Introduction to Differentiable Stacks (and gerbes, moduli spaces ...)

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Many interesting *geometric* objects in (algebraic or differential) geometry or mathematical/theoretical physics are *not* manifolds or do not carry the "correct" manifold structure. Typical such examples are spaces obtained as quotient of other spaces under some natural gauge equivalences/group actions.

Differentiable[1] stacks are a natural generalization of smooth manifolds encompassing, in addition, the following examples (with non-empty intersections) :

- *Quotient spaces:* let G be a compact Lie group acting (by diffeomorphisms) on a manifold M. When G acts freely, there is a natural smooth structure on the quotient space M/G such that the quotient map $M \to M/G$ is a G-principal bundle. In general, for non-free actions, however, M/G is either singular or may have a very boring smooth structure. For instance the quotient pt/G is just a point independently of the group. We'll see that there is a *stacky quotient* $M \to [M/G]$ which behaves much like a principal G-bundle. The quotient stack point of view also allows to treat equivariant problems as non-equivariant ones especially when there are several group actions involved.

- *Orbifolds:* Let G be a finite cyclic subgroup of $SO(n)$, acting on $\mathbb{R}^n$. The action is free on $\mathbb{R}^n \setminus \{0\}$ but not on 0. The quotient $\mathbb{R}^n/G$ looks like a cone. This is a typical example of an orbifold, that is of a space which locally is a quotient of a manifold by a finite group. A similar example adding a point at ∞ yields the "tear drop" orbifold. More generally, examples of (analytic) orbifolds include Riemann Surfaces with a finite set of orbifold points; that is Riemann Surfaces in which one has removed a finite set of holomorphic disks and replaced them by quotients of a holomorphic disk in $\mathbb{C}$ by a finite group of $SU(1)$ as above.

- *Moduli spaces:* moduli problems are concerned with classifying algebro-geometric objects in families. For instance, one can study moduli spaces of lines through the origin in $\mathbb{R}^{n+1}$ which is parametrized by $\mathbb{R}P^n$. Heavily studied examples are given by moduli spaces of elliptic curves or genus g Riemann surfaces with marked points. Unlike in the aforementioned case of lines, many of these problems are not classify by a space (that is they do not form a fine moduli space), but by a stack. We will see elementary examples below.

All the objects above give naturally rise to *differentiable stacks*. One important feature is that they are **non-singular**, *when viewed as stacks* (even though their associated coarse spaces, that is their naive quotients, are typically singular). For this reason, one can still do differential geometry with them and further, algebraic topology of stacks behaves much like for manifolds.

Underlying philosphy: stacks arise whenever one deals with or want to study "spaces" in which points have been identified (by some equivalence relations), in a ***non-unique*** *way.*

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†the notes are the notes I prepared for the lecture but contains more material than was actually covered; they do not claim to be original

[1]which are really C^∞ objects and not just differentiable

Contents

1 Stacks over Diff

We start with some abstract non-sense: the notion of (pre-)stacks, before defining the actual notion of differentiable stacks in § 2.

Let Diff ***be the category of smooth manifolds*** whose objects are smooth manifolds and morphisms smooth maps between smooth manifolds. We write *pt* for the manifold given by a single point.

1.1 PreStacks over Diff

Definition 1.1 A *category fibered in groupoids over* Diff[2] is a category $\mathfrak{X}$ together with a functor $\pi : \mathfrak{X} \to$ Diff satisfying:

i) For every arrow $f : V \to U$ in Diff, and for every object $\overline{U}$ in $\mathfrak{X}$ such that $\pi(\overline{U}) = U$, there is an arrow $F : \overline{V} \to \overline{U}$ in $\mathfrak{X}$ such that $\pi(F) = f$. In other words, in the above partial lift, the object $\overline{V}$ and the dotted arrow exist:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, equal] & U \arrow[d, equal] \\ \overline{V} \arrow[r, dashed, "\pi(F)"] & \overline{U}. \end{tikzcd}$$

[2]the category Diff can be replaced by analytic spaces, or schemes, or topological spaces

ii) Given a commutative triangle in Diff, and a partial lift for it to $\mathfrak{X}$ as in the diagram:

$$\begin{tikzcd} V \arrow[dr, "f"] \arrow[dd, "h"'] & & & & \overline{V} \arrow[dr, "F"] \arrow[dd, dashed, "H"] & \\ & U & & \arrow[ll, mapsto, "\pi"'] & & \overline{U} \\ W \arrow[ur, "g"'] & & & & \overline{W} \arrow[ur, "G"'] & \end{tikzcd},$$

there is a *unique* morphism $H : \overline{V} \to \overline{W}$ such that the right triangle commutes and $\pi(H) = h$.

We will sometimes *call a category fibered in groupoids a* ***prestack*** since stacks will be a special sub-class of such.

Terminology: for $M \in$ Diff, call $\mathfrak{X}(M) := \pi^{-1}(M)$ the M-points of $\mathfrak{X}$. It is a subcategory of $\mathfrak{X}$; its morphisms are given by $\pi^{-1}(M \stackrel{id}{\to} M)$, that is all morphisms F in $\mathfrak{X}$ such that $\pi(F) = id : M \to M$.

Lemma 1.2 $\mathfrak{X}(M)$ *is a groupoid, that is, a category in which every morphism is invertible.*

The proof of the Lemma can be seen as follow: for any $F : X \to Y \in \mathfrak{X}(M)$, we have a partial lift:

$$\begin{tikzcd} M \arrow[dr, "id"] \arrow[dd, "id"'] & & & & Y \arrow[dr, "id"] \arrow[dd, dashed, "G"] & \\ & M & & \arrow[ll, mapsto, "\pi"'] & & Y \\ M \arrow[ur, "id"'] & & & & X \arrow[ur, "F"'] & \end{tikzcd}$$

which defines the (unique) right inverse $G : Y \to X$ of F. To check that $G : Y \to X$ is also a left inverse, we consider the diagram

$$\begin{tikzcd} X \arrow[dr, "F"] \arrow[dd, "G\circ F"'] & \\ & Y \\ X \arrow[ur, "F"'] & \end{tikzcd}$$

which commutes since $F \circ G = id_Y$. By uniqueness of the lift in **ii**), we have $G \circ F = id_X$.

Remark 1.3 (Restrictions) A similar proof shows that any two choices of lifts $F : \overline{V} \to \overline{U}$, $F' : \overline{V}' \to \overline{U}$ in **i**) are *canonically* isomorphic: that is there is a *unique* isomorphism $\psi : \overline{V} \xrightarrow{\cong} \overline{V}'$ such that $F' \circ \phi = F$.

In particular, let $f : M \to N$ be a map in Diff. By **i**), for any object in $X \in \mathfrak{X}(N)$, we can choose an object $f^*(X) \in \mathfrak{X}(M)$ together with a map $\rho_X : f^*(X) \to X$ in $\pi^{-1}(f)$. By **ii**), this map extends canonically into a functor $f^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$ which, to $F : Y \to X \in \mathfrak{X}(N)$, associates the unique lift $f^*(F)$ as given below:

$$\begin{tikzcd} M \arrow[dr, "f"] \arrow[dd, "id"'] & & & & f^*(X) \arrow[dr, "\rho_X"] \arrow[dd, dashed, "f^*(F)"'] & \\ & N & & \arrow[ll, mapsto, "\pi"'] & & X. \\ M \arrow[ur, "g"'] & & & & f^*(Y) \arrow[ur, "F\circ\rho_Y"'] & \end{tikzcd}$$

Again by **ii**), different choices of $f^*(X)$ yields canonically equivalent functors. This allows to define a contravariant (lax-)functor Diff $\to \mathfrak{X}$, $M \mapsto \mathfrak{X}(M)$ (which depends on some choices but is well defined up to unique natural equivalence).

Notation: let $f : U \to M \in$ Diff and $\phi : A \to B \in \mathfrak{X}(M)$. We will simply *denote* $A_{|U}, B_{|U}$ *the **pullbacks*** $f^*(A), f^*(B) \in \mathfrak{X}(U)$ *and* $\phi_{|U} := f^*(\phi) : A_{|U} \to B_{|U}$ when there is no ambiguity about f.

Example 1 (Manifolds as stacks) Let M be a manifold. We define the category $[M]$ with objects all smooth maps $f : U \to M$ (where U is arbitrary in Diff) and with morphisms (say from $f : U \to M$ to $g : V \to M$) the commutative diagrams
$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\psi"'] & M \\ V \arrow[ur, "g"'] & \end{tikzcd}$$
(in other words all maps $\psi : U \to V \in$ Diff such that $g \circ \psi = f$). We have an obvious functor $\psi : [M] \to$ Diff mapping $f : U \to M$ to U and a commutative diagram as above to ψ. Note that all lifts here are uniquely defined so that the conditions of Definition 1.1 are trivial to check.

Example 2 (Classifying stack of a Lie group) Let G be a Lie group. We define a category $[pt/G]$ with objects all principal G-bundles $P \to U$ (in Diff). Morphisms (from $P \to U$ to $Q \to V$) are all *cartesian* diagrams
$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ U \arrow[r, "\psi"] & V \end{tikzcd}$$
such that the top arrow is G-equivariant. The functor $[pt/G] \to Diff$ given by $(P \to U) \mapsto U$ makes $[pt/G]$ a category fibered in groupoids over Diff. It is *called the classifying stack of* G.

Example 3 (=1+2) Let G be a Lie group acting by diffeomorphisms on a smooth manifold M. Let $[M/G]$ be the category of pairs $(P \to U, P \stackrel{f}{\to} M)$ where $P \to U$ is a principal G-bundle and $f : P \to M$ is G equivariant. A morphism from $(P \to U, P \stackrel{f}{\to} M)$ to $(Q \to V, Q \stackrel{g}{\to} M)$ is a cartesian diagram
$$\begin{tikzcd} P \arrow[r, "\beta"] \arrow[d] & Q \arrow[d] \\ U \arrow[r, "\psi"] & V \end{tikzcd}$$
such that the top arrow β is G-equivariant and further $g \circ \beta = f$. Again, the functor $[M/G] \to$ Diff is just given by forgetting all the data but U (and $\phi : U \to V$ for morphisms). The properties of lifts in Definition 1.1 follows from the ones of pullback of principal G-bundles.

1.2 Descent data

Definition 1.4 (Stacks over Diff**)** A category $\mathfrak{X}$ fibered in groupoids over Diff[3] is a **stack**, if, for any $M \in$ Diff and any open cover $(U_i)_{i\in I}$ of M, the following two conditions are satisfied:

1. *Gluing morphisms:* Given two objects $A, B \in \mathfrak{X}(M)$ and any family $(\phi_i : A_{|U_i} \to B_{|U_i})_{i \in I}$ of maps such that $\phi_{i|U_i \cap U_j} = \phi_{j|U_i \cap U_j}$, there exist a *unique* morphism $\phi : A \to B \in \mathfrak{X}(M)$ such that $\phi_{|U_i} = \phi_i$ (for all i's). (In other words the presheaf of sets $U \mapsto Hom_{\mathfrak{X}(U)}(A_{|U}, B_{|U})$ is a sheaf.)

2. *Gluing objects:* Assume we are given objects $A_i \in \mathfrak{X}(U_i)$, together with isomorphisms $\varphi_{ij}\ A_j|_{U_i \cap U_j} \to A_i|_{U_i \cap U_j}$ in $\mathfrak{X}(U_i \cap U_j)$ which satisfy the cocycle condition
$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$$
on $U_i \cap U_j \cap U_k$ (for every triple of indices i, j and k). Then, there is an object $A \in \mathfrak{X}(M)$, together with isomorphisms $\varphi_i\ A|_{U_i} \to A_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$.

