Algebraic Stacks

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1 Introduction

This introduction (just like the whole course) aims to explain two things: the meaning of the word *stack*, and its connection with *algebraic geometry*. We begin by discussing three typical examples of stacks, two of which take place in the category of topological spaces, while the third is of arithmetic nature.

1.1 What is a stack?

Glueing of vector bundles

The archetypical example of a stack is well-known to every mathematician. In order to describe the data of a rank n vector bundle E on a topological space X one often resorts to a so-called *cocycle description*. This amounts to choosing an open covering $\{U_i\}_{i\in I}$ of X, and for each pair (i,j) of indices, a vector bundle automorphism ϕ_{ij} of the *trivial* vector bundle

$$(U_i \cap U_j) \times \mathbb{R}^n$$

on the open subset $U_i \cap U_j$. One then hopes to find a vector bundle E on X, together with vector bundle isomorphisms (called local trivializations)

$$\phi_i \colon U_i \times \mathbb{R}^n \xrightarrow{\cong} E|_{U_i},$$

such that the identities

$$\phi_{ij} = \phi_i \circ \phi_j^{-1} \tag{1}$$

are satisfied on $U_i \cap U_j$. A necessary condition for the existence of $(E, (\phi_i)_{i\in I})$ is that the *cocycle condition*

$$\phi_{ij} \circ \phi_{jk} = \phi_{ik} \tag{2}$$

is satisfied for every triple (i,j,k). Note that this identity implies for $i = j = k$ that $\phi_{ii} = \mathrm{id}$, and hence for $i = k$ that $\phi_{ji} = \phi_{ij}^{-1}$. It is essential to the theory of topological vector bundles that the cocycle condition is also sufficient. Indeed, we can define E by glueing the trivial vector bundles $U_i \times \mathbb{R}^n$ with respect to the isomorphisms ϕ_{ij}: let $\widetilde{E}$ be the disjoint union of the topological spaces $U_i \times \mathbb{R}^n$; we define an equivalence relation, by declaring $(x,v) \in U_i \times \mathbb{R}^n$ and $(y,w) \in U_j \times \mathbb{R}^n$ as equivalent if $x = y$ and $v = \phi_{ij}(w)$. The topological quotient space will be denoted by E and has a canonical map $E \longrightarrow X$, and canonical trivializations $(\phi_i)_{i\in I}$, which satisfy the identity (1) above.

In modern language we would say that the theory of vector bundles on topological spaces is a stack. Let's be more precise and actually nail down what kind of mathematical object the stack is in this example. It is the following assignment, which associates to an open subset $U \subset X$ the *groupoid* of vector bundles on U. Groupoid is a fancy word for a simple thing. In fact, it simply denotes a category where every morphism is invertible. Hence, the groupoid of vector bundles on U denotes the category of vector bundles on U, where we discard non-invertible morphisms.

Remark 1.1. *It is possible to formulate a theory of stacks (or rather* sheaves of categories*), which takes values in honest categories and not just groupoids. However, it will not be necessary for us to pursue that level of generality. As we have seen in the example of vector bundles on topological spaces, we only need vector bundle isomorphisms to describe the glueing data. Hence it is sensible to discard all the other morphisms of vector bundles.*

The cocycle condition (2) can be elegantly reformulated. This requires the usage of *fibre products.*

Definition 1.2. *For continuous maps of topological spaces $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$, we denote by $X \times_Z Y$ the subspace of $X \times Y$, consisting of tuples (x, y) with $f(x) = g(y)$.*

We denote by

$$Y = \coprod_{i \in I} U_i,$$

and by $\pi\colon Y \longrightarrow X$ the canonical map, induced by the inclusions $U_i \hookrightarrow X$. Let $\widetilde{E}$ be the trivial rank n vector bundle on Y. The fibre product $Y \times_X Y$ is equivalent to the disjoint union

$$\coprod_{(i,j) \in I^2} U_i \cap U_j.$$

The cocycle $(\phi_{ij})_{(i,j) \in I^2}$ is captured by an isomorphism of vector bundles on $Y \times_X Y$, namely

$$\phi\colon p_2^* \widetilde{E} \xrightarrow{\cong} p_1^* \widetilde{E}. \tag{3}$$

Here, we use the suggestive notation which denotes the projection to the first component of a fibre product $X \times_Z Y \longrightarrow X$ by p_1, and similarly for the projection to the second component. Let's pause a second to see that this is indeed equivalent to a collection of automorphisms (ϕ_{ij}) of the trivial vector bundles on $U_i \cap U_j$. The trivial vector bundle always pulls back to the trivial vector bundle, no matter which map we considered. Hence, ϕ is really an automorphism of the trivial vector bundle on $Y \times_X Y$, which is, as remarked above, equivalent to the disjoint union of all the spaces $U_i \cap U_j$.

The cocycle condition amounts to the identity of vector bundle isomorphisms on $Y \times_X Y \times_X Y$:

$$p_{12}^* \phi \circ p_{23}^* \phi = p_{13}^* \phi\colon p_3^* \widetilde{E} \longrightarrow p_1^* \widetilde{E}. \tag{4}$$

The family of local trivializations $(\phi_i)_{i \in I}$ amounts to

$$\pi^* E \cong \widetilde{E} = Y \times \mathbb{R}^n. \tag{5}$$

We will re-encounter a similar formulation below.

Descending vector bundles to quotients

One of the first non-trivial vector bundles one encounters in topology is a *Möbius strip* of infinite width over the circle $\mathbf{S}^1$. A rigorous description of this bundle can be given as follows. At first we observe the existence of a homeomorphism

$$\mathbf{S}^1/\mu_2 \cong \mathbf{S}^1,$$

where $\mu_2 = \{1, -1\}$ denotes the group of second order roots of unity, acting on $\mathbf{S}^1 \subset \mathbb{C}$ multiplicatively. The homeomorphism is induced by the μ_2-invariant map $\mathbf{S}^1 \longrightarrow \mathbf{S}^1$, $z \mapsto z^2$. Although both spaces are homeomorphic, we will distinguish between both sides, and denote the canonical projection by

$$\pi\colon Y = \mathbf{S}^1 \longrightarrow X \cong \mathbf{S}^1/\mu_2 \cong \mathbf{S}^1.$$

The Möbius bundle E on $\mathbf{S}^1$ can be defined by extending the action of μ_2 on $\mathbf{S}^1$ to the total space $\widetilde{E}$ of the trivial rank 1 bundle $\mathbf{S}^1 \times \mathbb{R} \longrightarrow \mathbb{R}$. Namely, we define the action of $\zeta \in \mu_2$ on $(z, \lambda) \in \mathbf{S}^1 \times \mathbb{R}$ to be

$$\zeta \cdot (z, \lambda) = (\zeta x, \zeta\lambda).$$

The quotient $\widetilde{E}/\mu_2$ will be denoted by E. It defines a rank 1 vector bundle on $\mathbf{S}^1/\mu_2 = X \cong \mathbf{S}^1$.

One observes that $\pi^* E \cong \mathbf{S}^1 \times \mathbb{R}$, which should be compared to equation (5). But there are even more similarities with the glueing procedure described in the proceeding subsection. Let's take a look at the fibre products $Y \times_X Y$ and $Y \times_X Y \times_X Y$.

Lemma 1.3. *The map*

$$Y \times \mu_2 \longrightarrow Y \times_X Y,$$

sending (z, ζ) *to* $(z, \zeta z)$ *is a* homeomorphism*. Similarly, the map*

$$Y \times \mu_2 \times \mu_2 \longrightarrow Y \times_X Y \times_X Y,$$

which sends (z, ζ_1, ζ_2) *to* $(z, \zeta_1 z, \zeta_2 z)$*, is a homeomorphism.*

Proof. We prove the first assertion, the second one is totally analogous. Since both sides of the map are compact and Hausdorff (and every continuous map from a compact space to a Hausdorff space is *closed*), it suffices to show that the map is a bijection of sets. This follows from the definition of X as the quotient Y/μ_2. $\square$

With respect to this canonical homeomorphism between $Y \times_X Y$ and $Y \times \mu_2$, the projection p_1 corresponds to the map $(z, \zeta) \mapsto z$, and p_2 is equivalent to the map describing the group action $Y \times \mu_2$, sending (z, ζ) to ζz.

The extension of the group action of μ_2 to the total space $\widetilde{E}$, can be understood as a vector bundle isomorphism

$$\phi\colon p_2^* \widetilde{E} \xrightarrow{\cong} p_1^* \widetilde{E}$$

on $Y \times_X Y$. The map ϕ is given as follows: since $\widetilde{E}$ is the trivial rank 1 bundle, the fibrewise identity gives an isomorphism. We choose ϕ to be this map over the component of $Y \times_X Y = Y \times \mu_2$, corresponding to $1 \in \mu_2$, and modify this map by -1 over the component corresponding to $-1 \in \mu_2$.

To complete the circle, we observe that identity (4) is satisfied.

Lemma 1.4. *The cocycle condition*

$$p_{12}^*\phi \circ p_{23}\phi^* = p_{13}^*\phi$$

is satisfied on $Y \times_X Y \times_X Y$.

Proof. In Lemma 1.3 we have seen that the triple fibre product $Y \times_X Y \times_X Y$ can be identified with $Y \times \mu_2 \times \mu_2$. With respect to this identification, the projection $p_{12} \colon Y \times \mu_2 \times \mu_2 \longrightarrow Y \times \mu_2$ is given by $p_{12}(z, \zeta_1, \zeta_2) = (z, \zeta_1)$, while we have $p_{23}(z, \zeta_1, \zeta_2) = (\zeta_1 z, \zeta_2)$, and $p_{13}(z, \zeta_1, \zeta_2) = (z, \zeta_1\zeta_2)$.

The topological space $Y \times \mu_2 \times \mu_2$ has four connected components indexed by $\mu_2 \times \mu_2$. On the connected component corresponding to the pair (ζ_1, ζ_2), the maps $p_{12}^*\phi$ is given by fibrewise multiplication with ζ_1, while $p_{23}^*\phi$ agrees with fibrewise multiplication with ζ_2. Similarly, $p_{13}^*\phi$ is fibrewise multiplication with $\zeta_1\zeta_2$. This implies the cocycle condition asserted above. □

With the proof of this lemma concluded we should pause a second to contemplate about the remarkable similarity between *glueing* of locally defined vector bundles, and *descending* a so-called *equivariant* vector bundle to the quotient. We see that the theory of vector bundles on topological spaces almost doesn't see a difference between an open covering, and the map $\pi \colon Y \longrightarrow X$, given by the canonical projection to the quotient of a free group action. The theory of *Grothendieck topologies*, which we will encounter later in this course, provides a common framework to treat open coverings, and such maps to quotients, on an equal footing.

