THE BSD CONJECTURE, REGULATORS AND SPECIAL VALUES OF L-FUNCTIONS

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Contents

1. Introduction

Much of what is written in this note follows the line of argument in the excellent survey paper by Ramakrishnan [Ram89]. The aim is to understand why the BSD conjecture (10) presents so accurately the residue at the pole $s = 1$ and why so many global invariants of an elliptic curve appear in this residue. A similar statement for the Dedekind zeta function attached to a number field appears in the class number formula proven by Dirichlet and Dedekind over one hundred years ago. After the formulation of the Birch-Swinnerton-Dyer conjecture in the early 1960s, the question remained what generalisations of formulas of this type could appear. Tate initiated this search by generalising to abelian varieties. The work of Borel on regulators followed by the work of Quillen and Bass-Milnor-Serre on K-theory played a key role in the creation of a platform suitable for generalisation. Interpreting regulator maps and their connection to cycle maps through the development of higher Chow rings and higher K-theory by Bloch and others formed the backbone of Beilinson's influential paper [Bei84] that led to vast generalisations of conjectures of the BSD type, most of which remain unproven in generality.

2. From Riemann to Dedekind

Let $K/\mathbb{Q}$ be a number field of degree n and let $\mathcal{O}_k$ be its ring of integers. Denote by r_1, r_2 the number of inequivalent real embeddings of K and the number of non-conjugate complex embeddings of K respectively. Classical results in algebraic number theory give us $n = r_1 + 2r_2$. Let $N_{K/\mathbb{Q}}(I) = [\mathcal{O}_K : I]$ denote the norm of an ideal I in $\mathcal{O}_K$.

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DEFINITION 2.1. The **Dedekind zeta function** is defined for $s \in \mathbb{C}$ as

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} N_{K/\mathbb{Q}}(I)^{-s}$$

As with the Riemann zeta function, the above converges absolutely for $Re(s) > 1$ and admits a (Euler) product formula

$$\zeta_K(s) = \prod_{P \subset \mathcal{O}_K} (1 - N_{K/\mathbb{Q}}(P)^{-s})^{-1}$$

where the product ranges over all prime ideals in $\mathcal{O}_K$. Next, define

$$\begin{aligned} K \to K_{\mathbb{C}} &= \prod_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{C} : a \mapsto (\tau_1 a, \ldots, \tau_n a) \\ j : K \to K_{\mathbb{R}} &= K_{\mathbb{C}}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}. \end{aligned}$$

$K_{\mathbb{R}}$ is called the Minkowski space of K and is a Euclidean real vector space. We have isomorphisms $K_{\mathbb{R}} \cong K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1+2r_2}$ and $K_{\mathbb{C}} \cong K \otimes_{\mathbb{Q}} \mathbb{C}$. We then have that the image of every ideal I in $\mathcal{O}_K$ under the map j is a lattice in $K_{\mathbb{R}}$ and one can prove that the volume (induced by the metric on this Euclidean vector space) of this lattice is given by the formula

$$\mathrm{vol}(jI) = \sqrt{|D_K|}[\mathcal{O}_K : I] = \sqrt{|D_K|} N_{K/\mathbb{Q}}(I)$$

With these notions in mind, one can set up a Haar measure and a suitable Melin transform and more importantly, prove that $\zeta_K(s)$ can be meromorphically continued to a function on the entire complex plane with a simple pole at $s = 1$ which has a functional equation (Hecke, 1917). Namely, if we let

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(\tfrac{s}{2}) \tag{2}$$

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s) \tag{3}$$

and write the "archimedean part" as $\zeta_{K_\infty}(s) = (\Gamma_{\mathbb{R}}(s))^{r_1} (\Gamma_{\mathbb{C}}(s))^{2r_2}$ and let $\zeta_K^*(s) = \zeta_{K_\infty}(s)\zeta_K(s)$ then the functional equation reads

$$\zeta_K^*(s) = D_K^{\frac{1}{2}-s} \zeta_K^*(1-s)$$

where D_K is the discriminant of the number field K.