[3]again, one can work in context different than smooth manifolds; for instance over analytic spaces, schemes, or topological spaces. In that cases one has to use a correct notion of covers which can be more tricky

The data given in (**2**) is usually called a *gluing data* or a *descent data*. It follows from (**1**) that the object A in (**2**) is unique up to a unique isomorphism.

Definition 1.4 is sometimes summarized into the catch phrase

" A stack (over Diff*) is a sheaf* $M \mapsto \mathfrak{X}(M)$ *of groupoids".*

It is indeed a categorical extension of the notion of sheaves of sets.

Proposition 1.5 *Examples 1, 2 and 3 are stacks over* Diff.

The above proposition is a consequence of the fact that a map $f : M \to N$ is determined by its value on a cover of M and that fiber bundles are obtained by gluing local data (i.e. trivializations) satisfying a cocycle condition.

Example 4 (Moduli stack of Riemann surfaces) Define $\mathfrak{M}_g$ to be the following category: objects are fiber bundles $X \to U$ endowed with a smoothly varying fiberwise complex structure, such that all fibers are Riemann surfaces of genus g. Morphisms are commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

such that $X \to Y \times_V U$ is a conformal isomorphism. The functor $\mathfrak{M}_g \to \mathrm{Diff}$ is again $(X \to U) \mapsto U$. This is a stack over Diff called the *moduli stack* of Riemann surfaces of genus g. An object $X \to U$ of $\mathfrak{M}_g$ is a family of Riemann surfaces parametrized by S.

1.3 The (2-)category of stacks

Categories fibered in groupoids over Diff (and thus stacks over Diff as well) are naturally organized into a *2-category.* The latter means that in addition of the structure of a category, we also have morphisms (often called transformations or 2-morphisms) between morphisms (with same source and target). More precisely, this 2-category is given by:

A ***morphism*** f between two fibered categories (or stacks) $\mathfrak{X} \overset{\pi_{\mathfrak{X}}}{\to} \mathrm{Diff}$ and $\mathfrak{Y} \overset{\pi_{\mathfrak{Y}}}{\to} \mathrm{Diff}$ is a functor $f : \mathfrak{X} \to \mathfrak{Y}$ between the underlying categories such that $\pi_{\mathfrak{Y}} \circ f = \pi_{\mathfrak{X}}$. Given two such morphisms $f, g : \mathfrak{X} \to \mathfrak{Y}$, a ***2-morphism*** $\varphi : f \Rightarrow g$ between them is a natural transformation of functors φ from f to g such that the composition $\pi_{\mathfrak{Y}} \circ \varphi$ is the identity transformation from $\pi_{\mathfrak{X}}$ to itself[4].

Since all maps in $\mathfrak{X}(M)$ are invertible, one has

Lemma 1.6 *With morphisms and 2-morphisms as above, categories fibered in groupoids over* Diff *form a 2-category* $\mathfrak{pst}$[5]*. The 2-morphisms in* $\mathfrak{pst}$ *are automatically invertible.*

In particular, the morphisms $\mathrm{Hom}_{\mathfrak{pst}}(\mathfrak{Y}, \mathfrak{X})$ between two stacks form a *groupoid.*

The classical Yoneda lemma from category theory has an analogue for stacks.

Lemma 1.7 (Yoneda lemma) *Let M be manifold and $\mathfrak{X}$ a category fibered in groupoids over* Diff*. The natural functor* $\mathrm{Hom}_{\mathfrak{pst}}([M], \mathfrak{X}) \to \mathfrak{X}(M)$ *is an equivalence of groupoids.*

[4] in other words, it is given by morphisms $\eta_A : f(A) \to g(A) \in \mathfrak{Y}(M)$ for every $M \in \mathrm{Diff}$ and object $A \in \mathfrak{X}(M)$ such that $\pi_{\mathfrak{Y}}(\eta_A) = id : M \to M$ and for any $H : A \to B \in \mathfrak{X}$, one has $g(H) \circ \eta_A = \eta_B \circ f(H) \in \mathfrak{Y}$

[5] the notation is chosen to remind of category fibered in groupoids as a *pre*-stack

The natural functor maps $f : [M] \to \mathfrak{X}$ to $f(M \overset{id}{\to} M)$ using the canonical object $id : M \to M$ in the stack $[M]$ (Example 1). Note that by Remark 1.3, given an object $\phi : U \to M$ in $[M]$ and an object $A \in \mathfrak{X}(M)$, we can construct an object $A_{|U} \in \mathfrak{X}(U)$; this is how one construct a morphism $[M] \to \mathfrak{X} \in \mathfrak{pst}$ out of the groupoid $\mathfrak{X}(M)$.

Remark 1.8 The Yoneda lemma implies that the functor Diff $\to \mathfrak{pst}$ is *fully faithful*. That is, we can think of **the category of smooth manifolds as a full subcategory of stacks** over Diff. In particular:

$$\mathrm{Hom}_{\mathfrak{pst}}([M],[N]) \;\cong\; C^\infty(M,N).$$

Stacks are thus a generalization of smooth manifolds.

Example 5 (Principal G-bundles) Let G be a Lie group and recall example 2. By the Yoneda lemma we have a *bijection between the groupoid of stack maps $[M] \to [pt/G]$ and the groupoid $[pt/G](M)$ of principal G-bundles over M*:

$$\mathrm{Hom}_{\mathfrak{pst}}([M],[pt/G]) \cong [pt/G](M).$$

In particular, isomorphisms classes of principal G-bundles are in one-to-one correspondance with (natural equivalence classes of) maps of stacks $[M] \to [pt/G]$. If $P \to M$ is a principal G-bundle over M, the associated functor $[M] \to [pt/G]$ sends an object $U \overset{\phi}{\to} M \in [M](U)$ to the pullback principal bundle $\phi^*(P) \to U$. We will see later that $[pt] \to [pt/G]$ is a principal G-bundle and that the latter construction is nothing more than the pullback of this principal bundle under the stack morphism $[M] \to [pt/G]$.

The above construction is a “geometric” version of the following result in topology: isomorphism classes of principal G-bundles over a CW-complex X are in bijection with homotopy classes of maps from X to BG. Here BG is the classifying space of G (which is in general an infinite dimensional CW-complex). There is a (contractible) G-bundle EG over BG. The isomorphism is obtained by pulling back this bundle along a map $X \to BG$. The stack point of view in some sense avoids dealing with homotopy classes of continuous maps and infinite dimensional spaces.

We now define the correct notion of pullback for stacks which is really a kind of “homotopy pullback”. Recall that the groupoid $\mathfrak{X}(U)$ models a quotient in which we identify two objects of $\mathfrak{X}(U)$ whenever there is a (necessarily invertible) morphism between them. Thus, when defining the fiber product of two stacks $\mathfrak{X}, \mathfrak{Y}$ over a third one $\mathfrak{Z}$, objects of $\mathfrak{Z}$ shall be thought as being the same when they are isomorphic (and not just equal). The precise definition is:

Definition 1.9 (Fiber product of stacks) Let $F : \mathfrak{X} \to \mathfrak{Z}$ and $G : \mathfrak{Y} \to \mathfrak{Z}$ be two (pre-)stacks morphisms. We define $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ to be the category fibered in groupoids with objects all triples

$$\big\{\big(x, y, F(x) \overset{\alpha}{\to} G(y)\big) \,|\, U \in \mathrm{Diff},\ x, y \text{ are objects in } \mathfrak{X}(U), \mathfrak{Y}(U) \text{ and } \alpha \text{ an arrow in } \mathfrak{Z}(U)\big\}$$

and with morphisms

$$\mathrm{Mor}_{\mathfrak{X}\times_{\mathfrak{Z}}\mathfrak{Y}}\big((x,y,\alpha),(x',y',\alpha')\big) = \left\{ \begin{array}{ccc} (u,v) & | & u : x \to x',\ v : y \to y' \text{ s.t.:} \\ & & \begin{array}{ccc} F(x) & \xrightarrow{F(u)} & F(x') \\ \alpha\downarrow & \circlearrowleft & \downarrow\alpha' \\ G(y) & \xrightarrow[G(u)]{} & G(y') \end{array} \end{array} \right\}.$$

The functor $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathrm{Diff}$ is the obvious one.

We do have a 2-commutative diagram:

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \arrow[r] \arrow[d] & \mathfrak{Y} \arrow[d, "G"] \arrow[dl, Rightarrow] \\ \mathfrak{X} \arrow[r, "F"] & \mathfrak{Z} \end{tikzcd}$$

and $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ *satisfy a (2-categorical) universal property similar to usual fibered product.*

Note that the fiber product of stacks over Diff is always a stack.

2 Differentiable stacks and Lie groupoids

2.1 Differentiable stacks *via* atlases

Stacks over Diff are, in general, *not* geometric enough in order to do differential geometry, see Example 7 below. This is why we need the following definition.

Definition 2.1 A stack $\mathfrak{X}$ over Diff is called a **differentiable stack** if there exists a *representable epimorphism* $p : [X] \to \mathfrak{X}$, where X is a manifold. That p is a representable epimorphism is equivalent to the the condition that, *for all maps* $[U] \to \mathfrak{X}$*, where* $U \in$ Diff*, the fibered product* $[X] \times_{\mathfrak{X}} [U]$ *is isomorphic to a manifold* (seen as a stack and denoted $X \times_{\mathfrak{X}} U$) and *the induced map* $X \times_{\mathfrak{X}} U \to U$ *is a surjective submersion* (in Diff).

In that case, $[X] \to \mathfrak{X}$ is called an **atlas for** $\mathfrak{X}$.

One can define similarly other classes of *geometric or topological* stacks.

One shall be careful that the fibered product of differentiable stacks is not necessarily a differentiable stack (though it is a stack over Diff).

Remark 2.2 (about representability) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of stacks. That f *is representable* precisely means that, for every morphism $U \to \mathfrak{Y}$ from a manifold U, the fiber product $U \times_{\mathfrak{X}} \mathfrak{Y}$ is equivalent to a manifold.

Representable maps are very useful for the following reason: *any property* $\mathbf{P}$ *of smooth maps between manifolds which is invariant under base change can be defined for an arbitrary representable morphism of stacks.* More precisely, we say that a representable morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is $\mathbf{P}$, if for every map $[U] \to \mathfrak{Y}$ (with U a manifold), the base extension $f_U : U \times_{\mathfrak{Y}} \mathfrak{X} \to U$ is $\mathbf{P}$ as a map in Diff[6].

For instance, one can define embedding, open maps, etale maps and so on. In particular, this allows to mimick most construction of differential geometry for *differentiable* stacks.

Remark 2.3 (about epimorphisms) The condition (for f) of being an *epimorphism* is a 2-categorical analogue of the usual notion in category theory. It is equivalent to requiring that the image $\mathrm{im}(f)$, that is the smallest substack of $\mathfrak{Y}$ through which f factorizes, is equal to $\mathfrak{Y}$.

Equivalently, it means, that for every manifold U, there exists an open cover V, such that every object $Y \in \mathfrak{Y}(V)$ can be lifted, up to isomorphism, to some $X \in \mathfrak{X}(V)$.

For example, in the case where $\mathfrak{X} = [X]$ and $\mathfrak{Y} = [Y]$ are honest manifolds, it is equivalent to requiring that the induced map $X \to Y$ admits local sections.