Galois descent

Consider the field extension $\mathbb{C}/\mathbb{R}$. Its Galois group Γ is cyclic of order 2, generated by complex conjugation $z \mapsto \bar{z}$. We will focus on this particular extension for simplicity, but the reader should be feel free to work with an arbitrary finite Galois extension $\mathbb{L}/\mathbb{K}$ instead.

It is well-known and ubiquitous principle that the Γ-action on $\mathbb{C}$ extends to an array of algebraic objects, e.g. to the ring of of polynomials $\mathbb{C}[X]$. Moreover, this principle extends to the assertion that imposing Γ-invariance is tantamount to working over the smaller field $\mathbb{R}$. For example, a polynomial $f \in \mathbb{C}[X]$ has *real coefficients*, if and only if $f \in \mathbb{C}[X]^\Gamma$.

The theory of *Galois descent* is of a similar flavour, since it allows to describe mathematical objects over $\mathbb{R}$, by *descending* objects over $\mathbb{C}$. This allows to study the *arithmetic* of non-algebraically closed fields like $\mathbb{R}$, by constructing non-trivial (but "locally trivial") objects by descent. Since this process would be boring if the Galois group Γ was uninteresting, we obtain at least a certain amount of information about the Galois group.

We begin with a slightly boring but important example, in the sense that it doesn't allow to extract any arithmetic information, but still contains all the main features of Galois descent. Namely, we will describe the datum of a $\mathbb{R}$-vector space, by *descending* a $\mathbb{C}$-vector space. The statement below seems to be purely algebraic at first, but we will give a geometric interpretation subsequently.

Proposition 1.5. *The category of finite-dimensional real vector space* V *is equivalent to the category of pairs* $(V_{\mathbb{C}}, f)$*, where* $V_{\mathbb{C}}$ *is a finite-dimensional complex vector space, and* f *an anti-linear map, i.e.* $f(\lambda v) = \bar{\lambda} v$*, such that* $f^2 = \mathrm{id}_{V_{\mathbb{C}}}$*. A morphism of such pairs consists of a complex linear* *map $\phi\colon V_{\mathbb{C}} \longrightarrow W_{\mathbb{C}}$, such that the diagram*

$$\begin{array}{ccc} V_{\mathbb{C}} & \xrightarrow{\phi} & W_{\mathbb{C}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V_{\mathbb{C}} & \xrightarrow{\phi} & W_{\mathbb{C}} \end{array}$$

commutes.

Proof. Let $\mathsf{Vect}_{\mathbb{R}}$ denote the category of finite-dimensional real vector spaces. We denote the category of pairs, as described above, by C. We begin by defining a functor $F\colon \mathsf{Vect}_{\mathbb{R}} \longrightarrow \mathsf{C}$. It sends V to the pair $(V_{\mathbb{C}}, f)$, where $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, and f is the $\mathbb{R}$-linear map induced by complex conjugation $\mathbb{C} \longrightarrow \mathbb{C}$.

We define an inverse $G\colon \mathsf{C} \longrightarrow \mathsf{Vect}_{\mathbb{R}}$, by sending $(V_{\mathbb{C}}, f)$ to the real subspace of f-invariants $V = (V_{\mathbb{C}})^f$. By definition, it consists of the vectors $v \in V_{\mathbb{C}}$, satisfying $f(v) = v$.

The map $\mathbb{R} \longrightarrow \mathbb{C}$ induces a map of real vector spaces $V \longrightarrow V_{\mathbb{C}}$. Conjugation-invariance of this map implies that it lands inside of $(V_{\mathbb{C}})^f$. Choosing a real basis $(v_1, \dots, v_n)$ for V, we obtain a basis $(v_1 \otimes 1, \dots, v_n \otimes 1)$ for $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. This implies immediately, that the image of $V \longrightarrow V_{\mathbb{C}}$ agrees with $(V_{\mathbb{C}})^f$. Hence, we have constructed a natural isomorphism between the functor $G \circ F$, and the identity functor of the category $\mathsf{Vect}_{\mathbb{R}}$.

Conversely, let $(V_{\mathbb{C}}, f)$ be an object of C. We have a canonical map $(V_{\mathbb{C}})^f \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V_{\mathbb{C}}$, which sends $v \otimes \lambda$ to λv (with respect to the complex structure of $V_{\mathbb{C}}$). We claim that this map is bijective. To see this, note that it certainly is injective when restricted to $(V_{\mathbb{C}})^f \otimes i$, since multiplication with i has zero kernel. Moreover, the image of this map agrees with the linear subspace of elements $v \in V_{\mathbb{C}}$, such that $f(v) = -v$ (anti-invariants). It is an elementary fact that every element $v \in V_{\mathbb{C}}$ can be decomposed uniquely as a sum of an invariant and an anti-invariant:

$$v = \frac{1}{2}(v + f(v)) + \frac{1}{2}(v - f(v)).$$

This yields a natural isomorphism between $F \circ G$ and the identity functor on C. $\square$

In this course, we will associate to any ring R, in particular also fields, a geometric object, denoted by $\operatorname{Spec} R$. A ring homomorphism $R \longrightarrow S$ will induce a map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. We will argue, that for a Galois extension of fields, e.g. $\mathbb{R} \subset \mathbb{C}$, the resulting map $\operatorname{Spec} \mathbb{C} \longrightarrow \mathbb{R}$ behaves just like a covering map with a group of deck transformations equivalent to the Galois group μ_2. Vector bundles on $\operatorname{Spec} K$, for K a field will correspond to K-vector spaces. The above proposition can then be understood as comparing Galois-equivariant vector bundles on $\operatorname{Spec} \mathbb{C}$ with vector bundles on the quotient $\operatorname{Spec} \mathbb{R}$.

1.2 What is algebraic geometry? A functorial approach.

Algebraic geometry is often described as the study of solutions to systems of polynomial equations in several variables. Although this is a perfectly fine explanation, the modern, scheme-theoretic approach, seems to be quite remote from solving polynomial equations. In fact, one needs to go through a lot of definitions in order to explain how to pass from a system of equations to a scheme. In the post-modern, functorial framework, this step becomes easier, as we explain below.

The case of integral equations

Let $f = (f_1, \dots, f_n)$ be an n-tuple of polynomials $f_i \in \mathbb{Z}[t_1, \dots, t_m]$. We view f as a system of equation, i.e. we are interested in solving

$$f(x_1, \dots, x_m) = 0.$$

Since the coefficients have been chosen to be integral, it is possible to define the solution set

$$Z_f(R) = \{(x_1, \dots, x_m) \in R^m | f(x_1, \dots, x_m) = 0\}$$

with respect to an arbitrary ring R (every ring is understood to be commutative and unital). Every ring homomorphism $\phi\colon R \longrightarrow S$ induces a map of sets

$$Z_f(\phi)\colon Z_f(R) \longrightarrow Z_f(S),$$

which sends $(x_1, \dots, x_m) \in R^m$ to $(\phi(x_1), \dots, \phi(x_m)) \in S^m$. By definition, we have $Z_f(\mathrm{id}_R) = \mathrm{id}_{Z_f(R)}$, and

$$Z_f(\psi) \circ Z_f(\phi) = Z_f(\psi \circ \phi)$$

for a ring homomorphism $\psi\colon S \longrightarrow T$. In other words, Z_f defines a *functor* from the category Rng of rings to the category Set of sets.

Coefficients in an arbitrary ring

If we want our coefficients of the system f to lie in an arbitrary ring A, we can solve the system $f = 0$, i.e. define the set $Z_f(R)$ if we can make sense of the coefficients (which are elements of A), as elements of R. In order for this to be possible, we have to consider rings R, endowed with the structure of an A-algebra. By definition this amounts to a ring homomorphism $A \longrightarrow R$. A morphism of A-algebras $\phi\colon R \longrightarrow S$ is simply a ring homomorphism, such that the following triangle commutes:

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & \\ R \arrow[r, "\phi"] & S. \end{tikzcd}$$

We denote the category of A-algebras by Alg_A.

If $f \in A[t_1, \dots, t_m]^n$ is an n-tuple of polynomials in m variables, then we have a well-defined set

$$Z_f(R) = \{(x_1, \dots, x_m) \in R^m | f(x_1, \dots, x_m) = 0\}$$

for every A-algebra R. As before, one checks that this construction gives rise to a functor

$$Z_f\colon \mathsf{Alg}_A \longrightarrow \mathsf{Set}.$$

Subsequently, we postulate that algebraic geometry is concerned with the study of functors

$$X\colon \mathsf{Rng} \longrightarrow \mathsf{Set}.$$

We will call such a functor a *space*, and denote the category of spaces by Spaces. During this course, we will restrict the class of functors X, such that it becomes meaningful to think of X as geometric object, locally glued from solutions Z_f to a system of equations.

Affine schemes

Just as quickly as to an equation we can associate to every ring R a functor

$$\operatorname{Spec} R\colon \ \mathsf{Rng} \longrightarrow \mathsf{Set}\,.$$

By definition, it is the functor $\mathsf{Hom}_{\mathsf{Rng}}(R,-)$, which sends a ring S to the set of ring morphisms $R \longrightarrow S$. We will also say that the object $R \in \mathsf{Rng}$ *co-represents* the functor $\operatorname{Spec} R$. We call the functor $\operatorname{Spec} R$ the affine scheme associated to the ring R.

Lemma 1.6. *Let $f \in \mathbb{Z}[t_1,\dots,t_m]^n$ be a system of polynomial equations. Let $R = \mathbb{Z}[t_1,\dots,t_m]/(f)$, where (f) denotes the ideal generated by the polynomials $(f_1,\dots,f_n)$. Then, we have a natural isomorphism of functors*

$$Z_F \simeq \operatorname{Spec} R.$$

Proof. The ring $\mathbb{Z}[t_1,\dots,t_m]$ satisfies a universal property. For every ring S, and every m-tuple of elements $(x_1,\dots,x_m) \in S^m$, there exists a unique morphism $\mathbb{Z}[t_1,\dots,t_m] \longrightarrow S$, such that $t_i \mapsto x_i$. We have a commutative diagram

$$\begin{tikzcd} \mathbb{Z}[t_1,\dots,t_m] \arrow[d] \arrow[dr] & \\ R \arrow[r] & S, \end{tikzcd}$$

if and only if $(x_1,\dots,x_m) \in Z_f(S)$. Hence, we see that the set of morphisms $\mathsf{Hom}_{\mathsf{Rng}}(R,S)$ is naturally isomorphic to $Z_f(S)$. □

This lemma shows that the functors Z_f are a special case of the Spec-construction. We therefore want to think of the functor $\operatorname{Spec} R$ for a general ring R, to describe a generalized system of equations. Solutions in a ring S correspond to a variable $x_r \in S$ for every $r \in R$, such that

$$x_{r_1+r_2} = x_{r_1} + x_{r_2},$$

and similarly

$$x_{r_1 r_2} = x_{r_1} x_{r_2},$$

and $x_1 = 1$.

