{\frac{p-3}{2}} \, i\,\sigma_2 \otimes \gamma_{\alpha_0...\alpha_p}\epsilon & \text{(IIB)} \end{cases} \tag{2.A.4}$$

Equality holds if and only if the embedding is supersymmetric. Now the right-hand side of (2.A.4) may be written as the pull-back of a differential form defined

in space-time:

$$\mathcal{K} = \frac{1}{(p+1)!} e^{a_0 \dots a_p} \begin{cases} \epsilon^\dagger (\Gamma^{11})^{\frac{p-2}{2}} \Gamma_{a_0 \dots a_p} \epsilon & \text{(IIA)} \\ \epsilon^\dagger \sigma_3^{\frac{p-3}{2}} \, i \, \sigma_2 \otimes \Gamma_{a_0 \dots a_p} \epsilon & \text{(IIB)} \end{cases} \qquad (2.A.5)$$

where $\mathcal{K}$ is known as the calibration form. An alternative criterion for supersymmetry of an embedding to the one of (2.A.1) is then given by

$$\imath^* (\mathcal{K}) = \sqrt{-\hat{g}_{(p+1)}} \, \mathrm{d}^{p+1} \xi \,, \qquad (2.A.6)$$

that is, the pull-back of the calibration form onto the world-volume is equal to the induced volume form. One may obtain $\mathcal{K}$ directly from its definition (2.A.5) and the knowledge of the projections imposed onto the supersymmetric spinors.

A more formal definition   Formally one defines a calibration on a Riemannian manifold as a $(p+1)$-form $\mathcal{K}$ satisfying

$$\mathrm{d}\mathcal{K} = 0 \,, \qquad \mathcal{K}|_{\xi^{p+1}} \leq \eta_{(p+1)}|_{\xi^{p+1}} \,. \qquad (2.A.7)$$

Here ξ^{p+1} is a set of vectors specifying a tangent $(p+1)$-plane to a $(p+1)$-cycle Σ_{p+1} while $\eta_{(p+1)} = \sqrt{-\hat{g}_{(p+1)}} \, \mathrm{d}^{p+1}\xi$ is the volume form induced on that cycle. The cycle Σ_{p+1} is calibrated if the above bound is saturated, $i.e.$ if $\mathcal{K}|_{\xi^{p+1}} = \eta_{(p+1)}|_{\xi^{p+1}}$.

As we have seen above in (2.A.6), κ-symmetric brane embeddings satisfy the volume bound, which can be thought of as a BPS bound. In this and in the next paragraphs, we shall turn to the issue of the closure of (2.A.5). For a background without fluxes, the issue is rather easily resolved. From the gravitino variation

$$\delta_\epsilon \psi_M = D_M \epsilon = 0 \,, \qquad (2.A.8)$$

it follows that the supersymmetric spinor ϵ is covariantly constant. As the covariant derivative of both the vielbein and the tangent-space gamma matrices does also vanish, it follows that

$$\mathrm{d}\mathcal{K} = \nabla \wedge \mathcal{K} = 0 \,. \qquad (2.A.9)$$

$\nabla \wedge \mathcal{K}$ is to be thought of as a formal expression: the wedge product antisymmetrises over the relevant indices and, as the Levi-Civita connection is symmetric in two of its indices, it follows that the first equality holds. Since all the ingredients of (2.A.5) are covariantly constant, the exterior derivative is closed.

There is a nice interpretation of the closure of the calibration form. Let us assume that we deform the calibrated cycle Σ_{p+1} to Σ'_{p+1}. The two cycles differ by a boundary $\Sigma_{p+1} - \Sigma'_{p+1} = \delta\Sigma_{p+2}$. More formally we would not consider Σ'_{p+1} as a deformation, yet as a cycle within the homology class defined by Σ_{p+1}. We use Stokes theorem to establish

$$\mathrm{Vol}(\Sigma_{p+1}) = \int_{\Sigma_{p+1}} \mathcal{K} = \int_{\Sigma_{p+2}} \mathrm{d}\mathcal{K} + \int_{\Sigma'_{p+1}} \mathcal{K} = \int_{\Sigma'_{p+1}} \mathcal{K} \leq \mathrm{Vol}(\Sigma'_{p+1})\,. \qquad (2.A.10)$$

The final inequality uses (2.A.4). It follows that the calibrated cycle Σ_{p+1} is a minimal-volume cycle. This matches nicely with our experience from string theory where, in the absence of fluxes, branes wrap cycles of minimal volume.

Generalised calibrations   The κ-symmetry matrix (2.A.2) does not change in the presence of Ramond-Ramond background fields and thus neither does the definition of the calibration form or the supersymmetry condition (2.A.6). Background fluxes however deform branes such that they do not wrap minimal-volume cycles any longer. Therefore, for a background with fluxes, we do not expect the calibration form (2.A.5) to be closed. Rather, its exterior differential should be related to the flux. Indeed, in all the examples studied in Section 2.2.1, the calibration satisfied

$$\mathrm{d}\left(e^{\frac{p-3}{4}\Phi}\mathcal{K}\right) = F_{(p+2)}\,. \qquad (2.A.11)$$

In this case, one speaks of a generalised calibration, a concept which was first introduced in [42].

There are several ways to prove (2.A.11). For all the examples of Section 2.2.1, the equality held after we imposed the BPS equations, so it should be no surprise that (2.A.11) is intimately linked to the supersymmetry of the background. The original proof [42] showed that the expression $(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)})$ appears as the central charge of a supersymmetry algebra and must therefore be topological and thus exact. It is also possible to verify (2.A.11) in terms of the dilatino and gravitino supersymmetry transformations.

Let us now take a look at the appropriate generalisation of (2.A.10). To do so we shall assume that both the brane and the background fields are static. It follows that the energy of the system is proportional to its action – with the proportionality constant being infinity. Minimum energy configurations therefore minimise the brane action. Let Σ_{p+1} be the supersymmetric cycle wrapped by the

brane and $\Sigma'_{p+1} = \Sigma_{p+1} + \delta\Sigma_{(p+2)}$ a deformation. Then (setting $T_p = 1$)

$$
\begin{aligned}
\Delta E &\propto S_{\Sigma'_{p+1}} - S_{\Sigma_{p+1}} \\
&= \int_{\Sigma'_{p+1}} \left(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)} \right) - \int_{\Sigma_{p+1}} \left(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)} \right) \\
&\geq \int_{\delta\Sigma_{p+2}} \left(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)} \right) \\
&= \int_{\Sigma_{p+2}} \mathrm{d}\left(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)} \right) = 0 \,.
\end{aligned}
\tag{2.A.12}
$$

The inequality in the second line used again (2.A.4). It follows that supersymmetric, static embeddings are minimum energy configurations.

Chapter 3

Flavour D6-branes and lift to M-theory

3.1 Introduction

This chapter deals with two issues: the flavouring of some Type IIA supergravity background, and the understanding of how the link between Type IIA string theory and M-theory can be made compatible with the presence of smeared sources. It is based on [8], done in collaboration with Schmude.

In the context of M-theory, the relations between Type IIA string theory and eleven-dimensional supergravity are by now standard textbook material (see for example [56, 57, 58, 59]). The M2-brane gives rise to the D2-brane and the fundamental string, the M5 to the D4 and NS5-branes. The D0 and D6-branes on the other hand have a slightly different origin. Not being related to any brane-like object in eleven dimensions, they result from the Kaluza-Klein (KK) reduction relating the two theories; the former being a particle-like, localised gravitational excitation on the KK-circle, the latter a peculiar fibration of said circle over the ten-dimensional base, known as a Kaluza-Klein monopole (a good review is given by [60]). In this chapter, we are concerned with a small gap in this formalism that becomes apparent when one tries to consider the M-theory lift of smeared D6-branes.

The problem can be explained quite easily. The bosonic sector of eleven-dimensional supergravity contains only the graviton $\hat{g}_{MN}$ and a four-form field $\hat{F}_{(4)}$. Upon KK reduction, $\hat{F}_{(4)}$ gives rise to the Kalb-Ramond three-form field $H_{(3)}$ as well as the Ramond-Ramond four-form $F_{(4)}$. From $\hat{g}_{MN}$, one obtains the

ten-dimensional metric $g_{\mu\nu}$, the dilaton Φ, and a one-form gauge potential $C_{(1)}$, with an associated field strength $F_{(2)} = \mathrm{d}C_{(1)}$. If we assume the KK circle to be parametrised by z, the standard KK ansatz relating the two geometries is[1]

$$\mathrm{d}s_{\mathrm{M}}^2 = e^{-\frac{2}{3}\Phi}\mathrm{d}s_{\mathrm{IIA}}^2 + e^{\frac{4}{3}\Phi}(C_{(1)} + \mathrm{d}z)^2 \,,$$
$$\hat{F}_{(4)} = F_{(4)} + H_{(3)} \wedge \mathrm{d}z \,. \tag{3.1.1}$$

Given any solution of the equations of motion of Type IIA supergravity, one can use (3.1.1) to lift to eleven dimensions and vice versa. However, as $C_{(1)}$ plays the role of a gauge potential, it is actually $F_{(2)} = \mathrm{d}C_{(1)}$ that contains the physically relevant degrees of freedom. Thus, given a set $\{g_{\mu\nu}, \Phi, F_{(2)}, H_{(3)}, F_{(4)}\}$, one first has to find a gauge potential prior to lifting. Now assume that for some reason $\mathrm{d}F_{(2)} \neq 0$. Clearly $C_{(1)}$ cannot be globally defined and we are unable to find a gauge potential. Therefore we cannot use (3.1.1) to perform the lift. This is the apparent gap in the standard formalism we alluded to earlier.

The problem is not a purely formal one. D6-branes couple magnetically to $C_{(1)}$. As seen previously, the inclusion of sources – in this case D6-branes – violates the Bianchi identity $\mathrm{d}F_{(2)} = 0$ at the position of the sources. While this is not a problem for localised sources – as a matter of fact it is the reason why the KK monopole is a gravitational instanton – one encounters the problem at hand once one distributes the branes continuously and thus violates the Bianchi identity on an open subset of space-time.

As an aside it is worthwhile to point out that the relation between D6-branes and the RR two-form is much the same as that between magnetic monopoles and the $F_{\mathrm{E\&M}}$ in standard electro-magnetism. The inclusion of magnetic sources restores the symmetry of the Maxwell equations. Schematically

$$\mathrm{d} * F_{\mathrm{E\&M}} = *j_{\mathrm{E}} \,, \qquad \mathrm{d}F_{\mathrm{E\&M}} = *j_{\mathrm{M}} \,. \tag{3.1.2}$$

Thus, the Bianchi identity is violated by the magnetic current j_{M}.

In this chapter, we do not resolve the issue in full generality, but we focus on the inclusion of D6-sources in Type IIA backgrounds of the form

$$\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{M}_6 \,, \tag{3.1.3}$$

---

[1]Where the distinction is necessary, hats and tildes denote eleven-dimensional quantities. Capital letters describe eleven-dimensional indices. The M-theory circle is parametrised by either z, ψ_+ or ψ.

without three of four-form flux, that preserve four supercharges. More precisely, we are interested in the construction of string duals to four-dimensional $SU(N_c)$ gauge theories with $\mathcal{N} = 1$ supersymmetry and N_f flavours using D6-branes.

This work was originally born out of an interest in studying the addition of flavour branes to Type IIA backgrounds dual to $\mathcal{N} = 1$, $SU(N_c)$ super Yang-Mills. Before flavouring, the geometry we start with is that of N_c D6-branes wrapping a three-cycle in the deformed conifold[2]. In the limit $N_c\, g_{\text{YM}}^2 \gg 1$, the backreaction of the colour branes causes the system to undergo a geometric transition. The system is now best described in terms of the resolved conifold with the branes having been replaced by N_c units of two-form flux over a two-cycle. This was originally studied in [61, 62] and the geometric transition is based on the work of [63, 64]; an attempt at generalising the duality to include finite-temperature duals was made in [65]. The resulting ten-dimensional background consists of metric, dilaton and RR two-form $(g_{\mu\nu}, \Phi, F_{(2)})$. Referring back to (3.1.1), one sees that it lifts to pure geometry in M-theory, as both $H_{(3)}$ and $F_{(4)}$ are set to zero. It is for this reason that it is particularly simple and interesting to study these geometries and dualities from the perspective of eleven-dimensional supergravity. There, the equations of motion and supergravity variations simplify to

$$\hat{R}_{MN} = 0\,, \qquad \delta_{\hat\epsilon}\hat\psi_M = \partial_M \hat\epsilon + \frac{1}{4}\hat\omega_{MAB}\hat\Gamma^{AB}\hat\epsilon\,. \qquad (3.1.4)$$

The eleven-dimensional geometry is of the form

$$\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathcal{M}_7\,. \qquad (3.1.5)$$

As the seven-dimensional manifold $\mathcal{M}_7$ preserves one-eighth of the supersymmetry and is Ricci flat, it is a manifold of G_2-holonomy. The concept of M-theory compactifications on such manifolds (see [66]) is pretty much the same as that of the old heterotic string models on Calabi-Yau three-folds used in string phenomenology. Mathematically this is reflected by the presence of a three-form ϕ that is closed and co-closed

$$\mathrm{d}\phi = 0\,, \qquad \mathrm{d}\left(*_7\,\phi\right) = 0\,, \qquad (3.1.6)$$

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[2]To be precise, we are dealing with conifolds deformed by the presence of branes or $F_{(2)}$ flux. They do carry an $SU(3)$-structure but are not of $SU(3)$-holonomy. Therefore, they are not Calabi-Yau and strictly speaking we should not refer to them as (deformed/resolved) conifolds. For the lack of a better term however, we shall refer to the internal six-dimensional manifolds in this chapter by that name though, as their topology is the same as that of their Calabi-Yau cousins.

where $*_7$ denotes the seven-dimensional Hodge dual on the internal space.

From the point of view of Type IIA string theory, the flavouring procedure is reasonably straightforward. Following Chapter 2, we consider the action

$$S = S_{IIA} + S_{\text{sources}}. \qquad (3.1.7)$$

The brane action can be written as

$$S_{\text{sources}} = -T_6 \int_{\mathcal{M}_{10}} \left(e^{-\Phi}\mathcal{K} - C_{(7)}\right) \wedge \Xi, \qquad (3.1.8)$$

where $\mathcal{K}$ is the calibration form and Ξ takes the role of a source density for the D6-branes. The presence of S_{sources} in the modified action (3.1.7) gives source term contributions to the equations of motion. Most prominent among these is the appearance of a magnetic source term for the RR two-form,

$$\mathrm{d}F_{(2)} = -(2\kappa_{10}^2 T_6)\Xi, \qquad (3.1.9)$$

that violates the standard Bianchi identity. In Type IIA, one accommodates for this simply by adding a flavour contribution to the RR form,

$$F_{(2)} = \mathrm{d}C_{(1)} + G_{(2)}, \qquad (3.1.10)$$

with $G_{(2)} \to 0$ as $N_f \to 0$. As it was shown in Chapter 2, the choice of smearing form is not arbitrary, as supersymmetry and the modified Bianchi identities require it to satisfy

$$\mathrm{d} *_{10} \mathrm{d}\left(e^{-\Phi}\mathcal{K}\right) = -(2\kappa_{10}^2 T_6)\Xi. \qquad (3.1.11)$$

It is *a priori* not obvious how to accommodate the violation of the Bianchi identity (3.1.9) in M-theory. However, as the sources do not only modify the Bianchi identity, yet also the dilaton and Einstein equations, it is reasonable to expect that the eleven-dimensional geometry is not Ricci flat. Instead, the Einstein equations should be supplemented by the presence of a source term,

$$\hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} = \hat{T}_{MN}. \qquad (3.1.12)$$

From the loss of Ricci flatness, it follows that the manifold can no longer be of G_2-holonomy; as it preserves the same amount of supersymmetry however, it is fair to expect it to carry a G_2-structure. Therefore, there is still a three-form ϕ that now

fails to be (co-)closed. One can anticipate that the failure of the manifold to be of G_2-holonomy is parametrised by N_f and thus ultimately by the $G_{(2)}$ contribution to $F_{(2)}$, *i.e.*

$$\mathrm{d}\phi \sim \left(F_{(2)} - \mathrm{d}C_{(1)} \right),$$

$$\mathrm{d}\left(*_7\phi \right) \sim \left(F_{(2)} - \mathrm{d}C_{(1)} \right). \tag{3.1.13}$$

These expressions, relating forms of different degrees, are to be understood in such a way that the left-hand side vanishes when the right-hand side does, and vice versa. Now for a manifold carrying a G-structure, its failure to be of G-holonomy is measured by its intrinsic torsion[3]. Therefore, we expect the flavours in eleven dimensions to appear in the form of intrinsic torsion. A detailed study of the relation between the eleven and ten-dimensional supersymmetry variations prompts us to consider eleven-dimensional backgrounds with torsion $\hat{\tau}$, where the torsion is related to $F_{(2)} - \mathrm{d}C_{(1)} = G_{(2)}$.

Finally, we see that an uplift of our ten-dimensional equations of motion is given by the relation

$$R_{MN}^{(\tau)} + \frac{1}{2} R_{KLRN}^{(\tau)} (*_7\phi)_M{}^{KLR} = 0, \tag{3.1.14}$$

which is the solution of our initial problem. $R^{(\tau)}$ is the eleven-dimensional Riemann (Ricci) tensor with torsion – we have discarded the use of hats to avoid an overly cluttered notation. As one can always rewrite the Riemann tensor as a combination of a torsion free Riemann tensor with additional terms depending on the torsion, it is possible to recast the above equation in the form of (3.1.12) with the energy-momentum tensor depending only on the torsion.

At first glance, equation (3.1.14) appears like a modification of M-theory and violates all intuition that eleven-dimensional supergravity is unique. However, one must not forget that we never pretended that we would solve the problem in its full generality. As a matter of fact, (3.1.14) has to be taken cautiously – which might not be a surprise, as the inclusion of source terms in theories of gravity is always a rather difficult business. First of all, (3.1.14) assumes the background to be of topology $\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathcal{M}_7$, with the internal manifold carrying a G_2-structure. Furthermore, this means that we are not dealing with maximal eleven-dimensional supergravity, but with a situation with reduced supersymmetry – 1/8 BPS – in

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[3]For intrinsic torsion in the context of string theory see [48].

which case the theory is no longer unique. Still, equation (3.1.14) manages what the standard KK ansatz (3.1.1) does not. It gives the correct source-modified equations of motion in Type IIA supergravity.

The structure of this chapter is as follows. In Section 3.2, we begin with a review of the unflavoured geometries in ten and eleven dimensions and then continue by studying the flavouring problem from the perspective of Type IIA. Following this, we turn to the issue of the M-theory lift in Section 3.3. For illustrative and motivational purposes, we use a specific case of an M-theory G_2-holonomy manifold and its Type IIA reduction in Section 3.2. However, the results of Section 3.3 on the M-theory lift of smeared D6-branes do not depend on this example or the Type IIA reduction chosen. They only depend on the presence of a G_2-structure, four-dimensional Minkowski space and the absence of M-theory fluxes.

Note that (3.1.14) is not the only result presented here. As we are studying the flavouring problem in Type IIA in order to find an answer to the issue of the M-theory lift, this chapter makes also considerable progress towards the construction of a dual to four-dimensional, $\mathcal{N} = 1$ $SU(N_c)$ super Yang-Mills with backreacting flavours. For the specific ansatz of Section 3.2, we are able to derive a set of very generic first-order equations – (3.2.32) and (3.2.36) – that have to be satisfied by smeared D6-sources in this geometry. We proceed to derive an analytic one-parameter family of solutions in Section 3.2.3. While the fluxes in this solution satisfy the flux quantisation necessary for a string dual, the geometry is that of a cone over $S^2 \times S^3$ with a singularity at the origin. So we expect the interpretation of this solution as a suitable dual to be difficult.

3.2   Flavoured $\mathcal{N} = 1$ string duals from D6-branes

In this section, we review the source-free string duals in their ten and eleven-dimensional formulations. Subsequently, we turn to the issue of adding sources to the Type IIA background. Let us once more emphasise that the particular choices of eleven-dimensional geometry (and its dimensional reduction) are of no direct consequence for our results concerning the M-theory lift of smeared D6-branes. The concrete geometry presented here is chosen due to its relevance to the flavouring problem in Type IIA.

3.2.1   The eleven-dimensional dual without sources

Building on the work of Brandhuber [67] (see also [68, 69]), we consider the purely gravitational M-theory background given by the vielbein

$$
\begin{aligned}
\tilde{e}^{\mu} &= \mathrm{d}x^{\mu}\,, & \tilde{e}^{\rho} &= E(\rho)\mathrm{d}\rho\,, \\
\tilde{e}^{1,2} &= A(\rho)\sigma_{1,2}\,, & \tilde{e}^{3,4} &= C(\rho)[\Sigma_{1,2} - f(\rho)\sigma_{1,2}]\,, & (3.2.1) \\
\tilde{e}^{5} &= B(\rho)\sigma_{3}\,, & \tilde{e}^{6} &= D(\rho)[\Sigma_{3} - g(\rho)\sigma_{3}]\,.
\end{aligned}
$$

σ_i, Σ_i are left-invariant Maurer-Cartan forms which we choose to be

$$
\begin{aligned}
\sigma_1 &= \cos\psi\,\mathrm{d}\theta + \sin\psi\sin\theta\,\mathrm{d}\varphi\,, & \Sigma_1 &= \cos\tilde{\psi}\,\mathrm{d}\tilde{\theta} + \sin\tilde{\psi}\sin\tilde{\theta}\,\mathrm{d}\tilde{\varphi}\,, \\
\sigma_2 &= -\sin\psi\,\mathrm{d}\theta + \cos\psi\sin\theta\,\mathrm{d}\varphi\,, & \Sigma_2 &= -\sin\tilde{\psi}\,\mathrm{d}\tilde{\theta} + \cos\tilde{\psi}\sin\tilde{\theta}\,\mathrm{d}\tilde{\varphi}\,, & (3.2.2) \\
\sigma_3 &= \mathrm{d}\psi + \cos\theta\,\mathrm{d}\varphi\,, & \Sigma_3 &= \mathrm{d}\tilde{\psi} + \cos\tilde{\theta}\,\mathrm{d}\tilde{\varphi}\,.
\end{aligned}
$$

The solutions we are interested in are 1/8-BPS; therefore, one can impose the following constraints onto the supersymmetric spinor $\tilde{\epsilon}$:

$$
\tilde{\Gamma}^{1234}\tilde{\epsilon} = \tilde{\epsilon}\,, \qquad \tilde{\Gamma}^{1356}\tilde{\epsilon} = -\tilde{\epsilon}\,, \qquad \tilde{\Gamma}^{\rho 126}\tilde{\epsilon} = -\tilde{\epsilon}\,. \tag{3.2.3}
$$

As a direct consequence, we can calculate the following spinor bilinear, which turns out to be the G_2-structure form:

$$
\begin{aligned}
\phi &= (\bar{\tilde{\epsilon}}\tilde{\Gamma}_{A_0 A_1 A_2}\tilde{\epsilon})\tilde{e}^{A_0 A_1 A_2} \\
&= \tilde{e}^{\rho 13} + \tilde{e}^{\rho 24} + \tilde{e}^{\rho 56} + \tilde{e}^{146} + \tilde{e}^{345} - \tilde{e}^{125} - \tilde{e}^{236}\,.
\end{aligned} \tag{3.2.4}
$$

In the absence of four-form flux, the preservation of four supercharges is equivalent to the manifold being of G_2-holonomy. A necessary and sufficient condition is the closure and co-closure of the G_2-structure form. By imposing $\mathrm{d}\phi = 0$ and $\mathrm{d}(*_7\phi) = 0$, we obtain the BPS equations

$$
\begin{aligned}
A' &= \frac{E[BD(g-f^2) + ACf(1-g)]}{2AB}\,, & B' &= \frac{ECf(1-g)}{A}\,, \\
D' &= \frac{E[A^2(2C^2 - D^2) + C^2 D^2(f^2 - g)]}{2A^2 C^2}\,, & C' &= \frac{E[ABD - C^3 f(1-g)]}{2ABC}\,, \\
f &= \frac{BC}{2AD}\,, & g &= 1 - 2f^2\,.
\end{aligned}
$$

$$
\tag{3.2.5}
$$

The same BPS system follows from demanding that $\delta_{\tilde{\epsilon}}\tilde{\psi}_M = 0$.

The best known solution of (3.2.5) is the Bryant-Salamon metric [70]. With

$$A^2 = B^2 = \frac{\rho^2}{12}, \qquad C^2 = D^2 = \frac{\rho^2}{9}\left(1 - \frac{\rho_0^3}{\rho^3}\right),$$

$$E^2 = \left(1 - \frac{\rho_0^3}{\rho^3}\right)^{-1}, \qquad f = g = \frac{1}{2}, \tag{3.2.6}$$

the metric takes the form

$$ds^2 = dx_{1,3}^2 + \left(1 - \frac{\rho_0^3}{\rho^3}\right)^{-1} d\rho^2 + \sum_{i=1}^{3}\left[\frac{\rho^2}{12}\sigma_i^2 + \frac{\rho^2}{9}\left(1 - \frac{\rho_0^3}{\rho^3}\right)\left(\Sigma_i - \frac{1}{2}\sigma_i\right)^2\right].$$
$$\tag{3.2.7}$$

The seven-dimensional G_2 cone actually turns out to be the cotangent bundle T^*S^3. The geometry is that of a cone over $S^3 \times S^3$, with each sphere being parametrised by a set of Maurer-Cartan forms. At $\rho = \rho_0$, the minimum of the radial parameter, one of the spheres (Σ) collapses, while the other (σ) remains of finite size. M-theory dynamics on this type of manifold were discussed in [66]. Fluctuations in ρ_0 and the gauge potential $C_{(3)}$ can be combined into a complex parameter. However, as these fluctuations turn out to be non-normalisable, they do not parametrise a moduli space of vacua, yet rather a moduli space of theories.

There are three $U(1)$ isometries in (3.2.1) given by ∂_φ, $\partial_{\tilde\varphi}$ and $\partial_\psi + \partial_{\tilde\psi}$ and there are therefore three different dimensional reductions to Type IIA supergravity. In each case, one obtains a conifold geometry with flux, with the conifold singularity being resolved by a deformation or resolution. *I.e.* there is a cone over $S^2 \times S^3$ and one of the spheres vanishes at the minimal radius while the other remains of finite size. Furthermore, if we choose to reduce along an isometry embedded in the vanishing sphere, we need to recall that the vanishing of the M-theory circle indicates the presence of D6-branes. Thus, the reduction along $\partial_{\tilde\varphi}$ yields a deformed conifold with a D6-brane at $\rho = \rho_0$ extending along the Minkowski directions and wrapping the non-vanishing S^3. If one mods out the $U(1)$ by $\mathbb{Z}_{N_c}$ before reducing, the corresponding geometry is that of N_c branes. The other two reductions include non-singular $U(1)$'s, so we end up with resolved conifolds. As the M-theory circle is non-singular, there are no D6-branes. There is $F_{(2)}$ flux though on the finite-size two-sphere. The different geometries are related by a flop transition between the resolved conifolds and the conifold transition between the deformed and the resolved ones.

In the context of gauge/string duality, the deformed conifold corresponds to the weak 't Hooft coupling regime, while the resolved one is to be considered for

large 't Hooft coupling. Thus the latter provides the appropriate supergravity dual. M-theory realises the conifold dualities via the aforementioned moduli space of solutions. See [61, 62, 66].

Scherk-Schwarz gauge   In what follows, we study the reduction along $\partial_\psi + \partial_{\tilde\psi}$. In the context of the flavouring problem of Section 3.2.2, one expects the system to be best described by one of the resolved conifold geometries with additional flavour branes. Therefore, out of the three isometries discussed, ∂_φ and $\partial_\psi + \partial_{\tilde\psi}$ are the obvious choices. We selected the latter as it leads to simpler equations in Type IIA supergravity. The choice made here does affect the flavouring problem, yet not our results on the M-theory lift. As we are interested in the reduction of tangent-space quantities, we need to transform the vielbein to Scherk-Schwarz gauge

$$
\hat{e}^A_M = \begin{pmatrix} e^{-\frac{1}{3}\Phi}e^a_\mu & e^{\frac{2}{3}\Phi}A_\mu \\ 0 & e^{\frac{2}{3}\Phi} \end{pmatrix}_{MA} ,
$$

$$
\hat{E}^M_A = \begin{pmatrix} e^{\frac{1}{3}\Phi}E^\mu_a & -e^{\frac{1}{3}\Phi}A_a \\ 0 & e^{-\frac{2}{3}\Phi} \end{pmatrix}_{AM} .
\tag{3.2.8}
$$

To obtain the gauge (3.2.8) from (3.2.1), we perform the following gauge transformation:

$$
\Lambda = \Lambda^{(3)}\Lambda^{(2)}\Lambda^{(1)} ,
$$

with the individual transformations $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}$ being

$$
\Lambda^{(1)} = \begin{pmatrix} \mathbb{I}_{9\times 9} & \\ & \begin{matrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{matrix} \end{pmatrix} ,
$$

$$
\Lambda^{(2)} = \begin{pmatrix} \mathbb{I}_{5\times 5} & & & \\ & \begin{matrix} \cos\frac{\psi_+}{2} & -\sin\frac{\psi_+}{2} \\ \sin\frac{\psi_+}{2} & \cos\frac{\psi_+}{2} \end{matrix} & & \\ & & \begin{matrix} \cos\frac{\psi_+}{2} & -\sin\frac{\psi_+}{2} \\ \sin\frac{\psi_+}{2} & \cos\frac{\psi_+}{2} \end{matrix} & \\ & & & \mathbb{I}_{2\times 2} \end{pmatrix} ,
\tag{3.2.9}
$$

$$
\Lambda^{(3)} = \begin{pmatrix} \mathbb{I}_{6\times 6} & \\ & \begin{matrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{matrix} & \\ & & \mathbb{I}_{2\times 2} \end{pmatrix} ,
$$

and all other entries zero. Here we defined

$$
\cos\alpha(\rho) = \frac{D(1-g)}{\sqrt{B^2 + (1-g)^2 D^2}} , \qquad\qquad \psi_+ = \psi + \tilde\psi ,
$$

$$\sin \alpha(\rho) = \frac{B}{\sqrt{B^2 + (1-g)^2 D^2}} , \qquad \psi_- = \psi - \tilde{\psi} . \qquad (3.2.10)$$

In principle one needs only $\Lambda^{(1)}$ and $\Lambda^{(2)}$ to obtain Scherk-Schwarz gauge; yet, without $\Lambda^{(3)}$, the new projections satisfied by the supersymmetric spinor would be linear combinations of the old ones (3.2.3) with coefficients $\cos \alpha$, $\sin \alpha$. As it is, the form of the supersymmetric projections remains invariant under Λ. *I.e.*

$$\hat{\Gamma}^{1234}\hat{\epsilon} = \hat{\epsilon} ,$$
$$\hat{\Gamma}^{1356}\hat{\epsilon} = -\hat{\epsilon} , \qquad\qquad (3.2.11)$$
$$\hat{\Gamma}^{\rho126}\hat{\epsilon} = -\hat{\epsilon} .$$

Thus the G_2-structure (3.2.4) remains formally the same, with the vielbeins $\tilde{e}^A$ now replaced by $\hat{e}^A$. A disadvantage of the reducible gauge is that the new vielbein is rather complicated.

Dimensional reduction and Type IIA string theory   The resulting ten-dimensional vielbein is given by

$$e^\mu = e^{\frac{1}{3}\Phi}\mathrm{d}x^\mu ,$$
$$e^\rho = e^{\frac{1}{3}\Phi}E\,\mathrm{d}\rho ,$$
$$e^1 = e^{\frac{1}{3}\Phi}A \left(\cos\frac{\psi_-}{2}\,\mathrm{d}\theta + \sin\theta \sin\frac{\psi_-}{2}\,\mathrm{d}\varphi \right) ,$$
$$e^2 = e^{\frac{1}{3}\Phi}A\cos\alpha \left(\cos\frac{\psi_-}{2}\sin\theta\,\mathrm{d}\varphi - \sin\frac{\psi_-}{2}\,\mathrm{d}\theta \right)$$
$$\qquad + e^{\frac{1}{3}\Phi}C\sin\alpha \left[\cos\frac{\psi_-}{2}(\sin\tilde{\theta}\,\mathrm{d}\tilde{\varphi} - f\sin\theta\,\mathrm{d}\varphi) + \sin\frac{\psi_-}{2}(\mathrm{d}\tilde{\theta} + f\mathrm{d}\theta) \right] , \qquad (3.2.12)$$
$$e^3 = e^{\frac{1}{3}\Phi}C \left[\cos\frac{\psi_-}{2}(\mathrm{d}\tilde{\theta} - f\mathrm{d}\theta) - \sin\frac{\psi_-}{2}(f\sin\theta\,\mathrm{d}\varphi + \sin\tilde{\theta}\,\mathrm{d}\tilde{\varphi}) \right] ,$$
$$e^4 = -e^{\frac{1}{3}\Phi}A\sin\alpha \left(\cos\frac{\psi_-}{2}\sin\theta\,\mathrm{d}\varphi - \sin\frac{\psi_-}{2}\,\mathrm{d}\theta \right)$$
$$\qquad + e^{\frac{1}{3}\Phi}C\cos\alpha \left[\cos\frac{\psi_-}{2}(\sin\tilde{\theta}\,\mathrm{d}\tilde{\varphi} - f\sin\theta\,\mathrm{d}\varphi) + \sin\frac{\psi_-}{2}(\mathrm{d}\tilde{\theta} + f\mathrm{d}\theta) \right] ,$$
$$e^5 = e^{\frac{1}{3}\Phi}D\sin\alpha \left(\cos\theta\,\mathrm{d}\varphi - \cos\tilde{\theta}\,\mathrm{d}\tilde{\varphi} + \mathrm{d}\psi_- \right) ,$$

while the dilaton and gauge potential are

$$e^{\frac{2}{3}\Phi} = \frac{B}{2\sin\alpha} = \frac{D(1-g)}{2\cos\alpha} ,$$

$$C_{(1)} = \cos\theta \, d\varphi + \cos\tilde{\theta} \, d\tilde{\varphi} + \frac{B^2 - D^2(1 - g^2)}{B^2 + (1 - g)^2 D^2} (\cos\theta \, d\varphi - \cos\tilde{\theta} \, d\tilde{\varphi} + d\psi_-)$$

$$= \cos\theta \, d\varphi + \cos\tilde{\theta} \, d\tilde{\varphi} + \left(\sin^2\alpha - \frac{1 + g}{1 - g} \cos^2\alpha \right) (\cos\theta \, d\varphi - \cos\tilde{\theta} \, d\tilde{\varphi} + d\psi_-) \, .$$

$$\text{(3.2.13)}$$

Using $\hat{\Gamma}^{10} = \Gamma^{11}$, the reduction of the supersymmetric projections takes a more pleasing form:

$$\Gamma^{1234}\epsilon = \epsilon \, , \qquad \Gamma^{135}\Gamma^{11}\epsilon = -\epsilon \, , \qquad \Gamma^{\rho12}\Gamma^{11}\epsilon = -\epsilon \, . \tag{3.2.14}$$

This allows us to calculate the generalised calibration form for D6-branes in this background:

$$\mathcal{K} = (\bar{\epsilon}\Gamma_{a_0\dots a_6}\epsilon)e^{a_0\dots a_6} = e^{x^0 x^1 x^2 x^3} \wedge \left(e^{125} - e^{345} - e^{\rho24} - e^{\rho13} \right) . \tag{3.2.15}$$

Note that the internal three-form part of this is, up to some overall dilaton factor, identical to that part of the G_2-structure (3.2.4) independent of $\hat{e}^6$.

G-structures   In terms of G-structures, the situation in Type IIA supergravity is the following. Because we preserve four supercharges, we expect space-time to carry an $SU(3)$-structure. As it was shown in [71], it can be directly derived from the G_2-structure of the KK lift. Centrepiece of that reduction are the relations

$$J = \phi_{ab6} \, e^{ab} \, , \qquad \Psi = \phi_{abc} \, e^{abc} \, . \tag{3.2.16}$$

For the six-dimensional internal manifold, J defines an almost complex structure, with respect to which we can define from Ψ a $(3,0)$-form Ω as

$$\Omega = \Psi - i *_6 \Psi \, . \tag{3.2.17}$$

These satisfy the equations

$$J \wedge \Omega = 0 \, , \qquad J \wedge J \wedge J = \frac{3i}{4}\Omega \wedge \bar{\Omega} \, . \tag{3.2.18}$$

In the case at hand, we have

$$J = e^{\rho5} + e^{14} - e^{23} \, ,$$
$$\Psi = e^{\rho13} + e^{\rho24} + e^{345} - e^{125} \, , \tag{3.2.19}$$

which gives

$$\Omega = \Psi - i *_6 \Psi = \left(e^{\rho} + ie^5\right) \wedge \left(e^1 + ie^4\right) \wedge \left(e^3 + ie^2\right). \tag{3.2.20}$$

Thinking about lifting from ten to eleven dimensions, we can invert equations (3.2.16) to express the eleven-dimensional G_2-structure in terms of the ten-dimensional quantities:

$$\phi = e^{-\Phi}\Psi + e^{-\frac{2}{3}\Phi} J \wedge \hat{e}^6,$$
$$*_7\phi = -\frac{1}{2}e^{-4\Phi/3} J \wedge J + e^{-\Phi}(*_6\Psi) \wedge \hat{e}^6. \tag{3.2.21}$$

As previously stated, Ricci flatness, preservation of four supercharges and absence of four-form flux in eleven dimensions guarantee the G_2-holonomy of the internal manifold. This translates in the closure and co-closure of ϕ. As the fibration of the M-theory circle over the ten-dimensional base is non-trivial, one obtains non-vanishing two-form flux upon reduction to Type IIA supergravity. Hence the internal six-dimensional manifold is not of $SU(3)$-holonomy due to its intrinsic torsion. This means that the forms J and Ω are not both closed. The relations they obey can be derived from the closure and co-closure of ϕ thanks to (3.2.21):

$$0 = \mathrm{d}\phi = \mathrm{d}\left(e^{-\Phi}\Psi\right) + \mathrm{d}J \wedge \left(C_{(1)} + \mathrm{d}\psi_+\right) + J \wedge \mathrm{d}C_{(1)},$$
$$0 = \mathrm{d} *_7 \phi = -\frac{1}{2}\mathrm{d}\left(e^{-4\Phi/3} J \wedge J\right) + \mathrm{d}\left(e^{-\Phi/3} *_6 \Psi\right) \wedge \left(C_{(1)} + \mathrm{d}\psi_+\right) \tag{3.2.22}$$
$$- e^{-\Phi/3}\left(*_6\Psi\right) \wedge \mathrm{d}C_{(1)}.$$

We know that none of the ten-dimensional quantities depend on ψ_+. Hence, the contribution to the previous equations coming from $\mathrm{d}\psi_+$ must cancel by itself. It gives

$$0 = \mathrm{d}J, \qquad\qquad 0 = -\frac{1}{2}\mathrm{d}\left(e^{-4\Phi/3} J \wedge J\right) - e^{-\Phi/3}\left(*_6\Psi\right) \wedge \mathrm{d}C_{(1)}.$$
$$0 = \mathrm{d}\left(e^{-\Phi/3} *_6 \Psi\right), \qquad 0 = \mathrm{d}\left(e^{-\Phi}\Psi\right) + J \wedge \mathrm{d}C_{(1)},$$
$$\tag{3.2.23}$$

These equations can be rephrased (following [71] for example) as

$$\mathrm{d}J = 0,$$
$$\mathrm{d}\Phi = \frac{3}{4}e^{\Phi}\mathrm{d}C_{(1)}\lrcorner(*_6\Psi), \tag{3.2.24}$$
$$J\lrcorner\mathrm{d}C_{(1)} = 0,$$

where

$$G_{(p)} \lrcorner H_{(p+q)} = \frac{1}{p!} G^{\mu_1 \cdots \mu_p} H_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_{p+q}} \mathrm{d}x^{\mu_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_{p+q}} . \tag{3.2.25}$$

We described in this section the construction of a Type IIA supergravity background from the reduction of eleven-dimensional supergravity. We also derived the equations imposed on the structure by supersymmetry. Now we turn to the problem of adding backreacting flavours in this ten-dimensional context.

3.2.2 Smeared sources in Type IIA supergravity

The source-modified first-order system

Applying the method developed in Chapter 2, we now address the problem of flavouring the Type IIA supergravity background obtained in the previous section. It means that we look for a solution of the following action, describing the backreaction of smeared D6-brane sources in a ten-dimensional background:

$$S = S_{IIA} - T_6 \int \left(e^{-\Phi} \mathcal{K} - C_{(7)} \right) \wedge \Xi , \tag{3.2.26}$$

where S_{IIA} is the Type IIA supergravity action, $\mathcal{K}$ is the calibration form corresponding to supersymmetric D6-branes, $C_{(7)}$ is the seven-form potential and Ξ is the smearing form, representing the distribution of sources. The sources in (3.2.26) modify the standard Type IIA equations of motion and Bianchi identities to

$$\mathrm{d}F_{(2)} = -(2\kappa_{10}^2 T_6)\Xi ,$$

$$0 = \mathrm{d} *_{10} F_{(2)} ,$$

$$0 = \frac{1}{\sqrt{-g}} \partial_\kappa (\sqrt{-g} g^{\kappa\lambda} e^{-2\Phi} \partial_\lambda \Phi) - \frac{3}{8} F_{(2)}^2 - \frac{3}{4} e^{-\Phi} \Xi \lrcorner (*_{10}\mathcal{K}) , \tag{3.2.27}$$

$$R_{\mu\nu} = -2\nabla_\mu \partial_\nu \Phi + \frac{e^{2\Phi}}{2} (F_{\mu\kappa} F_\nu{}^\kappa - \frac{1}{4} g_{\mu\nu} F_{(2)}^2)$$

$$+ \frac{e^\Phi}{4} \left((*_{10}\mathcal{K})_\mu{}^{\kappa\lambda} \Xi_{\nu\kappa\lambda} - g_{\mu\nu} \Xi \lrcorner (*_{10}\mathcal{K}) \right) .$$

Fortunately, the flavouring procedure does not require us to explicitly solve the complete second-order system. Due to the standard integrability arguments ([50, 72, 51]), it is sufficient to satisfy the Bianchi identities along with the first-order

BPS equations[4]. However, in Section 3.3, we show how to derive the second-order system directly from M-theory.

The metric ansatz is given by the vielbein (3.2.12) and the dilaton is assumed to depend only on the radial coordinate ρ. The calibration associated with κ-symmetric D6-branes is given by (3.2.15) which is

$$\mathcal{K} = e^{x^0 x^1 x^2 x^3} \wedge \Psi. \tag{3.2.28}$$

Supersymmetry requires the two-form flux to obey the generalised calibration condition

$$*_{10} \mathrm{d}\left(e^{-\Phi}\mathcal{K}\right) = F_{(2)}. \tag{3.2.29}$$

This tells us that the most general ansatz for $F_{(2)}$ is

$$F_{(2)} = e^{-4\Phi/3}\left(F_{\rho 5}(\rho)e^{\rho 5} + F_{12}(\rho)e^{12} + F_{14}(\rho)e^{14} + F_{23}(\rho)e^{23} + F_{34}(\rho)e^{34}\right). \tag{3.2.30}$$

The conditions given by supersymmetry on this $SU(3)$-structure geometry with intrinsic torsion are still given by (see end of Section 3.2.1)

$$\mathrm{d}J = 0, \qquad \mathrm{d}\Phi = \frac{3}{4}e^{\Phi}F_{(2)}\lrcorner(*_6\Psi), \qquad J\lrcorner F_{(2)} = 0, \tag{3.2.31}$$

where we have now replaced $\mathrm{d}C_{(1)}$ by $F_{(2)}$, thus enabling for $\mathrm{d}F_{(2)} \neq 0$, as necessary for D6-sources. Together with the generalised calibration condition (3.2.29), these equations give the first-order equations the system must satisfy:

$$f = \frac{A}{C\tan\alpha},$$

$$\alpha' = \frac{E}{2}\left(\frac{2}{D\tan\alpha} - \frac{D}{C^2\tan\alpha} + \frac{D\cos\alpha\sin\alpha}{A^2} - 2F_{23}\right),$$

$$A' = \frac{E}{2}\left(\frac{A}{D\tan^2\alpha} + \frac{D}{A} - \frac{AD}{C^2\tan^2\alpha} - \frac{2AF_{23}}{\tan\alpha} - AF_{34}\right),$$

$$C' = \frac{E}{2}\left(-\frac{C}{D\tan^2\alpha} + \frac{D}{C} - CF_{34}\right),$$

$$D' = \frac{E}{2}\left(-\frac{2D^2}{C^2} - \frac{D^2}{A^2} + \frac{DF_{23}}{\tan\alpha} + 2\right), \tag{3.2.32}$$

$$\Phi' = \frac{3E}{2}\left(-\frac{D\cos^2\alpha}{2A^2} + \frac{D}{2C^2\tan^2\alpha} + \frac{F_{23}}{\tan\alpha} + F_{34}\right),$$

---

[4]Technically there are further mild assumptions to satisfy. *I.e.* one needs the $(0, \mu)$ components of the Einstein equation to vanish explicitly.

$$F_{\rho 5} = \frac{D \cos \alpha \sin \alpha}{A^2} - \frac{D}{C^2 \tan \alpha} \, ,$$

$$F_{12} = \frac{D \cos^2 \alpha}{A^2} - \frac{D}{C^2 \tan^2 \alpha} - \frac{2F_{23}}{\tan \alpha} - F_{34} \, ,$$

$$F_{14} = \frac{D}{C^2 \tan \alpha} - \frac{D \cos \alpha \sin \alpha}{A^2} + F_{23} \, .$$

As mentioned before, the modified equations of motion relate the smearing form to the two-form flux:

$$\mathrm{d}F_{(2)} = -2\kappa_{10}^2 T_6 \, \Xi \, . \tag{3.2.33}$$

This equation, combined with (3.2.30) and (3.2.32), tells us that the most general ansatz for Ξ is

$$\Xi = e^{-5\Phi/3} \Big(\Xi_1(\rho) e^{\rho 34} + \Xi_2(\rho) \left(e^{\rho 23} + e^{\rho 14} \right) + \Xi_3(\rho) e^{\rho 12} + \Xi_4(\rho) \left(e^{135} + e^{245} \right) \Big) \, , \tag{3.2.34}$$

with

$$\Xi_3 = -\Xi_1 - \frac{2\Xi_2}{\tan \alpha} \, , \qquad \Xi_4 = \frac{F_{34}}{2\kappa_{10}^2 T_6 D \sin^2 \alpha} \, , \tag{3.2.35}$$

and the additional conditions

$$\begin{aligned}
F_{23}' = E \Bigg(& -\frac{F_{34}}{D \tan \alpha} - \frac{DF_{23} \cos^2 \alpha + DF_{34} \cos \alpha \sin \alpha}{A^2} - \frac{D^2 \cos \alpha \sin \alpha}{A^2 C^2} + \frac{2F_{23}^2}{\tan \alpha} \\
& + \frac{DF_{23} \cos(2\alpha) + DF_{34} \sin \alpha \cos \alpha}{C^2 \sin^2 \alpha} + \frac{D^2 \cos \alpha}{C^4 \sin \alpha} + 3F_{34}F_{23} - 2\kappa_{10}^2 T_6 \Xi_2 \Bigg) \, , \\
F_{34}' = E \Bigg(& \frac{F_{34}}{D \tan^2 \alpha} - \frac{DF_{34}}{2A^2} + \frac{DF_{34} \cos(2\alpha)}{2C^2 \sin^2 \alpha} + \frac{F_{34}F_{23}}{\tan \alpha} + 2F_{34}^2 - 2\kappa_{10}^2 T_6 \Xi_1 \Bigg) \, .
\end{aligned} \tag{3.2.36}$$

One can verify explicitly that any solution of equations (3.2.32) and (3.2.36) automatically solves the source-modified equations of motion (3.2.27).

As we want to interpret the two-form flux $F_{(2)}$ as created by brane sources, we need the flux to be quantised, obeying $\int_{S^2} F_{(2)} = 2\pi N_c$. S^2 is a suitable two-cycle surrounding the branes in the transverse, three-dimensional space. This adds constraints on Ξ and $F_{(2)}$:

$$\Xi_1 = \Xi_2 \tan \alpha \, ,$$

$$F_{23} = \frac{-A^2 D + C^4 F_{34} \sin^2 \alpha + C^2 (2N_c e^{\frac{2}{3}\Phi} \sin \alpha \tan \alpha + D \sin^2 \alpha + A^2 F_{34})}{(A^2 C^2 + C^4 \sin^2 \alpha) \tan \alpha} \, , \tag{3.2.37}$$

that are compatible with the equation (3.2.36).

3.2.3   Finding a solution

In this section, we present an analytic solution of the previous system of first-order equations. First, we can directly solve one of the equations in (3.2.32):

$$D = e^{\frac{2}{3}\Phi} \frac{N_c C^2 \sin\alpha \tan\alpha}{A^2} \,. \tag{3.2.38}$$

Let us now specialise to the case $\Xi_2 = 0$. We see that this reduces the freedom of the smearing form to

$$\Xi = \frac{e^{-5\Phi/3} F_{34}}{2\kappa_{10}^2 T_6 D \sin^2\alpha} \left(e^{135} + e^{245} \right) \,. \tag{3.2.39}$$

The branes smeared with this particular form would correspond to branes extended in the radial direction ρ in a trivial way. This simplification allows us to solve the equation for the last unknown component of the two-form flux $F_{(2)}$:

$$F_{34} = e^{\frac{2}{3}\Phi} \frac{N_f \sin\alpha}{AC} \,, \tag{3.2.40}$$

where N_f is a constant of integration related to the number of flavours in the dual field theory. We now suppose that the two-form flux is independent of the radial coordinate ρ, a property verified in other examples of string duals. This imposes that

$$A^2 = C^2 \sin^2\alpha \,. \tag{3.2.41}$$

Finally, we assume f to be constant. A look at the original metric (3.2.12) tells us that f parametrises the fibration between the two spheres – this becomes rather more obvious in (3.2.1). Thus if f is independent of ρ, the fibration does not change when we flow along the radial direction. Then we can solve the full BPS system analytically, and we find:

$$D^2 = e^{\frac{4}{3}\Phi} \frac{N_c^2}{f^2} \,, \qquad\qquad A^2 = e^{\frac{4}{3}\Phi} \frac{4N_c^2(1-f^2)^2}{3f^2} \,,$$

$$C^2 = e^{\frac{4}{3}\Phi} \frac{4N_c^2(1-f^2)}{3f^2} \,, \qquad E^2 = \frac{16N_c^2(1-f^2)^2}{f^2} \left[\left(e^{\frac{2}{3}\Phi} \right)' \right]^2 \,, \tag{3.2.42}$$

$$\cos\alpha = f \,, \qquad\qquad N_f = \pm \frac{N_c(4f^2 - 1)}{3f} \,,$$

where $0 < f < 1$. The two-form flux is

$$
\begin{aligned}
F_{(2)} = &- N_c \Big(\sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi + \sin\tilde\theta \, \mathrm{d}\tilde\theta \wedge \mathrm{d}\tilde\varphi \Big) \\
&+ N_f \sin\psi_- \Big(\mathrm{d}\theta \wedge \mathrm{d}\tilde\theta + \sin\theta \sin\tilde\theta \, \mathrm{d}\varphi \wedge \mathrm{d}\tilde\varphi \Big) \\
&+ N_f \cos\psi_- \Big(\sin\tilde\theta \, \mathrm{d}\theta \wedge \mathrm{d}\tilde\varphi + \sin\theta \, \mathrm{d}\tilde\theta \wedge \mathrm{d}\varphi \Big) .
\end{aligned}
\tag{3.2.43}
$$

At this point, we notice that we can write the metric explicitly as a cone upon redefinition of the radial coordinate. We take

$$
r = \frac{4 N_c (1 - f^2)}{f} e^{2\Phi/3} ,
\tag{3.2.44}
$$

then $\mathrm{d}r^2 = E^2 \mathrm{d}\rho^2$ and the metric is

$$
\mathrm{d}s^2_{IIA} = e^{2\Phi/3} \Big(\mathrm{d}x^2_{1,3} + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{int} \Big) ,
\tag{3.2.45}
$$

where

$$
\begin{aligned}
\mathrm{d}\Omega^2_{int} = &\frac{1}{12} \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \right) + \frac{1}{12(1 - f^2)} \left[(\omega_1 - f \mathrm{d}\theta)^2 + (\omega_2 - f \sin\theta \, \mathrm{d}\varphi)^2 \right] \\
&+ \frac{1}{16(1 - f^2)} \left(\omega_3 - \cos\theta \, \mathrm{d}\varphi \right)^2 ,
\end{aligned}
\tag{3.2.46}
$$

with

$$
\begin{aligned}
\omega_1 &= \cos\psi_- \mathrm{d}\tilde\theta - \sin\psi_- \sin\tilde\theta \, \mathrm{d}\tilde\varphi , \\
\omega_2 &= \sin\psi_- \mathrm{d}\tilde\theta + \cos\psi_- \sin\tilde\theta \, \mathrm{d}\tilde\varphi , \\
\omega_3 &= -\mathrm{d}\psi_- + \cos\tilde\theta \, \mathrm{d}\tilde\varphi .
\end{aligned}
\tag{3.2.47}
$$

We can first notice that taking the limit $N_f \to 0$ for this solution gives a singular background. It indeed corresponds to taking $f \to \frac{1}{2}$, giving

$$
\begin{aligned}
\mathrm{d}s^2_{N_f \to 0} = &\frac{r}{6 N_c} \bigg(\mathrm{d}x^2_{1,3} + \mathrm{d}r^2 + \frac{r^2}{12} \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \right) \\
&+ \frac{r^2}{9} \left[\left(\omega_1 - \frac{1}{2}\mathrm{d}\theta \right)^2 + \left(\omega_2 - \frac{1}{2}\sin\theta \mathrm{d}\varphi \right)^2 \right] + \frac{r^2}{12} \left(\omega_3 - \cos\theta \mathrm{d}\varphi \right)^2 \bigg) .
\end{aligned}
\tag{3.2.48}
$$

Secondly, we have quantisation of the two-form colour flux, which is necessary for the gauge/string duality.

The interpretation of the additional flavour terms to the flux is not clear. A

look at the solution prompts us to suspect that the interpretation of the sources as being due to flavour branes is more straightforward if one reduces along ∂_φ. It should be interesting to consider the solution at hand in the context of conifold transitions though. Of course, this is just one solution of the BPS equations of this particular dimensional reduction. Other solutions might also present interesting properties. Anyway, we stop for now the study and interpretation of flavoured solutions, and turn back, in the following, to the problem of the M-theory lift.

3.3 Back to M-theory

Having studied the flavouring problem of D6-branes in the background (3.2.12) in the previous section, we have sufficient intuition to turn back towards the more general case of smeared D6-sources in M-theory. The discussion here is fairly generic and requires only the presence of the various G-structures as well as the overall topology $\mathbb{R}^{1,3} \times \mathcal{M}_7$.

3.3.1 Lifting the supersymmetric variations

The G_2-structure

Our considerations in the introduction about the loss of Ricci flatness prompted us to consider the appearance of intrinsic torsion. So we begin our attempt at finding a candidate M-theory lift with magnetic $C_{(1)}$ sources by studying the ten and eleven-dimensional G-structures. Originally, we were dealing with a G_2-holonomy manifold in eleven dimensions. Then, we reduced it to an $SU(3)$-structure in ten dimensions, following the equations (3.2.16). After this, we flavoured the theory, which changed the structure equations in ten dimensions (3.2.23) by replacing $\mathrm{d}C_{(1)}$ by $F_{(2)}$. However, after adding sources in Type IIA supergravity, we have $F_{(2)} \neq \mathrm{d}C_{(1)}$. So, if we try to lift back to eleven dimensions, we start from

$$
\begin{aligned}
0 &= \mathrm{d}J\,, & 0 &= -\frac{1}{2}\mathrm{d}(e^{-4\Phi/3}J \wedge J) - e^{-\Phi/3}(*_6\Psi) \wedge F_{(2)}\,. \\
0 &= \mathrm{d}(e^{-\Phi/3} *_6 \Psi)\,, & 0 &= \mathrm{d}(e^{-\Phi}\Psi) + J \wedge F_{(2)}\,,
\end{aligned}
\tag{3.3.1}
$$

When we then look at the G_2-structure we find, combining (3.2.22) with the above,

$$
\begin{aligned}
\mathrm{d}\phi &= -J \wedge \left(F_{(2)} - \mathrm{d}C_{(1)}\right)\,, \\
\mathrm{d} *_7 \phi &= e^{-\Phi/3}(*_6\Psi) \wedge \left(F_{(2)} - \mathrm{d}C_{(1)}\right)\,.
\end{aligned}
\tag{3.3.2}
$$

So, sources in Type IIA supergravity translate, in eleven dimensions, into the loss of G_2-holonomy and the appearance of torsion proportional to $F_{(2)} - dC_{(1)} = G_{(2)}$.

The supersymmetric variations

The previous section gave a first confirmation of our suspicion that geometric torsion should allow us to accommodate for the sources in M-theory. This suggests that all geometric quantities, such as covariant derivatives and curvature tensors, should be replaced by their torsion-modified relatives. Simplest among these is the covariant derivative, which makes an explicit appearance in the eleven-dimensional supergravity variation $\delta_{\hat{\epsilon}} \hat{\psi}_M = D_M \hat{\epsilon}$, which yields the Type IIA supergravity variations upon KK reduction. In Appendix 3.A, we therefore study how this equation and its Kaluza-Klein reduction change upon inclusion of a torsion tensor[5] $\hat{\tau}$

$$\delta_{\hat{\epsilon}} \hat{\psi}_M = \partial_M \hat{\epsilon} + \frac{1}{4} \hat{\omega}_{MAB} \hat{\Gamma}^{AB} \hat{\epsilon} + \frac{1}{4} \hat{\tau}_{MAB} \Gamma^{AB} \hat{\epsilon} \equiv D_M^{(\tau)} \epsilon. \qquad (3.3.3)$$

The result is given in (3.A.20) and we proceed by investigating what constraints we have to impose on $\hat{\tau}_{MAB}$ in order for the lower-dimensional variations to include magnetic sources.

Now from the form of the dilatino variation (Einstein frame),

$$\delta_{\epsilon} \lambda = \frac{3}{16} \frac{1}{\sqrt{2}} e^{\frac{3}{4}\Phi} (dC_{bc} + 2e^{-\frac{3}{2}\Phi} \hat{\tau}_{zbc}) \Gamma^{bc} \epsilon + \frac{\sqrt{2}}{4} (\partial_b \Phi + \frac{3}{2} e^{-\frac{3}{4}\Phi} \hat{\tau}_{zbz}) \Gamma^b \Gamma^{11} \epsilon, \qquad (3.3.4)$$

it follows that we have to demand $\hat{\tau}_{zaz} = 0$ and $\hat{\tau}_{zbc} = \frac{1}{2} e^{\frac{3}{2}\Phi} G_{bc}$, as (3.3.4) then takes the form

$$\delta_{\epsilon} \lambda = \frac{3}{16} \frac{1}{\sqrt{2}} e^{\frac{3}{4}\Phi} F_{bc} \Gamma^{bc} \epsilon + \frac{\sqrt{2}}{4} \partial_b \Phi \Gamma^b \Gamma^{11} \epsilon, \qquad (3.3.5)$$

with the two-form now no longer closed, $F_{(2)} = dC_{(1)} + G_{(2)}$.

Substituting $\hat{\tau}_{zaz}$ and $\hat{\tau}_{zbc}$ into the gravitino variation of (3.A.20) we see that if we impose

$$\hat{\tau}_{zaz} = 0, \qquad\qquad\qquad \hat{\tau}_{zbc} = \frac{e^{\frac{3}{2}\Phi}}{2} G_{bc},$$

---

[5]Of course, once we include the torsion and proceed from $D_M \hat{\epsilon}$ to $D_M^{(\tau)} \hat{\epsilon}$, it is not certain whether this defines a supersymmetric variation of a supergravity theory. What we do know for certain however – and show in the following – is that the naive dimensional reduction of the usual eleven-dimensional supersymmetric variation does not yield the correct Type IIA one and that (3.3.3) gives a first-order differential equation on the spinor that does reduce to the correct equations. With this in mind, we write $\delta_{\hat{\epsilon}} \hat{\psi}_M = D_M^{(\tau)} \epsilon$.

$$\hat{\tau}_{\mu bc} = \frac{e^{\frac{3}{2}\Phi}}{2}C_\mu G_{bc}\,, \qquad\qquad \hat{\tau}_{\mu bz} = -\frac{e^{\frac{3}{4}\Phi}}{2}G_{\mu b}\,, \qquad (3.3.6)$$

the gravitino variation turns also to the desired form

$$\delta_\epsilon \psi_\mu = \partial_\mu \epsilon + e_\mu^a \frac{1}{4}\omega_{abc}\Gamma^{bc}\epsilon + \frac{1}{64}e^{\frac{3}{4}\Phi}e_\mu^a F_{cd}\left(\eta_{ab}\Gamma^{bcd} - 14\delta_a^c\Gamma^d\right)\Gamma^{11}\epsilon\,. \qquad (3.3.7)$$

Equations (3.3.5) and (3.3.7) are important results. If one performs a KK reduction of the original supergravity variation without torsion, $\delta_{\hat{\epsilon}}\hat{\psi}_M = D_M\hat{\epsilon}$, one obtains supergravity variations including $dC_{(1)}$, yet not $G_{(2)} = F_{(2)} - dC_{(1)}$. By adding the torsion term to the eleven-dimensional supergravity variation, we are able to directly derive the ten-dimensional variations with $F_{(2)}$ instead of $dC_{(1)}$. Looking back at (3.3.3), it is fair to say that the spin connection $\hat{\omega}_{MAB}$ contains $dC_{(1)}$, while the torsion carries the $G_{(2)}$ term necessary to complete $F_{(2)}$. The right-hand side of (3.3.3) is constituted of two parts. The first two terms are the ones coming from the lift of the Type IIA part, and are exactly the terms present in eleven-dimensional supergravity. The last term, which is the only one involving the torsion, corresponds to the lift of the contribution of the sources to the ten-dimensional supergravity variations. Thus, it seems that, mimicking what happens in ten dimensions, we are in presence of the usual eleven-dimensional supergravity plus some sources.

Using the torsion-modified covariant derivative for spinors (3.3.3), we can also define such an operator $\nabla^{(\tau)}$ for tensors. The relevant connection coefficients Γ are

$$\Gamma^K{}_{LM} = \left\{{}^K_{LM}\right\} + K^K{}_{LM}\,,$$
$$K_{AMB} = \hat{\tau}_{MAB}\,, \qquad\qquad (3.3.8)$$

where $\left\{{}^K_{LM}\right\}$ is the Levi-Civita connection. With the help of $\nabla^{(\tau)}$, we can rewrite equations (3.3.2) as

$$\nabla^{(\tau)}_M \phi = 0\,,$$
$$\nabla^{(\tau)}_M (*_7\phi) = 0\,. \qquad\qquad (3.3.9)$$

One should remember that the original BPS equations could be written geometrically as $\nabla_M\phi = 0$ and $\nabla_M *_7\phi = 0$, yet that these ceased to be valid once we included the sources in ten dimensions – as we discussed in Section 3.3.1. Equations (3.3.9) show however that these geometric BPS equations remain formally invariant once we include torsion.

