THE GEOMETRY OF THE HOPF FIBRATIONS

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This is an elementary exposition of the geometry, and especially the symmetries, of the Hopf fibrations of spheres by great spheres.

Using the complex numbers C, the quaternions H and the Cayley numbers Ca, we will describe the Hopf fibrations

$$S^1 \hookrightarrow S^{2n-1} \to CP^{n-1} = \text{complex projective } n-1 \text{ space,}$$

$$S^3 \hookrightarrow S^{4n-1} \to HP^{n-1} = \text{quaternionic projective } n-1 \text{ space, and}$$

$$S^7 \hookrightarrow S^{15} \to S^8 .$$

These fibrations were introduced by Heinz Hopf [Ho 1, 2] about fifty years ago. Even the littlest one, $S^1 \hookrightarrow S^3 \to CP^1 = S^2$, had a powerful effect on topology: it provided the first example of a homotopically nontrivial map from one sphere to another of lower dimension, spurring the development of both homotopy theory and fibre spaces in their infancy.

The Hopf fibrations have many beautiful properties. For example, we will see that their fibres are parallel, in the sense of having constant distance from one another. This actually characterizes the Hopf fibrations among all fibrations of round spheres by great subspheres, as was proved by Wong [Won] and Wolf [Wol 1, 2] and later by Escobales [Es] and Ranjan [Ra]. See [GWZ] for an expository account.

Another striking property of the Hopf fibrations is that they all have a large group of symmetries, acting transitively on the total space and in particular on the fibres. We will determine these groups here. The finale is the calculation that the symmetry group of the "exceptional" Hopf fibration $S^7 \hookrightarrow S^{15} \to S^8$ is isomorphic to $Spin(9)$, the simply connected double cover of the special orthogonal group $SO(9)$. This will involve us in details about the arithmetic of Cayley numbers and the "Triality Principle" for $SO(8)$.

14

1. HOPF FIBRATIONS WITH FIBRE S^1

We describe the Hopf fibration

$$H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$$

as follows. Choose orthonormal coordinates in real $2n$-space R^{2n} and write

$$(x_1, x_2, ..., x_{2n-1}, x_{2n}) = (x_1 + ix_2, ..., x_{2n-1} + ix_{2n})$$
$$= (u_1, ..., u_n),$$

thus identifying R^{2n} with complex n-space C^n.

The complex lines in C^n, each looking like a real 2-plane, form the complex projective space CP^{n-1} and fill out C^n, with any two meeting only at the origin. The unit circles on these complex lines give us the *Hopf fibration* of S^{2n-1}.

The simplest case occurs for $n = 2$. The complex lines in C^2 are of the form

$$L_m = \{(u, mu): u \in C\} \quad \text{for each} \quad m \in C,$$

and
$$L_\infty = \{(0, v): v \in C\}.$$

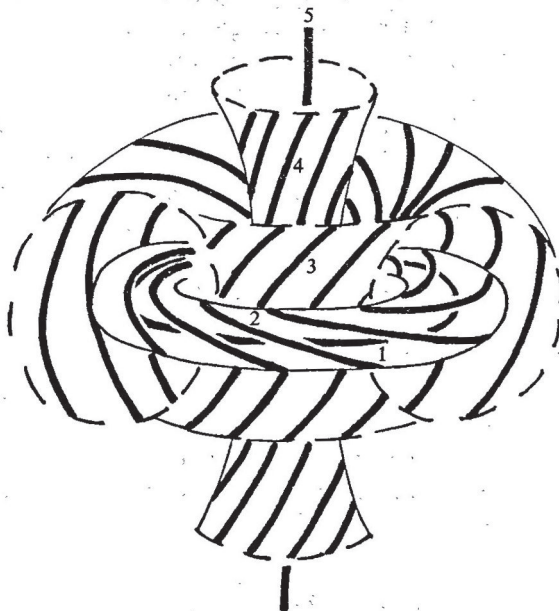FIGURE 1

Note that there is one Hopf circle for each complex number m, and one for the number ∞. So the set of Hopf circles is topologically a 2-sphere.

Above is a sketch of the Hopf fibration $H: S^1 \hookrightarrow S^3 \to S^2$, due to Roger Penrose [Pe].

The portions of the this sketch may be identified as follows:

1) Circle $x_1^2 + x_2^2 = 1$, $x_3 = 0, x_4 = 0$ fibre

2) Torus $x_1^2 + x_2^2 = 3/4$, $x_3^2 + x_4^2 = 1/4$ union of fibres

3) Torus $x_1^2 + x_2^2 = 1/2$, $x_3^2 + x_4^2 = 1/2$ union of fibres

4) Torus $x_1^2 + x_2^2 = 1/4$, $x_3^2 + x_4^2 = 3/4$ union of fibres

5) Circle $x_1 = 0, x_2 = 0$, $x_3^2 + x_4^2 = 1$ fibre

In the construction of the Hopf fibration of S^{2n-1} by great circles, we began by choosing orthonormal coordinates for R^{2n}. A different choice of such coordinates simply turns the picture of the Hopf fibration around by a rigid motion of S^{2n-1}, and we refer to all of these as "Hopf fibrations".

A key geometric feature of the Hopf fibrations is given by

PROPOSITION 1.1. *The Hopf circles on S^{2n-1} are parallel to one another.*

What do we even mean by this? Two subsets P and Q of a metric space will be said to be *parallel* if there is some real number d such that each point of P has minimum distance d from Q, and vice versa. If P and Q are parallel great circles on S^{2n-1} at distance d apart, then each lies on the boundary of a tubular neighborhood of radius d about the other.

To see this with more precision, first suppose that P and Q are arbitrary great circles on S^{2n-1}, and use the same symbols to denote the 2-planes through the origin that they span in R^{2n}. Let α_1 denote the smallest angle that any line in P makes with Q, and let α_2 denote the largest such angle. Then $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \pi/2$. These angles are called the *principal angles* between P and Q.