Note that $\zeta_K(s)$ is a product over all the non-archimedean places whereas $\zeta_K^*(s)$ encodes all the "local information", that is archimedean and non-archimedean. The factors $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$ appear in a similar nature for the functional equation for the Riemann zeta function

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

whose proof goes back to Riemann himself who used a Melin transform argument.

What is remarkable here is that the functional equation applies when one considers every single place of the number field and not just the non-archimedean ones. It is exactly this type of "local to global" construction that Tate investigated in his famous thesis where he develops an abstract Fourier theory and Pontryagin duality for locally compact Hausdorff topological groups. As I will try and outline in the rest of this note, these results fall in a huge category of results and deep conjectures, linking the geometry or topology of various objects with many of their invariants.

The first non-trivial appearance of the above principle is majestically encoded in the following theorem of Dedekind and Dirichlet which is often called the class number formula, even though it gives us much more than simply a method of calculating the class number!

Theorem 2.2. *The residue at* 1 *of* $\zeta_K(s)$ *is given by the following formula*

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{|D_K|}} \tag{4}$$

Remark 2.3. A similar result exists for the Riemann zeta function. It is however trivial since the residue of $\zeta(s)$ at 1 is simply 1. One way of seeing this would also be to use the above theorem, since by letting $K=\mathbb{Q}$, we have $\zeta_{\mathbb{Q}}(s)=\zeta(s)$ and noticing what each of the terms on the right mean in the case.

Now I will try and briefly explain what each of these factors appearing on the right hand side are, so that it is more obvious later how one can generalise these to objects other than number fields. Firstly, h_K is the class number, a very important invariant of a number field which is defined as the order of the group of fractional ideals in $\mathcal{O}_K$ modulo the principal fractional ideals. In a sense this measures how far the ring of integers is from being a principal ideal domain. In the case that $K=\mathbb{Q}$, we have $h_{\mathbb{Q}}=1$. Next, w_K denotes the number of roots of unity $\mu(K)$ in K, that is the number of roots of polynomials of the form x^n-1 for $n\in\mathbb{N}$. The discriminant, which also appeared before in the functional equation, is the determinant of the trace form induced by the map $\mathrm{Tr}_{K/\mathbb{Q}}: K\to\mathbb{Q}$. As Milne describes (see [Mil08a]) the discriminant is an invariant associated to $\mathcal{O}_K$, whereas the regulator described below plays the same role for $\mathcal{O}_K^*$.

The important term appearing in the right hand side of (4) that remains to be discussed is the regulator R_K. I will here digress a bit further since the notion of "regulator" generalises vastly. What we aim is to gain information on the group of units $\mathcal{O}_K^*$ of the ring of integers of our algebraic number field. A famous theorem of Dedekind gives us the structure of this group as follows

$$\mathcal{O}_K^* = \mu(K)\times\mathbb{Z}^{r_1+r_2-1}$$

Remember that we have a total of $n=[K:\mathbb{Q}]$ embeddings $K\to\mathbb{C}$, r_1 of which are in fact into the real numbers $\mathbb{R}$. The regulator R_K of K is defined as the determinant of the (r_1+r_2-1, r_1+r_2-1)-submatrix of a matrix whose terms are the $||\log e_i||_j$ for $1\le i,j\le r_1+r_2$ for e_i the generators of the non-torsion of the unit group $\mathcal{O}_K^*$ and $||.||_j=|\sigma_j(.)|$ the norm induced by each embedding $\sigma_j: K^*\to\mathbb{R}$.

Now consider the free $\mathbb{Z}$-module generated by all the embeddings and denote it by X_K. Define the **regulator map**

$$r : \mathcal{O}_K^* \rightarrow X_K \otimes_{\mathbb{Q}} \mathbb{R} \tag{5}$$

$$u \mapsto \sum_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} \log |u|_\sigma \sigma \tag{6}$$

It turns out that this regulator map admits vast generalisations, mostly due to the work of Borel leading on from the foundations of K-theory of Grothendieck (see [BG02] for example). I will not digress too far into higher K-theory, but it is worth seeing the connection in elementary terms. For R a Dedekind domain, define the lower K-groups as

$$\begin{aligned} K_0(R) &= \mathrm{Pic}(R) \\ K_1(R) &= GL(R)/[GL(R), GL(R)], \text{ where } GL(R) = \mathrm{colim}_n\, GL_n(R) \end{aligned}$$