3.3.2 The equations of motion

We shall finally turn to the search for equations of motion in M-theory that reduce to the source-modified second-order equations in Type IIA supergravity as given in equation (3.2.27). To find these equations, we actually reverse the integrability argument that allowed us to consider the first instead of the second-order equations in Sections 3.2.2 and 3.2.3.

To get an idea of what we are about to do, let us briefly come back to the simple case without any flavours or sources. The Bianchi identities are the usual ones, the equation of motion is simple Ricci flatness, $\hat{R}_{MN} = 0$, and the G_2-structure form is closed and co-closed. Thus the latter satisfies $\nabla_M \phi = 0$. Taking the commutator

$$
\begin{aligned}
0 &= [\nabla_K, \nabla_L] \phi_{MNP} \\
&= -\hat{R}^S{}_{MKL} \phi_{SNP} - \hat{R}^S{}_{NKL} \phi_{MSP} - \hat{R}^S{}_{PKL} \phi_{MNS} \,.
\end{aligned}
\tag{3.3.10}
$$

Upon contraction of (3.3.10) with ϕ, we find[6]

$$
0 = 2\hat{R}_{KL} + \hat{R}_{MNPL}(*_7\phi)_K{}^{MNP} \,.
\tag{3.3.11}
$$

In the absence of torsion, $\hat{R}_{MNPL}(*_7\phi)_K{}^{MNP} = 0$, due to the well-known symmetries satisfied by the Riemann tensor,

$$
\begin{aligned}
\hat{R}_{K[LMN]} &= 0 \,, \\
\hat{R}_{KLMN} &= \hat{R}_{MNKL} = -\hat{R}_{MNLK} \,.
\end{aligned}
\tag{3.3.12}
$$

Therefore, our space-time is Ricci flat and the equations of motion are satisfied.

After this brief digression, we return to the original problem. Our aim is to find a suitable equation of motion in M-theory, that reduces to (3.2.27) upon dimensional reduction. For consistency this equation of motion needs to reduce to simple Ricci flatness in the limit where the ten-dimensional source density Ξ – equivalently the torsion $\hat{\tau}$ in M-theory – vanishes. Contrary to our considerations in the previous paragraph, the G_2-structure does no longer satisfy $\nabla_M \phi = 0$,

---

[6] As one can verify by direct calculation using (3.2.4), the G_2-structure satisfies

$$
\phi_{lmn}\phi^{kmn} = 6\delta_l^k \,,
$$
$$
\phi^{klp}\phi_{mnp} = (*_7\phi)_{mn}{}^{kl} + \delta_m^k \delta_n^l - \delta_n^k \delta_m^l \,,
$$

where k, l, m, n, p denote indices of the seven-dimensional internal manifold.

but instead satisfies $\nabla_M^{(\tau)} \phi = 0$. So we can once more consider the commutator of covariant derivatives. The identities of Footnote 6, used to derive (3.3.11), still hold, yet equations (3.3.12) do not, and we arrive at the main result of this chapter, the M-theory lift of the source-modified equations of motion:

$$0 = 2\hat{R}_{KL}^{(\tau)} + \hat{R}_{MNPL}^{(\tau)}(*_7\phi)_K{}^{MNP}, \qquad (3.3.13)$$

where $\hat{R}^{(\tau)}$ is the Riemann (Ricci) tensor in the presence of torsion.

As we pointed out before, the BPS equations in their geometric form, $\nabla_M^{(\tau)} \phi = 0$, are equivalent to those obtained from the supersymmetric spinor ϵ, $D_M^{(\tau)} \epsilon = 0$. Therefore we could have derived (3.3.13) also using (3.3.3). A commutator of covariant derivatives acting on the supersymmetric spinor yields

$$0 = \hat{R}_{CDML}^{(\tau)} \hat{\Gamma}^{CD} \hat{\epsilon}. \qquad (3.3.14)$$

We then contract with $\bar{\hat{\epsilon}} \hat{\Gamma}_K{}^M$ and make use of the identity

$$\Gamma^A \Gamma^B \Gamma^C \Gamma^D = \Gamma^{ABCD} + \eta^{AB}\Gamma^{CD} - \eta^{CB}\Gamma^{DA} + \eta^{CD}\Gamma^{AB} + \eta^{DA}\Gamma^{BC}$$
$$- \eta^{AC}\Gamma^{BC} - \eta^{BD}\Gamma^{AC} + \eta^{AB}\eta^{CD} - \eta^{AC}\eta^{BD} + \eta^{AD}\eta^{BC}. \qquad (3.3.15)$$

It follows that

$$0 = 2(\bar{\hat{\epsilon}}\hat{\epsilon})\hat{R}_{KL}^{(\tau)} + (\bar{\hat{\epsilon}}\hat{\Gamma}_K{}^{MNP}\hat{\epsilon})\hat{R}_{MNPL} + \mathcal{O}\left(\bar{\hat{\epsilon}}\hat{\Gamma}^{AB}\hat{\epsilon}\right). \qquad (3.3.16)$$

The assumptions made about the supersymmetric spinor $\hat{\epsilon}$ imply that there is a G_2-structure that can be expressed as

$$*_7\phi = \left(\bar{\hat{\epsilon}}\,\Gamma_{ABCD}\,\hat{\epsilon}\right)\hat{e}^{ABCD}. \qquad (3.3.17)$$

They also imply that all terms of the form $\bar{\hat{\epsilon}}\hat{\Gamma}^{AB}\hat{\epsilon}$ vanish. Hence (3.3.13) follows from (3.3.16).

The equations of motion (3.3.13) can be rewritten in a more typical and enlightening fashion, using the Einstein tensor:

$$\hat{R}_{KL} - \frac{1}{2}\hat{g}_{KL}\hat{R} = \hat{T}_{KL}, \qquad (3.3.18)$$

where $\hat{T}_{KL}$ is the energy-momentum tensor of the sources. It can be written in

terms of the torsion as

$$\hat{T}_{KL} = \nabla_L K^M{}_{MK} - \nabla_M K^M{}_{LK} + K^M{}_{LP} K^P{}_{MK} - K^M{}_{MP} K^P{}_{LK}$$

$$+ \frac{1}{2} \left(\nabla_L K_{MPN} - \nabla_P K_{MLN} + K_{MLQ} K^Q{}_{PN} - K_{MPQ} K^Q{}_{LN} \right) (*_7 \phi)_K{}^{MNP}$$

$$+ \frac{1}{2} \left(\nabla_L K_{MPN} - \nabla_P K_{MLN} + K_{MLQ} K^Q{}_{PN} - K_{MPQ} K^Q{}_{LN} \right) (*_7 \phi)_K$$

$$+ \frac{1}{2} \hat{g}_{KL} \left(\nabla_M K^M{}_Q{}^Q - \nabla_Q K^M{}_M{}^Q + K^M{}_{MP} K^P{}_Q{}^Q - K^M{}_{QP} K^P{}_M{}^Q \right)$$

$$+ \frac{1}{2} \hat{g}_{KL} \left(\nabla_P K_{MQN} + K_{MPR} K^R{}_{QN} \right) (*_7 \phi)^{QMNP} , \qquad (\text{3.19})$$

where K_{MNP} is the contorsion tensor (see (3.3.8)). From (3.3.13), we can see that
the Einstein equation we are proposing contains two terms: on the left-hand side,
one has the Einstein tensor one would get from varying the eleven-dimensional
supergravity action with no four-form flux; on the right-hand side, one has an
energy-momentum tensor that vanishes when the torsion is set to zero. When
the torsion vanishes, so does $\hat{T}$ and one recovers the M-theory Einstein equation.
Writing the equation in this form makes very clear the fact that the lift of Type
IIA supergravity with sources is eleven-dimensional supergravity supplemented by
some sources. Unfortunately, we were not able to find an action that would be
responsible for this energy-momentum tensor. To summarise, we claim that having
sources in ten dimensions corresponds to having an energy-momentum tensor in
eleven dimensions, of the form presented above.

To verify our claim, we now perform the explicit dimensional reduction of
(3.3.13), and show that we recover all the equations of motion of Type IIA su-
pergravity with sources. The calculations are – as so often in supergravity –
straightforward yet tedious. We found [73] quite helpful, yet not essential. No-
tice that, in the following, despite the fact that we dropped the superscript (τ) for
simplicity of notation, all hatted Riemann and Ricci tensor are considered in the
presence of torsion.

Let us start with the zz-component of (3.3.11). We find

$$\hat{R}_{zz} = -\frac{2}{3} e^{4\Phi} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} e^{-2\Phi} \partial^\mu \Phi) + \frac{1}{4} e^{4\Phi} F^2_{(2)} ,$$

$$(*_7 \phi)_z{}^{SPK} \hat{R}_{SPKz} = e^{3\Phi} (*_6 \Psi) \lrcorner dG_{(2)} , \qquad (3.3.20)$$

from which it follows that

$$0 = 2\hat{R}_{zz} + (*_7 \phi)_z{}^{SPK} \hat{R}_{SPKz}$$

$$0 = -\frac{4}{3}e^{4\Phi}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}e^{-2\Phi}\partial^\mu\Phi) + \frac{1}{2}e^{4\Phi}F^2_{(2)} + e^{3\Phi}(*_6\Psi)\lrcorner dG_{(2)}$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}e^{-2\Phi}\partial^\mu\Phi) - \frac{3}{8}F^2_{(2)} - \frac{3}{4}e^{-\Phi}(*_{10}\mathcal{K})\lrcorner\Xi. \qquad (3.3.21)$$

Here we used that $*_6\Psi = -*_{10}\mathcal{K}$ and $dG_{(2)} = dF_{(2)} = -\Xi$ (for simplicity, we reabsorbed the coefficient $2\kappa^2_{10}T_6$ in the definition of Ξ). We notice that we find exactly the source-modified ten-dimensional equation of motion for the dilaton as in (3.2.27).

Now we investigate the μz-component of (3.3.11). We get

$$\hat{R}_{\mu z} = -\frac{1}{2}e^{2\Phi}\nabla^\nu F_{\nu\mu} + C_\mu\hat{R}_{zz},$$

$$(*_7\phi)_\mu{}^{SPK}\hat{R}_{SPKz} = -\frac{1}{6}e_{a\mu}(*_7\phi)^{abcd}(dG)_{bcd} + C_\mu(e^{3\Phi}(*_6\Psi)\lrcorner dG_{(2)}). \qquad (3.3.22)$$

Now we have

$$\frac{1}{6}e_{a\mu}(*_7\phi)^{abcd}(dG)_{bcd} = \frac{1}{6}(*_6J)^{\mu bcd}(dG)_{bcd} = \frac{1}{12}\sqrt{-g_{(6)}}\epsilon^{\alpha\beta\mu\nu\rho\sigma}J_{\alpha\beta}(dG)_{\nu\rho\sigma}$$

$$= \frac{1}{12}\frac{1}{6!}*_6\left(dx^\mu \wedge J \wedge dG_{(2)}\right) = \frac{1}{12}\frac{1}{6!}*_6\left[dx^\mu \wedge d\left(J \wedge G_{(2)}\right)\right]$$

$$= 0, \qquad (3.3.23)$$

because supersymmetry tells us that $dJ = 0$ and $d(J \wedge G_{(2)}) = d(d\phi) = 0$. Thus

$$0 = 2\hat{R}_{\mu z} + (*_7\phi)_\mu{}^{SPK}\hat{R}_{SPKz}$$

$$= -e^{2\Phi}\nabla^\nu F_{\nu\mu} + 2C_\mu\hat{R}_{zz} - \frac{1}{6}e_{a\mu}(*_7\phi)^{abcd}(dG)_{bcd} + C_\mu(e^{3\Phi}(*_6\Psi)\lrcorner dG_{(2)})$$

$$= -e^{2\Phi}\nabla^\nu F_{\nu\mu} + C_\mu\left[2\hat{R}_{zz} + e^{3\Phi}(*_6\Psi)\lrcorner dG_{(2)}\right]. \qquad (3.3.24)$$

The term in square brackets is equal to the zz-component of (3.3.11) and the remaining part corresponds to the Maxwell equation for $F_{(2)}$.

The $z\nu$-component[7] of (3.3.11) gives

$$\hat{R}_{z\nu} = -\frac{1}{2}e^{2\Phi}\nabla^\rho F_{\rho\nu} + C_\nu\hat{R}_{zz} + \frac{2}{3}e^{2\Phi}(dC_{(1)} - F_{(2)})_{\nu\rho}\partial^\rho\Phi,$$

---

[7]One might suspect this to be identical to the μz-component. Due to the presence of torsion however, the Ricci tensor is no longer symmetric and one has to check this independently. Interestingly, the Kaluza-Klein reductions of μz and $z\nu$ are already different in the torsion-free case. Here the two differ by $F_{(2)} - dC_{(1)}$ however, which vanishes in source and torsion-free geometries.

$$(*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu} = e^{3\Phi}G_{\nu\beta}[F_{(2)}\lrcorner(*_6\Psi)]^{\beta} + C_{\nu}(e^{3\Phi}(*_6\Psi)\lrcorner dG_{(2)})$$

$$= \frac{4}{3}e^{2\Phi}G_{\nu\beta}\partial^{\beta}\Phi + C_{\nu}(e^{3\Phi}(*_6\Psi)\lrcorner dG_{(2)}), \qquad (3.3.25)$$

with $d\Phi = \frac{3}{4}e^{\Phi}F_{(2)}\lrcorner(*_6\Psi)$ due to supersymmetry. Putting things together

$$0 = 2\hat{R}_{z\nu} + (*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu}$$

$$= -e^{2\Phi}\nabla^{\rho}F_{\rho\nu} + C_{\nu}\left[2\hat{R}_{zz} + e^{3\Phi}(*_6\Psi)\lrcorner dG_{(2)}\right]. \qquad (3.3.26)$$

This agrees with the μz-component. Let us finally look at the $\mu\nu$-component of (3.3.11). We have

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + 2\nabla_{\mu}\partial_{\nu}\Phi - \frac{e^{2\Phi}}{2}(F_{\mu\rho}(dC)_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{(2)}^2) - \frac{1}{2}C_{\nu}\nabla^{\rho}F_{\rho\mu} + C_{\mu}\hat{R}_{z\nu} - \frac{e^{-2\Phi}}{2}g_{\mu\nu}\hat{R}_{zz}, \qquad (3.3.27)$$

and

$$(*_7\phi)_{\mu}{}^{SPK}\hat{R}_{SPK\nu} = C_{\mu}[(*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu}] + \frac{4}{3}e^{\Phi}(*_6\Psi)_{\mu}{}^{cd}G_{\nu d}\partial_c\Phi$$

$$- \frac{1}{6}C_{\nu}e^{2\Phi}e_{a\mu}(*_7\phi)^{abcd}(dG)_{bcd} - e^{\Phi}(*_6\Psi)_{\mu}{}^{cd}\nabla_d G_{\nu c} \qquad (3.3.28)$$

$$+ \frac{1}{2}e^{2\Phi}e_{a\mu}(*_7\phi)^{abcd}G_{\nu d}F_{cb} - \frac{1}{2}e^{\Phi}(*_6\Psi)_{\mu}{}^{cd}\nabla_{\nu}G_{cd}.$$

Let us first notice that

$$(*_6\Psi)_{\mu}{}^{cd}(\nabla_d G_{\nu c} + \frac{1}{2}\nabla_{\nu}G_{cd}) = \frac{1}{2}(*_6\Psi)_{\mu}{}^{cd}(dG)_{\nu cd}. \qquad (3.3.29)$$

Then, from a previous computation, we know that

$$e_{a\mu}(*_7\phi)^{abcd}(dG)_{bcd} = 0. \qquad (3.3.30)$$

Here are formulas that are useful in the following calculations:

$$(*_6\Psi)^f{}_{ab}(*_6\Psi)_{cd}{}^f = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + J_{ac}J_{db} + J_{ad}J_{bc},$$

$$\frac{1}{2}(J \wedge J)_{abcd} = J_{ab}J_{cd} + J_{ac}J_{db} + J_{ad}J_{bc}, \qquad (3.3.31)$$

and, once again,

$$\partial_a\Phi = \frac{3}{4}e^{\Phi}(F_{(2)}\lrcorner(*_6\Psi))_a = \frac{3}{8}e^{\Phi}F^{bc}(*_6\Psi)_{bca}. \qquad (3.3.32)$$

So

$$
\begin{aligned}
(*_7\phi)^{abcd}F_{cb} &= -\frac{1}{2}(J \wedge J)^{abcd}F_{cb} \\
&= F_{cb}(J^{ab}J^{cd} + J^{ac}J^{db} + J^{ad}J^{bc}) \\
&= 2J^{ab}F_{bc}J^{cd},
\end{aligned}
\tag{3.3.33}
$$

because supersymmetry dictates that $F_{(2)}\lrcorner J = 0$. And we have

$$
\begin{aligned}
(*_6\Psi)_\mu{}^{cd}\partial_c\Phi &= -\frac{3}{8}e^\Phi F^{fg}(*_6\Psi)_\mu{}^{dc}(*_6\Psi)_{fgc} \\
&= -\frac{3}{4}e^\Phi(F_\mu{}^d + J_\mu{}^f F_{fg}J^{gd}).
\end{aligned}
\tag{3.3.34}
$$

So if we now put everything together, we get

$$
\begin{aligned}
(*_7\phi)_\mu{}^{SPK}\hat{R}_{SPK\nu} &= C_\mu[(*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu}] + e^{2\Phi}e_{a\mu}G_{\nu d}J^{ab}F_{bc}J^{cd} \\
&\quad - \frac{1}{2}e^\Phi(*_6\Psi)_\mu{}^{cd}(\mathrm{d}G)_{\nu cd} - e^{2\Phi}G_{\nu d}(F_\mu{}^d + J_\mu{}^f F_{fg}J^{gd}) \\
&= C_\mu[(*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu}] - \frac{1}{2}e^\Phi(*_6\Psi)_\mu{}^{cd}(\mathrm{d}G)_{\nu cd} - e^{2\Phi}F_\mu{}^d G_{\nu d}.
\end{aligned}
\tag{3.3.35}
$$

So looking finally at the whole picture

$$
\begin{aligned}
0 &= 2\hat{R}_{\mu\nu} + (*_7\phi)_\mu{}^{SPK}\hat{R}_{SPK\nu} \\
&= 2R_{\mu\nu} + 4\nabla_\mu\partial_\nu\Phi - e^{2\Phi}(F_{\mu\rho}(\mathrm{d}C)_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{(2)}^2) - C_\nu\nabla^\rho F_{\rho\mu} - e^{2\Phi}F_\mu{}^d G_{\nu d} \\
&\quad - e^{-2\Phi}g_{\mu\nu}\hat{R}_{zz} + C_\mu[(*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu}] - \frac{1}{2}e^\Phi(*_6\Psi)_\mu{}^{cd}(\mathrm{d}G)_{\nu cd} + C_\mu 2\hat{R}_{z\nu} \\
&= 2R_{\mu\nu} + 4\nabla_\mu\partial_\nu\Phi - e^{2\Phi}(F_{\mu\rho}(\mathrm{d}C_{(1)} + G_{(2)})_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{(2)}^2) - C_\nu\nabla^\rho F_{\rho\mu} \\
&\quad + C_\mu[2\hat{R}_{z\nu} + (*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu}] - e^{-2\Phi}g_{\mu\nu}[\hat{R}_{zz} + \frac{1}{2}e^{3\Phi}(*_6\Psi)\lrcorner \mathrm{d}G_{(2)}] \\
&\quad - \frac{1}{2}e^\Phi(*_{10}\mathcal{K})_\mu{}^{\rho\sigma}\Xi_{\nu\rho\sigma} + \frac{1}{2}e^{-2\Phi}g_{\mu\nu}e^{3\Phi}(*_6\Psi)\lrcorner \mathrm{d}G_{(2)},
\end{aligned}
\tag{3.3.36}
$$

which gives

$$
\begin{aligned}
0 &= 2R_{\mu\nu} + 4\nabla_\mu\partial_\nu\Phi - e^{2\Phi}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{(2)}^2) \\
&\quad - \frac{1}{2}e^\Phi\left((*_{10}\mathcal{K})_\mu{}^{\rho\sigma}\Xi_{\nu\rho\sigma} - g_{\mu\nu}(*_{10}\mathcal{K})\lrcorner\Xi\right) \\
&\quad - C_\nu\nabla^\rho F_{\rho\mu} + C_\mu[2\hat{R}_{z\nu} + (*_7\phi)_z{}^{SPK}\hat{R}_{SPK\nu}] \\
&\quad - \frac{e^{-2\Phi}}{2}g_{\mu\nu}[2\hat{R}_{zz} + e^{3\Phi}(*_{10}\mathcal{K})\lrcorner\Xi],
\end{aligned}
\tag{3.3.37}
$$

where we recognise the first two lines of this equation as being the Einstein equa-

tion of Type IIA supergravity with sources, and the rest vanishes thanks to other components of (3.3.11). This completes the reduction of the eleven-dimensional Einstein equations to the Type IIA supergravity equations of motion.

To summarise, in this section we showed that the equation of motion of eleven-dimensional supergravity with torsion (3.3.13), which is given to us by integrability, reduces to the source-modified Type IIA supergravity equations of motion (3.2.27). It thus shows that adding torsion to eleven-dimensional supergravity reduces to adding smeared D6-sources in Type IIA supergravity.

3.4 Conclusions

In this chapter, we have been interested in two related issues: the addition of D6-branes as smeared sources to a Type IIA supergravity background, and the lifting of such a system to eleven dimensions. We considered these in the context of one-eighth BPS solutions of the form $\mathbb{R}^{1,3} \times \mathcal{M}$, a fact represented by the presence of a G_2 or $SU(3)$-structure.

Concerning the problem of the M-theory lift, we showed that ordinary eleven-dimensional supergravity cannot accommodate for the presence of the additional sources and argued that a possible solution might lie in the inclusion of geometric torsion. While our argument was founded on the observed loss of Ricci flatness in the higher-dimensional theory, we were able to show by explicit calculation that the supersymmetry variations take the required form upon addition of torsion. Moreover, the torsion must take the form (3.3.6), related to the distribution Ξ of the sources in the reduced theory. Subsequently, we derived a set of second-order equations that could be the equations of motion of some eleven-dimensional supergravity with torsion, and proved that they reduce to the Type IIA equations of motion with smeared D6-branes. As we pointed out, this work is not in contradiction with the uniqueness of supergravity in eleven dimensions, because we are only considering a theory that preserves four supercharges. We did not of course show that there is a well-defined theory in eleven dimensions that is supersymmetric and has the field content of both eleven-dimensional supergravity as well as of the additional torsion. One should not forget however, that we do not study the uplift of S_{IIA}, which is well known, but of

$$S = S_{\text{IIA}} + S_{\text{D6-sources}} \,. \tag{3.4.1}$$

The problem was first addressed in [74], whose authors found a seven-dimensional gauged-sigma-model action that reduces to the DBI term of the D6-brane. They were unable to find a suitable uplift of the Wess-Zumino term however. While this chapter does not solve the problem in the sense of [74], it does succeed in lifting the ten-dimensional equations of motion to pure eleven-dimensional geometry. The question whether the results are just an accidental rewriting of Type IIA supergravity dynamics in higher-dimensional notation or do actually point to a higher-dimensional supersymmetric theory that includes torsion is still unanswered.

While there is a long history of the uses of torsion in the context of string theory, the torsion used in papers such as [75] and [48] is related to the presence of fluxes, not of sources. Therefore, the addition of further torsion is a rather unorthodox concept. So it is necessary to wonder if we would not have been able to solve the problem at hand with simpler methods. As mentioned before, our argument was based on the loss of Ricci flatness in eleven dimensions. One might guess that it is possible to use the four-form in M-theory, $\hat{F}_{(4)}$, to obtain a suitable energy-momentum tensor to supplement the Einstein equations. This however leads to four and three-form flux in Type IIA supergravity, in contradiction with our results of Section 3.2. Another possibility would be to use the KK monopole action of [74]. There, the authors constructed a gauged-sigma-model action that is the dimensional uplift of the DBI term of a D6-brane. Using this, one could try to lift the action (3.1.7) to M-theory. Yet, considered in connection with the standard Kaluza-Klein mechanism, (3.1.7) is an action in terms of $dC_{(1)}$, not $F_{(2)}$. So, even if one were able to lift the brane contribution to (3.1.7), the supergravity part would still be lacking the source contribution. Still, it might be interesting to try to match the sigma-model action [74] with the inclusion of torsion.

The other problem studied in this chapter is the construction of a gravity dual to $\mathcal{N} = 1$, $SU(N_c)$ super Yang-Mills with flavours. We addressed this in Section 3.2. There we found a system of first-order BPS equations that describes the addition of D6-sources to the Type IIA supergravity background (3.2.12). At the end of Section 3.2, we presented a family of exact solutions. The detailed study of these, especially concerning the physics of their gauge-theory dual, has not been made and could be of interest, as well as finding other solutions.

In the next chapter, we once again use an $SU(3)$-structure and the smearing technique to construct new supergravity solutions, that we believe are dual to field theories exhibiting a Kutasov-like duality.

3.A Supergravity variations with torsion

We review the dimensional reduction of the supersymmetric variations – with an additional torsion term – from eleven to ten-dimensional supergravity. Conceptually, we follow [58], but our conventions are slightly different. We assume a space-time with topology $\mathcal{M}_{10} \times S^1$ and label the eleventh coordinate as z. Naturally, all fields are independent of z. Further assuming the eleven-dimensional background to be purely gravitational, we only need to consider the variation of the gravitino,

$$\delta_{\hat{\epsilon}} \hat{\psi}_M = \partial_M \hat{\epsilon} + \frac{1}{4} \hat{\omega}_{MAB} \Gamma^{AB} \hat{\epsilon} + \frac{1}{4} \hat{\tau}_{MAB} \hat{\Gamma}^{AB} \hat{\epsilon} \,, \tag{3.A.2}$$

which we have modified by the presence of the torsion term $\hat{\tau}$. As in Section 3.2.1, we take the vielbein to be in Scherk-Schwarz gauge (3.2.8).

We shall perform the reduction of (3.A.2) step by step and our first aim shall be the reduction of the spin connection

$$\hat{\omega}_{ABC} = \frac{1}{2} \left(\hat{\Omega}_{CAB} - \hat{\Omega}_{BAC} - \hat{\Omega}_{ABC} \right) \,, \tag{3.A.3}$$

with the objects of anholomorphicity defined as

$$\hat{\Omega}_{ABC} = \left(\partial_M \hat{e}_N^K - \partial_N \hat{e}_M^K \right) \hat{\eta}_{KA} \hat{E}_B^N \hat{E}_C^M \,. \tag{3.A.4}$$

Then

$$\hat{\omega}_{zbc} = + \frac{e^{\frac{4}{3}\Phi}}{2} (\mathrm{d}C)_{bc} \,, \qquad \hat{\omega}_{abc} = \frac{e^{\frac{1}{3}\Phi}}{3} \left(\eta_{ab} \partial_c \Phi - \eta_{ac} \partial_b \Phi \right) + e^{\frac{1}{3}\Phi} \omega_{abc} \,,$$

$$\hat{\omega}_{abz} = - \frac{e^{\frac{4}{3}\Phi}}{2} (\mathrm{d}C)_{ab} \,, \qquad \hat{\omega}_{zaz} = \frac{2}{3} e^{\frac{1}{3}\Phi} \partial_a \Phi \,. \tag{3.A.5}$$

Note that we use $\mathrm{d}C_{\mu\nu}$ instead of $F_{\mu\nu}$ as we are anticipating the inclusion of sources such that $F_{(2)}$ is no longer exact.

Turning to the gravitino, one could make an ansatz

$$\hat{\psi}_M = \left(e^{m\Phi} \psi_\mu, e^{n\Phi} \lambda \right) \,, \tag{3.A.6}$$

and

$$\hat{\epsilon} = e^{l\Phi} \epsilon \,, \tag{3.A.7}$$

with $l, m, n \in \mathbb{C}$. Yet, for reasons that are obvious later, we need to consider linear

combinations such as $\hat{\psi}_\mu = e^{m\Phi}\psi_\mu + e^{n\Phi}\Gamma_\mu\lambda + e^{p\Phi}\Gamma_\mu\Gamma^{11}\lambda$.

We begin with the covariant derivative of the supersymmetric spinor, looking first at the vector components:

$$e^{-l\Phi}\hat{D}_\mu\hat{\epsilon} = (l\partial_\mu\Phi\epsilon + \partial_\mu\epsilon) + e_\mu^a\left[\frac{1}{4}\omega_{abc} + \frac{1}{12}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\right]\Gamma^{bc}\epsilon$$

$$-\frac{1}{4}e^\Phi e_\mu^a(dC)_{ab}\Gamma^b\Gamma^{11}\epsilon + \left(\frac{1}{8}e^{2\Phi}C_\mu(dC)_{bc}\Gamma^{bc} + \frac{1}{3}e^\Phi C_\mu\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon$$

$$+\frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon\,.$$

$$(3.A.8)$$

The scalar component satisfies

$$e^{-l\Phi}\hat{D}_z\hat{\epsilon} = \frac{e^{2\Phi}}{8}(dC)_{bc}\Gamma^{bc}\epsilon + \frac{e^\Phi}{3}\partial_b\Phi\Gamma^b\Gamma^{11}\epsilon + \frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11}\epsilon\,. \qquad (3.A.9)$$

Equations (3.A.8) and (3.A.9) hold in string frame. To convert to Einstein frame, we need to recall that the gamma matrices are defined in tangent space, from which it follows that only the curved-space gamma matrices are affected by Weyl transformations. For a generic Weyl transformation, we have

$$e_\mu^a \mapsto e^{\delta\Phi}e_\mu^a\,, \qquad\qquad \Omega_{abc} \mapsto e^{-\delta\Phi}\Omega_{abc} + e^{-\delta\Phi}\delta(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\,,$$

$$E_a^\mu \mapsto e^{-\delta\Phi}E_a^\mu\,, \qquad\qquad \omega_{abc} \mapsto e^{-\delta\Phi}\omega_{abc} - \delta e^{-\delta\Phi}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\,,$$

$$\partial_a \mapsto e^{-\delta\Phi}\partial_a\,, \qquad (dA)_{a_1\ldots a_p} \mapsto e^{-p\delta\Phi}(dA)_{a_1\ldots a_p}\,, \qquad\qquad (3.A.10)$$

$$\Gamma^a \mapsto \Gamma^a\,, \qquad\qquad \eta_{ab} \mapsto \eta_{ab}\,,$$

$$\hat{\tau}_{\mu bc} \mapsto \hat{\tau}_{\mu bc}\,.$$

This leads to having, with $(e^S)_\mu^a = e^{\frac{1}{4}\Phi}(e^E)_\mu^a$, $\delta = \frac{1}{4}$,

$$e^{-l\Phi}\hat{D}_\mu\hat{\epsilon} = (l\partial_\mu\Phi\epsilon + \partial_\mu\epsilon) + e_\mu^a\left[\frac{1}{4}\omega_{abc} + \frac{1}{48}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\right]\Gamma^{bc}\epsilon$$

$$-\frac{1}{4}e^{\frac{3}{4}\Phi}e_\mu^a(dC)_{ab}\Gamma^b\Gamma^{11}\epsilon + \left(\frac{1}{8}e^{\frac{3}{2}\Phi}C_\mu(dC)_{bc}\Gamma^{bc} + \frac{1}{3}e^{\frac{3}{4}\Phi}C_\mu\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon$$

$$+\frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon\,,$$

$$e^{-l\Phi}\hat{D}_z\hat{\epsilon} = \frac{e^{\frac{3}{2}\Phi}}{8}(dC)_{bc}\Gamma^{bc}\epsilon + \frac{e^{\frac{3}{4}\Phi}}{3}\partial_b\Phi\Gamma^b\Gamma^{11}\epsilon + \frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11}\epsilon\,.$$

$$(3.A.11)$$

One needs to compare (3.A.8) and (3.A.9) or (3.A.11) respectively to the su-

persymmetric variations of the ansatz (3.A.6)

$$\hat{D}_\mu \hat{\epsilon} = \delta_{\hat{\epsilon}} \hat{\psi}_\mu = e^{m\Phi} \left(m \delta_{\hat{\epsilon}} \Phi \psi_\mu + \delta_{\hat{\epsilon}} \psi_\mu \right) = e^{m\Phi} \delta_{\hat{\epsilon}} \psi_\mu \,,$$
$$\hat{D}_z \hat{\epsilon} = \delta_{\hat{\epsilon}} \hat{\psi}_z = e^{n\Phi} \left(n \delta_{\hat{\epsilon}} \Phi \lambda + \delta_{\hat{\epsilon}} \lambda \right) = e^{n\Phi} \delta_{\hat{\epsilon}} \lambda \,. \tag{3.A.12}$$

The last equalities follow from the fact that we assume the spinor fields to vanish. However, the resulting variations explicitly depend on the gauge potential $C_{(1)}$. We therefore replace the original ansatz (3.A.6) with

$$\psi_\mu = \hat{\psi}_\mu - x_2 e_\mu^a \eta_{ab} \Gamma^b \Gamma^{11} \hat{\psi}_z - x_3 C_\mu \hat{\psi}_z \,, \qquad \lambda = x_1 \hat{\psi}_z \,, \qquad \hat{\epsilon} = e^{l\Phi} \epsilon \,, \tag{3.A.13}$$

which amounts to a field redefinition in ten dimensions. If one were to work properly, one would have to perform the dimensional reduction of the action as well in order to make sure that the fermion terms have the proper normalisations. The supersymmetric variations of (3.A.13) are

$$\delta_\epsilon \psi_\mu = \delta_\epsilon \hat{\psi}_\mu - x_2 e_\mu^a \eta_{ab} \Gamma^b \Gamma^{11} \delta_\epsilon \hat{\psi}_z - x_3 C_\mu \delta_\epsilon \hat{\psi}_z$$
$$= e^{-l\Phi} \hat{D}_\mu \hat{\epsilon} - x_2 e_\mu^a \eta_{ab} \Gamma^b \Gamma^{11} e^{-l\Phi} \hat{D}_z \hat{\epsilon} - x_3 C_\mu e^{-l\Phi} \hat{D}_z \hat{\epsilon} \,, \tag{3.A.14}$$
$$\delta_\epsilon \lambda = x_1 e^{-l\Phi} \hat{D}_z \hat{\epsilon} \,.$$

Note that the variations of the bosonic fields all vanish, as we have set the fermions explicitly to zero. Our aim is to compare (3.A.14) with the Type IIA Einstein-frame supersymmetric variations as taken from [76]:

$$\delta \lambda = \frac{\sqrt{2}}{4} \partial_\mu \Phi \Gamma^\mu \Gamma^{11} \epsilon + \frac{3}{16} \frac{1}{\sqrt{2}} e^{\frac{3}{4}\Phi} (\mathrm{d}C)_{\mu_1 \mu_2} \Gamma^{\mu_1 \mu_2} \epsilon \,, \tag{3.A.15a}$$

$$\delta \psi_\mu = D_\mu \epsilon + \frac{1}{64} e^{\frac{3}{4}\Phi} (\mathrm{d}C)_{\mu_1 \mu_2} \left(\Gamma_\mu{}^{\mu_1 \mu_2} - 14 \delta_\mu^{\mu_1} \Gamma^{\mu_2} \right) \Gamma^{11} \epsilon \,. \tag{3.A.15b}$$

Before evaluating (3.A.14), we calculate

$$x_2 e_\mu^a \eta_{ab} \Gamma^b \Gamma^{11} e^{-l\Phi} \hat{D}_z \hat{\epsilon} = x_2 \frac{1}{8} e^{\frac{3}{2}\Phi} e_\mu^a \eta_{ab} (\mathrm{d}C)_{cd} (\Gamma^{bcd} + 2\eta^{bc} \Gamma^d) \Gamma^{11} \epsilon$$

$$- x_2 \frac{1}{3} e^{\frac{3}{4}\Phi} e_\mu^a \partial_a \Phi \epsilon - x_2 \frac{1}{6} e^{\frac{3}{4}\Phi} e_\mu^a (\eta_{ab} \partial_c \Phi - \eta_{ac} \partial_b \Phi) \Gamma^{bc} \epsilon$$

$$- \frac{1}{2} x_2 e_\mu^a \eta_{ab} \hat{\tau}_{zaz} \epsilon - \frac{1}{2} x_2 e_\mu^a \eta_{ab} \hat{\tau}_{zcz} \Gamma^{bc} \epsilon$$

$$+ \frac{1}{4} x_2 e_\mu^a \hat{\tau}_{zcd} (\eta_{ab} \Gamma^{bcd} + 2\delta_a^c \Gamma^d) \Gamma^{11} \epsilon \,. \tag{3.A.16}$$

where we used

$$\Gamma^a\Gamma^b = \Gamma^{ab} + \eta^{ab}, \qquad \Gamma^a\Gamma^b\Gamma^c = \Gamma^{abc} + \eta^{ab}\Gamma^c - \eta^{ca}\Gamma^b + \eta^{bc}\Gamma^a. \qquad (3.A.17)$$

Putting things together, we use equations (3.A.11) and (3.A.14) to find

$$\begin{aligned}
\delta_\epsilon\psi_\mu = {}& (l\partial_\mu\Phi\epsilon + \partial_\mu\epsilon) + e_\mu^a\left[\frac{1}{4}\omega_{abc} + \frac{1}{48}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\right]\Gamma^{bc}\epsilon \\
& - \frac{1}{4}e^{\frac{3}{4}\Phi}e_\mu^a(dC)_{ab}\Gamma^b\Gamma^{11}\epsilon - x_2\frac{1}{8}e^{\frac{3}{2}\Phi}e_\mu\eta_{ab}(dC)_{cd}(\Gamma^{bcd} + 2\eta^{bc}\Gamma^d)\Gamma^{11}\epsilon \\
& + x_2\frac{1}{6}e^{\frac{3}{4}\Phi}e_\mu^a(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\Gamma^{bc}\epsilon - x_3 C_\mu(\frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc} + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11})\epsilon \\
& - x_3\left(\frac{e^{\frac{3}{2}\Phi}}{8}e_\mu^a C_a(dC)_{bc}\Gamma^{bc} + \frac{e^{\frac{3}{4}\Phi}}{3}e_\mu^a C_a\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon + x_2\frac{1}{3}e^{\frac{3}{4}\Phi}e_\mu^a\partial_a\Phi\epsilon \\
& + \frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon + \left(\frac{1}{8}e^{\frac{3}{2}\Phi}C_\mu(dC)_{bc}\Gamma^{bc} + \frac{1}{3}e^{\frac{3}{4}\Phi}C_\mu\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon \\
& + \frac{1}{2}x_2 e_\mu^a\eta_{ab}\hat{\tau}_{zaz}\epsilon + \frac{1}{2}x_2 e_\mu^a\eta_{ab}\hat{\tau}_{zcz}\Gamma^{bc}\epsilon - \frac{1}{4}x_2 e_\mu^a\hat{\tau}_{zcd}(\eta_{ab}\Gamma^{bcd} + 2\delta_a^c\Gamma^d)\Gamma^{11}\epsilon, \\
\delta_\epsilon\lambda = {}& x_1\frac{e^{\frac{3}{2}\Phi}}{8}(dC)_{bc}\Gamma^{bc}\epsilon + x_1\frac{e^{\frac{3}{4}\Phi}}{3}\partial_b\Phi\Gamma^b\Gamma^{11}\epsilon + x_1(\frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc} + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11})\epsilon.
\end{aligned}$$

$$(3.A.18)$$

Investigating this and comparing with (3.A.15), one sets $l = \frac{1}{24}$ and

$$x_1 = \frac{3\sqrt{2}}{4}e^{-\frac{3}{4}\Phi}, \qquad x_2 = -\frac{1}{8}e^{-\frac{3}{4}\Phi}, \qquad x_3 = 1, \qquad (3.A.19)$$

to obtain the standard Type IIA supersymmetric variations garnished with some additional torsion terms:

$$\begin{aligned}
\delta_\epsilon\psi_\mu = {}& \partial_\mu\epsilon + e_\mu^a\frac{1}{4}\omega_{abc}\Gamma^{bc}\epsilon + \frac{1}{64}e^{\frac{3}{4}\Phi}e_\mu^a(dC)_{cd}\left(\eta_{ab}\Gamma^{bcd} - 14\delta_a^c\Gamma^d\right)\Gamma^{11}\epsilon \\
& + \frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon \\
& - \frac{1}{16}e^{-\frac{3}{4}\Phi}e_\mu^a\eta_{ab}\hat{\tau}_{zaz}\epsilon - \frac{1}{16}e^{-\frac{3}{4}\Phi}e_\mu^a\eta_{ab}\hat{\tau}_{zcz}\Gamma^{bc}\epsilon \\
& + \frac{1}{32}e^{-\frac{3}{4}\Phi}e_\mu^a\hat{\tau}_{zcd}(\eta_{ab}\Gamma^{bcd} + 2\delta_a^c\Gamma^d)\Gamma^{11}\epsilon \\
& - C_\mu(\frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc} + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11})\epsilon, \\
\delta_\epsilon\lambda = {}& \frac{3}{16}\frac{1}{\sqrt{2}}e^{\frac{3}{4}\Phi}\left((dC)_{bc} + 2e^{-\frac{3}{2}\Phi}\hat{\tau}_{zbc}\right)\Gamma^{bc}\epsilon + \frac{\sqrt{2}}{4}(\partial_b\Phi + \frac{3}{2}e^{-\frac{3}{4}\Phi}\hat{\tau}_{zbz})\Gamma^b\Gamma^{11}\epsilon.
\end{aligned}$$

$$(3.A.20)$$

Chapter 4

Seeing Kutasov duality in supergravity

4.1 Introduction

We construct in this chapter new Type IIB supergravity solutions, and find that their field-theory duals exhibit a Kutasov-like duality. This chapter is based on [9] which has been done in collaboration with Conde.

We look at supergravity solutions corresponding to branes wrapping compact cycles, an approach set forth by [77], and we choose those cycles to have a non-zero genus. Wrapping branes on cycles of non-trivial homology does not seem to be a much explored avenue (see however [77, 78, 76, 79]), although a lot of mathematical structure appears, that relates to interesting physics. One recent example is the construction by Gaiotto and Maldacena [80, 81], using M5-branes wrapping Riemann surfaces of arbitrary genus, of the gravity duals of certain $\mathcal{N} = 2$ super-conformal theories previously found by Gaiotto [82]. We are interested here in a related configuration, yielding different physics though: we wrap D5-branes on Riemann surfaces with genus $g > 1$, preserving only four supercharges. We refer to such surfaces as *hyperbolic cycles* since we build them as quotients of hyperbolic spaces. Our main motivation for doing this is to look for the gravity duals of theories displaying Kutasov duality [83, 84, 85], that appears when one has a non-trivial adjoint matter content in an $SU(N_c)$ gauge theory with fundamental matter. The fact that the adjoint content is non-trivial is directly related to having $g > 1$. One formal way to explain this is using the index theorem like in [77], that determines the number of fermion zero-modes from the topology of the space. As

shown there, having a non-trivial genus $g > 1$ implies the existence of $(g - 1)$ massless adjoint fermions. Another way to think about those adjoints is that they roughly correspond to the zero-modes of the $B_{(2)}$ field on the cycles of different homology within the Riemann surface.

As we said, Kutasov duality involves the presence of fundamental matter in the gauge theory. The way to implement this on the gravity side is to introduce a smeared distribution of branes. As we know from Chapter 2, one has to study the action

$$S = S_{IIB} + S_{\text{sources}} \,. \tag{4.1.1}$$

For the configurations we want to study, the action of Type IIB supergravity reduces to (in Einstein frame):

$$S_{IIB} = \frac{1}{2\kappa_{10}^2} \left[\int \sqrt{-g} \left(R - \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi \right) - \frac{1}{2} \int \left(e^\Phi F_{(3)} \wedge *F_{(3)} \right) \right] \,, \tag{4.1.2}$$

while the action for the smeared sources is:

$$S_{\text{sources}} = -T_{D5} \int \left(e^{\Phi/2}\mathcal{K} - C_{(6)} \right) \wedge \Xi \,, \tag{4.1.3}$$

where Ξ is the smearing form that accounts for the distribution of the flavour branes, and $\mathcal{K}$ is the calibration form for the D5-branes.

The introduction of S_{sources} in (4.1.1) modifies the equations of motion of Type IIB supergravity. In addition, it is responsible for the violation of the Bianchi identity for $F_{(3)}$, which allows us to relate the smearing form to the RR flux:

$$\mathrm{d}F_{(3)} = 2\kappa_{10}^2 T_{D5} \, \Xi \,. \tag{4.1.4}$$

In this chapter, we decide to look for solutions of (4.1.3) that are dual to field theories exhibiting a Kutasov-like duality. Kutasov duality is a generalisation of Seiberg duality [17]. It relates two four-dimensional $\mathcal{N} = 1$ gauge theories. One has gauge group $SU(N_c)$, with N_f chiral multiplets in the fundamental representation, and one adjoint chiral superfield X with the following superpotential:

$$W(X) = \mathrm{Tr} \sum_{l=1}^{k} g_l X^{l+1} \,, \tag{4.1.5}$$

where k is an integer. The second gauge theory, related by Kutasov duality to the one we just described, is very similar: it has gauge group $SU(kN_f - N_c)$ with

N_f fundamental chiral superfields and one adjoint one Y. In addition it has N_f^2 mesons. The details of the construction of the mesons and the superpotential for Y in terms of quantities of the first gauge theory can be found in [83, 84, 85]. It can be generalised to the case where we have multiple generations of adjoint chiral superfields [86]. In this chapter, we show the way one can see this Kutasov duality in our supergravity solutions. Especially, we identify the parameter k of the duality with some gravity quantities.

The structure of the chapter is as follows. In Section 4.2, we find the supergravity differential equations describing branes wrapping Riemann surfaces with higher genus. We are able to reduce the study of this system of equations to the study of a simple ordinary second-order differential equation. We systematically investigate the solutions of this differential equation in Section 4.3. In Section 4.4, we critically analyse several features of the dual gauge theory to our brane configuration. We show, among other things, that we do see a realisation of Kutasov duality in the supergravity picture. Section 4.5 can be read independently; it deals with a generalisation of the ansatz previously used, allowing for more general brane configurations, as well as with the study of its solutions. We did not however study the field theories dual to those additional solutions. Finally, we sum up the results of this chapter.

4.2 The $\mathbb{H}_2 \times \widetilde{SL_2}$ ansatz

Our goal is to find Type IIB supergravity solutions that correspond to D5-branes wrapping Riemann surfaces of higher genus. As we know from the uniformisation theorem (see Section 4.2.5 for details), these particular spaces admit a geometric structure modelled on the hyperbolic plane $\mathbb{H}_2$, this being the reason why we often refer to these surfaces as hyperbolic two-cycles. As we argue later (see Section 4.4), our motivation for looking for such configurations is that of finding gravity duals to supersymmetric gauge theories with massless adjoint matter. For the moment, in these first sections, we focus only on the gravity side and the quest for these new Type IIB supergravity solutions.

We are interested in finding geometries dual to four-dimensional $\mathcal{N} = 1$ gauge theories. One simple way to achieve this is by imposing on the geometries an $SU(3)$-structure. Additionally, it tells us that we should wrap our D5-branes on a two-cycle as mentioned before so that, for energies that appear small compared to the inverse size of the cycle, the six-dimensional theory on the branes reduces to

a four-dimensional one. There is a close example that achieves exactly this, which is the so-called Maldacena-Núñez model [13, 87]. So it is interesting to revisit it as a starting point for motivating the ansatz we later use. In fact, for our present purpose, it is far more appropriate to have a look at a generalisation of the MN solution: the one found in [21] by Casero, Núñez and Paredes, that has come to be known as the CNP solution, and which accounts for the inclusion of dynamical massless flavours into the MN background (see also [28] for a more precise matching with the dual field theory). Let us then recall how this CNP geometry looks like.

4.2.1   The CNP solution

By wrapping a large number N_c of D5-branes on a two-sphere inside a Calabi-Yau threefold, and adding a smeared set of N_f $(\sim N_c)$ D5-branes overlapping with the former along Minkowski space-time, one finds a Type IIB supergravity solution dual to an $\mathcal{N} = 1$, $SU(N_c)$ SQCD-like theory with N_f flavours.

In Einstein frame, and with the conventions $\alpha' = 1 = g_s$, the metric, RR three-form and dilaton cast as:

$$\mathrm{d}s^2 = e^{2f}\Big[\mathrm{d}x_{1,3}^2 + e^{2k}\mathrm{d}r^2 + e^{2h}\big(\sigma_1^2 + \sigma_2^2\big)$$

$$+ \frac{e^{2g}}{4}\big((\omega_1 - A_1)^2 + (\omega_2 - A_2)^2\big) + \frac{e^{2k}}{4}(\omega_3 - A_3)^2\Big], \tag{4.2.1}$$

$$F_{(3)} = -\frac{N_c}{4}\bigwedge_i (\omega_i - B_i) + \frac{N_c}{4}\sum_i G_i \wedge (\omega_i - B_i) - \frac{N_f}{4}\sigma_1 \wedge \sigma_2 \wedge (\omega_3 - B_3)\,,$$

$$\tag{4.2.2}$$

$$\Phi = 4f\,, \tag{4.2.3}$$

where f, g, h, k are all functions of the radial/holographic coordinate r; $\sigma_{1,2}$ parametrise a two-sphere S^2 and $\omega_{1,2,3}$ parametrise a three-sphere S^3. These $\omega_{1,2,3}$ are $SU(2)$ left-invariant one-forms satisfying the Maurer-Cartan relations:

$$\mathrm{d}\omega_i = -\frac{1}{2}\epsilon_{jik}\,\omega_j \wedge \omega_k\,. \tag{4.2.4}$$

The set of S^2 one-forms $\sigma_{1,2}$ can be completed with a third one σ_3, such that they mimic the S^3 Maurer-Cartan algebra, $\mathrm{d}\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$, although they are obviously not independent. The one-forms A_i, B_i entering the fibration and the

RR form then read:

$$A_{1,2} = a\,\sigma_{1,2}\,, \quad A_3 = \sigma_3\,, \qquad\qquad B_{1,2} = b\,\sigma_{1,2}\,, \quad B_3 = \sigma_3\,, \qquad (4.2.5)$$

where a, b are also functions of r. Finally the two-forms G_i appearing in $F_{(3)}$ can be written as a gauge field strength for B_i:

$$G_i = \mathrm{d}B_i + \frac{1}{2}\epsilon_{ijk}B_j \wedge B_k\,. \qquad (4.2.6)$$

For concreteness, let us show a coordinate representation for the left-invariant one-forms used above. If we choose the usual coordinate system for the S^2 and S^3, $\{\theta_1, \varphi_1\}$ and $\{\theta_2, \varphi_2, \psi\}$ respectively, we have:

$$
\begin{aligned}
\sigma_1 &= -\mathrm{d}\theta_1\,, & \omega_1 &= \cos\psi\,\mathrm{d}\theta_2 + \sin\psi\,\mathrm{d}\varphi_2\,, \\
\sigma_2 &= \sin\theta_1\,\mathrm{d}\varphi_1\,, & \omega_2 &= -\sin\psi\,\mathrm{d}\theta_2 + \cos\psi\,\mathrm{d}\varphi_2\,, \\
\sigma_3 &= -\cos\theta_1\,\mathrm{d}\varphi_1\,, & \omega_3 &= \mathrm{d}\psi + \cos\theta_2\,\mathrm{d}\varphi_2\,.
\end{aligned}
\qquad (4.2.7)
$$

The CNP background is 1/8-supersymmetric and has consequently four Killing spinors that satisfy the following projections:

$$\epsilon = \tau_1\,\epsilon\,, \qquad \Gamma_{12}\epsilon = \Gamma_{34}\epsilon\,, \qquad \Gamma_{r345}\epsilon = \cos\alpha\,\epsilon + \sin\alpha\,\Gamma_{24}\epsilon\,, \qquad (4.2.8)$$

where τ_1 is the first Pauli matrix, $\alpha = \alpha(r)$, and the $\Gamma_{a_1 a_2 \dots}$ are antisymmetrised products of constant Dirac matrices in the natural vielbein frame for the metric (4.2.1):

$$
\begin{aligned}
e^{x^i} &= e^f \mathrm{d}x^i\,, \quad (i = 0,1,2,3)\,, & e^r &= e^{f+k}\mathrm{d}r\,, \\
e^1 &= e^{f+h}\sigma_1\,, & e^2 &= e^{f+h}\sigma_2\,, \\
e^3 &= \frac{e^{f+g}}{2}(\omega_1 - A_1)\,, & e^4 &= \frac{e^{f+g}}{2}(\omega_2 - A_2)\,, \quad e^5 = \frac{e^{f+k}}{2}(\omega_3 - A_3)\,.
\end{aligned}
$$
$$(4.2.9)$$

The functions f, g, h, k, a, b, α characterising the background are known[1] as the solution of a system of first-order ordinary differential equations, the so-called BPS system. This BPS system can be reduced to one second-order ordinary differential equation, which we call "master equation" since, once it is solved, all the previous functions follow. This master equation is simpler if we perform the reparametrisa-

---

[1]For general N_c and N_f, the full solutions are only known numerically. Only the asymptotic UV and IR behaviours are known analytically.

tion of the ansatz that was originally proposed in [29]. After this reparametrisation, the geometry is not as transparent as in (4.2.1), where we can clearly see an S^3 fibred over an S^2 (reason why we refer to this CNP solution as the $S^2 \times S^3$ case), but in turn, the analytic treatment of the solution is much simpler. The change of variables reads as follows:

$$
e^{2h} = \frac{1}{4}\frac{P^2 - Q^2}{P\cosh\tau - Q}\,, \qquad a = \frac{P\sinh\tau}{P\cosh\tau - Q}\,, \qquad \cos\alpha = \frac{P - Q\cosh\tau}{P\cosh\tau - Q}\,,
$$

$$
e^{2g} = P\cosh\tau - Q\,, \qquad b = \frac{\sigma}{N_c}\,, \qquad \sin\alpha = -\frac{\sinh\tau\sqrt{P^2 - Q^2}}{P\cosh\tau - Q}\,,
$$

$$
e^{2k} = 4Y\,, \qquad\qquad e^{2\Phi} = \frac{D}{Y^{1/2}(P^2 - Q^2)}\,,
$$

(4.2.10)

where, of course, the new functions P, Q, Y, τ, σ, D depend only on r. Note there is one function less than before. This occurs because α could be written in terms of the others as a consequence of supersymmetry. In these new variables, the CNP solution reads:

$$
\sigma = \tanh\tau\left(Q + \frac{2N_c - N_f}{2}\right)\,, \qquad\qquad \sinh\tau = \frac{1}{\sinh(2r - 2r_0)}\,,
$$

$$
D = e^{2\Phi_0}\sqrt{P^2 - Q^2}\cosh(2r_0)\sinh(2r - 2r_0)\,, \qquad Y = \frac{1}{8}\left(P' + N_f\right)\,,
$$

$$
Q = \left(Q_0 + \frac{2N_c - N_f}{2}\right)\coth(2r - 2r_0) + \frac{2N_c - N_f}{2}\left(2r\coth(2r - 2r_0) - 1\right)\,,
$$

(4.2.11)

where the prime denotes differentiation with respect to r, the terms with a zero index are constants, and P is the solution of the following second-order differential equation:

$$
P'' + (P' + N_f)\left(\frac{P' + Q' + 2N_f}{P - Q} + \frac{P' - Q' + 2N_f}{P + Q} - 4\coth(2r - 2r_0)\right) = 0\,.
$$

(4.2.12)

We call (4.2.12) the master equation for the $S^2 \times S^3$ case.

4.2.2   The ansatz

Inspired by (4.2.1), we write down an ansatz for a Type IIB supergravity solution representing D5-branes wrapping a hyperbolic two-cycle (recall that by this we mean a Riemann surface with genus $g > 1$), plus a smeared set of N_f flavour D5-branes. The first guess would be to substitute the S^2 appearing in (4.2.1) by

an $\mathbb{H}_2{}^2$. However, we know that this S^2 is not the two-cycle wrapped by the D5-branes. The latter actually involves another S^2 inside the S^3 as well (see [88]). It then makes sense to think that we also need to substitute the S^3 by some three-dimensional manifold that can accommodate the hyperbolic two-cycle inside it.

This substitution can be achieved by keeping basically the same ansatz as in the $S^2 \times S^3$ case:

$$
ds^2 = e^{2f}\Big[dx_{1,3}^2 + e^{2k}dr^2 + e^{2h}\big(\underline{\sigma}_1^2 + \underline{\sigma}_2^2\big)
$$
$$
+ \frac{e^{2g}}{4}\big((\underline{\omega}_1 - A_1)^2 + (\underline{\omega}_2 - A_2)^2\big) + \frac{e^{2k}}{4}(\underline{\omega}_3 - A_3)^2\Big]\,, \tag{4.2.13}
$$

$$
F_{(3)} = -\frac{\widetilde{N}_c}{4}\bigwedge_i(\underline{\omega}_i - B_i) + \frac{\widetilde{N}_c}{4}\sum_i \underline{G}_i \wedge (\underline{\omega}_i - B_i) - \frac{\widetilde{N}_f}{4}\sigma_1 \wedge \sigma_2 \wedge (\underline{\omega}_3 - B_3)\,,
$$
$$
\tag{4.2.14}
$$

$$
\Phi = 4f\,, \tag{4.2.15}
$$

where f, g, h, k are all functions of the radial/holographic coordinate r; but now we are using a different set of left-invariant one-forms $\underline{\omega}_i$, such that they satisfy the following Maurer-Cartan relations:

$$
d\underline{\omega}_1 = -\underline{\omega}_2 \wedge \underline{\omega}_3\,, \qquad d\underline{\omega}_2 = -\underline{\omega}_3 \wedge \underline{\omega}_1\,, \qquad d\underline{\omega}_3 = +\underline{\omega}_1 \wedge \underline{\omega}_2\,. \tag{4.2.16}
$$

Notice the flip of the last sign with respect to (4.2.4). This choice enforces the presence of hyperbolic cycles. We also use a different set of one-forms $\underline{\sigma}_i$, that characterise the $\mathbb{H}_2$ in the same way as the σ_i characterised the S^2, and once again mimic the algebra (4.2.16) of their $\underline{\omega}_i$ counterparts: $d\underline{\sigma}_1 = -\underline{\sigma}_2 \wedge \underline{\sigma}_3$, $d\underline{\sigma}_2 = -\underline{\sigma}_3 \wedge \underline{\sigma}_1$ and $d\underline{\sigma}_3 = +\underline{\sigma}_1 \wedge \underline{\sigma}_2$. The one-forms A_i, B_i entering the fibration and the RR form stay as in the $S^2 \times S^3$ case:

$$
A_{1,2} = a\,\underline{\sigma}_{1,2}\,, \quad A_3 = \underline{\sigma}_3\,, \qquad\qquad B_{1,2} = b\,\underline{\sigma}_{1,2}\,, \quad B_3 = \underline{\sigma}_3\,, \tag{4.2.17}
$$

with $a = a(r)$, $b = b(r)$, but we have to modify slightly the definition of the gauge field strength:

$$
\underline{G}_i = dB_i + \frac{1}{2}\epsilon_{ijk}B_j \wedge B_k\,, \quad (i = 1, 2)\,, \qquad \underline{G}_3 = -\,(dB_3 - B_1 \wedge B_2)\,. \tag{4.2.18}
$$

---

[2]Recall the Riemann surface can be later obtained from $\mathbb{H}_2$ by quotienting by a Fuchsian group Γ, and this leaves locally the same metric as that of $\mathbb{H}_2$. See Appendix 4.B.

In what follows, we use this vielbein base for the metric (4.2.13):

$$
e^{x^i} = e^f \mathrm{d}x^i, \quad (i = 0, 1, 2, 3), \qquad e^r = e^{f+k}\mathrm{d}r,
$$

$$
e^1 = e^{f+h}\underline{\sigma}_1, \qquad\qquad\qquad e^2 = e^{f+h}\underline{\sigma}_2,
$$

$$
e^3 = \frac{e^{f+g}}{2}(\underline{\omega}_1 - A_1), \qquad e^4 = \frac{e^{f+g}}{2}(\underline{\omega}_2 - A_2), \qquad e^5 = \frac{e^{f+k}}{2}(\underline{\omega}_3 - A_3).
$$

$$
(4.2.19)
$$

Let us exhibit a definite coordinate representation for the one-forms $\underline{\omega}_i$ and $\underline{\sigma}_i$ above. First, if we choose the metric of the Poincaré half-plane $\mathbb{H}_2$ as it is customary, $\mathrm{d}s^2 = \frac{\mathrm{d}z_1^2 + \mathrm{d}y_1^2}{y_1^2}$, the following one-forms:

$$
\underline{\sigma}_1 = -\frac{\mathrm{d}y_1}{y_1}, \quad \underline{\sigma}_2 = -\frac{\mathrm{d}z_1}{y_1}, \quad \underline{\sigma}_3 = -\frac{\mathrm{d}z_1}{y_1}, \qquad\qquad (4.2.20)
$$

play the same role as the one the σ_i played for the S^2. Note that the $\underline{\sigma}_i$ are clearly not independent, as it happened with the σ_i.

Then, to specify some coordinate representation of $\underline{\omega}_i$, we should first know which three-manifold they parametrise. This is a squashed version of the universal cover of $SL_2(\mathbb{R})$, that we denote by $\widetilde{SL_2}$, as we discuss in Section 4.2.5. $\widetilde{SL_2}$ can be built as an S^1 fibre bundle over $\mathbb{H}_2$, which shows that a hyperbolic two-cycle can be accommodated inside it. Choosing z_2, y_2 for the coordinates of $\mathbb{H}_2$ as before, and ψ as the coordinate for the fibre, the $\underline{\omega}_i$ read:

$$
\underline{\omega}_1 = \cos\psi\,\frac{\mathrm{d}y_2}{y_2} - \sin\psi\,\frac{\mathrm{d}z_2}{y_2}, \quad \underline{\omega}_2 = -\sin\psi\,\frac{\mathrm{d}y_2}{y_2} - \cos\psi\,\frac{\mathrm{d}z_2}{y_2}, \quad \underline{\omega}_3 = \mathrm{d}\psi + \frac{\mathrm{d}z_2}{y_2}.
$$

$$
(4.2.21)
$$

The range of these coordinates $\{z_1, y_1, z_2, y_2, \psi\}$ does not bother us for the moment, since we eventually take a quotient of both $\mathbb{H}_2$ and $\widetilde{SL_2}$ by some freely acting discrete isometry groups Γ and G respectively. These quotients need to be taken in order to generate the higher genus surface from $\mathbb{H}_2$ and a compact space out of $\widetilde{SL_2}$. This is reflected on the fact that, in the ansatz for $F_{(3)}$ (4.2.14), neither N_c nor N_f appear directly, but rather some related quantities $\widetilde{N}_c$, $\widetilde{N}_f$. We investigate what the relation is in Section 4.2.4.