Note that every ring morphism $R \longrightarrow S$ induces a map of functors $\mathsf{Hom}_{\mathsf{Rng}}(S,-) \longrightarrow \mathsf{Hom}_{\mathsf{Rng}}(R,-)$. It sends $S \longrightarrow T$ to the composition $R \longrightarrow S \longrightarrow T$. This is summarized by the following assertion.

Lemma 1.7. *The construction* Spec *defines a co-functor from the category of rings* Rng*, to the category of spaces* Spaces*, i.e. a functor*

$$\operatorname{Spec}\colon \ \mathsf{Rng}^{\mathrm{op}} \longrightarrow \mathsf{Spaces}\,.$$

Yoneda's lemma

Yoneda's lemma provides both moral justification for the functorial approach to algebraic geometry, but also implies the following statement, to which we will refer again and again: giving a map $\operatorname{Spec} R \longrightarrow X$, where R is an arbitrary ring, and X a space, is equivalent to giving an element of $X(R)$.

Lemma 1.8 (Yoneda)**.** *Let* C *be a category, and* $X\colon \mathsf{C} \longrightarrow \mathsf{Set}$ *a functor. If* $A \in \mathsf{C}$ *is an element, then the set of natural transformations*

$$\alpha\colon \mathit{Hom}_{\mathsf{C}}(A, -) \longrightarrow X$$

corresponds to elements of $X(A)$*. We send* α *to* $\alpha(\mathrm{id}_A) \in X(A)$*.*

Proof. [1] We need to define the map in the other direction. For $x \in X(A)$ we define $\bar{x}_B : \mathsf{Hom}(A, B) \to X(B)$ by $f \mapsto (X(f))(x)$. We need to check two things: that $\bar{x}_B$ are the components of a natural transformation $\bar{x}$, and that the two constructions $\alpha \mapsto \alpha_A(1_A)$ and $x \mapsto \bar{x}$ are mutually inverse.

To check naturality, we need to check that for $g : B \to C$ in C the diagram

$$\begin{tikzcd}
\mathsf{Hom}(A,B) \arrow[r, "\bar{x}_B"] \arrow[d, "g\circ-"'] & X(B) \arrow[d, "X(g)"] \\
\mathsf{Hom}(A,C) \arrow[r, "\bar{x}_C"] & X(C)
\end{tikzcd}$$

commutes. To see this, we chase some $f : A \to B$ around the diagram: one way we get $(X(gf))(x)$ and the other way we get $(X(g))(X(f))(x)$ – these are of course equal.

To show our constructions are mutually inverse, we first note that for $x \in X(A)$ we have $\bar{x}_A(1_A) = (X(1_A))(x) = x$ by definition. For the other direction, let $\alpha : \mathsf{Hom}(A, -) \to X$ be a natural transformation and $f \in \mathsf{Hom}(A, B)$ arbitrary. We have the naturality square for α

$$\begin{tikzcd}
\mathsf{Hom}(A,A) \arrow[r, "\alpha_A"] \arrow[d, "f\circ-"'] & X(A) \arrow[d, "X(f)"] \\
\mathsf{Hom}(A,B) \arrow[r, "\alpha_B"] & X(B)
\end{tikzcd}$$

from which we see, chasing $1_A \in \mathsf{Hom}(A, A)$ around, that $\alpha_B(f) = (X(f))(\alpha_A(1_A)) = \overline{\alpha_A(1_A)}_B(f)$. This means that $\alpha = \overline{\alpha_A(1_A)}$ as desired.

To show naturality in A, let $f : A \to B$ be an arrow in C. This induces a map $- \circ f : \mathsf{Hom}(B, -) \to \mathsf{Hom}(A, -)$, and thence a map $-\circ(-\circ f) : Nat(\mathsf{Hom}(A, -), X) \to Nat(\mathsf{Hom}(B, -), X)$. To show naturality, we want to show that

$$\begin{tikzcd}
Nat(\mathsf{Hom}(A,-),X) \arrow[r, "-\circ(-\circ f)"] \arrow[d, "\sim"'] & Nat(\mathsf{Hom}(B,-),X) \arrow[d, "\sim"'] \\
X(A) \arrow[r, "X(f)"] & X(B)
\end{tikzcd}$$

commutes. To do this, we chase some $\alpha : \mathsf{Hom}(A, -) \to X$ around the diagram: one way we get $\alpha_B \circ (- \circ f)_B(1_B) = \alpha_B(f)$, and the other way around we get $(X(f))(\alpha_A(1_A))$ – we saw above that these are equal.

Naturality in X is easier. Suppose $\beta : X \to Y$ is a natural transformation. We want to show that

[1] This proof was provided by Alexander Betts.

$$\begin{array}{ccc} Nat(\mathsf{Hom}(A,-),X) & \xrightarrow{\beta\circ -} & Nat(\mathsf{Hom}(A,-),Y) \\ \sim\downarrow & & \sim\downarrow \\ X(A) & \xrightarrow{\beta_A} & YA \end{array}$$

commutes, which is now trivial. □

Standard open subfunctors

For a ring R and an element $h \in R$, we denote by $R_h = R[h^{-1}] = R[t]/(th-1)$ the ring obtained by adjoining a formal inverse for h to R. We say that R_h is obtained from R by *localization* at the element h. The functor $\operatorname{Spec} R_h$ can be described in terms of the ring R. For every ring S, the set $(\operatorname{Spec} R_h)(S)$ is given by the set of ring homomorphisms $R \longrightarrow S$, which send $h \in R$ to an invertible element of S.

The geometric meaning of this construction becomes evident, when considering the inclusion of sets

$$\operatorname{Spec} R_h(k) \longrightarrow \operatorname{Spec} R(k),$$

where k is a field, and $R = \mathbb{Z}[x_1, \ldots, x_n]/(f)$.

Lemma 1.9. *We know from Lemma 1.6 that* $\operatorname{Spec} R(k) = Z_f(k)$. *With respect to this identification, the subset* $\operatorname{Spec} R_h(k)$ *is given by the difference of sets* $Z_f(k) \setminus Z_h(k)$.

In algebraic geometry, the subfunctors $\operatorname{Spec} R_h \longrightarrow \operatorname{Spec} R$ are often called the *standard open subfunctors*. They generate what is classically known as the Zariski topology on $\operatorname{Spec} R$.

Definition 1.10. *If $h_1, \ldots, h_\ell \in R$ are elements generating the unit ideal, i.e. $(h_1, \ldots, h_\ell) = R$, then we say that the collection of standard open subfunctors* $(\operatorname{Spec} R_{h_i} \longrightarrow \operatorname{Spec} R)_{i=1}^{\ell}$ *is a* standard open covering *of* $\operatorname{Spec} R$.

To see why the terminology covering is justified, we consider again what happens on the level of the set $\operatorname{Spec} R(k)$, where k is a field. Since there exist elements $r_1, \ldots, r_\ell \in R$, such that we have $r_1 h_1 + \cdots r_\ell h_\ell = 1$ we see that every point $x \in \operatorname{Spec} R(k)$ lies in at least one of the subsets $\operatorname{Spec} R_h(k)$. Otherwise, if all $(h_i)_{i=1}^{\ell}$ induced the zero element in k, we would obtain the contradiction $0 = 1$.

An example of a moduli problem

The set of 1-dimensional subspaces of a 2-dimensional complex vector space $\mathbb{C}^2$ has the structure of a complex manifold, often denoted by $\mathbb{CP}^1$. The analogue of a 2-dimensional complex vector space over a general ring R, is the free module R^2 of rank 2.

Definition 1.11. *Recall that a submodule $M \subset V$ of an R-module V is called a direct summand, if there exists a submodule $N \subset V$, such that we have $M \oplus N = V$. The functor $\mathbb{P}^1_{\mathbb{Z}} \colon \mathsf{Rng} \longrightarrow \mathsf{Set}$ sends a ring R to the set of direct summands $M \subset R^2$, for which there exists a standard open covering, corresponding to elements $(h_i)_{i=1}^{\ell} \in R$, such that $M_{h_i} = M \otimes_R R_h$ is a free R-module of rank 1, for all i.*

If $R \longrightarrow S$ is a ring homomorphism, then then resulting map $\mathbb{P}^1_{\mathbb{Z}}(R) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}(S)$ is given by $M \mapsto M \otimes_R S$.

Definition 1.11 is the correct one for a couple of reasons. First of all one sees that $\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C})$ is the set of 1-dimensional subspaces $M \subset \mathbb{C}^2$. This follows from the fact that every subspace of a vector space is a direct summand.

An element of $\mathbb{P}^1_{\mathbb{Z}}(\mathbb{Z}[t])$ induces for every complex number $z \in \mathbb{C}$ a 1-dimensional subspace $M \subsetneq \mathbb{C}^2$, by virtue of functoriality applied to the ring homomorphism $\mathbb{Z}[t] \longrightarrow \mathbb{C}$, which sends t to z. We may therefore think of it as a family of 1-dimensional subspaces, parametrized by a formal variable t. One of the pleasant consequences of Definition 1.11 is that it guarantees that the dimension of all induced complex subspaces (for any value of the parameter t) is equal to 1. Moreover, it somehow guarantees that the subspaces in a given family vary continuously.

The third reason is that this definition allows us to describe $\mathbb{P}^1_{\mathbb{Z}}$ by glueing two copies of $\operatorname{Spec} \mathbb{Z}[t] = \mathbb{A}^1_{\mathbb{Z}}$, just like the complex manifold $\mathbb{CP}^1$ is obtained by glueing two copies of $\mathbb{C}$.