One can always choose an orthonormal basis $e_1, ..., e_{2n}$ for R^{2n} so that e_1 and e_2 form an orthonormal basis for P, while $\cos \alpha_1 \, e_1 + \sin \alpha_1 \, e_3$ and $\cos \alpha_2 \, e_2 + \sin \alpha_2 \, e_4$ form an orthonormal basis for Q. Then P and Q are parallel if and only if the two principal angles α_1 and α_2 are equal.
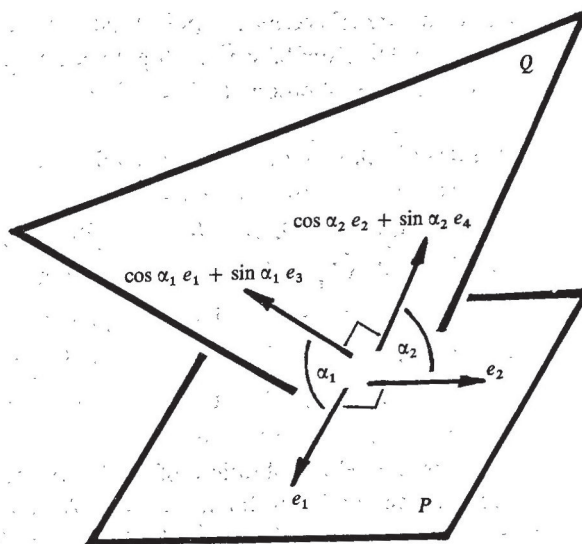
FIGURE 2

Note that with respect to these bases, the matrix for orthogonal projection of P onto Q (or vice versa) is given by

$$\begin{pmatrix} \cos \alpha_1 & 0 \\ 0 & \cos \alpha_2 \end{pmatrix}.$$

Thus P and Q are parallel if and only if orthogonal projection of P to Q is a conformal map. For future use, we also note that if A is the matrix of a linear map with respect to orthonormal bases, then that map is conformal if and only if $A A^t = \lambda I$.

To prove the proposition, let P and Q be two Hopf circles on S^{2n-1}. If u is any unit vector in the 2-plane P, then u and iu form an orthonormal basis for P. Likewise we get an orthonormal basis v and iv for Q. With respect to these bases, the matrix A of orthogonal projection of P onto Q is given by

$$\begin{pmatrix} a = <u, v> & b = <u, iv> \\ c = <iu, v> & d = <iu, iv> \end{pmatrix}.$$

But multiplication by i is an isometry; hence $a = d$ and $b = -c$. Thus

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{and} \quad A\,A^t = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix},$$

showing that A is conformal, and hence that P and Q are parallel.   QED

We remark here that, unlike the usual situation in Euclidean space, being parallel is not a transitive relation in spherical geometry. Consider, for example, the following three great circles in S^3:

$$P_1 = \{(x, y, x, y): x^2 + y^2 = 1/2\}$$
$$P_2 = \{(x, y, 0, 0): x^2 + y^2 = 1\}$$
$$P_3 = \{(x, y, x, -y): x^2 + y^2 = 1/2\}.$$

Then P_1 and P_3 are each parallel to P_2, but certainly not to each other, since they meet in two points.

Since the Hopf fibrations of S^{2n-1} have parallel fibres, they can be viewed as Riemannian submersions as follows.

Let $\pi: M \to N$ be a smooth map between smooth manifolds. This map is said to be a *submersion* if its differential π_* has maximal rank at each point. A submersion between closed manifolds must be a fibration.

If in addition M and N are Riemannian manifolds, then a submersion between them is said to be a *Riemannian submersion* if its differential preserves the lengths of tangent vectors orthogonal to the fibres $\pi^{-1}(y)$, $y \in N$.

Suppose now that $\pi: M \to N$ is a submersion of one complete connected smooth manifold onto another. The following facts are easy to deduce:

1) *If M and N have Riemannian metrics which make π a Riemannian submersion, then the fibres of π are parallel in M.*

2) *If M has a Riemannian metric in which the fibres of π are parallel, then one can choose a Riemannian metric on N in terms of which π becomes a Riemannian submersion.*

In particular, there is a Riemannian metric on CP^{n-1} which makes the Hopf projection $\pi: S^{2n-1} \to CP^{n-1}$ into a Riemannian submersion. This is known as the *canonical metric* on CP^{n-1}. The distance between points on CP^{n-1} equals the distance between corresponding Hopf fibres on S^{2n-1}.

PROPOSITION 1.2. *The canonical metric on CP^1 makes it into a round two-sphere of radius 1/2.*

We've already noted that for the lowest dimensional Hopf fibration $H: S^1 \hookrightarrow S^3 \to CP^1$, the base space is topologically a two-sphere. Let P denote one of the fibres of H, say the unit circle on the $x_1 x_2$-plane. Let $P^\perp$ denote the orthogonal fibre, in this case the unit circle on the $x_3 x_4$-plane. We let P correspond to the north pole and $P^\perp$ to the south pole on a round two-sphere $S^2(1/2)$ of radius 1/2.

For each quarter circle on S^3 from P to $P^\perp$, orthogonal to P and $P^\perp$, we obtain a family of fibres of H, one through each point of the quarter circle. These will correspond to the points of a semicircle on $S^2(1/2)$ from the north pole to the south pole.

Now consider all the fibres of H which are at distance α from P, $0 < \alpha < \pi/2$. They fill out the torus

$$T_\alpha = \{x_1^2 + x_2^2 = \cos^2 \alpha, \; x_3^2 + x_4^2 = \sin^2 \alpha\}$$
$$= S^1(\cos \alpha) \times S^1(\sin \alpha).$$

Every fibre on this torus is the graph of a "linear" bijection from $S^1(\cos \alpha)$ to $S^1(\sin \alpha)$. Each such fibre meets a small circle $(\cos \alpha, 0, 0, 0) \times S^1(\sin \alpha)$ at a single point. But these points are further apart than the actual distances between the fibres. The following diagram shows that in the limit, as q' approaches q, the scale correction factor is $\cos \alpha$.
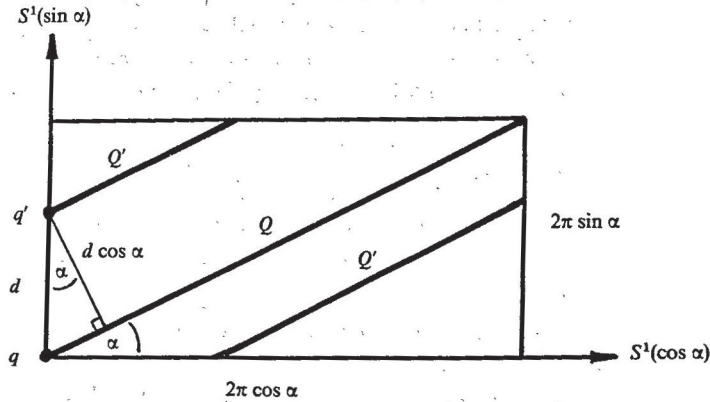

FIGURE 3

Hence the fibres of H which lie on the torus T_α form a circle of radius $\sin \alpha \cos \alpha$. But a circle of latitude on $S^2(1/2)$, located at distance α from the north pole, has radius $(1/2) \sin 2\alpha = \sin \alpha \cos \alpha$. It follows that there is a correspondence between fibres of H and points of $S^2(1/2)$ which is a Riemannian isometry, proving the proposition.   QED

Besides being parallel, the fibres of the Hopf fibration are assembled in a very regular way. The following two geometric features give an expression of this regularity, and were important in [GWZ].