The second condition is often the easiest to check in practice. In particular, it implies it is sufficient to check the condition of being a representable epimorphism on an open cover.

Example 6 *Examples 1, 2, 3 and 4 above are differentiable stacks.*

An atlas for the stack $[M]$ is given by the identity map $M \to M$ or by any surjective submersion $U \to M$.

Let see with a bit more detail the case of Example 2. We claim that there is a canonical map $\epsilon : [pt] \to [pt/G]$ which is an atlas. An object of $[pt]$ is simply a manifold U. We define $\epsilon(U) := U \times G \to U$, the trivial principal G-bundle over U, and similarly $\epsilon(U \stackrel{\phi}{\to} V)$ is the obvious map between trivial bundles.

[6] here we use the Yoneda lemma to identify smooth maps and maps between the associated stacks

Let M be a manifold and $[M] \to [pt/G]$ a stack morphism. By example 5, we identify it with a principal G-bundle $q : P \to M$. Then, the fibered product $[pt] \times_{[pt/G]} [M]$ is the category with objects

$$\left\{ \left(\phi : U \to M, \quad \begin{array}{ccc} U \times G & \xrightarrow{\simeq} & \phi^*(P) \\ \Big\downarrow{\scriptstyle \epsilon(U)} & & \Big\downarrow{\scriptstyle \phi^*(q)} \\ U & \xrightarrow{id} & U \end{array} \right) \mid \phi \text{ is smooth and the diagram is } \textit{cartesian} \text{ and the top map is } G\text{-equivariant} \right\}.$$

In particular, $\phi^*(P) \to U$ is trivialisable (which puts some restriction on ϕ). Assume first $q : P \to M$ is trivial, then the above fibered product has objects the set of maps $\phi : U \to M \times G$ and thus $[pt] \times_{[pt/G]} [M] \cong [M \times G]$. In general, we can cover M by open subsets U_i, such that $q_{|u_i} : P_{|U_i} \to U_i$ can be trivialised and thus $[pt] \times_{[pt/G]} [U_i] \cong [U_i \times G]$. Using the descent condition for stacks, we get that $[pt] \times_{[pt/G]} [M] \cong [P]$ (which is indeed a manifold).

In particular the *canonical map* $[pt] \to [pt/G]$ *exhibits* pt *as a "trivial principal* G*-bundle over the stack* $[pt/G]$*"* and the above analysis shows that a principal bundle $P \to M$ viewed as a map of stacks $f : [M] \to [pt/G]$ is the pullback of $pt \to [pt/G]$ along f.

Example 7 Let T be a topological space. Then we can define a category fibered in groupoids $[T]$ over Diff as in Example 1. Objects of $[T]$ are all *continuous* maps $f : U \to T$ and morphisms are commutative diagrams $\begin{array}{ccc} U & \xrightarrow{f} & T \\ \Big\downarrow{\scriptstyle \psi} & \nearrow{\scriptstyle g} & \\ V & & \end{array}$ in which ϕ is smooth. The functor $\pi : [T] \to$ Diff is as in Example 1 and one can check that $[T]$ is a stack over Diff. If T is not (diffeomorphic to) a manifold, then $[T]$ is *not* a differentiable stack[7].

For instance, assume P, Q are submanifolds of M whose intersection is *not* a manifold. Then the fibered product $[P] \times_{[M]} [Q]$ is isomorphic to $[P \cap Q] \in \mathfrak{pst}$ and is thus not a differentiable stack.

2.2 Differentiable stacks *via* Lie groupoids

Let $p : X \to \mathfrak{X}$ be an atlas for a differentiable stacks. Then we can make the fiber product

$$\begin{array}{ccc} [X] \times_{\mathfrak{X}} [X] & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & \mathfrak{X} \end{array} \tag{2.1}$$

and we know that $[X] \times_{\mathfrak{X}} [X]$ is isomorphic to a manifold $X \times_{\mathfrak{X}} X$ and, further, that the left down arrow and top arrow are *surjective submersions*. We respectively denote them $s, t : X \times_{\mathfrak{X}} X \to X$. In particular the induced maps $p \circ s, p \circ t : X \times_{\mathfrak{X}} X \to X \to \mathfrak{X}$ are also representable epimorphisms and the fiber product $[X \times_{\mathfrak{X}} X] \times_{s,\mathfrak{X},t} [X \times_{\mathfrak{X}} X]$ is also a manifold. By universal property of fibered products, we get two smooth

[7] it is however a *topological stack*

maps 1_X and m_X given by the diagrams:

$$\begin{tikzcd} {[X]} \arrow[r, dashed, "1_X"'] \arrow[rr, bend left, "id"] \arrow[dr, bend right, "id"'] & {[X] \times_{\mathfrak{X}} [X]} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ & X \arrow[r, "p"] & \mathfrak{X}, \end{tikzcd} \qquad \begin{tikzcd} {[X \times_{\mathfrak{X}} X] \times_{s,\mathfrak{X},t} [X \times_{\mathfrak{X}} X]} \arrow[r, dashed, "m_X"'] \arrow[rr, bend left] \arrow[dr, bend right] & {[X] \times_{\mathfrak{X}} [X]} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ & X \arrow[r, "p"] & \mathfrak{X}. \end{tikzcd}$$

Finally we also have the flip map $I : X \times_{\mathfrak{X}} X \to X \times_{\mathfrak{X}} X$ exchanging the two factors of $X \times_{\mathfrak{X}} X$.

Proposition 2.4 *The structure maps above make $X \times_{\mathfrak{X}} X \overset{t}{\underset{s}{\rightrightarrows}} X$ a Lie groupoid as in Definition 2.5 below.*

Definition 2.5 (Lie groupoids) A Lie groupoid $X_1 \overset{t}{\underset{s}{\rightrightarrows}} X_0$ is a groupoid in which the objects and morphisms have a structure of smooth manifolds such that s, t are surjective submersions (hence $X_2 := X_1 \times_{s,X_0,t} X_1$ inherits a smooth structure) and all structure maps (the multiplication $m : X_2 \to X_1$, the unit $1 : X_0 \to X_1$ and inverse map $I : X_1 \to X_1$) of the groupoid structure are smooth maps.

We now explain a converse of the Proposition 2.4 which is given by the stack of torsors of a groupoid:

Example 8 (the quotient stack of a Lie groupoid) Let $X_1 \overset{t}{\underset{s}{\rightrightarrows}} X_0$ be a Lie groupoid. We can consider a category fibered in groupoids $[X_0/X_1]$ whose objects are all principal X_1-bundles, that is the following data:

- a surjective submersion $q : P \to U \in \mathrm{Diff}$,
- a smooth map $\psi : P \to X_0$ together with an action $\rho : P \times_{\phi,X_0,s} X_1 \to P$ such that

$\psi(\phi(p,\gamma)) = t(\gamma)$, $q(\phi(p,\gamma)) = q(p)$ for all $(p,\gamma) \in P \times_{\phi,X_0,s} X_1$ and such that, for all pairs (p,p') with $q(p) = q(p')$, there exists an unique $\gamma \in X_1$ such that $p' = \rho(p,\gamma)$. The morphism in this category are given by commutative diagram of smooth maps $\begin{tikzcd} P \arrow[r, "F"] \arrow[d] & Q \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd}$ such that F is X_1-equivariant.

The functor $[X_0/X_1] \to \mathrm{Diff}$ maps $P \to U$ to U and a commutative diagram to f.

Note that any smooth map $\varphi : U \to X_0$ defines a principal X_1-bundle given by $\varphi^*(X_1) \xrightarrow{\varphi^*(t)} U$. Further, every principal X_1-bundle is locally of this form. It follows that $[X_0/X_1]$ is a stack over Diff.

One should note that this construction is just a generalisation of Example 3, which is the special case given by the transformation groupoid $M \times G \rightrightarrows M$ with source map the projection to M and target map the G-action.

The rule $(\varphi : U \to X_0) \mapsto \varphi^*(X_1) \xrightarrow{\varphi^*(t)} U$ defines a *stack morphism* $[X_0] \to [X_0/X_1]$.

Theorem 2.6 *Let $X_1 \overset{t}{\underset{s}{\rightrightarrows}} X_0$ be a Lie groupoid and $\mathfrak{X}$ be a differentiable stack.*

- *The map $[X_0] \to [X_0/X_1]$ is a representable epimorphism and the fibered product $[X_0] \times_{[X_0/X_1]} [X_0]$ is isomorphic to $[X_1]$.*
- *If $p : X \to \mathfrak{X}$ is an atlas, then $\mathfrak{X}$ is isomorphic to the stack of torsors $[X/X \times_{\mathfrak{X}} X]$ of the Lie groupoid given by the atlas in Proposition 2.4.*

In other words, ***differentiable stacks are those stacks which are isomorphic to quotient stacks of Lie groupoids****.*

For instance, if G acts on a group M, then the canonical map $[M] \to [M/G]$ is an atlas.

The **dimension of a differentiable stack** $\mathfrak{X}$ is $2\dim(X_0) - \dim(X_1)$ where $X_1 \rightrightarrows X_0$ is any Lie groupoid such that $\mathfrak{X} \cong [X_0/X_1]$ (this number is indeed indepent of the choice of a particular Lie groupoid and is constant on any connected component of the stack).

In particular, if G acts on M, $\dim([M/G]) = \dim(M) - \dim(G)$ which can be negative.

There is no uniqueness of the Lie groupoid in the Theorem. Indeed, a differentiable stack is isomorphic to the quotient stack of infinitely many differents Lie groupoids !

Example 9 (Čech groupoid) Let M be a manifold and $(U_i)_{i\in I}$ an open cover of M. We define the Čech groupoid $\left(\coprod_{i,j} U_i \cap U_j\right) \rightrightarrows \coprod_k U_k$ as follows. Denoting $x_{i,j}$ an element $x \in M$ which lies in both U_i and U_j and x_k a point $x \in M$ which lies in U_k, we set

$$s(x_{ij} = x_i), \qquad t(x_{i,j}) = x_j, \qquad m(x_{i,j}, x_{j,k}) = x_{i,k}, \qquad 1(x_k) = x_{k,k}.$$

Denote $u : \coprod_{k\in I} U_k \to M$ the canonical submersion. Then the functor $\left(V \stackrel{\phi}{\to} \coprod_{k\in I} U_k\right) \mapsto \left(V \stackrel{u\circ\phi}{\to} M\right)$ is an equivalence $[\coprod U_k / \coprod U_i \cap U_j] \cong [M]$ of stacks.

Example 10 (Orbifolds) In terms of differentiable stacks, *orbifolds can be defined as differentiable stacks, which, locally are isomorphic to differentiable stacks of the form $[M/G]$ where G is a finite group.* The latter definition is equivalent to the following one in terms of Lie groupoids:

Definition 2.7 A differentiable stack $\mathfrak{X}$ is an orbifold if it isomorphic to a quotient stack $[X_0/X_1]$ of a Lie groupoid such that the source and target maps $s, t : X_1 \to X_0$ are étale and further the map $X_1 \stackrel{(s,t)}{\longrightarrow} X_0 \times X_0$ is proper.