4.2.3   Supersymmetry analysis

We want our background (4.2.13)-(4.2.15) to possess four supersymmetries. That is, one-eighth of the thirty-two supercharges of Type IIB supergravity should be preserved. As one can see in (4.2.13), our space is of the form $\mathcal{M}_4 \times_w X_6$ where

$\mathcal{M}_4$ is four-dimensional Minkowski space, X_6 is a six-dimensional manifold and $\times_w$ means a warped product. One way to dictate the preservation of only four supercharges is to impose that our six-dimensional internal manifold X_6 is equipped with an $SU(3)$-structure. We are interested in having only the three-form flux $F_{(3)}$ non-zero, so our $SU(3)$-structure is parametrised by one two-form J and one three-form Ω. In the basis of (4.2.19), one can define the $SU(3)$-structure forms as

$$J = e^r \wedge e^5 + e^1 \wedge \left(\cos\alpha\, e^2 + \sin\alpha\, e^4\right) + e^3 \wedge \left(\sin\alpha\, e^2 - \cos\alpha\, e^4\right),$$

$$\Omega = \left(e^r + ie^5\right) \wedge \left(e^1 + i\left(\cos\alpha\, e^2 + \sin\alpha\, e^4\right)\right) \wedge \left(e^3 + i\left(\sin\alpha\, e^2 - \cos\alpha\, e^4\right)\right),$$
$$(4.2.22)$$

where, once again, α is a function of r only. G-structures are a way to express supersymmetry in a geometric form. So one can write the supersymmetry equations in terms of the $SU(3)$-invariant forms J and Ω. The BPS system of first-order differential equations is then given by [44]

$$d\left(e^{3f+\Phi/2}\Omega\right) = 0, \qquad d\left(e^{4f}J \wedge J\right) = 0,$$
$$d\left(e^{2f-\Phi/2}\right) = 0, \qquad d\left(e^{2f+\Phi}J\right) = -e^{2f+3\Phi/2} *_6 F_{(3)},$$
$$(4.2.23)$$

where $*_6$ indicates the Hodge dual in the internal manifold. In addition, the $SU(3)$-structure also plays a role when writing the action for the flavour branes. Indeed, supersymmetry is equivalent to the $SU(3)$-structure in the case at hand, and the flavour branes are supersymmetric. So it makes sense that the calibration form $\mathcal{K}$ appearing in (4.1.3) can be written in terms of the $SU(3)$-structure forms, namely:

$$\mathcal{K} = e^{4f}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge J.$$
$$(4.2.24)$$

The system found from these equations can be obtained from the one found in [21] by doing the following transformations:

$$e^g \to -ie^g, \quad e^h \to -ie^h, \quad a \to -ia, \quad b \to -ib, \quad N_c \to \widetilde{N}_c, \quad N_f \to \widetilde{N}_f.$$
$$(4.2.25)$$

However, we can directly study it after making the following redefinitions for our functions:

$$e^{2h} = -\frac{1}{4}\frac{P^2-Q^2}{P\cosh\tau-Q}, \qquad a = \frac{P\sinh\tau}{P\cosh\tau-Q}, \qquad \cos\alpha = -\frac{P-Q\cosh\tau}{P\cosh\tau-Q},$$

$$e^{2g} = -P\cosh\tau+Q, \qquad b = \frac{\sigma}{\widetilde{N}_c}, \qquad \sin\alpha = \frac{\sinh\tau\sqrt{P^2-Q^2}}{P\cosh\tau-Q},$$

$$e^{2k} = 4Y\,, \qquad\qquad e^{2\Phi} = \frac{D}{Y^{1/2}(P^2 - Q^2)}\,, \qquad\qquad (4.2.26)$$

where, of course, the new functions P, Q, Y, τ, σ, D depend only on r. Note the change of sign in the transformation of e^{2g} and e^{2h} as compared to (4.2.10).

In terms of those new functions, the BPS system can be written as

$$P' = 8Y - \widetilde{N}_f\,, \qquad\qquad \left(\frac{Q}{\cosh\tau}\right)' = \frac{2\widetilde{N}_c - \widetilde{N}_f}{\cosh^2\tau}\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{D}{\sqrt{P^2 - Q^2}}\right) = 2\cosh\tau\,, \qquad \frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{D}{\sqrt{Y}}\right) = \frac{16YP}{P^2 - Q^2}\,, \qquad (4.2.27)$$

$$\tau' + 2\sinh\tau = 0\,, \qquad\qquad \sigma = \tanh\tau\left(Q + \frac{2\widetilde{N}_c - \widetilde{N}_f}{2}\right)\,.$$

This BPS system is identical (barring the tildes in $\widetilde{N}_c, \widetilde{N}_f$) to the one of the $S^2 \times S^3$ case, and it is solved in the same manner:

$$\sigma = \tanh\tau\left(Q + \frac{2\widetilde{N}_c - \widetilde{N}_f}{2}\right)\,, \qquad\qquad \sinh\tau = \frac{1}{\sinh(2r - 2r_0)}\,,$$

$$D = e^{2\Phi_0}\sqrt{P^2 - Q^2}\cosh(2r_0)\,\sinh(2r - 2r_0)\,, \qquad Y = \frac{1}{8}\left(P' + \widetilde{N}_f\right)\,,$$

$$Q = \left(Q_0 + \frac{2\widetilde{N}_c - \widetilde{N}_f}{2}\right)\coth(2r - 2r_0) + \frac{2\widetilde{N}_c - \widetilde{N}_f}{2}\left(2r\coth(2r - 2r_0) - 1\right)\,.$$

$$(4.2.28)$$

And we then remain with a second-order differential equation:

$$P'' + (P' + \widetilde{N}_f)\left(\frac{P' + Q' + 2\widetilde{N}_f}{P - Q} + \frac{P' - Q' + 2\widetilde{N}_f}{P + Q} - 4\coth(2r - 2r_0)\right) = 0\,.$$

$$(4.2.29)$$

The search for solutions boils down to solving this master equation[3], which is, apart from the change $N_f \to \widetilde{N}_f$, identical to the master equation of the $S^2 \times S^3$ case (4.2.12). However, it is important to notice that, in the case at hand, in order for the transformation (4.2.26) and the solution (4.2.28) to be well defined, we look for solutions such that

$$Q \geq P\cosh\tau\,, \qquad P^2 \geq Q^2\,, \qquad P' + \widetilde{N}_f \geq 0\,, \qquad\qquad (4.2.30)$$

---

[3]It can be checked that, as expected, solving the equations of motion of Type IIB supergravity is implied by solving this master equation and the Bianchi identity violation [51].

which makes the solutions of this $\mathbb{H}_2 \times \widetilde{SL}_2$ case behave very differently from their $S^2 \times S^3$ relatives.

4.2.4 Brane setup

Let us briefly discuss the brane configuration our background (4.2.13)-(4.2.14) describes. The idea is that we have N_c D5-branes (the so-called colour branes), wrapping a hyperbolic two-cycle inside a Calabi-Yau threefold. When we take this number N_c to be very large, plus a near-horizon limit, the Calabi-Yau threefold undergoes a geometric transition and the branes dissolve into flux [64]. The resulting internal manifold preserves the $SU(3)$-structure, and topologically it is an interval times $\dfrac{\mathbb{H}_2}{\Gamma} \times \dfrac{\widetilde{SL}_2}{G}$, as sketched below:

(r_{IR}, r_{UV})	$\mathbb{H}_2/\Gamma$	$\widetilde{SL}_2/G$
r	z_1, y_1	z_2, y_2, ψ

From the general geometric transition picture, one would expect to find a vanishing hyperbolic two-cycle in the IR, which by analogy with what happens in the MN solution should read[4] $z_1 = z_2$, $y_1 = -y_2$, $\psi = \pi$, and a blown-up three-cycle pervaded by the three-from flux. A good choice for this three-cycle is $\widetilde{SL}_2$, and what remains from the initial N_c branes is the flux quantisation condition:

$$-N_c = \frac{1}{2\kappa_{(10)}^2 T_{D5}} \int_{\widetilde{SL}_2} \imath^*(F_{(3)}) = -\frac{\widetilde{N}_c \, \mathrm{vol}(\widetilde{SL}_2)}{2\pi^2}, \qquad (4.2.31)$$

where we abuse notation by denoting by $\widetilde{SL}_2$ the actual appropriate compact quotient $\widetilde{SL}_2/G$. The volume is to be understood as taking into account possible winding effects. The inclusion of this submanifold in the ten-dimensional background, used for the pull-back, has been denoted by $\imath$. Note that from there we get

$$\widetilde{N}_c = \frac{2\pi^2}{\mathrm{vol}(\widetilde{SL}_2)} N_c. \qquad (4.2.32)$$

As for the relation between $\widetilde{N}_f$ and N_f, it can be obtained by looking at the violation of the Bianchi identity. As in the CNP solution, the $\widetilde{N}_f$ in (4.2.14) accounts

---

[4]Actually there are two equivalent two-cycles, the other one being defined by $z_1 = -z_2$, $y_1 = y_2$, $\psi = \pi$. It can be checked that these two-cycles do indeed vanish in the IR when we remove the flavours from the solution. See Section 4.3.3.

for a set of N_f D5-branes extended along (r, ψ) plus Minkowski coordinates[5] (with the transverse coordinates being constant), and homogeneously smeared over the space transverse to them. Thus, the violation of the Bianchi identity should read

$$\mathrm{d}F_{(3)} = -2\kappa_{(10)}^2 T_{D5} \frac{N_f}{\mathrm{vol}(\mathbb{H}_2 \times \mathbb{H}_2)} \omega_{\mathrm{vol}(\mathbb{H}_2 \times \mathbb{H}_2)}, \qquad (4.2.33)$$

where by ω_{Vol} we denote the volume form, and we abuse notation once again by having $\mathbb{H}_2$ stand for the quotient $\mathbb{H}_2/\Gamma$. There are two $\mathbb{H}_2$'s in (4.2.33). Recalling the sketchy table above, one is characterised by (z_1, y_1), and the other one, being the base space of $\widetilde{SL_2}$ when thought of as a line bundle over $\mathbb{H}_2$, is characterised by the (z_2, y_2) coordinates. As explained later, it is possible to take simultaneously the same quotient $\mathbb{H}_2/\Gamma$ in both of them.

From (4.2.14) we obtain:

$$\mathrm{d}F_{(3)} = -\frac{\widetilde{N}_f}{4} \omega_{\mathrm{vol}(\mathbb{H}_2 \times \mathbb{H}_2)}, \qquad (4.2.34)$$

and the comparison with the previous equation (4.2.33) yields the relation

$$\widetilde{N}_f = \frac{(4\pi)^2}{\mathrm{vol}(\mathbb{H}_2)^2} N_f. \qquad (4.2.35)$$

4.2.5   A geometric remark

The way we substituted the S^2 wrapped by the D5-branes in the CNP solution (recall this S^2 was extended along both the topological two-sphere and three-sphere present in this solution) by a Riemann surface of genus $g > 1$, $\mathcal{C}_g$, was by replacing in (4.2.1) the metrics of the two-sphere and three-sphere by their "hyperbolic analogues":

$$\begin{aligned}
\mathrm{d}s_{S^2}^2 &= \sigma_1^2 + \sigma_2^2 \ \to \ \mathrm{d}s_{\mathbb{H}_2}^2 = \underline{\sigma}_1^2 + \underline{\sigma}_2^2, \\
\mathrm{d}s_{S^3}^2 &= \omega_1^2 + \omega_2^2 + \omega_3^2 \ \to \ \mathrm{d}s_{\widetilde{SL_2}}^2 = \underline{\omega}_1^2 + \underline{\omega}_2^2 + \underline{\omega}_3^2,
\end{aligned} \qquad (4.2.36)$$

where the one-forms σ_i, $\underline{\sigma}_i$, ω_i, $\underline{\omega}_i$ have been defined in the previous subsections. One can notice that the metrics on the right-hand side of (4.2.36) represent non-compact spaces. The way to get a hyperbolic compact space out of them is to perform a quotient by a discrete subgroup of isometries. Such a quotient leaves

---

[5]It is easy to see that this six-cycle is κ-symmetric, for instance by looking at the calibration six-form (4.2.24), and checking that $\imath^*(\mathcal{K}) = \omega_{\mathrm{vol}\imath^*(g)}$.

locally the very same metrics of (4.2.36), which are therefore the metrics we have to use for C_g and for the S^1 fibre bundle over C_g respectively. How to perform this quotient is not important for the supergravity analysis, and only some details of it are needed for the matching with the field theory, which have been moved to Appendix 4.B. This construction of subspaces as quotients by isometries of a bigger space is well known in Geometry, and from it we can deduce that, in our case, these bigger spaces are $\mathbb{H}_2$ and $\widetilde{SL_2}$ respectively. For the sake of completeness, we comment a few words on this topic.

All closed (compact and with an empty boundary) smooth two-manifolds can be given a metric of constant curvature. The uniformisation theorem for surfaces provides a way to realise this construction in terms of a so-called geometric structure. A geometric structure on a manifold M is a diffeomorphism between M and a quotient space X/Γ, where X is what one calls a model geometry, and Γ is a group of isometries, such that the projection $X \mapsto X/\Gamma$ is a covering map. In the case of two-manifolds, there are three model geometries (homogeneous and simply connected spaces with a "nice" metric): the two-sphere S^2, the Euclidean space E^2, and the hyperbolic plane $\mathbb{H}_2$. Any surface with genus $g > 1$ is obtained from the latter (see for instance [92]).

It is natural to ask whether there exists a similar classification in three dimensions. This question has only been recently, and positively, answered by Perelman[6], who proved the Thurston geometrisation conjecture [89, 90, 91]. One could naively think that the model geometries in three dimensions are in correspondence with the two-dimensional ones: S^3, E^3 and $\mathbb{H}_3$. But it is easy to see that these three are not enough, since all of them are isotropic, and there are three-manifolds like $S^2 \times \mathbb{R}$ that are not. In 1982, Thurston proposed eight model geometries for the classification of three-manifolds, and proved that a large part of them admit a geometric structure modelled on these eight geometries. The classification in three dimensions is more complicated than in two dimensions since not all three-manifolds admit a geometric structure, but it is always possible to "cut any three-manifold into pieces" such that each piece does admit a geometric structure. This is the content of the geometrisation conjecture. We found that a good account of these topics can be read in [92]; despite not being completely up-to-date, it deals with a

---

[6]Perelman's works have become famous because of proving the Poincaré conjecture, which says that the only simply connected three-manifold that exists is the three-sphere S^3, up to diffeomorphisms; this result however was just a corollary of the much stronger statement he proved, the Thurston geometrisation conjecture, which classifies all the possible geometric structures on three-manifolds.

lot of the mathematical constructions we use.

It is clear that the construction of a geometric structure is appealing to us, since the manifold parametrised by the $\underline{\omega}_i$'s in (4.2.13) is precisely realised as a quotient of a model geometry by a discrete group of isometries. In order to know which of the eight model geometries we deal with, we can resort to the relation between these eight geometries and the Bianchi groups: seven of the eight geometries can be realised as a simply-connected three-dimensional Lie group (which were classified by Bianchi) with a left-invariant metric. From this construction (see for instance [93] for details) it follows that the metric

$$\mathrm{d}s^2 = (\underline{\omega}_1)^2 + (\underline{\omega}_2)^2 + (\underline{\omega}_3)^2 \ , \tag{4.2.37}$$

corresponds to the Thurston model geometry $\widetilde{SL_2}$, since the algebra of the $\underline{\omega}_i$'s relates to the type VIII Bianchi algebra.

4.3 Solutions for the case $\mathbb{H}_2 \times \widetilde{SL_2}$

We have not been able to find a general analytic solution of the master equation (4.2.29). Of course, it is easy to find numerical solutions, but no matter what values we use for the initial conditions, the solutions always seem to exist only on a finite interval (r_0, r_{UV}). This in itself does not mean much since one can always perform a redefinition of the radial coordinate in order to have it go between 0 and ∞. However, the invariant length $\int_{r_0}^{r_{UV}} \mathrm{d}r\sqrt{g_{rr}}$ is finite for all the solutions, telling us that there is a fundamental difference between the solutions studied here and the ones of CNP. We identify r_0 with the deep IR, and $r \to r_{UV}$ with the UV. This identification is made precise in Section 4.4.3[7].

Despite the fact that we only found full solutions numerically, we were able to get analytic expansions both in the IR and in the UV . Actually, as we later show, this is enough to extract all the physically relevant information (about the dual field theory) we want.

Below, we present different expansions that correspond to different solutions of the case $\mathbb{H}_2 \times \widetilde{SL_2}$. As we prove in Appendix 4.A, because of the constraints (4.2.30), it is not possible to obtain solutions for this case that extend all the way to infinity. We are restricted to having the end of the space at a finite position r_{UV}

---

[7]Notice that r_{UV} denotes the place in the geometry where our solutions stop being valid. It is the furthest point along the RG flow we can probe in the dual field theory.

in the radial coordinate. Following the arguments made in [29] for the possible types of IR and UV expansions, we found one expansion for the IR situated at $r = r_0 > -\infty$ and three different expansions for the UV situated at $r = r_{UV} < \infty$. Restricting ourselves to Frobenius series, it seems that no other consistent expansions can be found. Without loss of generality, we choose $r_{UV} = 0$, so we automatically have $r_0 < 0$.

In addition to presenting each time the solution for the function P, we also translate the results back to the original functions a, g, h, k and Φ in order to make it easier to get an idea of the background and to compare with other results in the literature.

4.3.1 Expansions in the IR

Let us first start by describing the unique infrared expansion, around $r = r_0$. For Q not to have a pole there[8], one needs to impose first $Q_0 = -\frac{2\widetilde{N}_c - \widetilde{N}_f}{2}(1 + 2r_0)$. Then one finds that the expansion for the function P is:

$$
\begin{aligned}
P =& P_0 - \widetilde{N}_f(r - r_0) + \frac{4}{3}c_+^3 P_0^2(r - r_0)^3 - 2c_+^3 \widetilde{N}_f P_0(r - r_0)^4 \\
&+ \frac{4}{5}c_+^3 \left(\widetilde{N}_f^2 + \frac{4}{3}P_0^2 \right)(r - r_0)^5 + \mathcal{O}\left((r - r_0)^6\right) ,
\end{aligned}
\tag{4.3.1}
$$

where P_0 and c_+ are free constants that need to obey $P_0 < 0$ and $c_+ > 0$, in order to satisfy the consistency conditions (4.2.30) imposed on the solutions of the master equation. The functions in the metric then are

$$
e^{2h} = -\frac{P_0}{2}(r - r_0) + \frac{1}{2}\widetilde{N}_f(r - r_0)^2 + \frac{2}{3}P_0(r - r_0)^3 + \mathcal{O}\left((r - r_0)^4\right) ,
$$

$$
e^{2g} = -\frac{P_0}{2}(r - r_0)^{-1} + \frac{\widetilde{N}_f}{2} - \frac{2}{3}P_0(r - r_0) + \mathcal{O}\left((r - r_0)^2\right) ,
$$

$$
e^{2k} = 2c_+^3 P_0^2(r - r_0)^2 - 4c_+^3 \widetilde{N}_f P_0(r - r_0)^3 + \frac{2}{3}c_+^3(3\widetilde{N}_f^2 + 4P_0^2)(r - r_0)^4
$$
$$
+ \mathcal{O}\left((r - r_0)^5\right) ,
$$

$$
\begin{aligned}
e^{4\Phi - 4\Phi_{IR}} =& 1 + \frac{4\widetilde{N}_f}{P_0}(r - r_0) + \frac{10\widetilde{N}_f^2}{P_0^2}(r - r_0)^2 \\
&+ \left(\frac{20\widetilde{N}_f^3}{P_0^3} - \frac{8\widetilde{N}_f}{3P_0} - \frac{8c_+^3 P_0}{3} \right)(r - r_0)^3 + \mathcal{O}\left((r - r_0)^4\right) ,
\end{aligned}
$$

$$
a = 1 - 2(r - r_0)^2 - \frac{4}{3P_0}(\widetilde{N}_f - 2\widetilde{N}_c)(r - r_0)^3 + \mathcal{O}\left((r - r_0)^4\right) .
\tag{4.3.2}
$$

---

[8]Notice that this condition follows from the constraint on P and Q for this case: if Q has a pole, P must have a pole too, with negative residue, but this is not possible to achieve for finite r because of the $P' \geq -\widetilde{N}_f$ constraint.

Looking at these expressions, one notices that in the IR (at $r = r_0$) the dilaton is finite, e^{2h} and e^{2k} go to zero, while e^{2g} goes to infinity. The issue of the singularity of the solutions in the IR is addressed later in Section 4.3.3. Let us now present the different possibilities for the UV.

4.3.2　Expansions in the UV

In this section, we present three different possibilities for the UV expansions, that we can group into two classes, class I and class II, for reasons that become apparent when we look at the behaviour of the metric functions in each of them. The interpretation of the different UV's is discussed in Section 4.4.1. As we previously mentioned, all the UV's happen at finite r_{UV}, that we can choose to be $r_{UV} = 0$. So, in the following, the expansions are around 0 and for $r < 0$. As we look for a solution that has a space ending in $r = r_{UV}$, we search for solutions where some function in the metric either goes to zero, or to infinity at r_{UV}. Each of the following expansions has a different function having this behaviour.

Let us note that one can find numerical solutions interpolating between the previous IR and each of the following UV's (see Figure 4.1), so we are still working with $Q_0 = -\frac{2\widetilde{N}_c - \widetilde{N}_f}{2}(1 + 2r_0)$. Then we can expand Q as

$$Q = b_0 + b_1 r + b_2 r^2 + \mathcal{O}\left(r^3\right) ,\qquad(4.3.3)$$

where

$$
\begin{aligned}
b_0 &= \frac{1}{2}(2\widetilde{N}_c - \widetilde{N}_f)\big(2r_0 \coth(2r_0) - 1\big) , \\
b_1 &= \frac{1}{2}(2\widetilde{N}_c - \widetilde{N}_f)\frac{4r_0 - \sinh(4r_0)}{\sinh^2(2r_0)} , \\
b_2 &= (4\widetilde{N}_c - 2\widetilde{N}_f)\frac{2r_0 \cosh(2r_0) - \sinh(2r_0)}{\sinh^3(2r_0)} .
\end{aligned}
\qquad(4.3.4)
$$

Let us now detail the three different expansions and give their domain of validity.

First UV　The first possible expansion for P is:

$$
\begin{aligned}
P = {}&Q + h_1(-r)^{1/2} + \frac{1}{6b_0}\big(-h_1^2 + 12b_0(b_1 + \widetilde{N}_f)\big)(-r) \\
&+ \frac{h_1}{72b_0^2}\big(5h_1^2 - 6b_0(5b_1 + 2\widetilde{N}_f) + 72b_0^2 \coth(2r_0)\big)(-r)^{3/2} + \mathcal{O}\left((-r)^2\right) .
\end{aligned}
$$

$$(4.3.5)$$

With this, the functions in the metric are

$$e^{2h} = \frac{h_1}{2 + \coth(r_0) + \tanh(r_0)}(-r)^{1/2}$$
$$+ \frac{h_1^2 + 6b_0(b_1 + \widetilde{N}_f) + \coth(2r_0)\big(-2h_1^2 + 6b_0(b_1 + \widetilde{N}_f)\big)}{6b_0\big(1 + \coth(2r_0)\big)^2}(-r) + \mathcal{O}\left((-r)^{3/2}\right) ,$$

$$e^{2g} = b_0\big(1 + \coth(2r_0)\big) + h_1 \coth(2r_0)(-r)^{1/2} + \mathcal{O}\left((-r)\right) ,$$

$$e^{2k} = -\frac{h_1}{4}(-r)^{-1/2} + \frac{h_1^2 - 6b_0(b_1 + \widetilde{N}_f)}{12b_0} + \mathcal{O}\left((-r)^{1/2}\right) ,$$

$$e^{4\Phi - 4\Phi_{UV}} = 1 - \frac{4(b_1 + \widetilde{N}_f)}{h_1}(-r)^{1/2} + \mathcal{O}\left((-r)\right) ,$$

$$a = \cosh(2r_0) - \sinh(2r_0) + \frac{h_1}{b_0\big(\sinh(2r_0) + \cosh(2r_0)\big)^2}(-r)^{1/2} + \mathcal{O}\left((-r)\right) .$$

$$(4.3.6)$$

This is only valid for $\widetilde{N}_f > 2\widetilde{N}_c$ (which gives $b_0 < 0$) and $h_1 < 0$. We have $b_1 + \widetilde{N}_f > 0$, so the dilaton decreases towards the UV and is finite. We also have

$$e^{4\Phi_{IR} - 4\Phi_{UV}} = -\frac{b_0 h_1^2}{c_+^3 P_0^4 \sinh^2(2r_0)} . \tag{4.3.7}$$

Notice also that e^{2h} goes to zero while e^{2k} goes to infinity at the UV.

Second UV We present now the second possibility for the UV. The expansion for P in that case is

$$P = -Q + h_1(-r)^{1/2} + \frac{1}{6b_0}\big(h_1^2 + 12b_0(\widetilde{N}_f - b_1)\big)(-r)$$
$$+ \frac{h_1}{72b_0^2}\big(5h_1^2 - 6b_0(5b_1 - 2\widetilde{N}_f) + 72b_0^2 \coth(2r_0)\big)(-r)^{3/2} + \mathcal{O}\left((-r)^2\right) .$$

$$(4.3.8)$$

Looking at the metric, it gives that

$$e^{2h} = \frac{h_1}{-2 + \coth(r_0) + \tanh(r_0)}(-r)^{1/2}$$
$$+ \frac{h_1^2 + 6b_0(b_1 - \widetilde{N}_f) + 2\coth(2r_0)\big(h_1^2 + 3b_0(\widetilde{N}_f - b_1)\big)}{6b_0\big(-1 + \coth(2r_0)\big)^2}(-r) + \mathcal{O}\left((-r)^{3/2}\right) ,$$

$$e^{2g} = b_0\big(1 - \coth(2r_0)\big) + h_1 \coth(2r_0)(-r)^{1/2} + \mathcal{O}\left((-r)\right) ,$$

$$e^{2k} = -\frac{h_1}{4}(-r)^{-1/2} + \frac{-h_1^2 + 6b_0(b_1 - \widetilde{N}_f)}{12b_0} + \mathcal{O}\left((-r)^{1/2}\right),$$

$$e^{4\Phi - 4\Phi_{UV}} = 1 + \frac{4(b_1 - \widetilde{N}_f)}{h_1}(-r)^{1/2} + \mathcal{O}\left((-r)\right),$$

$$a = \cosh(2r_0) + \sinh(2r_0) + \frac{h_1}{b_0\left(-\sinh(2r_0) + \cosh(2r_0)\right)^2}(-r)^{1/2} + \mathcal{O}\left((-r)\right).$$

$$(4.3.9)$$

This is only valid for $\widetilde{N}_f < 2\widetilde{N}_c$ (which gives $b_0 > 0$) and $h_1 < 0$. In that case, we have $-\widetilde{N}_f < b_1 - \widetilde{N}_f < 2\widetilde{N}_c - 2\widetilde{N}_f$. So, if $\widetilde{N}_c < \widetilde{N}_f$, then $b_1 - \widetilde{N}_f < 0$ and the dilaton decreases towards the UV. Otherwise, if $\widetilde{N}_f < \widetilde{N}_c$, $b_1 - \widetilde{N}_f$ can be positive or negative depending on the value of r_0. So the dilaton either decreases or increases towards the UV. In any case, we have

$$e^{4\Phi_{IR} - 4\Phi_{UV}} = \frac{b_0 h_1^2}{c_+^3 P_0^4 \sinh^2(2r_0)}. \qquad (4.3.10)$$

We also have e^{2h} going to 0 while e^{2k} goes to infinity at the UV. We see that the qualitative behaviour of the metric functions in this UV is the same as that in the first UV. It makes sense then to group them under one common class, that we call class I.

Third UV   We now write the last possibility for the UV. The expansion for P is

$$P = -b_0 + \widetilde{N}_f(-r) + P_2(-r)^2 + \frac{P_2}{3b_0(\widetilde{N}_f - b_1)}\left[b_1^2 - \widetilde{N}_f^2 + 2b_0(b_2 - 3P_2)\right.$$
$$\left. - 8b_0(b_1 - \widetilde{N}_f)\coth(2r_0)\right](-r)^3 + \mathcal{O}\left((-r)^4\right). \qquad (4.3.11)$$

This leads to

$$e^{2h} = \frac{b_1 - \widetilde{N}_f}{2 - \coth(r_0) - \tanh(r_0)}(-r) + \mathcal{O}\left((-r)^2\right),$$

$$e^{2g} = b_0\left(1 - \coth(2r_0)\right) + \left(2b_0\coth^2(2r_0) - 2b_0 + \widetilde{N}_f\coth(2r_0) - b_1\right)(-r)$$
$$+ \mathcal{O}\left((-r)^2\right),$$

$$e^{2k} = -P_2(-r) - \frac{P_2}{2b_0(\widetilde{N}_f - b_1)}\left[b_1^2 - \widetilde{N}_f^2 + 2b_0(b_2 - 3P_2)\right.$$
$$\left. - 8b_0(b_1 - \widetilde{N}_f)\coth(2r_0)\right](-r)^2 + \mathcal{O}\left((-r)^3\right),$$
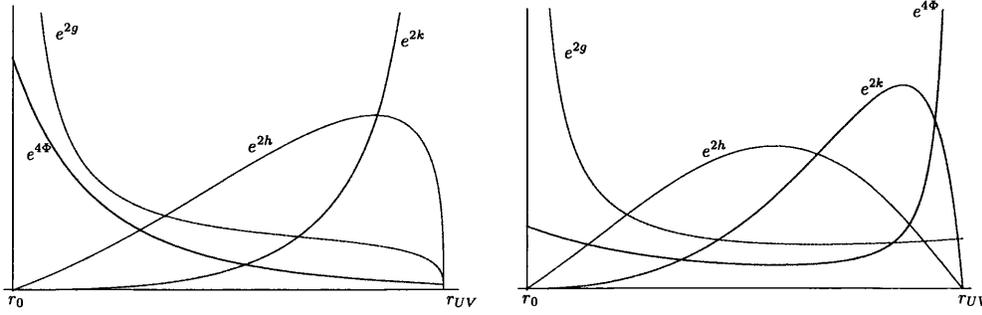
Figure 4.1: Plots of the functions e^{2g}, e^{2h}, e^{2k} and $e^{4\Phi}$. On the left, the plots are of class I solutions, while on the right they are of class II.

$$e^{4\Phi} = \frac{c_+^3 P_0^4}{8} e^{4\Phi_0} \frac{8\cosh^2(r_0)\sinh^2(r_0)}{b_0 P_2(\widetilde{N}_f - b_1)} (-r)^{-2}$$

$$+ \frac{c_+^3 P_0^4}{8} e^{4\Phi_0} \frac{2\left(\widetilde{N}_f^2 - b_1^2 + 2b_0(P_2 - b_2)\right)\sinh^2(2r_0)}{b_0^2 P_2 (b_1 - \widetilde{N}_f)^2} (-r)^{-1} + \mathcal{O}\left((-r)^0\right),$$

$$a = \cosh(2r_0) + \sinh(2r_0) + \frac{\widetilde{N}_f - b_1 + 2b_0\left(-1 + \coth(2r_0)\right)}{b_0\sinh(2r_0)\left(-1 + \coth(2r_0)\right)^2} (-r) + \mathcal{O}\left((-r)^2\right).$$

$$(4.3.12)$$

This case is valid only for $\widetilde{N}_f < 2\widetilde{N}_c$ (which gives $b_0 > 0$), $P_2 < 0$ and $b_1 - \widetilde{N}_f > 0$. This second condition requires $\widetilde{N}_f < \widetilde{N}_c$ and depends on the value of r_0 (see previous section). For this UV, e^{2h} and e^{2k} both go to zero while the dilaton diverges. Notice that this is a qualitatively very different UV behaviour than the one we found in the UV's of class I. That is why we put the third UV in a different class: class II. Figure 4.1 shows the difference of behaviour of the functions in the metric between the two classes of solutions.

4.3.3 Comments on the IR singularity

In order to know whether the solutions for the $\mathbb{H}_2 \times \widetilde{SL}_2$ case are singular or not in the IR, we can look at the behaviour of several curvature invariants around $r = r_0$:

$$R = \frac{\widetilde{N}_f^2 e^{-\Phi_0/2}}{2c_+^3 P_0^4}(r - r_0)^{-2} + \frac{7\widetilde{N}_f^3 e^{-\Phi_0/2}}{4c_+^3 P_0^5}(r - r_0)^{-1} \qquad (4.3.13)$$

$$+ e^{-\Phi_0/2} \frac{189\widetilde{N}_f^4 - 16\widetilde{N}_f(8\widetilde{N}_c P_0^2 - 9c_+^3 P_0^4) + 128\widetilde{N}_c^2 P_0^2}{48c_+^3 P_0^6} + \mathcal{O}\left((r - r_0)\right),$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{20e^{-\Phi_0}}{c_+^6 P_0^4}(r - r_0)^{-8} + \frac{52\widetilde{N}_f e^{-\Phi_0}}{c_+^6 P_0^5}(r - r_0)^{-7} + \mathcal{O}\left((r - r_0)^{-6}\right).$$

From that, one can see that a generic solution is indeed singular in the IR, since the Ricci scalar $R \sim (r - r_0)^{-2}$. This was to be expected since we deal here with backreacting massless flavours. Indeed, in our setup, we smear D5-branes that are extended in the radial coordinate r from $r = r_0$ to $r = r_{UV}$. As the branes extend all the way to the IR, at $r = r_0$, their density diverges. Thus they must create a curvature singularity in the space. Notice though that this is a good singularity in the sense that the metric component $g_{tt} = e^{\Phi/2}$ is bounded [77], but since $P_0 < 0$, g_{tt} grows towards the IR. We comment on this point in Section 4.4.5.

However, in the unflavoured case $\widetilde{N}_f = 0$, we see that the Ricci scalar goes to a constant in the IR, meaning that the solution is better behaved than the flavoured one. The same happens for $R_{\mu\nu}R^{\mu\nu}$. Indeed, the problem of the infinite density of branes is not present anymore since we do not consider the addition of sources. Nevertheless, the solution is still singular, as one can see by looking at $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. This singularity could have been expected because of the presence of vanishing higher genus manifolds (which contain non-contractible cycles) in the deep IR. It is a "better" singularity than the flavoured one since, this time, g_{tt} decreases towards the IR. The field theories dual to both the flavoured and the unflavoured cases are studied in the following section.

4.4   Field Theory

In this section we would like to interpret several features of our $\mathbb{H}_2 \times \widetilde{SL_2}$ solution in the gauge/gravity correspondence picture. We argue that the field theory dual is of the SQCD-type plus adjoint matter charged under the gauge field and self-interacting through a dangerously irrelevant polynomial superpotential, and correspondingly displays a Kutasov-like duality. Notice that this interpretation is only valid for energies smaller than the inverse size of the cycle wrapped by the branes. Moreover, we compute several observables of the field theory, that give us some insight on its IR and UV behaviours.

4.4.1   RG flow

In the gravity solutions presented in Section 4.3, we have one IR expansion but two possible classes of UV asymptotics. Each possibility should correspond to

$$\text{IR} \begin{cases} \left. \begin{array}{l} \text{1st UV} \\ \text{2nd UV} \end{array} \right\} \text{Class I} \\ \\ \text{3rd UV } \} \text{Class II} \end{cases}$$
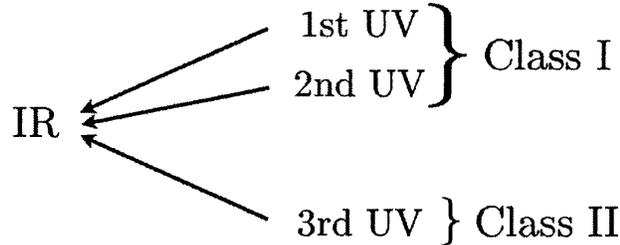
Figure 4.2: On this picture is represented schematically the classification of solutions in the $\mathbb{H}_2 \times \widetilde{SL_2}$ case and their RG flow.

a different six-dimensional UV dynamics[9]. That is, we have different solutions, each with the same IR behaviour. This situation is once again an example of the universality principle. Indeed, looking at the UV, we have different theories. But if one follows their RG flow, one notices that they all go to the same IR theory (see Figure 4.2). As mentioned in Section 4.3.2, each expansion is valid only for a given range of parameters, like $\widetilde{N}_f$ and $\widetilde{N}_c$. For example, the fact that the third UV is valid only for $\widetilde{N}_f < \widetilde{N}_c$ means that its dual field theory cannot exhibit Kutasov duality. The differences between the two classes of solutions are made clear in the following sections, when studying some of the properties of their field-theory duals.

4.4.2 Seeing Kutasov duality

Our $\mathbb{H}_2 \times \widetilde{SL_2}$ solutions describe D5-branes wrapping Riemann surfaces with genus $g > 1$. In the IR, one expects the theory on the branes to become effectively a four-dimensional gauge theory and, as explained in the introduction, to have $(g - 1)$ massless adjoint fermions. We provide in what follows some arguments indicating that we deal indeed with gauge theories with adjoint matter. Note that our solutions are not dual to Kutasov-like theories all the way to the UV, since they become eventually dual to six-dimensional field theories.

Kutasov duality [83, 84, 85] is a generalisation of Seiberg duality [17]. It states the equivalence of two different $\mathcal{N} = 1$ gauge theories in the IR. One is the "electric theory", with gauge group $SU(N_c)$, N_f quarks in the fundamental representation

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[9]Since we deal with wrapped branes, as we move towards the UV, we start to see the compact directions the branes wrap, and the effective four-dimensional theory living on them becomes six-dimensional. *I.e.* the field theory in the IR is "completed" with the dynamics of the KK modes to a different theory in the UV. Notice that, since the space has a UV singularity, one would ultimately need to use string-theory operations to make the theory UV complete.

(and of course the corresponding N_f antiquarks in the antifundamental), and a chiral adjoint superfield X with superpotential

$$W(X) = \text{Tr} \sum_{l=1}^{k} g_l X^{l+1} \,, \tag{4.4.1}$$

where k is an integer. The other one is the "magnetic theory". It is similar, having N_f quarks (and N_f antiquarks), and an adjoint chiral superfield Y, but the gauge group is $SU(kN_f - N_c)$, and we also have N_f^2 mesons. Kutasov duality gives a prescription for what the superpotential for Y is (it is of the type (4.4.1)), and for how to build the magnetic mesons out of the electric quarks.

If one sets $k = 1$, then (4.4.1) is just a mass term for X, implying that X can be integrated out in the IR. One is then left with usual SQCD, for which Seiberg duality applies. The way one was able to see a geometric realisation of Seiberg duality in the CNP solution was to notice that the BPS equations of the supergravity system remained the same[10] under the change

$$N_c \to N_f - N_c \,, \qquad N_f \to N_f \,. \tag{4.4.2}$$

Indeed, under this change, the only functions changing are $Q \to -Q$, $\sigma \to -\sigma$, and this clearly leaves the master equation (4.2.12) invariant. In the ten-dimensional geometry, Seiberg duality is equivalent to a swap of the two-spheres present in the $S^2 \times S^3$ geometry.

It is easy to see that, in our case, the master equation (4.2.29) possesses the symmetry:

$$\widetilde{N}_c \to \widetilde{N}_f - \widetilde{N}_c \,, \qquad \widetilde{N}_f \to \widetilde{N}_f \,. \tag{4.4.3}$$

If we take into account relations (4.2.32) and (4.2.35), we can rephrase this symmetry as:

$$N_c \to \frac{8\,\text{vol}(\widetilde{SL_2})}{\text{vol}(\mathbb{H}_2)^2} N_f - N_c \,, \qquad N_f \to N_f \,. \tag{4.4.4}$$

Calling $k = \frac{8\,\text{vol}(\widetilde{SL_2})}{\text{vol}(\mathbb{H}_2)^2}$, we see that we get precisely the transformation needed for Kutasov duality. Taking into account the way we perform the quotients, we find

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[10]There is a little subtlety here. In principle Seiberg duality relates two different theories in the IR, while here it would seem that the two theories related by Seiberg duality are the same. In fact, the CNP solution is dual to $SU(N_c)$ SQCD with a quartic superpotential (generated after integrating out the KK modes), and this theory is actually Seiberg self-dual (see [94] for a nice review). We expect a similar phenomenon for Kutasov duality to happen in this case.

that

$$k = \frac{q}{g-1} \,, \tag{4.4.5}$$

where g is the genus and q is a rational number. The details of this derivation are in Appendix 4.B. k can be made an integer by choosing g and q appropriately. Unfortunately, the relation between the quotienting and the generation of the $\text{Tr}\, X^{k+1}$ superpotential is not completely clear to us; we think it might be related to the number of times the colour branes wrap the hyperbolic cycle, as explained in the appendix. The geometric interpretation of Kutasov duality here would be the swap of the two $\mathbb{H}_2$'s (their quotients to be more precise) present in the geometry (4.2.13). Notice that this duality only makes sense when $\widetilde{N}_f > \widetilde{N}_c$, and exchanges $2\widetilde{N}_c - \widetilde{N}_f \to \widetilde{N}_f - 2\widetilde{N}_c$. In particular, it means that it takes one solution with the asymptotics of the first UV (4.3.6) into one with the asymptotics of the second UV (4.3.9), and that it is not possible to perform Kutasov duality on a solution with the asymptotics of class II. Moreover, performing a second duality gives back the original solution, analogous to what happens for Seiberg duality in CNP.

4.4.3 UV behaviour of the theory

From the field-theory side, not much is known about the UV of the theories displaying Kutasov duality. The fact that, in (4.4.1), $\text{Tr}\, X^{k+1}$ is an irrelevant operator puts these theories in need of a UV completion if they are to be well defined. Moreover, the general expectation from the NSVZ β-function is that we might come across a Landau pole. Since

$$\frac{\partial g_{YM,4}}{\partial \log \mu} \propto g_{YM,4}^{-3} \left(3N_c - N_{\text{adj}}(1 - \gamma_{\text{adj}}) - N_f(1 - \gamma_f) \right) \,, \tag{4.4.6}$$

where N_{adj} is the number of chiral adjoints, and the γ's are the anomalous dimensions, we see that the adjoints generically push towards a Landau pole, in the same direction as the flavours. It is not surprising then that our solutions are always singular in the UV. Let us make a more precise statement.

The gauge/gravity duality provides us with a way of computing the β-function of a gauge theory by examining the action of a brane probing the dual supergravity solution. In our case, the computation is analogous to that carried out in [53, 88]. Take a D5-brane that extends on $\mathcal{M}_4 \times \Sigma_2$, where Σ_2 is the two-cycle defined in Section 4.2.4. We also add a gauge field on the world-volume of this brane $F_{\mu\nu}$,

only along the Minkowski directions. Let us first recall that Σ_2 is defined as

$$z_1 = z_2, \quad y_1 = -y_2, \quad \psi = \pi. \tag{4.4.7}$$

A D5-brane wrapped on Σ_2 has an induced metric on its world-volume that (in string frame) reads

$$\mathrm{ds}^2_{ind} = e^{\Phi} \left[\mathrm{d}x^2_{1,3} + \left(e^{2h} + \frac{e^{2g}}{4}(1-a)^2 \right) \frac{\mathrm{d}z_2^2 + \mathrm{d}y_2^2}{y_2^2} \right], \tag{4.4.8}$$

and the brane action then is

$$S = -T_{D5} \int \mathrm{d}^6x\, e^{-\Phi} \sqrt{-\det[g_{ab} + F_{ab}]} + T_{D5} \int \left(C_{(6)} + \frac{1}{2} C_{(2)} \wedge F_{(2)} \wedge F_{(2)} \right). \tag{4.4.9}$$

The WZ term $C_{(2)} \wedge F_{(2)} \wedge F_{(2)}$ gives a theta term for the gauge field, since the $C_{(2)}$ is localised on the Σ_2 manifold. Now, we compute the determinant and expand it to second order in the gauge field to get, looking at that $F^2_{(2)}$ term,

$$\begin{aligned} S &= -T_{D5} \int \mathrm{d}^6x\, e^{-\Phi} \frac{\sqrt{g_6}}{2} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \\ &= -\left(T_{D5} \int_{\Sigma_2} \frac{\mathrm{d}z_2 \mathrm{d}y_2}{y_2^2} \right) \left[e^{2h} + \frac{e^{2g}}{4}(1-a)^2 \right] \int \mathrm{d}^4x\, F^2_{(2)}. \end{aligned} \tag{4.4.10}$$

So, from here, we read the gauge coupling of the dual field theory that, up to a constant, is

$$\frac{1}{g^2_{YM,4}} \sim \left[e^{2h} + \frac{e^{2g}}{4}(1-a)^2 \right]. \tag{4.4.11}$$

If we now apply the change of functions from (4.2.26), we find

$$\frac{1}{g^2_{YM,4}} \sim -Pe^{-\tau}. \tag{4.4.12}$$

Starting from this expression, we can calculate the β-function for the inverse of the gauge coupling:

$$\beta_{\frac{1}{g^2_{YM,4}}} = \frac{\mathrm{d}r}{\mathrm{d}\log\mu} \left(\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{g^2_{YM,4}} \right). \tag{4.4.13}$$

We do not take any precise expression for the relation between the radial coordinate r of the gravity solution and the energy scale μ of the dual field theory, but different choices would lead to different renormalisation schemes. However, for consistency

reasons, it has to be a monotonically increasing function. Just looking at the derivative of the inverse of the coupling with respect to the radial coordinate, we can see two different UV behaviours, depending on the solution from Section 4.3.2 we are considering:

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{g_{YM,4}^2} = -\frac{h_1 \tanh r_0}{2\sqrt{-r}} + \mathcal{O}\left((-r)^0\right) \qquad \text{for class I.}$$

$$= \left(\frac{b_0}{\cosh^2 r_0} - \widetilde{N}_f \tanh r_0\right) + \mathcal{O}\left((-r)^1\right) \qquad \text{for class II.}$$

(4.4.14)

We can then notice that, for class I UV asymptotics, the β-function goes to infinity at $r = r_{UV} = 0$, which could indicate the presence of a Landau pole in the field theory. Notice this happens regardless of the presence of flavours, in accordance with the expectation that the adjoints might overshoot the β-function. The third UV on the contrary leads to a finite β-function even in the UV. Nevertheless, we can see that increasing $\widetilde{N}_f$ has the effect of raising the asymptotic value of the β-function, once more agreeing with the field theory expectation that the flavours should push towards a Landau pole.

In the discussion above, it is important to take into account the following remark: the field theory is never concerned with the part of the space close to $r = 0$ because of the behaviour of the holographic c-function [95, 96]. The latter is a quantity that was first found by reducing the ten-dimensional action to five dimensions. It is related to the number of degrees of freedom in the theory, which means that it must always increase when going from the IR to the UV. Another way to obtain it, as explained in [97], is through the calculation of the holographic entanglement entropy, where it appears as a prefactor. The holographic entanglement entropy is computed as the volume of a minimal nine-manifold within a time slice of the ten-dimensional background, where one of the Minkowski spatial directions spans an interval. For computational details, it might be useful to have a look at [98]. In our case, this volume is given by

$$S_{ent} \sim \int \mathrm{d}x \, e^{2\Phi + 2h + 2g + k} \sqrt{1 + e^{2k}\left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)^2}.$$

(4.4.15)

From here, one can read the so-called a-charge, related to the factor in front of the square root as

$$3A = 2\Phi + 2h + 2g + k.$$

(4.4.16)

Curiously, we have that $3A = \log(D/2)$, where D was defined in the change of variables (4.2.26). Finally, the c-function is defined in terms of the a-charge as

$$c = \frac{1}{(A')^3} . \qquad (4.4.17)$$

Here, we can first look at its behaviour in the IR. It goes as

$$c = 27(r - r_0)^3 + \mathcal{O}\left((r - r_0)^4\right) . \qquad (4.4.18)$$

So one can see that it starts growing from the IR, and it is actually independent of the number of flavours at first order. Then one can look at the behaviour of the c-function numerically. For every solution, the c-function becomes infinite at some finite radius strictly before $r = 0$ which we considered as the UV. It means that the field theories dual to our solutions do not know about the whole geometry, but rather only about the part between $r = r_0$ and the position where the c-function blows up.

4.4.4   Domain walls

We first look at the possibility of having domain walls in our theory, and study their tension. We model a domain wall (separating different vacua in the dual field theory) by considering a fivebrane that wraps a three-cycle inside the internal geometry. We take this three-cycle to be

$$\Sigma_3 = [z_2, y_2, \psi] , \qquad (4.4.19)$$

and the brane also extends along t, x_1, x_2 among the Minkowski directions. Then the induced metric on the D5-brane is

$$ds_{ind}^2 = e^{\Phi/2} \left[dx_{1,2}^2 + \frac{e^{2g}}{4} \frac{dz_2^2 + dy_2^2}{y_2^2} + \frac{e^{2k}}{4} \left(d\psi + \frac{dx_2}{y_2} \right)^2 \right] , \qquad (4.4.20)$$

and its action is

$$S = - \left[T_{D5} \frac{e^{2\Phi + 2g + k}}{8} \int \frac{dz_2 dy_2 d\psi}{y_2^2} \right] \int d^{2+1}x . \qquad (4.4.21)$$

The tension of this domain-wall object is given by the value in the IR (as these

objects exist in the IR) of the function inside square brackets above:

$$T_{DW} = \frac{T_{D5}\text{vol}(\widetilde{SL_2})\, e^{2\Phi_0}\cosh(2r_0)}{4} \frac{\sinh(2r - 2r_0)(Q - P\coth(2r - 2r_0))}{\sqrt{P^2 - Q^2}} \, .$$

(4.4.22)

Using the IR expansion from Section 4.3.1, one can study the behaviour of the tension of the domain wall in the IR. It goes as

$$T_{DW} \sim \frac{T_{D5}\text{vol}(\widetilde{SL_2})\, e^{2\Phi_0}\cosh(2r_0)}{4}\left(1 + 2(r - r_0)^2 + \mathcal{O}\left((r - r_0)^3\right)\right) . \quad (4.4.23)$$

So the tension of the domain wall goes to a non-zero constant in the IR. The presence of an IR singularity casts some doubts on the validity of this result. If we believe the fact that a good IR singularity does not spoil the physical meaning of this computation, the result would mean that our theory has isolated vacua. It is interesting to notice that the IR behaviour of the domain-wall tension does not depend on the number of flavours N_f. The reason for the existence of isolated vacua in our field theory is less obvious than in the spherical case of CNP, where it was interpreted as a breaking of the translation invariance along ψ. In our case, the function a goes to a non-zero constant in the UV, so this translation invariance does not strictly exist even in the UV of our theory. But, as the constant towards which a goes can be taken as small as one wants by moving r_0 closer and closer to $-\infty$, the translation invariance along ψ is still present approximately. Thus it is understandable that the domain walls behave in the same way in both the spherical and the hyperbolic cases.

4.4.5 Wilson loops

Another observable of the dual field theory that should be captured by our geometry is the Wilson loop. Wilson loops provide information about the long-distance behaviour of the field theory, whether it is confining, screening, etc. Through the gauge/gravity correspondence, it gives us some insight about the IR geometry.

In a gauge theory, from the expectation value of the Wilson loop in a particular configuration, it is possible to extract the quark-antiquark potential. The standard lore [99] is that this expectation value can be computed from the area of a certain fundamental string in the supergravity dual to the gauge theory. The idea is to introduce a probe flavour brane (non-compact and spanning Minkowski space-time) sitting at some $r = r_Q$ ($r_Q \sim m_Q$ is related to the mass of the test quarks). We

attach a string to this brane, whose ends correspond to the quark and the antiquark, that hangs into the ten-dimensional geometry, reaching a minimum radial distance $\hat{r}_0$. We can then compute the energy E of the flux-tube between the quarks as the renormalised area of the string worldsheet, and the separation L of the quarks at the end-points of the string (measured in the Minkowski space-time) for different $\hat{r}_0$'s. We briefly summarise the relevant formulas. For details one can have a look at [100] (see also [36, 41, 101, 102] for related examples).

Define

$$
\hat{f}^2 = g_{tt}g_{x^i x^i} = e^{2\Phi}, \qquad \hat{g}^2 = g_{tt}g_{rr} = e^{2\Phi + 2k}, \qquad V = \frac{\hat{f}}{C\hat{g}}\sqrt{\hat{f}^2 - C^2}, \quad (4.4.24)
$$

where $C = \hat{f}(\hat{r}_0)$ and we use string frame. Then,

$$
L = 2\int_{\hat{r}_0}^{r_Q} \frac{\mathrm{d}r}{V}, \qquad E = 2\int_{\hat{r}_0}^{r_Q} \mathrm{d}r \frac{\hat{g}\,\hat{f}}{\sqrt{\hat{f}^2 - C^2}} - 2\int_0^{r_Q} \mathrm{d}r\,\hat{g}. \quad (4.4.25)
$$

Several comments are in order. First, note that the formulas in (4.4.25) depend on r_Q, that can be interpreted as a UV regulator. Ideally, one would like to take $r_Q \to \infty$, so that the test quarks are infinitely massive and become non-dynamical. However, since our solution never reaches infinity, we can set at most $r_Q = r_{UV}$. Actually, as shown in Section 4.4.3, the connection with the dual field theory finishes before $r = r_{UV}$. Nevertheless, one expects the long-distance behaviour of the Wilson loop to be independent of any UV cut-off. We critically analyse this claim in what follows.

Second, from the supergravity point of view, attaching a string to the probe flavour brane we introduce can be done whenever it is possible to impose Dirichlet conditions on the string end-points. Notice that this condition is somehow also accounting for the stability of the configuration, since it guarantees that we can place a flavour brane at $r = r_Q$, regardless of supersymmetry considerations. For the type of ansatz of our geometry, as discussed in [100], this is only possible when $\lim_{r \to r_Q} V(r) = \infty$. Notice that this only happens when the asymptotics are those of the second class of UV's (comprised just by the so-called third UV). However, since at r_{UV} we have a singularity, it is not clear that for $r_Q \to r_{UV}$ this condition is very trustworthy, so we drop it when performing the numerical computations and analyse the ensuing results.

There are two clearly differentiated regimes in which we can compute the Wilson
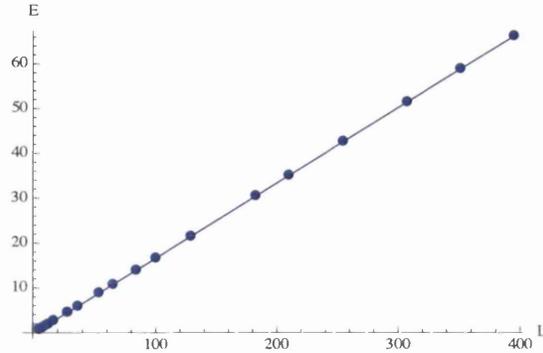
Figure 4.3: Plot of the energy E of the Wilson loop, as a function of the quark separation L. We can see a linear confining behaviour. This plot corresponds to a solution with second UV asymptotics, with $\widetilde{N}_c = 1$ and $\widetilde{N}_f = 0$.

loop. One is the unflavoured background, and the other is such that $N_f \neq 0$. The field-theory expectations are different, and thus we analyse them separately.

$N_f = 0$ **geometry**   The results are plotted in Figure 4.3 for the asymptotics of the first class of UV's (necessarily the second UV type, since $\widetilde{N}_f = 0$) and in Figure 4.4 (a) for the class II UV. There is a striking difference between the two, since at first sight, one displays confinement, and the other one does not. This difference is spurious though, as we now argue.

Recall the discussion in Section 4.4.1. As we move towards the UV, the wrapped compact directions of the D5-branes are not invisible anymore, and the gauge theory living on the stack becomes six-dimensional. The different UV asymptotics we have found should be related to different UV dynamics of this six-dimensional gauge theory. Although one would not expect the details of the UV of the theory to affect its IR properties from a field-theory point of view, our supergravity computation of the Wilson loop is quite sensitive to these UV details; imagine this six-dimensional dynamics is not negligible anymore from some scale on, given by r_{split} with $r_{IR} < r_{\text{split}} < r_{UV}$. In the plots of Figures 4.3 and 4.4 (a), we take $r_{\text{split}} \ll r_Q \approx r_{UV}$, so the string giving the Wilson loop is probing a large region of the geometry concerned with this UV dynamics, thus rendering the results UV dependent.

The way to get rid of this issue is to shift r_Q so that $r_Q \lesssim r_{\text{split}}$. One problem with this is that the test quarks become dynamical. In addition, we do not know in practice how to determine the value of r_{split}. We think that it might be possible to estimate its value looking at Figure 4.4 (b): the fact that the c-function shows
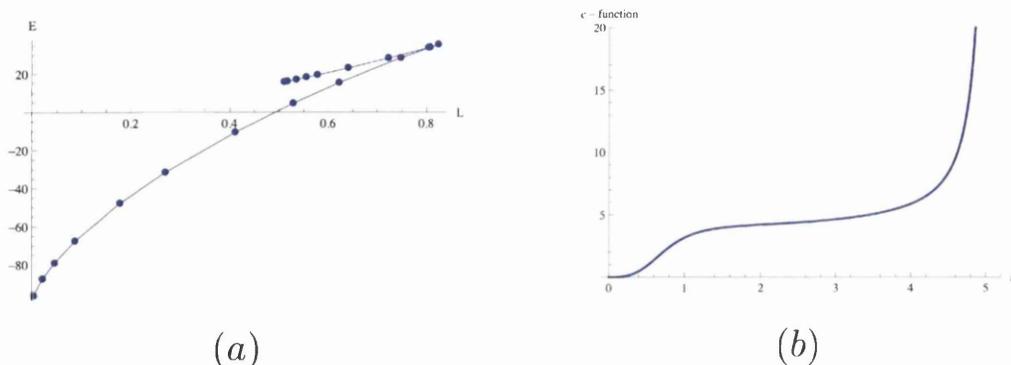
Figure 4.4: In (a) we plot E vs. L for a solution with third UV asymptotics. In (b) we plot the corresponding c-function for this solution. Notice the plateau it shows. Both figures are with $\widetilde{N}_c = 1$ and $\widetilde{N}_f = 0$.

a plateau might be signalling that, at the beginning of it, something is changing in the dual field theory. If we identify this point with the point where the six-dimensional UV dynamics is taking over, we have a definition for r_{split}. Performing the numerical integration taking $r_Q = r_{\text{split}}$, it appears that we recover in class II the linear confining behaviour in the quark-antiquark potential, observed in class I.

However, once the effects of the six-dimensional UV dynamics are separated, we still need to perform a more thorough analysis of the deep IR. We have not been able to reach this region with our numerical integration, which requires high computational precision. This would not be very useful nonetheless: as we approach the IR singularity $\hat{r}_0 \to r_0$, from the asymptotics (4.3.2), it follows that V would behave as $V \sim (r - r_0)^{-1/2}$. As proved in [100], this implies that the hanging string develops a cusp near the singularity, making the corresponding results unreliable.

As an aside note, we can also notice that, as the distance between the quarks tends to zero, we observe a Coulombic behaviour in Figure 4.4 (a), but not in Figure 4.3. We believe this is intimately related to the discussion above about Dirichlet boundary conditions. When it is not possible to impose those conditions, the string end-points might not be representing quarks, and then the universal Coulombic behaviour is not necessarily observed. This remark is only useful from a pure supergravity point of view, since the connection with the four-dimensional field theory is finishing much before the region contributing to this effect, $r_{UV} - \epsilon < r < r_{UV}$.
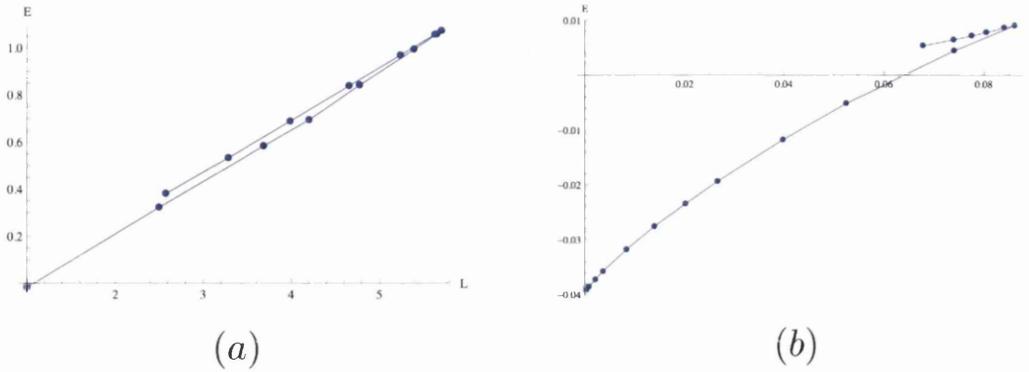
Figure 4.5: Both plots are of E vs. L for a Wilson loop. The plot in (a) corresponds to the class I of UV asymptotics, while the one in (b) corresponds to the UV of class II; both are for $\widetilde{N}_c = 2$ and $\widetilde{N}_f = 1$.

$N_f \neq 0$ **geometry** We give an example of the typical behaviour for flavoured solutions in Figure 4.5, where plots corresponding to both classes of asymptotics have been gathered. Let us remember that for $\widetilde{N}_f > \widetilde{N}_c$ we do not have a class II solution though. In this flavoured case, we expect a string-breaking length related to the string breaking into the lightest mesons by pair production, which we observe both in Figures 4.5 (a) and (b). For a discussion of the effect of smearing on this breaking length, see [40]. Most of the comments made in the previous $N_f = 0$ case apply here. So we might expect again that these plots are "contaminated" by the six-dimensional UV dynamics. Here, unfortunately, we have not found a way to decouple this effect, since the c-function is monotonically increasing, not showing any plateau. So the previous result and the corresponding interpretation are not completely certain.

Moreover, recall that the flavoured solutions also have an IR singularity. According to the criterion proposed in [77], the singularity is good in the weak formulation of the criterion (since $g_{tt} \sim e^{\phi/2}$ is bounded), but it is bad in its strong formulation, since the dilaton starts increasing as we move very closely towards the singularity (see (4.3.2)). Looking at equations (4.4.24)-(4.4.25), we see that the string is not able to probe this region in which the dilaton increases towards the singularity, indicating the presence of some kind of IR wall. Unfortunately, this is a very delicate effect and our numerics are not precise enough to see it.

Let us emphasise the main lessons we can draw from this section. Contrary to the usual computation, the test quarks we use are dynamical because our space does not extend to infinity. The results we obtain indicate a confining behaviour of

the unflavoured theory. In the case with flavours, we observe the expected string-breaking phenomenon due to the pair production of quarks. However, no definite conclusion can be made due to the presence of the IR singularity.

In the next section, we leave the treatment of the $\mathbb{H}_2 \times \widetilde{SL_2}$ case to explore other possible internal spaces. The results that we present in this following section are independent of what we have done so far.