In the Dedekind domain case (for example the ring of integers of a number field, see [Wei05]), the Picard group (otherwise can be viewed as the Picard group of the affine scheme $Spec(R)$) in this case is the class group of R. On the other hand, we have that $K_1(\mathcal{O}_K)$ fits into the following sequence of isomorphisms

$$K_1(\mathcal{O}_K) \xrightarrow{\sim} H_1(GL_n(\mathcal{O}_K), \mathbb{Z}) \xrightarrow{\sim} \mathcal{O}_K^*$$

for any $n \geq 3$. The above isomorphisms are a result of Bass-Milnor-Serre in [BMS67].

What is now more obvious is that we can interpret the regulator map of (6) as a map from a K-group.

3. Hasse-Weil and the BSD

I briefly recall the definitions concerning the Hasse-Weil zeta function associated to an elliptic curve $\mathcal{E}/\mathbb{Q}$. The definitions generalise naturally to any smooth projective variety over a number field k. Denote by $\widetilde{\mathcal{E}}/\mathbb{F}_p$ the reduction modulo p of a global minimal model of $\mathcal{E}/\mathbb{Q}$ and let $\Delta_{\mathcal{E}/\mathbb{Q}}$ be the discriminant of $\mathcal{E}$.

Definition 3.1. let $\mathcal{E}/\mathbb{Q}$ be an elliptic curve and let $\mathbb{F}_q$ be a finite field of order $q = p^n$ for some prime number p. The local Hasse-Weil zeta function is defined as

$$\zeta_{\mathcal{E}/\mathbb{F}_q}(T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#\mathcal{E}(\mathbb{F}_{q^n})}{n} T^n\right)$$

The global zeta function over $\mathbb{Q}$ is defined as

$$\zeta_{\mathcal{E}/\mathbb{Q}}(s) = \prod_{p \text{ prime}} \zeta_{\widetilde{\mathcal{E}}/\mathbb{F}_p}(p^{-s})$$

The Hasse-Weil theorem states that for primes of good reduction, $\#\mathcal{E}(\mathbb{F}_q^n) = q^n - \alpha^n - \beta^n + 1$ for all $n \geq 1$ where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha| = |\beta| = \sqrt{q}$. For $n = 1$ it is common to denote

$a_q = \alpha + \beta$ and so we have the following equalities for p a prime of good reduction for $\mathcal{E}$

$$\zeta_{\mathcal{E}/\mathbb{F}_q}(T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#\mathcal{E}(\mathbb{F}_{q^n})}{n} T^n\right) = \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)} = \frac{1-a_qT+qT^2}{(1-T)(1-qT)} \tag{7}$$

$$\zeta_{\mathcal{E}/\mathbb{Q}}(s) = \prod_p \zeta_{\widetilde{\mathcal{E}}/\mathbb{F}_p}(p^{-s}) = \prod_p \frac{F_p(p^{-s})}{(1-p^{-s})(1-p^{1-s})} = \frac{\zeta(s)\zeta(s-1)}{L(\mathcal{E}/\mathbb{Q},s)} \tag{8}$$

where

$$F_p(T) = \begin{cases} 1 - a_pT + pT^2, & p \nmid \Delta \\ 1 - a_pT, & p \mid \Delta \end{cases}$$

The function appearing in the denominator of the global zeta function above is the L-function of $\mathcal{E}/\mathbb{Q}$ which converges for $Re(s) > \frac{3}{2}$ and is an isogeny invariant.

Similarly, in the case of a number field $K/\mathbb{Q}$, we have an L-function defined by

$$L(\mathcal{E}/K,s) = \prod_P F_P(q^{-s})^{-1}$$

where $q = N_{K/\mathbb{Q}}(P)$ and P ranges over the prime ideals in $\mathcal{O}_K$. In this case we have defined $F_P(T) = 1 - a_PT + qT^2$ for primes P of good reduction or $F_P(T) = 1 - a_PT$ for bad primes, where $a_P = q + 1 - \#\mathcal{E}(\mathbb{F}_q)$.