We recall that a common definition of orbifolds[8] is the following: an orbifold is a topological space X endowed with an (equivalence class of) orbifold atlas. That is, X has an open covering $(U_i)_{i\in I}$ together with homeomorphisms $U_i \cong Y_i/G_i$, where Y_i are open connected subspaces of $\mathbb{R}^n$, G_i are finite subgroups of the group of diffeomorphisms of Y_i, satisfying the following condition: for any $x \in U_i \cap U_j$, there exists diffeomorphic neighborhoods W_{ij}, W_{ji} of Y_i, Y_j (containing the pre-image of x) such that the following diagram commutes:

$$\begin{tikzcd} Y_i \supset W_{ij} \arrow[rr, "\cong"] \arrow[dr] & & W_{ji} \subset Y_j \arrow[dl] \\ & U_i \cap U_j & \end{tikzcd}$$

Out of an atlas, one construct a groupoid $Y \rightrightarrows \coprod Y_i$. The space Y is, roughly speaking, the space of all triples $(x, y\phi_{ij})$ where $x \in Y_i$, $y \in Y_j$ and $\phi_{ij} : W_{ij} \to W_{ji}$ is a germ of an homeomorphism as above, mapping x to y. The source and target maps are the obvious projections to Y_i, Y_j and the multiplication is given by $m\big((x, y, \phi_{ij}), (y, z, \phi_{jk})\big)$. One can check that the associated Lie groupoids satisfies the condition of Definition 2.7.

A ***morphism*** $F : \left(X_1 \rightrightarrows X_0\right) \to \left(Y_1 \rightrightarrows Y_0\right)$ ***of Lie groupoids*** (i.e. a morphism of groupoids such that the underlying maps $F_i : X_i \to Y_i$ on the manifolds of objects and morphisms are smooth) ***induces a morphism of stacks*** $F : [X_0/X_1] \to [Y_0/Y_1]$ in a canonical way.

Reciprocally, let $\mathfrak{X} \cong [X_0/X_1]$ and $\mathfrak{Y} \cong [Y_0/Y_1]$ be differentiable stacks. It is *not* true that every stack morphism $\mathfrak{X} \to \mathfrak{Y}$ comes from a Lie groupoid morphism from $\left(X_1 \rightrightarrows X_0\right) \to \left(Y_1 \rightrightarrows Y_0\right)$; the reason is that,

[8] à la Satake

because of the descent property, a morphism of stacks can be defined on a cover of $\mathfrak{X}$ and thus on a cover of X_0 as well. We thus have to add the possibility of refining a Lie groupoid (by passing to an open cover of X_0) before using a Lie groupoid morphism to recover all stacks morphisms. This motivates:

Definition 2.8 A morphism $F : \big(X_1 \rightrightarrows X_0\big) \to \big(Y_1 \rightrightarrows Y_0\big)$ of Lie groupoids is a **Morita morphism** if

- the induced map $F_0 : X_0 \to Y_0$ is a surjective submersion and
- the diagram $\begin{tikzcd} X_1 \arrow[r, "F_1"] \arrow[d, "(s,t)"'] & Y_1 \arrow[d, "(s,t)"] \\ X_0 \times X_0 \arrow[r, "(F_0,F_0)"] & Y_0 \times Y_0 \end{tikzcd}$ is cartesian.

A Morita morphism is thus an *equivalence of groupoids* which is induced by a smooth functor[9]. From this, one can get

Lemma 2.9 • *A* ***Morita morphism*** *$F : \big(X_1 \rightrightarrows X_0\big) \to \big(Y_1 \rightrightarrows Y_0\big)$ of Lie groupoids* ***induces an isomorphism of quotient stacks*** *$[X_0/X_1] \xrightarrow{\cong} [Y_0/Y_1]$.*

- *Two Lie groupoids morphisms $F, G : \big(X_1 \rightrightarrows X_0\big) \to \big(Y_1 \rightrightarrows Y_0\big)$ which are naturally equivalent induces equivalent stacks morphisms* $\begin{tikzcd} {[X_0/X_1]} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & {[Y_0/Y_1]} \end{tikzcd}$.

Example 11 The equivalence of stacks of Example 9 is induced by a Morita morphism of Lie groupoids.

Example 12 Let M be a manifold. The Lie groupoid of pairs is $M \times M \rightrightarrows M$ with source and target map given by the canonical projections, unit given by the diagonal and multiplication given by $m((x,y),(y,z)) = (x,z)$. There is a unique Lie groupoid morphism $\big(M \times M \rightrightarrows M\big) \to pt \rightrightarrows pt$ which is a Morita map. In particular, $[M/M \times M] \cong [pt]$.

Example 13 Let $P \to M$ be a principal G-bundle where G is a Lie group and let $(U_i)_{i\in I}$ be a trivializing cover. Then the stack morphism $[M] \to [pt/G]$ (given by the bundle according to Example 5) is given by the zigzag

$$\big(M \rightrightarrows M\big) \leftarrow \big(G \times \coprod U_i \cap U_j \rightrightarrows \coprod U_i\big) \to \big(G \rightrightarrows pt\big)$$

where the left arrow is the canonical map of Lie groupoids[10] given by the trivialization of P, which is a Morita morphism. The right arrow is simply given by the projection onto G.

Now we state more precisely the relationship between functors between Lie goupoids and maps of stacks.

Proposition 2.10 *Let $\mathfrak{X} \cong [X_0/X_1]$ and $\mathfrak{Y} \cong [Y_0/Y_1]$ be differentiable stacks. Any stack morphism $\mathfrak{X} \to \mathfrak{Y}$ is naturally equivalent to a zigzag*

$$\big(X_1 \rightrightarrows X_0\big) \xleftarrow{\simeq} \big(Z_1 \rightrightarrows Z_0\big) \xrightarrow{F} \big(Y_1 \rightrightarrows Y_0\big)$$

where the left arrow is a Morita morphism *and the right arrow a Lie groupoid morphism.*

3 Elementary examples of moduli problem and moduli stack

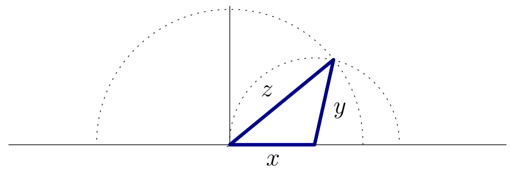
By a moduli space/problem of some algebro-geometric structure (call it $\mathcal{G}$), one usually means a topological space (or parametrization) of this structure which one would be allowed to study *in families*. A family of $\mathcal{G}$-objects over a manifold[11] S is then a fiber bundle $P \to S$ whose fibers are given the structure of an object

[9] it is not necessarily an equivalence of smooth groupoids since there are only local sections of F_0 in general

[10] the multiplication in the middle groupoid uses the 2-cocycle $U_{ij} \to G$ defined by the principal bundle $P \to M$

[11] or analytic space or topological space or so, depending on the precise geometric context one is interested in

Figure 1: The fiber (in U^{ord}) at a point (x, y, z) of T^{ord}



of $\mathcal{G}$, which is assumed to vary *smoothly.* More precisely, a *fine moduli space* is then a space/manifold $B_{\mathcal{G}}$ (whose points are isomorphism classes of objects of $\mathcal{G}$) together with a *universal family* $U \to B_{\mathcal{G}}$, that is a fiber bundle such that each fiber over m carries naturally the algebro-geometric structure defined by m. This data shall be organized so that, for any family $P \to S$ of $\mathcal{G}$-objects, there exists an *unique* smooth map $f : S \to B_{\mathcal{G}}$ such that the diagram
$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \arrow[d] \\ S \arrow[r, "f"] & B_{\mathcal{G}} \end{tikzcd}$$
is cartesian. In other words, P is the pullback along f of the universal family U.

In many cases[12], a fine moduli space do *not* exists, but we can find rather a differentiable stack (i.e. a moduli stack) representing the moduli problem.

The moduli problem is encoded in a category whose objects are families of $\mathcal{G}$-objects $P \to S$ and morphisms are cartesian diagrams
$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[d] \\ S \arrow[r, "f"] & S' \end{tikzcd}.$$

Example 14 We have already seen an example: the stack $[pt/G]$ is the stack of G-torsors, that is, the moduli stack encoding families of objects which are free G-sets of rank one. Indeed, such a family over M, that is a G-torsor over M, is uniquely determined by a stack morphisms $[M] \to [pt/G]$.

More generally, principal G-bundles over a stack $\mathfrak{X}$ are in bijection with the maps of stacks $\mathfrak{X} \to [pt/G]$; out of a map, one obtains a principal G-bundle as the pullback $\mathfrak{X} \times_{[pt/G]} [pt] \to \mathfrak{X}$.

Example 15 (Moduli of triangles) We wish to consider the problem of classifying (plane) triangles up to isometries. We are going to compare two such problem:

- the one in which we consider *ordered triangles*, that is in which we have specified the name of each edge of the triangle,
- and the one of *unordered triangle*, which really correspond to our initial problem.

To do this, first, we need to put a topology on isomorphism classes of triangles so that we consider

$$T^{ord} := \{(x, y, z) \in (0, +\infty)^3 \,|\, x + y > z, \quad y + z > x, \quad z + x > y\}$$

which correspond to a triangle whose first edge is of length x, the second y and the third one z. It is an open submanifold of $\mathbb{R}^3$, whose points are indeed in one to one correspondence with *isometry classes of ordered triangles.* The symmetric group Σ_3 on 3 letters acts on T^{ord} and we define $T := T^{ord}/\Sigma_3$, which is a topological space whose points are in one to one correspondence with *isometry classes of triangles.*

Now define U^{ord} to be the subspace of $T^{ord} \times \mathbb{R}^2$ consisting of tuples of a point $(x, y, z) \in T^{ord}$ and a standard triangle of respective lengths x, y, z in $\mathbb{C}$ such that x is on the real axis, starting at 0 and the triangle lies in the upper-half plane (see Figure 1). We also define $U := U^{ord}/\Sigma_3$ and we get canonical families $U^{ord} \to T^{ord}$ and $U \to T$ whose fibers are respectively ordered triangles and triangles.

[12] precisely when the objects we are studying have too many isomorphisms

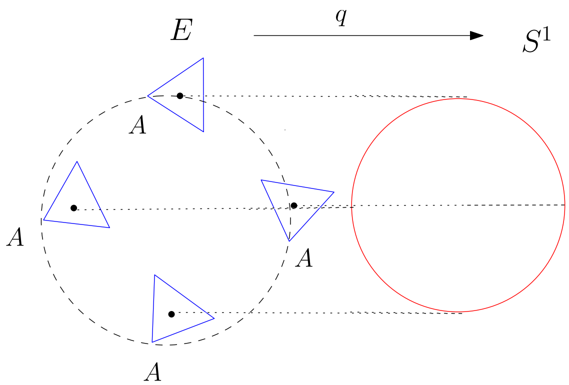


Figure 2: A non-trivial family of equilateral triangles over S^1, which are rotating by an angle of $2\pi/3$

The family $U^{ord} \to T^{ord}$ is a fine moduli space. Indeed, let $P^{ord} \to S^{ord}$ be a smooth family of ordered triangle, that is a fiber bundle together with a smoothly varying metric on the fibers such that each fiber is isometric to an ordered triangle. There is an unique smooth map $j : S^{ord} \to T^{ord}$ identifying P^{ord} with the pullback $j^*(U^{ord})$; this map j is given by $j(s) = U^{ord}_s$ the fiber of s. This construction works, because we have no choices to make when defining the ordered triangle of lengths (x, y, z).

However, the construction does *not* work for (unordered) triangles. Indeed, consider the family $q : E \to S^1$ (depicted in figure 2) which is a family of equilateral triangles over the circle S^1, such that the triangles makes a rotation of angle $2\pi/3$ when making one full revolution around S^1.