4.5 The $\mathbb{H}_2 \times S^3$ and $S^2 \times \widetilde{SL_2}$ ansätze

In Section 4.2, one considered a class of metrics of the form (4.2.1), with the forms $\underline{\sigma}_i$ and $\underline{\omega}_i$ obeying

$$d\underline{\omega}_1 = -\underline{\omega}_2 \wedge \underline{\omega}_3 , \qquad d\underline{\omega}_2 = -\underline{\omega}_3 \wedge \underline{\omega}_1 , \qquad d\underline{\omega}_3 = +\underline{\omega}_1 \wedge \underline{\omega}_2 .$$
$$d\underline{\sigma}_1 = -\underline{\sigma}_2 \wedge \underline{\sigma}_3 , \qquad d\underline{\sigma}_2 = -\underline{\sigma}_3 \wedge \underline{\sigma}_1 , \qquad d\underline{\sigma}_3 = +\underline{\sigma}_1 \wedge \underline{\sigma}_2 .$$

$$(4.5.1)$$

We saw that the resolution of the whole system of equations of motion of supergravity could be reduced to solving one single second-order equation, the master equation (4.2.29). But instead of changing the Maurer-Cartan relations for both $\underline{\sigma}_i$ and $\underline{\omega}_i$, one can think of altering them for only one set of forms. This leads to two new ansätze, that we call mixed cases. We preserve the same form for the metric, three-form and dilaton as in (4.2.13)-(4.2.15). But we take

$$d\underline{\omega}_1 = -\underline{\omega}_2 \wedge \underline{\omega}_3 , \qquad d\underline{\omega}_2 = -\underline{\omega}_3 \wedge \underline{\omega}_1 , \qquad d\underline{\omega}_3 = \gamma\underline{\omega}_1 \wedge \underline{\omega}_2 .$$
$$d\underline{\sigma}_1 = -\underline{\sigma}_2 \wedge \underline{\sigma}_3 , \qquad d\underline{\sigma}_2 = -\underline{\sigma}_3 \wedge \underline{\sigma}_1 , \qquad d\underline{\sigma}_3 = -\gamma\underline{\sigma}_1 \wedge \underline{\sigma}_2 .$$

$$(4.5.2)$$

where $\gamma^2 = 1$. If $\gamma = 1$, then we have $S^2 \times \widetilde{SL_2}$, while if $\gamma = -1$ we have $\mathbb{H}_2 \times S^3$. Although we preserve the same functional form for $F_{(3)}$ as in (4.2.14), we generically[11] denote the parameters $\widetilde{N}_c \to \hat{N}_c$ and $\widetilde{N}_f \to \hat{N}_f$, since their proportionality relation with N_c and N_f respectively are different than in the $\mathbb{H}_2 \times \widetilde{SL_2}$ case. We can apply the same treatment as in Section 4.2.3, but this time we define our change of functions as

$$e^{2h} = \frac{\gamma}{4} \frac{P^2 - Q^2}{P\cos\tau - Q} , \qquad\qquad a = \frac{P\sin\tau}{P\cos\tau - Q} ,$$
$$e^{2g} = \gamma \left(-P\cos\tau + Q \right) , \qquad\qquad b = \frac{\sigma}{\hat{N}_c} ,$$

---

[11]Notice that the relation between for instance $\hat{N}_c$ and N_c is not the same in the $S^2 \times \widetilde{SL_2}$ and the $\mathbb{H}_2 \times S^3$ cases. We just use a common notation for convenience.

$$e^{2k} = 4Y, \qquad\qquad e^{2\Phi} = \frac{D}{Y^{1/2}(Q^2 - P^2)}. \qquad (4.5.3)$$

One can, as previously, write the BPS system of differential equations in these cases in terms of the newly defined functions:

$$P' = 8Y - \hat{N}_f, \qquad\qquad \sigma = \tan\tau \left(Q + \frac{2\hat{N}_c - \hat{N}_f}{2} \right),$$

$$\tau' + 2\sin\tau = 0, \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{D}{\sqrt{Y}} \right) = \frac{16YP}{P^2 - Q^2}, \qquad (4.5.4)$$

$$\left(\frac{Q}{\cos\tau} \right)' = \frac{2\hat{N}_c - \hat{N}_f}{\cos^2\tau}, \qquad \frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{D}{\sqrt{Q^2 - P^2}} \right) = 2\cos\tau.$$

This system is quite similar to the one of Section 4.2.3, and it can be solved as follows:

$$\sigma = \tan\tau \left(Q + \frac{2\hat{N}_c - \hat{N}_f}{2} \right), \qquad\qquad \sin\tau = \frac{1}{\cosh(2r - 2r_0)},$$

$$D = e^{2\Phi_0}\sqrt{Q^2 - P^2}\,\cosh(2r_0)\,\cosh(2r - 2r_0), \qquad Y = \frac{1}{8}\left(P' + \hat{N}_f \right),$$

$$Q = \left(Q_0 + \frac{2\hat{N}_c - \hat{N}_f}{2} \right)\tanh(2r - 2r_0) + \frac{2\hat{N}_c - \hat{N}_f}{2}\left(2r\tanh(2r - 2r_0) - 1 \right).$$

$$(4.5.5)$$

We are then left with one second-order differential equation:

$$P'' + (P' + \hat{N}_f)\left(\frac{P' + Q' + 2\hat{N}_f}{P - Q} + \frac{P' - Q' + 2\hat{N}_f}{P + Q} - 4\tanh(2r - 2r_0) \right) = 0.$$

$$(4.5.6)$$

So in the end, in the mixed cases as well, the whole problem reduces to finding solutions of the second-order differential equation (4.5.6). However, not all solutions of (4.5.6) are valid. Indeed, they need to obey some consistency relations:

$$P' \geq -\hat{N}_f, \qquad \gamma Q \geq \gamma P\cos\tau, \qquad Q^2 \geq P^2. \qquad (4.5.7)$$

The second condition depends on γ, so it means that solutions valid for $S^2 \times \widetilde{SL_2}$ are not valid for $\mathbb{H}_2 \times S^3$, and vice versa.

4.5.1   Exact solutions

In this section, we present different exact solutions of the two mixed cases introduced above. Each case has been shown to reduce to the study of the same master equation (4.5.6) for the function $P(r)$. Note, however, that the conditions that P has to verify for each case are different, forbidding applying one solution directly to a different case.

Exact solutions extending all the way to infinity seem to exist only in the particular case where $\hat{N}_f = 2\hat{N}_c$. Under this assumption, we found two exact solutions for each of the mixed cases: one that can be defined on the whole real line and that we call type A solution by analogy with the analysis carried out in [29], and another that starts at a finite value r_0 of the radial coordinate, that we call type N solution accordingly. We also present another exact solution already known in the literature [76, 103], that is valid only for the $\mathbb{H}_2 \times S^3$ case and $\hat{N}_f = 0$.

Type A solutions

If we first look at the case where $r_0 \to -\infty$ (that is, we allow the radial coordinate r to take any value in $\mathbb{R}$), we realise that the master equation reduces to:

$$P'' + (P' + \hat{N}_f) \left(\frac{P' + Q' + 2\hat{N}_f}{P - Q} + \frac{P' - Q' + 2\hat{N}_f}{P + Q} - 4 \right) = 0, \qquad (4.5.8)$$

where

$$Q = Q_0. \qquad (4.5.9)$$

The solutions we found for both mixed cases can be cast in the following way:

$$P = \hat{N}_c - \sqrt{\hat{N}_c^2 + Q_0^2}, \qquad Q_0 = 4\gamma\,\hat{N}_c \frac{2 + \xi}{\xi(4 + \xi)}, \qquad (4.5.10)$$

where ξ is a strictly positive constant. When $\gamma = 1$, then the solution corresponds to having $S^2 \times \widetilde{SL_2}$, while if $\gamma = -1$ the solution is valid for the $\mathbb{H}_2 \times S^3$ case. This translates in terms of the functions in the background as

$$
\begin{aligned}
e^{2h} &= \frac{\hat{N}_c}{\xi + 2(1 + \gamma)}, & e^{2g} &= \frac{4\hat{N}_c}{\xi + 2(1 - \gamma)}, \\
e^{2k} &= \hat{N}_c, & e^{4(\Phi - \Phi_0)} &= \frac{\xi(4 + \xi)}{4\hat{N}_c^3} e^{4r}.
\end{aligned}
\qquad (4.5.11)
$$

These solutions are exact solutions defined for $-\infty < r < \infty$. They are singular in the IR with $e^{\Phi} \to 0$. In the UV, we have $e^{\Phi} \to \infty$. Despite $e^{\Phi} \to 0$ in the IR, the singularity is a "good" one according to the criterion of [77]. That is, the term g_{tt} in the Einstein-frame metric is bounded and decreasing when approaching the IR. One could then use those solutions to learn about the IR of their potential field-theory dual. Let us now look at solutions of type N, where r_0 is finite.

Type N solutions

In the following, we write two exact solutions for the case where $r_0 > -\infty$, one for each mixed case. Recall we take $\hat{N}_f = 2\hat{N}_c$, which seems to be the only scenario where exact solutions with a good UV exist.

The solutions of the master equation read:

$$P = -\hat{N}_c \tanh(2r - 2r_0) , \qquad Q_0 = \gamma\sqrt{3}\hat{N}_c . \qquad (4.5.12)$$

Notice that taking the limit $r_0 \to -\infty$ gives a particular solution of the type A mentioned previously. Once again, the correspondence is $\gamma = 1$ with $S^2 \times \widetilde{SL_2}$, and $\gamma = -1$ with $\mathbb{H}_2 \times S^3$. This means that the functions in the metric are

$$e^{2h} = \gamma\frac{\hat{N}_c}{2}\frac{\tanh(2r - 2r_0)}{\tanh(2r - 2r_0) + \gamma\sqrt{3}} ,$$

$$e^{2g} = \gamma\hat{N}_c \tanh\left(2r - 2r_0\right)\left(\tanh\left(2r - 2r_0\right) + \gamma\sqrt{3}\right) ,$$

$$e^{2k} = \hat{N}_c \tanh^2(2r - 2r_0) , \qquad (4.5.13)$$

$$e^{4\Phi - 4\Phi_0} = \frac{2\cosh^2(2r - 2r_0)\coth^4(2r - 2r_0)}{\hat{N}_c^3 \cosh^2(2r_0)} ,$$

$$a = \frac{1}{\sinh(2r - 2r_0) + \gamma\sqrt{3}\cosh(2r - 2r_0)} .$$

These solutions are exact solutions defined for $r_0 < r < +\infty$. However the dilaton goes to infinity both in the IR $(r \to r_0)$ and in the UV. This time, the former is responsible for a bad singularity for both mixed cases. So, despite being exact, which is never easy to find, these solutions are of very little interest for our purpose, since they cannot have a well-defined field-theory dual.

Dropping the $\hat{N}_f = 2\hat{N}_c$ simplification, the following exact solution can also be found.

A solution without flavours

Putting $\hat{N}_f = 0$ in the master equation, one can find the following exact solution:

$$P = 2\hat{N}_c \, r \,, \qquad Q_0 = -\hat{N}_c \,. \qquad (4.5.14)$$

Looking at the constraints (4.5.7), it is obvious that this solution only works for the $\mathbb{H}_2 \times S^3$ case. Moreover, these constraints also imply that this solution terminates at a finite value of r. In terms of the functions in the metric, the solution reads

$$e^{2h} = \frac{\hat{N}_c}{4} - \hat{N}_c \, r \, \frac{2r + \sinh(4r - 4r_0)}{2\cosh^2(2r - 2r_0)} \,,$$

$$e^{2g} = \hat{N}_c = e^{2k} \,,$$

$$e^{4\Phi - 4\Phi_0} = \frac{8 \cosh^4(2r - 2r_0) \cosh^2(2r_0)}{\hat{N}_c^3 \left(1 - 8r^2 + \cosh(4r - 4r_0) - 4r \sinh(4r - 4r_0)\right)} \qquad (4.5.15)$$

$$= \frac{\cosh^2(2r_0)}{\hat{N}_c^2} \cosh^2(2r - 2r_0) e^{-2h} \,,$$

$$a = \frac{2r}{\cosh(2r - 2r_0)} \,.$$

This solution is defined on an interval $[r_{IR}, r_{UV})$. It is fully regular in the IR, and it ends at $r = r_{UV}$, where $e^{2h} = 0$ and the dilaton blows up.

4.5.2   Asymptotic expansions in the IR

We found, for both mixed cases, four possibilities for the IR behaviour of the solutions. We arranged them so that the first three expansions of each case all have a dilaton that diverges in the IR. That creates a bad singularity, that is, none of those solutions are gravity duals to a field theory in the IR. Fortunately, the last expansion is better behaved in both cases. All of the expansions stop before reaching r_0, that is, $r_{IR} > r_0$.

In this section, we only present the different expansions for the function P. The corresponding results for the functions g, h, k, a of the metric and the dilaton Φ can be found in Appendix D of [9]. There, the origin of the conditions imposed on the integration constants appear clearly.

$S^2 \times \widetilde{SL_2}$ case

We look at expansions around r_{IR}, that is such that $r_{IR} > r_0$. Without loss of generality, we choose $r_{IR} = 0$. So the following expansions are around 0 and for $r > 0$, and we have $r_0 < 0$.

First IR Let us first look at the function Q. We parametrise its expansion as follows for convenience:

$$Q = b_0 + (b_1 - \hat{N}_f)r + b_2 r^2 + \mathcal{O}\left(r^3\right) , \qquad (4.5.16)$$

where

$$
\begin{aligned}
b_0 &= \frac{1}{2}\left(\hat{N}_f - 2\hat{N}_c + (\hat{N}_f - 2\hat{N}_c - 2Q_0)\tanh(2r_0)\right) , \\
b_1 &= \frac{1}{2\cosh^2(2r_0)}\left(4\hat{N}_c - \hat{N}_f + 4Q_0 + \hat{N}_f\cosh(4r_0) + (\hat{N}_f - 2\hat{N}_c)\sinh(4r_0)\right) , \\
b_2 &= \frac{2}{\cosh^3(2r_0)}\left((2\hat{N}_c - \hat{N}_f)\cosh(2r_0) + (2\hat{N}_c - \hat{N}_f + 2Q_0)\sinh(2r_0)\right) .
\end{aligned}
$$

$$(4.5.17)$$

As before, we first solve for the function P. Its expansion is

$$P = b_0 - \hat{N}_f r + P_2 r^2 + P_2 \frac{b_1^2 - 2b_1\hat{N}_f + 2b_0(b_2 + 3P_2 - 4b_1\tanh(2r_0))}{3b_0 b_1}r^3 + \mathcal{O}\left(r^4\right) , $$

$$(4.5.18)$$

where P_2 is an integration constant that has to be taken positive. This solution exists only in the case of $b_0 > 0$ and $b_1 > 0$, which corresponds to having

$$Q_0 > \frac{e^{4r_0}}{1 - e^{4r_0}}(2\hat{N}_c - \hat{N}_f) . \qquad (4.5.19)$$

Second IR In this paragraph, we study a second possible behaviour of the functions in the IR. Once again we start with the function Q. The expansion is parametrised differently from before, again for convenience. Notice however that the function Q is the same:

$$Q = b_0 + (b_1 + \hat{N}_f)r + b_2 r^2 + \mathcal{O}\left(r^3\right) , \qquad (4.5.20)$$

where

$$b_0 = \frac{1}{2}\left(\hat{N}_f - 2\hat{N}_c + (\hat{N}_f - 2\hat{N}_c - 2Q_0)\tanh(2r_0)\right) ,$$

$$b_1 = \frac{1}{2\cosh^2(2r_0)}\left(4\hat{N}_c - 3\hat{N}_f + 4Q_0 - \hat{N}_f\cosh(4r_0) + (\hat{N}_f - 2\hat{N}_c)\sinh(4r_0)\right) ,$$

$$b_2 = \frac{2}{\cosh^3(2r_0)}\left((2\hat{N}_c - \hat{N}_f)\cosh(2r_0) + (2\hat{N}_c - \hat{N}_f + 2Q_0)\sinh(2r_0)\right) . \quad (4.5.21)$$

Then we solve for the function P. Its expansion is

$$P = -b_0 - \hat{N}_f r + P_2 r^2 + P_2\frac{b_1^2 + 2b_1\hat{N}_f + 2b_0(b_2 - 3P_2 - 4b_1\tanh(2r_0))}{3b_0 b_1}r^3 + \mathcal{O}\left(r^4\right) ,$$

$$(4.5.22)$$

where P_2 is an integration constant that has to be taken positive. This solution exists only in the case of $b_0 > 0$ and $b_1 > 0$, which corresponds to having

$$Q_0 > \frac{e^{4r_0}}{1 - e^{4r_0}}(2\hat{N}_c - \hat{N}_f) , \quad (4.5.23)$$

when $\hat{N}_c < \hat{N}_f < \hat{N}_c(1 + e^{4r_0})$, or

$$Q_0 > \frac{\hat{N}_f(3 + \cosh(4r_0)) - 4\hat{N}_c + (2\hat{N}_c - \hat{N}_f)\sinh(4r_0)}{4} , \quad (4.5.24)$$

when $\hat{N}_c(1 + e^{4r_0}) \leq \hat{N}_f < 2\hat{N}_c$.

Third IR   Let us now look at yet another possible IR behaviour. We use the same expansion for Q as in the second IR discussion. Looking at P we see

$$P = -b_0 + (b_1 - \hat{N}_f)r + \frac{b_1^2 + b_1\hat{N}_f + b_0(b_2 - 4b_1\tanh(2r_0))}{3b_0}r^2 + P_3 r^3 + \mathcal{O}\left(r^4\right) ,$$

$$(4.5.25)$$

where P_3 is an integration constant. This solution exists only in the case of $b_0 > 0$ and $b_1 > 0$, that is, for the same ranges of values for Q_0 as in the second IR case.

Fourth IR   In this paragraph, we study the last possible IR behaviour. We use the same expansion for Q as in the second and third IR discussions. But, this time, we change our ansatz for the function P by taking

$$P = -b_0 + P_1 r^{1/2} + \frac{6b_0(b_1 - \hat{N}_f) + P_1^2}{6b_0}r$$

$$+ P_1\frac{6b_0(5b_1 + 3\hat{N}_f) + 5P_1^2 - 72b_0^2\tanh(2r_0)}{72b_0^2}r^{3/2} + \mathcal{O}\left(r^2\right) ,$$

$$(4.5.26)$$

where P_1 is necessarily a positive integration constant. This solution exists only in the case of $b_0 > 0$, which corresponds to

$$Q_0 > -\frac{1}{2}(2\hat{N}_c - \hat{N}_f)(1 + \coth(2r_0)).   \tag{4.5.27}$$

$\mathbb{H}_2 \times S^3$ case

We now focus on the $\mathbb{H}_2 \times S^3$ case. As we said, we also have four different IR expansions around $r_{IR} = 0$, for $r > 0$ and with $r_0 < 0$. Here we compile just the different expansions for the function P.

First IR   We choose to expand Q as in (4.5.16)-(4.5.17). Solving for the function P, we find that its expansion is naturally given by (4.5.18) (recall the master equation is the same for both mixed cases). What changes are the conditions we have to impose on the integration constants. The integration constant P_2 has to be taken positive. The corresponding solution exists only in the case of $b_0 < 0$ and $b_1 < 0$, which in this case amounts to having

$$Q_0 < \frac{e^{4r_0}}{1 - e^{4r_0}}(\hat{N}_f - 2\hat{N}_c),   \tag{4.5.28}$$

when $\hat{N}_c(1 + e^{-4r_0}) < \hat{N}_f$, or

$$Q_0 < \frac{1}{4}\left(\hat{N}_f - 4\hat{N}_c - \hat{N}_f \cosh(4r_0) + (2\hat{N}_c - \hat{N}_f)\sinh(4r_0)\right),   \tag{4.5.29}$$

when $\hat{N}_c(1 + e^{-4r_0}) \geq \hat{N}_f$.

Second IR   If we choose to expand Q as in (4.5.20)-(4.5.21), we find a second possible behaviour of the functions in the IR. Of course, the function P is given by (4.5.22). Here P_2 has to be taken positive, and this solution exists only in the case of $b_0 < 0$ and $b_1 < 0$, which corresponds to having

$$Q_0 < \frac{e^{4r_0}}{1 - e^{4r_0}}(\hat{N}_f - 2\hat{N}_c).   \tag{4.5.30}$$

Third IR   Let us now look at another possible IR behaviour. We use the same expansion for Q as in the first IR. Looking at P we find that

$$P = b_0 - (b_1 + \hat{N}_f)r - \frac{b_1^2 - b_1 \hat{N}_f + b_0(b_2 - 4b_1 \tanh(2r_0))}{3b_0} r^2 + P_3 r^3 + \mathcal{O}\left(r^4\right) ,$$

(4.5.31)

where P_3 is an integration constant. This solution exists only in the case of $b_0 < 0$ and $b_1 < 0$, that is for the same ranges of values for Q_0 as in the first IR case.

Fourth IR   We finally study the last possible IR behaviour. We use the same expansion for Q as in the first and third IR's. We take for the function P the following ansatz:

$$\begin{aligned}
P =& b_0 + P_1 r^{1/2} - \frac{6b_0(b_1 + \hat{N}_f) + P_1^2}{6b_0} r \\
&+ P_1 \frac{6b_0(5b_1 - 3\hat{N}_f) + 5P_1^2 - 72b_0^2 \tanh(2r_0)}{72b_0^2} r^{3/2} + \mathcal{O}\left(r^2\right) ,
\end{aligned}$$

(4.5.32)

where P_1 is necessarily a positive integration constant. This solution exists only in the case of $b_0 < 0$, which corresponds to

$$Q_0 < -\frac{1}{2}(2\hat{N}_c - \hat{N}_f)(1 + \coth(2r_0)) .$$

(4.5.33)

4.5.3   Asymptotic expansions in the UV

The solutions for the mixed cases can reach infinity, so we only focus on these good UV's, *i.e.* those reaching the region $r \to \infty$. The results for both mixed cases are quite similar. Contrary to what happened in the $\mathbb{H}_2 \times \widetilde{SL}_2$ case, we have this time one good UV, and only one. However, it is present only in the case where $\hat{N}_c < \hat{N}_f < 2\hat{N}_c$ for the $S^2 \times \widetilde{SL}_2$ case, and when $\hat{N}_f > 2\hat{N}_c$ for the other mixed case $\mathbb{H}_2 \times S^3$.

As for the IR expansions, we gather in this section just the expansions for P. To get a better feeling of these solutions, one could look at Appendix D of [9] for the asymptotic behaviour of the functions g, h, k, a of the metric and the dilaton Φ.

$S^2 \times \widetilde{SL_2}$ case

We now present the UV behaviour of the system. We look at the functions P and Q to find the following expansions, valid for $r \to \infty$:

$$Q = (2\hat{N}_c - \hat{N}_f)\, r + Q_0 + \mathcal{O}\left(r^{-1}\right) ,$$

$$P = -Q + (\hat{N}_f - \hat{N}_c)\left(1 - \frac{\hat{N}_f}{4Q} + \hat{N}_f \frac{2\hat{N}_c - \hat{N}_f}{8Q^2} - \hat{N}_f \frac{16\hat{N}_c^2 - 13\hat{N}_c\hat{N}_f + 2\hat{N}_f^2}{32Q^3}\right)$$

$$+ \mathcal{O}\left(Q^{-4}\right) .$$

$$(4.5.34)$$

$\mathbb{H}_2 \times S^3$ case

We deal with this other case in a similar fashion as above. The UV asymptotics we found for the functions P and Q, valid for $r \to \infty$, are

$$Q = (2\hat{N}_c - \hat{N}_f)\, r + Q_0 + \mathcal{O}\left(r^{-1}\right) ,$$

$$P = Q + \hat{N}_c\left(1 + \frac{\hat{N}_f}{4Q} + \frac{\hat{N}_f(\hat{N}_f - 2\hat{N}_c)}{8Q^2} + \frac{\hat{N}_f(16\hat{N}_c^2 - 19\hat{N}_c\hat{N}_f + 5\hat{N}_f^2)}{32Q^3}\right)$$

$$+ \mathcal{O}\left(Q^{-4}\right) .$$

$$(4.5.35)$$

4.5.4   Some comments on the solutions

Notice that, in the mixed cases, we have two quantities $\hat{N}_c$, and $\hat{N}_f$ which should be proportional to N_c and N_f respectively, but we never mentioned what the relation is. Although one could expect to have relations like (4.2.32) and (4.2.35), the reason for not having written them down is that we are not sure of what the brane setup is in these mixed cases. In both the $S^2 \times S^3$ and $\mathbb{H}_2 \times \widetilde{SL_2}$ cases, the colour branes wrap a cycle that mixes some coordinates of the two S^2's or $\mathbb{H}_2$'s present in the geometries. If this general feature holds, it is not clear to us how to entangle the coordinates of an S^2 and an $\mathbb{H}_2$. As a consequence, not exactly knowing what $\hat{N}_c$ and $\hat{N}_f$ stand for, we have not pursued a further analysis of the connection of these solutions with their field-theory duals. Despite this fact, we would like to make a couple of remarks about them.

   A good place to start a possible investigation of the field theory could be the solutions of Section 4.5.1. Indeed, they are analytic, well-behaved solutions, very similar to the one found in Section 6 in CNP. The fact that the metric functions are

constant might make it easier to find a way to compute the gauge coupling, even if the cycle on which to wrap a probe D5-brane is not clear. In the aforementioned solution of CNP, the gauge coupling is constant, in accordance to the field-theory expectation of a "conformal point" $2N_c = N_f$. One could try to learn about the field theory dual to our solution looking for an analogue of this fact.

Regarding the UV, the solutions for these cases are better behaved than their non-mixed relatives, for it is possible for some of them to reach infinity, at least for some values of $\hat{N}_c$ and $\hat{N}_f$. In the IR, they can always flow to a geometry with a singularity of the good type. It is also an interesting fact that exact solutions could be found in this case, at least in the case $\hat{N}_f = 2\hat{N}_c$. When considering the unflavoured setup $\hat{N}_f = 0$, other exact solutions exist, at least in the $\mathbb{H}_2 \times S^3$ case; this can be somewhat related to the fact that one can uplift on an S^3 a seven-dimensional $SO(4)$ gauged supergravity solution (see Section 7.2.3 of [76]). However, this solution does not go all the way to infinity in the UV. We are not aware of a seven-dimensional gauged supergravity generated by the compactification of Type IIB supergravity on a Bianchi group (for M-theory this construction was done in [93]).

Finally, one can wonder if, due to the presence of hyperbolic cycles in the geometries, there is some Kutasov-like duality here. In principle, the transformation $Q \to -Q, \sigma \to -\sigma$ leaves the master equation invariant. However, if we look at the solutions with nice IR and UV behaviours, we see that the Kutasov-like duality $\hat{N}_c \to \hat{N}_f - \hat{N}_c$, $\hat{N}_f \to \hat{N}_f$ interchanges a solution of the $\mathbb{H}_2 \times S^3$ case with one of the $S^2 \times \widetilde{SL_2}$ case, and vice versa. This complies with the geometric interpretation of this Kutasov-like duality as a swap in a given geometry of the $\mathbb{H}_2$ and the S^2. The fact that we lack the correct identification of $\hat{N}_c$ and $\hat{N}_f$ in these mixed cases makes the field-theoretical interpretation of this fact far from obvious.

4.6   Conclusions

In this chapter, we looked into the possibility of finding gravity duals to field theories exhibiting a Kutasov-like duality. The existence of chiral adjoint superfields in the field theory was ensured by having branes wrapped around cycles of higher genus on the gravity side. That is why we studied supergravity solutions where the internal space contains hyperbolic subspaces that, once properly quotiented, have submanifolds of non-trivial homology. More precisely, we investigated three possible types of internal manifolds containing a fibred product of either $\mathbb{H}_2 \times \widetilde{SL_2}$,

$S^2 \times \widetilde{SL_2}$ or $\mathbb{H}_2 \times S^3$. We showed that the search for solutions in each case could be reduced to solving a "master" equation, that is only one second-order ordinary differential equation for a function P obeying some constraints. For the first case, despite the fact that the master equation was the same as in previously studied cases, we found that it was not possible to get solutions going all the way to infinity in the UV. The end of the space introduces a singularity in the supergravity solution; that was expected from the field theory which needs a UV completion. We presented several asymptotic solutions. The solutions are singular in the IR, but it is always a good singularity. For the mixed cases, we found several exact and asymptotic solutions. In the case $N_f \neq 0$, all of them have good UV's, but they are singular in the IR.

In Section 4.4, we presented some features of the field theories dual to the solutions of the background $\mathbb{H}_2 \times \widetilde{SL_2}$. After discussing the way our different supergravity solutions are related through RG flow, we looked at how Kutasov duality is implemented by a quotienting of the hyperbolic spaces by subgroups. We showed that, depending on how these subgroups are chosen, the k parameter of the Kutasov duality can take different values. We studied the gauge coupling of the theory in the UV, matching some qualitative expectations from the field theory, as well as the holographic c-function, discovering that one needs to put a UV cutoff before the end of the space, due to the divergence of the c-function at a finite point in the radial direction. We also investigated the domain walls and the Wilson loops. Those calculations are not fully reliable because of the IR singularity. The domain-wall tension, which does not depend at first order on the number of flavours, indicates the existence of isolated vacua. Concerning the Wilson loop calculation, the presence of the UV singularity forced us to use dynamical test quarks. The results are different for the flavoured and the unflavoured solutions. For $N_f = 0$, we obtain indications of confinement. For $N_f \neq 0$, the flux-tube between the quarks can decay into mesons, which is reflected in the string-breaking phenomenon we observe.

In the next two chapters, we study how one can use G-structures in order to find solution-generating techniques in supergravity. First we look at $SU(3)$-structure in Type IIB supergravity, and then at G_2-structure in Type IIA.

4.A   UV problem of the $\mathbb{H}_2 \times \widetilde{SL_2}$ case

We prove here that any solution of the master equation for the $\mathbb{H}_2 \times \widetilde{SL_2}$ case (4.2.29) always breaks down at some finite value of r. Recall that, in order for the solutions to be consistent, we need the following conditions to hold:

$$P \leq 0, \qquad |Q| \leq |P|, \qquad P' + \widetilde{N}_f \geq 0. \qquad (4.\text{A}.1)$$

Let us proceed by contradiction.

Assuming we have a solution extending all the way from some finite r_{IR} to ∞, if we look at the conditions (4.A.1) for large enough r, we easily deduce that

$$-\widetilde{N}_f \leq \lim_{r \to \infty} P' \leq 0. \qquad (4.\text{A}.2)$$

Now, let us focus our attention on the $r \to \infty$ limit of the following piece of the master equation:

$$\frac{P' + Q' + 2\widetilde{N}_f}{P - Q} + \frac{P' - Q' + 2\widetilde{N}_f}{P + Q}. \qquad (4.\text{A}.3)$$

We want to see that the limit of this piece is not positive. When $2\widetilde{N}_c = \widetilde{N}_f$, which implies that Q is constant, it is immediate that this limit is negative or zero in virtue of the constraints (4.A.1). In the $2\widetilde{N}_c \neq \widetilde{N}_f$ case, we can notice that these constraints imply that both denominators are always negative, and also that the $P' + \widetilde{N}_f$ piece is always positive. Since asymptotically we have $Q' + \widetilde{N}_f \sim 2\widetilde{N}_c$, the first summand gives a non-positive contribution. The second summand is a little bit more troublesome, since $-Q' + \widetilde{N}_f \sim 2(\widetilde{N}_f - \widetilde{N}_c)$ asymptotically, and this could be negative if $\widetilde{N}_f > \widetilde{N}_c$. But actually, when $\widetilde{N}_f > \widetilde{N}_c$ holds, one can see that, because of the last constraint in (4.A.1), the denominator $P + Q$ goes to $-\infty$, and the contribution of this summand is null.

So we conclude that the $r \to \infty$ limit of (4.A.3) is not positive. We can then have a look at the limit of the whole master equation (4.2.29).

Assuming that P is monotonic for large ρ, which is a sensible physical condition to impose, one can rigorously prove that (4.A.2) implies $\lim_{r \to \infty} P'' = 0$. Then

$$0 = \lim_{r \to \infty} P''$$
$$= -\lim_{r \to \infty} \left[(P' + \widetilde{N}_f) \left(\frac{P' + Q' + 2\widetilde{N}_f}{P - Q} + \frac{P' - Q' + 2\widetilde{N}_f}{P + Q} - 4\coth(2r - 2r_0) \right) \right]$$

$$\leq -4(\lim_{r\to\infty} P' + \widetilde{N}_f).\qquad\qquad(4.A.4)$$

The only possibility for satisfying this equation is to have $\lim_{r\to\infty} P' = -\widetilde{N}_f$. But actually this is ruled out by the master equation as well. This can be seen by writing $P = -\widetilde{N}_f r + p(r)$, with $p(r)$ tending to zero as $r \to \infty$. The master equation could be solved asymptotically and the leading behaviour for p would be $p \sim e^{4r}$: a contradiction.

So the assumption that a solution of the master equation satisfying the constraints (4.A.1) would exist all the way till $r \to \infty$ leads us to a contradiction. Thus, any solution of the master equation fulfilling our requirements eventually breaks down.

4.B How to quotient $\mathbb{H}_2$ and $\widetilde{SL_2}$

We briefly discuss, in this appendix, the possible quotients by discrete groups of isometries we can perform on $\mathbb{H}_2$ and $\widetilde{SL_2}$, and what the resulting value is for the ratio

$$k = \frac{8\,\mathrm{vol}(\widetilde{SL_2})}{\mathrm{vol}(\mathbb{H}_2)^2}\,,\qquad\qquad(4.B.1)$$

which we have associated in Section 4.4.2 with the integer number appearing in (4.4.1), relevant for Kutasov duality. Recall that, in (4.B.1), the volumes stand for the finite volumes of the quotients $\mathbb{H}_2/\Gamma$ and $\widetilde{SL_2}/G$.

The quotients of $\mathbb{H}_2$ are very well known. The discrete subgroups Γ of its isometry group $PSL(2,\mathbb{R})$ are the so-called Fuchsian groups, and the resulting quotients $\mathbb{H}_2/\Gamma$ are Riemann surfaces of genus $g > 1$ of constant negative curvature $R = -1$. The volume of such a quotient can be straightforwardly computed from the Gauss-Bonnet theorem:

$$\mathrm{vol}(\mathbb{H}_2) = \int \omega_{\mathrm{vol}(\mathbb{H}_2)} = -\int R\,\omega_{\mathrm{vol}(\mathbb{H}_2)} = -2\pi\chi(g) = 4\pi(g-1)\,,\qquad(4.B.2)$$

where χ is the Euler characteristic of the resulting Riemann surface.

The isometry group of $\widetilde{SL_2}$ might be less well known, but its structure can be deduced from the exact sequence

$$0 \to \mathbb{R} \to \mathcal{I} \to PSL(2,\mathbb{R}) \to 1\,,\qquad\qquad(4.B.3)$$

where $\mathcal{I}$ is standing for the identity component[12] of the isometry group of $\widetilde{SL_2}$. This means that, basically, there are two types of isometries acting on $\widetilde{SL_2}$, that can be thought of as an S^1 bundle over $\mathbb{H}_2$. One type comprises the isometries that rotate the S^1, *i.e.* that rotate the fibres by a constant angle, while covering the identity map of $\mathbb{H}_2$. This type is parametrised by $\mathbb{R}$. The other type is composed of those isometries that "rotate" the base $\mathbb{H}_2$, and it is therefore parametrised by $PSL(2,\mathbb{R})$. This "rotation" of the base also induces a constant-angle rotation in each fibre S^1.

The idea to retain from the discussion of the paragraph above is that each quotient of $\widetilde{SL_2}$ is roughly a quotient of the base $\mathbb{H}_2$ times a quotient of S^1. The quotient we have to perform on the base $\mathbb{H}_2$ has to be equal to the one we performed on the other $\mathbb{H}_2$ of the geometry. The only freedom left is to perform an extra discrete quotient in S^1. We compute the volume of $\widetilde{SL_2}$, including the effect of a winding number m of the colour branes, as

$$\mathrm{vol}(\widetilde{SL_2}) = m \int \omega_{\mathrm{vol}(\widetilde{SL_2})} = m \int \omega_{\mathrm{vol}(\mathbb{H}_2)}\, \omega_{\mathrm{vol}(S^1)} = m\, \mathrm{vol}(\mathbb{H}_2)\, \mathrm{vol}(S^1)\,. \quad (4.B.4)$$

We already know the volume of the base (4.B.2). The volume of the S^1, taking into account the quotienting, is $\mathrm{vol}(S^1) = \frac{2\pi}{n}$, where n is an integer. Then

$$\mathrm{vol}(\widetilde{SL_2}) = 2\pi^2 q\, (g-1)\,, \quad (4.B.5)$$

where $q = \frac{4m}{n}$ is a rational number. Coming back to (4.B.1), the k of Kutasov duality is, in terms of the quotient parameters,

$$k = \frac{q}{g-1}\,. \quad (4.B.6)$$

In general $q \in \mathbb{Q}$, but for some particular configurations, this k becomes an integer. As we see, k is proportional to the winding number m of the colour branes wrapping the hyperbolic cycle. We think this might be the reason k appears in the superpotential for the adjoint fermions in the dual field theory; as an adjoint can be thought of as a zero-mode of the $B_{(2)}$ field wrapping a particular cycle on the Riemann surface, the winding of the brane would correspond to the adjoint self-interacting $k \sim m$ times.

---

[12]The isometry group of $\widetilde{SL_2}$ has two connected components and the other one simply contains the isometries induced from the orientation-reversing isometries of $\mathbb{H}_2$.

Chapter 5

$SU(3)$-structure and rotating solutions

5.1 Introduction

In this chapter, we develop a solution-generating technique in Type IIB supergravity with sources, based on the presence of an $SU(3)$-structure. The following is based on [10] which has been done in collaboration with Martelli, Núñez and Papadimitriou.

We consider a class of supersymmetric backgrounds of Type IIB supergravity characterised by an $SU(3)$-structure [104], and we discuss a simple solution-generating method applicable to these geometries. In particular, we show that, starting from a solution of the "torsional superstring" equations of [75], a more general interpolating solution may be generated, that includes the simple class of warped Calabi-Yau solutions [105] in a limit. In most cases, this procedure is equivalent to the chain of U-dualities discussed in [106], as was also showed in [107]. However, exploiting the relation to generalised calibrations, this method can be applied as well to geometries which include the backreaction of supersymmetric sources. In fact, in the following, we apply the procedure to a supersymmetric solution describing N_c D5-branes wrapped on the S^2 inside the resolved conifold, plus N_f D5-branes sharing the $\mathbb{R}^{1,3}$ Minkowski directions and infinitely extended along a transverse cylinder [21]. We study the case in which the N_f sources are smeared over their transverse compact directions and we can have $N_f/N_c \sim \mathcal{O}(1)$.

The "seed" solution, on which the mentioned solution-generating technique is

applied, was discussed in [29][1]. The large radius (UV) behaviour of this solution is such that the dilaton asymptotes to a constant. In Section 5.3.2, we study the small radius (IR) asymptotics suitable for the purposes of this work, extending the analysis in [29, 28]. The solution we discuss is singular at the origin of the radial direction and we may view this singularity as one that can be resolved, in a fashion similar to the Klebanov-Tseytlin solution [108] where, away from the singularity, the solution captures the correct Physics. We do not resolve this singularity in the present chapter, but we anticipate that one way to do this is to consider a profile for the smeared flavour branes that vanishes smoothly at the origin, as explained in Chapter 7. The solution depends on two integers N_c and N_f, and we fix the boundary conditions in the IR by requiring that setting $N_f = 0$ gives the smooth solution discussed in [106].

After applying the solution-generating technique to the "seed" solution, we find a new background with non-zero Ramond-Ramond and Neveu-Schwarz fields. In addition to the integers N_c and N_f, the solution depends on three continuous parameters. One of them is the string coupling at infinity. Then there is a parameter we denote c, which is related to the size of the S^2 as measured from infinity [106], hence to the amount of resolution in the geometry. In the unflavoured case, this parameter is related to the VEV of baryonic operators in the baryonic branch of the Klebanov-Strassler theory [109]. Finally, we have a parameter introduced by the transformation[2]. The solution may be viewed as a "flavoured" version of the resolved deformed conifold solution [110]. Sending to zero the resolution parameter $(c \to \infty)$, and N_f as well, we obtain a solution closely related to the Klebanov-Strassler solution.

The rest of the chapter is organised as follows: in Section 5.2, we discuss a solution-generating method applicable to supersymmetric Type IIB geometries characterised by an $SU(3)$-structure. We also explain how to incorporate supersymmetric sources (flavour branes). In Section 5.3, we give some details of the new solutions. First we extend previous studies on fivebrane solutions with asymptotically constant dilaton, and then we construct explicitly the new flavoured resolved deformed conifold solution. Finally, we summarise the results of this chapter. Notice that a field-theory interpretation of our new solutions has been proposed in Section 4 of [10].

---

[1]More precisely, the solutions in [29] had different IR asymptotics to the ones we present in this chapter.

[2]As explained later, this parameter corresponds to the boost parameter in [106].

5.2 Generating solutions from $SU(3)$-structures

We start this section by presenting the BPS equations of an $SU(3)$-structure background, derived from the general setup in [104]. While the authors of [104] work with pure spinors, in a particular case, their results can be formulated in terms of the two differential forms characterising the $SU(3)$-structure. First, let us state precisely the ansatz and conventions. We work with Type IIB supergravity in Einstein frame and consider a ten-dimensional space which is a warped product of a four-dimensional Minkowski space and a six-dimensional space equipped with an $SU(3)$-structure. For the metric, we take

$$ds^2 = e^{2\Delta}\big[dx_{1,3}^2 + ds_6^2\big]. \tag{5.2.1}$$

We also have several fluxes: the RR forms $F_{(1)}$, $F_{(3)}$, $F_{(5)}$ and the NS three-form $H_{(3)}$. These have components only in the internal six-dimensional space, except for $F_{(5)}$ that is self-dual. Generically, we can write

$$F_{(5)} = e^{4\Delta + \Phi}(1 + *_{10})\text{vol}_{(4)} \wedge f\,,$$
$$H_{(3)} = dB_{(2)}\,, \tag{5.2.2}$$

where f is a one-form. The general Type IIB supersymmetry conditions for these geometries were derived in [104, 44] as equations for the two pure spinors (multiforms) Ψ_1, Ψ_2 and read

$$e^{-2\Delta + \Phi/2}(d - H_{(3)}\wedge)\big[e^{2\Delta - \Phi/2}\Psi_1\big] = d\left(\Delta + \frac{\Phi}{4}\right) \wedge \bar{\Psi}_1$$
$$+ \frac{i\,e^{\Delta + 5\Phi/4}}{8}\big[f - *_6 F_{(3)} + e^{4\Delta + \Phi} *_6 F_{(1)}\big]\,,$$
$$(d - H_{(3)}\wedge)\big[e^{2\Delta - \Phi/2}\Psi_2\big] = 0\,. \tag{5.2.3}$$

We then specialise these to the case of $SU(3)$-structure. This means that the two pure spinors take the form

$$\Psi_1 = -\frac{e^{i\zeta}}{8}e^{\Delta + \Phi/4}\left(1 - i\,e^{2\Delta + \Phi/2}J - \frac{1}{2}e^{4\Delta + \Phi}J \wedge J\right)\,, \qquad \Psi_2 = -\frac{e^{4\Delta + \Phi}}{8}\Omega\,. \tag{5.2.4}$$

Here J is the (would-be) Kähler two-form and Ω is the holomorphic three-form. Together, they define an $SU(3)$-structure on the six-dimensional geometry. In

addition, we have a function ζ arising as a phase in the pure spinor Ψ_1. For this reason, the $SU(3)$-structure is referred to as interpolating[3].

Equating terms involving forms of the same degree, we obtain the BPS equations of the system written as

$$
\mathrm{d}\left(e^{6\Delta+\Phi/2}\Omega\right) = 0\,,
$$

$$
\mathrm{d}\left(e^{8\Delta}J\wedge J\right) = 0\,,
$$

$$
\mathrm{d}\left(e^{2\Delta-\Phi/2}\cos\zeta\right) = 0\,,
$$

$$
-e^{-4\Delta-\Phi}\mathrm{d}\left(e^{4\Delta}\sin\zeta\right) = f\,,
$$

$$
-e^{\Phi}\cos\zeta *_6 F_{(3)} - e^{2\Delta+3\Phi/2}\sin\zeta\,\mathrm{d}\left(e^{-\Phi}\sin\zeta\right)\wedge J = e^{-2\Delta-\Phi/2}\mathrm{d}\left(e^{4\Delta+\Phi}J\right)\,,
$$

$$
-\sin\zeta e^{\Phi} *_6 F_{(3)} + \cos\zeta e^{2\Delta+3\Phi/2}\mathrm{d}\left(c^{-\Phi}\sin\zeta\right)\wedge J = H_{(3)}\,,
$$

$$
-\frac{1}{2}\mathrm{d}\left(e^{-\Phi}\sin\zeta\right)\wedge J\wedge J = *_6 F_{(1)}\,.
$$

$$(5.2.5)$$

Manipulating these equations a little more, one can show that

$$
H_{(3)} = \mathrm{d}\left(\tan\zeta e^{2\Delta+\Phi/2}J\right) \quad \rightarrow \quad B_{(2)} = \tan\zeta e^{2\Delta+\Phi/2}J\,, \qquad (5.2.6)
$$

thus the (non-closed part of the) $B_{(2)}$ field is completely determined by the $SU(3)$-structure. Notice that the first equation in (5.2.5) implies that the geometry is *complex*, in the usual sense, as opposed to the general case discussed in [104, 44]. This gives a useful characterisation of the geometries we are interested in. These equations were also derived in [107], which discussed first the results presented in this section.

In the rest of this chapter, we impose that $F_{(1)} = 0$. The last equation in (5.2.5) then implies that $\mathrm{d}\left(e^{-\Phi}\sin\zeta\right) = 0$ and the system simplifies further, reducing to

$$
\mathrm{d}\left(e^{6\Delta+\Phi/2}\Omega\right) = 0, \qquad \mathrm{d}\left(e^{2\Delta-\Phi/2}\cos\zeta\right) = 0, \qquad \mathrm{d}\left(e^{8\Delta}J\wedge J\right) = 0,
$$

$$
-e^{\Phi}\cos\zeta *_6 F_{(3)} = e^{-2\Delta-\Phi/2}\mathrm{d}\left(e^{4\Delta+\Phi}J\right)\,, \qquad H_{(3)} = -\sin\zeta e^{\Phi} *_6 F_{(3)},
$$

$$
-e^{-4\Delta-\Phi}\mathrm{d}\left(e^{4\Delta}\sin\zeta\right) = f.
$$

$$(5.2.7)$$

It is instructive to specialise the system (5.2.7) to the case $\zeta = 0$:

$$
\mathrm{d}\left(e^{6\Delta+\Phi/2}\Omega\right) = 0\,, \qquad \mathrm{d}\left(e^{2\Delta-\Phi/2}\right) = 0\,, \qquad \mathrm{d}\left(e^{8\Delta}J\wedge J\right) = 0\,,
$$

$$
-e^{\Phi} *_6 F_{(3)} = e^{-2\Delta-\Phi/2}\mathrm{d}\left(e^{4\Delta+\Phi}J\right)\,, \qquad (5.2.8)
$$

---

[3]See [111, 112] for earlier work on interpolating geometries.

$$f = 0 \,, \qquad H_{(3)} = 0 \,, \qquad F_{(1)} = 0 \,.$$

The only non-zero flux is then $F_{(3)}$ and the BPS system describes a configuration of D5-branes. These are simply the S-dual version of the "torsional superstring" equations of [75, 113] and they were written in this form in [47, 48]. A notable solution of these equations was discussed in [13, 87].

We now show that, from a solution of the system (5.2.8), one can generate a solution of the more complicated system (5.2.7) for a non-vanishing ζ. This is then a simple solution-generating technique. We sometimes refer to this procedure as *rotation*[4]. Precisely, if one defines

$$\Phi = \Phi^{(0)} \,,$$

$$e^{2\Delta} = \frac{1}{\cos\zeta} e^{2\Delta^{(0)}} = \frac{\kappa_1}{\cos\zeta} e^{\Phi/2} \,,$$

$$\Omega = \left(\frac{\cos\zeta}{\kappa_1}\right)^3 \Omega^{(0)} \,,$$

$$J = \left(\frac{\cos\zeta}{\kappa_1}\right)^2 J^{(0)} \,, \tag{5.2.9}$$

$$F_{(3)} = \frac{1}{\kappa_1^2} F_{(3)}^{(0)} \,,$$

$$F_{(5)} = -(1 + *_{10}) \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}\left(\frac{\sin\zeta}{\cos^2\zeta} e^{4\Delta^{(0)}}\right) \,,$$

where the quantities with a superscript $^{(0)}$ obey the equations in (5.2.8), then the (new) quantities on the left-hand side obey the equations in (5.2.7). κ_1 is here an integration constant. We require that the condition $F_{(1)} = 0$ is preserved, which implies that $\mathrm{d}\left(e^{-\Phi}\sin\zeta\right) = 0$. We can then solve this equation obtaining

$$\sin\zeta = \kappa_2 e^{\Phi} \,, \tag{5.2.10}$$

where κ_2 is another integration constant. This formula requires the dilaton to be bounded from above at any position in space. To summarise, let us write the background after the "rotation" in terms of the initial one (in Einstein frame):

$$\mathrm{d}s^2 = e^{-\Phi/2} \left[h^{-1/2} \mathrm{d}x_{1,3}^2 + e^{2\Phi} h^{1/2} \mathrm{d}s_6^{(0)2} \right] \,,$$

---

[4]There are different motivations for this: firstly, it is a rotation in the space of Killing spinors, parametrised by ζ. Secondly, in the particular case discussed in [106], this corresponds to an actual rotation of NS5-branes in a T-dual Type IIA brane picture.

$$F_{(3)} = \frac{1}{\kappa_1} e^{-2\Phi} *_6 \mathrm{d} \left(e^{2\Phi} J^{(0)} \right) ,$$

$$B_{(2)} = \frac{\kappa_2}{\kappa_1} e^{2\Phi} J^{(0)} ,$$

$$F_{(5)} = -\kappa_2 (1 + *_{10}) \mathrm{vol}_{(4)} \wedge \mathrm{d}h^{-1} , \tag{5.2.11}$$

where

$$h = \frac{1}{\kappa_1^2} \left(e^{-2\Phi} - \kappa_2^2 \right) . \tag{5.2.12}$$

Any solution of the system (5.2.5), supplemented by the Bianchi identities for the fluxes, is a solution of the equations of motion of Type IIB supergravity [104, 44]. One can then show that imposing the Bianchi identities for the simplified (seed) system (5.2.8) implies also the Bianchi identities, and hence the full equations of motion, of the more complicated system. Thus, starting from a solution of the system (5.2.8) (with a bounded dilaton), one can generate a solution of the system (5.2.7) using the formulas in (5.2.9). This result was also discussed in [107].

5.2.1   Adding D5-brane sources

In this subsection, we show how the generating technique discussed above may be also applied to supersymmetric solutions for a combined system of supergravity plus smeared sources. The key observations are the following. Firstly, when the sources are smeared in a supersymmetric way, the D-brane action can be written in terms of generalised calibrations and their net effect is captured by simple modifications of the Bianchi identities for the fluxes. In particular, the non-closed part of the fluxes is identified with the so-called smearing form – see Chapter 2. The supersymmetry equations, however, do not change in form. Then, by using the results of [51], the computation of the previous subsection can be applied to the case with sources. The interest of including the backreaction of such explicit branes is that these may be interpreted as flavours in the context of the gauge/gravity duality.

We consider the case of D5-brane sources. One then needs to study the combined action of Type IIB supergravity with the DBI and WZ terms for the source branes. Using the $SU(3)$-structure calibration conditions, the combined action can be written, as shown in Chapter 2, as

$$S = S_{IIB} - \int \left(e^{4\Delta + \Phi/2} \mathrm{vol}_{(4)} \wedge \left(\cos\zeta\, e^{2\Delta} J + \sin\zeta\, e^{-\Phi/2} B_{(2)} \right) - C_{(6)} + C_{(4)} \wedge B_{(2)} \right) \wedge \Xi_{(4)} , \tag{5.2.13}$$

where S_{IIB} is the action of Type IIB supergravity. Here $\Xi_{(4)}$ is the smearing

form, characterising the distribution of sources. It is proportional to the number of flavour branes N_f and has no components along the Minkowski directions. $C_{(4)}$ and $C_{(6)}$ are defined via

$$F_{(5)} = dC_{(4)} + B_{(2)} \wedge F_{(3)},$$
$$F_{(7)} = -e^{\Phi} *_{10} F_{(3)} = dC_{(6)} + B_{(2)} \wedge F_{(5)}. \tag{5.2.14}$$

Using the results of [51], we know that the addition of sources, even when smeared, does not modify the form of the BPS system (5.2.5) but only the Bianchi identities. These now read

$$dF_{(3)} = \Xi_{(4)},$$
$$dF_{(5)} = H_{(3)} \wedge F_{(3)} + B_{(2)} \wedge \Xi_{(4)}. \tag{5.2.15}$$

As shown in [51], if one imposes the Bianchi identities, every solution of the BPS system is a solution of the equations of motion coming from (5.2.13) (see Appendix 5.A). As previously described, one can generate a solution of the equations of motion of the action (5.2.13) from the case $\zeta = 0$. Setting $\zeta = 0$ in (5.2.13) and using the fact that $B_{(2)} = 0$, we find

$$S_{\text{D5-sources}} = -\int \left(e^{6\Delta + \Phi/2} \text{vol}_{(4)} \wedge J - C_{(6)} \right) \wedge \Xi_{(4)}. \tag{5.2.16}$$

This is the action for supersymmetric D5-brane sources in a background characterised by the equations (5.2.8).

5.2.2   The limit of D3-brane sources

The limit $\zeta \to \pi/2$, in which the supergravity background goes over to the warped Calabi-Yau geometry, is slightly more subtle [106]. Here, we determine how the action for the source branes behaves in this limit, and we find that the limiting action indeed corresponds to smeared D3-brane sources, with a particular smearing form arising in the limit. Considering the action (5.2.13) as generated from the $\zeta = 0$ case (5.2.16), we can work out the dependence on ζ of every quantity from equations (5.2.7) and (5.2.9). Recalling that the RR potentials are defined using (5.2.14), we have

$$e^{2\Delta} = \frac{1}{\cos\zeta} e^{2\Delta^{(0)}} = \frac{\kappa_1}{\cos\zeta} e^{\Phi/2} \,,$$
$$e^{\Phi} = \frac{1}{\kappa_2} \sin\zeta \,,$$

$$J = \frac{\cos^2 \zeta}{\kappa_1^2} J^{(0)},$$

$$B_{(2)} = \frac{1}{\kappa_1^2 \sqrt{\kappa_2}} \sin^{3/2} \zeta \, e^{2\Delta^{(0)}} J^{(0)}, \tag{5.2.17}$$

$$C_{(6)} = \frac{1}{\kappa_1^2 \sqrt{\kappa_2}} \sqrt{\sin \zeta} \, e^{6\Delta^{(0)}} \mathrm{vol}_{(4)} \wedge J^{(0)},$$

$$C_{(4)} = -\frac{\sin \zeta}{\cos^2 \zeta} e^{4\Delta^{(0)}} \mathrm{vol}_{(4)}.$$

Looking at the dependence on $\cos \zeta$, the action for the sources can be written as

$$S_{\text{sources}} = -\int \left(\left[\cos^2 \zeta \sin \zeta e^{4\Delta} \mathrm{vol}_{(4)} \wedge B_{(2)} + \cos^2 \zeta C_{(4)} \wedge B_{(2)} \right] + \mathcal{O}(\cos^2 \zeta) \right) \wedge \frac{\Xi_{(4)}}{\cos^2 \zeta}. \tag{5.2.18}$$

In this formula, the quantity in square brackets does not scale with $\cos \zeta$, implying that this is approaching a finite non-zero value when ζ goes to $\pi/2$, and additional terms in $\mathcal{O}(\cos^2 \zeta)$ go to zero in the limit. However, there is an overall factor $\cos^{-2} \zeta$. Therefore, if we want the action to be finite in the limit $\cos \zeta \to 0$, then we need to scale $\Xi_{(4)}$ accordingly. However, $\Xi_{(4)} = N_f \, \omega_{(4)}$ where $\omega_{(4)}$ cannot depend on ζ. We then conclude that we need to impose the following condition:

$$\frac{N_f}{\cos^2 \zeta} \to \text{constant} \quad \text{when} \quad \zeta \to \frac{\pi}{2}. \tag{5.2.19}$$

In this case the limit of the D5-brane source action is

$$S_{\text{sources}} \to S_{\text{D3 sources}} = -\int \left(e^{4\Delta^{(0)}} \mathrm{vol}_{(4)} + \tilde{C}_{(4)} \right) \wedge \widetilde{\Xi}_{(6)}, \tag{5.2.20}$$

where the tilded quantities correspond to the limit of the untilded ones. We have defined $\widetilde{\Xi}_{(6)}$ as

$$B_{(2)} \wedge \Xi_{(4)} \to \widetilde{\Xi}_{(6)} \quad \text{when} \quad \zeta \to \frac{\pi}{2}. \tag{5.2.21}$$

In the limiting case, we can then identify the action as the smearing of supersymmetric D3-branes with smearing form $\widetilde{\Xi}_{(6)}$.

5.3   D3-branes in the flavoured D5-brane solution

Here, we apply the procedure discussed in the previous section to a solution representing D5-branes wrapped on the S^2 of the resolved conifold, with the addition of explicit smeared D5-brane sources. The resulting Type IIB solution with D3-brane

charge and $B_{(2)}$ field is a "flavoured" version of the warped resolved deformed conifold solution originally derived in [110]. A field-theory interpretation of this new background can be found in Section 4 of [10].

5.3.1 D5-branes on the resolved conifold with flavour D5-branes

The setup corresponding to D5-branes wrapped on the S^2 of the resolved conifold, with addition of smeared D5-sources, was described in [21, 29, 28]. The metric in Einstein frame takes the form

$$\mathrm{d}s^2 = e^{2\Delta} \left[\mathrm{d}x_{1,3}^2 + e^{\Phi - 4\Delta} \mathrm{d}s_6^2 \right], \tag{5.3.1}$$

where here the internal metric $\mathrm{d}s_6^2$ does not change under the "rotation" procedure. As we have seen in the previous section, the unrotated metric is obtained by setting $e^{2\Delta} = e^{2\Delta^{(0)}} = \kappa_1 e^{\Phi/2}$ in (5.3.1). The solution can then be completely described in terms of the vielbein e^a, $a = 1, \ldots, 6$, parametrising the internal metric $\mathrm{d}s_6^2$, i.e. $\mathrm{d}s_6^2 = \delta_{ab} e^a e^b$. In the notation of [21], the vielbein are

$$
\begin{aligned}
& e^\rho = e^k \mathrm{d}\rho, && e^\theta = e^q \omega_1, && e^\varphi = e^q \omega_2, \\
& e^1 = \frac{1}{2} e^g \left(\tilde{\omega}_1 + a\omega_1 \right), && e^2 = \frac{1}{2} e^g \left(\tilde{\omega}_2 - a\omega_2 \right), && e^3 = \frac{1}{2} e^k (\tilde{\omega}_3 + \omega_3),
\end{aligned}
\tag{5.3.2}
$$

where the one-forms ω_i, $\tilde{\omega}_i$, $i = 1, 2, 3$, are defined as

$$
\begin{aligned}
\omega_1 &= \mathrm{d}\theta, & \tilde{\omega}_1 &= \cos\psi \, \mathrm{d}\tilde{\theta} + \sin\psi \sin\tilde{\theta} \, \mathrm{d}\tilde{\varphi}, \\
\omega_2 &= \sin\theta \, \mathrm{d}\varphi, & \tilde{\omega}_2 &= -\sin\psi \, \mathrm{d}\tilde{\theta} + \cos\psi \sin\tilde{\theta} \, \mathrm{d}\tilde{\varphi}, \\
\omega_3 &= \cos\theta \, \mathrm{d}\varphi, & \tilde{\omega}_3 &= \mathrm{d}\psi + \cos\tilde{\theta} \, \mathrm{d}\tilde{\varphi}.
\end{aligned}
\tag{5.3.3}
$$

Moreover, the RR three-form is given by

$$
\begin{aligned}
F_{(3)} = & -2N_c \, e^{-2g-k} e^1 \wedge e^2 \wedge e^3 + \frac{N_c}{2} \left(a^2 - 2ab + 1 - \frac{N_f}{N_c} \right) e^{-2g-k} e^\theta \wedge e^\varphi \wedge e^3 \\
& + N_c(b-a) e^{-g-q-k} \left(e^1 \wedge e^\varphi + e^2 \wedge e^\theta \right) \wedge e^3 \\
& + \frac{N_c}{2} b' e^{-g-q-k} e^\rho \wedge \left(-e^\theta \wedge e^1 + e^\varphi \wedge e^2 \right).
\end{aligned}
\tag{5.3.4}
$$

As was shown in [29], there is a set of variables that decouples the BPS equations described in the previous section for this ansatz and leads to a single second-order

ordinary differential equation, whose solution completely determines the super-gravity background. This set of variables is introduced defining

$$e^{2q} = \frac{1}{4}\left(\frac{P^2 - Q^2}{P\cosh\tau - Q}\right), \qquad e^{2g} = P\cosh\tau - Q, \quad e^{2k} = 4Y,$$

$$a = \frac{P\sinh\tau}{P\cosh\tau - Q}, \qquad b = \frac{\sigma}{N_c}.$$

$$(5.3.5)$$

Solving the resulting set of decoupled BPS equations, one then finds

$$\sinh\tau = \frac{1}{\sinh(2(\rho - \rho_o))},$$

$$Q = \left(Q_o + \frac{2N_c - N_f}{2}\right)\cosh\tau + \frac{2N_c - N_f}{2}\left(2\rho\cosh\tau - 1\right),$$

$$\sigma = \tanh\tau\left(Q + \frac{2N_c - N_f}{2}\right),$$

$$(5.3.6)$$

$$e^{4(\Phi - \Phi_o)} = \frac{\cosh^2(2\rho_o)}{(P^2 - Q^2)Y\sinh^2\tau},$$

$$Y = \frac{1}{8}(P' + N_f),$$

while the only remaining unknown function, $P(\rho)$, is determined by the equation

$$P'' + (P' + N_f)\left(\frac{P' + Q' + 2N_f}{P - Q} + \frac{P' - Q' + 2N_f}{P + Q} - 4\cosh\tau\right) = 0. \qquad (5.3.7)$$

Here ρ_o, Q_o and Φ_o are constants of integration and we set $Q_o = -N_c + N_f/2$. The $SU(3)$-structure for this class of backgrounds is specified by a (would-be) Kähler form J and a holomorphic three-form Ω, which may be written explicitly as follows (cf. [106] for the unflavoured case):

$$J^{(0)} = e^\rho \wedge e^3 + e^\theta \wedge \left(-\cos\mu\, e^\varphi + \sin\mu\, e^2\right) + e^1 \wedge \left(-\sin\mu\, e^\varphi - \cos\mu\, e^2\right),$$

$$\Omega^{(0)} = \left(e^\rho + i\, e^3\right) \wedge [e^\theta + i\left(-\cos\mu\, e^\varphi + \sin\mu\, e^2\right)] \wedge [e^1 + i\left(-\sin\mu\, e^\varphi - \cos\mu\, e^2\right)],$$

$$(5.3.8)$$

where the angle $0 < \mu < \pi/2$ corresponds to a rotation in the $e^\varphi - e^2$ plane and is given by

$$\cos\mu = \frac{P - \cosh\tau\, Q}{P\cosh\tau - Q}. \qquad (5.3.9)$$

The $SU(3)$-structure for the transformed solution is now obtained simply from (5.3.8) via the rescalings (5.2.9).