Remark 3.2. One of the key steps in the proof of the Weil conjectures is the Lefschetz trace formula which expresses $N_m = \#X/F_{q^m}$, for a non-singular projective variety over $\mathbb{F}_q$ of dimension d, as follows (see Milne [Mil08b])

$$N_m = \sum_r (-1)^r \operatorname{Tr}(F^m | H^r_{et}(X, \mathbb{Q}_l))$$

where F denotes the Frobenius map. One can then deduce the rationality of the local zeta function in the following form

$$Z(X,t) = \frac{P_1(X,t)P_3(X,t)\cdots P_{2d-1}(X,t)}{P_0(X,t)P_2(X,t)\cdots P_{2d}(X,t)}$$

where $P_i(X,t) = \det\left(1 - Ft|H^i_{et}(X,\mathbb{Q}_l)\right)$. In the case of elliptic curves, looking at the equations given in (7) and (8) we see that we only have a $P_1(X,t)$ term on the numerator which corresponds to the first étale cohomology group of X and the product of all these terms gives us the L-function, whereas the terms in the numerator come from the 0th and 2nd étale cohomology groups and form $\zeta(s)$ and $\zeta(s-1)$ respectively.

In correspondence with the case of the Dedekind zeta function, to obtain a suitable functional equation one must consider the terms arising from the archimedean places. One defines

$$L^*(\mathcal{E}/K,s) = \left(\Gamma(s)\left(\frac{\sqrt{N}}{2\pi}\right)^s\right)^{[K:\mathbb{Q}]} L(\mathcal{E}/K,s)$$

where N is the conductor of $\mathcal{E}/K$, defined in table (1).

Conjecture 1. *(Hasse-Weil Conjecture) Let $\mathcal{E}/K$ be an elliptic curve. The L-function $L(\mathcal{E}/K, s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation*

$$L^*(\mathcal{E}/K, s) = w(\mathcal{E}/K)L^*(\mathcal{E}/K, 2-s)$$

Remark 3.3. The case $K = \mathbb{Q}$ has been proven in various papers of Taylor, Wiles, Breuil, Conrad and Diamond.

Remark 3.4. One would expect a functional equation from Poincaré duality in étale cohomology, at least in the local case (that is over the reduction to a finite field). Namely, for a smooth projective variety $X/\mathbb{F}_q$ of dimension d, we have a perfect pairing of groups

$$H^{2d-r}_{et}(X, \mathbb{Q}_l) \times H^r(X, \mathbb{Q}_l(d)) \to H^{2d}(X, \mathbb{Q}_l) \to \mathbb{Q}_l$$

which implies the functional equation

$$Z(X, \frac{1}{q^d t}) = \pm(q^{d/2}t)^{\chi} Z(X, t)$$

where $\chi = \sum_r (-1)^r \beta_r$ is the Euler characteristic given by the Betti numbers β_r.

On the other hand, the Hasse-Weil conjecture should come as no surprise to the keen eye. In fact it fits in suitably with the aforementioned "local to global" principle described in the first section on the Dedekind zeta function. One would thus naturally expect a result similar to (4) in the Hasse-Weil settings. This is exactly the Birch-Swinnerton-Dyer conjecture.

For $K/\mathbb{Q}$ a number field and $\mathcal{E}/K$ an elliptic curve and $r = \operatorname{rk} \mathcal{E}/K$ its rank

Conjecture 2. *(Birch-Swinnerton-Dyer)*

$$\operatorname{ord}_{s=1} L(\mathcal{E}/K, s) = r \tag{9}$$

$$\lim_{s \to 1} \frac{L(\mathcal{E}/K, s)}{(s-1)^r} = \frac{R \cdot |\text{Ш}(\mathcal{E}/K)| \cdot \prod_v c_v}{\sqrt{|\Delta_K|} \cdot |\mathcal{E}_{tors}|^2} \tag{10}$$

Remark 3.5. The above conjecture generalises almost word to word for general abelian varieties and as far as I could find, was first stated in Tate's paper [Tat95]. I will not include it here as it would involve talking about the dual of an abelian variety which would not be in alignment with the general motivation of this note.