The family $q : E \to S^1$ is *not* a constant family nor isomorphic to a constant one. However, the canonical map (induced by j as above) $S^1 \to T$ given by $t \mapsto E_t$ the fiber at t is *constant* (and equal to the unique equilateral triangle of length 1 on each edge). Thus the *pullback of U along this map is the constant family* $S^1 \times \Delta$ where Δ is the equilateral triangle of legnth 1. This shows that $U \to T$ is not a fine moduli space and in fact no such fine moduli space could exists for this problem (it would have to be given by a similar construction).

Nevertheless, we can construct the *differentiable stack* $\mathfrak{T} := [T^{ord}/\Sigma_3]$ (that is the stacky quotient of T^{ord} by the action of Σ_3 instead of the naive quotient) as defined by Example 2. By the Yoneda lemma, for any manifold M, $\mathrm{Hom}_{\mathtt{pst}}([M], \mathfrak{T})$ is the groupoid of principal Σ_3-bundles (i.e. covering space) $Q \to M$ over M, together with a Σ_3 equivariant map $Q \to T^{ord}$. In fact, this stack is indeed the stack encoding the moduli problem of triangles as we now explain.

First, note that we could also encode triangles (up to isometry) in a slightly different way. Define

$$B^{ord} := \{(A, B, C) \in (\mathbb{R}^2)^3, \mid (A, B, C) \text{ is an affine base of } \mathbb{R}^2\}$$

which is an open submanifold of $\mathbb{R}^6$. Let Iso be the group of affine isometry of $\mathbb{R}^2$ (that is the group generated by translations, rotations and symmetries), which acts diagonally on B^{ord}. Then an isomorphism class of ordered triangles is a point in the quotient B^{ord}/Iso. This action is compatible with the natural Σ_3 action, so that an isomorphism class of triangle is also given by a point in the quotient $B^{ord}/\mathrm{Iso} \times \Sigma_3$.

Finally, let us consider the moduli problem of triangles, that is the category $\mathfrak{M}_T$ whos objects are smooth families $P \to S$ of triangles over the manifold S and with morphisms given by cartesian squares
$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[d] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$
where f is smooth. This is a stack over Diff, with the obvious functor $\mathfrak{M}_T \to \mathrm{Diff}$ given by $(P \to S) \mapsto S$. We define similarly the stack $\mathfrak{M}^{ord}_T$ over Diff of moduli problems of ordered triangles.

The following proposition relates all the above (differentiable) stacks.

Proposition 3.1 *There are isomorphisms of stacks over* Diff

- $\mathfrak{M}_T^{ord} \cong [T^{ord}] \cong [B^{ord}/\mathrm{Iso}]$;
- $\mathfrak{M}_T \cong [T^{ord}/\Sigma_3] \cong [B^{ord}/\mathrm{Iso} \times \Sigma_3]$

The first isomorphisms were explain above.

To see that $\mathfrak{M}_T \cong [T^{ord}/\Sigma_3]$, we recall that objects of $[T^{ord}/\Sigma_3](M)$ are principal Σ_3-bundles $Q \to M$ over M, together with a Σ_3 equivariant map $\phi : Q \to T^{ord}$. We thus get a principal bundle $\phi^*(U^{ord}) \to Q$. Since ϕ is equivariant, we can mod out by the Σ_3 action to get the bundle $\big(\phi^*(U^{ord})/\Sigma_3\big) \to M$, which is precisely an object of $\mathfrak{M}_T(M)$. This extends trivially into a functor which is seen to be an equivalence of groupoids. Indeed, given a family of triangle $P \to S$, one construct the covering space $Q \to S$ whose fiber Q_s consists of all possibles orderings of the triangles in the fiber P_s of $P \to S$; this yields the reverse equivalence of groupoids.

Finally we have a canonical surjective submersion $B^{ord} \to T^{ord}$ (which to an affine base associates the lengths of the edges between the vectors). This map induces a Morita map $\big(B^{ord} \times (\mathrm{Iso} \times \Sigma_3) \rightrightarrows B^{ord}\big) \longrightarrow \big(T^{ord} \times \Sigma_3 \rightrightarrows T^{ord}\big)$ hence an equivalence of differentiable stacks.

Example 16 (The moduli stack of elliptic curves) An elliptic curve (over $\mathbb{C}$) is a Riemann surface of genus 1 together with a choice of 0. Every elliptic curves is isomorphic to the quotient $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ of the group $\mathbb{C}$ by a lattice. In the above formula, one can assume τ is in the upper half plane $\mathbb{H}$. Two quotients $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, $\mathbb{C}/\mathbb{Z} \oplus \tau'\mathbb{Z}$ are isomorphic if and only if there exists $g \in SL_2(\mathbb{Z})$ such that $\tau' = g(\tau)$. Here g acts on the upper-half plane $\mathbb{H}$ by $g(\tau) = \frac{a\tau+b}{c\tau+d}$ ($a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$). This suggests to define the ***moduli stack of elliptic curves as*** $\mathfrak{M}_{\mathfrak{ell}} := [\mathbb{H}/SL_2(\mathbb{Z})]$. Note that one uses the action of $SL_2(\mathbb{Z})$ and not of $PSL_2(\mathbb{Z})$ because every elliptic curve has an automorphism of order 2 induced by $z \mapsto -z$ and so we wish to take it into account in the stack structure.

One can also define a moduli stack of elliptic curves as in example 4: let $\tilde{\mathfrak{M}}_1$ be the stack over Diff with objects fiber bundles $X \to U$ endowed with a smoothly varying fiberwise complex structure, whose fibers are Riemann surfaces of genus 1, and equipped with a smooth section (prescribing the 0 of the elliptic curves) $\sigma_U : U \to X$. Morphisms are cartesian squares
$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d] & Y \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd}$$
which also commutes with sections, that is: $\overline{f} \circ \sigma_U = \sigma_V \circ f$.

Similarly, but more complicated, than the case of triangles, we have:

Proposition 3.2 *There is a stack isomorphism* $\mathfrak{M}_{\mathfrak{ell}} \cong \tilde{\mathfrak{M}}_1$.

A moduli stack has an underlying space of isomorphisms classes of objects:

Definition 3.3 (Coarse moduli space of a stack) Let $\mathfrak{X}$ be a differentiable stack and assume $\mathfrak{X} \cong [X_0/X_1]$. The naive quotient space $\mathfrak{X}_{coarse} := X_0/X_1$ is called the *coarse space of* $\mathfrak{X}$. It is independent (up to homeomorphisms) of the choice of a groupoid presenting the stack $\mathfrak{X}$.

The coarse moduli space of a differentiable stack $[X_0/X_1]$ is precisely a topological space whose points are all isomorphisms classes of objects of the groupoid $X_1 \rightrightarrows X_0$.

Proposition 3.4 *There is a canonical map (of topological stacks)* $p : \mathfrak{X} \cong [X_0/X_1] \to X_0/X_1 = \mathfrak{X}_{coarse}$ *which has the* universal property *that, for any map* $f : \mathfrak{X} \to Y$*, there is an unique map* $f_{coarse} : \mathfrak{X}_{coarse} \to Y$ *such that* $f_{coarse} \circ p = f$.

For instance $\mathfrak{M}_{\mathfrak{ell}\,coarse} \cong \mathbb{H}/SL_2(\mathbb{Z}) \cong \mathbb{C}$ and the map $\mathbb{H} \to \mathfrak{M}_{\mathfrak{ell}} \xrightarrow{p} \mathbb{C}$ is the j-invariant of an elliptic curve.

Remark 3.5 One can always see/think of a differentiable stack $\mathfrak{X}$ as a *moduli stack/space*; precisely as the moduli stack classifying the objects of $\mathfrak{X}(pt)$.

4 Algebraic Topology of stacks

One can do algebraic topology for stacks in a way similar to algebraic topology for manifolds.

4.1 Classifying spaces of differentiable stacks

A manifold has an underlying topology; the same is true for differentiable stacks. Indeed, replacing Diff by the category of (nice enough) topological spaces, one obtains the notion of topological stacks. We do not really need to detail this notion here. The category of topological spaces embed in topological stacks and we have seen in example 7 how to interpret them as stacks over Diff. Similarly to the way one defines atlas for differentiable stacks, one can define classifying space/homotopy type for them:

Proposition 4.1 *Every differentiable stack $\mathfrak{X}$ has a* ***classifying space****, that is a* representable[13] epimorphism $[B] \to \mathfrak{X}$ *from a CW-complex B which is an* universal weak equivalence*: that is for any map $[T] \to \mathfrak{X}$ from a topological space T, the fiber product $[T] \times_{\mathfrak{X}} [B]$ is isomorphic to a topological space and the induced continuous map $[T] \times_{\mathfrak{X}} [B] \to T$ is a weak homotopy equivalence.*

A classifying space is unique up to homotopy equivalence (and isomorphisms of stacks). We can thus call it the **homotopy type of $\mathfrak{X}$**.

General construction: One can define homotopy groups, various (co)homology theories and so on for differentiable stacks by simply defining it to be the same as the one of its classifying space. For instance, one defines the singular homology groups $H_i(\mathfrak{X}, \mathbb{Z}) := H_i(B, \mathbb{Z})$. In fact, one can prove:

Proposition 4.2 *there is a bivariant theory for differentiable stacks; that is a theory encompassing homology, cohomology with their algebraic structure, intersection theory and Poincaré duality for stacks, and which generalizes the usual structures and constructions available for manifolds.*

Example 17 The classifying space of the stack $[pt/G]$ is (homotopy) equivalent to the usual classifying space BG of the group which is defined as the quotient of a contractible free G-space by the G-action. In particular

$$H^*([pt/G]) \cong H^*(BG) \cong H^*_G(pt).$$

More generally, for any G-manifold M, one has a ring isomorphism $H^*([M/G]) \cong H^*_G(M)$ the equivariant cohomology of M.

4.2 The de Rham complex for differentiable stacks

One can also define de Rham forms for *differentiable stacks.* These can be defined in purely intrinsic terms[14] of a differentiable stacks but there is also a nice explicit construction associated to each atlas, that is Lie groupoid presenting a differentiable stack. For a manifold M, we let $\Omega^i_{dR}(M)$ *be the degree i differential forms on M* and we write $d : \Omega^*_{dR}(M) \to \Omega^{*+1}_{dR}(M)$ for *de Rham differential.*

Let $\mathfrak{X}$ be differentiable stack and take a Lie groupoid $X_1 \rightrightarrows X_0$ such that $\mathfrak{X} \cong [X_0/X_1]$. We set $X_n = \{(x_1, \ldots, x_n) | t(x_i) = s(x_{i+1}) \ i = 1, \ldots, n-1\}$, the space of composables n-many arrows. Since s, t are submersions, X_n is naturally a smooth manifold and for any $i = 0, \ldots, n$, we get smooth maps (called the *face operators*)

$$\begin{tikzcd}
X_{n-1} & X_n \arrow[l, "d_0 = t \times id^{n-1}_{X_1}"'] \arrow[r, "d_n = id^{n-1}_{X_1} \times s"] \arrow[dl, "d_1 = m \times id^{n-2}_{X_1}"] \arrow[d, "d_i = id^{i-1}_{X_1} \times m \times id^{n-i-1}_{X_1}"] & X_{n-1} \\
X_{n-1} \quad \cdots & X_{n-1} & \cdots
\end{tikzcd}.$$

[13] in the sense of topological stacks

[14] namely by defining a complex of sheaves of forms over the stack $\mathfrak{X}$, which to $x \in \mathfrak{X}(U)$ associates $\Omega^*_{dR}(U)$

This way, we get a simplicial manifold, called the nerve[15]:

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_1 \rightrightarrows X_0$$

The space of *de Rham k-forms* of $X_1 \rightrightarrows X_0$ is

$$\Omega^k_{dR}(X_\bullet) := \bigoplus_{p+q=k} \Omega^p(X_q).$$

For each i, the face operator d_i induces a linear map $d_i^* : \Omega^p(X_q) \to \Omega^p(X_{q+1}) \subset \Omega^{p+q+1}_{dR}(X_\bullet)$, which commutes with de Rham differential of each X_n. Define $b := \sum(-1)^{i+p} d_i^* : \Omega^p_{dR}(X_\bullet) \to \Omega^p_{dR}(X_{\bullet+1})$ and let $\partial : \oplus\Omega^p(X_*) \xrightarrow{d} \Omega^{p+1}(X_*)$ be the differential induced by the de Rham differential on each X_n.