Charge quantisation

Given a solution of (5.3.7), one immediately obtains the full string background via the above relations. In particular, the new background is obtained as in (5.2.11) and it depends on the parameters N_c and N_f, which, in the case $\zeta = 0$, can be interpreted respectively as the number of colour and flavour D5-branes. However, this interpretation should be reconsidered for the transformed backgrounds. Notice that, in the presence of sources, one should be careful with the D5 charge-quantisation condition for the original background. In particular, since $F_{(3)}$ is not closed, its integral over the three-cycle at infinity depends on the representative submanifold, hence it cannot be quantised. We therefore *define* the number of colour D5-branes by integrating, over the three-cycle, $F_{(3)}$ evaluated at $N_f = 0$. Since the latter is closed, this definition makes sense and we have

$$\frac{1}{2\kappa_{10}} \int_{S^3} F_{(3)}|_{N_f=0} = N_c T_5 \,, \tag{5.3.10}$$

where S^3 is any representative of the unique three-cycle at infinity. For the transformed background, this should be modified to

$$\frac{1}{2\kappa_{10}} \int_{S^3} F_{(3)}|_{N_f=0} = \widetilde{N}_c T_5 = \frac{N_c}{\kappa_1} T_5 \in \mathbb{N} \,. \tag{5.3.11}$$

We also redefine the number of flavours using

$$\mathrm{d}F_{(3)} = \frac{\widetilde{N}_f}{4} \sin\theta \sin\tilde\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}\tilde\theta \wedge \mathrm{d}\tilde\varphi = \frac{N_f}{4\kappa_1} \sin\theta \sin\tilde\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}\tilde\theta \wedge \mathrm{d}\tilde\varphi. \tag{5.3.12}$$

Noticing that P, Q and $J^{(0)}$ are homogeneous in N_c and N_f of degree one, so it follows that the rotated background, in *string frame*, becomes

$$\begin{aligned}
\mathrm{d}s^2_{str} &= h^{-1/2}\mathrm{d}x^2_{1,3} + e^{2\Phi}h^{1/2}\kappa_1 \mathrm{d}s^2_6(\widetilde{N}_c, \widetilde{N}_f) \,, \\
F_{(3)} &= e^{-2\Phi} *_6 \mathrm{d}\left(e^{2\Phi} J^{(0)}(\widetilde{N}_c, \widetilde{N}_f) \right) \,, \\
B_{(2)} &= \kappa_2 \, e^{2\Phi} J^{(0)}(\widetilde{N}_c, \widetilde{N}_f) \,, \\
F_{(5)} &= -\kappa_2(1 + *_{10})\mathrm{d}h^{-1} \wedge \mathrm{vol}_{(4)} \,.
\end{aligned} \tag{5.3.13}$$

Using the expression (5.2.12) for h, it is clear that the constant κ_1 can be absorbed into the rescaled charges $\widetilde{N}_c$ and $\widetilde{N}_f$. Namely, defining

$$\hat{h} = e^{-2\Phi} - \kappa_2^2 \,, \tag{5.3.14}$$

and absorbing κ_1 by a trivial rescaling of the world-volume coordinates $x^i \to \kappa_1^{-1} x^i$, we have

$$
\begin{aligned}
\mathrm{d}s_{str}^2 &= \hat{h}^{-1/2}\mathrm{d}x_{1,3}^2 + e^{2\Phi}\hat{h}^{1/2}\mathrm{d}s_6^2(\widetilde{N}_c, \widetilde{N}_f)\,, \\
F_{(3)} &= e^{-2\Phi} *_6 \mathrm{d}\left(e^{2\Phi} J^{(0)}(\widetilde{N}_c, \widetilde{N}_f)\right)\,, \\
B_{(2)} &= \kappa_2\, e^{2\Phi} J^{(0)}(\widetilde{N}_c, \widetilde{N}_f)\,, \\
F_{(5)} &= -\kappa_2(1 + *_{10})\mathrm{d}\hat{h}^{-1} \wedge \mathrm{vol}_{(4)}\,.
\end{aligned}
\tag{5.3.15}
$$

It follows that the effect of the rotation described in the previous section is simply the introduction of the parameter κ_2, with $0 \le \kappa_2 \le \max\{e^{-\Phi}\}$. For $\kappa_2 = 0$, we recover the original background. To make contact with the discussion in [106], we may parametrise κ_2 as

$$
\kappa_2 = e^{-\Phi_\infty} \tanh\beta,
\tag{5.3.16}
$$

where Φ_∞ is the asymptotic value of the dilaton. In [106], this transformation was derived as a simple chain of U-dualities, and the constant β arose as a boost parameter in eleven dimensions. However, the derivation presented here (see also [107]) may be readily applied to cases with sources.

5.3.2   The flavoured resolved deformed conifold

We now present a deformation of the solution of Butti *et al.* [110], describing the baryonic branch of the Klebanov-Strassler theory [109], induced by the backreaction of D5-brane sources. We start by first reviewing some of the material in [106].

Review of the unflavoured solution

Before we present the flavoured solution, let us recall the unflavoured solution [110, 21, 29, 106]. This solution is obtained by setting $\widetilde{N}_f = 0$ in (5.3.15), $Q_o = -\widetilde{N}_c$, $\rho_o = 0$ and picking a specific solution of the differential equation (5.3.7) for P. The solution is only known numerically, but one can easily determine its IR and UV asymptotic forms, which are specified in terms of two arbitrary constants, h_1 and c:

$$
P = h_1\rho + \frac{4h_1}{15}\left(1 - \frac{4\widetilde{N}_c^2}{h_1^2}\right)\rho^3 + \frac{16h_1}{525}\left(1 - \frac{4\widetilde{N}_c^2}{3h_1^2} - \frac{32\widetilde{N}_c^4}{3h_1^4}\right)\rho^5 + \mathcal{O}(\rho^7)\,, \quad \rho \to 0,
$$

$$
P = c\,e^{4\rho/3} + \frac{4\widetilde{N}_c^2}{c}\left(\rho^2 - \rho + \frac{13}{16}\right)e^{-4\rho/3} + \mathcal{O}(\rho e^{-8\rho/3})\,, \qquad \rho \to \infty. \tag{5.3.17}
$$

In the full solution, the two constants are related in a non-trivial way [106][5], given in (B.41) of [10]. What is important for the present discussion is that $h_1(c)$ as a function of c takes values in $[2, +\infty)$, while $c \in [0, +\infty)$, with $h_1(0) = 2$. One can also construct the solution in an expansion for large c. One then finds, via (5.3.6), that the dilaton takes the form

$$e^{2\Phi} = e^{2\Phi_\infty} \left(1 - \frac{1}{c^2} h_{KS}(\rho) + \mathcal{O}\left(c^{-4}\right) \right), \qquad (5.3.18)$$

where

$$e^{2\Phi_\infty} \equiv \sqrt{\frac{3}{2}} \frac{e^{2\Phi_o}}{c^{3/2}}, \qquad (5.3.19)$$

and

$$h_{KS} = 2^{1/3} \widetilde{N}_c^2 \int_\rho^\infty \frac{d\rho'}{\sinh^2(2\rho')} \left(2\rho' \coth(2\rho') - 1 \right) \left(\sinh(4\rho') - 4\rho' \right)^{1/3}, \qquad (5.3.20)$$

is the Klebanov-Strassler warp factor (cf. equation (90) in [109]).

Let us recall some limits of this two-parameter family of solutions discussed in [106]:

- **$\beta \to 0$**

  This is the original background before adding the D3-brane charge [21], namely it describes *wrapped D5-branes.* The interpretation of the parameter c was discussed in [29, 106]. Taking $c \to 0$ corresponds to going to the near-brane limit, which is the solution discussed in [13, 87]. In this decoupling limit, the theory on the fivebranes was argued to flow to pure $\mathcal{N} = 1$ super Yang-Mills in the IR [13, 87]. In the opposite $c \to \infty$ limit, the metric approaches the deformed conifold with three-form flux.

- **$\beta \to \infty$**

  This limit ensures that the constant term in the warp factor in (5.3.14) is removed and the leading term in the UV is dominated by h_{KS}. The expansion

---

[5]These constants are related to the parameters γ, t_∞ and U in [106] as follows:

$$h_1 = 2\gamma^2 \widetilde{N}_c, \qquad c = \frac{\widetilde{N}_c}{6} e^{-\frac{2}{3} t_\infty}, \qquad U = \frac{2\widetilde{N}_c}{c}.$$

Moreover, $\widetilde{N}_c$ here is $\widetilde{M}$ in [106]. The complete map to the variables used in [106] includes $t_{MM} = 2\rho$, $\tau_{MM} = t$, $c_{MM} = P/\widetilde{N}_c$, and $f_{MM} = 4P/\widetilde{N}_c$.

in large c does not terminate and c remains as the only non-trivial parameter of the solution in addition to Φ_∞. This solution describes the *baryonic branch* of the Klebanov-Strassler theory and the parameter c is related to the baryonic branch VEV as $U \propto c^{-1}$ [110].

- $\beta \to \infty$ *and* $c \to \infty$

  In this case, the warp factor $\hat{h}$ in (5.2.11) is replaced by h_{KS} and hence the background is the *Klebanov-Strassler* background [109]. In particular, the unwarped internal metric is the deformed conifold metric. The only free parameter is the asymptotic value of the dilaton Φ_∞. The deformation parameter ϵ of the deformed conifold may be reabsorbed by a rescaling of the metric.

- $\beta \to \infty$ *and* $c \to 0$

  This is the limit of large VEVs on the baryonic branch solution. In [106], it was shown that there is a large region where the dilaton is approximately constant and the solution is well approximated by the solution of [114]. It was then argued that, in this limit, one approaches the wrapped fivebrane theory, but with a $B_{(2)}$ field on the two-sphere, induced by the "rotation" procedure. Therefore, in this case, the theory is well described by fivebranes wrapped on a *fuzzy two-sphere*.

The new flavoured solution

Let us now present the new flavoured resolved deformed conifold solution. Note that the analysis so far is general enough to allow for smeared D5-sources and, therefore, we only need to find a new solution of the "master equation" (5.3.7) with $\widetilde{N}_f \neq 0$, subject to the condition that it reduces to the solution of [110] in the limit $\widetilde{N}_f \to 0$. We have found this new solution numerically, but again one can systematically determine the IR and UV asymptotics:

$$
\begin{aligned}
P \underset{\rho \to 0}{\sim}\ & h_1 \rho + \frac{4\widetilde{N}_f}{3}\left(-\rho \log \rho - \frac{1}{12}\rho \log(-\log \rho) + \mathcal{O}\left(\frac{\rho \log(-\log \rho)}{\log \rho}\right)\right) \\
& + \mathcal{O}(\rho^3 \log \rho)\,, \\
P \underset{\rho \to \infty}{\sim}\ & c\,e^{4\rho/3} + \frac{9\widetilde{N}_f}{8} + \frac{1}{c}\left((2\widetilde{N}_c - \widetilde{N}_f)^2\left(\rho^2 - \rho + \frac{13}{16}\right) - \frac{81\widetilde{N}_f^2}{64}\right)e^{-4\rho/3} \\
& + \mathcal{O}(\rho e^{-8\rho/3})\,.
\end{aligned}
\tag{5.3.21}
$$

These solutions are different from those discussed in [29] and, in particular, they reduce to their unflavoured counterparts in (5.3.17) in the limit $\widetilde{N}_f \to 0$. However, contrary to the unflavoured solution, the flavoured solution is singular in the IR. Moreover, note that the flavours, $i.e.$ terms proportional to $\widetilde{N}_f$, dominate the IR, as well as the UV, when $\beta \to \infty$ (see (5.3.24) below). Via (5.3.6), we see that these asymptotics imply that the dilaton goes to $-\infty$ in the IR and not to a constant as in the unflavoured case. In particular, for small values of the radial coordinate we have[6]

$$e^{2\Phi} = \frac{3e^{2\Phi_\infty}c^{3/2}}{\widetilde{N}_f^{3/2}(-\log\rho)^{3/2}}\left(1 - \frac{\log(-\log\rho)}{8\log\rho} + \mathcal{O}\left(\frac{1}{\log\rho}\right)\right). \tag{5.3.22}$$

Thus, the warp factor has the following expansion in the IR:

$$\hat{h} = \frac{1}{3}e^{-2\Phi_\infty}\left(\frac{2\widetilde{N}_f}{c}\right)^{3/2}(-\log\rho)^{3/2}\left(1 + \frac{\log(-\log\rho)}{8\log\rho} + \mathcal{O}\left(\frac{1}{\log\rho}\right)\right). \tag{5.3.23}$$

The numerical solution interpolating between the asymptotic behaviours in (5.3.21) is plotted in Figure 5.1. In Figure 5.2, the numerical solution is explicitly compared with the asymptotic solutions (5.3.21) by zooming in the IR and UV regions.

Figure 5.1: Plot of the function $P(\rho)$ and the dilaton for the numerical solution interpolating between the two asymptotic behaviours in (5.3.21). The plots correspond to the values $\widetilde{N}_c = 10$, $\widetilde{N}_f = 20$, $c = 30$, and $h_1 = 100$.

Let us now reconsider the various limits discussed in the previous section for the unflavoured solution:

[6]To recover the limit $\widetilde{N}_f = 0$ at fixed c, the expansions (5.3.22) and (5.3.23) are not useful. This limit is completely smooth as can be seen from the expansions of P in (5.3.21) and the various plots of the numerical solutions.

Figure 5.2: In these graphs, we plot the same numerical solution as in Figure 5.1, but we zoom in on the IR region (left) and on the UV region (right), and we compare the numerical solution (black) with the corresponding asymptotic solutions given in (5.3.21). These are plotted in red (IR solution) and in blue (UV solution).

- $\beta \to 0$

  Again, this is the original D5-brane background before adding the D3-branes. Further taking $c \to 0$ is the near-brane (decoupling) limit. In this case, the resulting solution interpolates between the IR asymptotics given in (5.3.21) and the linear dilaton asymptotics $P \sim |2\widetilde{N}_c - \widetilde{N}_f|\rho + P_o$ (with $P_o = \frac{\widetilde{N}_f}{2}$). The solution is plotted in Figures 5.3 and 5.4, and it is the flavoured generalisation of the wrapped D5 solution of [13, 87].

- $\beta \to \infty$

  Although in this limit we remove the constant term from the warp factor in the UV, the leading form of the warp factor is not dominated by h_{KS} any more, but by the term introduced by the sources:

$$\hat{h} = e^{-2\Phi_\infty}\left(\frac{2^{2/3}\widetilde{N}_f}{c}\int_\rho^\infty d\rho'\,(\sinh(4\rho') - 4\rho')^{-1/3} + \mathcal{O}(1/c^2)\right). \qquad (5.3.24)$$

  It follows that the UV asymptotic behaviour of the flavoured solution is *different* from the Klebanov-Strassler asymptotics. This leads to a different field-theory picture, as discussed in [10]

- $c \to \infty$ *and* $\beta \to \infty$

  This limit cannot be taken naively in the flavoured case, due to the fact that the flavours dominate the UV after the leading constant term in the warp factor is removed. The reason why we cannot go to the Klebanov-Strassler limit, while keeping the flavour D5-branes, is that this is not a

supersymmetric configuration [115]. Therefore, we consider the limit of $c \to \infty$ and $\widetilde{N}_f \to 0$ at fixed $c\,\widetilde{N}_f$. This limit is the subject of the next subsection. In Section 4 of [10], it is argued that the field-theory interpretation of this solution is a modification (by Higgsing) of the Klebanov-Strassler cascade.

- $\beta \to \infty$ *and* $c \to 0$

  One can perform an analysis similar to that in [106] for the solution in this range of parameters, and show that there is again a large region where the solution is well approximated by a resolved conifold metric, with addition of fluxes *and* sources. This suggests that there should exist an exact Type IIB solution analogous to that in [114], modified by the presence of sources. It would be interesting to find this solution.
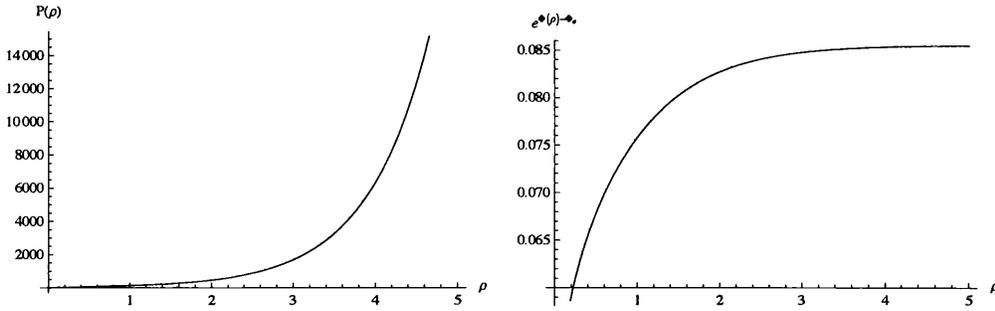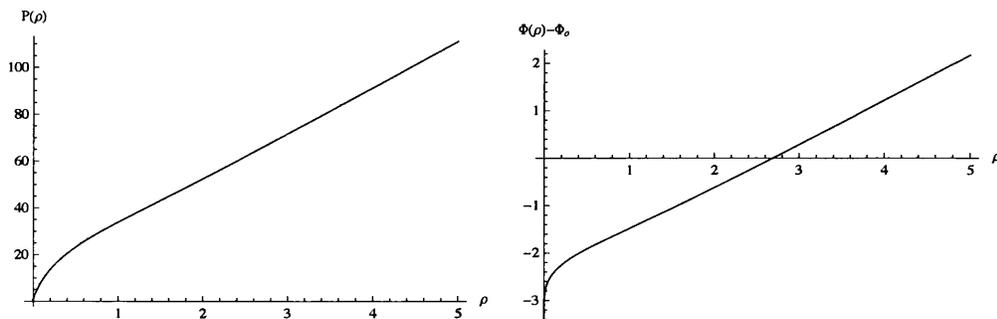

Figure 5.3: Plot of the function $P(\rho)$ and the dilaton for the numerical solution interpolating between the IR asymptotic behaviour in (5.3.21) and the linear UV asymptotics $P \sim |2\widetilde{N}_c - \widetilde{N}_f|\rho$. The plots correspond to the values $\widetilde{N}_c = 20$, $\widetilde{N}_f = 20$, $P_o = 10$ and $h_1 = 25.93$. Contrary to the $\widetilde{N}_f = 0$ case, we do not have an analytic expression for the value of h_1 leading to linear dilaton asymptotics in the UV.

Summary

Let us summarise the effects of the addition of the flavour D5-branes to the unflavoured solution. The ansatz for the metric and fluxes is essentially unchanged and may be parametrised completely in terms of the function P. This function for various values of $\widetilde{N}_f$ is plotted on the left in Figure 5.5. Notice that the leading UV behaviour of P is not affected by the flavours. However, the flavoured solution is actually singular at $\rho = 0$. After introducing the D3-branes, the six-dimensional metric ds_6^2 is unchanged, and is warped by the warp factor (5.3.14). This picks up
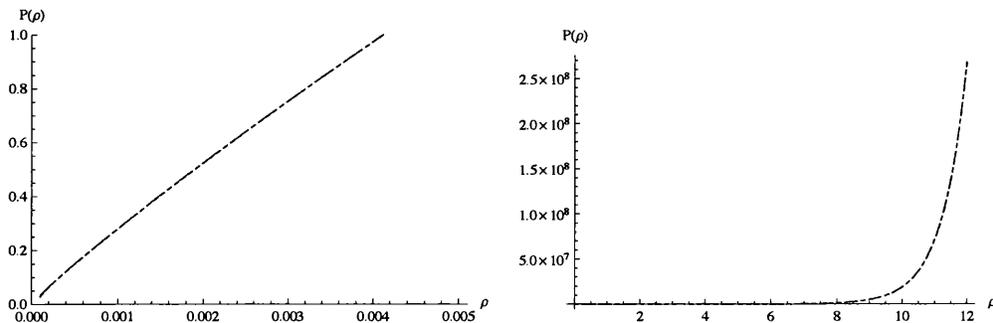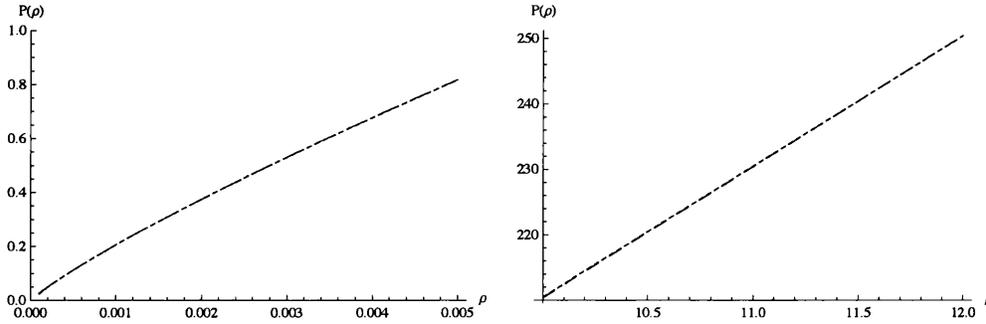
Figure 5.4: Here, we plot the same numerical solution as in Figure 5.3, but we zoom in on the IR region (left) and on the UV region (right), and we compare the numerical solution (black) with the IR asymptotic solution given in (5.3.21) and the UV asymptotic solution $P \sim |2\widetilde{N}_c - \widetilde{N}_f|\rho$. These are plotted in red (IR solution) and in blue (UV solution).
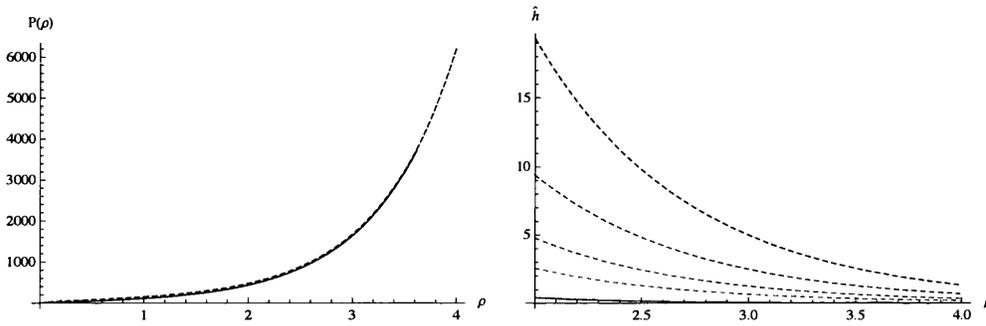
Figure 5.5: On the left: plots of $P(\rho)$ for fixed values $c = 30$, $\widetilde{N}_c = 10$ and different values of $\widetilde{N}_f$ (and h_1). The continuous curve is $\widetilde{N}_f = 0$. Superimposed on this are the curves for the following values: $\widetilde{N}_f = 5$ (dotted green), $\widetilde{N}_f = 10$ (dotted red), $\widetilde{N}_f = 20$, (dotted blue), $\widetilde{N}_f = 40$ (dotted black). On the right: different plots of $\hat{h}(\rho)$ for the same values of $\widetilde{N}_f$.

the subleading behaviour of the dilaton in the UV and, therefore, it is sensitive to the $\widetilde{N}_f$ flavours – see the plots of $\hat{h}$ on the right of Figure 5.5. The divergence of $\hat{h}$ at $\rho = 0$ is due to the singularity in the IR. The fall-off at infinity is noticeably slower with respect to the unflavoured case, and we expect that this persists after resolving the IR singularity. In order to understand the physical origin of the UV behaviour, we next discuss the solution in a limit in which the six-dimensional (unwarped) metric becomes an ordinary deformed conifold.

5.3.3 Adding smeared D3-branes to the Klebanov-Strassler theory

We now discuss a solution obtained in the limit $c \to \infty$ (with $\tanh \beta = 1$) of the flavoured solution. This limit can be obtained by inserting the warp factor (5.3.24) in (5.3.14) and sending $c \to \infty$. However, the fact that $\hat{h} \sim 1/c$, and not $\hat{h} \sim 1/c^2$ as is the case for the unflavoured solution, does not allow us to take this limit directly. To obtain a well-defined limit, we set $c\widetilde{N}_f = \nu$, and keep ν fixed in the limit $c \to \infty$. This gives

$$\hat{h} = \frac{1}{c^2}e^{-2\Phi_\infty}\left(2^{2/3}\nu\int_\rho^\infty \mathrm{d}\rho'\,(\sinh(4\rho') - 4\rho')^{-1/3} + h_{KS}\right) + \mathcal{O}(1/c^3)\,. \quad (5.3.25)$$

Inserting this in (5.3.15) and sending $c \to \infty$, we obtain an exact solution (notice that the expressions for $B_{(2)}$ and $F_{(3)}$ below do not scale with the parameter c):

$$\begin{aligned}
\mathrm{d}s_{str}^2 &= e^{\Phi_\infty}\left[h_\nu^{-1/2}\mathrm{d}x_{1,3}^2 + h_\nu^{1/2}\mathrm{d}s_6^2(\widetilde{N}_c,0)\right]\,, \\
F_{(3)} &= *_6\mathrm{d}J^{(0)}(\widetilde{N}_c,0)\,, \\
B_{(2)} &= e^{\Phi_\infty}J^{(0)}(\widetilde{N}_c,0)\,, \\
F_{(5)} &= -e^{-\Phi_\infty}(1 + *_{10})\mathrm{d}h_\nu^{-1}\wedge \mathrm{vol}_{(4)}\,,
\end{aligned} \qquad (5.3.26)$$

where $\mathrm{d}s_6^2(\widetilde{N}_c,0)$ is the *deformed conifold* metric and we defined

$$h_\nu = 2^{2/3}\nu\int_\rho^\infty \mathrm{d}\rho'\,(\sinh(4\rho') - 4\rho')^{-1/3} + h_{KS}\,. \qquad (5.3.27)$$

To understand the significance of this solution, notice that $\nu \neq 0$ leads to

$$\mathrm{d}F_{(5)} - H_{(3)}\wedge F_{(3)} = B_{(2)}\wedge \Xi_{(4)} \neq 0\,, \qquad (5.3.28)$$

where the term on the right-hand side of this equation may be interpreted as the contribution from D3-brane sources smeared on the transverse directions. Indeed, the world-volume action for these sources arises in this limit, as discussed in Subsection 5.2.2.

Going back to the solution *before* taking the $c \to \infty$ limit, we can think of the term $B_{(2)}\wedge \Xi_{(4)}$ in the second equation in (5.2.15) as a D3-brane charge density induced on the D5-brane sources by the $B_{(2)}$ field on their world-volume. Then, we can compute the density of D3-branes by integrating the $B_{(2)}$ field pulled back

onto the world-volume cylinder wrapped by the D5-branes, where we put a cut-off at some radial distance. The result is also valid in the $c \to \infty$ limit. Namely, we may define a running number[7] of D3-brane sources as

$$n_f = \frac{\widetilde{N}_f}{(2\pi)^2} \int_{\text{cylinder}} B_{(2)} \propto \widetilde{N}_f \, e^{-\Phi_\infty} \int^\rho e^{2\Phi + 2k} \mathrm{d}\rho' \,, \qquad (5.3.29)$$

where the factor of $\widetilde{N}_f$ comes from the overall factor in front of the action for the flavour D5-branes. Expanding this in the UV, we get ($g_s = e^{\Phi_\infty}$)

$$n_f \sim g_s \nu \, e^{4\rho/3} \qquad \text{for} \quad \rho \to \infty \,. \qquad (5.3.30)$$

The interpretation of this quantity becomes clear if we look at the asymptotic form of the warp factor in the standard radial coordinate $r \sim e^{2\rho/3}$. The leading term of the warp factor in the UV goes like

$$h_\nu \sim \frac{\nu r^2 + \widetilde{N}_c^2 \log r}{r^4} \qquad \text{for} \quad r \to \infty \,. \qquad (5.3.31)$$

Expressing this in terms of the running number of D3-brane sources n_f, and running number of bulk D3-branes [109]

$$n \sim k \widetilde{N}_c \sim g_s \widetilde{N}_c^2 \log r \qquad \text{for} \quad r \to \infty \,, \qquad (5.3.32)$$

this takes the form

$$g_s h_\nu \sim \frac{n_f + n}{r^4} \qquad \text{for} \quad r \to \infty \,. \qquad (5.3.33)$$

This shows that there are precisely $n_f + n$ D3-branes in the background and reduces to the Klebanov-Strassler expression for $\nu = 0$. Notice that the running of the source and bulk D3-branes is quite different, and in particular the former dominates the UV.

The limiting solution is again singular near $\rho = 0$. However, this singularity comes entirely from the warp factor (5.3.27), while the metric $\mathrm{d}s_6^2(\widetilde{N}_c, 0)$ is the smooth deformed conifold metric. This singularity is due to the fact that the D3-sources are distributed uniformly along the radial direction down to $\rho = 0$.

---

[7]In the following formulas we ignore numerical factors.

5.4 Discussion

In this chapter, we have used a solution-generating transformation, applicable to a large class of supersymmetric Type IIB backgrounds, to construct a new family of solutions generalising the resolved deformed conifold of Butti *et al.* [110]. Using this method, we can take any solution of the torsional superstring equations [75, 113, 48] and generate a solution where various RR and NS fields are turned on. The method may be applied to solutions which include the backreaction of smeared source branes, usually referred to as flavour branes. In particular, we have applied this procedure to a solution representing a system of N_c D5-branes wrapped on the two-sphere inside the resolved conifold, with addition of N_f flavour D5-branes wrapped on a transverse infinitely extended cylinder. The final solution is then a warped resolved deformed conifold, modified by the backreaction of the extra flavour branes.

The flavoured solution differs qualitatively from the unflavoured one in two ways. Firstly, it is singular in the IR. Secondly, the UV asymptotics is not the (logarithmic) Klebanov-Strassler one. Although we have not addressed the resolution of the IR singularity in this chapter, we expect it may be resolved by considering a profile for the flavour D5-branes that vanishes smoothly in the IR (see Chapter 7). The different behaviour in the UV is induced by the presence of a uniform distribution of D3-brane sources, smeared on the transverse geometry, up to infinity. We have explained that these D3-branes are induced by the presence of the flavour D5-branes in a geometry with a non-trivial $B_{(2)}$ field. In other words, the "rotation" procedure has the effect of adding bulk D3-branes, coming from the original colour D5-branes, and smeared D3-brane sources, coming from the original flavour D5-branes.

In the next chapter, we present a similar solution-generating technique, but this time applied in the case of Type IIA supergravity backgrounds with a G_2-structure.

5.A   Type IIB equations of motion with sources

In this appendix, we state the equations of motion coming from the action (5.2.13). Let us first rewrite the total action, comprising the Type IIB supergravity action plus the supersymmetric source action:

$$S = S_{IIB} + S_{\text{sources}} \,, \tag{5.A.1}$$

where

$$S_{IIB} = \int \sqrt{-g} \left(R - \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi \right) + \frac{1}{2} \int C_{(4)} \wedge F_{(3)} \wedge H_{(3)}$$
$$- \frac{1}{2} \int \left(e^{2\Phi} F_{(1)} \wedge *F_{(1)} + e^{-\Phi} H_{(3)} \wedge *H_{(3)} + e^{\Phi} F_{(3)} \wedge *F_{(3)} + \frac{1}{2} F_{(5)} \wedge *F_{(5)} \right), \tag{5.A.2}$$

and

$$S_{\text{sources}} = - \int \left(e^{4\Delta + \Phi/2} \text{vol}_{(4)} \wedge \left(\cos\zeta e^{2\Delta} J + \sin\zeta e^{-\Phi/2} B_{(2)} \right) - C_{(6)} + C_{(4)} \wedge B_{(2)} \right) \wedge \Xi_{(4)}. \tag{5.A.3}$$

In the following, we set $F_{(1)} = 0$. The modified Bianchi identities for the fluxes read

$$\mathrm{d}H_{(3)} = 0 \,,$$
$$\mathrm{d}F_{(3)} = \Xi_{(4)} \,, \tag{5.A.4}$$
$$\mathrm{d}F_{(5)} = H_{(3)} \wedge F_{(3)} + B_{(2)} \wedge \Xi_{(4)} \,.$$

The equations of motion for the fluxes are given by

$$\mathrm{d}\left(e^{-\Phi} * H_{(3)}\right) = F_{(3)} \wedge F_{(5)} + \sin\zeta \, e^{4\Delta} \text{vol}_{(4)} \wedge \Xi_{(4)} \,,$$
$$\mathrm{d}\left(e^{\Phi} * F_{(3)}\right) = -H_{(3)} \wedge F_{(5)} \,. \tag{5.A.5}$$

Notice that, in the equation of motion for $H_{(3)}$, the term coming from $C_{(4)} \wedge B_{(2)} \wedge \Xi_{(4)}$ in the source action is exactly cancelled by a contribution from the Chern-Simons term of the Type IIB supergravity equations. We then define the following notation:

$$\omega_{(p)} \lrcorner \lambda_{(p)} = \frac{1}{p!} \omega^{\mu_1 \cdots \mu_p} \lambda_{\mu_1 \cdots \mu_p} \,. \tag{5.A.6}$$

Using it, we can show that

$$\int \omega_{(p)} \wedge \lambda_{(10-p)} = - \int \sqrt{-g} \, \lambda \lrcorner (*\omega) \,. \tag{5.A.7}$$

Then, we can write the dilaton equation of motion as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\Phi\right) = \frac{1}{12}e^\Phi F_{(3)}^2 - \frac{1}{12}e^{-\Phi}H_{(3)}^2 - \frac{1}{2}e^{\Phi/2}\Xi_{(4)}\lrcorner *\left(\cos\zeta\, e^{6\Delta}\mathrm{vol}_{(4)}\wedge J\right).$$

$$(5.A.8)$$

Finally, the Einstein equation is

$$
\begin{aligned}
R_{\mu\nu} =& \frac{1}{2}\partial_\mu\Phi\,\partial_\nu\Phi + \frac{1}{48}e^\Phi\left(12F_{\mu\rho\sigma}F_\nu{}^{\rho\sigma} - g_{\mu\nu}F_{(3)}^2\right) + \frac{1}{48}e^{-\Phi}\left(12H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - g_{\mu\nu}H_{(3)}^2\right) \\
&+ \frac{1}{96}F_{\mu\rho_1\rho_2\rho_3\rho_4}F_\nu{}^{\rho_1\rho_2\rho_3\rho_4} - \frac{1}{240}\sin\zeta\, e^{4\Delta}\left((B_{(2)}\wedge\Xi_{(4)})_{\mu\rho_1\dots\rho_5}\left(*\mathrm{vol}_{(4)}\right)_\nu{}^{\rho_1\dots\rho_5}\right) \\
&- \frac{1}{24}e^{6\Delta+\Phi/2}\cos\zeta\left(2\Xi_{\mu\rho_1\rho_2\rho_3} *\left(\mathrm{vol}_{(4)}\wedge J\right)_\nu{}^{\rho_1\rho_2\rho_3} - 3g_{\mu\nu}\Xi_{(4)}\lrcorner *\left(\mathrm{vol}_{(4)}\wedge J\right)\right) \\
&+ \frac{1}{4}\sin\zeta\, e^{4\Delta}g_{\mu\nu}\left((B_{(2)}\wedge\Xi_{(4)})\lrcorner\left(*\mathrm{vol}_{(4)}\right)\right).
\end{aligned}
$$

$$(5.A.9)$$

Chapter 6

Rotation for G_2-structure

6.1 Introduction

In this chapter, we present a solution-generating technique similar to the one developed in the previous chapter, but this time concerning solutions of Type IIA supergravity with G_2-structure. This is based on [11], done in collaboration with Martelli.

The simplest and most studied Calabi-Yau singularity in string theory is the conifold [116]. There are two distinct desingularisations of this, in which the singularity at the tip is replaced by a three-sphere or a two-sphere. These are referred to as *deformation* and *resolution*, respectively. The relevance of the transition between the resolved and deformed conifold in string theory was emphasised in [117]. The deformed conifold geometry underlies the Klebanov-Strassler solution [109], which is dual to a four-dimensional $\mathcal{N} = 1$ field theory displaying confinement. A different supergravity solution, dual to a closely related field theory, was discussed in [13, 87]. This arises as the decoupling limit of a configuration of fivebranes wrapped on the two-sphere of the resolved conifold.

A solution of Type IIB supergravity that contains as special cases the Klebanov-Strassler and the Maldacena-Nuñez solutions was constructed in [110]. This was interpreted as the gravity dual to the baryonic branch of the Klebanov-Strassler theory, with a non-trivial parameter related to the VEV of the baryonic operators [118, 119]. It was later pointed out in [106] that the solution of [110] is related to a simpler solution [21] corresponding to fivebranes wrapped on the two-sphere of the resolved conifold, without taking any near-brane limit. In this context, the non-trivial parameter can roughly be viewed as the size of the two-sphere

wrapped by the branes. When this is very large, the solution looks like the resolved conifold with branes on the two-sphere and when this is very small, it looks like the deformed conifold with flux on the three-sphere. Therefore, it displays an explicit realisation of the geometric transition described in [64]. The key fact that allows the connection of the resolved and deformed conifolds at the classical level is that the solution is an example of non-Kähler, or *torsional*, geometry [75, 113]. For related work, see [120, 121].

In this chapter, we present a G_2 version of the picture advocated in [106]. In particular, we discuss supergravity solutions which correspond to M fivebranes wrapped on the three-sphere inside the G_2-holonomy manifold $S^3 \times \mathbb{R}^4$ [70, 122], without taking any near-brane limit. Solutions of this type were previously discussed in [36]. If we take the near-brane limit, we find the solutions discussed by Maldacena-Nastase in [123], based on [124]. These were argued in [123] to be the gravity dual of $\mathcal{N} = 1$ supersymmetric $U(M)$ Chern-Simons theories in three dimensions, with Chern-Simons level $|k| = M/2$.

The (classical) moduli space of asymptotically conical G_2-holonomy metrics on $S^3 \times \mathbb{R}^4$ comprises *three* branches, that we denote X_i [66], intersecting on the singular cone. These three branches are related to the three branches of the conifold moduli space, namely the deformation, the resolution and the flopped resolution [61, 66]. One therefore expects close analogies with the discussion in [106]. Indeed, by working in the context of *torsional G_2 manifolds* [24], we find a set of one-parameter families of solutions that pairwise interpolate between the three classical branches of G_2-holonomy. The non-trivial parameter in these solutions can roughly be viewed as the size of the three-sphere wrapped by the fivebranes. When this is very large, the solution looks like a G_2 manifold X_i with branes on a three-sphere and when this is very small, it looks like a distinct G_2 manifold X_j ($i \neq j$) with flux on a different three-sphere. This then realises a G_2 *geometric transition*. However, we do not attempt to relate this to a "large N duality" as in the original discussion in [64]. More concretely, we find *six* distinct solutions connecting the three classical branches X_i, that we denote X_{ij}. In contrast to the conifold case, we can go from X_i with branes to X_j with flux or, conversely, from X_j with branes to X_i with flux, hence $X_{ij} \neq X_{ji}$. A rather different connection of the three classical branches was discussed in [66], where it was related to quantum effects in M-theory. Our discussion, on the other hand, is purely classical and ten-dimensional.

Starting from these relatively simple solutions, we construct Type IIA solutions with D2-brane charge and RR $C_{(3)}$ field. This can be done by applying a simple

transformation analogous to the one discussed in [106] and further studied in Chapter 5 (see also [107]). The solutions that we obtain in this way have a warp factor (in the string-frame metric) that becomes constant at infinity, thus the geometry merges into an ordinary G_2-holonomy manifold. By taking a scaling limit, we obtain solutions that become asymptotically AdS$_4 \times S^3 \times S^3$, albeit only in string frame. If we further tune the non-trivial parameter, we recover the backgrounds of [125], corresponding to D2-branes and fractional NS5-branes transverse to the G_2 manifold $S^3 \times \mathbb{R}^4$. Thus, our solutions may be thought of as one-parameter deformations of the latter and are analogous to the baryonic branch deformation [110] of the Klebanov-Strassler geometry [109]. It is therefore very tempting to think that there should be a close relation between the supersymmetric Chern-Simons theory discussed in [123] and the three-dimensional field theory on the D2-branes.

6.2   Review of the G_2-holonomy manifold $S^3 \times \mathbb{R}^4$

In this section, we review some aspects of the G_2-holonomy manifold $S^3 \times \mathbb{R}^4$ that are relevant for our discussion. We follow closely the presentation in [66].

6.2.1   The manifold and its topology

The non-compact seven-dimensional manifold defined by

$$X = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = \epsilon, \quad x_i, y_i, \epsilon \in \mathbb{R}\} \qquad (6.2.1)$$

is the spin bundle over S^3 and is topologically equivalent to the manifold $S^3 \times \mathbb{R}^4$. For $\epsilon > 0$, the S^3 corresponds to the locus $y_i = 0$ and the coordinates y_i parametrise the normal $\mathbb{R}^4$ directions. This manifold admits a Ricci-flat metric with G_2-holonomy, that at infinity approaches the cone metric

$$ds_{\text{cone}}^2 = dt^2 + t^2 ds^2(Y), \qquad (6.2.2)$$

where $Y \cong S^3 \times S^3$. The Einstein metric $ds^2(Y)$ is not the product of round metrics on the two three-spheres. It is in fact a nearly Kähler metric, which may be described in terms of $SU(2)$ group elements as follows [66]. Consider three elements $a_i \in SU(2)$ obeying the constraint

$$a_1 a_2 a_3 = 1. \qquad (6.2.3)$$

There is an $SU(2)^3$ action preserving this relation given by $a_i \rightarrow u_{i+1}a_iu_{i-1}^{-1}$, with $u_i \in SU(2)$, where the index i is defined modulo 3. There is also an action by a "triality" group Σ_3, which is isomorphic to the group of permutations of three elements. This is an outer automorphism of the group $SU(2)^3$ and may be generated by[1]

$$
\begin{aligned}
\sigma_{31} &: (a_1, a_2, a_3) \rightarrow \left(a_3^{-1}, a_2^{-1}, a_1^{-1}\right), \\
\sigma_{231} &: (a_1, a_2, a_3) \rightarrow (a_2, a_3, a_1).
\end{aligned}
\tag{6.2.4}
$$

The full list of group elements is given by $\Sigma_3 = \{e = \sigma_{123}, \sigma_{231}, \sigma_{312}, \sigma_{12}, \sigma_{31}, \sigma_{23}\}$, with actions on a_i following from (6.2.4).

There are three different seven-manifolds X_1, X_2, X_3, all homeomorphic to $S^3 \times \mathbb{R}^4$, which can be obtained smoothing out the cone singularity by blowing up three different three-spheres inside Y. These are permuted among each other by the action of Σ_3. This can be seen from the description of the base $Y \cong S^3 \times S^3$ in terms of triples of group elements $(g_1, g_2, g_3) \in SU(2)^3$ subject to an equivalence relation $g_i \cong g_ih$ with $h \in SU(2)$, and is related to the previous description by setting $a_i = g_{i+1}g_{i-1}^{-1}$. We can consider three different compact seven-manifolds X_i', bounded by Y, obtained in each case by allowing g_i to take values in the four-ball $\mathbb{B}^4$. The non-compact seven-manifolds obtained after omitting the boundary are precisely the X_i. By setting $h = g_{i-1}^{-1}$, we see that each X_i has topology $S^3 \times \mathbb{R}^4$, where S^3 and $\mathbb{R}^4$ are parametrised by g_{i+1} and g_i, respectively. We review explicit metrics with G_2-holonomy in the three different cases in the next subsection.

We can define three *sub-manifolds* of Y as

$$
C_i = \{a_i = 1\} \cong S^3,
\tag{6.2.5}
$$

which also extend to sub-manifolds in X_i, defined at some constant t. These are topologically three-spheres but, as cycles in Y and X_i, they are not independent since the third Betti numbers of these manifolds are $b_3(Y) = 2$ and $b_3(X_i) = 1$ respectively. In fact, we have the following homology relation:

$$
[C_1] + [C_2] + [C_3] = 0 \quad \text{in } Y.
\tag{6.2.6}
$$

As cycles in X_i, the $[C_i]$ must obey an additional relation, which, in view of their construction above, is simply given by $[C_i] = 0$ in X_i. Therefore the third homology group $H_{(3)}(X_i; \mathbb{Z})$ is generated by C_{i-1} or C_{i+1}, with $[C_{i-1}] = -[C_{i+1}]$.

---

[1]In the notation of [66], the generators of the group Σ_3 are denoted as $\sigma_{31} = \alpha$ and $\sigma_{231} = \beta$.

6.2.2   A triality of G_2-holonomy metrics

A seven-dimensional manifold is said to be a G_2-holonomy manifold if the holonomy group of the Levi-Civita connection ∇ is contained in $G_2 \subset SO(7)$. It is well known that these are characterised by the existence of a G_2-invariant three-form ϕ (associative three-form), together with its Hodge dual $*\phi$, which are both closed:

$$\mathrm{hol}(\nabla) \subseteq G_2 \qquad \text{iff} \qquad d\phi = d * \phi = 0 \,. \tag{6.2.7}$$

The metric compatible with these is Ricci-flat and there exists a covariantly constant spinor, $\nabla \eta = 0$. The G_2-invariant forms can be constructed from the constant spinor as bilinears $\phi_{abc} = \eta^T \gamma_{abc}\, \eta$, $*\phi_{abcd} = \eta^T \gamma_{abcd}\, \eta$, and the metric is then uniquely determined by these. More generally, the two invariant forms define a G_2-*structure* on the seven-dimensional manifold. See for example [24].

An explicit G_2-holonomy metric on the spin bundle over S^3 was constructed in [70, 122]. In [126] were presented G_2-holonomy metrics on each X_i, characterised by three distinct values of a parameter $\lambda = 0, \pm 1$. We rederive those results in a way that is suitable for a generalisation to be discussed in the next section. We define the following left-invariant $SU(2)$-valued one-forms[2] on $SU(2)^3$:

$$a_1^{-1}\mathrm{d}a_1 \equiv -\frac{i}{2}\sigma_i \tau_i \,, \qquad a_2\,\mathrm{d}a_2^{-1} \equiv -\frac{i}{2}\Sigma_i \tau_i \,, \qquad a_3^{-1}\mathrm{d}a_3 \equiv -\frac{i}{2}\gamma_i \tau_i \,, \tag{6.2.8}$$

where τ_i are the Pauli matrices. We can "solve" the constraint (6.2.3) by introducing two sets of angular variables parametrising the first two $SU(2)$ factors. Then, more explicitly, we have

$$\begin{aligned}
\sigma_1 + i\,\sigma_2 &= e^{-i\psi_1}(\mathrm{d}\theta_1 + i\sin\theta_1\,\mathrm{d}\varphi_1)\,, & \sigma_3 &= \mathrm{d}\psi_1 + \cos\theta_1\,\mathrm{d}\varphi_1\,, \\
\Sigma_1 + i\,\Sigma_2 &= e^{-i\psi_2}(\mathrm{d}\theta_2 + i\sin\theta_2\,\mathrm{d}\varphi_2)\,, & \Sigma_3 &= \mathrm{d}\psi_2 + \cos\theta_2\,\mathrm{d}\varphi_2\,,
\end{aligned} \tag{6.2.9}$$

obeying $\mathrm{d}\sigma_3 = -\sigma_1 \wedge \sigma_2, \mathrm{d}\Sigma_3 = -\Sigma_1 \wedge \Sigma_2$ and cyclic permutations. We also have $\gamma_i = M_{ij}(\Sigma_i - \sigma_i)$, where M_{ij} is an $SO(3)$ matrix. See Appendix 6.A for more details. Introducing the notation

$$\mathrm{d}a_1^2 \equiv 2\sum_{i=1}^{3}\sigma_i^2\,, \qquad \mathrm{d}a_2^2 \equiv 2\sum_{i=1}^{3}\Sigma_i^2\,, \qquad \mathrm{d}a_3^2 \equiv 2\sum_{i=1}^{3}(\Sigma_i - \sigma_i)^2\,, \tag{6.2.10}$$

---

[2]The one-form Σ_3 should not be confused with the triality group Σ_3.

we can write a metric ansatz in a manifestly Σ_3 covariant form as

$$\mathrm{d}s^2 = \mathrm{d}t^2 + f_1\,\mathrm{d}a_1^2 + f_2\,\mathrm{d}a_2^2 + f_3\,\mathrm{d}a_3^2\,, \tag{6.2.11}$$

where $f_i(t)$ are three functions. In order to write the G_2 forms compatible with this metric, it is convenient to pass to a different set of functions a, b, ω, defined by

$$f_1 = \frac{a^2}{2} + \frac{b^2}{8}(\omega^2 - 1)\,, \qquad f_2 = \frac{b^2}{4}(1 - \omega)\,, \qquad f_3 = \frac{b^2}{4}(1 + \omega)\,. \tag{6.2.12}$$

In terms of these, the metric then reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 + a^2 \sum_{i=1}^{3} \sigma_i^2 + b^2 \sum_{i=1}^{3} \left(\Sigma_i - \frac{1}{2}(1 + \omega)\sigma_i \right)^2 \tag{6.2.13}$$

and we can introduce an orthonormal frame defined as

$$e^t = \mathrm{d}t\,, \qquad \tilde{e}^a = a\,\sigma_a\,, \qquad e^a = b\left(\Sigma_a - \frac{1}{2}(1 + \omega)\sigma_a \right)\,, \tag{6.2.14}$$

where a is a tangent space index. The associative three-form ϕ may be conveniently written in terms of an auxiliary $SU(3)$-structure as follows:

$$\phi = e^t \wedge J + \mathrm{Re}[e^{i\theta}\Omega]\,. \tag{6.2.15}$$

Here, θ is a phase that needs not be constant and the differential forms J and Ω define the $SU(3)$-structure. In terms of the local frame, they read

$$J = \sum_{a=1}^{3} e^a \wedge \tilde{e}^a\,, \qquad \Omega = (e^1 + i\,\tilde{e}^1) \wedge (e^2 + i\,\tilde{e}^2) \wedge (e^3 + i\,\tilde{e}^3)\,. \tag{6.2.16}$$

We can also rewrite

$$\phi = e^t \wedge J + \cos\theta\,\mathrm{Re}[\Omega] - \sin\theta\,\mathrm{Im}[\Omega]\,,$$
$$*\phi = \frac{1}{2}J \wedge J + (\sin\theta\,\mathrm{Re}[\Omega] + \cos\theta\,\mathrm{Im}[\Omega]) \wedge e^t\,. \tag{6.2.17}$$

Imposing $\mathrm{d}\phi = \mathrm{d} * \phi = 0$ gives the following system of first-order differential equations:

$$Zf_1' = \frac{f_2 f_3}{\sqrt{2}(f_2 + f_3)}\,, \qquad Zf_2' = \frac{f_3 f_1}{\sqrt{2}(f_3 + f_1)}\,, \qquad Zf_3' = \frac{f_1 f_2}{\sqrt{2}(f_1 + f_2)}\,, \tag{6.2.18}$$

where the prime denotes derivative with respect to t and

$$Z \equiv \frac{f_1 f_2 + f_2 f_3 + f_3 f_1}{\sqrt{(f_1 + f_2)(f_2 + f_3)(f_3 + f_1)}}. \tag{6.2.19}$$

This system can be integrated explicitly in terms of three constants c_i defined as

$$c_1^2 = \frac{(f_2 - f_3)^2(f_2 + f_3)}{(f_1 + f_2)(f_1 + f_3)}(f_1 f_2 + f_2 f_3 + f_3 f_1) \tag{6.2.20}$$

and c_2, c_3 obtained by cyclic permutations of this expression. Although *a priori* there are three independent integration constants, at least one of them must vanish, and the other two are then equal. For example, assuming that $c_1 = 0$, then $f_2 = f_3$ and

$$\frac{r_0^6}{216} \equiv c_2^2 = c_3^2 = (f_1 - f_2)^2(f_1 + \frac{f_2}{2}). \tag{6.2.21}$$

After a change of radial coordinate, the metric can be written in the form [66]

$$ds^2(X_1) = \frac{dr^2}{1 - (r_0/r)^3} + \frac{r^2}{72}\left(1 - (r_0/r)^3\right)\left(2da_2^2 - da_1^2 + 2da_3^2\right) + \frac{r^2}{24}da_1^2. \tag{6.2.22}$$

This is a G_2-holonomy metric on the manifold X_1, where the three-sphere C_1 shrinks to zero at the origin and the three-sphere "at the centre" is homologous to C_2 or $-C_3$. In the notation of [126], this corresponds to the metric with $\lambda = 0$. This metric is invariant under σ_{23} or, equivalently, under the interchange of $f_2 \leftrightarrow f_3$. In fact, we can redefine the triality symmetry Σ_3 as acting on the functions f_1, f_2, f_3 in the obvious way[3]. The other two solutions may be obtained, for example, by acting with the cyclic permutation σ_{231}. In the notation of [126], the metric on X_2 corresponds to $\lambda = -1$ and the metric on X_3 to $\lambda = 1$. Notice that the phase θ that enters in the definition of the G_2-structure in (6.2.15) is not constant and, in particular, we have

$$\cos\theta = \lambda\frac{3\sqrt{3}\,r_0^3\,r^{3/2}}{(4r^3 - r_0^3)^{3/2}}, \tag{6.2.23}$$

with $\lambda = 0, \pm 1$. The conical metric

$$ds_{\text{cone}}^2 = dr^2 + \frac{r^2}{36}\left(da_1^2 + da_2^2 + da_3^2\right) \tag{6.2.24}$$

may be obtained from any of these by setting $r_0 = 0$ and is invariant under Σ_3.

---

[3]For the elements of order two, we must also reverse the orientation of the seven-dimensional space.

The parameter space of these metrics is depicted in Figure 6.1.
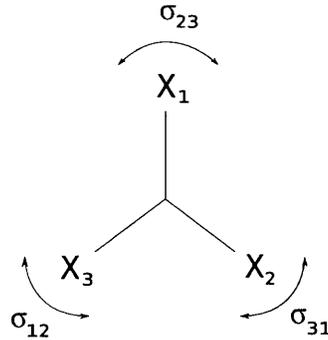
Figure 6.1: The moduli space of asymptotically conical G_2-holonomy metrics on $S^3 \times \mathbb{R}^4$ has three branches permuted by the action of the group Σ_3. The three G_2-holonomy metrics are invariant under elements of order two σ_{ij}, which are reflections about each of the three axes. The intersection point of the three branches corresponds to the singular G_2 cone. In the notation of [126], X_1 corresponds to $\lambda = 0$, X_2 corresponds to $\lambda = -1$ and X_3 corresponds to $\lambda = 1$.

In the figure, moving along an axis corresponds to changing the radius of the three-sphere at the centre of one of the X_i spaces, whose volume is $\frac{2}{3\sqrt{3}}\pi^2 r_0^3$. Hence, in analogy with the deformed conifold, r_0 measures the amount of "deformation" of the conical singularity. However, in analogy with the resolved conifold, we can also define a parameter measuring the amount of "resolution" of a space X_i. Recall that, in the resolved conifold, the resolution parameter may be defined as the difference of volumes of two two-spheres at large distances [106]. In particular, this parameter measures the breaking of a $\mathbb{Z}_2$ symmetry of the singular (and deformed) conifold, consisting in swapping these two-spheres. Here, we can define a triple of resolution parameters, each measuring the breaking of the $\mathbb{Z}_2$ symmetry given by reflection about one of the three axis in Figure 6.1. Following [66], we first consider the "volume defects" of the sub-manifolds C_i^∞ defined at a large constant value of r. In terms of the radial coordinate t, we have the following asymptotic form of the G_2 metrics:

$$\mathrm{d}s^2(X_i) = \mathrm{d}t^2 + \frac{t^2}{36}\left[\mathrm{d}a_1^2 + \mathrm{d}a_2^2 + \mathrm{d}a_3^2 - \frac{r_0^3}{2t^3}\left(\ell_1 \mathrm{d}a_1^2 + \ell_2 \mathrm{d}a_2^2 + \ell_3 \mathrm{d}a_3^2\right) + \mathcal{O}(r_0^6/t^6)\right],$$
$$(6.2.25)$$

where $t = r - r_0^3/(4r^2) + \mathcal{O}(r^{-5})$ and the constants (ℓ_1, ℓ_2, ℓ_3) take the values $(-2, 1, 1)$ for X_1, $(1, -2, 1)$ for X_2 and $(1, 1, -2)$ for X_3. Then, for the "volume

defects", we have

$$\mathrm{vol}(C_i^\infty) = \frac{16}{27}\pi^2 t^3 + \frac{2}{9}\pi^2 r_0^3 \ell_i \,, \tag{6.2.26}$$

and we may define the ith resolution parameter as

$$\begin{aligned} \alpha_i^{\mathrm{res}} &\equiv \mathrm{vol}(C_{i+1}^\infty) - \mathrm{vol}(C_{i-1}^\infty) \\ &= \frac{2}{9}\pi^2 r_0^3 (\ell_{i+1} - \ell_{i-1}) \,. \end{aligned} \tag{6.2.27}$$

To see that this is a sensible definition, let us evaluate α_1^{res} in the three cases X_i. We have

$$\alpha_1^{\mathrm{res}}(X_1) = 0 \,, \qquad \alpha_1^{\mathrm{res}}(X_2) = -\frac{2}{3}\pi^2 r_0^3 \,, \qquad \alpha_1^{\mathrm{res}}(X_3) = +\frac{2}{3}\pi^2 r_0^3 \,. \tag{6.2.28}$$

The interpretation is that the manifold X_1 preserves the $\mathbb{Z}_2$ reflection about the axis 1 hence, from this point of view, r_0 is a "deformation" parameter. On the other hand, the manifolds X_2 and X_3 break this symmetry in opposite directions. From the point of view of X_1, X_2 is a "resolution" and X_3 its flopped version. Notice in particular that we cannot have "resolution" and "deformation" at the same time, exactly as it happens for the conifold in six dimensions. Indeed, the relation to the conifold may be made very precise by considering the different G_2-holonomy metrics (times $\mathbb{R}^{1,3}$) as solutions of M-theory. Then, there exist three different reductions to Type IIA supergravity that give rise to manifolds with topologies of the deformed, resolved, and flopped resolved conifold [66, 61].

Finally, let us recall some facts about the cohomologies of these spaces. For each X_i, the third cohomology group is $H^3(X_i; \mathbb{Z}) = \mathbb{Z}$, so there is only one generator that can be chosen to integrate to one on the non-trivial three-cycle. However, it is convenient to introduce the following set of three-forms:

$$\eta^1 = -\frac{1}{16\pi^2}\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \,, \qquad \eta^2 = \frac{1}{16\pi^2}\Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 \,, \qquad \eta^3 = -\frac{1}{16\pi^2}\gamma_1 \wedge \gamma_2 \wedge \gamma_3 \,, \tag{6.2.29}$$

which are exchanged by the action of Σ_3. We also have that

$$\eta^3 = -\frac{1}{16\pi^2}(\Sigma_1 - \sigma_1) \wedge (\Sigma_2 - \sigma_2) \wedge (\Sigma_3 - \sigma_3) \,. \tag{6.2.30}$$

Integrating the η^j over the sub-manifolds C_i, we have the relation

$$\int_{C_i} \eta^j = \delta_{j,i+1} - \delta_{j,i-1} \,, \tag{6.2.31}$$

and, by Poincaré duality $\eta^i \to C_i$, the intersection numbers $C_i \cdot C_j = \delta_{j,i+1} - \delta_{j,i-1}$. Notice that, having fixed the orientation so that $C_1 \cdot C_2 = +1$, we then have that $C_3 \cdot C_1 = +1$, which gives $\int_{C_3} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = -16\pi^2$ so that the orientation of C_3 is opposite to that of C_2.

6.3 Fivebranes wrapped on a three-sphere in $S^3 \times \mathbb{R}^4$

In this section, we discuss solutions describing fivebranes wrapped on a three-sphere inside a G_2-holonomy manifold X_i. The backreaction of the fivebranes modifies the geometry, making the internal space a smooth *torsional G_2 manifold*. As we explain later, the topology is again $\mathbb{R}^4 \times S^3$, although we need to be careful about which S^3. Since we are interested in solutions arising from NS5-branes, we may work in Type I supergravity, and allow for a non-trivial three-form $H_{(3)}$ and dilaton profile. By applying an S-duality to the NS5-branes in Type IIB, these solutions may also be interpreted as arising from D5-branes.

6.3.1 Torsional G_2 solutions

General classes of supersymmetric solutions of Type I and heterotic supergravities have been studied in [24], extending the works of [75] and [113]. Here, we are interested in solutions where the non-trivial geometry is seven-dimensional and is characterised by a G_2-structure. We therefore refer to this class as *torsional G_2 solutions*. The ten-dimensional metric in string frame is unwarped

$$\mathrm{d}s_{str}^2 = \mathrm{d}x_{1+2}^2 + \mathrm{d}s_7^2\,. \tag{6.3.1}$$

The supersymmetry equations are equivalent to a system of exterior differential equations obeyed by the G_2-structure on the seven-dimensional space with metric $\mathrm{d}s_7^2$ and read [24, 127]

$$\phi \wedge \mathrm{d}\phi = 0\,,$$
$$\mathrm{d}\left(e^{-2\Phi} *_7 \phi\right) = 0\,, \tag{6.3.2}$$
$$e^{2\Phi} *_7 \mathrm{d}\left(e^{-2\Phi}\phi\right) = -H_{(3)}\,,$$

where Φ is the dilaton field and $*_7$ denotes the Hodge star operator with respect to the metric $\mathrm{d}s_7^2$. The Bianchi identity $\mathrm{d}H_{(3)} = 0$ implies that all remaining equations of motion are satisfied. See [24] for a more detailed discussion of this

type of G_2-structure. Examples of solutions of these equations were discussed in [128, 123, 24, 36] and we shall return to some of these later.

6.3.2 Ansatz and BPS equations

We now specify ansätze for the metric, G_2-structure and $H_{(3)}$ field. Although the G_2-structure determines the metric uniquely, and the $H_{(3)}$ field is then derived from the third equation in (6.3.2), we find more convenient to start with an ansatz for $H_{(3)}$ that is manifestly closed, $dH_{(3)} = 0$. Specifically, we use the ansatz for the metric and associative three-form discussed earlier

$$
\begin{aligned}
ds_7^2 &= M\left[dt^2 + f_1\,da_1^2 + f_2\,da_2^2 + f_3\,da_3^2\right], \\
\phi &= M^{3/2}\left[e^t \wedge J + \text{Re}[e^{i\theta}\Omega]\right],
\end{aligned}
\tag{6.3.3}
$$

where we inserted a factor of M in front of the metric. For the three-form flux, we take

$$
H_{(3)} = 2\pi^2 M\left[\gamma_1\,\eta^1 + \gamma_2\,\eta^2 + \gamma_3\,\eta^3 + \gamma_4\,dt \wedge \sum_{i=1}^{3}\sigma_i \wedge \Sigma_i\right],
\tag{6.3.4}
$$

where the factor of $2\pi^2 M$ is again for convenience. Imposing $dH_{(3)} = 0$ implies

$$
\gamma_i = \gamma + \alpha_i \qquad i = 1,2,3\,, \qquad \gamma_4 = \frac{\gamma'}{16\pi^2}\,,
\tag{6.3.5}
$$

where γ is a function and α_i are three integration constants. The ansatz then depends on four functions f_i, γ and three constants α_i, although we later see that the homology relation among the C_i implies that only two of these are significant, and are fixed by flux quantisation and regularity of the metric.

The action of the Σ_3 symmetry on the functions in the ansatz is given by

$$
\begin{aligned}
\sigma_{12} &: (f_1, f_2, f_3) \to (f_2, f_1, f_3)\,, & \sigma_{231} &: (f_1, f_2, f_3) \to (f_2, f_3, f_1)\,, \\
\sigma_{12} &: (\gamma_1, \gamma_2, \gamma_3) \to (-\gamma_2, -\gamma_1, -\gamma_3)\,, & \sigma_{231} &: (\gamma_1, \gamma_2, \gamma_3) \to (\gamma_2, \gamma_3, \gamma_1)\,,
\end{aligned}
\tag{6.3.6}
$$

with the rest following from group multiplication rules. The minus signs in the action of the order-two elements on the γ_i's arise because the orientation of the seven-dimensional space is reversed, hence the Hodge $*_7$ operator in (6.3.2) changes sign.