1) *Constancy Feature.* Refer again to the figure showing the Hopf fibration of S^3, in which we see S^3 decomposed into a pair of orthogonal great circles and a family of intermediating tori:

$$T_0 = S^1(1) \times 0$$
$$T_\alpha = S^1(\cos \alpha) \times S^1(\sin \alpha) \qquad 0 < \alpha < \pi/2$$
$$T_{\pi/2} = 0 \times S^1(1).$$

Any two of these intermediating tori are a constant distance apart, and hence parallel to one another. There is a natural "radial projection" map between them, which matches closest neighbors on the two surfaces. It is easy to see that this map also matches Hopf circles, and in this sense we regard the Hopf fibration as "constant" on the family of tori. A corresponding phenomenon can be observed in all the Hopf fibrations.

2) *Inductive Feature.* A Hopf fibration contains within itself copies of lower dimensional Hopf fibrations, and can be regarded as assembled from these in a certain way. For example, just as C^n contains C^{n-1}, so does the Hopf fibration of S^{2n-1} contain the Hopf fibration of S^{2n-3}.

2. SYMMETRIES OF THE HOPF FIBRATIONS WITH FIBRE S^1

Let $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ denote a Hopf fibration with fibre S^1. By a *symmetry* of H we mean a rigid motion of S^{2n-1} which takes Hopf circles to Hopf circles. We want to find these symmetries explicitly.

The unitary group

$$U(n) = Gl(n, C) \cap O(2n)$$
$$= \text{complex general linear group} \cap \text{orthogonal group}$$

consists of complex linear maps which are also rigid. Since these maps take complex lines to complex lines, they must be symmetries of the above Hopf fibration.

But there are other symmetries. Define *complex conjugation*

$$c: C^n \to C^n \quad \text{by} \quad c(z_1, ..., z_n) = (\overline{z_1}, ..., \overline{z_n}).$$

Note that c lies in $O(2n)$ but not in $Gl(n, C)$, yet takes complex lines to complex lines, hence must be a symmetry of the Hopf fibration. Note also that c reverses the natural orientations of the complex lines in C^n.

The next proposition indicates that there are no further symmetries.

PROPOSITION 2.1. *The group G of all symmetries of the Hopf fibration H is $G = U(n) \cup c\, U(n)$.*

Let g be a rigid motion of S^{2n-1} taking complex lines to complex lines. In case g reverses the natural orientations of complex lines, compose it with c so as to preserve these orientations. The new g commutes with multiplication by i, hence is complex linear. Since it is also rigid, it lies in $U(n)$.   QED

Remark. Note that all the symmetries are orientation preserving when n is even, while half are orientation reversing when n is odd.

The group of symmetries of the Hopf fibration is quite large, and this may be underscored by exhibiting symmetries with preassigned features. We collect some of these in the following proposition.

PROPOSITION 2.2. *Let $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ be a Hopf fibration. Then*

a) *There is a symmetry of H inducing the identity on the base space (and thus taking each Hopf circle to itself) and restricting to a preassigned rotation on a given Hopf circle.*

b) *If P and Q are any two fibres of the Hopf fibration, then any preassigned rigid motion of P onto Q can be extended to a symmetry of H.*

c) *The group of symmetries acts transitively on S^{2n-1}, and in particular acts transitively on fibres.*

By contrast, here is a limitation on the possible symmetries.

d) *There is no symmetry of H inducing the identity on the base space and reversing the orientations of the Hopf circles.*

Consider the symmetries $z \cdot Id$, $|z| = 1$, which multiply each coordinate in C^n by the complex number z of unit length. They induce the identity on the base space, and can be selected to take a fibre to itself by a preassigned rotation, proving a).

The transformations in $U(n)$ can take any complex line in C^n to any other by a preassigned orientation preserving rigid motion. Complex conjugation then adds the orientation reversing ones, proving b).

In particular, this implies c).

Suppose there were a symmetry of $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ taking each Hopf circle to itself with reversal of orientation. Then, by restriction to C^2, such a symmetry would also exist for $n = 2$. Its reversal of orientation on the total space S^3 would then contradict the remark following Proposition 2.1. This proves d).   QED

Remarks. 1) Note that the existence of symmetries of H rotating each Hopf circle within itself shows again that these circles must be parallel.

2) Also note that a symmetry of $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ induces an isometry of the base space CP^{n-1} in its canonical metric. We remark without proof that *all* isometries of CP^{n-1} can be produced this way.

3. HOPF FIBRATIONS WITH FIBRE S^3

Choose orthonormal coordinates in R^{4n} and identify this space with quaternionic n-space H^n. A little care is needed in dealing with H^n because the quaternions form a *non*-commutative division algebra:

1) Scalars $v \in H$ will act on vectors $(u_1, ..., u_n) \in H^n$ from the *right*.

$$(u_1, ..., u_n) v = (u_1 v, ..., u_n v).$$

2) H-linear transformations of H^n will be expressed by matrices of quaternions acting from the *left* (so as to commute with scalar multiplication).

The quaternionic lines in H^n, each looking like a real 4-plane, form the quaternionic projective space HP^{n-1} and fill out H^n, with any two meeting only at the origin. The unit 3-spheres on these quaternionic lines give us the *Hopf fibration*

$$H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}.$$

PROPOSITION 3.1. *The Hopf 3-spheres on S^{4n-1} are parallel to one another.*

The proof is similar to that of Proposition 1.1 for Hopf circles; it uses the fact that scalar multiplication by i, j and k are isometries of H^n. Alternatively, it will follow, as in Remark 1 above, from Proposition 4.2 a.

QED

The Riemannian metric on HP^{n-1} which makes the Hopf projection $S^{4n-1} \to HP^{n-1}$ into a Riemannian submersion is known as the *canonical metric* on HP^{n-1}. The canonical metric on HP^1 makes it into a round 4-sphere of radius $1/2$. This follows by the same argument given in Proposition 1.2 for the case $H: S^1 \hookrightarrow S^3 \to CP^1$.

4. SYMMETRIES OF THE HOPF FIBRATIONS WITH FIBRE S^3

We now investigate the symmetries of the Hopf fibration

$$H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}.$$

The symplectic group

$$Sp(n) = Gl(n, H) \cap O(4n)$$

consists of quaternionically linear maps which are also rigid. Since these maps take quaternionic lines to quaternionic lines, they must be symmetries of the above Hopf fibration.