Before I explain what all the terms appearing in the quotient are, I should mention that there is another more restricted conjecture which would be implied by the BSD which has applications to rank calculations for elliptic curves.

Conjecture 3. *(Parity Conjecture)*

$$(-1)^r = w(\mathcal{E}/K) \tag{11}$$

Remark 3.6. The parity conjecture is known to hold for elliptic curves over a number field assuming the finiteness of $\text{Ш}(\mathcal{E}/K)$.

In tables (1) and (2) I summarise the various invariants that appear, assuming for brevity that $K = \mathbb{Q}$. Note however that similar calculations can be performed for number fields which are in general not much more cumbersome. I will also make the assumption (wherever this applies) that $p \neq 2, 3$ as these cases tend to be a bit more involved, especially in the case of additive reduction. In general, the source for such matters are Silverman's books [Sil86],[Sil94].

Symbol	Name	Definition	Notes
$w(\mathcal{E}/\mathbb{Q})$	Global root number	$(-1)^{\#\{v\mid\infty\}}\prod_v w(\mathcal{E}/\mathbb{Q}_p)$	$p=2,3$ harder
$w(\mathcal{E}/\mathbb{Q}_p)$	Local root number	See table (2)	Additive case harder
$N_{\mathcal{E}/\mathbb{Q}}$	Conductor of $\mathcal{E}/\mathbb{Q}$	$\prod_p p^{n_p}$	Tate's algorithm
N	Conductor of $L(\mathcal{E}/\mathbb{Q},s)$	$N_{\mathcal{E}/\mathbb{Q}}\left\|\Delta_{\mathcal{E}/\mathbb{Q}}\right\|^2$	
$Ш(\mathcal{E}/\mathbb{Q})$	Shafarevich-Tate group	See below	Difficult
c_v	Tamagawa numbers	See below	Depend on differentials

TABLE 1. Invariants associated to an elliptic curve $\mathcal{E}/\mathbb{Q}$

Reduction Type	Roots of $\widetilde{f}(x)$	$F_p(T)$	n_p	$w(\mathcal{E}/\mathbb{Q}_p)$	c_v
Good	3	$1-a_pT+pT^2$	0	1	1
Split multiplicative	2	$1-T$	1	-1	$v_p(\Delta_{\mathcal{E}/\mathbb{Q}})$
Non-split mult.	2	$1+T$	1	1	2 if $2\mid v_p(\Delta_{\mathcal{E}/\mathbb{Q}})$, 1 o/w
Additive	1	1	2	$(-1)^{\lfloor\frac{p}{7}\rfloor}$	≤ 4, Tate's algo.

TABLE 2. Local invariant factors for $\mathcal{E}/\mathbb{Q}: y^2=f(x)=x^3+Ax+B$ where $p\neq 2,3$

The remaining pieces to the remarkable formula predicted by the BSD are the order of the Shafarevich-Tate group Ш and the regulator R. I have singled these out as they are much more complicated and of deeper meaning. Firstly the Shafarevich-Tate group is defined as follows, for $G=\mathrm{Gal}(\overline{K}/K)$ and $G_p=\mathrm{Gal}(\overline{K}_p/K_p)$

$$Ш(\mathcal{E}/\mathbb{Q})=\ker\left(H^1(G,\mathcal{E}(\overline{K}))\to\prod_p H^1(G_p,\mathcal{E}(\overline{K}))\right)$$

Note here that $H^1(G,\mathcal{E}(\overline{K}))$ is as a group isomorphic to the Weil-Châtelet group of $\mathcal{E}$ which is defined as the group of torsors (principal homogeneous spaces) of $\mathcal{E}$ modulo isomorphism. The group law on the Weil-Châtelet group comes from Galois cohomology. A standard result in Weil-Châtelet groups states that a torsor $\mathcal{C}$ is trivial in $H^1(G,\mathcal{E}(\overline{K}))$ if and only if it has a rational point over K (see [Sil86]). Thus a more intuitive way of thinking about elements of Ш is as certain other genus 1 curves $\mathcal{C}$ with a map to $\mathcal{E}$, which have points everywhere locally (ie over every field K_p). This group is remarkably hard to compute in general. The following conjecture is however one of the big unsolved problems in this area.