Inserting the ansatz into the equations (6.3.2), after some computations, we arrive at a system of first-order ordinary differential equations. We have four

coupled differential equations for the functions f_i, γ, while an additional decoupled equation determines the dilaton profile in terms of the other functions. Although the explicit form of the equations is rather complicated, its presentation can be simplified slightly by organising it in terms of the Σ_3 symmetry. We have

$$
\begin{aligned}
D(f_i, \gamma_i)\, f_1' &= F(f_1, f_2, f_3, \gamma_1, \gamma_2, \gamma_3)\,, \\
D(f_i, \gamma_i)\, f_2' &= F(f_2, f_1, f_3, -\gamma_2, -\gamma_1, -\gamma_3)\,, \\
D(f_i, \gamma_i)\, f_3' &= F(f_3, f_2, f_1, -\gamma_3, -\gamma_2, -\gamma_1)\,, \\
D(f_i, \gamma_i)\, \gamma' &= G(f_i, \gamma_i)\,,
\end{aligned}
\tag{6.3.7}
$$

where, defining $Q \equiv f_1 f_2 + f_2 f_3 + f_3 f_1$, the functions F, G and D are given by

$$
\begin{aligned}
F(f_1, f_2, f_3, \gamma_1, \gamma_2, \gamma_3) =\ & 768 f_2 f_3 (f_1 + f_2)(f_1 + f_3) + \gamma_1^2 (f_2 + f_3)^2 + \gamma_2 \gamma_3 (3 f_1^2 - Q) \\
& + \gamma_1 \gamma_2 (Q + 2 f_1 f_3 - f_3^2) + \gamma_1 \gamma_3 (Q + 2 f_1 f_2 - f_2^2) \\
& + 32 \gamma_1 Q (f_3 - f_2) - 32 \gamma_2 Q (f_1 + f_3) + 32 \gamma_3 Q (f_1 + f_2)\,, \\
G(f_i, \gamma_i) =\ & -256 \big[\gamma_1 (f_2 + f_3)(Q + f_2 f_3) + \gamma_2 (f_3 + f_1)(Q + f_3 f_1)\,,
\end{aligned}
\tag{6.3.8}
$$

and

$$
\begin{aligned}
\sqrt{2}\, D(f_i, \gamma_i) =\ & 32 \big(\gamma_1^2 (f_2 + f_3)^3 + \gamma_2^2 (f_3 + f_1)^3 + \gamma_3^2 (f_1 + f_2)^3 \\
& + 2 \gamma_1 \gamma_2 f_3 (3Q - f_3^2) + 2 \gamma_1 \gamma_3 f_2 (3Q - f_2^2) + 2 \gamma_2 \gamma_3 f_1 (3Q - f_1^2) \\
& + 96 Q \left(\gamma_1 (f_3^2 - f_2^2) + \gamma_2 (f_1^2 - f_3^2) + \gamma_3 (f_2^2 - f_1^2) \right) \\
& + 2304 Q (f_1 + f_2)(f_2 + f_3)(f_3 + f_1) \big)^{1/2} Q^{1/2}\,.
\end{aligned}
\tag{6.3.9}
$$

The decoupled equation for the dilaton reads

$$
2\sqrt{2}\, Q D(f_i, \gamma_i)\, \Phi' = P(f_i, \gamma_i)\,,
\tag{6.3.10}
$$

where

$$
\begin{aligned}
P(f_i, \gamma_i) =\ & 2\gamma_1^2 (f_2 + f_3)^3 + 2\gamma_2^2 (f_3 + f_1)^3 + 2\gamma_3^2 (f_1 + f_2)^3 \\
& + 4\gamma_1 \gamma_2 f_3 (3Q - f_3^2) + 4\gamma_2 \gamma_3 f_1 (3Q - f_1^2) + 4\gamma_3 \gamma_1 f_2 (3Q - f_2^2) \\
& + 96 Q \left(\gamma_1 (f_3^2 - f_2^2) + \gamma_2 (f_1^2 - f_3^2) + \gamma_3 (f_2^2 - f_1^2) \right)\,.
\end{aligned}
\tag{6.3.11}
$$

Once a solution for f_i, γ is determined (for example numerically), then the dilaton can be obtained integrating (6.3.10). Notice that D and P are invariant under

Σ_3 and G is invariant up to an overall change of sign under transformations of order-two elements. It follows that Φ is invariant under Σ_3. The phase θ in the associative three-form is a non-trivial function of f_i, γ_i, whose explicit form can be found in Appendix B of [11].

From the BPS system, it is clear that generically, for any given solution, we have in fact *six* different solutions, obtained acting with Σ_3. To study the system we can therefore focus on one particular case. Notice that, if we formally set $\gamma_i = 0$ in (6.3.7), then we recover the G_2-holonomy BPS equations (6.2.18). Solutions of this system were presented in [128, 123, 124] and [36]. In particular, the solution of [123] corresponds to the near-brane limit of a configuration of M NS5-branes wrapped on an S^3 inside the G_2 manifold $S^3 \times \mathbb{R}^4$. Below, we are more precise about which G_2 manifold X_i is relevant for a particular solution of the type discussed in [123].

Maldacena-Nastase solutions

The basic solution of [123] may be recovered from our general ansatz by setting

$$f_2 + f_3 = 1/8 \,,$$
$$\alpha_1 = \alpha_2 = -\alpha_3 = -1 \,. \tag{6.3.12}$$

For consistency, these conditions impose also $\gamma_2 = \gamma - 1 = -16 f_2$. Then, we are left with two unknown functions, f_1 and γ. To make contact with the variables in [123], one has to set

$$f_1 = \frac{4R^2 + \gamma^2 - 1}{32} \,, \tag{6.3.13}$$

and then, passing to the variables a, b, ω, we have

$$a^2 = \frac{R^2}{4} \,, \qquad b^2 = \frac{1}{4} \,, \qquad \omega = \gamma \,, \tag{6.3.14}$$

so that the metric reduces to the ansatz[4] in [123], namely

$$\mathrm{d}s_7^2 = M \left[\mathrm{d}t^2 + \frac{R^2}{4} \sum_{i=1}^{3} \sigma_i^2 + \frac{1}{4} \sum_{i=1}^{3} \left(\Sigma_i - \tfrac{1}{2}(1+\omega)\sigma_i \right)^2 \right] \,. \tag{6.3.15}$$

Similarly, the $H_{(3)}$ reduces to that in [123]. In terms of the functions R and ω, the system (6.3.7) reduces to the system in the appendix of [123]. As discussed in

---

[4]Our one-forms are related to those in [123] as $\sigma_i^{\text{here}} = w_i^{\text{there}}$ and $\Sigma_i^{\text{here}} = \tilde{w}_i^{\text{there}}$.

[123, 124], there exists a unique non-singular solution of the differential equations. In the interior[5], the topology of the solution is $S^3 \times \mathbb{R}^4$, where the three-sphere[6] C_3 smoothly shrinks to zero and the three-cycle is represented by C_1 or $-C_2$. Then, more precisely, the topology of this solution is that of the G_2 manifold X_3. The authors of [123] discussed also a second solution which can be obtained from the basic solution by acting with σ_{23}. Hence the topology of this solution is that of G_2 manifold X_2. It is clear that there are four more different solutions, obtained by acting with elements of Σ_3.

6.3.3   One-parameter families of solutions

Finding analytic solutions of the BPS system (6.3.7) seems very difficult. As usual in these cases, we then turn to a combination of numerical methods and asymptotic expansions. We are interested in non-singular solutions to the system, giving rise to spaces with topology $S^3 \times \mathbb{R}^4$. As for the G_2-holonomy manifolds X_i and the Maldacena-Nastase solutions, we then require that two functions f_i go to zero in the interior, while the third function approaches a constant value, parametrising the size of the non-trivial S^3 inside $S^3 \times \mathbb{R}^4$. We can restrict our attention to one particular case, for example we may require that f_1 and f_2 go to zero in the IR (at $t = 0$) while f_3 approaches a constant value $f_3(0) \equiv c > 0$. This then has the topology of X_3, where C_3 shrinks to zero. This solution was studied in [36].

More generally, we impose boundary conditions such that the topology of the solution is that of one of the manifolds X_i. This fixes the values of the integration constants α_i. Using the relation (6.2.31), we can evaluate the flux of the three-form $H_{(3)}$ (6.3.4) on the sub-manifolds C_i, defined exactly like in (6.2.5), and at some constant t. We have

$$q_i \equiv \frac{1}{4\pi^2} \int_{C_i} H_{(3)} = \frac{M}{2}(\gamma_{i+1} - \gamma_{i-1}) = \frac{M}{2}(\alpha_{i+1} - \alpha_{i-1}), \qquad (6.3.16)$$

where the result does not depend on t. The q_i then obey the relation $q_1 + q_2 + q_3 = 0$, reflecting the homology relation $[C_1] + [C_2] + [C_3] = 0$. Hence, we can parametrise the constants α_i by taking for example

$$\alpha_1 = -k_1, \qquad \alpha_2 = -k_2, \qquad \alpha_3 = k_2, \qquad (6.3.17)$$

---

[5]Asymptotically the geometry is a linear dilaton background.
[6]Notice that as $t \to 0$ we have $f_1 \to 0$, $f_2 \to 0$ and $f_3 \to 1/8$ [123, 124].

so that

$$q_1 = -Mk_2\,, \qquad q_2 = \frac{M}{2}(k_1 + k_2)\,, \qquad q_3 = \frac{M}{2}(-k_1 + k_2)\,. \qquad (6.3.18)$$

The constants k_1 and k_2 are determined for a given solution as follows. Suppose that we require the manifold to have the topology of X_3. Then the flux of $H_{(3)}$ through C_1 is minus the flux through C_2, namely $q_1 = -q_2$. In terms of the constants k_1 and k_2, we must then have $k_1 = k_2 = k$ and k can be reabsorbed in the definition of M. There are then essentially two choices for k, namely $k = \pm 1$, corresponding to two different solutions, both with topology of X_3. We denote these two solutions as X_{31} and X_{32}, respectively. More generally, there are six different solutions and we denote the corresponding spaces as X_{ij}. The topology of the spaces X_{ij} is the same as the G_2-holonomy manifolds X_i.

In each case the flux through the non-trivial cycle must be quantised, thus we require that

$$\mathcal{N}(X_{ij}) = |\epsilon_{ijk}|\frac{1}{4\pi^2}\int_{C_k} H_{(3)} = M\,. \qquad (6.3.19)$$

The signs have been chosen to always give a positive number and are consistent with the action of Σ_3. In conclusion, flux quantisation, together with the condition that the flux through the vanishing three-sphere vanishes, fixes the integration constants k_1 and k_2 in all cases. We summarise the values of k_1, k_2 and the q_i's in each of the six solutions in Table 6.1.

	k_1	k_2	q_1	q_2	q_3
X_{31}	1	1	$-M$	M	0
X_{21}	-1	1	$-M$	0	M
X_{12}	-2	0	0	$-M$	M
X_{32}	-1	-1	M	$-M$	0
X_{23}	1	-1	M	0	$-M$
X_{13}	2	0	0	M	$-M$

Table 6.1: Values of k_1, k_2 and q_i for the six different solutions X_{ij}. The basic Maldacena-Nastase solution is the $c = 1/8$ limit of X_{31}, while the second Maldacena-Nastase solution is the $c = 1/8$ limit of X_{21}.

Expansions in the IR

For definiteness, let us concentrate on the case of X_{31}. To discuss the expansions, it is convenient to first rescale the radial coordinate by a constant factor as $t \to \sqrt{c}\, t$. The boundary conditions that we impose at $t = 0$ determine the expansions[7] of the functions f_i and γ around $t = 0$ as follows:

$$f_1 + f_2 = \frac{1}{8}c\,t^2 + \frac{1 - 384c^2}{147456c}\,t^4 + \mathcal{O}\left(t^6\right),$$

$$f_1 - f_2 = \frac{1}{96}t^2 + \frac{3 - 256c^2}{589824c^2}\,t^4 + \mathcal{O}\left(t^6\right),$$

$$f_3 = c + \frac{-5 + 192c^2}{6144c}\,t^2 + \frac{-3 + 224c^2 + 2048c^4}{6291456c^3}\,t^4 + \mathcal{O}\left(t^6\right),$$

$$\gamma = 1 - \frac{1}{24}t^2 + \frac{-1 + 128c^2}{49152c^2}\,t^4 + \mathcal{O}\left(t^6\right).$$

(6.3.20)

The corresponding expansion for the dilaton reads

$$\Phi = \Phi_0 - \frac{7}{12288c^2}t^2 + \frac{-293 + 21504c^2}{452984832c^4}t^4 + \mathcal{O}\left(t^6\right),$$

(6.3.21)

where Φ_0 is an (IR) integration constant. We therefore have a family of non-singular solutions, parametrised by the constant c, measuring the size of the non-trivial S^3. Using numerical methods, one can then check that, for any value of $c \geq 1/8$, there exists a non-singular solution approaching (6.3.20) as $t \to 0$. The special value $c = 1/8$ corresponds precisely to the Maldacena-Nastase solution. Hence we have a one-parameter family of solutions with topology of X_3 (in the interior), generalising the solution discussed in [123].

Expansions in the UV

Towards infinity, we find two different types of asymptotic expansions of the functions. In one expansion, the functions have the following behaviour for large t:

$$f_1 = \frac{c\,t^2}{36} + \frac{1}{4} - \frac{21}{16\,c\,t^2} + \mathcal{O}\left(t^{-4}\right),$$

$$f_2 = \frac{c\,t^2}{36} - \frac{1}{4} - \frac{21}{16\,c\,t^2} + \mathcal{O}\left(t^{-4}\right),$$

$$f_3 = \frac{c\,t^2}{36} + \frac{69}{16\,c\,t^2} + \mathcal{O}\left(t^{-4}\right),$$

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[7]If we had left the constants k_1 and k_2 arbitrary, the IR expansions of the functions f_i and γ in power series would impose $\gamma_{(0)} = k_1 = k_2$, that is $q_3 = 0$.

$$\gamma = \frac{1}{3} + \mathcal{O}\left(t^{-4}\right),$$

$$\Phi = \Phi_\infty + \mathcal{O}\left(t^{-4}\right), \tag{6.3.22}$$

where Φ_∞ is an (UV) integration constant. Notice that the constant c appears here trivially because of the rescaling $t \to \sqrt{c}\,t$ we made, therefore at this order we do not see a genuine UV integration constant. After this particular order, the expansion in inverse powers of t is not valid anymore and one would need to use other types of series to gain more precision. This expansion can be matched numerically to the IR expansions for all values of $c > 1/8$.

Already, from these few orders, we can extract some useful information. The functions f_i all have the same leading behaviour in t^2 towards infinity, corresponding to the G_2-holonomy cone. From the subleading terms, we can also read off an effective "resolution parameter", measuring the amount of $\mathbb{Z}_2$ symmetry breaking in each case. The asymptotic form of the metric here is

$$\mathrm{d}s^2(X_{ij})_7 = Mc\Big[\mathrm{d}t^2 + \frac{t^2}{36}(\mathrm{d}a_1^2 + \mathrm{d}a_2^2 + \mathrm{d}a_3^2) \\ + \frac{1}{4c}\left(\ell_1 \mathrm{d}a_1^2 + \ell_2 \mathrm{d}a_2^2 + \ell_3 \mathrm{d}a_3^2\right) + \mathcal{O}(1/t)\Big], \tag{6.3.23}$$

where $(\ell_1, \ell_2, \ell_3) = (1, -1, 0)$ for X_{31}, and the remaining ones are determined by the Σ_3 action. The "volume defects" are given by

$$\mathrm{vol}(C_i^\infty) = (Mc)^{3/2}\left(\frac{16}{27}\pi^2 t^3 - 4\pi^2 \ell_i \frac{t}{c}\right). \tag{6.3.24}$$

Notice that, even after subtracting the leading divergent part, these volumes are now "running". This running is analogous to the running volume of the two-sphere at infinity in the resolved deformed conifold, although here the running is linear, rather than logarithmic. Then the ith effective resolution parameter may be defined as

$$\alpha_i^{\mathrm{res}} \equiv \mathrm{vol}(C_{i+1}^\infty) - \mathrm{vol}(C_{i-1}^\infty) = (Mc)^{3/2} 4\pi^2 (\ell_{i-1} - \ell_{i+1})\frac{t}{c}. \tag{6.3.25}$$

In Section 6.2.2, we saw that, in the G_2-holonomy manifold X_i, the resolution parameter α_i^{res} vanished, reflecting a $\mathbb{Z}_2 \subset \Sigma_3$ symmetry of the geometry. Hence, the relevant[8] resolution parameter to consider for the manifolds X_{ij} is α_i^{res}. For

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[8]Of course $\alpha_{i+1}^{\mathrm{res}}$ and $\alpha_{i-1}^{\mathrm{res}}$ are also non-zero, since the solutions do not preserve any $\mathbb{Z}_2$ symmetry. However, these are less interesting parameters, since they are non-zero also in the G_2-

example, we find that

$$\alpha_3^{\mathrm{res}}(X_{31}) = -\alpha_3^{\mathrm{res}}(X_{32}) = 8\pi^2 (Mc)^{3/2}\frac{t}{c}, \tag{6.3.26}$$

which is again running. Notice that, keeping Mc fixed, a non-zero value of the parameter c^{-1} may then be interpreted as turning on a "resolution" in the manifold X_3. Indeed, we show below that the limit $c \to \infty$ gives the G_2-holonomy manifold X_3.

We also find a second type of expansion at large t, in which the behaviour of the functions is different and we have

$$f_1 = \frac{\sqrt{2}}{16}\,t + \kappa - \frac{\sqrt{2}}{32\,t} + \frac{1 - 16\,\kappa}{32\,t^2} + \frac{7 - 64\,\kappa + 512\,\kappa^2}{64\sqrt{2}\,t^3} + \mathcal{O}\left(t^{-4}\right),$$

$$f_2 = \frac{1}{16} - \frac{\sqrt{2}}{32\,t} + \frac{-1 + 8\,\kappa}{16\,t^2} + \frac{-9 + 64\,\kappa - 256\,\kappa^2}{32\sqrt{2}\,t^3} + \mathcal{O}\left(t^{-4}\right),$$

$$\gamma = \frac{\sqrt{2}}{2\,t} + \frac{1 - 8\,\kappa}{t^2} + \frac{9 - 64\,\kappa + 256\,\kappa^2}{2\sqrt{2}\,t^3} + \mathcal{O}\left(t^{-4}\right), \tag{6.3.27}$$

$$\Phi = -\frac{\sqrt{2}}{4}\,t + \frac{3}{8}\log t + \Phi_\infty + \frac{3(1 + 32\,\kappa)}{16\sqrt{2}\,t} + \frac{29 - 192\,\kappa - 3072\,\kappa^2}{128\,t^2},$$

$$+ \frac{9 - 928\,\kappa + 3072\,\kappa^2 + 32768\,\kappa^3}{128\sqrt{2}\,t^3} + \mathcal{O}\left(t^{-4}\right).$$

In this case, the expansions remain valid at high orders, hence presumably they do not break down. These expansions may be matched numerically to the IR expansions for the particular case $c = 1/8$, thus they correspond to the Maldacena-Nastase solution[9]. Despite the fact that κ seems a free constant, it can be determined numerically to be $\kappa \approx -0.2189$.

Numerical solutions

As can be seen from the expansions, while the behaviour of the functions in the IR changes smoothly as we vary the parameter c, the behaviour in the UV changes discontinuously if we choose the extremal[10] value for the parameter $c = 1/8$. Here, we present some plots of the numerical solutions to illustrate the qualitative behaviour

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holonomy manifold X_i.

[9]The factor of $\sqrt{8}$ difference with respect to the UV behaviour of the functions in [123] is due to the rescaling $t \to \sqrt{c}\,t$.

[10]This behaviour is analogous to that of the one-parameter family of solutions discussed in [110, 106]. In that case, for the special value of the integration constant $\gamma^2 = 1$, one obtains the solution of [13], which has linear dilaton asymptotics.

Figure 6.2: Plots of the functions $c^{-1}f_i$ for the X_{31} solution for different values of c. f_1 is in red, f_2 in purple and f_3 in blue. The factor c^{-1} is there for normalisation purposes. From the top left to the bottom right, the values of c are increasing and are 0.2, 0.3, 0.4 for the first three. The bottom-right plot corresponds to the space X_3 where $f_1 = f_2$. This is formally the plot for $c \to \infty$.

of the metric functions f_i for various values of c.

In Figure 6.2, we show plots of the functions for large values of c. We see that, in the IR, the behaviour is that of the space X_3. However, despite starting below f_3 (at zero), the function f_1 eventually crosses f_3, in agreement with the UV expansions. The crossing point moves further and further along the radial direction as c is increased. In Figure 6.3, we plot the functions for values of c close to the minimum. We see that in the IR the functions are all very close to the special case $c = 1/8$. However, when c is not exactly equal to its minimum value, the functions start to deviate at some point. For values of c closer and closer to the special one, there is a larger and larger region where the functions are well approximated by the profiles of the Maldacena-Nastase solution. Finally, in Figure 6.4, we plot the dilaton and the function γ for various values of the constant c. We see that, generically, e^Φ goes to a constant e^{Φ_∞} at infinity, while in the particular case of Maldacena-Nastase, e^Φ vanishes in the UV.
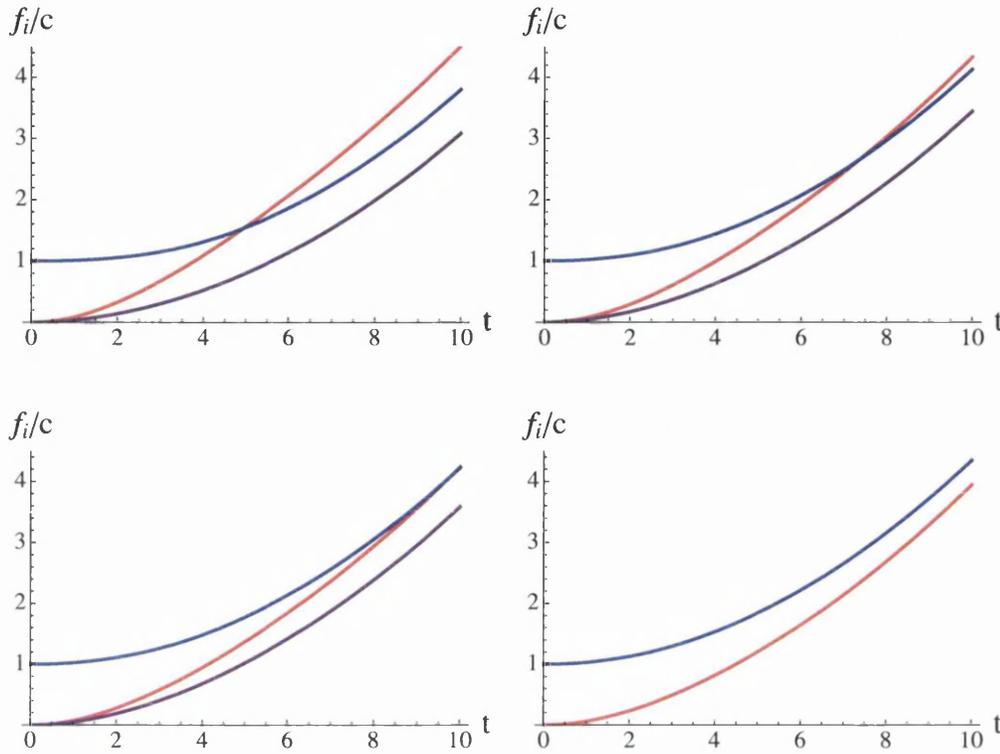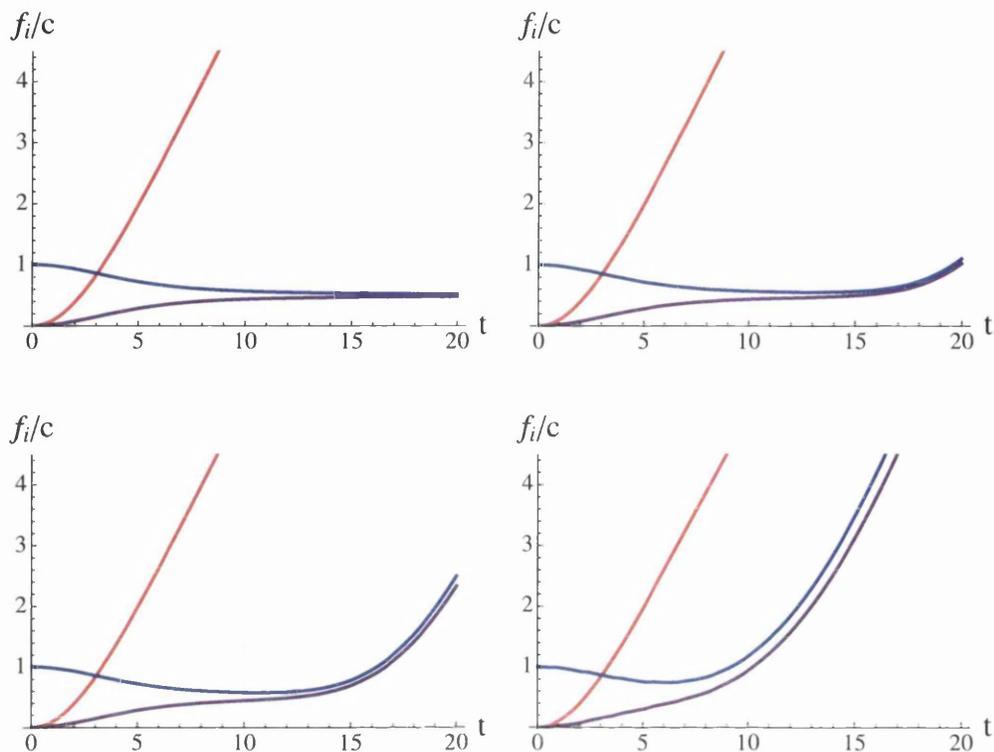
Figure 6.3: Plots of the functions $c^{-1}f_i$ for the X_{31} solution for different values of c. f_1 is in red, f_2 in purple and f_3 in blue. The factor c^{-1} is there for normalisation purposes. From the top left to the bottom right, the values of c are increasing and are 0.125 (the exact minimum value), 0.125001, 0.12501 and 0.126. The first plot is the Maldacena-Nastase solution.

6.3.4   Limits

In this subsection, we analyse two special limits of the one-parameter solutions, namely $c \to \infty$ and $c \sim 1/8$.

The solution for $c \to \infty$: G_2-holonomy with flux

The numerical solutions show that, by increasing the value of c, the solution X_{31} looks more and more like the G_2 manifold X_3. To see this more precisely, we consider an expansion of the functions f_i and γ in inverse powers[11] of c of the form:

$$f_i = c \sum_{n=0}^{\infty} \frac{1}{c^n} f_{i(n)} , \qquad \gamma = \sum_{n=0}^{\infty} \frac{1}{c^n} \gamma_{(n)} . \tag{6.3.28}$$

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[11]From (6.3.29), it can be checked *a posteriori* that the first few terms reproduce the UV expansions (6.3.22). Therefore, the series (6.3.28) is certainly not valid for $c = 1/8$. In any case, we only need this for large c here.
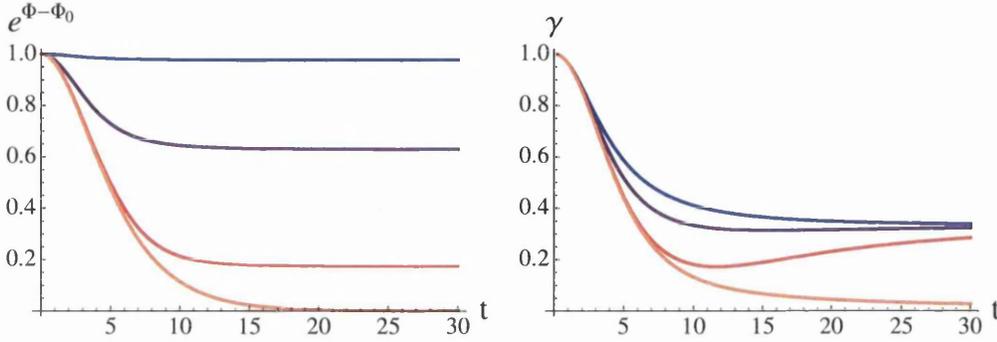
Figure 6.4: On the left are plots of the function $e^{\Phi - \Phi_0}$ for different values of c. On the right, plots of the function γ. The orange plots correspond to the minimum value $c = 0.125$, the red ones to $c = 0.126$, the purple ones to $c = 0.15$ and the blue ones to $c = 0.5$. In the Maldacena-Nastase solution at infinity, there is a linear dilaton and the $H_{(3)}$ vanishes.

For small t, it can be checked that this agrees with the IR expansions (6.3.20). Then one can solve the system (6.3.7) order by order in powers of c^{-1}. There are, of course, different solutions depending on the boundary conditions and here we concentrate on the boundary conditions already treated in Section 6.3.3. For our purpose, we only need the first few orders of the expansion (6.3.28). These read

$$f_{1(0)} = f_{2(0)} = \frac{r^3 - r_0^3}{36\, r}\,, \qquad\qquad f_{3(0)} = \frac{2\, r^3 + r_0^3}{72\, r}\,,$$

$$f_{1(1)} = -f_{2(1)} = -\frac{r_0^3 + r_0^2\, r + r_0\, r^2 - 3\, r^3}{12\, r(r_0^2 + r_0\, r + r^2)}\,, \qquad f_{3(1)} = 0\,, \tag{6.3.29}$$

$$\gamma_{(0)} = \frac{r^3 + r_0\, r^2 + r_0^2\, r + 6 r_0^3}{3\, r(r^2 + r_0\, r + r_0^2)}\,,$$

where r is a function of the radial coordinate t defined as in Section 6.2.2, namely

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{1 - \frac{r_0^3}{r^3}}\,. \tag{6.3.30}$$

The functions $f_{i(0)}$ at the lowest order in the expansion solve a simplified version of (6.3.7) where $\gamma_i = 0$ and $f_1 = f_2$, which are simply the differential equations for the G_2-holonomy metric X_3. The metric, in terms of the expansion (6.3.28), reads

$$\mathrm{d}s_7^2 = Mc\left[\mathrm{d}s_7^2(X_3) + \mathcal{O}\left(c^{-1}\right)\right]\,. \tag{6.3.31}$$

Thus, at leading order in c, the solution looks like the G_2 manifold X_3 with a very large S^3, and M units of $H_{(3)}$ flux through it. One could of course take $c \to \infty$ while keeping Mc fixed, by taking $M \to 0$ at the same time. This is an exact solution, where the flux $H_{(3)}$ vanishes.

The solution for $c \sim 1/8$: G_2-holonomy with branes

Here, we show that when c is very close to the minimum value $c = 1/8$, there is a region where the solution X_{ij} looks like a G_2-holonomy manifold X_j with M fivebranes wrapped on the non-trivial three-sphere. The calculation is analogous to that appearing in Section A.2 of [106].

The solution stays close to the Maldacena-Nastase solution up to large values of t. To analyse the behaviour of the solution where it starts departing from this, we consider the following ansatz for an approximate solution:

$$f_1 = \frac{\sqrt{2}}{16} t + \mu_1 \,, \qquad\qquad f_2 = \frac{1}{16} + \mu_2 \,,$$
$$f_3 = \frac{1}{16} + \mu_3 \,, \qquad\qquad \gamma = \frac{\sqrt{2}}{2t} \,. \tag{6.3.32}$$

The leading terms are those of the Maldacena-Nastase solution and we require that $\mu_1 \ll t$, $\mu_2, \mu_3 \ll 1$. Anticipating the form of the metric that we are after, we change coordinates as follows:

$$ds_7^2 = M \left[f_2 \left(\frac{1}{8} dy^2 + da_2^2 \right) + f_3 da_3^2 + f_1 da_1^2 \right] \,. \tag{6.3.33}$$

We could also have taken f_3 in front, but since we later find that $\mu_2 = \mu_3$, this does not matter. Then, we plug the ansatz (6.3.32) into the BPS equations (6.3.7) and expand to first order in the μ_i's. The equation for γ' is satisfied automatically at leading order. Working at large y, we can solve the equations for μ_i and we find

$$\mu_1 = \frac{\beta_1}{2} (e^{\sqrt{2}y/8} - 1) + \beta_2 \,,$$
$$\mu_2 = \mu_3 = \beta_1 e^{\sqrt{2}y/8} \,, \tag{6.3.34}$$

where β_1 and β_2 are two integration constants. We can determine the dilaton with the same precision by considering the ansatz

$$\Phi = -\frac{\sqrt{2}}{4} t + \mu_4 \,, \tag{6.3.35}$$

where $\mu_4 \ll t$. Then we find

$$\mu_4 = 8\beta_1(e^{\sqrt{2}y/8} - 1) + \Phi_4\,, \tag{6.3.36}$$

where Φ_4 is another integration constant. Inserting these back into the metric and changing coordinates as $r = 4\sqrt{M\beta_1}e^{\sqrt{2}y/16}$, we find

$$ds_7^2 \approx \left(1 + \frac{M}{r^2}\right)\left[dr^2 + \frac{r^2}{8}\sum_{i=1}^{3}\left(\Sigma_i^2 + (\Sigma_i - \sigma_i)^2\right)\right] + M\frac{\sqrt{2}}{32}y\sum_{i=1}^{3}\sigma_i^2\,,$$

$$e^{2(\Phi-\Phi_4)} \approx 16\beta_1 e^{-16\beta_1}\left(1 + \frac{M}{r^2}\right)\,.$$
$$\tag{6.3.37}$$

This is the approximate solution for M fivebranes in flat space, wrapped on the three-sphere parametrised by σ_i. More precisely, we see that the topology is that of $S^3 \times \mathbb{R}^4$, where the three-sphere C_1 transverse to the branes (defined by $\sigma_i = 0$) vanishes smoothly. Hence, this is the same topology as the one of the G_2-holonomy manifold X_1. The fivebranes then can wrap C_2 or $-C_3$.

This approximation however requires that y is large, but at the same time $y \ll y_5$, where y_5 is defined by $\beta_1 = e^{-\sqrt{2}y_5/8}$. Presumably, around $y \sim y_5$. the solution looks more accurately like X_1 [106], but this seems difficult to analyse in the linearised approximation. We can also estimate the relation between c and y_5 by extrapolating to zero the value of $f_2 + f_3$. This gives

$$c - \frac{1}{8} \approx e^{-\frac{\sqrt{2}}{8}y_5}\,. \tag{6.3.38}$$

6.3.5   Summary

In this section, we have discussed a set of gravity solutions characterised by a non-trivial parameter c. The additional parameters of the solutions are the M integral units of NS three-form flux $H_{(3)}$ and the asymptotic value of the dilaton Φ_∞. The constant Φ_0 is a function of Φ_∞ and c, that may be determined numerically. There are six different solutions, exchanged by the action of the triality group Σ_3. In each case, the internal seven-dimensional manifold is an asymptotically conical space, with topology $S^3 \times \mathbb{R}^4$, that we have denoted X_{ij}, with $i, j = 1, 2, 3$ and $i \neq j$. The base of the asymptotic cone is the nearly Kähler manifold $S^3 \times S^3$ with metric $ds^2(Y) = \frac{1}{36}(da_1^2 + da_2^2 + da_3^2)$ [66]. More precisely, the topology of the space X_{ij} is that of the G_2-holonomy manifold X_i, that we reviewed in Section 6.2.

In each case, the parameter c gives the size of the non-trivial S^3 at the origin, hence this is analogous to the *deformation* in the deformed conifold. On the other hand, one can also define a resolution parameter by looking at how the metric breaks a $\mathbb{Z}_2 \subset \Sigma_3$ symmetry at large distances. In particular, we have argued that the parameter $1/c$ gives a measure of how much the space X_{ij} deviates from the X_i geometry. Hence, from this point of view, $1/c$ can be interpreted as an effective *resolution* parameter. In the case of G_2-holonomy, the moduli space of metrics on $S^3 \times \mathbb{R}^4$ has three different branches, meeting at the origin. With an abuse of language[12], we can say that the singular G_2 cone over $S^3 \times S^3$ may be deformed, resolved or flopped-resolved, with the three possibilities mutually exclusive. The six solutions that we discussed may be said to be deformed *and* resolved, analogously to the resolved deformed conifold geometry [110, 106].

When c is very large, the solution approaches a G_2-holonomy manifold with flux on a large three-sphere. When c hits the lower bound $c = 1/8$, the X_{ij} geometry becomes a solution of the type discussed by [123], corresponding to the near-brane limit of a large number of fivebranes wrapped on the S^3 inside a G_2 manifold with topology $S^3 \times \mathbb{R}^4$. These have a finite size three-sphere at the origin, but are asymptotically linear dilaton backgrounds. When c is very close to the critical value $c = 1/8$, the solution stays close to the near-brane Maldacena-Nastase one up to large values of t (see the plots in Figure 6.3) and, when it starts deviating from this behaviour, the geometry becomes approximately that of the G_2 manifold X_j with M fivebranes wrapping the non-trivial three-sphere inside it.

For each X_{ij} solution, the parameter c interpolates between the G_2-holonomy manifold X_i with M units of flux on a large three-sphere, and the G_2-holonomy manifold X_j with M fivebranes wrapped on a (different) three-sphere. Hence, this may be interpreted as a realisation of a G_2 geometric transition, purely in the context of supergravity. Notice that this is different from the closely related setup in [61, 66], where the relevant geometric transition involved D6-branes wrapped in the conifold, although this was embedded in the G_2-holonomy context by uplifting to M-theory.

From the point of view of the G_2-holonomy manifold X_1 (say) with M units of flux through the three-sphere $C_2 \cong -C_3$, the two solutions X_{12} and X_{13} break the $\mathbb{Z}_2$ symmetry generated by σ_{23} in two opposite directions. These look like a "resolution" of the manifold X_1 and its flopped version. The breaking of this $\mathbb{Z}_2$ symmetry is analogous to the breaking of the $\mathbb{Z}_2$ symmetry in the resolved deformed

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[12]In particular, we do not use these words here in the sense of complex or symplectic geometry.
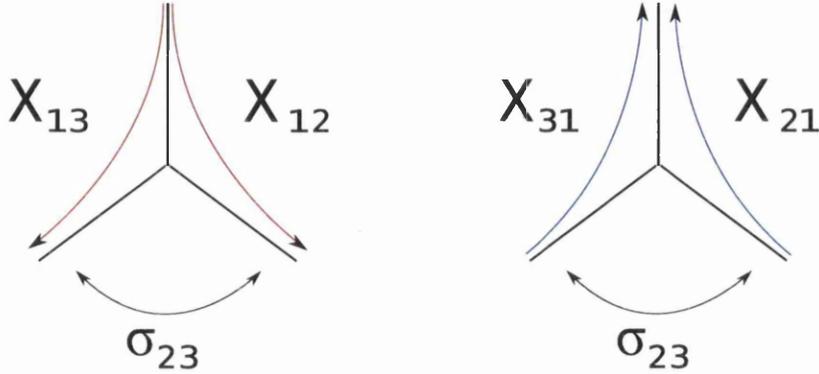
Figure 6.5: On the left: the solutions X_{13} and X_{12} interpolate continuously between the G_2 manifold X_1 with flux and the G_2 manifolds X_3 and X_2 with branes, respectively. The two solutions are related by the $\mathbb{Z}_2$ symmetry σ_{23} and both have topology of $X_1 \cong S^3 \times \mathbb{R}^4$. On the right: the solution X_{31} interpolates continuously between the G_2 manifold X_3 with flux and the G_2 manifold X_1 with branes, while the solution X_{21} interpolates between the G_2 manifold X_2 with flux and the G_2 manifold X_1 with branes. The two solutions are related by the same $\mathbb{Z}_2$ symmetry σ_{23}, however X_{31} has the topology of $X_3 \cong S^3 \times \mathbb{R}^4$ while X_{21} has the topology of $X_2 \cong S^3 \times \mathbb{R}^4$.

conifold. On the other hand, from the point of view of the branes wrapped on the three-sphere $C_2 \cong -C_3$ in X_1, the two solutions X_{21} and X_{31} break the $\mathbb{Z}_2$ symmetry generated by σ_{23} by "deforming" the original X_1 manifold in two different ways. In other words, the G_2 geometric transition may proceed from branes in X_1 to flux in X_2 *or* from branes in X_1 to flux in X_3. There is no analogue of this in the conifold case.

Moreover, depending on which three-sphere of X_1 the branes wrap, in the geometry after the backreaction this three-sphere may become contractible or not. For example, if the fivebranes were wrapped on $C_2 \subset X_1$ and after the geometric transition we have the manifold X_2 (with flux), the sphere wrapped by the branes is contractible. Whereas, if the fivebranes were wrapped on $C_2 \subset X_1$ and after the geometric transition we have the manifold X_3 (with flux), the sphere wrapped by the branes is *not* contractible. This phenomenon has no analogue in the conifold case, where the two-sphere wrapped by the branes always becomes contractible in the backreacted geometry. In the context of the discussion in [123], these two possibilities led to two different values of the quantity denoted k_6, defined as the flux of $H_{(3)}$ through the three-sphere wrapped by the branes. In particular, for the basic solution of [123] (with X_3 topology), it is assumed that the three-sphere

wrapped by the branes is $C_2 \subset X_1$ and therefore $k_6 = q_2 = M$. The second solution
of [123] (with X_2 topology) is interpreted as arising from fivebranes still wrapped
on $C_2 \subset X_1$ and therefore here $k_6 = q_2 = 0$ (*cf.* Table 6.1). This ambiguity may
also be understood as related to different gauge choices for the connection on the
normal bundle to the wrapped three-sphere. In [123], it was explained how this
corresponds to changing the number of fermionic zero-modes on the brane world-
volume, with all choices leading to equivalent results for the physical Chern-Simons
level of the dual gauge theory, namely $|k| = M/2$.

6.4   Solutions with interpolating G_2-structure

In this section, we discuss solutions of Type IIA supergravity of the type $\mathbb{R}^{1,2} \times_w$
$\mathcal{M}_7$, where the internal seven-dimensional manifold $\mathcal{M}_7$ has a G_2-structure and
there are various fluxes. We then show that, starting from a torsional G_2 geometry,
one can obtain a more general solution, interpolating between the original solution
and a warped G_2-holonomy solution. The method that we use is quite general and
can be applied to supergravity solutions different from the ones of the previous
section.

6.4.1   Supersymmetry conditions

We write the metric ansatz in string frame as

$$ds^2_{str} = e^{2\Delta + 2\Phi/3}\big(dx^2_{1+2} + ds^2_7\big)\,. \tag{6.4.1}$$

The solutions are characterised by a G_2-structure on the internal space, namely an
associative three-form ϕ (and its Hodge dual) and a non-trivial phase ζ. The non-
zero fluxes are the RR four-form $F_{(4)}$ and the NS three-form $H_{(3)}$. The equations
characterising the geometry may be written in the form of generalised calibration
conditions [42], and can be obtained straightforwardly by reducing the equations
presented in [111] from eleven to ten dimensions. Some details of this reduction
are presented in Appendix 6.B. The equations read

$$d\big(e^{6\Delta} *_7 \phi\big) = 0\,, \qquad 2\,d\zeta - e^{-3\Delta}\cos\zeta\,d\big(e^{3\Delta}\sin\zeta\big) = 0\,,$$
$$\phi \wedge d\phi = 0\,, \qquad d\big(e^{2\Delta + 2\Phi/3}\cos\zeta\big) = 0\,. \tag{6.4.2}$$

In addition, the fluxes are determined as follows:

$$H_{(3)} = \frac{1}{\cos^2 \zeta} e^{-4\Delta + 2\Phi/3} *_7 \mathrm{d}\left(e^{6\Delta} \cos \zeta \, \phi\right),$$

$$F_{(4)} = \mathrm{vol}_3 \wedge \mathrm{d}\left(e^{3\Delta} \sin \zeta\right) + F_{(4)}^{\mathrm{int}}, \qquad F_{(4)}^{\mathrm{int}} = -\frac{\sin \zeta}{\cos^2 \zeta} e^{-3\Delta} \mathrm{d}\left(e^{6\Delta} \cos \zeta \, \phi\right). \tag{6.4.3}$$

Notice the relation

$$\sin \zeta \, e^{\Delta - 2\Phi/3} H_{(3)} + *_7 F_{(4)}^{\mathrm{int}} = 0. \tag{6.4.4}$$

From the results of [111], we have that any solution of these conditions, supplemented by the Bianchi identities $\mathrm{d}H_{(3)} = \mathrm{d}F_{(4)} = 0$, solves also the equations of motion. This geometry is the G_2 analogue of the interpolating $SU(3)$-structure geometry discussed in Chapter 5 (see also [107]). Notice that this case is not contained in the equations presented in [129], which instead describe an $SU(3)$-structure in seven dimensions. Although the ansatz for the bosonic fields in the latter reference is equivalent to ours, the ansatz for the Killing spinors does not allow the structure that we are interested in here. A discussion of spinor ansätze can be found in Appendix C.2 of [11].

The conditions (6.4.2), (6.4.3) include the Type I torsional geometries as a special case, which are obtained by simply setting $\zeta = \pi$. The warp factor is related[13] to the dilaton as $e^{2\Phi} = e^{-6\Delta}$ so that the ten-dimensional metric in string frame is unwarped. The limit $\cos \zeta \to 0$ is slightly singular since, in this case, the G_2-structure in eleven dimensions from which our equations have been obtained breaks down[14]. However, going back to the equations in [111], one can see that, in this limit, the internal eight-dimensional geometry becomes a warped Spin(7) manifold with self-dual flux [130]. A careful analysis then shows that, in the $\cos \zeta \to 0$ limit, we obtain the warped G_2-holonomy solutions derived in [125]. The warp factor is again related to the dilaton

$$e^{4\Phi} = e^{-3\Delta} \equiv h, \tag{6.4.5}$$

and, rescaling the internal metric as $\mathrm{d}s_7^2 = h \, \mathrm{d}\hat{s}_7^2$, the full metric becomes

$$\mathrm{d}s_{str}^2 = h^{-1/2} \mathrm{d}x_{1+2}^2 + h^{1/2} \mathrm{d}\hat{s}_7^2, \tag{6.4.6}$$

where the rescaled metric now has G_2-*holonomy*, namely $\mathrm{d}\hat{\phi} = \mathrm{d}\hat{*}_7\hat{\phi} = 0$. Taking

---

[13] An integration constant can always be reabsorbed in a scaling of the coordinates.

[14] In particular, the one-form K does not exist. See Appendix 6.B and [111].

directly the limit on the relation (6.4.4) gives $H_{(3)} + \hat{\ast}_7 F^{\text{int}}_{(4)} = 0$. Hence the four-form flux can be written as

$$F_{(4)} = \text{vol}_3 \wedge \mathrm{d}h^{-1} - \hat{\ast}_7 H_{(3)} \, . \tag{6.4.7}$$

The equation of motion for $F_{(4)}$ implies that the warp factor is harmonic with respect to the G_2 metric, namely

$$\hat{\Box}_7 h = -\frac{1}{6} H^2_{(3)} \, . \tag{6.4.8}$$

6.4.2   Solution generating method

We now discuss two different methods to generate solutions of the equations presented above, starting from a solution of the Type I torsional system. One method, analogous to the procedure discussed in [106], involves a simple chain of dualities. Another method exploits the form of the supersymmetry conditions. We refer to this second method as "rotation" (see Chapter 5, or [107]).

Dualities   We start with a solution of (6.3.2). The non-trivial fields are the dilaton Φ, the three-form flux $H_{(3)}$ and the metric $\mathrm{d}s^2_{str} = \mathrm{d}x^2_{1+2} + \mathrm{d}s^2_7$. First we uplift to eleven dimensions. We rescale the new eleventh dimension by a constant factor $e^{-\Phi_\infty}$, boost along x_{11} with parameter β, and finally undo the rescaling of x_{11}. This gives the transformation

$$t \;\rightarrow\; \cosh\beta\, t - \sinh\beta\, e^{\Phi_\infty} x_{11} \, , \qquad x_{11} \;\rightarrow\; -\sinh\beta\, e^{-\Phi_\infty} t + \cosh\beta\, x_{11} \, . \tag{6.4.9}$$

Then, we reduce back to Type IIA supergravity along the transformed x_{11}, and we perform two T-dualities along the two spatial directions of the $\mathbb{R}^{1,2}$ part. At the level of brane charges, the steps in the transformation may be summarised as

$$\text{NS5} \;\rightarrow\; \text{M5} \;\rightarrow\; \text{M5}, p_{KK} \;\rightarrow\; \text{NS5, D0} \;\rightarrow\; \text{NS5, D2} \, .$$

Notice that a non-zero magnetic $\hat{C}_{(3)}$ field is generated in the process. The dualities above result in the following Type IIA supergravity solution:

$$\mathrm{d}\hat{s}^2_{str} = h^{-1/2} \mathrm{d}x^2_{1+2} + h^{1/2} \mathrm{d}s^2_7 \, , \qquad\qquad h = 1 + \sinh^2\beta (1 - e^{-2(\Phi - \Phi_\infty)}) \, ,$$

$$\hat{H}_{(3)} = \cosh\beta\, H_{(3)} \, , \qquad\qquad e^{2\hat{\Phi}} = e^{2\Phi} h^{1/2} \, ,$$

$$\hat{F}_{(4)} = -\frac{e^{-\Phi_\infty}}{\tanh\beta} \text{vol}_3 \wedge \mathrm{d}(h^{-1}) + \sinh\beta\, e^{\Phi_\infty} e^{-2\Phi} \ast_7 H_{(3)} \, ,$$

$$\tag{6.4.10}$$

where the hatted quantities denote the new solution while the unhatted ones denote the initial background. Notice that, in contrast to the case in [106], the dilaton is changed in the transformation. This can be understood because the procedure here introduces D2-branes, to which the dilaton couples. Notice also that we need $h > 0$, which imposes $e^{2\Phi - 2\Phi_\infty} > \tanh^2 \beta$. Thus the transformation may be applied only if, in the initial solution, the dilaton is a bounded function. We can write the transformed fluxes as

$$
\hat{H}_{(3)} = -\cosh\beta\, e^{2\Phi} *_7 \mathrm{d}\left(e^{-2\Phi}\phi\right),
$$
$$
\hat{F}_{(4)} = -\frac{e^{-\Phi_\infty}}{\tanh\beta}\mathrm{vol}_3 \wedge \mathrm{d}h^{-1} - \sinh\beta\, e^{\Phi_\infty}\mathrm{d}\left(e^{-2\Phi}\phi\right),
$$
(6.4.11)

from which we can read off the internal $\hat{C}_{(3)}$ field in terms of the associative three-form ϕ, namely

$$
\hat{C}_{(3)} = -\sinh\beta\, e^{\Phi_\infty - 2\Phi}\phi.
$$
(6.4.12)

From these expressions, it is clear that the Bianchi identities of the initial solution imply the ones of the transformed solution. In principle, this method may be also applied to non-supersymmetric solutions.

Rotation   The same transformation can be done directly on the supersymmetry equations, without doing any dualities. One advantage of this method is, for example, that it is applicable to configurations with sources (see Chapter 5). Suppose that $\Phi^{(0)} = -3\Delta^{(0)}$ and a three-form $\phi^{(0)}$ are a solution of the system (6.3.2). Then one can define

$$
\hat{\phi} = \left(\frac{\cos\zeta}{c_1}\right)^3 \phi^{(0)},
$$
$$
e^{2\hat{\Phi}} = \frac{\cos\zeta}{c_1}e^{2\Phi^{(0)}},
$$
(6.4.13)
$$
e^{3\hat{\Delta}} = \left(\frac{c_1}{\cos\zeta}\right)^2 e^{-\Phi^{(0)}},
$$

and a new seven-dimensional metric $\mathrm{d}\hat{s}_7^2 = c_1^{-2}\cos^2\zeta\, \mathrm{d}s_7^{(0)2}$. It is easy to check that the new quantities $\hat{\Phi}$, $\hat{\Delta}$ and $\hat{\phi}$ are a solution of the first three equations of the general system (6.4.2). The fourth one can be solved and it gives a relation between ζ and the dilaton of the original solution:

$$
\sin\zeta = c_2\, e^{-\Phi^{(0)}}.
$$
(6.4.14)

Here c_1 and c_2 are integration constants. The rotated background, in terms of unrotated quantities, reads

$$\mathrm{d}\hat{s}_{str}^2 = h^{-1/2}\mathrm{d}x_{1+2}^2 + h^{1/2}\mathrm{d}s_7^{(0)2}\,, \qquad h = \frac{1}{c_1^2}\left(1 - c_2^2 e^{-2\Phi^{(0)}}\right)\,,$$

$$\hat{H}_{(3)} = \frac{1}{c_1}e^{2\Phi^{(0)}} *_7^{(0)} \mathrm{d}\left(e^{-2\Phi^{(0)}}\phi^{(0)}\right)\,, \qquad e^{2\hat{\Phi}} = e^{2\Phi^{(0)}}h^{1/2}\,, \qquad (6.4.15)$$

$$\hat{F}_{(4)} = \frac{1}{c_2}\mathrm{vol}_3 \wedge \mathrm{d}h^{-1} - \frac{c_2}{c_1}\mathrm{d}\left(e^{-2\Phi^{(0)}}\phi^{(0)}\right)\,.$$

In order to match the result of this method to the previous one, one has to identify

$$c_1 = -\frac{1}{\cosh\beta}\,, \qquad c_2 = -e^{\Phi\infty}\tanh\beta\,. \qquad (6.4.16)$$

As before, the Bianchi identities of the general solution follow immediately from the ones of the unrotated solution.

6.4.3   Deformations of the warped G_2-holonomy solutions

We can now apply the above transformation to the solutions of Section 6.3. Notice that, indeed, in those solutions the dilaton is bounded from below. For each solution of Section 6.3, we then obtain a one-parameter family of solutions of Type IIA supergravity, with D2-brane charge and an internal $C_{(3)}$ field. The background is simply obtained by plugging the solutions of Section 6.3 into the equations (6.4.10) or (6.4.15). Notice that the warp factor h in (6.4.10) goes to one at infinity. However, for AdS/CFT applications, one would like to take a decoupling limit in which the warp factor goes to zero at infinity. In this way, the asymptotically Minkowski region is removed and replaced by a boundary. Later we are more precise about the asymptotics. To proceed, first recall that one should quantise the transformed three-form $\hat{H}_{(3)}$ as

$$\widetilde{M} = \frac{1}{4\pi^2}\int_{S^3}\hat{H}_{(3)} = M\cosh\beta \in \mathbb{N}\,, \qquad (6.4.17)$$

where S^3 is the appropriate non-trivial three-sphere in each case. Then, we rescale the Minkowski coordinates as

$$x^\mu \rightarrow \left(\frac{\widetilde{M}\cosh\beta}{c}\right)^{1/2} x^\mu\,. \qquad (6.4.18)$$

In the limit $\beta \to \infty$, keeping $\widetilde{M}$ fixed, the metric is finite and reads

$$d\hat{s}_{str}^2 = \widetilde{M} \left[\tilde{h}^{-1/2} c^{-1} dx_{1+2}^2 + \tilde{h}^{1/2} d\tilde{s}_7^2 \right]. \tag{6.4.19}$$

Here $d\tilde{s}_7^2$ does not have a factor of M and the new warp factor $\tilde{h} = 1 - e^{-2(\Phi - \Phi_\infty)}$ goes to zero at infinity. The factor of c makes sure that the asymptotic form of the metric is independent of c and, in addition, allows us to take the further limit $c \to \infty$. From the expressions in (6.4.10), we see that this limit is problematic for the transformed $\hat{F}_{(4)}$ and dilaton $\hat{\Phi}$. To obtain a finite limit, we also send $e^{\Phi_\infty} \to 0$, while keeping fixed

$$e^{2\Phi_\infty} \sinh \beta = c. \tag{6.4.20}$$

The factor c on the right-hand side is again inserted to allow to take a further $c \to \infty$ limit in the solution. Now, taking $\beta \to \infty$, the solution is completed with[15]

$$e^{2\hat{\Phi}} = c\, e^{2(\Phi - \Phi_\infty)} \tilde{h}^{1/2}, \qquad \hat{H}_{(3)} = -\widetilde{M} e^{2(\Phi - \Phi_\infty)} \tilde{*}_7 d\left(e^{-2(\Phi - \Phi_\infty)} \tilde{\phi} \right),$$

$$\hat{F}_{(4)} = -\widetilde{M}^{3/2} \left[c^{-2} \text{vol}_3 \wedge d\tilde{h}^{-1} + c^{-1/2} d\left(e^{-2(\Phi - \Phi_\infty)} \tilde{\phi} \right) \right]. \tag{6.4.21}$$

Here, tildes on $\tilde{*}_7$ and $\tilde{\phi}$ indicate that the expressions are computed with respect to the metric $d\tilde{s}_7^2$. We can now show that, in this solution, the limit $c \to \infty$ gives a solution of the type found in [125]. Firstly, as we saw in Section 6.3.4, for large c the metric for each X_{ij} solution reads

$$d\hat{s}_{str}^2 = \widetilde{M} \left[\tilde{h}^{-1/2} c^{-1} dx_{1+2}^2 + \tilde{h}^{1/2} c \left(ds_7^2(X_i) + \mathcal{O}(c^{-1}) \right) \right]. \tag{6.4.22}$$

From the differential equation for the dilaton, we find that

$$\Phi' = c^{-2} \left(\frac{1}{2} H' + \mathcal{O}\left(c^{-1} \right) \right), \tag{6.4.23}$$

where H is the warp factor of the solution found in [125], which reads

$$H = \frac{3(r_0 + r)}{4 r_0^3 r^3 (r^2 + r_0 r + r_0^2)^3} \left(16\, r^7 + 24\, r_0\, r^6 + 48\, r_0^2\, r^5 + 47\, r_0^3\, r^4 + 54\, r_0^4\, r^3 \right.$$
$$\left. + 36\, r_0^5\, r^2 + 18\, r_0^6\, r + 9\, r_0^7 \right) + \frac{8\sqrt{3}}{r_0^4} \arctan \frac{2r + r_0}{\sqrt{3}\, r_0} + q.$$
$$\tag{6.4.24}$$

---

[15]In the expressions below Φ_∞ enters only in the combination $\Phi - \Phi_\infty$, which is finite in the limit.

q is an integration constant and, taking $q = -4\sqrt{3}\,\pi\,r_0^{-4}$, we have that $H \sim 81/(4\,r^4)$ when $r \to \infty$. Solving for the dilaton in an expansion in c^{-1}, we find

$$e^{2(\Phi-\Phi_\infty)} = 1 + c^{-2}H + \mathcal{O}\left(c^{-3}\right) . \qquad (6.4.25)$$

Notice that, although this was obtained in [125] for the G_2-holonomy metric on X_1, it follows from our discussion in Section 6.3.4 that this expression is invariant under Σ_3 and hence the same function H in (6.4.24) appears for any X_i. Thus taking the limit $c \to \infty$ on a X_{ij} solution gives the following solution:

$$
\begin{aligned}
&\mathrm{d}\hat{s}^2_{str} = \widetilde{M}\left[H^{-1/2}\mathrm{d}x^2_{1+2} + H^{1/2}\mathrm{d}s^2_7(X_i)\right] , && e^{2\hat{\Phi}} = H^{1/2} , \\
&\hat{F}_{(4)} = -\widetilde{M}^{3/2}\left[\mathrm{vol}_3 \wedge \mathrm{d}\tilde{H}^{-1} - *_7 L_3\right] , && \hat{H}_{(3)} = L_3 ,
\end{aligned}
\qquad (6.4.26)
$$

where L_3 is a harmonic three-form[16] on X_i. This is precisely the warped G_2 solution presented in [125]. Notice that, asymptotically, the string-frame metric goes to $\mathrm{AdS}_4 \times Y$, where $Y \cong S^3 \times S^3$; however the dilaton vanishes like $e^{2\hat{\Phi}} \sim 9/(2r^2)$. In fact, by setting $r_0 = 0$, we have the exact solution with metric

$$\mathrm{d}\hat{s}^2_{str} = \frac{9}{2}\widetilde{M}\left[\mathrm{d}s^2(\mathrm{AdS}_4) + \mathrm{d}s^2(Y)\right] , \qquad (6.4.27)$$

$e^{2\hat{\Phi}} = 9/(2r^2)$ and non-trivial $F_{(4)}$ and $H_{(3)}$ fluxes. However, the solution does not have conformal symmetry because the dilaton depends on the radial coordinate r. In (6.4.27), we can replace $Y \cong S^3 \times S^3$ with another nearly Kähler metric, provided there exists the appropriate harmonic three-form L_3 on the G_2 cone, thus obtaining a solution generically preserving the same amount of supersymmetry. These metrics are in fact solutions of massive Type IIA supergravity [131]. This is a curious fact that might be relevant for AdS/CFT applications [132].

In conclusion, for any solution of the type of [125], arising from configurations of D2-branes and fractional NS5-branes transverse to a G_2 manifold X_i, we have constructed a one-parameter family of deformations, with the same $\mathrm{AdS}_4 \times Y$ asymptotics. These are analogous to the baryonic branch deformations [110] of the Klebanov-Strassler solution [109]. In particular, they break the $\mathbb{Z}_2 \subset \Sigma_3$ symmetry of a G_2-holonomy manifold X_i.

---

[16]This can be extracted from the $c \to \infty$ limit of $c^{-1/2}\tilde{*}_7\mathrm{d}\tilde{\phi}$.

6.5 Discussion

In this chapter, we have discussed various supergravity solutions related to configurations of fivebranes wrapping a three-sphere in a G_2-holonomy manifold $X_i \cong S^3 \times \mathbb{R}^4$. Our basic solutions are examples of torsional G_2 manifolds [24] and comprise some cases previously studied in [128, 123, 36]. There are six solutions characterised by a non-trivial parameter. As we change this parameter, each solution interpolates between a G_2 manifold with (NS5 or D5) branes on a three-sphere and a distinct G_2 manifold with (NS or RR) flux on a different three-sphere. This is then an explicit realisation of a geometric transition between a pair of G_2 manifolds, analogous to the version of the conifold transition described in [106]. The six solutions pairwise connect the three branches of the classical moduli space of G_2-holonomy metrics on $S^3 \times \mathbb{R}^4$ [66]. It would be interesting to see if the picture that we discussed, which is purely classical, may be related to a "large N duality" similar to [64].

From each of the basic solutions, we constructed new Type IIA supergravity backgrounds with D2-brane charge by employing a simple generating method applicable to a class of geometries with interpolating G_2-structure. The solutions constructed in this way are one-parameter deformations of the solutions presented in [125], corresponding to D2-branes and fractional NS5-branes transverse to the G_2 manifold $S^3 \times \mathbb{R}^4$. Therefore, they are analogous to the baryonic branch deformation [110] of the Klebanov-Strassler solution [109].

In the next chapter, we come back to the problem of flavouring. In particular, we develop a method for avoiding the IR singularity present in all backgrounds with flavours we investigated in previous chapters.