There are other symmetries. For each unit quaternion v, consider the action of right scalar multiplication by v on H^n,

$$R_v(u_1, ..., u_n) = (u_1 v, ..., u_n v).$$

This map is certainly *not* H-linear, since

$$R_v[(u_1, ..., u_n)w] = (u_1 w v, ..., u_n w v),$$

while

$$[R_v(u_1, ..., u_n)] w = (u_1 v w, ..., u_n v w).$$

Nevertheless, R_v takes each quaternionic line in H^n to itself. Thus the group S^3 of unit quaternions, acting on H^n from the right, must also be counted among the symmetries of our Hopf fibration.

Since the symplectic group $Sp(n)$ acts on S^{4n-1} from the left, while the group S^3 of unit quaternions acts from the right, these two actions

commute. The actions also overlap, because they both contain multiplication by -1. Hence they combine to give an action of the group

$$\frac{Sp(n) \times S^3}{2}$$

on S^{4n-1}, where this group is obtained from $Sp(n) \times S^3$ by dividing out by the two-element subgroup consisting of the identity and the antipodal map. The following lemma asserts that there are no further symmetries.

PROPOSITION 4.1. *The group G of all symmetries of the Hopf fibration H is*

$$G = \frac{Sp(n) \times S^3}{2}.$$

Let g be a symmetry of the Hopf fibration, i.e., a rigid map of H^n taking quaternionic lines to quaternionic lines. Composing g with an appropriate element of $Sp(n)$, we can arrange that the new g be invariant on each quaternionic coordinate line $0 \times \ldots \times H \times \ldots \times 0$.

We claim this new g is orientation preserving on $H \times 0 \times \ldots \times 0$. Suppose not. Then composing it with appropriate elements of $Sp(n)$ and S^3, we can further arrange that $g(u, \ldots) = (\bar{u}, \ldots)$. Here we use the fact that left and right multiplication by unit quaternions generates the group $SO(4)$. Since g takes quaternionic lines to quaternionic lines, we must have

$$g(u, u, \ldots) = (\bar{u}, m\bar{u}, \ldots), \quad \text{for some} \quad m \neq 0.$$

Then for any s,

$$g(u, su, \ldots) = \big(\bar{u}, m\overline{(su)}, \ldots\big) = (\bar{u}, m\,\bar{u}\,\bar{s}, \ldots).$$

As u varies, these image points must also fill out a quaternionic line, hence $m\,\bar{u}\,\bar{s} = t\,\bar{u}$. Putting $u = 1$, we get $t = m\,\bar{s}$. Thus $m\,\bar{u}\,\bar{s} = m\,\bar{s}\,\bar{u}$. Cancelling the m, we get $\bar{u}\,\bar{s} = \bar{s}\,\bar{u}$. Since both u and s are arbitrary, this is impossible, establishing the claim.

Thus g is orientation preserving on $H \times 0 \times \ldots \times 0$, and we compose it with appropriate elements of $Sp(n)$ and S^3 so as to make it the identity there. Then we again use the fact that g takes quaternionic lines to quaternionic lines to conclude that

$$g(u, u, \ldots, u) = (u, m_2\,u, \ldots, m_n\,u).$$

Hence

$$g(u_1, u_2, ..., u_n) = (u_1, m_2 u_2, ..., m_n u_n),$$

so the current version of g must lie in $Sp(n)$.   **QED**

Remark. Note that all the symmetries are orientation preserving, since the group G is connected.

Let $H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}$ denote our current Hopf fibration, and let us orient the fibres in a consistent fashion. The next proposition shows that this fibration is highly symmetric, yet slightly less so than the Hopf fibrations by circles.

PROPOSITION 4.2. *Let* $H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}$ *be a Hopf fibration. Then*

a) *The only symmetries of H inducing the identity on the base space are the right multiplications by unit quaternions. This is just a 3-parameter subgroup of the 6-parameter group $O(4)$ of all rigid motions of a fibre.*

b) *If P and Q are any two fibres, then any preassigned orientation preserving rigid motion of P onto Q can be extended to a symmetry of H. But no orientation reversing one can.*

c) *The group of symmetries acts transitively on S^{4n-1}, and in particular acts transitively on fibres.*

It follows easily from the non-commutativity of the quaternions that the only transformations in $Sp(n)$ which take each quaternionic line to itself are $\pm Id$. Then a) follows immediately from the description of the symmetry group given in Proposition 4.1.

Even the subgroup $Sp(n)$ of G acts transitively on S^{4n-1}, and c) follows.

To prove b), we can now assume that P and Q both coincide with the unit 3-sphere on $H \times 0 \times ... \times 0$. Then left and right multiplication by unit quaternions takes this fibre to itself, and generates $SO(4)$. No orientation reversing transformation of this fibre can be achieved, since the group of symmetries is connected. This proves b).   **QED**

Remarks. 1) Note that the existence of symmetries of H taking each fibre to itself and acting transitively on a given fibre shows that these fibres must be parallel.

2) Also note that a symmetry of $H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}$ induces an

isometry of the base space HP^{n-1} in its canonical metric. It is easy to check that when $n = 2$, every orientation preserving isometry of the base $HP^1 = S^4(1/2)$ can be produced this way, while no orientation reversing one can (since the group is connected). We remark without proof that *all* isometries of HP^{n-1}, $n > 2$, can be produced this way, and that they are all orientation preserving.

5. Normed Division Algebras and the Cayley Numbers

In order to describe the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$ in the next section, we first review here some facts about normed division algebras and the arithmetic of Cayley numbers. More can be found in two excellent references, [Cu] and [H-L, pp. 140-145].

A *normed division algebra* B is a finite dimensional algebra over the reals R, with multiplicative unit 1, and equipped with an inner product $<, >$ whose associated norm $| \ |$ satisfies

$$|xy| = |x| |y| \quad \text{for all} \quad x, y \in B.$$

By Hurwitz' Theorem ([Hu 1], 1898), a proof of which we will outline here, every normed division algebra is isomorphic to either the reals R, the complex numbers C, the quaternions H or the Cayley numbers Ca. Actually, what Hurwitz proved is that normed division algebras can only occur in dimensions 1, 2, 4 and 8. He stated the corresponding uniqueness result without proof. In [Hu 2], published in 1923 after his death, Hurwitz credits E. Robert [Ro] with writing out the details of the uniqueness argument in a 1912 Zurich thesis.

Now let B denote a given normed division algebra. Let $\text{Re } B$ denote the one-dimensional linear subspace spanned by the identity 1, and $\text{Im } B$ the orthogonal complement of $\text{Re } B$. Then each $x \in B$ has a unique orthogonal decomposition,

$$x = x_1 + x', \quad x_1 \in \text{Re } B \quad \text{and} \quad x' \in \text{Im } B,$$

into its real and imaginary parts. *Conjugation* in B is defined by:

$$\bar{x} = x_1 - x'.$$

Here are some basic facts about arithmetic in any normed division algebra B:

1) $<xw, yw> = <x, y> |w|^2 = <wx, wy>$.