Lemma 4.3 *Let $\mathfrak{X}$ be differentiable stack and $X_1 \rightrightarrows X_0$ a Lie groupoid such that $\mathfrak{X} \cong [X_0/X_1]$.*

- *$b+\partial$ is a differential on $\Omega^n_{dR}(X_\bullet)$; that is $(b+d)^2 = 0$. We call $(\Omega^*_{dR}(X_\bullet), b+\partial)$ the* de Rham complex *of $X_\bullet$ and $H^*_{dR}(X_\bullet) := H^*(\Omega^*_{dR}(X_\bullet), b+\partial)$ is called the* de Rham cohomology *of $X_\bullet$*
- *If $\mathfrak{X} \cong [Y_0/Y_1]$, then $\Omega^n_{dR}(Y_\bullet)$ is canonically quasi-isomorphic to $\Omega^n_{dR}(X_\bullet)$.*

In particular one can set $(\Omega^*_{dR}(\mathfrak{X}), b+\partial) = (\Omega^*_{dR}(X_\bullet), b+\partial)$ in the derived category and the de Rham cohomology of $\mathfrak{X}$ is canonically isomorphic to $H^*_{dR}(X_\bullet)$.

Remark 4.4 One can also do similar construction with singular (co)chain complexes with any coefficient in place of de Rham complex.

Example 18 Let G be a discrete group. Then $\Omega^p_{dR}([pt/G]) \cong \mathbb{R}[G^p]$ and the de Rham complex is precisely the standard complex computing ($\mathbb{R}$-valued) the cohomology of the group. One recovers group cohomology with coefficient in $\mathbb{Z}$ by replacing de Rham complex by the singular cochain complex.

Example 19 Let (U_i) be a *good*[16] cover of a manifold M. We have the Čech groupoid $\coprod U_i \cap U_j \rightrightarrows \coprod U_k$ (example 9) whose quotient stack is $[M]$, the manifold itself. Then the de Rham complex of the groupoid $\coprod U_i \cap U_j \rightrightarrows \coprod U_k$ is the Čech complex associated to the cover (U_i) and the sheaf given by the de Rham forms, which, by Poincaré Lemma is quasi-isomorphic to the usual Čech complex of the cover (U_i). We just recovered one of the standard proof that Čech cohomology of a manifold is isomorphic to (de Rham) cohomology. For general cover, the same observation gives a proof of the *generalized Mayer-Vietoris principle*.

4.3 Short digression on inertia stack

The fact that torsion is killed in characteristic zero has the following consequence for orbifolds:

Proposition 4.5 *Let $\mathfrak{X}$ be an orbifold presented by a proper étale groupoid $[X_0/X_1]$. Then the de Rham complex $\Omega^*_{dR}(X_\bullet) \stackrel{\sim}{\hookleftarrow} \Omega^*_{dR}(X_0)^{X_1}$ is quasi-isomorphic to its sub-complex given by the X_1-invariant forms.*

Example 20 If G is a finite group acting on a manifold M, then, $H^*_{dR}([M/G]) \cong H^*_{dR}(M/G) \cong H^*_{dR}(M)^G$.

The above proposition reduces the cohomology of an orbifold to the cohomology of its coarse space which loose a lot of information. This is why the following philosophy is useful:

Philosophy: *the correct characteristic zero (co)homology invariant (or representation theory) of an orbifold $\mathfrak{X}$ are those of its inertia orbifold $\Lambda\mathfrak{X}$ (possibly up to some regrading):*

[15] a geometric realization this simplicial manifold provides a classifying space for the differentiable stack

[16] i.e. all finite intersections of opens are contractible

Definition 4.6 (Inertia stack) If $\mathfrak{X}$ is a differentiable stack, its inertia stack $\Lambda\mathfrak{X}$ is the stack with objects $\phi_x : x \to x \in \mathfrak{X}(U)$ for any manifold U, that is all automorphisms of objects of $\mathfrak{X}(U)$. Its arrows are given by arrows $f : x \to y \in \mathfrak{X}(U)$ (which maps the automorphism ϕ_x to $f \circ \phi_x \circ f^{-1} : y \to y$).

Thus the inertia stack of $\mathfrak{X}$ is just the stack of all automorphisms in $\mathfrak{X}$.

Note that the inertia stack $\Lambda\mathfrak{X}$ *is isomorphic to the fiber product* $\Lambda\mathfrak{X} \cong \mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$ where the maps $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ are the diagonal; in other words, the inertia stack of $\mathfrak{X}$ is the self intersection of $\mathfrak{X}$ inside the diagonal (which is non-trivial in the differentiable stack world unless the stack is a manifold).

Example 21 $\Lambda[M/G] = \coprod_{[g]}[M^g/C(g)]$ where the disjoint union is over all conjugacy classes in G and $C(g)$ is the centralizer of g. In particular,

$$H^*_{dR}(\Lambda[M/G]) = \bigoplus_{[g]} H^*(M^g)^{C(G)}$$

which contains $H^*_{dR}([M/G])$ as the summand associated to $[g] = [e]$ the conjugacy class of the unit.

As an example of the above philosophy let us mention that the Chen-Ruan orbifold cohomology of an (almost complex) orbifold $\mathfrak{X}$ is the cohomology of $\Lambda\mathfrak{X}$ endowed with the ghost shift regrading and a twisted cup-product.

Also, for any compact Lie group, the cohomology $H^*(\Lambda[pt/G])$ has an natural structure of Frobenius algebra (or more precisely a graded version involving a degree shifting by the dimension of G).

Example 22 The simplest non-trivial example of Chen-Ruan orbifold cup product is given by $\mathfrak{X} = [pt/G]$ for a finite group G. In that case, $H^*_{dR}([pt/G]) \cong Z(k[G])$ the center of the group algebra of G and the Frobenius structure is the one associated to Dijkgraaf-Witten Frobenius algebra.

5 Vector Bundles and Tangent Stacks

To define vector bundles on differentiable stacks, one can proceed as follow:

Definition 5.1 (vector bundles over stacks) Let $\mathfrak{X}$ be a differentiable stack. A vector bundle on $\mathfrak{X}$ is a representable morphism of stacks $\mathfrak{E} \to \mathfrak{X}$ such that, for every $f : [U] \to \mathfrak{X}$ with U a manifold, the pullback $[U \times_{\mathfrak{X}} \mathfrak{E}] \to U$ is endowed with the structure of a vector bundle. We also require, for every $a : V \to U$, that the natural isomorphism $\varphi_a : (f \circ a)^*\mathfrak{E} \to a^*(f^*\mathfrak{E})$ is a bundle map. A complex vector bundle is defined analogously.

One can show that this definition is equivalent to the data of a representable morphism of stacks $\mathfrak{E} \to \mathfrak{X}$ which makes $\mathfrak{E}$ a vector space object relative to $\mathfrak{X}$. That is, we have an addition morphism $\mathfrak{E} \times_{\mathfrak{X}} \mathfrak{E} \to \mathfrak{E}$ and an $\mathbb{R}$-action $\mathbb{R} \times \mathfrak{E} \to \mathfrak{E}$, both relative to $\mathfrak{X}$, which satisfy the usual axioms.

Also if $\mathfrak{X} \cong [X_0/X_1]$, then, a vector bundle on $\mathfrak{X}$ is an X_1-equivariant vector bundle. That is, a vector bundle E over X_0, and an isomorphism $\psi s^*E \to t^*E$ of vector bundles over X_1 such that the three restrictions of ψ to $X_1 \times_{X_0} X_1$ satisfy the natural cocycle condition.

The standard operations/structure on vector bundles on spaces (e.g., direct sum, tensor product, exterior powers, metric, orientation and so on) can be carried out on vector bundles on stacks *mutatis mutandis.*

Let $\mathfrak{X} \cong [X_0/X_1]$ be a differentiable stack. Taking tangent bundles gives rise to the Lie groupoid $TX_1 \rightrightarrows TX_0$. We call $[TX_0/TX_1]$ the *tangent stack* $\mathfrak{T}\mathfrak{X}$[17] of $\mathfrak{X}$. The base map induces a Lie groupoid morphism $(TX_1 \rightrightarrows TX_0) \to (X_1 \rightrightarrows X_0)$ and thus a morphism of stacks $\mathfrak{T}\mathfrak{X} \to \mathfrak{X}$.

Lemma 5.2 *If $\mathfrak{X}$ is an orbifold, then $\mathfrak{T}\mathfrak{X} \to \mathfrak{X}$ is a vector bundle over $\mathfrak{X}$.*

[17] which is independent up to isomorphism of the choice of $X_1 \rightrightarrows X_0$

However, $\mathfrak{T}\mathfrak{X} \to \mathfrak{X}$ is *not* a vector bundle over $\mathfrak{X}$ in general.

Example 23 Let G be a Lie group and $[*/G]$ be its classifying stack. Let $\mathfrak{g}$ be the Lie algebra of G, viewed as a group via its vector space addition. We thus get the quotient stack $[*/\mathfrak{g}]$. Then, $\mathfrak{T}[*/G] = [pt/\mathfrak{g} \rtimes G]$, where G is acting on $\mathfrak{g}$ by the adjoint action. The fiber of the base map $\mathfrak{T}[*/G] \to [*/G]$ (over the unique point $pt \to [pt/G]$) is $[*/\mathfrak{g}]$ which is a stack, but certainly not a manifold; in particular, not a linear space ! In fact $\mathfrak{T}[*/G] \to [*/G]$ is an example of a what is called a $\mathfrak{g}$-gerbe !

In general, the tangent stack is a 2-categorical analogue of a vector bundle. In fact, if $\mathfrak{X} = [X_0/X_1]$, we have a *tangent complex*: $LX_\bullet \to TX_0$ a length 2-complex of vector bundles over X_0. Here $LX_\bullet$ is the Lie algebroid of the groupoid $X_1 \rightrightarrows X_0$ defined as the normal bundle of the unit map $1 : X_0 \to X_1$. It is naturally identified with the relative tangent bundle T_t of the target map $t : X_1 \to X_0$ and the anchor map ρ is the composition $\rho : LX_\bullet \cong 1^*(T_t) \hookrightarrow 1^*(T_t) \oplus TX_0 \cong 1^*(TX_1) \cong 1^*(T_s) \oplus TX_0 \to TX_0$ where T_s is the relative tangent bundle of the source map $s : X_1 \to X_0$.

In Example 23, the tangent stack $\mathfrak{T}[pt/G]$ can be represented by the complex of vector spaces $\mathfrak{g} \to 0$ where $\mathfrak{g}$ is in cohomological grading -1.

6 Gerbes as stacks

We just have seen an example of a gerbe, namely the tangent stack of $[pt/G]$. Another important class of examples of (central) gerbes in mathematical physics arise from Chern-Simons. We wish now to explain a differentiable (2-)stack point of view on gerbes. This will present gerbes in a way very similar to the usual notion of principal bundles.