6.A $SU(2)$-invariant one-forms

Consider three elements $a_1, a_2, a_3 \in SU(2)$ obeying the constraint $a_1 a_2 a_3 = 1$. We define the following $SU(2)^3$ Lie-algebra-valued one-forms

$$a_1^{-1} \mathrm{d}a_1 \equiv \frac{i}{2}\alpha_i \tau_i\,, \qquad a_2\,\mathrm{d}a_2^{-1} \equiv \frac{i}{2}\beta_i \tau_i\,, \qquad a_3^{-1}\mathrm{d}a_3 \equiv -\frac{i}{2}\gamma_i \tau_i\,, \qquad (6.\mathrm{A}.1)$$

where τ_i are Pauli matrices. We can invert these, obtaining

$$\alpha_i = -i\,\mathrm{Tr}[\tau_i\, a_1^{-1}\mathrm{d}a_1]\,, \qquad \beta_i = -i\,\mathrm{Tr}[\tau_i\, a_2\,\mathrm{d}a_2^{-1}]\,, \qquad \gamma_i = i\,\mathrm{Tr}[\tau_i\, a_3^{-1}\mathrm{d}a_3]\,.$$
$$(6.\mathrm{A}.2)$$

Parametrising the group elements explicitly in terms of angular variables as

$$a_1 = e^{-i\phi_1\tau_3/2}e^{-i\theta_1\tau_1/2}e^{-i\psi_1\tau_3/2}\,,$$

$$a_2 = e^{i\psi_2\tau_3/2}e^{i\theta_2\tau_1/2}e^{i\phi_2\tau_3/2}\,, \qquad\qquad (6.\mathrm{A}.3)$$

$$a_3 = a_2^{-1}a_1^{-1} = e^{-i\phi_2\tau_3/2}e^{-i\theta_2\tau_1/2}e^{-i(\psi_2-\psi_1)\tau_3/2}e^{i\theta_1\tau_1/2}e^{i\phi_1\tau_3/2}\,,$$

we get after some computation

$$\alpha_1 + i\,\alpha_2 = -e^{-i\psi_1}(\mathrm{d}\theta_1 + i\sin\theta_1\,\mathrm{d}\phi_1)\,, \qquad \alpha_3 = -(\mathrm{d}\psi_1 + \cos\theta_1\,\mathrm{d}\phi_1)\,,$$
$$\beta_1 + i\,\beta_2 = -e^{-i\psi_2}(\mathrm{d}\theta_2 + i\sin\theta_2\,\mathrm{d}\phi_2)\,, \qquad \beta_3 = -(\mathrm{d}\psi_2 + \cos\theta_2\,\mathrm{d}\phi_2)\,.$$
$$(6.\mathrm{A}.4)$$

Notice α_i and β_i are $SU(2)$ left-invariant one-forms, obeying

$$\mathrm{d}\alpha_3 = +\alpha_1 \wedge \alpha_2\,, \qquad \mathrm{d}\beta_3 = +\beta_1 \wedge \beta_2\,, \qquad (6.\mathrm{A}.5)$$

and cyclic permutations. We can also define the following Lie-algebra-valued one-forms:

$$a_1\,\mathrm{d}a_1^{-1} \equiv \frac{i}{2}\tilde{\alpha}_i \tau_i\,, \qquad a_2^{-1}\mathrm{d}a_2 \equiv \frac{i}{2}\tilde{\beta}_i \tau_i\,. \qquad (6.\mathrm{A}.6)$$

A similar computation gives

$$\tilde{\alpha}_1 + i\,\tilde{\alpha}_2 = e^{i\phi_1}(\mathrm{d}\theta_1 - i\sin\theta_1\,\mathrm{d}\psi_1)\,, \qquad \tilde{\alpha}_3 = \mathrm{d}\phi_1 + \cos\theta_1\,\mathrm{d}\psi_1\,,$$
$$\tilde{\beta}_1 + i\,\tilde{\beta}_2 = e^{i\phi_2}(\mathrm{d}\theta_2 - i\sin\theta_2\,\mathrm{d}\psi_2)\,, \qquad \tilde{\beta}_3 = \mathrm{d}\phi_2 + \cos\theta_2\,\mathrm{d}\psi_2\,.$$
$$(6.\mathrm{A}.7)$$

These are $SU(2)$ right-invariant one-forms, obeying

$$\mathrm{d}\tilde{\alpha}_3 = +\tilde{\alpha}_1 \wedge \tilde{\alpha}_2\,, \qquad \mathrm{d}\tilde{\beta}_3 = +\tilde{\beta}_1 \wedge \tilde{\beta}_2\,, \qquad (6.\mathrm{A}.8)$$

and cyclic permutations. Computing the γ_i, we obtain

$$-\gamma_i = \tilde{\alpha}_i + M_{ij}\beta_j \,, \tag{6.A.9}$$

where M_{ij} is the following $SO(3)$ matrix:

$$M_{ij} = \begin{pmatrix} \cos\phi_1\cos\psi_1 - \cos\theta_1\sin\phi_1\sin\psi_1 & -\cos\theta_1\cos\psi_1\sin\phi_1 - \cos\phi_1\sin\psi_1 & \sin\theta_1\sin\phi_1 \\ \cos\psi_1\sin\phi_1 + \cos\theta_1\cos\phi_1\sin\psi_1 & \cos\theta_1\cos\phi_1\cos\psi_1 - \sin\phi_1\sin\psi_1 & -\cos\phi_1\sin\theta_1 \\ \sin\theta_1\sin\psi_1 & \cos\psi_1\sin\theta_1 & \cos\theta_1 \end{pmatrix}.$$
$$\tag{6.A.10}$$

We note the following identities:

$$\sum_i \alpha_i^2 = \sum_i \tilde{\alpha}_i^2 \,, \qquad \sum_i \beta_i^2 = \sum_i \tilde{\beta}_i^2 \,, \qquad \sum_i \gamma_i^2 = \sum_i (\alpha_i - \beta_i)^2 \,. \tag{6.A.11}$$

To prove the third one, we have to use $M_{ij}M_{ik} = \delta_{ik}$ and $\alpha_i = -M_{ji}\tilde{\alpha}_j$. We identify the above with the (left-invariant) one-forms σ_i and Σ_i used in the main text

$$\sigma_i = -\alpha_i = i\,\mathrm{Tr}[\tau_i\, a_1^{-1}\mathrm{d}a_1] \,, \qquad \Sigma_i = -\beta_i = i\,\mathrm{Tr}[\tau_i\, a_2\,\mathrm{d}a_2^{-1}] \,, \tag{6.A.12}$$

where the minus signs have been included in order to match with our conventions on the Lie-algebra relations $\mathrm{d}\sigma_1 = -\sigma_2 \wedge \sigma_3$, and so on. Notice that

$$\gamma_i = i\,\mathrm{Tr}[\tau_i\, a_3^{-1}\mathrm{d}a_3] = \tilde{\sigma}_i + M_{ij}\Sigma_j = M_{ij}(\Sigma_j - \sigma_j) \,. \tag{6.A.13}$$

We also define

$$\mathrm{d}a_1^2 = -2\sum_i (\mathrm{Tr}[\tau_i\, a_1^{-1}\mathrm{d}a_1])^2 \,,$$
$$\mathrm{d}a_2^2 = -2\sum_i (\mathrm{Tr}[\tau_i\, a_2\, \mathrm{d}a_2^{-1}])^2 \,, \tag{6.A.14}$$
$$\mathrm{d}a_3^2 = -2\sum_i (\mathrm{Tr}[\tau_i\, a_3^{-1}\mathrm{d}a_3])^2 \,.$$

6.B   Type IIA supersymmetry conditions from M-theory

General conditions characterising $\mathcal{N} = 1$ solutions of eleven-dimensional supergravity of the warped-product type $X_{1+2} \times_w M_8$, where X_{1+2} is either $\mathbb{R}^{1,2}$ or AdS_3, were presented in [111]. Here, we are interested in the case where $X_{1+2} = \mathbb{R}^{1,2}$.

The eleven-dimensional metric is written as

$$d\hat{s}_{11}^2 = e^{2\Delta}(dx_{1+2}^2 + ds_8^2)\,,$$ (6.B.1)

and the four-form flux reads

$$G_{(4)} = e^{3\Delta}(F_{(4)} + \text{vol}_3 \wedge f)\,.$$ (6.B.2)

Thus $F_{(4)}$ is a four-form and f is a one-form. Upon setting $m = 0$, the equations (3.11) - (3.16) of [111] become

$$d(e^{3\Delta} K \cos\zeta) = 0\,,$$

$$K \wedge d(e^{6\Delta} *_7 \phi) = 0\,,$$ (6.B.3)

$$d(e^{12\Delta} \text{vol}_7 \cos\zeta) = 0\,,$$

$$d\phi \wedge \phi \cos\zeta = 2 * (\cos\zeta\, f - 2d\zeta)\,.$$

Here ϕ is a three-form, K is a one-form and ζ is a function, all defined as spinor bilinears, that characterise the G_2-structure in eight dimensions. The Hodge star operator in seven dimensions is defined as $*_7 = i_K*$ and $\text{vol}_7 = \frac{1}{7}\phi \wedge *_7\phi$. The electric and magnetic fluxes are then determined in terms of the G_2-structure as

$$e^{-3\Delta}d(e^{3\Delta} \sin\zeta) = f\,,$$ (6.B.4)

$$e^{-6\Delta}d(e^{6\Delta} \cos\zeta\, \phi) = - * F_{(4)} + \sin\zeta\, F_{(4)}\,.$$

The latter equation obeyed by the magnetic flux $F_{(4)}$ may be inverted, giving

$$\cos^2\zeta\, F_{(4)} = -e^{-6\Delta}\left[\sin\zeta\, d(e^{6\Delta} \cos\zeta\, \phi) + *d(e^{6\Delta} \cos\zeta\, \phi)\right]\,.$$ (6.B.5)

The one-form K in general does not correspond to a Killing vector. However, in order to reduce to Type IIA supergravity, we *assume* that the dual vector $K^{\#}$ is Killing. In particular, by writing $K = e^{2\Phi/3-\Delta}dy$, the eleven-dimensional metric takes the form

$$d\hat{s}_{11}^2 = e^{2\Delta}(dx_{1+2}^2 + ds_7^2 + e^{4\Phi/3-2\Delta}dy^2)\,,$$ (6.B.6)

and its reduction to ten dimensions then can be simply read off:

$$ds_{str}^2 = e^{2\Delta+2\Phi/3}(dx_{1+2}^2 + ds_7^2)\,.$$ (6.B.7)

Then we write

$$\mathrm{d} = \mathrm{d}_7 + \mathrm{d}y\nabla_y\,, \qquad f = f_7 + \mathrm{d}y f_y\,. \tag{6.B.8}$$

Looking first at the fluxes, we find

$$f_y = 0\,, \qquad f_7 = e^{-3\Delta}\mathrm{d}_7(e^{3\Delta}\sin\zeta)\,. \tag{6.B.9}$$

Using these equations, from (6.B.3), we obtain

$$
\begin{aligned}
\mathrm{d}_7\big(e^{6\Delta} *_7 \phi\big) &= 0\,, & \phi \wedge \mathrm{d}_7\phi &= 0\,, \\
2\mathrm{d}_7\zeta - e^{-3\Delta}\cos\zeta\,\mathrm{d}_7\big(e^{3\Delta}\sin\zeta\big) &= 0\,, & \mathrm{d}_7\big(e^{2\Delta+2\Phi/3}\cos\zeta\big) &= 0\,.
\end{aligned}
\tag{6.B.10}
$$

Finally, the reduction of the four-form $G_{(4)}$ gives the NS three-form $H_{(3)}$ and the RR four-form $F_{(4)}$:

$$
\begin{aligned}
H_{(3)} &= \frac{1}{\cos^2\zeta}e^{-4\Delta+2\Phi/3} *_7 \mathrm{d}_7\big(e^{6\Delta}\cos\zeta\,\phi\big)\,, \\
F_{(4)} &= \mathrm{vol}_3 \wedge \mathrm{d}_7\big(e^{3\Delta}\sin\zeta\big) - \frac{\sin\zeta}{\cos^2\zeta}e^{-3\Delta}\mathrm{d}_7\big(e^{6\Delta}\cos\zeta\,\phi\big)\,.
\end{aligned}
\tag{6.B.11}
$$

Chapter 7

Massive flavours in the MN model

7.1 Introduction

We study, in this chapter, the flavouring of the Maldacena-Núñez background [13] with massive flavours. We show that the main issue in doing so is finding embeddings for the flavour branes whose backreaction is compatible with our ansatz for the background metric. The holomorphic structure of our complex space proves very useful in addressing this problem. In addition, we propose a method for detecting which embeddings have an appropriate backreaction. This technique also provides an easy way to calculate the smearing form corresponding to those suitable embeddings. This chapter is based on [12] which was done in collaboration with Conde and Ramallo.

So we are interested in adding backreacting flavours in the holographic dual of $\mathcal{N} = 1$ super Yang-Mills. In this case, one has to compute the backreaction by solving the equations of motion of a system of gravity with brane sources. Generically, these sources modify the Einstein equations and the Bianchi identities of some Ramond-Ramond field strengths. We follow the approach initiated in [21], in which one has a large number of flavour-brane sources which are delocalised and one has to deal with a continuous smeared distribution of branes (see [20] for an earlier implementation of this idea in the context of non-critical string theory). In this approach, the sources do not contain Dirac δ-functions, which greatly simplifies the task of solving the equations of motion. On the field-theory side, this setup corresponds to the so-called Veneziano limit, in which both N_c and N_f are large but their ratio is kept fixed. In [28, 29], different aspects of the supergravity duals of $\mathcal{N} = 1$ super Yang-Mills with smeared flavour branes were studied, whereas this

approach has also been successfully applied to other types of backgrounds (see [22] for a detailed review).

The $\mathcal{N} = 1$ flavours added in [21, 28, 29] are massless, which amounts to considering flavour branes extended along the full range of the holographic coordinate r. The corresponding supergravity solutions are singular in the IR. This is actually a common feature of most massless flavoured solutions found so far with the smearing technique (see, for example, [30, 32, 34] for the D3-D7 systems on the conifold). This curvature singularity can be qualitatively understood as due to the fact that, for massless flavours, all branes extend all the way to the origin $r = 0$ and, therefore, the brane density is highly peaked at $r = 0$ (an exception to this behaviour is the solution found in [133] for the gravity dual of Chern-Simons-matter theories with flavours).

To remove the IR singularity, one can consider massive quarks or, equivalently, a family of flavour branes which do not reach the origin (another possibility is adding temperature and hiding the singularity behind a horizon, as was done in [134]). For the D3-D7 system, these regular solutions for massive flavours were found in [35, 41, 102]. As argued in [30], going from the massless to the massive case in these systems just amounts to substituting in the ansatz N_f by $N_f S(r)$, where $S(r)$ is a profile function that interpolates between 0 in the IR and 1 in the UV. To calculate $S(r)$, one has to perform a microscopic calculation of the flavour-brane charge density, whose result is not universal since it depends both on the characteristics of the unflavoured system and on the particular family of flavour-brane embeddings.

In this chapter, we find supergravity backgrounds dual to $\mathcal{N} = 1$ super Yang-Mills theories with backreacting massive quarks. The first step in our analysis is finding the precise deformation of the background which corresponds to the backreaction induced by the massive flavours. We show that the compatibility with the $\mathcal{N} = 1$ supersymmetry implies a certain type of deformation which is also parametrised by a profile function $S(r)$. When this function S is identically equal to one, we recover the results of [21] for massless quarks. However, it is important to point out that, in this D5-brane case, the massive quark ansatz cannot be recovered by performing the $N_f \rightarrow N_f S(r)$ substitution in the massless ansatz of [21].

From our ansatz, we are able to obtain a consistent system of first-order BPS equations, which can be partially integrated and reduced to a second-order master equation which is the generalisation to this massive case of the equation derived in

[29] for massless quarks. In order to solve this master equation (and the full BPS system), one first needs to know the profile function $S(r)$ which, as mentioned above, is not universal and depends on the family of embeddings of the flavour D5-branes. Such families are generated by acting with isometries on a specific embedding. It turns out that only a particular set of these families produce a backreaction which is compatible with our ansatz. For this reason, we must generalise the results of [49] and find new classes of supersymmetric embeddings of flavour D5-branes. To carry out this analysis, we introduce a convenient set of complex coordinates suitable for representing the metric and forms of the $SU(3)$-structure of our geometry. Using these variables, we are able to find a family of compatible embeddings and to compute the corresponding profile function $S(r)$.

For massive quarks, the function $S(r)$ vanishes when r is less than a certain value r_0, which is related to the mass of the quarks. When $r \leq r_0$, the BPS system coincides with the unflavoured one, which corresponds to the fact that the quarks are effectively integrated out in this low-energy region. As shown in [21, 29], there exists a one-parameter family of solutions of the unflavoured system which are regular at $r = 0$. Our flavoured solutions coincide with these, studied in [21, 29], in this $0 \leq r \leq r_0$ region and, although a potential threshold singularity could appear at $r = r_0$, we show how to engineer brane distributions which give rise to geometries that are regular everywhere.

The rest of this chapter is organised as follows. In Section 7.2, we study the addition of massive flavours to the $\mathcal{N} = 1$ background and we introduce our ansatz for the backreaction induced by a smeared distribution of flavour branes. In that section, we also present the result of the partial integration of the BPS system, as well as the master equation for massive flavours. The holomorphic structure of the model is worked out in Section 7.3. In Section 7.4, we develop a technique to compute the charge-distribution function $S(r)$. With this method, $S(r)$ is obtained by comparing the Wess-Zumino action for the continuous set of branes and that of a single representative embedding. Applying this procedure, we discover that not all the families of embeddings produce a backreaction compatible with our ansatz. In Section 7.5, we find a simple class of compatible embeddings and we compute the corresponding profile function. The problem of the threshold singularities is analysed in Section 7.6, where we show how to avoid them and how one can construct regular flavoured backgrounds. Finally, in Section 7.7, we integrate numerically the master equation and we provide numerical solutions for the different functions of the ansatz.

7.2   Addition of massive flavours

In this section, we deal again with the flavouring of the Maldacena-Núñez background [13], briefly reviewed at the beginning of Section 2.2.1. But, this time, we consider massive quarks. Let us introduce flavours in the field theory by means of pairs of chiral multiplets Q and $\widetilde{Q}$ transforming in the fundamental and antifundamental representations of both the gauge group $SU(N_c)$ and the flavour group $SU(N_f)$. The Lagrangian for the $(Q, \widetilde{Q})$ fields is given by the usual kinetic terms and the Yukawa interaction between the quarks and the KK modes, which can be schematically written as

$$L_{Q,\widetilde{Q}} = \int \mathrm{d}^4\theta \left(Q^\dagger e^{-V} Q + \widetilde{Q}^\dagger\, e^V \widetilde{Q} \right) + \int \mathrm{d}^2\theta\, \widetilde{Q}\, \Phi_k Q\,. \qquad (7.2.1)$$

In the effective low-energy theory obtained by integrating out the massive modes, the Yukawa coupling between $(Q, \widetilde{Q})$ and the Φ_k gives rise to a quartic term for the quark fields (see [21, 28, 29] for details).

On the gravity side, as we know from Chapter 2, the addition of quarks can be performed by means of flavour branes, which add an open-string sector to the unflavoured closed-string background. For the case at hand, the flavour branes are D5-branes extended along a non-compact cycle of the six-dimensional internal space. If the branes reach the origin $r = 0$ of the geometry, the corresponding flavour fields are massless. If, on the contrary, the D5-branes do not reach $r = 0$, the quark fields are massive (the minimal value of r attained by the brane is related to the mass of the quark fields).

In this chapter, we are interested in getting a holographic dual of the $\mathcal{N} = 1$ model with matter, in which the dynamics of fundamentals is encoded in the background. To achieve this goal, we must go beyond the probe approximation and take into account the backreaction of the flavour branes. As presented in Chapter 2, we use for that purpose the technique of smearing. Let us briefly recall here, for completeness, how this method is applied in the particular case we are interested in (see Section 2.2.1). We want to study solutions of the action

$$S = S_{IIB} + S_{\text{sources}}\,, \qquad (7.2.2)$$

where S_{IIB} is the action of ten-dimensional Type IIB supergravity and S_{sources} denotes the sum of the DBI and WZ actions for the flavour branes. Generically,

the branes act as sources for the different supergravity fields. In particular, the WZ term of S_sources is a source term for the RR fields which induces a violation of the Bianchi identity of the corresponding RR field strength. In our case, the WZ term of the action of a set of D5-branes is

$$S_{WZ} = T_5 \sum_{i=1}^{N_f} \int_{\mathcal{M}_6^{(i)}} \imath^* \big(C_{(6)} \big) \,, \tag{7.2.3}$$

where $C_{(6)}$ is the RR six-form potential and $\imath^* \big(C_{(6)} \big)$ denotes its pull-back to the D5-brane world-volume. Let us rewrite (7.2.3) as a ten-dimensional integral, in terms of a charge-distribution four-form Ξ:

$$S_{WZ} = T_5 \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Xi \,. \tag{7.2.4}$$

The term (7.2.4) induces a violation of the Bianchi identity for $F_{(3)}$, which is just

$$\mathrm{d} F_{(3)} = 4\pi^2 \, \Xi \,. \tag{7.2.5}$$

The four-form Ξ is simply the RR charge distribution due to the presence of the flavour D5-branes. Clearly, Ξ is non-zero at the location of the sources. In a localised setup, in which the N_f flavour branes are on top of each other, Ξ contains Dirac δ-functions and finding the corresponding backreacted geometry is technically a very complicated task. For this reason, we separate the N_f branes and we distribute them homogeneously along the internal manifold in such a way that, in the limit in which N_f is large, they can be described by a continuous charge distribution Ξ.

As we detail below, the continuous set of flavour branes that we use in our construction can be generated by acting with the isometries of the background on a particular embedding and, therefore, all the branes of the continuous set are physically equivalent. Actually, we do not choose an arbitrary distribution of branes. First of all, we require that all branes are mutually supersymmetric (and thus they do not exert force on each other) and that they preserve the same supercharges as the ones of the unflavoured background. Moreover, we also require the deformation induced on the metric to be mild enough, in such a way that it reduces to squashing the unflavoured metric with functions depending only on the radial coordinate r. One can prove that the most general squashing of this type compatible with the $\mathcal{N} = 1$ supersymmetry of the unflavoured background is the

one used in the following ansatz for the Einstein-frame metric of the flavoured theory:

$$
ds^2 = e^{2f(r)} \Big[dx_{1,3}^2 + e^{2k(r)}dr^2 + e^{2h(r)} \big(d\theta^2 + \sin^2\theta \, d\varphi^2 \big)
$$

$$
+ \frac{e^{2g(r)}}{4} \Big(\big(\tilde{\omega}^1 + a(r)d\theta \big)^2 + \big(\tilde{\omega}^2 - a(r)\sin\theta \, d\varphi \big)^2 \Big) + \frac{e^{2k(r)}}{4} \big(\tilde{\omega}^3 + \cos\theta \, d\varphi \big)^2 \Big],
$$
$$(7.2.6)$$

where

$$
\tilde{\omega}_1 = \cos\psi \, d\tilde{\theta} + \sin\psi \sin\tilde{\theta} \, d\tilde{\varphi} \,,
$$
$$
\tilde{\omega}_2 = -\sin\psi \, d\tilde{\theta} + \cos\psi \sin\tilde{\theta} \, d\tilde{\varphi} \,, \qquad (7.2.7)
$$
$$
\tilde{\omega}_3 = d\psi + \cos\tilde{\theta} \, d\tilde{\varphi} \,.
$$

Notice that the ansatz (7.2.6) is exactly the same as the one considered in [21] for the case of massless flavours.

Let us next consider the deformation of the RR three-form $F_{(3)}$. Clearly, due to the modified Bianchi identity (7.2.5) that must be satisfied in the flavoured case, $F_{(3)}$ cannot have the same form as in the unflavoured case. We then adopt the following ansatz for the RR three-form $F_{(3)}$:

$$
F_{(3)} = -\frac{N_c}{4} \big(\tilde{\omega}^1 - B^1 \big) \wedge \big(\tilde{\omega}^2 - B^2 \big) \wedge \big(\tilde{\omega}^3 - B^3 \big) + \frac{N_c}{4} \sum_a \big(F^a + f^a \big) \wedge \big(\tilde{\omega}^a - B^a \big) \,,
$$
$$(7.2.8)$$

where B^a is an $SU(2)$ one-form gauge connection and F^a is its two-form field strength, defined as

$$
F^a = dB^a + \frac{1}{2}\epsilon^{abc}B^b \wedge B^c \,. \qquad (7.2.9)
$$

In (7.2.8), the f^a are two-forms that parametrise the violation of the Bianchi identity and thus the flavour deformation of the RR three-form. Indeed, when $f^a = 0$, the three-form $F_{(3)}$ is closed by construction, due to the relation (7.2.9) between F^a and B^a. We take, as in [21], the following ansatz for B^a:

$$
B^1 = -b(r)d\theta \,, \qquad B^2 = b(r)\sin\theta \, d\varphi \,, \qquad B^3 = -\cos\theta \, d\varphi \,, \qquad (7.2.10)
$$

where $b(r)$ is different from the fibring function $a(r)$ of the metric (they are equal in the background of [13]). By applying the definition (7.2.9), we get that the different components of the two-form field strength F^a are

$$
F^1 = -b'dr \wedge d\theta \,, \qquad F^2 = b'\sin\theta \, dr \wedge d\varphi \,, \qquad F^3 = (1-b^2)\sin\theta \, d\theta \wedge d\varphi \,.
$$
$$(7.2.11)$$

We adopt for the flavour-deformation two-forms f^a an ansatz that parallels F^a, namely:

$$f^1 = -L_1(r)\mathrm{d}r \wedge \mathrm{d}\theta\,, \qquad f^2 = L_1(r)\sin\theta\,\mathrm{d}r \wedge \mathrm{d}\varphi\,, \qquad f^3 = L_2(r)\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi\,,$$

$$(7.2.12)$$

where L_1 and L_2 are two functions of the radial variable to be determined. Actually, after a detailed study of the realisation of supersymmetry for the metric ansatz (7.2.6), one can show that (7.2.12) gives rise to the most general form of $F_{(3)}$. By computing the exterior derivative of (7.2.8) and applying (7.2.5), one gets the following expression for the smearing form Ξ:

$$\Xi = -\frac{N_c}{16\pi^2}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \left[L_2\,\tilde{\omega}^1 \wedge \tilde{\omega}^2 - L_2'\,\mathrm{d}r \wedge \tilde{\omega}^3 \right]$$
$$+ \frac{N_c L_1}{16\pi^2}\,\mathrm{d}r \wedge \left[\mathrm{d}\theta \wedge \tilde{\omega}^2 \wedge \tilde{\omega}^3 + \mathrm{d}\varphi \wedge \left(\sin\theta\,\tilde{\omega}^1 \wedge \tilde{\omega}^3 + \cos\theta\,\mathrm{d}\theta \wedge \tilde{\omega}^2 \right) \right]\,.$$

$$(7.2.13)$$

One can now study the realisation of $\mathcal{N} = 1$ supersymmetry in Type IIB supergravity for a background with metric and RR three-form given by the ansatz written in (7.2.6) and (7.2.8). Following the conventions of Chapter 5, we define the $SU(3)$-invariant forms as

$$J = e^{r3} + \left(\cos\alpha\,e^\varphi + \sin\alpha\,e^2\right) \wedge e^\theta + \left(-\sin\alpha\,e^\varphi + \cos\alpha\,e^2\right) \wedge e^1\,,$$

$$\Omega = \left(e^r + i\,e^3\right) \wedge \left(\left(\cos\alpha\,e^\varphi + \sin\alpha\,e^2\right) + i\,e^\theta\right) \wedge \left(\left(-\sin\alpha\,e^\varphi + \cos\alpha\,e^2\right) + i\,e^1\right)\,,$$

$$(7.2.14)$$

where

$$e^r = e^k \mathrm{d}r\,, \qquad\qquad e^\theta = e^h \mathrm{d}\theta\,,$$

$$e^\varphi = e^h \sin\theta\,\mathrm{d}\varphi\,, \qquad e^1 = \frac{e^g}{2}(\tilde{\omega}^1 + a(r)\mathrm{d}\theta)\,, \qquad (7.2.15)$$

$$e^2 = \frac{e^g}{2}(\tilde{\omega}^2 - a(r)\sin\theta\,\mathrm{d}\varphi)\,, \qquad e^3 = \frac{e^k}{2}(\tilde{\omega}^3 + \cos\theta\,\mathrm{d}\varphi)\,.$$

In terms of those forms, the BPS system can be written as (see (5.2.8))

$$\mathrm{d}\left(e^{6f+\Phi/2}\Omega\right) = 0\,, \qquad\qquad \mathrm{d}\left(e^{2f-\Phi/2}\right) = 0\,,$$
$$\mathrm{d}\left(e^{8f}J \wedge J\right) = 0\,, \qquad -e^{-2f-3\Phi/2}\mathrm{d}\left(e^{4f+\Phi}J\right) = *_6 F_{(3)}\,,$$

$$(7.2.16)$$

where $*_6$ denotes the Hodge dual with respect to the internal part of the metric (7.2.6). This system can be reduced and a partial integration is possible. Let us summarise in this section the results of the study of the BPS equations (additional

details can be found in Appendix A of [12]). First of all, one can verify that the functions L_1 and L_2 parametrising f^a and Ξ are not independent. Actually, L_1 can be written in terms of the derivative of L_2 as follows:

$$L_1 = -\frac{L_2'}{2\cosh(2r)} \, . \tag{7.2.17}$$

Therefore, if we define the function $S(r)$ as

$$N_f\, S(r) \equiv -N_c\, L_2(r) \, , \tag{7.2.18}$$

then the two-forms f^a of (7.2.12) become

$$f^1 = -\frac{N_f}{2N_c}\frac{S'(r)}{\cosh(2r)}\, \mathrm{d}r \wedge \mathrm{d}\theta \, ,$$

$$f^2 = \frac{N_f}{2N_c}\frac{S'(r)}{\cosh(2r)}\sin\theta\, \mathrm{d}r \wedge \mathrm{d}\varphi \, , \tag{7.2.19}$$

$$f^3 = -\frac{N_f}{N_c}S(r)\sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi \, ,$$

and the smearing form Ξ can be written in terms of S as

$$\Xi = \frac{N_f}{16\pi^2}\sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \left[S\, \tilde\omega^1 \wedge \tilde\omega^2 - S'\mathrm{d}r \wedge \tilde\omega^3 \right]$$

$$+ \frac{N_f}{32\pi^2}\frac{S'}{\cosh(2r)}\mathrm{d}r \wedge \left[\mathrm{d}\theta \wedge \tilde\omega^2 \wedge \tilde\omega^3 + \mathrm{d}\varphi \wedge \left(\sin\theta\, \tilde\omega^1 \wedge \tilde\omega^3 + \cos\theta\, \mathrm{d}\theta \wedge \tilde\omega^2 \right) \right] . \tag{7.2.20}$$

Moreover, for the function b parametrising the one-forms B^a, we find

$$b(r) = \frac{2r + \eta(r)}{\sinh(2r)} \, , \tag{7.2.21}$$

where $\eta(r)$ is defined as the following integral involving S:

$$\eta(r) = -\frac{N_f}{2N_c}\left[\tanh(2r)S(r) + 2\int_0^r \tanh^2(2\rho)S(\rho)\mathrm{d}\rho \right] . \tag{7.2.22}$$

It follows from these results that the RR three-form $F_{(3)}$ in (7.2.8) is determined in terms of a unique function $S(r)$. Notice that the case of massless flavours studied in [21] is recovered by taking $S = 1$ in our formulas. Indeed, in this case, only the first term of the right-hand side of (7.2.20) is non-zero and the charge density distribution Ξ is independent of the radial variable. Moreover, by computing the

integral in (7.2.22), one can show that our ansatz for $F_{(3)}$ is reduced to the one adopted in [21].

In the case of massive flavours, one expects the charge distribution to depend non-trivially on the radial coordinate and, actually, to vanish for values of r smaller than a certain scale related to the mass of the quarks. In our approach, this non-trivial structure is encoded in the dependence of the function S on the radial variable. Notice also that S should approach the massless value $S = 1$ as $r \to \infty$ since the quarks are effectively massless in the deep UV. The way in which the profile function S interpolates between the IR and UV values depends on the particular set of D5-branes that constitutes our delocalised source and should be obtained by means of a microscopic calculation of the charge density (see below).

Another interesting observation is that, contrary to the backgrounds with massive flavours studied in [35, 41, 102], passing from the massless to the massive case is not, in our current example, equivalent to substituting N_f by $N_f S(r)$ in the massless ansatz. Indeed, it is immediate to check that making this substitution only generates the first line in (7.2.20), while the last two components of Ξ (which are essential for the consistency of the approach) are missing. Notice that these last two terms are precisely those in (7.2.13) which are proportional to the function L_1 which, according to (7.2.17), always vanishes when $r \to \infty$. This means that, in the UV, the two-form f^a that implements the flavour deformation of $F_{(3)}$ is non-vanishing only along the third $SU(2)$ direction, while its other two components are excited when we move towards the IR.

Actually, it turns out that one can also integrate partially the BPS system for the functions of the metric in terms of $S(r)$. First of all, the function f is related to the dilaton Φ as $f = \Phi/4$. Moreover, the dilaton can be expressed in terms of the other functions h, g and k as

$$e^{-2\Phi} = 2e^{-2\Phi_0} \frac{e^{h+g+k}}{\sinh(2r)}, \qquad (7.2.23)$$

where Φ_0 is a constant. In order to solve for the remaining functions in the metric (7.2.6), let us define, following [29], the functions $P(r)$ and $Q(r)$ in terms of a and g as

$$Q = \big(a \cosh(2r) - 1\big)e^{2g}, \qquad P = a\, e^{2g} \sinh(2r). \qquad (7.2.24)$$

The inverse of this relation is

$$e^{2g} = P \coth(2r) - Q, \qquad a = \frac{P}{P \cosh(2r) - Q \sinh(2r)}. \qquad (7.2.25)$$

Again from the BPS system, one can express h and k in terms P, Q and S, namely:

$$e^{2h} = \frac{1}{4} \frac{P^2 - Q^2}{P \coth(2r) - Q}, \qquad e^{2k} = \frac{P' + N_f S(r)}{2}. \qquad (7.2.26)$$

It follows from equations (7.2.23)-(7.2.26) that the dilaton and the functions of the metric are determined in terms of P, Q and S. Actually, the function Q can be integrated in terms of the profile S as

$$Q = \coth(2r) \left[\int_0^r \frac{2N_c - N_f S(\rho)}{\coth^2(2\rho)} d\rho + q_0 \right], \qquad (7.2.27)$$

where q_0 is a constant of integration. Moreover, as in [29], one can find a master equation:

$$P'' + N_f S' + (P' + N_f S) \left(\frac{P' - Q' + 2N_f S}{P + Q} + \frac{P' + Q' + 2N_f S}{P - Q} - 4\coth(2r) \right) = 0. \qquad (7.2.28)$$

One can first notice that, in the case $S = 1$, (7.2.28) reduces to the equation found in [29]. Otherwise, knowing the function S (from a microscopic description of the smearing), one can get Q from (7.2.27) and solve the second-order master equation (7.2.28) for P. As argued above, each solution of this equation gives a complete solution of the problem, as long as it satisfies the following conditions:

$$P \coth(2r) \geq Q, \qquad P^2 \geq Q^2, \qquad P' + N_f S \geq 0. \qquad (7.2.29)$$

Moreover, in Appendix 5.A, we have explicitly written the equations of motion derived from the Type IIB supergravity plus sources action. One can check that any solution of the BPS system also solves the second-order equations of motion written in Appendix 5.A.

Finding an analytic solution of this master equation is probably not possible, but we are able to find numerical solutions, and study their asymptotics. In order to achieve this goal, we first have to identify a family of supersymmetric embeddings whose backreaction on the background is compatible with our ansatz, and then we must be able to compute the corresponding profile function S. In the next section, we start to develop the necessary machinery to carry out this computation.

7.3   Holomorphic structure

As stated at the end of Section 7.2, in order to find the profile function $S(r)$, we must analyse the families of supersymmetric embeddings of the flavour D5-branes.

This problem was addressed in [49] by looking at the realisation of κ-symmetry for probe D5-branes in the unflavoured background of [13]. The analysis of [49] was performed in terms of the angular coordinates of the metric (7.2.6) and some particularly interesting embeddings were found. For our present purposes, we clearly need a more systematic approach, which could allow us to study different families of embeddings and to determine whether or not their backreaction is consistent with our ansatz (7.2.6)-(7.2.12). As the internal manifold of our background is complex, it is quite natural to work in a system of complex coordinates. The purpose of this section is to introduce these coordinates and to uncover the holomorphic structure of our background.

Let us begin by introducing a set of four complex variables z_i $(i = 1 \ldots 4)$ parametrising a deformed conifold, i.e. satisfying the following quadratic equation:

$$z_1 z_2 - z_3 z_4 = 1 \,. \tag{7.3.1}$$

We also introduce a radial variable r, related to the z_i's, as

$$\sum_{i=1}^{4} |z_i|^2 = 2 \cosh(2r) \,. \tag{7.3.2}$$

In order to find a useful parametrisation of the z_i's, let us arrange them as the following 2×2 complex matrix Z:

$$Z = \begin{pmatrix} z_3 & z_2 \\ -z_1 & -z_4 \end{pmatrix} \,. \tag{7.3.3}$$

Then, the defining equations (7.3.1) and (7.3.2) can be written in matrix form as

$$\det(Z) = 1 \,, \qquad \mathrm{Tr}\left(ZZ^\dagger\right) = 2 \cosh(2r) \,. \tag{7.3.4}$$

It is immediate to verify that the matrix

$$Z_0 = \begin{pmatrix} 0 & e^r \\ -e^{-r} & 0 \end{pmatrix} \tag{7.3.5}$$

is a particular solution of (7.3.4). The general solution of this equation can be found by realising that the equations in (7.3.4) exhibit the following $SU(2)_L \times SU(2)_R$ symmetry:

$$Z \to LZR^\dagger \,, \qquad \text{with} \quad L \in SU(2)_L \,, \quad R \in SU(2)_R \,. \tag{7.3.6}$$

A generic point in the conifold can be obtained by acting with these isometries on the point (7.3.5). Actually, if we parametrise the $SU(2)$ matrices above in terms of Euler angles as

$$
L = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}
\qquad
\begin{aligned}
a &= \cos\tfrac{\theta}{2}\, e^{i\frac{\psi_1+\varphi}{2}} \,, \\
b &= \sin\tfrac{\theta}{2}\, e^{i\frac{\psi_1-\varphi}{2}} \,,
\end{aligned}
$$

$$
R = \begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix}
\qquad
\begin{aligned}
k &= \cos\tfrac{\tilde\theta}{2}\, e^{i\frac{\psi_2+\tilde\varphi}{2}} \,, \\
l &= -\sin\tfrac{\tilde\theta}{2}\, e^{i\frac{\psi_2-\tilde\varphi}{2}} \,,
\end{aligned}
\tag{7.3.7}
$$

then the four complex variables z_1, z_2, z_3, z_4 that solve (7.3.4) are given by

$$
z_1 = -e^{-\frac{i}{2}(\varphi+\tilde\varphi)} \left(e^{r+i\frac{\psi}{2}} \sin\frac{\theta}{2} \sin\frac{\tilde\theta}{2} - e^{-r-i\frac{\psi}{2}} \cos\frac{\theta}{2} \cos\frac{\tilde\theta}{2} \right) \,,
$$

$$
z_2 = e^{\frac{i}{2}(\varphi+\tilde\varphi)} \left(e^{r+i\frac{\psi}{2}} \cos\frac{\theta}{2} \cos\frac{\tilde\theta}{2} - e^{-r-i\frac{\psi}{2}} \sin\frac{\theta}{2} \sin\frac{\tilde\theta}{2} \right) \,,
$$

$$
z_3 = e^{\frac{i}{2}(\varphi-\tilde\varphi)} \left(e^{r+i\frac{\psi}{2}} \cos\frac{\theta}{2} \sin\frac{\tilde\theta}{2} + e^{-r-i\frac{\psi}{2}} \sin\frac{\theta}{2} \cos\frac{\tilde\theta}{2} \right) \,,
$$

$$
z_4 = -e^{-\frac{i}{2}(\varphi-\tilde\varphi)} \left(e^{r+i\frac{\psi}{2}} \sin\frac{\theta}{2} \cos\frac{\tilde\theta}{2} + e^{-r-i\frac{\psi}{2}} \cos\frac{\theta}{2} \sin\frac{\tilde\theta}{2} \right) \,,
$$

$$\tag{7.3.8}$$

where $\psi = \psi_1 + \psi_2$. We show below that these holomorphic coordinates are very convenient to analyse the supersymmetric embeddings in our flavoured backgrounds. It is also useful to introduce a new set of complex variables w_i, related to the z_i's by means of the following linear combinations:

$$
w_1 = \frac{z_1 + z_2}{2} \,, \qquad
w_2 = \frac{z_1 - z_2}{2i} \,, \qquad
w_3 = \frac{z_3 - z_4}{2} \,, \qquad
w_4 = \frac{z_3 + z_4}{2i} \,. \tag{7.3.9}
$$

These variables satisfy

$$
(w_1)^2 + (w_2)^2 + (w_3)^2 + (w_4)^2 = 1 \,. \tag{7.3.10}
$$

There is an obvious $SO(4)$ invariance that is obtained by rotating the w_i's. The so-called $SO(4)$-invariant $(1,1)$-forms are defined as (see [135])

$$
\eta_1 = \delta^{ij} dw_i \wedge d\bar{w}_j \,, \qquad
\eta_2 = \left(\delta^{ij} w_i d\bar{w}_j \right) \wedge \left(\delta^{kl} \bar{w}_k dw_l \right) \,, \qquad
\eta_3 = \epsilon^{ijkl} w_i \bar{w}_j d\bar{w}_k \wedge dw_l \,.
$$

$$\tag{7.3.11}$$

In terms of the radial and angular coordinates, these forms are given by

$$\eta_1 = -i\bigg(\cosh(2r)\, dr \wedge \big(\tilde{\omega}^3 + \cos\theta\, d\varphi\big)$$

$$-\frac{1}{2}\sinh(2r)\Big(\sin\theta\, d\theta \wedge d\varphi + \sin\tilde{\theta}\, d\tilde{\theta} \wedge d\tilde{\varphi}\Big)\bigg),$$

$$\eta_2 = i\sinh^2(2r)\, dr \wedge \big(\tilde{\omega}^3 + \cos\theta\, d\varphi\big),$$

$$\eta_3 = -i\bigg(\frac{1}{4}\sinh(4r)\Big(\sin\theta\, d\theta \wedge d\varphi - \sin\tilde{\theta}\, d\tilde{\theta} \wedge d\tilde{\varphi}\Big)$$

$$-\frac{1}{2}\sinh(2r)\big(d\theta \wedge \tilde{\omega}^2 + \sin\theta\, d\varphi \wedge \tilde{\omega}^1\big)\bigg). \qquad (7.3.12)$$

The fundamental two-form J of the $SU(3)$-structure can be written in terms of the η_i forms, which is very useful in what follows (we use the conventions of Chapter 5):

$$J = \frac{e^{2k}}{2}dr \wedge \big(\tilde{\omega}^3 + \cos\theta\, d\varphi\big) + \frac{e^{2g}}{4}\frac{a\,\cosh(2r) - 1}{\sinh(2r)}\big(d\theta \wedge \tilde{\omega}^2 + \sin\theta\, d\varphi \wedge \tilde{\omega}^1\big)$$

$$-\frac{e^{2g}}{4}\bigg(\frac{a\,\cosh(4r) - \cosh(2r)}{\sinh(2r)}\sin\theta\, d\theta \wedge d\varphi + \frac{\cosh(2r) - a}{\sinh(2r)}\sin\tilde{\theta}\, d\tilde{\theta} \wedge d\tilde{\varphi}\bigg). \qquad (7.3.13)$$

In terms of the η_i's, J becomes

$$J = \frac{1}{2i}\bigg[\frac{e^{2k}}{\sinh^2(2r)}\eta_2 - a\, e^{2g}\bigg(\eta_1 + \frac{\cosh(2r)}{\sinh^2(2r)}\eta_2\bigg) + e^{2g}\frac{a\,\cosh(2r) - 1}{\sinh^2(2r)}\eta_3\bigg]. \qquad (7.3.14)$$

Let us now check that the complex variables z_i defined in (7.3.8) are good holomorphic coordinates for the internal manifold. Indeed, since our six-dimensional internal manifold is a complex manifold, we can write its metric in terms of the $(1,1)$-form J. Actually, if one writes J as

$$J = \frac{i}{2}h_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^{\bar{\beta}}, \qquad (7.3.15)$$

which is allowed thanks to the fact that J is a $(1,1)$-form, then one can prove that the metric of the internal space is

$$ds_6^2 = \frac{1}{2}h_{\alpha\bar{\beta}}\big(dz^\alpha \otimes d\bar{z}^{\bar{\beta}} + d\bar{z}^{\bar{\beta}} \otimes dz^\alpha\big), \qquad (7.3.16)$$

where we have split the ten-dimensional metric (7.2.6) as $ds^2 = e^{\Phi/2}[dx_{1,3}^2 + ds_6^2]$.

The $h_{\alpha\bar{\beta}}$ coefficients appearing in (7.3.15) and (7.3.16) can be read from (7.3.14) by using the relation between the η_i's and the z_i coordinates (see (7.3.11) and (7.3.9)). Moreover, the $SU(3)$-invariant three-form Ω is written in complex coordinates as:

$$\Omega = -\frac{1}{\sinh(2r)}e^{g+h+k}\frac{1}{z_3}\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \,. \qquad (7.3.17)$$

In addition, taking into account (7.2.23), the previous equation becomes

$$\Omega = -\frac{e^{2\Phi_0-2\Phi}}{2}\frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3}{z_3} \,, \qquad (7.3.18)$$

which shows that Ω is, indeed, a holomorphic $(3,0)$-form for the complex structure corresponding to the coordinates (7.3.8).

The RR six-form potential $C_{(6)}$, defined as $F_{(7)} = -e^\Phi * F_{(3)} = \mathrm{d}C_{(6)}$, can also be written in terms of the η_i one-forms. In fact, it follows from the $SU(3)$-structure equations that

$$C_{(6)} = e^{2\Phi}\mathrm{d}^4x \wedge J \,, \qquad (7.3.19)$$

where $\mathrm{d}^4x = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. Obviously, since J can be written in terms of the z^i variables, the six-form $C_{(6)}$ can also be written as a $(1,1)$-form in the internal space. Notice that $C_{(6)}$ is related to the calibration form of a D5-brane, whose pull-back onto the world-volume determines if the embedding is supersymmetric or not. Having $C_{(6)}$ written in complex coordinates is very convenient from the technical point of view, since it allows us to analyse the different supersymmetric embeddings by employing the full machinery of the complex variables.

Another relevant quantity that should be invariant under the $SO(4)$ isometry is the smearing form Ξ in (7.2.20), since it gives the charge distribution of the system. It is a $(2,2)$-form which can be cast in terms of $(1,1)$-forms as follows:

$$16\pi^2\Xi = -\frac{2N_f\,S}{\sinh^2(2r)}\eta_1 \wedge \left(\eta_1 + \frac{2\cosh(2r)}{\sinh^2(2r)}\eta_2\right) + \frac{N_f\,S'}{\sinh^3(2r)}\eta_2 \wedge \left(\eta_1 - \frac{1}{\cosh(2r)}\eta_3\right) \,. \qquad (7.3.20)$$

Supersymmetric embeddings

It is now straightforward to show that any embedding defined with holomorphic functions of the complex coordinates is supersymmetric. Let us study the case of an embedding extended in the Minkowski directions, and defined in the internal

space in the following way:

$$z_2 = F(z_1), \qquad z_3 = G(z_1), \qquad \bar{z}_2 = \bar{F}(\bar{z}_1), \qquad \bar{z}_3 = \bar{G}(\bar{z}_1), \qquad (7.3.21)$$

where, for definiteness, we have chosen z_1 and $\bar{z}_1$ as world-volume coordinates in the internal space. The calibration form $\mathcal{K}$ for a D5-brane in Einstein frame, is given by

$$\mathcal{K} = e^{3\Phi/2}\mathrm{d}^4 x \wedge J = e^{-\Phi/2}C_{(6)}. \qquad (7.3.22)$$

By using (7.3.15), one can easily get the pull-back of this calibration form onto the world-volume of the embedding, namely

$$\imath^* (\mathcal{K}) = i\, e^{3\Phi/2} K\, \mathrm{d}^4 x \wedge \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1, \qquad (7.3.23)$$

where we have defined the function K as

$$K = \frac{1}{2}\left(h_{1\bar{1}} + \bar{F}'h_{1\bar{2}} + \bar{G}'h_{1\bar{3}} + F'h_{2\bar{1}} + F'\bar{F}'h_{2\bar{2}} + F'\bar{G}'h_{2\bar{3}} + G'h_{3\bar{1}} + G'\bar{F}'h_{3\bar{2}} + G'\bar{G}'h_{3\bar{3}}\right). \qquad (7.3.24)$$

Now, we look at the induced metric $\mathrm{d}\hat{s}_6^2$ on the world-volume of the embedding. From (7.3.16), we get

$$\mathrm{d}\hat{s}_6^2 = e^{\Phi/2}\left[\mathrm{d}x_{1,3}^2 + 2K\,\mathrm{d}z_1\mathrm{d}\bar{z}_1\right]. \qquad (7.3.25)$$

Therefore, $\det \hat{g} = e^{3\Phi}K^2$, and one has

$$\sqrt{-\det \hat{g}}\,\mathrm{d}^4 x \wedge \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 = i\, e^{3\Phi/2}K\mathrm{d}^4 x \wedge \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 = \imath^* (\mathcal{K}). \qquad (7.3.26)$$

This means that the embedding is supersymmetric, proving explicitly that all holomorphic embeddings are supersymmetric.

7.4 Charge distributions

The supersymmetric D5-brane embeddings we are interested in are characterised by two algebraic equations of the type

$$F_1(z_i) = 0, \qquad F_2(z_i) = 0, \qquad (7.4.1)$$

which define a non-compact two-cycle $\mathcal{C}_2$ in the internal six-dimensional manifold. As argued above, the preservation of supersymmetry is ensured if the two functions

in (7.4.1) are holomorphic. However, in the brane setup we consider, we do not deal with a particular embedding of the flavour D5-branes but, instead, with a family of equivalent embeddings. This family can be generated from a particular representative of the form (7.4.1) by acting with the $SU(2)_L \times SU(2)_R$ isometries of the conifold. Let us recall how these symmetries act on the holomorphic coordinates. Under $SU(2)_L$, the holomorphic coordinates transform as $z_i \to \tilde{z}_i$, where

$$\begin{pmatrix} \tilde{z}_3 & \tilde{z}_2 \\ -\tilde{z}_1 & -\tilde{z}_4 \end{pmatrix} = \begin{pmatrix} \alpha z_3 + \bar{\beta} z_1 & \alpha z_2 + \bar{\beta} z_4 \\ -\bar{\alpha} z_1 + \beta z_3 & -\bar{\alpha} z_4 + \beta z_2 \end{pmatrix} , \qquad (7.4.2)$$

with $|\alpha|^2 + |\beta|^2 = 1$. Similarly, the $SU(2)_R$ transformation is

$$\begin{pmatrix} \tilde{z}_3 & \tilde{z}_2 \\ -\tilde{z}_1 & -\tilde{z}_4 \end{pmatrix} = \begin{pmatrix} \bar{\gamma} z_3 - \delta z_2 & \bar{\delta} z_3 + \gamma z_2 \\ -\bar{\gamma} z_1 + \delta z_4 & -\bar{\delta} z_1 - \gamma z_4 \end{pmatrix} , \qquad (7.4.3)$$

where the complex constants γ and δ satisfy the condition $|\gamma|^2 + |\delta|^2 = 1$. We now want to determine the charge distribution four-form Ξ (parametrised by the profile function $S(r)$) for a given family of embeddings. For that purpose, we employ a procedure which does not require performing the detailed analysis of the whole family and that allows to extract the function $S(r)$ by studying one single particular embedding belonging to the family [136]. This method is based on the comparison between the action for the whole set of N_f flavour branes and the one corresponding to a representative embedding. We can choose to compare either the DBI or WZ part of the actions, since supersymmetry guarantees that they are the same. The WZ term of the action of the full set of D5-branes is given by the following ten-dimensional integral:

$$S_{WZ}^{smeared} = T_5 \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Xi , \qquad (7.4.4)$$

whereas the action of one of the embeddings is simply

$$S_{WZ}^{single} = T_5 \int_{\mathcal{M}_6} \imath^* \left(C_{(6)} \right) , \qquad (7.4.5)$$

with $\mathcal{M}_6$ being the world-volume of the chosen representative embedding and $\imath^* \left(C_{(6)} \right)$ denotes the pull-back of $C_{(6)}$ to $\mathcal{M}_6$. Since all the embeddings of the family are related by isometries, they are equivalent and their actions should be

the same. Thus, we should have

$$S_{WZ}^{smeared} = N_f \, S_{WZ}^{single} \, . \tag{7.4.6}$$

The left-hand side of (7.4.6) can be obtained by plugging the expressions of Ξ and $C_{(6)}$ written in (7.2.20) and (7.3.19) respectively. After integrating over the angular coordinates, one gets a remarkably simple expression, namely

$$S_{WZ}^{smeared} = 2\pi N_f \, T_5 \int d^4 x \, dr \, e^{2\Phi} \left(e^{2k} S + \frac{1}{2} e^{2g} \tanh(2r) S' \right) . \tag{7.4.7}$$

The non-compact two-cycle $\mathcal{C}_2$ that the D5-branes wrap can be parametrised by the radial coordinate r and an angular variable. After integrating over the latter, the WZ action (7.4.5) becomes

$$S_{WZ}^{single} = 2\pi \, T_5 \int d^4 x \, dr \, e^{2\Phi} \mathcal{S}(r) \, , \tag{7.4.8}$$

where the function $\mathcal{S}(r)$ is related to the integral of the pull-back of J along the two-cycle by means of the expression

$$\int_{\mathcal{C}_2} \imath^* (J) = 2\pi \int dr \, \mathcal{S}(r) \, . \tag{7.4.9}$$

By plugging (7.4.7) and (7.4.8) into (7.4.6), we arrive at the following relation between the profile $S(r)$ and the function $\mathcal{S}(r)$:

$$e^{2k} S + \frac{1}{2} e^{2g} \tanh(2r) S' = \mathcal{S}(r) \, . \tag{7.4.10}$$

The function $\mathcal{S}$ appearing on the right-hand side of (7.4.10) depends both on the embedding and on the different functions of our ansatz. In the case in which $\mathcal{S}$ depends only on the functions k and g and this dependence is the same as on the left-hand side of (7.4.10), it is possible to obtain the profile function S from (7.4.10). However, this is a highly non-trivial condition which most families of embeddings do not satisfy. To illustrate this fact, let us consider the families of massive embeddings obtained by acting with the $SU(2)_L \times SU(2)_R$ isometries on the two non-compact two-cycles found in [49]. The first of these two-cycles is the so-called unit-winding embedding (see Section 6.1 of [49]), which has the following representation in terms of the real coordinates of the metric (7.2.6):

$$\sinh r = \frac{\sinh r_q}{\sin \theta} \, , \qquad \tilde{\theta} = \theta \, , \qquad \tilde{\varphi} = \varphi \, , \qquad \psi = \pi \, , \tag{7.4.11}$$

where r_q is a constant. In terms of the complex coordinates (7.3.8), one can easily show that (7.4.11) is a particular solution of the following two holomorphic equations:

$$z_1 z_2 = \cosh^2 r_q \,, \qquad z_3 + z_4 = 0 \,. \qquad (7.4.12)$$

By using (7.4.11) and (7.3.13), it is straightforward to compute the pull-back of J and to obtain $\mathcal{S}(r)$. One gets

$$\mathcal{S}(r) = \sqrt{1 - \frac{\sinh^2 r_q}{\sinh^2 r}} \left(e^{2k} + e^{2g} a \frac{\sinh^2 r}{\sinh^2 r - \sinh^2 r_q} \cosh^2 r \right) \,. \qquad (7.4.13)$$

Notice that the right-hand side of (7.4.13) contains the function $a(r)$, which is not present on the left-hand side of (7.4.10). Therefore, the determination of the profile S is not possible in this case. Similarly, one can consider the so-called zero-winding embeddings of Section 6.2 of [49]. In that case, the cycle is characterised by the equations

$$\sinh(2r) = \frac{\sinh(2r_q)}{\sin\theta} \,, \qquad \sin\tilde\theta = -\frac{\cos\theta}{\cosh(2r_q)} \,, \qquad \tilde\varphi = \tilde\varphi_0 \,, \qquad \psi = \pi \,, \quad (7.4.14)$$

which solve the following system of two complex equations:

$$z_1 z_2 = \frac{1}{2} \,, \qquad z_1 - e^{-2r_q} e^{-i\tilde\varphi_0} z_4 = 0 \,. \qquad (7.4.15)$$

In (7.4.14) and (7.4.15), $\tilde\varphi_0$ is a constant. Computing $\mathcal{S}(r)$ for the zero-winding embeddings, one gets

$$\mathcal{S}(r) = \sqrt{1 - \frac{\sinh^2(2r_q)}{\sinh^2(2r)}} \left(e^{2k} + e^{2g} \frac{\sinh^2(2r)}{\sinh^2(2r) - \sinh^2(2r_q)} (2a\cosh(2r) - 1) \right) \,, \qquad (7.4.16)$$

which, as in (7.4.13), contains the function $a(r)$ and, as a consequence, is not of the form displayed in (7.4.10).

Our interpretation of the fact that $\mathcal{S}$ is not of the form (7.4.10) for the embeddings (7.4.11) and (7.4.14) is that their backreaction is not compatible with our ansatz. Notice that our $F_{(3)}$ in (7.2.8), as well as the charge-density four-form Ξ in (7.2.20), is dictated by the $SU(3)$-structure of the $\mathcal{N} = 1$ supersymmetry and is highly asymmetric with respect to the exchange $(\theta, \varphi) \leftrightarrow (\tilde\theta, \tilde\varphi)$. This interpretation is supported by an independent microscopic calculation of Ξ, which we present in Appendix 7.A. Indeed, we show in this appendix that a simple embedding whose

$\mathcal{S}$ is not of the form (7.4.10) gives rise to a charge density Ξ which does not fit into our ansatz. Thus, in order to proceed further with our formalism, we have to find a concrete example of compatible embeddings and we have to determine the corresponding charge profile.

Fortunately, we have been able to find a simple family of embeddings for which $\mathcal{S}(r)$ depends on the functions k and g in the same way as the left-hand side of (7.4.10) and, as a consequence, one can directly read the profile function $S(r)$ for this configuration. The profile $S(r)$ obtained in this way can be used to get $Q(r)$ from (7.2.27), as well as an input for solving the master equation (7.2.28). In the next section, we present these embeddings and we determine the corresponding profile function.

7.5 A simple class of embeddings

As shown above, the massive embeddings found in [49] do not seem to produce a backreaction compatible with our ansatz and with its underlying $SU(3)$-structure. In principle, we should consider the logical possibility that such a compatible set of embeddings does not exist. In this section, we discard this possibility by finding a family of embeddings for which (7.4.10) can be solved and a simple expression for the profile function S can be found. We confirm this fact in Appendix 7.A by means of an explicit microscopic calculation, in the UV region of large r, of the charge density four-form Ξ. By considering the full distribution of flavour branes, we indeed show that, for the embeddings discussed in this section, the density Ξ is of the form displayed in (7.2.20), and we find an expression of $S(r)$ which is just the large-r limit of the one found by solving (7.4.10).

In terms of the holomorphic coordinates (7.3.8), the simplest embeddings one can think of are those characterised by two linear relations of the z_i's. Many of these embeddings are related by the action of the $SU(2)_L \times SU(2)_R$ symmetry and they belong to the same set. Instead of considering the full set, we only deal with a particular representative. Using the machinery developed previously, it is rather easy to consider systematically the different linear embeddings, to compute the pull-back of the fundamental two-form J and to verify if the function $\mathcal{S}(r)$ depends on the functions of the ansatz in the same way as the left-hand side of (7.4.10). Most of these linear embeddings do not give rise to a compatible charge density. Let us now study one example for which everything works fine. The representative embedding we want to focus on can be written in terms of the

holomorphic coordinates (7.3.8) as the following two linear equations:

$$z_3 = A z_1 , \qquad\qquad z_4 = B z_2 , \qquad\qquad (7.5.1)$$

where A and B are two complex constants. We can parametrise the two-surface defined by (7.5.1) in terms of, for example, z_1:

$$z_2 = \frac{1}{1 - AB} \frac{1}{z_1} , \qquad z_3 = A z_1 , \qquad z_4 = \frac{B}{1 - AB} \frac{1}{z_1} . \qquad (7.5.2)$$

This allows us to get the relation between r and z_1:

$$2 \cosh(2r) = (1 + |A|^2)|z_1|^2 + \frac{1 + |B|^2}{|1 - AB|^2} \frac{1}{|z_1|^2} , \qquad (7.5.3)$$

where we have used the relation between r and the holomorphic coordinates written in (7.3.2). From (7.5.3), we can compute the minimum distance r_q that this embedding reaches, namely:

$$\cosh(2r_q) = \frac{\sqrt{1 + |A|^2}\sqrt{1 + |B|^2}}{|1 - AB|} . \qquad (7.5.4)$$

Notice that this minimum distance depends on the modulus of the constants A and B, as well as on the phase of AB. In order to compute the function $\mathcal{S}(r)$ for these embeddings, let us compute the pull-back of J. It is quite useful to work with the complex coordinates z_i and to obtain first the pull-back of the η_i forms. Actually, the pull-backs of the $SO(4)$ invariant (1,1)-forms can be cast nicely as

$$\imath^*(\eta_1) = \frac{1}{2}\left((1 + |A|^2)|z_1|^2 + \frac{1 + |B|^2}{|1 - AB|^2} \frac{1}{|z_1|^2} \right) \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} = \cosh(2r) \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} ,$$

$$\imath^*(\eta_2) = -\frac{1}{4} \frac{\left((1 + |B|^2) - (1 + |A|^2)|1 - AB|^2|z_1|^2\right)}{|1 - AB|^4 |z_1|^4} \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2}$$

$$= \left(\cosh^2(2r_q) - \cosh^2(2r)\right) \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} ,$$

$$\imath^*(\eta_3) = \frac{|A + \bar{B}|^2}{|1 - AB|^2} \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} = \cosh^2(2r_q) \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} .$$

$$(7.5.5)$$

From these pull-backs, we can readily compute the pull-back of J, namely:

$$\imath^*(J) = \left(e^{2k} \frac{\cosh(4r) + 1 - 2\cosh^2(2r_q)}{\sinh^2(2r)} + e^{2g} \frac{2\cosh^2(2r_q) - 2}{\sinh^2(2r)} \right) \frac{i}{4} \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} .$$

$$(7.5.6)$$

Magically, the pull-back of J does not contain the function a, and it is ready for comparison with the smeared action. In order to obtain the actual value of $\mathcal{S}(r)$, we need to express $\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 = \mathrm{d}r \wedge \mathrm{d}(\text{angular})$. With this purpose in mind, we parametrise z_1 as

$$z_1 = u\, e^{i\alpha}. \tag{7.5.7}$$

Then, one has

$$\frac{\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1}{|z_1|^2} = -2i\frac{\mathrm{d}u}{u} \wedge \mathrm{d}\alpha, \tag{7.5.8}$$

and, since from (7.5.3) it follows that

$$\frac{\mathrm{d}u}{u} = \pm \frac{\sinh(2r)}{\sqrt{\cosh^2(2r) - \cosh^2(2r_q)}}, \tag{7.5.9}$$

we can write

$$\int_{C_2} \imath^*(J) = 2\pi \int \mathrm{d}r \left(e^{2k} \frac{\sqrt{\cosh(4r) - \cosh(4r_q)}}{\sqrt{2}\sinh(2r)} \right.$$
$$\left. + e^{2g}\tanh(2r)\frac{\sqrt{2}\cosh(2r)\sinh^2(2r_q)}{\sinh^2(2r)\sqrt{\cosh(4r) - \cosh(4r_q)}} \right). \tag{7.5.10}$$

Thus, the function $\mathcal{S}(r)$ in this case is given by

$$\mathcal{S} = e^{2k}\frac{\sqrt{\cosh(4r) - \cosh(4r_q)}}{\sqrt{2}\sinh(2r)} + e^{2g}\tanh(2r)\frac{\sqrt{2}\cosh(2r)\sinh^2(2r_q)}{\sinh^2(2r)\sqrt{\cosh(4r) - \cosh(4r_q)}}. \tag{7.5.11}$$

Plugging this result in the right-hand side of (7.4.10), and taking into account that the coefficients of e^{2k} and $e^{2g}\tanh(2r)$ are related by a derivative,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\sqrt{\cosh(4r) - \cosh(4r_q)}}{\sqrt{2}\sinh(2r)} \right] = 2\frac{\sqrt{2}\cosh(2r)\sinh^2(2r_q)}{\sinh^2(2r)\sqrt{\cosh(4r) - \cosh(4r_q)}}, \tag{7.5.12}$$

one immediately gets that the profile function $S(r)$ for this family is given by

$$S(r) = \frac{\sqrt{\cosh(4r) - \cosh(4r_q)}}{\sqrt{2}\sinh(2r)}\Theta(r - r_q) = \sqrt{1 - \frac{\sinh^2(2r_q)}{\sinh^2(2r)}}\,\Theta(r - r_q), \tag{7.5.13}$$

where we have taken into account that $r \geq r_q$ on the cycle. Notice that $S(r) \to 1$ as $r \to \infty$, and the massive solution becomes the solution of [21] in the far UV, as it should (see Figure 7.1). Notice also that $S(r) = 1$ in (7.5.13) for the massless case

$r_q = 0$ and, therefore, we recover the results of [21] for that limit. As mentioned above, in Appendix 7.A, we have checked the form of Ξ and the expression of S for these embeddings by means of a microscopic calculation in the UV, where the unflavoured model reduces to the "abelian" model.