    Thus right or left multiplication by a unit vector w is an isometry of B.

2) Every nonzero $x \in B$ has a unique left and right inverse:

$$x^{-1} = \bar{x}/|x|^2.$$

3) Given x and y in B with $x \neq 0$, the equations

$$xw = y \quad \text{and} \quad wx = y$$

can each be solved uniquely, with

$$w = x^{-1}y \quad \text{and} \quad w = yx^{-1}$$

respectively.

4) $\quad xy = \bar{y}\,\bar{x}$.

5) If x is imaginary (that is, $x \in \text{Im } B$), then $x^2 = -|x|^2$.

6) Orthogonal imaginaries anti-commute. That is,

$$x, y \in \text{Im } B \quad \text{and} \quad <x, y> = 0 \quad \text{imply} \quad xy = -yx.$$

7) The Moufang identities, the first two of which say that left and right multiplication by xyx can be performed successively:

$$(xyx)z = x(y(xz))$$
$$z(xyx) = ((zx)y)x$$
$$x(yz)x = (xy)(zx).$$

Given three elements $x, y, z \in B$, their *associator* is defined by

$$[x, y, z] = (xy)z - x(yz).$$

The following weak form of associativity always holds in a normed division algebra: the trilinear form $[x, y, z]$ is *alternating*, i.e., it vanishes whenever two of its arguments are equal. Such an algebra is said to be *alternative*.

    The Cayley-Dickson process generalizes the way in which the complex numbers are built up from the reals, and begins with the following

PROPOSITION 5.1. (see [Cu] or [H-L]). *Let A be a subalgebra (containing 1) of the normed division algebra B. Let ε be an element of B orthogonal to A with $|\varepsilon| = 1$. Then*

i) *$A\varepsilon$ is orthogonal to A, and*

ii) *$(a+b\varepsilon)(c+d\varepsilon) = (ac-\bar{d}b) + (da+b\bar{c})\varepsilon$ for all a, b, c, d in A.*

The proof makes use of the commutation rules given in Fact 6 above.

We note for future use that the above proposition implies that any subalgebra of Ca generated by two elements must be isomorphic to R, C or H, and hence must be associative.

Suppose now that we start with a normed division algebra A and define a product on $A \oplus A$ by

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}).$$

The new algebra $B = A \oplus A$ is said to be obtained from A via the *Cayley-Dickson process*. In particular,

$$C = R \oplus R, \quad H = C \oplus C, \quad Ca = H \oplus H$$

via the Cayley-Dickson process.

PROPOSITION 5.2. (Jacobson [Ja], 1958). *Suppose* $B = A \oplus A$ *is obtained from* A *by the Cayley-Dickson process. Then*

1) B *is commutative* $\Leftrightarrow$ $A = R$.

2) B *is associative* $\Leftrightarrow$ A *is commutative*.

3) B *is alternative* $\Leftrightarrow$ A *is associative*.

See [Cu] or [H-L] for details.

From this proposition, we have:

$C = R \oplus R$ is commutative;

$H = C \oplus C$ is associative, but not commutative;

$Ca = H \oplus H$ is alternative, but not associative;

$Ca \oplus Ca$ is not alternative, *hence not a normed division algebra.*

THEOREM 5.3. (Hurwitz [Hu 1]). *The only normed division algebras are* R, C, H *and* Ca.

One can check directly that R, C, H and Ca are normed, though the calculation for Ca is somewhat lengthy. An alternative argument can be found in [Cu]. That there are no other normed division algebras follows from Propositions 5.1 and 5.2.

We end this section with the following description of all possible automorphisms of the Cayley numbers.

PROPOSITION 5.4. *Suppose* e_1, e_2 *and* e_3 *are orthonormal imaginary Cayley numbers with* e_3 *orthogonal to* $e_1 e_2$. *Then there exists a unique automorphism of* Ca *sending* $i = (i, 0) \mapsto e_1$, $j = (j, 0) \mapsto e_2$ *and* $\varepsilon = (0, 1) \mapsto e_3$.

This follows from three applications of Proposition 5.1.

From Proposition 5.4, one concludes that the group of all automorphisms of the Cayley numbers (a Lie group known as G_2) is 14-dimensional.

6. THE HOPF FIBRATION $S^7 \hookrightarrow S^{15} \to S^8$

Choose orthonormal coordinates in R^{16} and identify it with Cayley 2-space Ca^2. In Ca^2 consider subsets of the form

$$L_m = \{(u, mu): u \in Ca\} \quad \text{for each} \quad m \in Ca,$$
$$L_\infty = \{(0, v): v \in Ca\}.$$

They are 8-dimensional real linear subspaces of R^{16}, but *not* Cayley subspaces of Ca^2 because they are not closed under Cayley multiplication. This is the effect of the nonassociativity of the Cayley numbers. Nevertheless, we call L_m and L_∞ *Cayley lines* for simplicity.

We need to check that these Cayley lines fill out Ca^2, with any two meeting only at the origin. Given $(u, v) \in Ca^2$, if $u = 0$ then this point is on the Cayley line L_∞. If $u \neq 0$, let $m = v u^{-1}$. Then $m u = (v u^{-1}) u = v$ by Fact 3 of the preceding section. Hence the point (u, v) lies on the Cayley line L_m. Thus the Cayley lines fill out Ca^2.

Clearly L_∞ meets each other Cayley line only at the origin. And if the point (u, v), with $u \neq 0$, lies on the Cayley lines L_m and L_n, then $v = m u = n u$. Hence $m = n$. Thus any two Cayley lines meet only at the origin.

The unit 7-spheres on these Cayley lines then define for us the *Hopf fibration* $S^7 \hookrightarrow S^{15} \to S^8$. Note that the base space is clearly homeomorphic to an 8-sphere, since there is one Cayley line for each Cayley number m, and one for the number ∞.

In a similar fashion, if we start with any k-dimensional normed division algebra K, we obtain a Hopf fibration

$$S^{k-1} \hookrightarrow S^{2k-1} \to S^k.$$

Note by Hurwitz's theorem that K is isomorphic to R, C, H or Ca, so there are really no new cases.

PROPOSITION 6.1.   *The Hopf 7-spheres on S^{15} are parallel to one another.*

We must show that the 8-planes

$$P = L_v = \{(u, vu)\} \quad \text{and} \quad Q = L_w = \{(u, wu)\}$$

intersect S^{15} in parallel great 7-spheres.