Proposition 1.12. *Let $\phi_1 \colon \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{P}^1_{\mathbb{Z}}$ be the map, which corresponds to the element of $\mathbb{P}^1_{\mathbb{Z}}(\mathbb{Z}[t])$, given by the surjection, corresponding to the matrix $(t,1) \colon \mathbb{Z}[t]^2 \twoheadrightarrow \mathbb{Z}[t]$. Similarly, ϕ_2 corresponds to the matrix $(1,t)$. Then, for every ring R, and every $M \in \mathbb{P}^1_{\mathbb{Z}}(R)$, there exist finitely many elements $(h_i)_{i\in I} \in R$, such that the ideal generated by $(h_i)_{i\in I} = R$, and $M \otimes_R R_{h_i} \in \mathbb{P}^1_{\mathbb{Z}}(R_{h_i})$ is induced by an element of $\mathbb{A}^1_{\mathbb{Z}}(R)$ via the map ϕ_j, for $j = 1,2$.*

The rest of this subsection deals with the proof of this proposition.[2] We have the functor $\mathbb{P}^1_{\mathbb{Z}}$: $(\mathsf{Rng} \to \mathsf{Set}$, which sends a ring R into the set $\mathbb{P}^1_{\mathbb{Z}}(R)$ made up of direct summands M of R^2 which are projective modules of rank 1. In other words, a submodule $M \subseteq R^2$ is an element of $\mathbb{P}^1_{\mathbb{Z}}(R)$ if and only if there exist a submodule $N \subseteq R^2$ and elements $h_1, \dots h_n \in R$ such that $R^2 = M \oplus N$, $Rh_1 + \cdots + Rh_n = R$ and $M[h_i^{-1}]$ is a free $R[h_i^{-1}]$-module of rank 1 for $i = 1, \dots, n$.

Let $\phi_1 \colon \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{P}^1_{\mathbb{Z}}$ the scheme morphism associated to the element of $\mathbb{P}^1_{\mathbb{Z}}(\mathbb{Z}[t])$ corresponding to the submodule

$$L_1 := \ker \left(\mathbb{Z}[t]^2 \xrightarrow{(t,1)} \mathbb{Z}[t] \right)$$

of $\mathbb{Z}[t]^2$. We see that L_1 is the cyclic $\mathbb{Z}[t]$-submodule of $\mathbb{Z}[t]^2$ generated by the vector $(1,-t)$. For every ring R, the set $\mathbb{A}^1_{\mathbb{Z}}(R) = \mathsf{Hom}_{(\mathsf{Rng})}(\mathbb{Z}[t], R)$ is canonically identified with the set R, because to the element $r \in R$ we associate the unique ring homomorphism $\mathbb{Z}[t] \longrightarrow R$ which maps t to r. If we use this identification, for every ring R, the function $\phi_1(R) \colon \mathbb{A}^1_{\mathbb{Z}}(R) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}(R)$ maps the element $r \in R$ into the R-submodule of R^2 generated by the vector $(1,-r)$. Therefore an element $M \in \mathbb{P}^1_{\mathbb{Z}}(R)$ is in the image of $\phi_1(R)$ if and only if there exists $r \in R$ such that M is generated by the vector $(1,-r)$.

In an analogous way, we can define the scheme morphism $\phi_2 \colon \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{P}^1_{\mathbb{Z}}$ corresponding to the submodule

$$L_2 := \ker \left(\mathbb{Z}[t]^2 \xrightarrow{(1,t)} \mathbb{Z}[t] \right)$$

of $\mathbb{Z}[t]^2$. Concretely, for every ring R, the function $\phi_2(R) \colon \mathbb{A}^1_{\mathbb{Z}}(R) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}(R)$ maps the element $r \in R$ into the R-submodule of R^2 generated by the vector $(-r,1)$.

[2]This part has been written by Andrea Petracci.

Lemma 1.13. *Let R be a ring and let M be a direct summand of R^2 such that M is a free R-module of rank 1. Then M is generated by a vector (p,q) with $p,q \in R$ such that $Rp + Rq = R$.*

Proof. Since M is free of rank 1, M must be generated by a single vector (p,q). Let $N \subseteq R^2$ such that $R^2 = M \oplus N$. Suppose by contradiction that the ideal $Rp+Rq$ is strictly smaller than R. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $Rp+Rq \subseteq \mathfrak{m}$. This shows that the image of $M \otimes_R k(\mathfrak{m})$ inside $R^2 \otimes_R k(\mathfrak{m}) = k(\mathfrak{m})^2$ is 0. Since N is a projective R-module, $\operatorname{Tor}_1^R(N, k(\mathfrak{m})) = 0$ and therefore $M \otimes_R k(\mathfrak{m}) \hookrightarrow R^2 \otimes_R k(\mathfrak{m}) = k(\mathfrak{m})^2$. Hence $M \otimes_R k(\mathfrak{m}) = 0$. But this is a contradiction, because M is a free R-module of rank 1. $\square$

Proposition 1.14. *Let R be a ring and let $M \in \mathbb{P}^1_{\mathbb{Z}}(R)$. Then there exist $f_1, \dots, f_m \in R$ such that $Rf_1 + \cdots + Rf_m = R$ and*

$$M[f_j^{-1}] \in \operatorname{im}\left(\phi_{k_j}(R[f_j^{-1}])\colon\ \mathbb{A}^1_{\mathbb{Z}}(R[f_j^{-1}]) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}(R[f_j^{-1}])\right)$$

for $j = 1, \dots, m$, with $k_j \in \{1,2\}$.

Proof. So we have a submodules $M \subseteq R^2$, $N \subseteq R^2$ and elements $h_1, \dots, h_n \in R$ such that $R^2 = M \oplus N$, $Rh_1 + \cdots + Rh_n = R$, and $M[h_i^{-1}]$ is a free $R[h_i^{-1}]$-module of rank 1 for $i = 1, \dots, n$.

Now, fix $i \in \{1, \dots, n\}$. We have $R[h_i^{-1}]^2 = M[h_i^{-1}] \oplus N[h_i^{-1}]$, so the $R[h_i^{-1}]$-module $M[h_i^{-1}]$ is a direct summand of $R[h_i^{-1}]^2$ and is free of rank 1. By Lemma 1.13, we can find $p_i, q_i \in R[h_i^{-1}]$ such that $R[h_i^{-1}] = R[h_i^{-1}]p_i + R[h_i^{-1}]q_i$ and $M[h_i^{-1}]$ is generated by the vector (p_i, q_i). Up to multiplying by a power of h_i we may assume that p_i and q_i come from elements of R. With little abuse of notation, we call p_i and q_i some liftings of p_i and q_i to R. From $R[h_i^{-1}] = R[h_i^{-1}]p_i + R[h_i^{-1}]q_i$ we deduce that $h_i \in \sqrt{Rh_ip_i + Rh_iq_i}$. It is clear that $M[h_i^{-1}p_i^{-1}] = R[h_i^{-1}p_i^{-1}](1, r_i)$, for some r_i, so it is in the image of $\phi_1(R[h_i^{-1}p_i^{-1}])$. In an analogous way, $M[h_i^{-1}q_i^{-1}]$ is in the image of $\phi_2(R[h_i^{-1}q_i^{-1}])$.

From $h_i \in \sqrt{Rh_ip_i + Rh_iq_i}$ for $i = 1, \dots, n$, we get $Rh_1p_1 + Rh_1q_1 + \cdots + Rh_np_n + Rh_nq_n = R$. $\square$

Remark 1.15. *Let R be a ring and let $M \in \mathbb{P}^1_{\mathbb{Z}}(R)$. We have that M is a submodule of R^2 and it corresponds to a scheme morphism* $\operatorname{Spec} R \longrightarrow \mathbb{P}^1_{\mathbb{Z}}$.

For $j = 1, 2$, let $\pi_j\colon R^2 \longrightarrow R$ denote the projection onto the j-th coordinate. Then $I_j = \pi_j(M)$ is an ideal of R. Let U_j be the open subscheme of $\operatorname{Spec} R$ *which is the complement of the closed subscheme defined by the ideal I_j. Then the following diagram is cartesian.*

$$\begin{tikzcd} U_j \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "\phi_j"] \\ \operatorname{Spec} R \arrow[r, "M"] & \mathbb{P}^1_{\mathbb{Z}} \end{tikzcd} \tag{6}$$

The principal open subschemes $\operatorname{Spec} R[h_i^{-1}p_i^{-1}]$ *give an affine cover of U_1 and the principal open subschemes* $\operatorname{Spec} R[h_i^{-1}q_i^{-1}]$ *give an affine cover of U_2. One could show that U_1 and U_2 are affine schemes.*

2 Faithfully flat descent

In this section we will recall the notion of *flat modules* over a ring. In a nutshell, flat modules are well-behaved with respect to base change. We will see examples and non-examples, and the basic properties of flat modules. In the second half of this section, we encounter *faithfully flat* maps of rings, and *descent* (which is the converse to base change).

2.1 Tensor products and flatness

2.1.1 A reminder on tensor products and base change

Recall that there is a functor

$$- \otimes_R -\colon \mathsf{Mod}(R) \times \mathsf{Mod}(R) \to \mathsf{Mod}(R),$$

which sends (M_R, N_R) to the tensor product $M_R \otimes_R N_R$. We are particularly interested in the case where either of the modules is an R-algebra. This case admits an alternative description as base change functor, which we discuss in this paragraph.

We fix a ring homomorphism

$$\alpha\colon R \to S,$$

where R and S are, as always in this course, assumed to be commutative and unital. In the words of the first section, S is endowed with the structure of an R-algebra. This section is concerned with passing between categories of R and S-modules. For the sake of clarity, we will therefore carefully indicate an R-module by the notation M_R, respectively, write N_S for an S-module. We denote the underlying abelian group by M, respectively N.

Definition 2.1. *For an S-module N_S we denote by N_R the R-module obtained by "forgetting the S-module structure" along the ring homomorphism α. Hence, N_R is the R-module given by choosing $(N, +)$ as underlying group, and defining scalar multiplication*

$$R \times N \longrightarrow N$$

by $(r, n) \mapsto \alpha(r)n$. The induced functor will be denoted by $\mathrm{res}_\alpha\colon \mathsf{Mod}(S) \longrightarrow \mathsf{Mod}(R)$, and referred to as restriction along α*.*

The functor res_α is an example of what is traditionally called a *forgetful functor*. In many cases, the ring S is of higher complexity, than the ring R. Applying the functor res_α corresponds to forgetting the extra structure imposed by being an S-module.

Example 2.2. *Let $\mathbb{K}$ be a field, and $\iota\colon \mathbb{K} \to \mathbb{K}[t]$ the canonical morphism of rings, which sends an element of $\mathbb{K}$ to the corresponding degree 0 polynomial in $\mathbb{K}[t]$. The category $\mathsf{Mod}(\mathbb{K}[t])$ is equivalent to the category of $\mathbb{K}$-vector spaces V, together with an endomorphism $f\colon V \to V$ (given by multiplication with t). The restriction functor*

$$\mathrm{res}_\iota\colon \mathsf{Mod}(\mathbb{K}[t]) \to \mathsf{Mod}(\mathbb{K})$$

sends a pair (V, f) to the $\mathbb{K}$-vector space V, i.e. forgets *the endomorphism f.*

Tensor products provide a functor from $\mathsf{Mod}(R)$ to $\mathsf{Mod}(S)$. We will begin with an explicit description, and then give a characterisation in terms of a universal property.