Near $r = r_q$, the profile $S(r)$ in (7.5.13) vanishes as

$$S(r) \sim 2\sqrt{\cosh(2r_q)}\sqrt{r - r_q}\,, \qquad (7.5.14)$$

which means that $S(r)$ is continuous at $r = r_q$. However, $S'(r)$ diverges as $1/\sqrt{r - r_q}$ when $r \to r_q$. Since $S'(r)$ enters into the energy momentum tensor of the branes through Ξ, it follows from the Einstein equations that this divergence induces the divergence of the Ricci tensor at $r = r_q$. This divergence is due to the hard-wall effect that we introduce in our configuration when the flavour branes are added, and it should be thought of as the gravitational analogue of the threshold effects of field theory. In the next section, we propose a way to resolve this singularity in our string duals.

7.6   Removing the threshold singularity

Let us consider the class of embeddings studied in Section 7.5. We show now how one can engineer a brane setup such that the unwanted singularity of $S'(r)$ at $r = r_q$ disappears. The idea is to consider branes whose tips reach different radial positions and perform an average over the value r_q of the radial coordinate of the tip of the flavour branes. Actually, this is the way in which the threshold singularity is removed in the Klebanov-Strassler model with massive flavours studied in [102]. Indeed, in Appendix D of [12], this last model is reconsidered with the tools developed here and it is shown explicitly how averaging over a certain phase is equivalent to a particular superposition of flavour branes ending at different radial positions. Moreover, the function $S(r)$ is also computed in that case for a set of branes ending at a fixed r_q (*i.e.* the analogue of (7.5.13) for the Klebanov-Strassler model).

Inspired by the resolution of the threshold singularity in the Klebanov-Strassler model, we consider a flavour-brane distribution containing branes with different r_q's. Furthermore, we allow r_q to vary in a certain finite interval and we weight the different values of r_q with a non-negative measure function $\rho(r_q)$, which should be conveniently normalised. In this way, the hard wall at $r = r_q$ is substituted by a

shell of non-vanishing width. If the resulting profile function S and its first radial derivative are continuous, the geometry is free of threshold singularities. As we see explicitly below, if the measure function is smooth enough, the resulting profile satisfies the conditions to have a regular supergravity solution.

For convenience let us redefine the radial coordinate as

$$x = \cosh(4r), \qquad x \geq 1. \tag{7.6.1}$$

We also denote $x_q = \cosh(4r_q)$. We consider distributions of branes having x_q's in the interval $x_0 \leq x_q \leq x_0 + \delta$. The resulting charge density distribution is additive and can be obtained by integrating over x_q the profile functions (7.5.13) multiplied by the measure $\rho(x_q)$. Since the branes with a given x_q only contribute to the charge density distribution $S(x)$ for $x \geq x_q$, one has

$$S(x) = \int_{x_0}^{x} dx_q \, \rho(x_q) \frac{\sqrt{x - x_q}}{\sqrt{x - 1}}. \tag{7.6.2}$$

The measure function $\rho(x_q)$ must obey the normalisation condition

$$\int_{1}^{\infty} dx_q \, \rho(x_q) = 1. \tag{7.6.3}$$

When the measure ρ is a δ-function of the type $\rho(x_q) = \delta(x_q - x_{\bar{q}})$, the profile (7.6.2) reduces to (7.5.13) which, as we have seen, leads to background with a threshold singularity. To resolve this singularity, we consider measures with a finite width δ, and we regard δ as a regularisation parameter of the threshold effect. As $\delta \to 0$, we recover (7.5.13). Below, we work out two simple prescriptions for the functional form of ρ. In both cases, $\rho(x_q)$ is non-vanishing only on a finite interval $x_0 \leq x_q \leq x_0 + \delta$, and the resulting $S(x)$ and $S'(x)$ are continuous. Thus they source a regular geometry. Moreover, the profile functions for both measures are actually very similar if one compares distributions with the same width, as one can appreciate in the right plot of Figure 7.1.

7.6.1 Flat measure

As a first example of a weighting measure, we consider the situation in which all the embeddings with different tips in the interval $x_0 \leq x_q \leq x_0 + \delta$ have the same weight. This prescription corresponds to choosing a rectangular step function in

the interval $x_0 \leq x_q \leq x_0 + \delta$ which, conveniently normalised, reads

$$\rho(x_q) = \frac{\Theta(x_q - x_0) - \Theta(x_q - x_0 - \delta)}{\delta} \,.$$ (7.6.4)

Performing the integral (7.6.2) for this measure, we get

$$S(x) = \frac{2}{3} \frac{(x - x_0)^{3/2}}{\delta \sqrt{x - 1}} \qquad \text{when } x_0 \leq x \leq x_0 + \delta \,,$$

$$S(x) = \frac{2}{3} \frac{(x - x_0)^{3/2} - (x - x_0 - \delta)^{3/2}}{\delta \sqrt{x - 1}} \qquad \text{when } x \geq x_0 + \delta \,,$$ (7.6.5)

and it is understood that $S(x) = 0$ for $x \leq x_0$. In Figure 7.1 (left), we plot the function $S(x)$ for different values of the width δ. As shown in this figure, when δ is increased, $S(x)$ grows slower in the transition region and, thus, $S(x)$ is a milder function of x.

Let us now consider the issue of the regularity of $S(x)$. The potentially dangerous points are $x = x_0$ and $x = x_0 + \delta$, where the measure ρ is discontinuous. It can be straightforwardly checked that S and its first derivative are continuous at these two points. Actually, one has

$$S(x_0) = 0\,, \qquad S(x_0 + \delta) = \frac{2}{3} \frac{\sqrt{\delta}}{\sqrt{x_0 + \delta - 1}}\,,$$

$$S'(x_0) = 0\,, \qquad S'(x_0 + \delta) = \frac{2\delta + 3x_0 - 3}{3\sqrt{\delta}(x_0 + \delta - 1)^{3/2}}\,.$$ (7.6.6)

Moreover, it follows from (7.6.5) that $S(x)$ vanishes as $(x - x_0)^{3/2}$ as we approach the endpoint of the charge distribution at $x = x_0$. Thus, this profile function gives rise to a solution without threshold singularities, as claimed.

7.6.2   Peaked measure

In our previous example, we have considered a weighting function ρ which is discontinuous at x_0 and $x_0 + \delta$. We now want to explore the possibility of having a measure which vanishes continuously at these endpoints. To choose this new measure, we think of taking a distribution that reproduces a mass peak with finite width for the quark. It means that we choose a distribution that looks like a peak of finite width, somewhat similar to a Gaussian function, but we want something
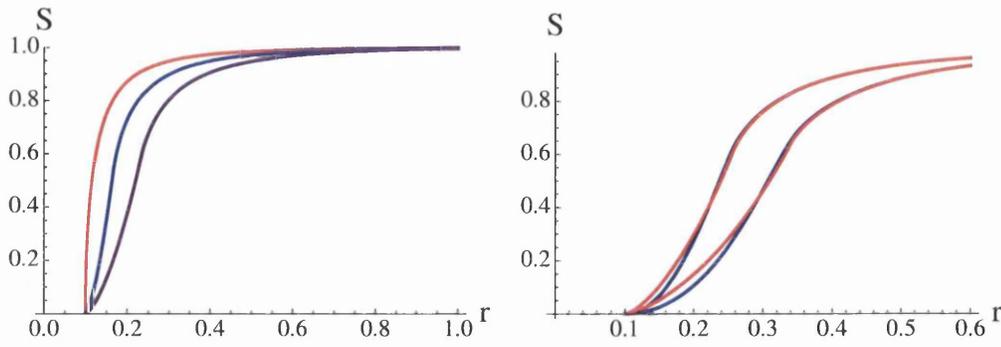
Figure 7.1: We show plots of the function S for the flat measure on the left. The red curve is the singular profile, the blue one is for $\delta = 0.15$ and the purple one is for $\delta = 0.4$. On the right, we plot, for $\delta = 0.5$ and $\delta = 1$, S for the flat measure (red) and the peaked one (blue) and we see that there is almost no difference.

simpler to be able to perform the integration. For that reason, we choose[1]

$$\rho(x_q) = \frac{8}{\pi \delta^2} \sqrt{(x_q - x_0)(x_0 + \delta - x_q)} \Big[\Theta(x_q - x_0) - \Theta(x_q - x_0 - \delta) \Big]. \qquad (7.6.7)$$

The integral (7.6.2) now gives

$$S(x) = \frac{16}{15\pi\delta^{3/2}} \frac{1}{\sqrt{x-1}} \left[2(x^2 + x_0^2 + \delta^2 + x_0\delta - 2xx_0 - x\delta)E\left(\frac{x - x_0}{\delta}\right) \right.$$

$$\left. + (x_0 + \delta - x)(x - x_0 - 2\delta)K\left(\frac{x - x_0}{\delta}\right) \right] \qquad \text{when } x_0 \le x \le x_0 + \delta,$$

$$S(x) = \frac{16}{15\pi\delta^2} \frac{\sqrt{x - x_0}}{\sqrt{x-1}} \left[2(x^2 + x_0^2 + \delta^2 + x_0\delta - 2xx_0 - x\delta)E\left(\frac{\delta}{x - x_0}\right) \right.$$

$$\left. + (x - x_0 - \delta)(2x_0 + \delta - 2x)K\left(\frac{\delta}{x - x_0}\right) \right] \qquad \text{when } x \ge x_0 + \delta,$$

$$(7.6.8)$$

where, again, it is understood that $S(x) = 0$ for $x \le x_0$. The functions K and E in (7.6.8) are complete elliptic integrals of the first and second kind respectively. They are defined as

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k\sin^2(t)}\, \mathrm{d}t\,, \qquad K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k\sin^2(t)}}\, \mathrm{d}t\,. \quad (7.6.9)$$

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[1]We thank Ángel Paredes for discussions on this point.

If we now look at the properties of our solution at $x = x_0$ and $x = x_0 + \delta$, we get

$$S(x_0) = 0\,, \qquad S(x_0 + \delta) = \frac{32\sqrt{\delta}}{15\pi\sqrt{x_0 + \delta - 1}}\,,$$
$$S'(x_0) = 0\,, \qquad S'(x_0 + \delta) = \frac{8(3\delta + 5x_0 - 5)}{15\pi\sqrt{(x_0 + \delta - 1)^3\delta}}\,, \tag{7.6.10}$$

which shows that S and S' are regular and, therefore, this measure leads to another solution free of threshold singularities. In order to compare the profile (7.6.8) with the one obtained with the flat measure (7.6.5), we plot in Figure 7.1 (right) the two functions $S(x)$ for two values of δ. It is rather clear from this figure that the choice of ρ does not influence much S (and even less the functions of the ansatz) and, therefore, the physically relevant parameter is the width. For this reason, in the numerical calculations of the next section, we use the simpler result (7.6.5).

7.7   Solutions of the master equation

After discussing the details of our setup in the previous sections, we now move on to the task of finding explicit Type IIB supergravity solutions. As we argued in Section 7.2, it is enough to solve the master equation (7.2.28), since all the other functions of our ansatz follow. For each P, we have a background preserving four supersymmetries that solves the equations of motion of Type IIB supergravity.

The master equation involves the profile $S(r)$, and the function $Q(r)$ that can be obtained in terms of S (see (7.2.27)). Notice that in the cases $S = 0$ and $S = 1$, this master equation has been extensively studied in the literature [29]. These cases are precisely the IR and UV limits of our profiles $S(r)$, so the asymptotics of our solutions are already known. What we have to find is a smooth matching between them.

We cannot provide an exact analytic solution of the master equation, but we can give analytic expansions in the relevant regions (around $r = 0$, $r = r_0$, and $r = \infty$), and solve numerically in between them.

7.7.1   Analytical matching

On general grounds, we expect $S(r)$ to be null up to a certain point $r = r_0$, where we have enough energy to start seeing the effects of virtual quarks running in the loops. Then it starts growing because, as the energy increases, it is easier

to produce quarks. Eventually, it stabilises around $S(r) = 1$ since we then have enough energy so that the quarks appear massless. Although we know the specific functional form of $S(r)$ in some cases, let us keep the discussion more general and assume that $S(r)$ can be expanded in a kind of power series around r_0 of the following form:

$$S(r) = \Theta(r - r_0) \left[S_1(r - r_0)^{1/2} + S_2(r - r_0) + S_3(r - r_0)^{3/2} + \mathcal{O}\left((r - r_0)^2\right) \right] .$$
(7.7.1)

It is important to notice that, according to this expansion, although $S(r)$ is continuous, the $\lfloor \frac{n+1}{2} \rfloor$-th derivative is not when S_n is the first non-zero coefficient of the expansion. Note that S calculated with both the flat measure and the peaked measure are included in this expansion (we only have the odd coefficients for the former, and the even ones for the latter).

Of course, up to $r = r_0$, the solution of the master equation is the unflavoured one. This solution was written close to $r = 0$ in [29][2]:

$$P_{\text{unfl}}(r) = N_c \left[2\beta r + \frac{8\beta}{15} \left(1 - \frac{1}{\beta^2}\right) r^3 + \frac{32\beta}{525} \left(1 - \frac{1}{3\beta^2} - \frac{2}{3\beta^4}\right) r^5 + \mathcal{O}\left(r^7\right) \right] ,$$
(7.7.2)

where $\beta \geq 1$. Unfortunately, far from this point, it is only known numerically (except for the case $\beta = 1$, where the previous expansion truncates to the exact solution $P = 2N_c r$ of [13]). Near $r = 0$, the different functions of this solution behave as

$$e^{2h} = N_c \left[\beta r^2 + \frac{4}{45\beta} \left(-12\beta^2 + 15\beta - 8\right) r^4 + \mathcal{O}\left(r^6\right) \right] ,$$

$$e^{2g} = N_c \left[\beta + \frac{4}{15\beta} \left(6\beta^2 - 5\beta - 1\right) r^2 + \frac{16}{1575\beta^3} \left(3\beta^4 + 35\beta^3 - 36\beta^2 - 2\right) r^4 + \mathcal{O}\left(r^6\right) \right] ,$$

$$e^{2k} = N_c \left[\beta + \frac{4}{5\beta} \left(\beta^2 - 1\right) r^2 + \frac{16}{315\beta^3} \left(3\beta^4 - \beta^2 - 2\right) r^4 + \mathcal{O}\left(r^6\right) \right] ,$$

$$e^{4(\Phi - \Phi_0)} = \frac{4}{N_c^3 \beta^3} \left[1 + \frac{16}{9\beta^2} r^2 + \frac{32}{405\beta^5} \left(-15\beta^2 + 31\right) r^4 + \mathcal{O}\left(r^6\right) \right] ,$$

$$a = 1 + \left(-2 + \frac{4}{3\beta}\right) r^2 + \frac{2}{45\beta^3} \left(75\beta^3 - 116\beta^2 + 40\beta + 8\right) r^4 + \mathcal{O}\left(r^6\right) .$$
(7.7.3)

This solution is regular in the IR. For small r, the different curvature invariants

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[2]The parameter h_1 found in [29] is related to β by $h_1 = 2N_c \beta$.

are

$$R = \frac{e^{-\Phi_0/2}}{3} \frac{2^{7/4}}{N_c^{5/8}\beta^{21/8}} + \mathcal{O}(r),$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{31e^{-\Phi_0}}{27} \frac{2^{5/2}}{N_c^{5/4}\beta^{21/4}} + \mathcal{O}(r), \qquad (7.7.4)$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{e^{-\Phi_0}}{45} \frac{976 - 3072\beta^2 + 3456\beta^4}{2^{1/2}N_c^{5/4}\beta^{21/4}} + \mathcal{O}(r).$$

From r_0 on, $S \neq 0$, and we have to solve the master equation with initial conditions given by the unflavoured solution: $P(r_0) = P_{\text{unfl}}(r_0)$, $P'(r_0) = P'_{\text{unfl}}(r_0)$. The form of the solution then depends on the form of $S(r)$ around $r = r_0$.

To solve the master equation in a power series around the matching point $r = r_0$, we need to know the expression for $Q(r)$, which can be obtained from (7.2.27):

$$Q(r) = N_c \left[2r_0 \coth(2r_0) - 1 + \frac{\sinh(4r_0) - 4r_0}{\sinh^2(2r_0)}(r - r_0) - \frac{2N_f}{3N_c}S_1 \tanh(2r_0)(r - r_0)^{\frac{3}{2}} \right.$$

$$\left. + \left(\frac{8r_0 \coth(2r_0) - 4}{\sinh^2(2r_0)} - \frac{N_f}{2N_c} \tanh(2r_0)S_2 \right) (r - r_0)^2 + \mathcal{O}\left((r - r_0)^{\frac{5}{2}} \right) \right].$$

$$(7.7.5)$$

To arrive at (7.7.5), we have fixed the integration constant q_0 in (7.2.27) to match the unflavoured solution at $r = r_0$, that is $q_0 = 0$. Notice that, in Q, N_f appears only through the combination N_f/N_c and N_c is just an overall factor. Actually, the master equation (7.2.28) can be written in terms of P/N_c, Q/N_c, N_f/N_c and S and no other term depends on N_c.

As there are no singular terms in the master equation at $r = r_0$, the uniqueness and existence theorem for ordinary differential equations guarantees the existence of a unique smooth solution (actually as smooth as $\int drS$) for this second-order differential equation. Therefore, let us propose an expansion for $P(r)$ as

$$N_c^{-1}P(r) = N_c^{-1}P_{\text{unfl}}(r_0) + N_c^{-1}P'_{\text{unfl}}(r_0)(r - r_0) + P_3(r - r_0)^{3/2} + P_4(r - r_0)^2$$

$$+ P_5(r - r_0)^{5/2} + \mathcal{O}\left((r - r_0)^3 \right).$$

$$(7.7.6)$$

Plugging the expansions (7.7.1), (7.7.5) and (7.7.6) in the master equation (7.2.28), we obtain the following solution:

$$P_3 = -\frac{2N_f}{3N_c}S_1, \qquad\qquad P_4 = \frac{1}{2}\left(N_c^{-1}P''_{\text{unfl}}(r_0) - \frac{N_f}{N_c}S_2 \right),$$

$$P_5 = -\frac{2N_f}{5N_c}S_3 + \frac{8N_f}{15N_c}S_1 \frac{N_c^{-1}P'_{\text{unfl}}(r_0)\left(N_c^{-1}P_{\text{unfl}}(r_0) - 2r_0 + \tanh(2r_0) \right)}{\left(2r_0\coth(2r_0) - 1 \right)^2 - N_c^{-2}P^2_{\text{unfl}}(r_0)}. \qquad (7.7.7)$$

An important lesson to extract from here is the following: our background presents no curvature discontinuity as long as P'' is continuous (if only P' is continuous, then the Ricci scalar has a finite jump at $r = r_0$). So in this case, no curvature singularity amounts to having $S_1 = S_2 = 0$.

In the UV, we have $S \to 1$. The asymptotic value $S = 1$ is reached exponentially, in a fashion that depends on the particular details of the measure used to compute S, although the first subleading term is universal (given by the abelian limit):

$$S = 1 - \left(\frac{2\cosh(4r_0) + \delta}{2} - 1 \right) e^{-4r} + \mathcal{O}\left(e^{-8r} \right). \qquad (7.7.8)$$

As we mentioned, the case $S = 1$ has been studied already in [29], where two possible analytic UV expansions were found, dubbed Class I (linearly growing P) and Class II (exponentially growing P). We also have two possible UV behaviours, and the analytic expansions have the same coefficients as those in [29] for the solutions with linearly growing P, and the same leading coefficients for the solutions with exponentially growing P. As argued in the next subsection, we are interested in the Class I behaviour.

7.7.2 Numerical matching

If we solve the master equation (7.2.28) numerically, we find, regardless of the specific profile $S(r)$ we use, two qualitatively different behaviours as we go to $r \to \infty$, which are in correspondence with the two classes of UV described in Section 4 of [29]. Indeed, in the deep UV, the massive flavours we introduce can be considered massless. We have checked that our numerical solutions comply with the UV asymptotic behaviours described in [29].

We find that, in general, the flavoured solution only matches nicely (meaning that the solution reaches infinity) with the unflavoured solution (7.7.2) if we choose β to be bigger than some critical value β_c, which is only known numerically (see Figure 7.3) and bigger than 1. This means in particular that the unflavoured solution cannot be that of [13]. We observe the following.

Assume the unflavoured P up to r_0 is given by the numerical solution characterised in the IR by (7.7.2). Then there exists a β_c such that:

- For $\beta < \beta_c$, P eventually starts decreasing, crossing Q at some finite value of the radial coordinate and making $e^{2h} = 0$ at that point. This solution is then singular.
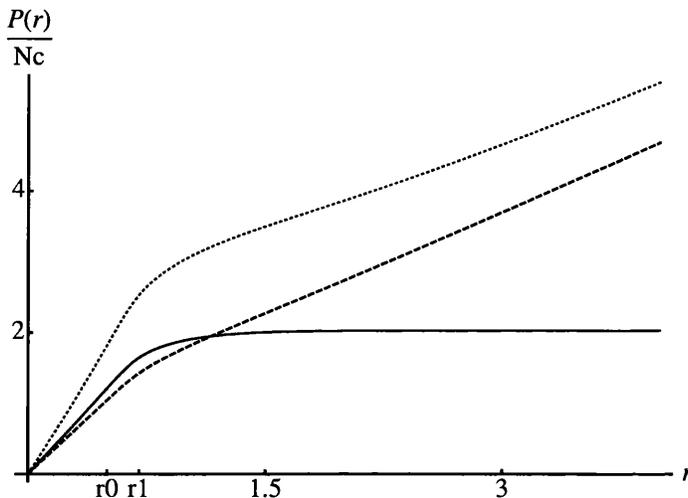
Figure 7.2: Numerical solutions for $N_c^{-1}P$ for different values of N_f/N_c, keeping fixed the profile (flat measure), r_0 and δ (in the plot, $\cosh(4r_1) = \cosh(4r_0) + \delta$). The blue dotted line corresponds to $N_f/N_c = 1$. The purple line corresponds to the conformal case $N_f/N_c = 2$. And the olive dotted line corresponds to $N_f/N_c = 3$. Notice the expected asymptotic UV behaviour.

- For $\beta = \beta_c$, P reaches infinity linearly (see Figure 7.2). This solution has precisely the same asymptotics as those described as Class I in [29], characterised by a linearly growing P and a linearly growing dilaton.

- For $\beta > \beta_c$, P reaches infinity exponentially. This solution possesses the asymptotics dubbed as Class II in the previous reference, characterised by an exponentially growing P, and an asymptotically constant dilaton.

So the IR expansion (7.7.2) can be connected with any of the two known UV behaviours as long as we choose the parameter β appropriately. For an interpretation of our solutions as gravity duals of $\mathcal{N} = 1$ SQCD, we are interested in the ones with asymptotically linear dilaton [21], i.e. the ones which have $\beta = \beta_c$. Notice that the IR effects of the flavours is codified in the dependence of β_c on N_f/N_c. We can then regard β_c as a measure of the deformation induced by the flavours in the IR. In Figure 7.3, we explore the dependence of β_c on the number of quarks and their mass.

Even if we fix $\beta = \beta_c$, and for a given ratio N_f/N_c, we can still play with several parameters in the profile $S(r)$, such as r_0 and δ, or even with the functional form of S itself. When doing that, we find that the qualitative behaviour of the metric functions does not change. For instance, varying the width δ of the mass distribution of the quarks just makes more or less sharp the transition from the
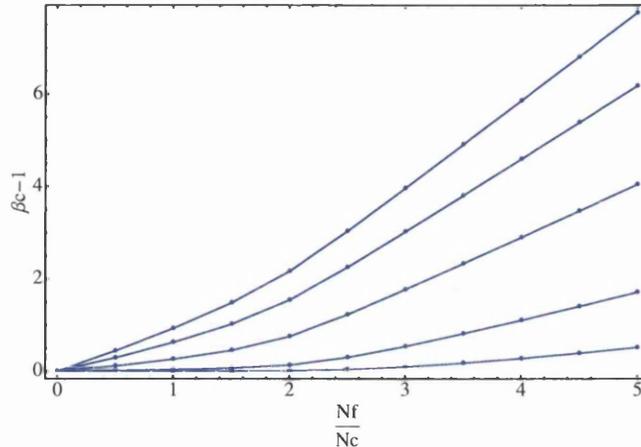
Figure 7.3: We plot the different values of $\beta_c - 1$ as one varies the ratio N_f/N_c. The curves are for fixed width $\delta = 0.2$ but different quark masses: moving from the upper curve to the lower one, the values used are $r_0 = 0, 0.15, 0.3, 0.7, 1.2$. Notice that, as r_0 increases (the mass increases), the growth of β_c with N_f/N_c is less and less noticeable, and the solution in the unflavoured region is almost that of [13] ($\beta_c \simeq 1$). This was to be expected since the more massive the flavours are, the less they affect the IR dynamics.

unflavoured region to the flavoured one. We gathered, in Figure 7.4, the plots of the various metric functions for some particular values of the parameters, just to exhibit explicitly this transition from unflavoured to flavoured background that happens around r_0.

7.7.3 The solution for massless flavours

Let us take $r_0 \to 0$ in our expressions, keeping a finite width δ for the measure (also taking $\delta \to 0$ gives back the singular solution of [21]). This makes the lightest quark we introduce massless. Nonetheless, due to the non-zero width, some of the quarks are massive; notice however that their mass can be chosen to be as small as one wants. In that respect, this solution is not a typical massless-flavour solution, as in [21].

Let us consider the following expansion for the profile function $S(r)$:

$$S(r) = S_1 r + S_2 r^2 + S_3 r^3 + \mathcal{O}(r^4). \tag{7.7.9}$$

We set the first coefficient to zero because we want to impose $S(0) = 0$.

For this $S(r)$, we have to integrate the differential equation for Q in a series
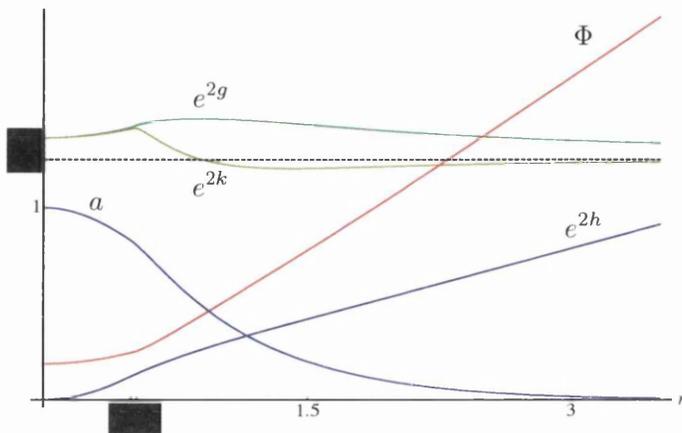
Figure 7.4: Metric functions for a case with $N_f \neq 2N_c$. We have used the flat measure profile with $r_0 = 0.5$, $\delta = 0.5$. All the functions have the expected asymptotics. Notice in particular the linearly growing dilaton, in red.

expansion. We get

$$Q(r) = N_c \left[\frac{4}{3}r^2 - \frac{N_f S_1}{2N_c}r^3 - \left(\frac{16}{45} + \frac{2N_f S_2}{5N_c} \right) r^4 + \left(\frac{2N_f S_1}{9N_c} - \frac{N_f S_3}{3N_c} \right) r^5 + \mathcal{O}\left(r^6\right) \right].$$

(7.7.10)

The expansion we get for P now is

$$P = N_c \left[2\beta\, r - \frac{5N_f}{4N_c} S_1\, r^2 + \frac{8\beta}{15} \left(1 - \frac{1}{\beta^2} + \frac{9}{256} \frac{N_f^2 S_1^2}{N_c^2 \beta^2} - \frac{9}{8} \frac{N_f S_2}{N_c \beta} \right) r^3 \right.$$
$$+ \left(-\frac{7N_f}{18N_c}S_3 + \frac{N_f}{N_c}S_1 \left(-\frac{34}{135} + \frac{7}{27\beta} - \frac{7}{45\beta^2} + \frac{7}{360}\frac{N_f S_2}{N_c\,\beta} + \frac{7}{1280}\frac{N_f^2 S_1^2}{N_c^2\beta^2} \right) \right) r^4$$
$$\left. + \mathcal{O}\left(r^5\right) \right],$$

(7.7.11)

where β is a free parameter. We then find the following asymptotics for the metric functions and the dilaton:

$$e^{2h} = N_c \left[\beta\, r^2 - \frac{5N_f}{8N_c}S_1\, r^3 - \frac{16\beta}{15} \left(1 + \frac{2}{3\beta^2} - \frac{5}{4\beta} - \frac{9}{1024}\frac{N_f^2 S_1^2}{N_c^2\beta^2} + \frac{9}{32}\frac{N_f S_2}{N_c\,\beta} \right) r^4 \right.$$
$$\left. + \mathcal{O}\left(r^5\right) \right],$$

$$e^{2g} = N_c \left[1 - \frac{5N_f}{8N_c}S_1\, r + \frac{8\beta}{5} \left(1 - \frac{1}{6\beta^2} - \frac{5}{6\beta} + \frac{3}{512}\frac{N_f^2 S_1^2}{N_c^2\beta^2} - \frac{3}{16}\frac{N_f S_2}{N_c\,\beta} \right) r^2 + \mathcal{O}\left(r^3\right) \right],$$

$$e^{2k} = N_c \left[1 - \frac{3N_f}{4N_c}S_1\, r + \frac{4\beta}{5} \left(1 - \frac{1}{\beta^2} + \frac{9}{256}\frac{N_f^2 S_1^2}{N_c^2\beta^2} - \frac{1}{2}\frac{N_f S_2}{N_c\,\beta} \right) r^2 + \mathcal{O}\left(r^3\right) \right],$$

$$e^{4(\Phi-\Phi_0)} = \frac{4}{N_c^3 \beta^3} \left[1 + 2\frac{N_f S_1}{N_c \beta} r + \left(\frac{16}{9\beta^2} + \frac{21}{8}\frac{N_f^2 S_1^2}{N_c^2 \beta^2} + \frac{N_f S_2}{N_c \beta} \right) r^2 + \mathcal{O}\left(r^3\right) \right] ,$$

$$a = 1 - \left(2 - \frac{4}{3\beta} \right) r^2 - \frac{N_f S_1}{6 N_c \beta} \left(3 - \frac{5}{\beta} \right) r^3 + \mathcal{O}\left(r^4\right) . \tag{7.7.12}$$

We have checked that the above solution presents no curvature singularity in the IR if we choose $S_1 = 0$. For instance, the Ricci scalar near $r = 0$ is given by

$$R = 3e^{-\Phi_0/2} \frac{N_f S_1}{2^{5/4} N_c^{13/8} \beta^{13/8}} \frac{1}{r} + \mathcal{O}(r^0) , \tag{7.7.13}$$

and the metric is clearly singular at $r = 0$ if $S_1 \neq 0$.

Note that what is done in this subsection might be thought as a regular way to introduce massless flavours, as opposite to what happens in [21], where the geometry is singular in the far IR. This statement should be taken with caution: if we interpret $N_f S(r)$ as giving the number of flavours that are effectively massless at a given scale r, we clearly read off $S(r = 0) = 0$ that there are no massless flavours in the far IR. But, given that the tips of some branes reach the origin of the space, there are certainly massless quark states in the dual theory. One could conjecture about the existence of some field-theoretical counterpart to the fact that the tips of the branes should be spread in order not to generate a curvature singularity. Unfortunately, we cannot make any strong claim in this direction.

7.8 Conclusions

Let us summarise our main results. We have considered the addition of backreacting massive fundamental matter to the gravity dual of the $\mathcal{N} = 1$ SQCD-like theory obtained when D5-branes wrap a two-cycle inside a Calabi-Yau threefold. The matter fields are added by means of D5-branes that wrap a non-compact two-dimensional submanifold in the internal space. These flavour branes do not reach, in general, the origin of the holographic coordinate r and their charge density depends on r. In order to incorporate consistently, in the backreacted background, the effects of this dependence, we have modified in a non-trivial way the ansatz of [21], including new terms in the RR three-form $F_{(3)}$ which depend on a profile function $S(r)$ and its derivative. We have shown that the BPS system reduces to a master equation, containing S and S', which is a generalisation of the one found in [29]. This equation can be integrated numerically and, by matching with the unflavoured solution at the scale at which S becomes non-zero, one finds a

supergravity solution in the whole range of the radial coordinate.

Our solutions solve the Einstein equations with sources, and one of the non-trivial points we have addressed is the determination of the distributions of branes whose charge density and backreaction have precisely the form that we have derived from our ansatz. We have verified this fact by means of a macroscopic calculation (comparing the action of the full set of branes with the one corresponding to a representative), as well as by a direct microscopic calculation of the charge density in the UV. We have also shown how to resolve the curvature singularities which appear at the position of the tip of the branes.

After all these developments, we were able to find regular supergravity backgrounds dual to $\mathcal{N} = 1$ SQCD-like theories with massive flavours. Our results generalise those of [21] in the sense that our solutions incorporate the effects of the mass scale introduced by the quark mass and, at the same time, they resolve the IR curvature singularity that limits the applicability of the geometry of [21] to explore holographically the $\mathcal{N} = 1$, $SU(N_c)$ gauge theory with flavours.

In the next chapter, we recall and summarise the results derived in this thesis.

7.A   Microscopic computation of Ξ

Let us show here how the results obtained for the smearing form in Section 7.5, using only the knowledge of one embedding of the family plus an ansatz for the functional form of Ξ (7.2.20), could be derived from a purely microscopic computation, *i.e.* by summing the contributions to the smearing form of all the embeddings in a given family.

Notice that this microscopic approach does not assume any specific ansatz for the smearing form. Obviously, one can expect it to be much harder to carry out. Indeed, except for some very simple cases ([35, 41, 102, 134]), a full reconstruction of the functional form of Ξ for massive quarks from the microscopic family of embeddings giving rise to it is not available in the literature. The use of the holomorphic structure of our internal manifold developed in Section 7.3 is instrumental in carrying out this microscopic computation.

7.A.1   Holomorphic structure in the abelian limit

For simplicity, we focus in this appendix on the UV limit ($r \to \infty$) of our backgrounds. This limit corresponds to the so-called abelian solution. The holomorphic structure simplifies a little bit in this limit, and one can define a new set of four complex variables ζ_i ($i = 1, \ldots, 4$) that parametrise now a singular conifold:

$$\zeta_1 \zeta_2 - \zeta_3 \zeta_4 = 0 \, . \tag{7.A.1}$$

The radial variable r is related to the ζ_i's in this case as

$$\sum_{i=1}^{4} |\zeta_i|^2 = e^{2r} \, . \tag{7.A.2}$$

The expression of these complex variables in terms of the coordinates of the internal manifold can be read from (7.3.8). One just needs to take the $r \to \infty$ limit there to obtain

$$\zeta_1 = -e^r \sin \frac{\theta}{2} \sin \frac{\tilde{\theta}}{2} e^{i \frac{\psi - \varphi - \tilde{\varphi}}{2}} \, , \qquad \zeta_2 = e^r \cos \frac{\theta}{2} \cos \frac{\tilde{\theta}}{2} e^{i \frac{\psi + \varphi + \tilde{\varphi}}{2}} \, ,$$

$$\zeta_3 = e^r \cos \frac{\theta}{2} \sin \frac{\tilde{\theta}}{2} e^{i \frac{\psi + \varphi - \tilde{\varphi}}{2}} \, , \qquad \zeta_4 = -e^r \sin \frac{\theta}{2} \cos \frac{\tilde{\theta}}{2} e^{i \frac{\psi - \varphi + \tilde{\varphi}}{2}} \, . \tag{7.A.3}$$

This abelian geometry inherits the $SU(2)_L \times SU(2)_R$ symmetry of the non-abelian one (actually the isometry group is enlarged to $SU(2)_L \times SU(2)_R \times U(1)$). Again, taking carefully[3] the limit $r \to \infty$ in the non-abelian expressions (7.3.13) and (7.2.20) for the fundamental two-form J and the smearing form Ξ, we get

$$
J = \frac{e^{2k}}{2}\mathrm{d}r \wedge (\tilde{\omega}_3 + \cos\theta\,\mathrm{d}\varphi) - \frac{e^{2g}}{4}\sin\tilde{\theta}\,\mathrm{d}\tilde{\theta} \wedge \mathrm{d}\tilde{\varphi} - e^{2h}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi\,,
$$
$$
\Xi = \frac{N_f}{16\pi^2}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \left(S\sin\tilde{\theta}\,\mathrm{d}\tilde{\theta} \wedge \mathrm{d}\tilde{\varphi} - S'\mathrm{d}r \wedge \left(\mathrm{d}\psi + \cos\tilde{\theta}\,\mathrm{d}\tilde{\varphi}\right)\right).
$$

$$(7.\mathrm{A}.4)$$

Then one can define $SO(4)$-invariant (1,1)-forms η_i ($i = 1,\ldots,4$) as in (7.3.11), and express both J and Ξ in this abelian setup as

$$
J = 2i\,e^{-2r}\left[e^{2h}\left(\eta_1 + 2e^{-2r}\eta_2 - 2e^{-2r}\eta_3\right) \right.
$$
$$
\left. + \frac{e^{2g}}{4}\left(\eta_1 + 2e^{-2r}\eta_2 + 2e^{-2r}\eta_3\right) - e^{2k}e^{-2r}\eta_2 \right],
$$

$$(7.\mathrm{A}.5)$$

$$
\frac{16\pi^2}{N_f}\Xi = -8e^{-4r}S\,\eta_1 \wedge \left(\eta_1 + 4e^{-2r}\eta_2\right) + 8e^{-6r}S'\eta_2 \wedge \left(\eta_1 - 2e^{-2r}\eta_3\right),
$$

where the η_i's are the abelian $(1,1)$ two-forms, which can be obtained from (7.3.12) by keeping the leading term when $r \to \infty$.

7.A.2 Abelian limit of the simple class of embeddings

Let us now calculate $S(r)$ for the abelian version of the class of embeddings discussed in Section 7.5. The first thing to notice is that the parametrisation (7.5.1) is not good in the UV limit. Indeed, as $z_4 = z_1 z_2/z_3$ when $r \to \infty$, the two equations in (7.5.1) become the same. For this reason, to study this cycle in the UV, it is better to use instead the first two equations in (7.5.2) and write the equations of the embedding as $z_1 = Cz_3$ and $z_1 z_2 = \tilde{\mu}$, with C and $\tilde{\mu}$ being arbitrary complex constants. By taking the UV limit in which $z_i \to \zeta_i$, one concludes that the abelian limit of the particular representative of the embedding studied in section (7.5) is

$$\zeta_1 = C\zeta_3\,, \qquad \zeta_1\zeta_2 = \tilde{\mu}\,. \qquad (7.\mathrm{A}.6)$$

---

[3]In the abelian limit: $a \to 0$, $\cosh(2r) \to \sinh(2r) \to \frac{e^{2r}}{2}$, and $a\,e^{2r} \to 1 + 4e^{2h-2g}$.

One nice thing about the abelian limit is that (7.A.6) can be easily solved in terms of coordinates:

$$\theta = \theta_0 \,, \qquad \varphi = \varphi_0 \,, \qquad \text{and} \qquad \frac{1}{2} \sin \tilde{\theta} \, e^{2r} e^{i\psi} = \mu \equiv \frac{1}{2} e^{2r_q} e^{i\gamma} \,, \tag{7.A.7}$$

where we have parametrised the constants above as

$$C = \tan \frac{\theta_0}{2} e^{-i\varphi_0} \,, \qquad \tilde{\mu} = 2\mu \sin^{-1} \theta_0 \,, \tag{7.A.8}$$

and r_q is the minimum radial distance this embedding reaches ($e^{2r_q} = |2\mu|$). If we now rotate this embedding with the $SU(2)_L \times SU(2)_R$ isometry group (see (7.3.6)), we obtain the expression of a generic embedding of the family as $f_1 = 0$ and $f_2 = 0$, with

$$f_1 = \zeta_1 - \frac{b + a\,C}{\bar{a} - \bar{b}\,C} \zeta_3 \,,$$

$$f_2 = \left((|k|^2 - |l|^2)\zeta_1\zeta_2 - k\,\bar{l}\,C^{-1}\zeta_1^2 + \bar{k}\,l\,C\,\zeta_2^2 \right) - \left(|a|^2 - |b|^2 - a\,\bar{b}\,C + \bar{a}\,b\,\bar{C} \right)^{-1} \tilde{\mu} \,. \tag{7.A.9}$$

The smearing form should be computed as an appropriately weighted sum of the transverse volume forms of each embedding. The formula for real constraints was first written in [35], and the generalisation to complex constraints like the ones we have now is immediate[4]:

$$\Xi = \frac{1}{(-2i)^2} \int_{\mathbb{C}^4} \mathrm{d}\rho \, \delta^{(2)} \left(f_1 \right) \delta^{(2)} \left(f_2 \right) \mathrm{d}f_1 \wedge \mathrm{d}\bar{f}_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\bar{f}_2 \,, \tag{7.A.10}$$

where ρ is the (normalised to the unity) measure of $SU(2)_L \times SU(2)_R$, multiplied by N_f, and is given by

$$\mathrm{d}\rho = \mathrm{d}a \, \mathrm{d}\bar{a} \, \mathrm{d}b \, \mathrm{d}\bar{b} \, \mathrm{d}k \, \mathrm{d}\bar{k} \, \mathrm{d}l \, \mathrm{d}\bar{l} \, \delta \left(|a|^2 + |b|^2 - 1 \right) \delta \left(|k|^2 + |l|^2 - 1 \right) \frac{N_f}{16\pi^4} \,. \tag{7.A.11}$$

A shortcut for computing (7.A.10) is to notice that all the embeddings of the present family, in virtue of the first equation in (7.A.9), sit at constant values of θ and φ. Since it turns out that the action of $SU(2)_L$ corresponds precisely to varying these constant values over a two-sphere, the smearing form Ξ necessarily exhibits a $\frac{1}{4\pi} \sin \theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi$ factor. We are not interested in getting this trivial part from (7.A.10), so we factor it out by defining an effective (complex) two-dimensional

---

[4]The complex Dirac δ-function should be understood as $\delta^{(2)}(f) = \delta(\mathrm{Re}(f))\delta(\mathrm{Im}(f))$. The $\frac{1}{-2i}$ prefactor is included because $\mathrm{d}f \wedge \mathrm{d}\bar{f} = -2i \, \mathrm{d}(\mathrm{Re}(f)) \wedge \mathrm{d}(\mathrm{Im}(f))$.

problem. We can define a new pair of effective complex variables as

$$\xi_1 = e^r \cos \frac{\tilde{\theta}}{2} \, e^{i\frac{\psi+\tilde{\varphi}}{2}} \,, \qquad \xi_2 = e^r \sin \frac{\tilde{\theta}}{2} \, e^{i\frac{\psi-\tilde{\varphi}}{2}} \,, \qquad (7.A.12)$$

and the family of embeddings over which we want to smear recasts as

$$f \equiv (|A|^2 - |B|^2)\xi_1\xi_2 + A\bar{B}\xi_2^2 - \bar{A}B\xi_1^2 - \mu = 0 \,, \qquad (7.A.13)$$

with $|A|^2 + |B|^2 = 1$ (Recall that $\mu = \frac{1}{2}e^{2r_q}e^{i\gamma}$, see (7.A.7)). Forgetting for the moment about the correct normalisation factors, the integral we want to compute is

$$W \equiv \int_{\mathbb{C}^2} dA \, d\bar{A} \, dB \, d\bar{B} \, \delta(|A|^2 + |B|^2 - 1)\delta^{(2)}(f) \, df \wedge d\bar{f} \,. \qquad (7.A.14)$$

Performing this integral requires some attention in dealing with the δ-functions but, other than that, it can be considered straightforward. Let us sketch how one could proceed. To simplify the calculation, we reparametrise the integration variables as follows:

$$A = \sqrt{\frac{u_1 + u_2}{2}} \, e^{i\alpha_1} \,, \qquad B = \sqrt{\frac{u_1 - u_2}{2}} \, e^{i\alpha_2} \,. \qquad (7.A.15)$$

Clearly, one has $|A|^2 + |B|^2 = u_1$ and $|A|^2 - |B|^2 = u_2$ and

$$\int_{\mathbb{C}^2} dA \, d\bar{A} \, dB \, d\bar{B} \, \delta(|A|^2 + |B|^2 - 1) = \frac{1}{2} \int_{-\infty}^{\infty} du_2 \int_{|u_2|}^{\infty} du_1 \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \, \delta(u_1 - 1).$$
$$\qquad (7.A.16)$$

The integral in u_1 is then immediate. Rewriting $e^{i\alpha_2} = x_2 + iy_2$, and using that

$$\int_0^{2\pi} d\alpha_2 = 2 \int_{\mathbb{R}^2} dx_2 \, dy_2 \, \delta(x_2^2 + y_2^2 - 1) \,, \qquad (7.A.17)$$

we can write

$$W = \int_0^{2\pi} d\alpha_1 \int_{-1}^1 du_2 \int_{\mathbb{R}^2} dx_2 \, dy_2 \, \delta(x_2^2 + y_2^2 - 1)\delta(R)\delta(I) \, df \wedge d\bar{f} \,, \qquad (7.A.18)$$

where $R \equiv \mathrm{Re}\,(f|_{u_1=1})$, $I \equiv \mathrm{Im}\,(f|_{u_1=1})$. In the new variables, one has

$$f|_{u_1=1} = -\frac{1}{2}e^{i(\alpha_1-\alpha_2)}\sqrt{1-u_2^2}\,\xi_1^2 + u_2\xi_1\xi_2 + \frac{1}{2}e^{i(-\alpha_1+\alpha_2)}\sqrt{1-u_2^2}\,\xi_2^2 - \mu \,. \quad (7.A.19)$$

Solving $R = I = 0$ for x_2 and y_2, the integral of the corresponding δ-functions produces the following factor:

$$
\left| \frac{\mathrm{d}R}{\mathrm{d}x_2} \frac{\mathrm{d}I}{\mathrm{d}y_2} \right|^{-1} = \frac{4}{1 - u_2^2} \frac{1}{\left| |\xi_1|^4 - |\xi_2|^4 \right|}, \tag{7.A.20}
$$

and leaves the argument of the remaining δ-function as

$$
\delta(x_2^2 + y_2^2 - 1) = \delta \left(\frac{(|\xi_1|^2 + |\xi_2|^2)^2}{(1 - u_2^2)(|\xi_1|^2 - |\xi_2|^2)^2} (u_2 - u_{2+})(u_2 - u_{2-}) \right), \tag{7.A.21}
$$

where $u_{2\pm}$ are given by

$$
u_{2\pm} = \frac{4 |\mu\, \xi_1\, \xi_2| \cos(\psi - \gamma) \pm \left| |\xi_1|^2 - |\xi_2|^2 \right| \sqrt{(|\xi_1|^2 + |\xi_2|^2)^2 - 4|\mu|^2}}{(|\xi_1|^2 + |\xi_2|^2)^2}. \tag{7.A.22}
$$

At this point, we have

$$
W = \int_0^{2\pi} \mathrm{d}\alpha_1 \int_{-1}^1 \mathrm{d}u_2 \frac{\frac{4}{1-u_2^2}}{\left| |\xi_1|^4 - |\xi_2|^4 \right|} \delta \left(\frac{(|\xi_1|^2 + |\xi_2|^2)^2 (u_2 - u_{2+})(u_2 - u_{2-})}{(1 - u_2^2)(|\xi_1|^2 - |\xi_2|^2)^2} \right) \mathrm{d}f \wedge \mathrm{d}\bar{f}. \tag{7.A.23}
$$

In this expression, nothing depends on α_1, so one can integrate it easily. Also, both u_{2+} and u_{2-} are between -1 and 1, so they both contribute to the integral. Using (7.A.2), and replacing u_{2+} and u_{2-} by their values (7.A.22), we finally get

$$
W = 4\pi \frac{1 - \cos\tilde{\theta} + 2e^{4r_q - 4r} \cos\tilde{\theta}}{\sqrt{e^{4r} - e^{4r_q}}} \mathrm{d}\xi_1 \wedge \mathrm{d}\bar{\xi}_1 + 4\pi \frac{1 + \cos\tilde{\theta} - 2e^{4r_q - 4r} \cos\tilde{\theta}}{\sqrt{e^{4r} - e^{4r_q}}} \mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_2
$$

$$
- 4\pi e^{-i\tilde{\varphi}} \frac{(1 - 2e^{4r_q - 4r}) \sin\tilde{\theta}}{\sqrt{e^{4r} - e^{4r_q}}} (\mathrm{d}\xi_1 \wedge \mathrm{d}\bar{\xi}_2 + \mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_1). \tag{7.A.24}
$$

Plugging the values of ξ_1 and ξ_2 in (7.A.12), and taking into account the proper normalisation factors, we find exactly

$$
\frac{N_f}{4\pi i} \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge W = 16\pi^2 \,\Xi, \tag{7.A.25}
$$

where Ξ is the one written in (7.A.4) with the following function $S(r)$:

$$
S(r) = \sqrt{1 - e^{4r - 4r_q}}\; \Theta(r - r_q). \tag{7.A.26}
$$

Notice that (7.A.26) is the limit of the function $S(r)$ written in (7.5.13) when r and r_q are large. This confirms our results of Section 7.5.

7.A.3   An example of a non-compatible embedding

As we saw in the previous subsection, one has to work quite hard in order to obtain the smearing form Ξ from the microscopic average over a family of embeddings. Certainly, the trick described in Section 7.4 gives a much faster and simpler way to get Ξ. One can wonder nevertheless about the reliability of the trick, since it assumes a given functional form for Ξ, and the only unknown is the radial profile of the brane distribution $S(r)$.

In principle, this trick can be run for any representative embedding. However, it is hard to think that any given family of embeddings, even if supersymmetric, generates (when we place flavour branes along the embeddings of the family) a backreaction of the metric that is compatible with the initial ansatz we assumed for this metric, if this is not the most general possible. It seems nonetheless that the trick is able to detect this "compatibility property", and we present in what follows some arguments in favour of that argument.

Recalling the discussion in Section 7.4, the trick was to compute the effective radial action of the smeared brane distribution, and to compare it with N_f times the WZ effective radial action of a single brane sitting at one of the embeddings of the family over which we smear. Both actions should be equal. The smeared action always contains two terms, one proportional to $S(r)$, and another proportional to $S'(r)$:

$$\mathcal{L}_{WZ}^{\text{smeared}} = F_1(r)S(r) + F_2(r)S'(r) \,, \tag{7.A.27}$$

where F_1 and F_2 depend on the functions of the ansatz. We conjecture that the way to detect if a family of embeddings generates a backreaction compatible with the ansatz is to take any representative embedding of this family and to compute its WZ effective radial action. We must then check whether or not the result depends on the functions of the ansatz in the same way as in (7.A.27). Let us assume that this is the case and that the effective WZ radial Lagrangian density for the representative embedding is of the form

$$\mathcal{L}_{WZ}^{\text{single}} = F_1(r)G(r) + F_2(r)H(r) \,, \tag{7.A.28}$$

where F_1 and F_2 are the same as in (7.A.27) and $G(r)$ and $H(r)$ are functions

of r which do not depend on the functions of the ansatz. In order to verify that (7.A.28) is of the form (7.A.27), one must finally check that

$$\frac{\mathrm{d}G(r)}{\mathrm{d}r} = H(r)\,. \tag{7.A.29}$$

If this is the case, we conjecture that the backreaction is compatible with the ansatz and, furthermore, that the profile function S is proportional to G.

In Section 7.5, we have worked out one example in which the compatibility condition is satisfied. Moreover, in Subsection 7.A.2, we checked explicitly with an independent calculation of Ξ that the trick gives the right result. In what follows, let us illustrate with an example the case in which the compatibility condition is not satisfied, and show with a microscopic calculation that indeed the resulting Ξ is incompatible with the ansatz for it.

We choose to work again in the abelian background, since it is simpler and therefore the explanation will be clearer. Let us focus on the following embedding:

$$\zeta_1 = C\zeta_4\,, \qquad\qquad \zeta_2 = \mu\,, \tag{7.A.30}$$

where the ζ_i's are the complex coordinates (7.A.3) and C and μ are constants that we parametrise as $C = \tan\frac{\tilde{\theta}_0}{2}e^{-i\tilde{\varphi}_0}$ and $\mu = \cos\frac{\tilde{\theta}_0}{2}e^{i\tilde{\varphi}_0/2}e^{i\beta}$. We can solve the embedding equations in (7.A.30) in terms of coordinates as

$$\tilde{\theta} = \tilde{\theta}_0\,, \quad \tilde{\varphi} = \tilde{\varphi}_0\,, \quad \text{and} \quad e^r\cos\frac{\theta}{2} = e^{r_q}\,, \quad \psi + \varphi = 2\beta\,. \tag{7.A.31}$$

It is then easy to compute the effective radial Lagrangians[5] of the smeared distribution and of a single brane extended along the embedding (7.A.30), with the result:

$$\mathcal{L}_{WZ}^{\text{smeared}} = 2\pi N_f\, T_{D5}\, e^{2\Phi}\left(e^{2k}S + \frac{e^{2g}}{2}S'\right)\,,$$
$$\mathcal{L}_{WZ}^{\text{single}} = 2\pi\, T_{D5}\, e^{2\Phi}\left(e^{2k}\left(1 - e^{2r_q - 2r}\right) + 4e^{2h}e^{2r_q - 2r}\right)\,, \tag{7.A.32}$$

where we have assumed that Ξ should be as in (7.A.4). As we see, the e^{2g} term in the smeared Lagrangian is not present in $\mathcal{L}_{WZ}^{\text{single}}$ (we have instead an e^{2h} term), so the compatibility condition is not satisfied.

Let us now check with a microscopic computation that, indeed, the family of

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[5]We compare the WZ actions but, since we have supersymmetry, the trick would also work if we compared the DBI actions.

embeddings generated by rotating (7.A.30) with the $SU(2)_L \times SU(2)_R$ symmetry gives a Ξ that is not of the form of (7.A.4). After using the relation (7.A.1), the family can be characterised by $f_1 = 0$ and $f_2 = 0$, where

$$\begin{aligned} f_1 &= \bar{a}\,\zeta_1 + \bar{b}\,\zeta_4 \,, \\ f_2 &= \bar{k}\,\zeta_2 + \bar{l}\,\zeta_4 + \bar{b}\,\mu \,, \end{aligned} \tag{7.A.33}$$

with $|a|^2 + |b|^2 = 1 = |l|^2 + |k|^2$. In this case, it is easy to perform the integral (7.A.10) by making use of the following two results:

$$I_1 = \int dz\,d\bar{z}\,\delta^{(2)}\left(w_1\,z - w_2\right) = -\frac{2i}{|w_1|^2}\,, \tag{7.A.34}$$

$$\begin{aligned} I_2 &= \int dx\,dy\,\left(x^2 + y^2 + \alpha_1 x + \beta_1 y + \gamma_1\right)\delta\left(x^2 + y^2 + \alpha_2 x + \beta_2 y + \gamma_2\right) \\ &= \pi\left(\gamma_1 - \gamma_2 + \frac{\alpha_2^2 + \beta_2^2 - \alpha_1\alpha_2 - \beta_1\beta_2}{2}\right)\,. \end{aligned} \tag{7.A.35}$$

The final result we get for the smearing form is

$$\Xi = \frac{N_f}{16\pi^2}\sin\tilde{\theta}\,d\tilde{\theta}\wedge d\tilde{\varphi}\wedge\left(\left(1 - e^{2r_q - 2r}\right)\sin\theta\,d\theta\wedge d\varphi - 2e^{2r_q - 2r}dr\wedge\left(d\psi + \cos\theta\,d\varphi\right)\right), \tag{7.A.36}$$

and we see that this is clearly incompatible with (7.A.4) (the roles of (θ, φ) and $(\tilde{\theta}, \tilde{\varphi})$ are exchanged in these two expressions of Ξ).

Chapter 8

Conclusion

Let us now summarise the results presented in this thesis. Our goal was trying to see how the mathematical concept of G-structures could be used in the context of gauge/gravity correspondence, and what we could learn from it. We have focused our attention on two main topics: the flavouring problem (see Chapters 2, 3, 4, 5 and 7) and solution-generating techniques (see Chapters 5 and 6). In the following, we sum up the results we found on those two issues, plus some others that do not exactly fit in one of those categories, but that are interesting and were discovered along the way.

The central result that the use of G-structures brought to the issue of flavouring was the fact that supersymmetric flavour branes wrap calibrated cycles. It was previously known that flavour branes were κ-symmetric, which gave conditions on the cycles those branes were allowed to wrap in the internal geometry. However, those conditions were quite difficult to solve, except in the most simple cases. In addition, the way the smeared DBI action for the flavour branes was written was quite obscure and apparently highly non-linear, despite the fact that the WZ term was linear and often very simple. Finally, writing an ansatz for the RR field strengths of flavoured background boiled down to physical intuition and guess work. As explained in Chapter 2 and illustrated in the rest of this thesis, realising that κ-symmetry was equivalent to calibrated geometry, and that the whole framework of flavouring could be recast in terms of G-structures, allowed a more systematic approach to the flavouring problem. Indeed, the number of supercharges one wants to preserve and the dimension of the dual field theory dictate which G-structure one has to impose on the internal manifold. From there, the κ-symmetric embeddings of the flavour branes become calibrated cycles, with their calibration forms provided by the G-structure. The smeared DBI action can be written linearly, in terms of

the appropriate calibration form, putting it on equal footing with the WZ action. Moreover, ansätze for the fluxes, and consequently for the smearing forms, that are compatible with the metric ansatz can be deduced easily by performing some simple operations on the G-invariant forms provided by the structure.

But, as we showed in this thesis, the power of the G-structure formulation of the flavouring problem goes beyond reproducing known results. In addition to improving the understanding of some of the known flavoured solutions, using G-structures made it possible to find new supergravity solutions. For example, in Chapters 2 and 3, we constructed two new flavoured backgrounds in Type IIB and Type IIA supergravities. We also saw one of the limitations of the formalism, which is that one still needs the input of a family of flavour embeddings to find a solution that can be interpreted with certainty as coming from the addition of flavour branes in a background. This issue was resolved in Chapter 7, where we studied what is necessary to get a flavoured solution. In that chapter, we saw that the knowledge of one particular embedding is sufficient to know if there can be a smeared solution involving this embedding, that is compatible with the ansatz for the metric. We showed as well that, once a compatible embedding is known, it is quite straightforward to get the exact expression for the smearing form it contributes to.

Through the formalism of G-structures, one can also study the lift to M-theory of flavoured Type IIA supergravity solutions. Indeed, in Chapter 3, we were concerned with the lift to eleven dimensions of a Type IIA supergravity background with flavour D6-branes. The relation between Type IIA and eleven-dimensional supergravities is well known, but it implies that the RR two-form of Type IIA supergravity is closed. However, having smeared D6-sources in the background means that this particular form is not closed. That lead to the question of the lift of such a background to M-theory. Once again, G-structures provided a very adequate framework to study this problem, and we proposed that the smeared D6-branes lift to geometric torsion in eleven dimensions. We were even able to write eleven-dimensional equations of motion that reduce to the Type IIA flavoured ones in ten dimensions.

In Chapter 4, we combined flavouring techniques with the use of hyperbolic geometry to construct supergravity solutions dual to field theories exhibiting a Kutasov-like duality. We saw that G-structures could deal with hyperbolic spaces as simply as with the usual spherical ones. We found that there were great similarities between the hyperbolic and spherical cases, all the way to the form of the

BPS equations. However, the apparently small differences ended up having a very non-trivial impact on the solutions. Indeed, we showed that it was not possible to find solutions of the BPS equations that would extend over an infinite range for the holomorphic radial coordinate. All the solutions we presented were consequently singular in the UV as well as in the IR. Nevertheless, we were still able to study their field-theory duals to find that they displayed a Kutasov-like duality.

With the flavouring of backgrounds comes the issue of singularities. Indeed, in almost all examples of adding backreacting massless flavours to supergravity solutions, one creates an IR singularity. It has been argued that in most cases it is a "good" singularity, meaning that theses solutions could still be used to understand some properties of their field-theory duals. Nevertheless, it seems that having completely regular supergravity backgrounds would put the solutions and their field-theory interpretation on a stronger footing. This has been done by going from massless to massive flavours and, in Chapter 7, we did exactly this for the Maldacena-Núñez background. We found that the whole setup depends on the choice of a profile function for the distribution of flavour branes in the internal space of the background. This function cannot be chosen arbitrarily, but can be deduced from the choice of embeddings for the sources. It is then possible to have solutions that are regular. In addition, we noticed that we could take the mass to zero and preserve the regularity of the solutions, providing an example of a regular solution with massless flavours. Moreover, the techniques applied in this particular example could easily be transposed to other cases.

In addition to its importance in dealing with flavouring, the concept of G-structures can also be useful in finding new supergravity solutions, regardless of the presence of sources. This is what we showed in Chapters 5 and 6. In both chapters, we used G-structures as a support for finding solution-generating methods in either Type IIA or Type IIB supergravity. In Chapter 5, we explained how some transformations on the $SU(3)$-invariant forms can be used as a way to create new Type IIB supergravity solutions from known ones. In particular, the new solutions are more complicated than the ones we start from, in the sense that they involve more fluxes. We proved that this solution-generating technique is equivalent to a certain chain of string dualities, but has the advantage of being able to include flavours. Indeed, we used it on a flavoured solution to find a new solution with sources, showing at the same time that G-structures have no problem in dealing with flavouring in a background where a non-zero $B_{(2)}$ field is turned on.

In Chapter 6, we saw that the method discovered in Chapter 5 for $SU(3)$-

structure could be adapted to the case of G_2-structure. Indeed, we provided a set of transformations on the G_2-invariant objects that generates new Type IIA supergravity solutions. As in the case of $SU(3)$-structure, there is also a chain of dualities that is equivalent to this method, at least as long as one does not consider the addition of sources. Thanks to this solution-generating technique, we found a family of new Type IIA supergravity solutions which would have been difficult to uncover directly because of the various fluxes that are turned on. In addition, this family has two interesting and quite well-known limits, one being the solutions of Maldacena-Nastase and the other one the so-called warped G_2-holonomy backgrounds.

Overall, we showed in this thesis the interest of using G-structures in the context of gauge/string duality. We saw that this mathematical concept offers a way to study supergravity solutions in a more systematic fashion, exhibiting relations between solutions that first seem very different. One of the main issues now would be to find out how this very general formalism and the results it provides can be transposed and understood on the field-theory side.

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