Let the vectors $e_i, i = 1, ..., 8$ form an orthonormal basis for Ca. Then the vectors $(e_i, v\, e_i), i = 1, ..., 8$ form an orthogonal basis for P, with each vector having length $(1+|v|^2)^{1/2}$. This is an immediate consequence of Fact 1 from the preceding section.

Likewise, the vectors $(e_j, w\, e_j), j = 1, ..., 8$ form an orthogonal basis for Q, with each vector having length $(1+|w|^2)^{1/2}$.

With respect to these bases, the matrix $A = (a_{ij})$ of orthogonal projection of P to Q is given by

$$a_{ij} = <e_i, e_j> + <v\, e_i, w\, e_j>,$$

or

$$A = I + B.$$

We want to show that A is conformal, i.e., that

$$A A^t = I + B + B^t + B B^t = \lambda I.$$

First note that

$$
\begin{aligned}
(B+B^t)_{ij} &= <ve_i, we_j> + <ve_j, we_i> \\
&= <(v+w)e_i, (v+w)e_j> - <ve_i, ve_j> - <we_i, we_j> \\
&= (|v+w|^2 - |v|^2 - |w|^2) <e_i, e_j> \\
&= 2<v, w> \delta_{ij},
\end{aligned}
$$

by repeated application of Fact 1 of the preceding section. Thus $B + B^t$ is a multiple of the identity.

Next note that

$$
\begin{aligned}
(B B^t)_{ij} &= \Sigma_r <ve_i, we_r> <ve_j, we_r> \\
&= <ve_i, ve_j> |w|^2 = |v|^2 |w|^2 \delta_{ij},
\end{aligned}
$$

since $we_r, r = 1, ..., 8$ is an orthogonal basis for Ca with each vector of length $|w|$. Thus $B B^t$ is also a multiple of the identity.

It follows that A is conformal, and hence that the 8-planes $P = L_v$ and $Q = L_w$ intersect S^{15} in parallel great 7-spheres. By continuity, the

same is true if one of these planes is L_∞. Thus the Hopf 7-spheres on S^{15} are parallel to one another, as claimed.   QED

The Riemannian metric on the base space S^8 which makes the Hopf projection $S^{15} \to S^8$ into a Riemannian submersion is that of a round 8-sphere of radius 1/2, which one sees directly just as in the previous cases.

7. SYMMETRIES OF THE HOPF FIBRATION $H: S^7 \hookrightarrow S^{15} \to S^8$

PROPOSITION 7.1. *The group G of all symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$ is isomorphic to $Spin(9)$, the simply connected double cover of $SO(9)$.*

The action is as follows:

1) *There is a $g \in G$ inducing any preassigned orientation preserving isometry of the round base S^8, but no orientation reversing ones.*

2) *Given such a g, there is exactly one other symmetry,*

$$-g = antipodal\ map \circ g\,,$$

which induces the same action on S^8.

It is likely that Élie Cartan was aware of this result, since in [Ca 2, esp. pp. 424 and 466] he identified $Spin(9)$ as the group of isometries fixing a point in the Cayley projective plane CaP^2. It is not hard to see that this is the same as the group of symmetries of our Hopf fibration. The symmetry groups of the other Hopf fibrations can likewise be identified with the groups of isometries fixing a point in complex and quaternionic projective spaces, also known to Cartan.

We give the proof of Proposition 7.1 in a series of lemmas.

LEMMA 7.2. *The only symmetries which take each fibre to itself are the identity and the antipodal map.*

Suppose $B: R^{16} \to R^{16}$ is such a symmetry. Since B maps

$$L_0 = \{(u, 0)\}\,, \quad L_\infty = \{(0, v)\} \quad \text{and} \quad L_1 = \{(u, u)\}$$

into themselves, we must have

$$B(u, v) = (A(u), A(v))$$

for some $A \in O(8)$. Since B maps $L_m = \{(u, mu)\}$ into itself, we get

$$B(u, mu) = (A(u), A(mu)) = (A(u), mA(u)).$$

Thus $\qquad A(mu) = mA(u), \quad \text{all} \quad m, u \in Ca.$

Now in this equation put $u = 1$ and keep m arbitrary:

$$A(m) = mA(1) = ma,$$

where we define $a = A(1)$. Insert this back into the previous equation, getting

$$(mu)a = m(ua), \quad \text{for all} \quad m, u \in Ca.$$

But then it follows from the nonassociativity of the Cayley numbers that the element a must be real. Since $A \in O(8)$, $a = \pm 1$. Thus $A(m) = \pm m$, and hence $B(u, v) = (\pm u, \pm v)$, that is, B is either the identity or the antipodal map, as claimed. $\qquad\qquad$ QED

If we compare Lemma 7.2 with the corresponding assertions about the earlier Hopf fibrations, we conclude that the current Hopf fibration is the least symmetric of all.

LEMMA 7.3. *There is a symmetry of our Hopf fibration inducing any preassigned orientation preserving isometry of the base which keeps L_0 fixed.*

Such a symmetry must also take the orthogonal fibre $L_\infty = \{(0, v)\}$ to itself, and hence must be of the form

$$(u, v) \mapsto (A(u), B(v)), \quad \text{where} \quad A, B \in O(8).$$

Given such a symmetry, the Cayley line $L_m = \{(u, mu)\}$ is taken to the set $\{(A(u), B(mu))\}$, which must itself be some Cayley line, say $L_{m'}$. Thus $B(mu) = m' A(u)$. Note that as a function of u, the left hand side is conformal with conformal factor $|m|$, while the right hand side is conformal with factor $|m'|$. Hence $|m| = |m'|$. Since the correspondence $m \mapsto m'$ is easily seen to be R-linear, it must be an isometry. Hence we can write $m' = C(m)$, with $C \in O(8)$.

Summarizing so far, a symmetry of our Hopf fibration which takes the fibre L_0 to itself must be of the form (A, B) with $A, B \in O(8)$, *and* there must exist a $C \in O(8)$ such that

$$B(mu) = C(m) A(u), \quad \text{for all} \quad m, u \in Ca.$$

Vice versa, if such a C exists, then the map (A, B) is indeed a symmetry of the Hopf fibration.

Since it is C which describes the induced action on the base space S^8, we need to be able to preassign $C \in SO(8)$. The possibility of doing this is the content of the "Triality Principle", as follows.

LEMMA 7.4. (Triality Principle for $SO(8)$, see [Ca 1, pp. 370 and 373] and [Fr]). *Consider the triples* A, B *and* C *in* $SO(8)$ *such that*

$$B(mu) = C(m) A(u), \quad for \ all \quad m, u \in Ca.$$

If any one of these three isometries is preassigned, then the other two exist and are unique up to changing sign for both of them.