Definition 2.3. *For an R-module M_R we denote by $S \otimes_R M_R$ the S-module S-linearly generated by formal expressions $s \otimes m$, where $m \in M$ and $s \in S$, satisfying the following identities:*

(a) $(s_1 + s_2) \otimes m = s_1 \otimes m + s_2 \otimes m$, *for $m \in M$, and $s_1, s_2 \in S$,*

(b) $s \otimes (m_1 + m_2) = (s \otimes m_1) + (s \otimes m_2)$, *for $m_1, m_2 \in M$, and $s \in S$,*

(c) $s \otimes (r \cdot m) = \alpha(r) \cdot (s \otimes m) = (\alpha(r)s) \otimes m$, *for* $m \in M$, $r \in R$, *and* $s \in S$.

The resulting functor will be denoted by

$$S \otimes_R -\colon \mathsf{Mod}(R) \to \mathsf{Mod}(S),$$

and referred to as tensor product *or* base change *functor.*

The overcautious reader may define $M_R \otimes_R S$ as the quotient

$$SM/I,$$

where SM denotes the free S-module generated by the set underlying M, and I is a submodule of SM, encoding the relations (a), (b), and (c) from the definition above.

Example 2.4. *We continue Example 2.2. For a* $\mathbb{K}$*-vector space* V*, the resulting* $\mathbb{K}[t]$*-module* $V \otimes_{\mathbb{K}} \mathbb{K}[t]$ *is isomorphic to the* $\mathbb{K}[t]$*-module* $V[t]$*, defined as follows. Its elements are formal polynomials* $a_n t^n + \cdots + a_0$ *with* $a_i \in V$*, with coefficient-wise addition. The usual method to multiply polynomials yields a scalar multiplication map*

$$\mathbb{K}[t] \otimes V[t] \to V[t].$$

The following example is an important tool for computations.

Example 2.5. *Assume that* α *is the canonical projection* $R \to R/I$*, where* $I \subset R$ *is an ideal. Then, for every* R*-module* M_R*, the abelian group underlying the base change* $M_R \otimes_R R/I$ *is naturally isomorphic to* M/IM*, where* IM *denotes the abelian subgroup, generated by* $\lambda \cdot m$*, with* $\lambda \in I$*, and* $m \in M$*. We have a well-defined scalar multiplication* $R/I \times M/IM \to M/IM$*, endowing the quotient* M/IM *with the structure of an* R/I*-module.*

In order to characterise tensor products in terms of a universal property, we use the notion of *adjoint functors.*

Definition 2.6. *Let* $F\colon \mathsf{C} \to \mathsf{D}$ *and* $G\colon \mathsf{D} \longrightarrow \mathsf{C}$ *be functors between categories* C *and* D*. We say that* F *is* left adjoint *to* G *(respectively that* G *is right adjoint to* F*), if we have a natural isomorphism*

$$\mathsf{Hom}_{\mathsf{C}}(X, G(Y)) \cong \mathsf{Hom}_{\mathsf{D}}(F(X), Y)$$

for all objects $X \in \mathsf{C}$ *and* $Y \in \mathsf{D}$*.*

In plain language this means that in order to describe morphisms from $F(X)$ to another object Y, it suffices to understand morphisms from Y to $G(Y)$. We will see below what this means in concrete terms for tensor products. Note that Yoneda's lemma 1.8 implies that $F(X)$ is characterised by the functor $\mathsf{Hom}_{\mathsf{C}}(F(X), -)$. In particular, we see that adjoint functors are unique up to a unique natural transformation, if they exist.

Proposition 2.7. *The base change functor* $- \otimes_R S\colon \mathsf{Mod}(R) \longrightarrow \mathsf{Mod}(S)$ *is left adjoint to the restriction functor* res_α*, i.e. we have a natural isomorphism*

$$\mathsf{Hom}_S(M_R \otimes_R S, N_S) \cong \mathsf{Hom}_R(M_R, N_R),$$

for an R*-module* M_R *and an* S*-module* N_S*.*

Proof. We have a map of R-modules

$$M_R \to (M_R \otimes_R S)_R,$$

which sends $m \in M$ to $m \otimes 1 \in M_R \otimes_R S$. Hence, given a map $M_R \otimes_R S \to N_S$, we may send it to the composition

$$\begin{tikzcd} M_R \arrow[r] \arrow[dr] & (M_R \otimes_R S)_R \arrow[d] \\ & N_S. \end{tikzcd}$$

Vice versa, a morphism of modules $f\colon M_R \to N_R$ induces a map

$$M_R \otimes_R S \to N_S,$$

by sending $m \otimes s$ to $s \cdot f(m)$. The maps

$$\mathsf{Hom}_S(M_R \otimes_R S, N_S) \leftrightarrows \mathsf{Hom}_R(M_R, N_R)$$

are mutually inverse. $\square$

Let $\beta\colon S \to T$ be a ring homomorphism. We will now investigate transitivity of base change.

Lemma 2.8. *For every R-module M_R we have natural equivalences*

$$(M_R \otimes_R S) \otimes_S T \cong M_R \otimes_R T.$$

Proof. We observe that the analogous statement for the restriction functor holds strictly, i.e. we have an equality of functors

$$\operatorname{res}_\alpha \circ \operatorname{res}_\beta = \operatorname{res}_{\beta\circ\alpha}.$$

To see this, choose a T-module M_T. The underlying abelian group M is not altered by the functors res_β, $\operatorname{res}_\alpha$, and $\operatorname{res}_{\beta\circ\alpha}$. In particular, it suffices to compare the S-module structures of $(\operatorname{res}_\alpha \circ \operatorname{res}_\beta)(M_T)$ and $\operatorname{res}_{\beta\circ\alpha}(M_T)$. By Definition 2.1, it is in both cases given by the map

$$S \times M \longrightarrow M,$$

which sends $(s, m) \in S \times M$ to $(\beta \circ \alpha)(s) \cdot m$.

By Proposition 2.7, $- \otimes_R T$ is a left adjoint to $\operatorname{res}_{\beta\circ\alpha}$. By virtue of the identity $\operatorname{res}_{\beta\circ\alpha} = \operatorname{res}_\beta \circ \operatorname{res}_\alpha$, we have that the composition $(- \otimes_R S) \otimes_S T$ is left adjoint to the same functor. In particular, we obtain a natural isomorphism between functors

$$(- \otimes_R S) \otimes_S T \simeq - \otimes_R T,$$

since adjoint functors are unique up to a unique natural isomorphism. $\square$

The following lemma is recorded for later purposes.

Lemma 2.9. *If $R \to S$ and $R \to T$ are ring homomorphisms, then the tensor product $S \otimes_R T$ carries a natural ring structure. We have natural ring homomorphisms $S \to S \otimes_R T$ (sending s to $s \otimes 1$), and $T \to S \otimes_R T$ (sending t to $1 \otimes t$), such that the diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & T \arrow[d] \\ S \arrow[r] & S \otimes_R T, \end{tikzcd} \tag{7}$$

commutes. Moreover, it is cocartesian *in the category* Rng *(a pushout diagram), i.e., for every other ring A, such that*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & T \arrow[d, "g"] \\ S \arrow[r, "f"'] & A, \end{tikzcd} \tag{8}$$

commutes, there exists a unique morphism $S \otimes_R T \to A$, rendering the resulting diagram commutative.

Proof. One checks easily that $(s \otimes t) \cdot (s' \otimes t') = ss' \otimes tt'$ yields a well-defined ring structure. For $r \in R$ we have $r \otimes 1 = (r \cdot 1) \otimes 1 = 1 \otimes r$, hence diagram (8) commutes indeed. If A is a ring, as in the assertion above, we may produce the required map $S \otimes_R T \to A$, by sending $s \otimes t$ to $f(s) \cdot g(t)$. $\square$

2.1.2 Base change invariant properties

The functor $- \otimes_R S$ preserves many properties of modules. We begin by recalling the definition of these properties, and then state that they are preserved by tensor products. Our list is by no means complete. A widely accepted dogma is that any reasonable property of modules should be preserved by base change.

Definition 2.10. *Let R be a ring, and M_R an R-module.*

(a) We say that M_R is finite, *if there exists a positive integer n, and a surjection $R^n \twoheadrightarrow M_R$.*

(b) The R-module M_R is called free *if there exists a basis, i.e. an isomorphism $M \cong R^{\oplus I}$, where I is a set.*

(c) The R-module M_R is called projective, *if there exists an R-module N_R, such that $M_R \oplus N_R$ is free (i.e. M_R is a direct summand of a free module).*

This leads us to the assertion that those properties of modules are preserved by base change.

Proposition 2.11. *If M_R satisfies property (x) from Definition 2.10, then so does $M_R \otimes_R S$.*

Proof. Finiteness of module is preserved, since $- \otimes_R S$ is a right exact functor, therefore sends a surjection $R^n \twoheadrightarrow M_R$ to a surjection $S^n \twoheadrightarrow M_R \otimes_R S$. The second assertion follows from the fact that $R_R \otimes_R S \cong S$, and that tensor products commute with direct sums. The third assertion follows from similar reasoning: we know that $M_R \oplus N_R$ is a free R-module. In particular, the direct sum

$$(M_R \otimes_R S) \oplus (N_R \otimes_R S) \cong (M_R \oplus N_R) \otimes_R S$$

is free as well. This proves that $M_R \otimes_R S$ is projective. $\square$

In Subsection 2.2 we investigate when properties *descend*, i.e., when can we conclude from $M_R \otimes_R S$ having property (x) that M_R has property (x)? This is the case if $R \to S$ is faithfully flat. We will see that all properties on the list, except from freeness, descend.

2.1.3 Flatness

The functors res_α and $- \otimes_R S$ are examples of so-called *additive* functors (i.e. preserving addition of morphisms). They satisfy additional properties, with respect to exact sequences of modules.

Definition 2.12. *Let $F\colon \mathsf{Mod}(R) \longrightarrow \mathsf{Mod}(S)$ be an additive functor between categories of modules.*

(a) We say that F is left exact*, if it sends an exact sequence*

$$0 \to U \to V \to W$$

of R-modules to an exact sequence

$$0 \to F(U) \to F(V) \to F(W).$$

(b) The functor F is called right exact*, if it sends an exact sequence*

$$U \to V \to W \to 0$$

to an exact sequence

$$F(U) \to F(V) \to F(W) \to 0.$$

(c) If F is both left and right exact, it is called an exact functor*. Equivalently, it sends a short exact sequence*

$$0 \to U \to V \to W \to 0$$

to a short exact sequence

$$0 \to F(U) \to F(V) \to F(W) \to 0.$$

A sequence of S-modules is exact if and only if the underlying sequence of abelian groups is exact. Since the restriction functor res_α doesn't alter the underlying abelian group, we obtain the following lemma.