6.1 Gerbes, non-abelian cocycles and groupoids extensions

Recall that a principal G-bundle is a fiber bundle $P \to M$ whose fibers are diffeomorphic to G and endowed with a G-action. We have a similar interpretation for gerbes: *"roughly, a G-gerbe is a stack epimorphism $\mathfrak{P} \to \mathfrak{X}$ whose fibers are diffeomorphic to the stack $[pt/G]$"*. However, one issue, here, is that in general the stack $[pt/G]$ is *not* a group.

Nevertheless, when G is abelian, then the multiplication $G \times G \to G$ is a group morphism, hence yields a map of stacks $[pt/G] \times [pt/G] \to [pt/G]$ which makes $[pt/G]$ a ***group-stack***[18]. In that case, one can define a notion of principal $[pt/G]$-bundles over a stack which is completely analogous to principal bundles over Lie groups. This special kind of gerbes are called central gerbes as we will see below.

Let us start with the general definition of gerbes on stacks.

Definition 6.1 A *gerbe on a stack* $\mathfrak{X}$ is stack epimorphism $\mathfrak{G} \to \mathfrak{X}$ such that the canonical map $\mathfrak{G} \to \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G}$ is also an epimorphism.

Let G be a Lie group. A ***G-gerbe*** over a stack $\mathfrak{X}$ is a gerbe $\mathfrak{G} \to \mathfrak{X}$ which locally is isomorphic to $[pt/G] \times \mathfrak{X}$.

For differentiable stacks, a gerbe can be thought of as implying that $\mathfrak{G}$ has the same (up to isomorphisms) objects as $\mathfrak{X}$ but more morphisms.

Let us focus on general G-gerbes $\mathfrak{P} \to [M]$ over say a manifold M. We first note that isomorphisms between different local trivializations (over an open subset U) $U \times [pt/G] \cong \mathfrak{P}_{|U}$ are given by a map $U \to \mathrm{Iso}_{\mathfrak{pst}}([pt/G])$. The stack of diffeomorphisms $\mathrm{Iso}_{\mathfrak{pst}}([pt/G])$ is isomorphic to the space of *group automorphisms* $\mathrm{Aut}(G)$. Hence, the choice of trivializations (U_i) yields a Čech 2-cocycle $U_{ij} \to \mathrm{Aut}(G)$. This data, of course, is not enough to determine a G-gerbe. We shall also take into account the differentiable stack structure, namely the fact that $U \times [pt/G] \cong \mathfrak{P}_{|U}$ is isomorphic to the quotient stack of a Lie groupoid $\mathbb{U}_1 \rightrightarrows \mathbb{U}_0$. The

[18] that is a group object in the category of stacks, which is to stacks what a Lie group is to manifolds

Groupoid structure will give an additional 3-cocycle condition with value in G and this is how one recovers the usual non-abelian cocycle conditions. In fact we have the following table relating non-abelian cocycles with geometric objects and their topological invariants:

Non abelian cohomology (Grothendieck,Giraud)	$H^1(M,G)$	"$H^2(M,G)$"
Geometric objects	principal G-bundles (up to iso)	G-gerbes
Topological invariants computed from geometric data	Char classes from Chern-Weil	For central gerbes only: Dixmier-Douday classes.

Let us (re)call that, for a smooth manifold M and G be a Lie group, *a non abelian 2-cocycle*[19] *on M with values in G* is an open covering $(U_i)_{i\in I}$ of M and a collection of smooth maps

$$\lambda_{ij} : U_i \cap U_j \to \mathrm{Aut}(G) \qquad \text{and} \qquad g_{ijk} : U_i \cap U_j \cap U_k \to G$$

satisfying the following relations:

$$\lambda_{ij} \circ \lambda_{jk} = \mathbf{Ad}_{g_{ijk}} \circ \lambda_{ik}$$
$$g_{ijl} g_{jkl} = g_{ikl} \lambda_{kl}^{-1}(g_{ijk}).$$

Over a manifold[20] M, ***isomorphisms classes of G-gerbes over $[M]$ are in bijection with non-abelian cohomology*** $H^2(M,G)$, that is non-abelian 2-cocycle up non-abelian coboundaries.

We now relate this with a Lie groupoid presentation of G-gerbes.

Proposition 6.2 *Let $\mathfrak{X}$ be any differentiable stack. Isomorphisms classes of G-gerbes over $\mathfrak{X}$ are in one to one correspondence with* G-extensions of Lie groupoids of $\mathfrak{X}$ *up to Morita equivalences.*

By a **Groupoid G-extension of $\mathfrak{X}$** we mean : a Lie groupoid $X_1 \rightrightarrows X_0$ such that $\mathfrak{X} \cong [X_0/X_1]$ and a *short exact* sequence of Lie groupoids

$$\begin{tikzcd} X_0 \times G \arrow[r, hook] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_1"] & Y_1 \arrow[r, two heads] \arrow[d, shift right] \arrow[d, shift left] & X_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ X_0 & X_0 & X_0 \end{tikzcd}$$

bundle of groups | extended groupoid | original groupoid

where $Y_1 \rightrightarrows Y_0$ is a Lie groupoid, the horizontal arrows are maps of Lie groupoids (which is the identity on the objects) and the left part is a bundle of group viewed as the trivial groupoid associated to the product of stacks $[X_0] \times [pt/G]$.

Example 24 From a non-abelian 2-cocycle over M associated to an open covering (U_i), we get the following G-extension:

$$\begin{tikzcd} \coprod U_i \times G \arrow[r, "1\times id"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_1"] & \coprod U_i \cap U_j \times G \arrow[r, "\phi"] \arrow[d, shift right] \arrow[d, shift left] & \coprod U_i \cap U_j \arrow[d, shift right] \arrow[d, shift left] \\ \coprod U_i \arrow[r, "id"'] & \coprod U_i \arrow[r, "id"'] & \coprod U_i \end{tikzcd}$$

The Lie groupoid structure on the right is the one of the Čech groupoid Example 9, while the one on the middle is given by as follows. Let x_{ij} denotes a point $x \in M$ seen as a point of the open subset $U_i \cap U_j$,

[19] in the sense of Giraud, Grothendieck, Dedecker...

[20] the definition and following result of non-abelian cohomology can actually be extended to all stacks

x_i denotes the point $x \in M$ seen as a point of the open subset U_i, and g, h arbitrary elements of G. The groupoid multiplication is given by

$$(x_{ij}, g) \cdot (x_{jk}, h) = (x_{ik}, g_{ijk}\lambda_{jk}^{-1}(g)h).$$

Let (U_i) be an open covering of $\mathfrak{X}$ by trivializations of a G-gerbe $\mathfrak{G} \to \mathfrak{X}$. Then we get non-abelian 2-cocycles as above. In particular the composition

$$\tilde{\lambda}_{ij} : U_i \cap U_j \xrightarrow{\lambda_{ij}} \mathrm{Aut}(G) \longrightarrow \mathrm{Out(G)} = \mathrm{Aut}(G)/G$$

is an ordinary Čech 2-cocyle whose cohomology class is denoted $[(\tilde{\lambda}_{ij})] \in H^1(\mathfrak{X}, \mathrm{Out(G)})$ and is called the *band of the G-gerbe*.

Definition 6.3 (central G-gerbes) A G-gerbe is said to be *central* if its band is trivial.

In terms of G-extensions of Lie groupoids, central gerbes are precisely the *central G-extensions*, meaning that the image of $X_0 \times G$ in Y_1 is central.

6.2 G-gerbes as principal bundles over a group-stack

We now wish to explain a bit more the previous comment that a central G-gerbe, with G *abelian*, is really like a principal bundle over the group stack $[pt/G]$. This is better understood by allowing ourselves to think in terms of 2-stacks. Indeed, as seen before, principal H-bundles over a manifold M are the same as stack morphisms $[M] \to [pt/H]$. The same will be true for a principal $[pt/G]$-bundle, except that we have to replace the quotient stack $[pt/H]$ by the "quotient 2-stack $\big[pt/\big([pt/G]\big)\big]$". This 2-stack shall really be think as the *stack* version of the naive quotient of the trivial group with one element $\{1\}$ by the image of the group G. This kind of quotient can be made in a more general situation that we now describe.

Definition 6.4 (Lie 2-groups and group stacks) A *Lie 2-group* is a group object in the category of Lie groupoids meaning it is a Lie groupoid $G_1 \rightrightarrows G_0$ where, both, G_1 and G_0 are Lie groups and all structure maps are Lie group morphisms. The ***quotient stack*** $[G_0/G_1]$ ***of a Lie 2-group is a differentiable group stack*** (that is a stack with a multiplication, unit and inverse map satisfying the usual axioms in the category of differentiable stacks).

Most of the group stacks arising in the literature are quotient stacks of Lie 2-groups. We have already seen the example of $[pt/G]$ with G abelian.

Lie 2-group can be equivalently described in terms of the following data : a **crossed module of Lie groups** is a homomorphism $G \xrightarrow{\rho} H$ of Lie groups together with an action of H on G by automorphisms: $g \mapsto g^h$ satisfying

$$\rho(g^h) = h^{-1} \cdot \rho(g) \cdot h$$
$$g_1^{\rho(g_2)} = g_2^{-1} \cdot g_1 \cdot g_2.$$

In fact, ***Lie 2-groups are in bijection with crossed module of Lie groups***; a crossed module $G \xrightarrow{\rho} H$ giving rise to the Lie 2-group $G \rtimes H \rightrightarrows H$. The underlying groupoid structure is the transformation groupoid given by the action $g \cdot h = \rho(g)h$. In particular the associated quotient group stack is $[H/G]$.

Note that the image $\rho(G) \subset H$ is a normal subgroup of H and that the kernel of ρ is central in G. In fact:

a crossed module of Lie group shall be thought as the "stacky quotient" of the Lie group $H/\rho(G)$ where we remember the non-trivial automorphism given by the elements of $\ker(\rho)$.

Example 25 A Lie group G gives rise to three different Lie 2-groups, i.e. crossed modules of Lie groups:

1. $1 \to G$, whose quotient group stack is just $[G]$, that is the Lie group G itself;

2. $G \xrightarrow{\mathbf{Ad}} \mathrm{Aut}(G)$ given by the adjoint action of G on itself;

3. $Z(G) \to 1$ where $Z(G)$ is the center of G. Its quotient group stack is $[pt/Z(G)]$ with its obvious group structure.

We have an evident map between the last two crossed modules, and thus a canonical group stack homomorphism $[1/Z(G)] \to [\mathrm{Aut}(G)/G]$.

Now, we can define[21] principal bundles over a group stack of the form $[H/G]$ (where $G \xrightarrow{\rho} H$ is a crossed module of Lie groups) in a way completely parallel to usual principal bundles over a manifold.

Definition 6.5 A *principal* $[H/G]$*-bundle over a differentiable stack* $\mathfrak{X}$ is a differentiable stack epimorphism $\mathfrak{P} \to \mathfrak{X}$ where $\mathfrak{P}$ is a differentiable stack endowed with a fiberwise action of the group stack $[H/G]$ and satisfying that the canonical map $\mathfrak{P} \times [H/G] \to \mathfrak{P} \times_{\mathfrak{X}} \mathfrak{P}$ is an isomorphism.