Conjecture 4. *The Shafarevich-Tate group* $Ш(\mathcal{E}/K)$ *is finite.*

The regulator R of $\mathcal{E}$ requires knowledge of the $P_1, \ldots, P_r$ (for $r = rk\mathcal{E}/K$) generators of the Mordell-Weil group $\mathcal{E}(K)$ modulo torsion. It is defined as the determinant of the matrix with i, j entries $\langle P_i, P_j \rangle$ where $\langle ., . \rangle$ denotes the Néron-Tate pairing

$$(\mathcal{E}(K) \otimes \mathbb{R}) \times (\mathcal{E}(K) \otimes \mathbb{R}) \to \mathbb{R}$$

4. Generalisations - A novice's survey

This section will attempt to very roughly sketch how the class number formula and BSD conjecture fit into a more general framework of conjectures on special values of L-functions formulated mostly by Beilinson and Deligne. A good source on this material are the survey paper by Schneider [Sch88] and Ramakrishnan [Ram89] but also Minhyong Kim's notes given at a summer school on motivic L-functions found on his website.

Fix X a smooth projective variety over $\mathbb{Q}$ and let $\overline{X} = X \times \overline{\mathbb{Q}}$. Let $d = \dim X$ and fix an integer $0 \leq i \leq 2d$. Let M denote the motive associated to X and i, which is the collection of the following cohomology groups

- $H^i_{et}(\overline{X}, \mathbb{Q}_l)$ the l-adic cohomology
- $H^i_{DR}(X(\mathbb{C}))$ the de Rham cohomology of the complex manifold $X(\mathbb{C})$
- $H^i(X(\mathbb{C}), \mathbb{Q})$ the singular cohomology.

Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and let $I_{\overline{p}}$ and $D_{\overline{p}}$ be the inertia and decomposition groups in G of a prime $\overline{p}$ lying over a prime number p. For $p \neq l$ let F denote geometric Frobenius (ie raises every coordinate of a point to the p-th power) and note that this is an element of $D_{\overline{p}}/I_{\overline{p}}$. I will not go into much detail about this but one can define an L-function for M as follows

$$L(M, s) = \prod_p P_p(p^{-s})^{-1} = \prod_p \frac{1}{\det\left(1 - Fp^{-s}; H^i_{et}(\overline{X}, \mathbb{Q}_l)^{I_{\overline{p}}}\right)}$$

Just as with the Dedekind zeta function and with the Hasse-Weil zeta function, $L(M, s)$ encodes the local information over all prime numbers. The trick is though that if one wants to set up a functional equation for a motivic zeta function, one considers contributions from the de Rham cohomology as the "infinity factor". In a nutshell, we have an isomorphism and a Hodge structure on $H^i_{DR}(X(\mathbb{C}))$

$$H^i_{DR}(X(\mathbb{C})) \xrightarrow{\sim} H^i(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{\substack{p+q=i \\ p,q \geq 0}} H^{pq}$$

such that the de Rham filtration agrees with the Hodge decomposition as follows

$$F^p H^i_{DR}(X(\mathbb{C})) = \bigoplus_{p' \geq p} H^{p'q}$$

There are standard conjectures generalising that of Hasse-Weil in the previous section which predict that $L(M, s)$ converges absolutely for $Re(s) > i/2 + 1$ and has a meromorphic continuation to $\mathbb{C}$ with a potential pole at $s = i/2 + 1$ if i is even. More importantly, we can form $\zeta(M, s) = L(M, s)L_\infty(M, s)$ where

$$L_\infty(M,s) = \begin{cases} \prod_{\substack{p<q \\ p+q=i}} (\Gamma_{\mathbb{C}}(s-p))^{h^{pq}} & \text{for odd } i \\ \prod_{\substack{p<q \\ p+q=i}} (\Gamma_{\mathbb{C}}(s-p))^{h^{pq}} (\Gamma_{\mathbb{R}}(s-i/2))^{h^{i/2+}} (\Gamma_{\mathbb{R}}(s-i/2+1))^{h^{i/2-}} & \text{for even } i \end{cases}$$