Lemma 2.13. *The functor* res_α *is* exact*.*

We will deduce from abstract nonsense that $- \otimes_R S$ is right exact.

Lemma 2.14. *Let $F\colon \mathsf{Mod}(R) \to \mathsf{Mod}(S)$ be an additive functor which is left adjoint to a functor $G\colon \mathsf{Mod}(S) \to \mathsf{Mod}(R)$. Then, F is right exact.*

Proof. [3] For the one direction, if the sequence is exact, then clearly $\mathsf{Hom}_R(W_R, M_R) \to \mathsf{Hom}_R(V_R, M_R)$ is injective and the composite $\mathsf{Hom}_R(W_R, M_R) \to \mathsf{Hom}_R(V_R, M_R) \to \mathsf{Hom}_R(U_R, M_R)$ is zero. To show exactitude, suppose that $f : V_R \to M_R$ becomes zero when precomposed with $U_R \to V_R$. Thus

[3] This proof was provided by Alexander Betts.

f vanishes on the image of U_R and so factors (uniquely) through the cokernel $W_R \cong V_R/\operatorname{image}(U_R)$ as desired.

For the converse direction, suppose that

$$0 \to \mathsf{Hom}_R(W_R, M_R) \to \mathsf{Hom}_R(V_R, M_R) \to \mathsf{Hom}_R(U_R, M_R)$$

is exact for any R-module M_R. Let W_R' be the cokernel of $U_R \to V_R$, so by the first part the same is true of W_R'. Applying the exact sequence when $M_R = W_R'$ tells us that there is a unique map $W_R \to W_R'$ making commute

$$\begin{tikzcd} U_R \arrow[r] \arrow[d, equal] & V_R \arrow[r] \arrow[d, equal] & W_R \arrow[r] \arrow[d, dotted] & 0 \\ U_R \arrow[r] & V_R \arrow[r] & W_R' \arrow[r] & 0 \end{tikzcd}$$

By exactly the same argument, there is also a unique map $W_R' \to W_R$ making an appropriate diagram commute. These are mutually inverse, since the composite $W_R \to W_R' \to W_R$ must be the *unique* map $W_R \to W_R$ making an appropriate diagram commute – this is of course the identity map on W_R. Similarly the composite $W_R' \to W_R \to W_R'$ is the identity on W_R' so W_R is isomorphic to the cokernel W_R' of $U_R \to V_R$ as desired.

For the second part, a small amount of care is needed. Suppose that

$$U_R \to V_R \to W_R \to 0$$

is exact. Then from naturality of the bijection in the adjunction, for each S-module N_S we have a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}_S(F(W_R), N_S) \arrow[r] \arrow[d, "\sim"] & \mathsf{Hom}_S(F(V_R), N_S) \arrow[r] \arrow[d, "\sim"] & \mathsf{Hom}_S(F(U_R), N_S) \arrow[d, "\sim"] \\ 0 \arrow[r] & \mathsf{Hom}_R(W_R, G(N_S)) \arrow[r] & \mathsf{Hom}_R(V_R, G(N_S)) \arrow[r] & \mathsf{Hom}_R(U_R, G(N_S)) \end{tikzcd}$$

where the bottom row is exact. This implies that $\mathsf{Hom}_S(F(W_R), N_S) \to \mathsf{Hom}_S(F(V_R), N_S)$ is injective, and that its image is precisely the preimage of $f \in \mathsf{Hom}_S(F(U_R), N_S)$, where f is the element corresponding to $0 \in \mathsf{Hom}_R(U_R, G(N_S))$ under the adjunction. However, $0 \in \mathsf{Hom}_S(F(V_R), N_S)$ is in the image of $\mathsf{Hom}_S(F(W_R), N_S) \to \mathsf{Hom}_S(F(V_R), N_S)$, and so we see that $f = 0$ and the top row is exact.

Now the first part tells us that

$$F(U_R) \to F(V_R) \to F(W_R) \to 0$$

is exact, i.e. that F is right exact. $\square$

Corollary 2.15. *The tensor product functor $- \otimes_R S$ is right exact.*

Proof. By Proposition 2.7 it is left adjoint to the restriction functor $\operatorname{res}_\alpha$. Since the latter is exact, thus left exact, we obtain from Lemma 2.14 that $- \otimes_R S$ is right exact. $\square$

Note that base change is not always a left exact functor, as we can see from the example below.

Example 2.16. *Consider for instance the ring homomorphism*

$$\alpha\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z},$$

and the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Tensoring this sequence with $\mathbb{Z}/2\mathbb{Z}$, we obtain (from Example 2.5)

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

which isn't exact.

The favourable case, where base change is an exact functor, deserves therefore a special name.

Definition 2.17. *An R-module M is said to be* flat*, if $- \otimes_R M\colon \mathsf{Mod}(R) \to \mathsf{Mod}(E)$ is an* exact *functor. An R-algebra S is* flat*, if it is flat as R-module, i.e. if $S \otimes_R -$ is exact.*

Free modules are the prime example of flat modules.

Example 2.18. *The ring R, as a module over itself, is flat, since the tensor product functor is naturally equivalent to the identity functor. Hence, a* free *R-module, i.e. a direct sum $R^{\oplus I}$, is flat. Since every vector space over a field $\mathbb{K}$ is a free $\mathbb{K}$-module, every $\mathbb{K}$-module or $\mathbb{K}$-algebra is flat.*

Recall that an R-module is called *projective*, if it is a direct summand of a free module. Since flatness is inherited to direct summands, and free modules are flat, we see that projective modules are flat.

Lemma 2.19. *The base change of a flat R-module is again flat. Hence, if M_R is flat, so is $S \otimes_R M_R$.*

Proof. This follows from the property

$$S \otimes_R (M_R \otimes_R N_R) \cong (S \otimes_R M_R) \otimes_S (S \otimes_R N_R).$$

□

2.2 Faithfully flat descent

2.2.1 Basic properties

Faithfully flat R-algebras S are flat R-algebras, which reflect if a module is *zero*.

Definition 2.20. *A flat R-algebra S is called* faithfully flat*, if for every R-module M_R we have that $S \otimes_R M_R = 0$ implies that M_R is the zero module.*

A faithfully flat R-algebra S allows us to check that module is zero after tensoring with S. This definition implies directly that many other properties of modules, and morphisms of modules, descend along faithfully flat maps.

Lemma 2.21. *Let $\alpha\colon R \to S$ be a faithfully flat ring homomorphism.*

(a) Let $f\colon M_R \to N_R$ a morphism of R-modules, such that $S \otimes_R M_R \to S \otimes_R M_R$ is an injection (respectively a surjection), then f is an injection (respectively a surjection).

(b) A sequence of R-modules

$$U_R \xrightarrow{f} V_R \xrightarrow{g} W_R,$$

with $g \circ f = 0$ is exact, if and only if the base change

$$S \otimes_R U_R \to S \otimes_R V_R \to S \otimes_R W_R$$

is exact.

(c) If M_R is an R-module, such that $S \otimes_R M_R$ is a finite S-module, then M_R is finite as well.

(d) If $S \otimes_R M_R$ is a finite projective S-module, then M_R is a finite projective R-module.

(e) Flatness of $S \otimes_R M_R$ as S-module implies flatness of M_R as R-module.

Proof. A morphism of modules $f\colon M_R \longrightarrow N_R$ is an injection if and only i $\ker f = 0$ (respectively if $\operatorname{coker} f = N_R/\operatorname{image} f = 0$). Flatness of S implies that $S \otimes_R -$ preserves ker and coker. Therefore, by the assumption that S is faithfully flat, we see that f is an injection (respectively a surjection) if and only if its base change is. This concludes the proof of (a).

Flatness implies that exactness is preserved, therefore it suffices to show that exactness of

$$S \otimes_R U_R \to S \otimes_R V_R \to S \otimes_R W_R$$

implies that

$$U_R \to V_R \to W_R$$

is exact. Since $g \circ f = 0$, we have to show that the induced map

$$\operatorname{coker} f \to \ker g$$

is an isomorphism. We know that this is true after applying the functor $S \otimes_R -$, this implies the assertion, using statement (a).

Assertion (c) follows directly from (a). Choose a finite basis $n_1, \dots, n_\ell$ for $S \otimes_R M_R$, where each n_i can be written as a sum $m_{i1} \otimes s_{i1} + \cdots m_{ik} \otimes s_{ik}$. We claim that the collection of elements m_{ij} yields a basis for M_R. This is the case, since the corresponding map $(R^{\ell k} \longrightarrow M_R) \otimes_R S$ is a surjection. Hence, by (a) $R^{\ell k} \longrightarrow M_R$ is already a surjection.

The proof of assertion (d) is left as an exercise.[4] Assertion (e) follows from (b). □

It is also true that projectivity descends without finiteness assumptions, but the proof requires Raynaud–Gruson's characterisation of projective modules.

Lemma 2.22. *Let $R \to S$ be faithfully flat, and $R \to T$ an arbitrary morphism of rings. Then the co-base change $T \to S \otimes_R T$ is faithfully flat.*

Proof. We have seen in Lemma 2.19 that co-base change preserves flatness. Hence, we only have to show that for a T-module M_T we have $M_T \otimes_T (S \otimes_R T) = 0$ if and only if $M_T = 0$. By virtue of $M_T \otimes_T (S \otimes_R T) \cong M_T \otimes_R S$, we see that M_T must be zero. This concludes the proof. □

[4] Jack volunteered.

2.2.2 Descending modules

So far our treatment of descent theory has focused on qualitative aspects of modules. We have seen that properties like finiteness, flatness, and projectivity descend along faithfully flat map of rings. One can do better. It is possible to describe the datum of an R-module M_R in terms of the S-module $S \otimes_R M_R$, endowed with extra structure, which we will pin down subsequently.

We refer the reader to Vistoli's chapter in [FGI+05, Thm. 4.21] for a more detailed version of the proofs below.

Definition 2.23. *For a ring homomorphism $R \to S$ we define a category $\mathsf{Desc}_{R\to S}$ as the category of pairs (M_S, ϕ), where M_S is an S-module, and ϕ is an isomorphism of $S \otimes_R S$-modules*

$$\phi\colon M_S \otimes_R S \xrightarrow{\cong} S \otimes_R M,$$

which satisfies the identity

$$\begin{tikzcd} M_S \otimes_R S \otimes_R S \arrow[r] \arrow[dr] & S \otimes_R M \otimes_R S \arrow[d] \\ & S \otimes_R S \otimes_R M_S \end{tikzcd} \tag{9}$$

of $(S \otimes_R S \otimes_R S)$-modules.