The condition $\mathfrak{P} \to \mathfrak{X}$ being an epimorphism ensures that, locally, there are 2-commutative diagram

$$\begin{tikzcd}
\mathfrak{U} \times [H/G] \arrow[r, "\text{projection}"] & \mathfrak{U} \subset \mathfrak{X}. \arrow[dl, phantom, "\Uparrow"] \\
\mathfrak{P}_{|\mathfrak{U}} \arrow[u, "\simeq"] \arrow[ur] &
\end{tikzcd}$$

When $G = 1$ and $\mathfrak{X} \cong [M]$, we simply recover the standard definition of a principal bundle over a manifold (or a stack as in Example 14).

Recall from Example 25 the two crossed modules of Lie groups $G \xrightarrow{\mathbf{Ad}} \mathrm{Aut}(G)$ and $Z(G) \to 1$ associated to a Lie group G and their associated group stacks $[\mathrm{Aut}(G)/G]$ and $[1/Z(G)]$. We now express gerbes and central gerbes as principal bundles over these group stacks.

Theorem 6.6 *Let $\mathfrak{X}$ be a differentiable stack.*

- *G-gerbes over $\mathfrak{X}$* $\overset{bijection}{\longleftrightarrow}$ *Principal $[\mathrm{Aut}(G)/G]$-bundles over $\mathfrak{X}$.*
- *central G-gerbes over $\mathfrak{X}$* $\overset{bijection}{\longleftrightarrow}$ *Principal $[\mathrm{Aut}(G)/G]$-bundles over $\mathfrak{X}$ whose structure 2-group can be reduced to $[1/Z(G)]$*

  " $\overset{bijection}{\longleftrightarrow}$ *Principal $[1/Z(G)]$-bundles over $\mathfrak{X}$.*

In particular, we recover that the tangent stack $\mathfrak{T}[pt/G] \to [pt/G]$ is a central $\mathfrak{g}$-gerbe.

Remark 6.7 (2-stack point of view) The analogue, for a group stack $[H/G]$ of the stack $[pt/G]$ associated to a Lie group is a differentiable 2-stack denoted $[pt/[H/G]]$. We do not wish to define all the theory of 2-stacks. We just mention that these objects can be defined similarly to differentiable stacks, replacing groupoids by 2-groupoids[22], that is 2-category in which all 1-morphisms and 2-morphisms are invertible. Differentiable 2-stacks are thus represented by Lie 2-groupoids and 2-stacks morphisms between two Lie 2-groupoids $X_\bullet$, $Y_\bullet$ are represented by zigzags:

$$X_\bullet \overset{\text{Morita equivalence}}{\underset{\sim}{\longleftarrow}} Z_\bullet \longrightarrow Y_\bullet$$

[21] one would have to be careful however, in the below definition, that stacks forms a 2-category and thus the relevant diagrams are commutative up to (coherent) 2-isomorphisms

[22] In the case we are interested in, one only needs to consider strict 2-categories

where the left arrrow is a Morita equivalence and a right arrow a morphism of Lie 2-groupoids.

Similarly to the fact that principal G-bundles are the same as maps of stacks to $[pt/G]$, we have the following proposition for group stacks.

Proposition 6.8 *Let $G \xrightarrow{\rho} H$ be a crossed module of Lie groups. Principal $[H/G]$-bundles over $\mathfrak{X}$ are in one to one correspondence with 2-stacks morphisms $\mathfrak{X} \to [pt/[H/G]]$.*

In particular, G-gerbes over $\mathfrak{X}$ *are in one to one correspondence with 2-stacks morphisms* $\mathfrak{X} \to [pt/[\mathrm{Aut}(G)/G]]$ *and* central G-gerbes *are in one to one correspondence with 2-stacks morphisms $\mathfrak{X} \to [pt/[pt/Z(G)]]$.*

6.3 Characteristic classes of gerbes and group stack bundles

Proposition6.8 allows to define *characteristic classes* for (central) gerbes and more generally 2-groups in an natural way. Recall that the characteristic classes of a principal G-bundle P over a manifold M are given as follows. The bundle determines the stack morphism[23] $f : [M] \to [pt/G]$. The (de Rham) cohomology $H^*_G(pt) \cong H^*([pt/G])$ is given by $\mathfrak{g}$-invariant polynomial functions $S(\mathfrak{g}^*[-2])$.

The *characteristic classes of P are given by the map $S(\mathfrak{g}^*[-2])^{\mathfrak{g}} \cong H^*([pt/G]) \xrightarrow{f^*} H^*(M)$ evaluated on the generators of the cohomology of $H^*([pt/G])$.*

This generalizes to group stack: let $\mathfrak{P} \to \mathfrak{X}$ be a principal $[H/G]$-bundle, and $\varphi : \mathfrak{X} \to [pt/[H/G]]$ be the associated 2-stack morphism.

The **characteristic map** of $\mathfrak{P} \to \mathfrak{X}$ is the composition

$$\begin{tikzcd} H^*([pt/[H/G]], \mathbb{R}) \arrow[r, "\varphi^*"] \arrow[rr, bend right, "\text{characteristic map}"'] & H^*(\mathfrak{X}, \mathbb{R}) \arrow[r, "\cong"] & H^*_{dR}(\mathfrak{X}) \end{tikzcd}$$

The same construction can be defined over arbitrary coefficient of course. This allows to define characteristic classes associated to generators of the cohomology of the classifying 2-stack $[pt/[H/G]]$ of the Lie 2-group $G \xrightarrow{\rho} H$.

We can in particular apply this to gerbes. Unfortunately, the cohomology of $[pt/[\mathrm{Aut}(G)/G]]$ is not known in general and further a lot of the interesting information seems to be concentrated in its torsion part:

Example 26 Let G be a compact Lie group and denote $Z(\mathfrak{g})$ the center of its Lie algebra $\mathfrak{g}$. If $\dim(Z(\mathfrak{g})) \leq 3$, then

$$H^p([G \to \mathrm{Aut}(G)], \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

However, if we are only interested in central gerbes, the situation is much easier.

Proposition 6.9 *$H^\bullet([pt/[pt/Z(G)]], \mathbb{R}) \cong \Lambda^* Z(\mathfrak{g})$ where $Z(\mathfrak{g})$ is in degree 3.*

Now apply that to a G-gerbe with trivial band over $\mathfrak{X}$. By the above theorem, we have a principal $[pt/Z(G)]$-bundle $\mathfrak{P} \to \mathfrak{X}$ and thus the characteristic map $H^*([pt/[pt/Z(G)]], \mathbb{R}) \xrightarrow{\text{characteristic map}} H^*_{dR}(\mathfrak{X})$. Dualizing the characteristic map in degree 3, we get a **characteristic class** $CC(\mathfrak{P}) \in H^3_{dR}(\mathfrak{X}) \otimes Z(\mathfrak{g})$.

Classicaly, the real valued cohomology classes of a principal bundles can be computed out of geometric data —connections and curvatures— by the Chern-Weil construction. There is a similar construction for central gerbes !

Let $\mathfrak{G} \to \mathfrak{X}$ be a central G-gerbe where G is a connected reductive Lie group so that we have a splitting $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{m}$.

[23] or more classicaly a continuous map $f : M \to BG$

Let $M \times G \to Y_1 \xrightarrow{\phi} X_1$ be an associated central G-extension of Lie groupoids (as given by Proposition 6.2). In particular $\mathfrak{X} \cong [X_0/X_1]$. Note that $Y_1 \xrightarrow{\phi} X_1$ is a principal G-G-bibundle.

Theorem 6.10 *1. Given a connection 1-form $\alpha \in \Omega^1(X_\bullet, \mathfrak{g})$ for the right principal G-bundle $Y_1 \xrightarrow{\phi} X_1$, then there exists $\Omega_\alpha \in Z^3_{dR}(\mathfrak{X}, Z(\mathfrak{g}))$ such that $p(\partial\alpha + b\alpha) = \phi^*(\Omega_\alpha)$.*

2. Given two connections α_1 and α_2, then $\Omega_{\alpha_1} - \Omega_{\alpha_2} \in B^3_{dR}(X_\bullet, Z(\mathfrak{g}))$ hence $[\Omega_\alpha]$ does not depends on the choice of connection.

The class $DD := [\Omega_\alpha] \in H^3_{DR}(X_\bullet) \otimes Z(\mathfrak{g})$ is called the **Dixmier-Douady class** *of the gerbe.*

This class yields the following Chern-Weil type theorem for gerbes:

Proposition 6.11 *The Dixmier-Douady class DD is equal to the universal class $CC(\mathfrak{P})$.*

As immediate corollaries, one sees that the Dixmier-Douady class only depends on the gerbe (and not the choice of extension) and also that the Dixmier-Douady class is integral.

Example 27 If $G = U(1)^n$ is a torus, then Proposition 6.9 follows from the following stronger result: The 2-stack $[pt/[pt/G]]$ is homotopy equivalent to the Eilenberg-Mac Lane space $K(\mathbb{Z}^n, 3)$. In that case, it follows in particular, that the characteristic/Dixmier-Douady class completely determines the central $U(1)^n$-gerbe (up to equivalences).

Example 28 (The string 2-group) Let G be a simply connected compact simple Lie group (for instance $G = Spin(n)$). There is an interesting (topological) 2-group, called the String 2-group associated to G. We have a left invariant *closed* 3*-form* ν on G which generates

$$H^3(G, \mathbb{Z}) \cong \mathbb{Z}\nu \cong H^4([pt/G]).$$

It determines the basic central extension

$$1 \to S^1 \to \widetilde{\Omega G} \xrightarrow{\tilde{p}} \Omega G \to 1$$

of the based (at identity) loop group ΩG of G. We have the associated Lie algebra extension

$$0 \to \mathbb{R} \to \widehat{\Omega\mathfrak{g}} \to \Omega\mathfrak{g} \to 0$$

which is quasi-isomorphic to the Lie 2-algebra $\mathbb{R}[1] \oplus \mathfrak{g}$. The String 2-group "integrates" this Lie 2-algebra and, denoting $PG = \{f : [0,1] \to G \,/\, f(0) = 1\}$ the path space of G, is given by the crossed module (of Fréchet) groups

$$\widetilde{\Omega G} \xrightarrow{p} PG$$

where p is the composition $\widetilde{\Omega G} \xrightarrow{\tilde{p}} \Omega G \hookrightarrow PG$ and PG acts on $\widetilde{\Omega G}$ by the lift of its conjugacy action on its normal subgroup ΩG.

We **denote** $String(G) := [PG/\widetilde{\Omega G}]$ **the induced (topological) group stack**.

Proposition 6.12 *The cohomology of the classifying* 2*-stack of* $String(G)$ *is given by*

$$H^\bullet([pt/[PG/\widetilde{\Omega G}]]) \cong H^\bullet([pt/G])/([\nu]).$$

It is thus a polynomial algebra on generators $y_2, \ldots, y_r$ (of degrees $2e_i + 2$ where $e_1, \ldots, e_r$ are the exponents of G).

The map $\big(PG \ni f\big) \mapsto f(1) \in G$ defines a map of group stack $String(G) \to [G]$. And thus any $String(G)$-principal bundle $\mathfrak{P} \to \mathfrak{X}$ gives rise to an associated principal G-bundle $G \times_{String(G)} \mathfrak{P} \to \mathfrak{X}$.

The proposition thus says that the characteristic classes of a principal $String(G)$-bundle are those of the associated principal G-bundles modulo the first Pontrjagin class. In particular, this class is an obstruction to lifting a principal G-bundle to a principal $String(G)$-bundle.

Some references related to the content of these notes or the applications of stacks and gerbes to mathematical physics are listed below. We mainly used [5, 6, 15, 26, 27].

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