We concentrate on preassigning C. Let G be the subset of $SO(8)$ consisting of all transformations C for which there exist A and B in $SO(8)$ satisfying the above equation for all $m, u \in Ca$. First note that G is actually a subgroup of $SO(8)$. For suppose that C and C' are in G, and correspond as above to A, B and A', B' respectively. Then

$$BB'(mu) = B\big(C'(m) A'(u)\big) = CC'(m) AA'(u),$$

showing that $CC' \in G$. And similarly for inverses.

We want to show that G is all of $SO(8)$. Let x be an imaginary Cayley number of unit length. We claim

(7.5)   *The right and left translations* R_x *and* L_x *are in* G.

To show this, we use the first two Moufang identities.

To satisfy $B(mu) = C(m) A(u)$ with $C = R_x$, choose $A = -L_x R_x$ and $B = R_x$. We must show that

$$(mu)x = -(mx)(xux).$$

To do this, simply take the Moufang identity

$$z(xyx) = ((zx)y)x$$

and put $x = x, y = u$ and $z = mx$, getting

$$(mx)(xux) = ((mxx)u)x = -(mu)x,$$

since $x^2 = -1$. Thus $R_x \in G$.

To satisfy $B(mu) = C(m) A(u)$ with $C = L_x R_x$, choose $A = L_x$ and $B = -L_x$. We must show that

$$-x(mu) = (xmx)(xu).$$

To do this, take the Moufang identity

$$(xyx)z = x(y(xz))$$

and put $x = x$, $y = m$ and $z = xu$, getting

$$(xmx)(xu) = x(m(xxu)) = -x(mu),$$

since $x^2 = -1$ as before. Thus $L_x R_x \in G$. Since we already know that G is a group and that it contains R_x, it must also contain L_x, establishing our claim.

Next we claim

(7.6)   *The transformations* R_x *and* L_x, *as* x *ranges over all imaginary unit Cayley numbers, generate* $SO(8)$.

Since the subgroup G contains these transformations, this will show that G is all of $SO(8)$.

First note that any unit vector can be mapped to any other unit vector by a composition of such transformations. To see this, first suppose that u and v are orthogonal unit vectors: $<u, v> = 0$. Then $<1, vu^{-1}> = 0$. Hence $x = vu^{-1}$ is an imaginary unit Cayley number such that $L_x(u) = (vu^{-1})u = v$. If u and v are unit vectors, but not necessarily orthogonal, just pick a unit vector w orthogonal to both. Find L_x and $L_{x'}$ such that $L_x(u) = w$ and $L_{x'}(w) = v$. Then $L_{x'}L_x(u) = v$, as desired.

So now it will be sufficient to show that any transformation in $SO(8)$ keeping 1 fixed is a composition of right and left translations by imaginary unit Cayley numbers. One such transformation is $-L_x R_x$ for any imaginary unit Cayley number x. Note that $-L_x R_x(x) = x$, so that this transformation also keeps x fixed. On the other hand, if y is an imaginary Cayley number orthogonal to x, then

$$-L_x R_x(y) = -xyx = xxy = -y,$$

since orthogonal imaginaries anti-commute by Fact 6. Thus $-L_x R_x$ is the identity on the 2-plane spanned by 1 and x, and is minus the identity on the orthogonal 6-plane. Viewed just on the imaginary Cayley numbers, this transformation is reflection about the line through x.

But it is easy to see that the set of reflections through all lines in R^7 generates $SO(7)$. Hence the transformations R_x and L_x, as x ranges over all imaginary unit Cayley numbers, generate $SO(8)$, as claimed.

Thus the subgroup G of transformations C in $SO(8)$, for which one can find A and B in $SO(8)$ satisfying $B(mu) = C(m) A(u)$ for all Cayley

numbers m and u, must be all of $SO(8)$. In a similar fashion, one can preassign either A or B and find the other two, completing the proof of existence for the Triality Principle.

To prove uniqueness up to sign change for the Triality Principle, suppose C is the identity. Thus $B(mu) = m A(u)$ for all $m, u \in Ca$. Put $m = 1$ to learn that $B(u) = A(u)$. So now $A(mu) = m A(u)$. Put $u = 1$ to get $A(m) = m A(1) = ma$, where we define $a = A(1)$. Then put this back in the previous equation to get $(mu)a = m(ua)$. Since this holds for all $m, u \in Ca$, the element a must be real. Since A is orthogonal, $a = \pm 1$. Thus $A = B = \pm I$, proving uniqueness up to sign change when $C = I$. Uniqueness up to sign change for all $C \in SO(8)$ follows by composition. A similar argument gives uniqueness up to sign change when A or B is preassigned, completing the proof of the Triality Principle.   QED

Preassigning C and using the Triality Principle to select A and B then completes the proof of Lemma 7.3: there is a symmetry of our Hopf fibration inducing any preassigned orientation preserving isometry of the base which keeps L_0 fixed.   QED

We next use Lemma 7.3 to sharpen itself.

LEMMA 7.7. *There is a symmetry of our Hopf fibration inducing any preassigned orientation preserving isometry of the base. In particular, there is a symmetry taking any fibre to any other.*

On the base space S^8, we take the north pole to be L_0 and the south pole to be L_∞. Then the equator will consist of all L_m for which $|m| = 1$. Now consider the circle consisting of the points L_m for real m. We plan to show that this circle is contained in the orbit of L_0 under the symmetry group of H. Since this circle meets the equator in two points, L_1 and L_{-1}, we can then use (7.3) to conclude that the orbit of L_0 is all of S^8. Combining again with (7.3) will yield (7.7).