Forgetting the isomorphism ϕ (a.k.a. the descent datum), we obtain a forgetful functor

$$\mathsf{Desc}_{R\to S} \to \mathsf{Mod}(R).$$

Base change always factors through this forgetful functor.

Lemma 2.24. *We have a commutative diagram of categories*[5]

$$\begin{tikzcd} \mathit{Mod}(R) \arrow[r] \arrow[dr, "S\otimes_R -"'] & \mathsf{Desc}_{R\to S} \arrow[d] \\ & \mathit{Mod}(S). \end{tikzcd}$$

By abuse of language, the resulting functor $\mathsf{Mod}(R) \to \mathsf{Desc}_{R\to S}$ will also be denoted by $S \otimes_R -$.

Proof. Let M_R be an R-module. We have to produce an isomorphism ϕ_M of $(S \otimes_R S)$-modules

$$(S \otimes_R M_R) \otimes_R S \xrightarrow{\phi} S \otimes_R (S \otimes_R M_R).$$

There is a natural choice for such a morphism, it sends the element $s_1 \otimes m \otimes s_2$ to $s_1 \otimes s_1 \otimes m$. We now have to check that (9) is satisfied. This amounts to

$$s_1 \otimes m \otimes s_2 \otimes s_3 \mapsto s_1 \otimes s_2 \otimes m \otimes s_3 \mapsto s_1 \otimes s_2 \otimes s_3 \otimes m$$

being the same map as

$$s_1 \otimes m \otimes s_2 \otimes s_3 \mapsto s_1 \otimes s_2 \otimes s_3 \otimes m.$$

This defines the required functor $\mathsf{Mod}(R) \to \mathsf{Desc}_{R\to S}$, such that the diagram above commutes. $\square$

[5] We adopt the convention that a diagram of functors which commutes up to a natural transformation is called commutative. A more precise formulation would be to call such diagrams 2-commutative.

Theorem 2.25 (Faithfully flat descent)**.** *Let $R \to S$ be a faithfully flat morphism of rings. The canonical functor*

$$- \otimes_R S \colon \mathit{Mod}(R) \longrightarrow \mathsf{Desc}_{R\to S}$$

is an equivalence of categories.

Proof. We denote the functor $- \otimes_R S$ by F. Let $G\colon \mathsf{Desc}_{R\to S} \to \mathsf{Mod}(R)$ be the functor, sending (M_S, ϕ) to the R-module

$$G(M_S, \phi) = \{m \in M | \phi(m \otimes 1) = 1 \otimes m\}.$$

We claim that F and G are mutually inverse functors. At first, we construct a natural transformation $\mathrm{id}_{\mathsf{Mod}(R)} \to GF$, i.e. for every R-module M_R a canonical map

$$\tau\colon \text{ M} \to G(S \otimes_R M_R).$$

Lemma 2.26. *Let $R \to S$ be faithfully flat, and M_R an R-module. For $i = 1, 2$ we denote by $e_i\colon S \to S \otimes_R S$ the maps $e_1(s) = s \otimes 1$, and $e_2(s) = 1 \otimes s$. The sequence*

$$0 \to M_R \xrightarrow{\delta} S \otimes_R M_R \xrightarrow{(e_1-e_2)\otimes_R \mathrm{id}_M} S \otimes_R S \otimes_R M_R$$

is an exact sequence.

We will prove this lemma at the end of this subsection. For now we note that $\ker((e_1-e_2)\otimes_R \mathrm{id}_M)$ can be identified with $G(S \otimes_R M_R)$, since we have

$$((e_1 - e_2) \otimes_R \mathrm{id}_M)(s \otimes b) = s \otimes 1 \otimes m - 1 \otimes s \otimes m = \phi_M(s \otimes m \otimes 1) - 1 \otimes s \otimes m.$$

By virtue of the lemma we have that $G(S \otimes_R M_R)$ is isomorphic to M_R.

Vice versa, if (N_S, ϕ) is an object in $\mathsf{Desc}_{R\to S}$, we have to produce a natural morphism

$$\gamma\colon S \otimes_R G(N_S, \phi) \to N_S.$$

By definition, we have that $G(N_S, \phi) \subset N_S$. In particular, we obtain a morphism γ by S-linear extension:

$$s \otimes n \mapsto s \cdot n.$$

As before, we have to check that γ is an isomorphism. In order to see this, we define morphisms of modules $f_i\colon N_S \longrightarrow S \otimes_R N_S$ for $i = 1, 2$. We set $f_1(n) = 1 \otimes n$, and $f_2(n) = \phi(n \otimes 1)$. The morphisms are chosen in a way, such that we have

$$G(N, \phi) = \ker(f_1 - f_2).$$

We then use the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_R \otimes_R S \arrow[r] \arrow[d, "\gamma\circ T"] & N \otimes_R S \arrow[r, "(e_1-e_2)\otimes_R \mathrm{id}_S"] \arrow[d, "\phi"] & S \otimes_R M_R \otimes_R S \arrow[d] \\
0 \arrow[r] & N_S \arrow[r] & S \otimes_R N_S \arrow[r, "(f_1-f_2)"] & S \otimes_R S \otimes_R N_S.
\end{tikzcd}$$

Here, T denotes the map exchanging the factors $M_R \otimes_R S \xrightarrow{\cong} S \otimes_R M$. Since the second and third vertical arrow are isomorphisms, so is the first. This implies that $S \otimes_R G(N_S, \phi) \cong N_S$. $\square$

It remains to prove Lemma 2.26. It could be considered at the key technical result which lies at the heart of descent theory. It is also the only place where we will visibly use the assumption that $\alpha\colon R \to S$ is faithfully flat.

Proof of Lemma 2.26. We assume that there exists a ring homomorphism $g\colon S \to R$, such that $g \circ \alpha = \mathrm{id}_R$. In plain language: g is a left inverse. This implies in particular that α is injective, hence deals with exactness at the first node from the left. We have to show that an element in the kernel of $(e_1 - e_2) \otimes \mathrm{id}_M$ lies in the image of δ. Let $s \otimes m$ be in the kernel, i.e. we have $s \otimes m \otimes 1 = 1 \otimes s \otimes m$. Apply the map g to the first factor, which yields the identity

$$g(s) \otimes m = s \otimes m.$$

Since $g(s) \in R$, we can rewrite the left hand side as $1 \otimes g(s)m$. This implies that $s \otimes m \in \mathrm{image}(\delta)$.

If $R \to S$ is a ring homomorphism, we observe that the base change

$$R \otimes_R S \cong S \to S \otimes_R S$$

has a section given by the multiplication map $S \otimes_R S \to S$. This implies directly that the sequence of Lemma 2.26 is exact, after tensoring with $- \otimes_R S$. Since $\alpha\colon R \to S$ is faithfully flat, we conclude from Lemma 2.21(b) that the original sequence is exact as well. $\square$

2.2.3 Descent for ring homomorphisms

Assume that we have a ring homomorphism $\beta\colon S \to T$, and a third ring R. We will see in this paragraph that ring homomorphisms from R to S can be described in terms of the composition $R \to T$, provided that β is faithfully flat. While this is a purely algebraic statement at this point, we will give a geometric interpretation of this result in a later section.

Proposition 2.27. *We have natural maps $e_1\colon T \to T \otimes_S T$, and $e_2\colon T \to T \otimes_S T$. The diagram of sets*

$$\mathit{Hom}_{\mathsf{Rng}}(R, S) \to \mathit{Hom}_{\mathsf{Rng}}(R, T) \rightrightarrows \mathit{Hom}_{\mathsf{Rng}}(R, T \otimes_S T)$$

is an equalizer diagram in the category of sets. I.e., the set of ring homomorphism $g\colon R \to T$, satisfying $e_1 \circ g = e_2 \circ g$, is in bijection with the set of ring homomorphisms $f\colon R \to S$.

Proof. Lemma 2.26 implies that we have an exact sequence

$$0 \to S \xrightarrow{e_1 - e_2} S \otimes_R S,$$

hence an equalizer diagram in the category of rings

$$S \to T \rightrightarrows T \otimes_S T.$$

Since $\mathsf{Hom}_{\mathsf{Rng}}(R, -)$ sends equalizers to equalizers, we obtain the assertion. $\square$

3 Sheaves and stacks

Grothendieck topologies provide a framework which allow to formalise the process of glueing global data from local data. The ability to glue, or descend, is the defining quality of sheaves and stacks.

3.1 Sheaves

3.1.1 Topological spaces

We fix a topological space X. There is a category, denoted by $\mathsf{Open}(X)$, whose objects are open subsets $U \subset X$, and morphisms are inclusions $U \subset V$.

Definition 3.1. *A (set-valued)* presheaf *on X is a functor $F\colon \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Set}$.*

In more concrete terms, we associate to every open subset $U \subset X$ a set $F(U)$, as well as a restriction map

$$r_U^V\colon F(V) \to F(U)$$

for every inclusion $U \subset V$. Moreover, the conditions

(a) $r_U^U = \mathrm{id}_{F(U)}$,

(b) $r_U^V \circ r_V^W = r_U^W$ for triples of open subsets $U \subset V \subset W$,

are satisfied.

If Y is a topological space, we denote by $\underline{Y}_X$ the presheaf on X, which associates to an open subset $U \subset X$ the set of continuous functions $U \to Y$, i.e.,

$$\underline{Y}_X(U) = \mathsf{Hom}_{\mathsf{Top}}(U, Y).$$

The restriction maps r_U^V are given by

$$f \mapsto f|_U,$$

i.e., sending a continuous map $f\colon V \to Y$ to the composition $f \circ i$, where $i\colon U \hookrightarrow V$ denotes the inclusion.

If $U = \bigcup_{i \in I} U_i$ is an open covering, we have for every pair of open subsets U_i, U_j two maps

$$U_i \hookleftarrow U_i \cap U_j \hookrightarrow U_j.$$

Hence, for every presheaf F we have a pair of restriction maps

$$F(U_i) \to F(U_i \cap U_j) \leftarrow F(U_j).$$

Taking a product over all pairs $(i, j) \in I^2$, and relabelling indices, we obtain

$$\prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I^2} F(U_i \cap U_j).$$

Definition 3.2. *A presheaf F is called a* sheaf*, if for every open subset $U \subset X$, and every open covering $U = \bigcup_{i \in I} U_i$, we have that*

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I^2} F(U_i \cap U_j)$$

is an equalizer diagram.