for the Hodge numbers $h^{pq} = \dim_{\mathbb{C}} H^{pq}$ and $h^{p\pm} = \dim_{\mathbb{C}} H^{p,\pm(-1)^p}$ and $\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{C}}$ as defined in (3). It is conjectured that there is a functional equation around $s \mapsto i + 1 - s$. To formulate the conjectures of Deligne and Beilinson one would have to develop Deligne cohomology, but the idea is that Deligne's conjecture gives us the the leading coefficient of $L(M, m)$ for an integer $m \leq i/2 + 1$ in terms of a higher regulator (Deligne periods, cycle class map). On the other hand Beilinson's conjectures (among other things) give a full framework for this line of argument and more specifically tells us the order of the vanishing of $L(M, s)$ at $s = m$ in terms of the dimension of absolute cohomology groups (Deligne cohomology, higher K-theory).

I would also like to outline how this line of thought fits in with the BSD. Denote by $C^m(X)$ the group of codimension m algebraic cycles which are defined as elements of the form $\sum_{i=1}^r n_i Z_i$ where Z_i are closed irreducible subvarieties of codimension m in X and $n_i \in \mathbb{Z}$. Note that Weil divisors are algebraic cycles of codimension 1. There is a cycle class map from $C^{2m}(X)$ into the singular cohomology group $H^{2m}(X(\mathbb{C}), \mathbb{Q}(m))$ which extends to a map into Hodge cycles, the image of which is explained by the Hodge conjectures. As with Weil divisors, there is an equivalence relation on algebraic cycles (in fact there are many). We say that two algebraic cycles in $C^m(X)$ are rationally equivalent if their difference in $C^m(X)$ is given by a sum of divisors of functions on various other closed irreducible subvarieties of X. Namely, we let for f in the coordinate ring of a closed irreducible subvariety $Y \subset X$ of codimension $m - 1$

$$\operatorname{div}(f) \quad = \quad \sum_{\text{all } Z} \operatorname{div}_Z(f) Z$$

where the sum ranges over closed irreducible subvarieties of X of codimension m and $\operatorname{div}_Z(f)$ is defined as the length of $\mathcal{O}_{Y,Z}/f\mathcal{O}_{Y,Z}$ if $Z \subset Y$ or 0 otherwise. Note here that $\mathcal{O}_{Y,Z}$ is the local ring at the unique generic point of Z. Define the Chow group $CH^m(X)$ of codimension m of X as $C^m(X)$ modulo rational equivalence. Note that $CH^0(X) = \mathbb{Z}$ and $CH^1(X) = \operatorname{Pic}(X)$. The cycle class map mentioned earlier factors through the Chow group and we define the kernel of this map to be $CH^m(X)^0$. The Beilinson-Bloch refined conjectures expect that

Conjecture 5. *(Beilinson-Bloch)*

(1) $CH^m(X)^0$ *is finite dimensional,*
(2) *there is a non-degenerate "height pairing"*

$$\langle .,. \rangle_m : CH^m(X)^0 \times CH^{d-m+1}(X)^0 \to \mathbb{Q},$$

(3) $\operatorname{ord}_{s=m} L(M, s) = \dim_{\mathbb{Q}} CH^m(X)^0$ *and*

(4) *the leading coefficient* $L^*(M,m)$ *of* $L(M,m)$ *is given by*

$$L^*(M,m) \equiv c_M(m) \det\langle .,. \rangle_m \mod \mathbb{Q}^*$$

where $c_M(m)$ *is the Deligne period.*

The first part of the conjecture would be a generalisation of the Mordell-Weil theorem on the finite generation of the group $\mathcal{E}(\mathbb{Q})$ for $\mathcal{E}$ an elliptic curve. The height pairing corresponds to the Néron-Tate pairing for elliptic curves, attempting to encode the notion of "regulator". The last two parts of the conjecture act as a generalised Birch-Swinnerton-Dyer conjecture.

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