Consider the map $A_\phi : Ca^2 \to Ca^2$ defined by

$$A_\phi(u, v) = (\cos\phi\, u - \sin\phi\, v, \sin\phi\, u + \cos\phi\, v).$$

These maps, for $0 \leq \phi \leq 2\pi$, provide a circle group of isometries of Ca^2. We claim

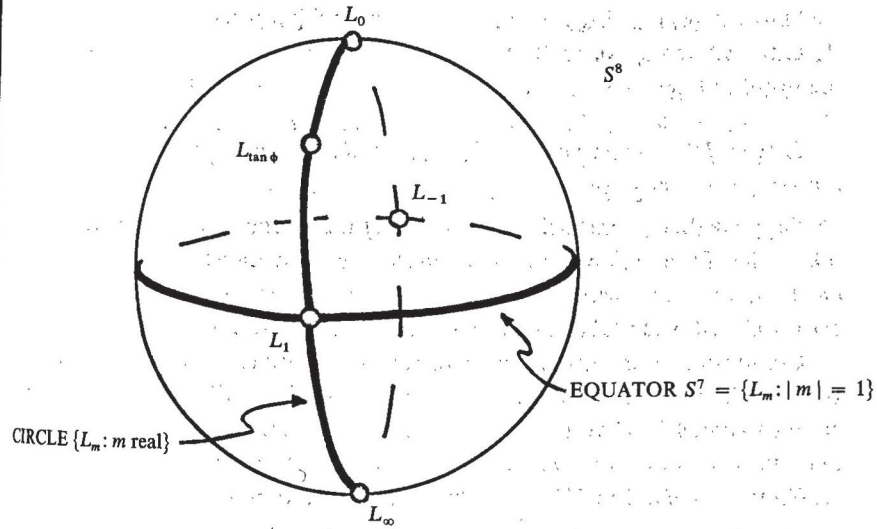(7.8)   *Each A_ϕ is a symmetry of our Hopf fibration.*

FIGURE 4

We must show that, given $m \in Ca$, there exists $m' \in Ca$ such that $A_\phi(L_m) = L_{m'}$. Now

$$A_\phi(u, mu) = (\cos\phi\, u - \sin\phi\, mu, \sin\phi\, u + \cos\phi\, mu)$$

$$= ((\cos\phi - \sin\phi\, m)\, u, (\sin\phi + \cos\phi\, m)\, u).$$

Let $\quad u' = (\cos\phi - \sin\phi\, m)\, u$

and $\quad m' = (\sin\phi + \cos\phi\, m)(\cos\phi - \sin\phi\, m)^{-1}.$

Then $\quad m'u' = [(\sin\phi + \cos\phi\, m)(\cos\phi - \sin\phi\, m)^{-1}]\,[(\cos\phi - \sin\phi\, m)\, u].$

The product on the right hand side may be reassociated because all the elements lie in the subalgebra of Ca spanned by the *two* elements m and u. As noted in section 5, such a subalgebra must be associative. But then clearly

$$m'u' = (\sin\phi + \cos\phi\, m)\, u,$$

so that we have

$$A_\phi(u, mu) = (u', m'u').$$

Thus $A_\phi(L_m) = L_{m'}$, so each A_ϕ is a symmetry of our Hopf fibration, as claimed.

Since $A_\phi(L_0) = L_{\tan\phi}$, we see that the orbit of L_0 under the various A_ϕ is the circle $(L_m: m \text{ real})$. As indicated above, this is enough to complete the proof of (7.7). QED

LEMMA 7.9. *No symmetry of our Hopf fibration can induce an orientation reversing isometry of the base.*

Suppose there were such a symmetry. Using Lemma 7.7, we can assume it takes the fibre $L_0 = \{(u, 0)\}$ to itself. Then it must be of the form $(u, v) \mapsto (A(u), B(v))$ with $A, B \in O(8)$, and as we saw in (7.3) there must exist a $C \in O(8)$ such that $B(mu) = C(m) A(u)$ for all $m, u \in Ca$.

Composing our symmetry with an appropriate one guaranteed by Lemma 7.3 we can assume that $C(m) = \bar{m}$. Thus $B(mu) = \bar{m} A(u)$. Put $m = 1$ to conclude that $A = B$. Thus $A(mu) = \bar{m} A(u)$. Put $u = 1$ to conclude that $A(m) = \bar{m} A(1) = \bar{m} a$. Then put this back in the previous equation to get $(\overline{mu})a = \bar{m}(\bar{u}a)$. But $\overline{mu} = \bar{u}\,\bar{m}$ by Fact 4 of section 5. Hence

$$(\bar{u}\,\bar{m})a = \bar{m}(\bar{u}\,a) .$$

Now replace $\bar{u}$ by u and $\bar{m}$ by m to get

$$(um)a = m(ua) \quad \text{for all} \quad u, m \in Ca .$$

But this equation is impossible, which we see as follows.

Simply choose an automorphism of the Cayley numbers, see (5.4), which moves the element a to a unit quaternion. Apply such an automorphism to the above equation, and now consider that equation only for the quaternions:

$$(um)a = m(ua) \quad \text{for all} \quad u, m \in H .$$

But the quaternions are associative, so we remove the parentheses, then cancel the a and learn that

$$um = mu \quad \text{for all} \quad u, m \in H ,$$

which is of course false. QED

Proof of (7.1). Let G again denote the group of all symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$. Consider the homomorphism $G \to O(9)$, which takes each $g \in G$ to its induced action on the base space S^8. By Lemma 7.9, the image lies in $SO(9)$. By Lemma 7.7, the homomorphism is onto. By Lemma 7.2, it is two-to-one. Thus G is a double covering of $SO(9)$. It remains to show that this covering is nontrivial.

It will be sufficient to look only at the symmetries of H which take the fibre $L_0 = \{(u, 0)\}$ to itself, and hence are of the form $(u, v) \mapsto \big(A(u), B(v)\big)$. We already know that there must be a $C \in SO(8)$ such that $B(mu) = C(m) A(u)$ for all $m, u \in Ca$. To show that G is a nontrivial double covering of $SO(9)$, we must find a *loop* of C's which lifts to a *non-loop* of (A, B)'s.

This can be done by using the Moufang identities, just as in the proof of the Triality Principle. Recall from that proof that if x is an imaginary Cayley number of unit length, then $A = L_x$, $B = -L_x$ and $C = L_x R_x$ "works", that is, $-L_x(mu) = L_x R_x(m) L_x(u)$. Now let x describe a semi-circular path in the i, j-plane from i to $-i$. At the beginning of the path, $C(m) = imi$, while at the end of the path $C(m) = (-i)m(-i) = imi$. Thus C describes a loop in $SO(8)$. At the beginning of the path, $\big(A(u), B(v)\big) = (iu, -iv)$, while at the end $\big(A(u), B(v)\big) = (-iu, iv)$. Hence (A, B) describes a non-loop in G. Thus G is the non-trivial double covering Spin(9) of $SO(9)$.

<div align="right">QED</div>

Here is a further indication of the extent of symmetry of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$. Orient the fibres.

PROPOSITION 7.10. *Let P and Q be any two fibres of H. Then a preassigned orientation preserving rigid motion of P onto Q can be extended to a symmetry of H. In particular, the symmetries act transitively on S^{15}.*

By Lemma 7.7, the symmetries act transitively on fibres, so we may take $P = Q = L_0$. To preassign an orientation preserving rigid motion of L_0 onto itself is to preassign the map $A \in SO(8)$ in the Triality Principle, which then promises the desired symmetry of H.

<div align="right">QED</div>

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