Unravelling the definition of equalizers, we see that a presheaf is a sheaf, if and only if for every $U = \bigcup_{i\in I} U_i$ as above, the following condition is satisfied: given a collection of local sections $s_i \in F(U_i)$, which agree on overlaps, i.e. satisfy $r^{U_i}_{U_i\cap U_j}(s_i) = r^{U_j}_{U_i\cap U_j}(s_j)$ for all pairs of indices, there exists a unique section $s \in F(U)$, such that $r^U_{U_i}(s) = s_i$.

Lemma 3.3. *The presheaf $\underline{Y}_X$ is a sheaf.*

Concrete proof. If $f_i \colon U_i \to Y$ are continuous functions, such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all pairs of indices, then there is a well-defined map of sets $f \colon U \to Y$, which sends $x \in U$ to $f_i(x)$, if $x \in U_i$. Since continuity is a local property, i.e. continuity at a point $x \in X$ depends only on the restriction $f|_{U_i}$, for $x \in U_i$, we see that f is a continuous function. □

Abstract proof. We can represent U as a co-equalizer

$$\coprod_{(i,j)\in I^2} U_i \cap U_j \rightrightarrows \coprod_{i\in I} U_i \to U,$$

i.e., as a colimit in the category Top of topological spaces. The universal property of colimits implies that $\mathsf{Hom}_{\mathsf{Top}}(-, Y)$ sends a co-equalizer to an equalizer. □

3.1.2 Grothendieck topologies and sheaves

Definition 3.1 of presheaves would work for any category C instead of Open_X. In fact, our prime example, the sheaf $\underline{Y}_X$, extends by definition to a functor

$$\begin{tikzcd} \mathsf{Top}^{\mathrm{op}} \arrow[r, "\mathsf{Hom}_{\mathsf{Top}}(-,Y)"] & \mathsf{Set} \\ \mathsf{Open}(X)^{\mathrm{op}}. \arrow[u] \arrow[ur, "\underline{Y}_X"'] & \end{tikzcd}$$

The only reason to prefer the category $\mathsf{Open}(X)$ over an abstract category C is the fact that we have a notion of open coverings in $\mathsf{Open}(X)$, coming from point-set topology. This is essential to introduce sheaves. Grothendieck topologies provide a remedy for general categories.

Definition 3.4. *Let C be a category. A* Grothendieck topology *$\mathcal{T}$ on C consists of a collection of sets of morphisms (called* coverings*) $\{U_i \to U\}_{i\in I}$ for each object $U \in \mathsf{C}$, satisfying:*

(a) For every isomorphism $U' \to U$, the singleton $\{U' \to U\}$ is a covering.

(b) Coverings are preserved by base change, i.e. if $\{U_i \to U\}_{i\in I}$ is a covering, and $V \to U$ a morphism in C, then $\{U_i \times_U V \to V\}_{i\in I}$ is well-defined, and a covering.

(c) Given a covering $\{U_i \to U\}$, and for each $i \in I$ a covering $\{U_{ij} \to U_i\}_{j\in J_i}$, then

$$\{U_{ij} \to U\}_{(i,j)\in\prod_{i\in I} J_i}$$

is a covering.

Originally, Grothendieck topologies were called *pre*-topologies. We drop the prefix, but will comment later on the reasons for this distinction.

A pair $(\mathsf{C}, \mathcal{T})$ is called a *site*. Implicitly, we have already seen an example of a site.

Example 3.5. *(a) For the category* $\mathsf{Open}(X)$*, for* X *a topological space, we have a natural choice for a Grothendieck topology. We define* $\mathcal{T}(X)$ *to be the collection of all* $\{U_i \subset U\}$*, such that* $\bigcup_{i \in I} U_i = U$*.*

(b) The Grothendieck topology $\mathcal{T}(X)$ *can be extended to* $\mathcal{T}$ *on* Top*. We say that* $\{U_i \xrightarrow{f_i} U\}_{i \in I} \in \mathcal{T}$*, if* $\bigcup_{i \in I} f(U_i) = U$*, and each* f_i *is an open map, which is a homeomorphism onto its image (in other words, it is equivalent to the inclusion of an open subset).*

(c) Let Aff *denote the category* $\mathsf{Rng}^{\mathrm{op}}$*. Recall, that we denote the object in* Aff*, corresponding to the ring* R *by* $\operatorname{Spec} R$*. We have a Grothendieck topology on* Aff*, consisting of singletons* $\{\operatorname{Spec} S \to \operatorname{Spec} R\}$*, where* $R \to S$ *is a faithfully flat map of rings.*

Proof. Statements (a) and (b) follow right from the definitions. We will therefore focus on assertion (c). We know that an isomorphism of rings $R \xrightarrow{\simeq} S$ is faithfully flat, hence axiom (a) of Definition 3.4 is satisfied. We now have to show that for a covering $\{\operatorname{Spec} S \to \operatorname{Spec} R\}$, and an arbitrary map $\operatorname{Spec} T \to \operatorname{Spec} R$, the base change

$$\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} T \to \operatorname{Spec} T$$

is also a covering. Note that $\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} T \cong \operatorname{Spec}(S \otimes_R T)$, since the tensor product is the coproduct in the category Rng (Lemma 2.9). For a faithfully flat R-algebra S, the base change $S \otimes_R T$ is a faithfully flat T-algebra (Lemma 2.22), which concludes the verification of axiom (b) of Definition 3.4. $\square$

If $(\mathsf{C}, \mathcal{T})$ is a site, we can make sense of sheaves on C. We define a presheaf on C to be a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. We will use the abstract coverings provided by the Grothendieck topology $\mathcal{T}$ to make sense of the sheaf condition. In order to imitate Definition 3.2, we have to make sense of the intersection $U_i \cap U_j$.

Definition 3.6. *Let* C *be a category, and* $f \colon X \to Z$*,* $g \colon Y \to Z$ *two morphisms. Consider the category of diagrams*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z. \end{tikzcd}$$

If it exists, we denote the top left corner (W*) of the final object in this category by* $X \times_Z Y$*, and call it the fibre product of the two morphisms* f *and* g*.*

By the definition of final objects, we see that whenever we have a commutative diagram as above, there exists a unique morphism $W \to X \times_Z Y$, such that the resulting diagram

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & Z.
\end{tikzcd}$$

commutes.

Example 3.7. *For a topological space X, and inclusions of open subsets $U \hookrightarrow X$, $V \hookrightarrow X$, we have that the fibre product $U \times_X V$ in the category $\mathsf{Open}(X)$, respectively Top, is given by the inclusion of the open subset $U \cap V \hookrightarrow X$.*

This motivates us to replace every occurrence of intersections, in the definition of a sheaf, by fibre products.

Definition 3.8. *Let C be a category. A functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is called a* presheaf*. The category of presheaves will be denoted by $\mathsf{Pr}(\mathsf{C})$. If $(\mathsf{C}, \mathcal{T})$ is a site, a presheaf is called a* sheaf*, if for every $\{U_i \to U\}_{i\in I}$ the diagram*

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I^2} F(U_i \times_U U_j)$$

is an equalizer. We denote the full subcategory of sheaves by $\mathsf{Sh}_{\mathcal{T}}(\mathsf{C})$.

The heuristics behind this definition is the same as for topological spaces. Given an abstract covering $\{U_i \to U\}_{i\in I}$ of $U \in \mathsf{C}$, and locally defined sections $s_i \in F(U_i)$, which agree when restricted (or pulled back) to the "intersections" $U_i \times_U U_j$, then there exists a unique section $s \in F(U)$, which agrees with s_i over each U_i.

3.1.3 Three examples

What makes the theory of sheaves over general sites slightly more harder to wrap ones head around is the fact that the maps of the abstract coverings $U_i \to U$ cannot be pictured as inclusions of open subsets, but could in principle be arbitrarily complicated. In this paragraph we discuss three abstract examples of Grothendieck topologies and sheaves, which illustrate that the glueing procedure, imposed by the sheaf condition, can be reasonable even in the absence of topological context.

Example 3.9. *Let $\mathsf{C} = \mathsf{Set}$ be the category of sets. We consider the Grothendieck topology $\mathcal{T}$, which consists of all collections $\{U_i \xrightarrow{f_i} U\}$, such that $\bigcup_{i\in I} f(U_i)$ is surjective. For a set X we have the presheaf $h_X = \mathit{Hom}_{\mathsf{Set}}(-, X)$, represented by X. We claim that h_X is a sheaf with respect to the topology $\mathcal{T}$.*

Example 3.10. *For C the category of open subsets $U \subset \mathbb{R}^n$ (where n is allowed to vary), and smooth maps as morphisms, we may choose $\mathcal{T}$ to consist of all sets $\{U_i \xrightarrow{f_i} U\}_{i\in I}$, where each f_i is* *a smooth submersion, and* $\bigcup_{i \in I} f(U_i) = U$*. Every smooth manifold* X *gives rise to a sheaf on* C*, by sending* $U \in \mathsf{C}$ *to the set of smooth maps* $U \to X$*. The resulting functor*

$$\mathsf{Mfd} \to \mathsf{Sh}_{\mathcal{T}}(\mathsf{C})$$

is an embedding of categories (i.e. fully faithful).

Proof. Exercise.[6] □

Example 3.11. *As in Example 3.5(c), we choose* C *to be* $\mathsf{Aff} = \mathsf{Rng}^{\mathrm{op}}$*, with the topology induced by* $\{\operatorname{Spec} S \to \operatorname{Spec} R\}$*, with* $R \to S$ *being a faithfully flat map of rings. For every* $\operatorname{Spec} T \in \mathsf{Aff}$*, the presheaf* $h_{\operatorname{Spec} T} = \mathit{Hom}_{\mathsf{Aff}}(-, \operatorname{Spec} T)$*, represented by* $\operatorname{Spec} T$*, is a sheaf.*

Proof. □

3.2 Stacks

3.2.1 Groupoids as generalised sets

Definition 3.12. *A category in which every arrow is invertible is called a groupoid.*

Every set can be viewed as a category, with every morphism being the identity morphism of an object. Hence, sets give rise to examples of groupoids. Groupoids are best visualised as a generalised set, where every element has a possibly non-trivial group of automorphisms.

Example 3.13. *Let* G *be a group acting on a set* X*. We denote by* $[X/G]$ *the so-called* quotient groupoid*. It is defined to be the category, whose set of objects is* X*. A morphism* $x \to y$ *is given by an element* $g \in G$*, such that* $g \cdot x = y$*. For every* $x \in X$*, we have* $\operatorname{Aut}_{[X/G]}(x) = G_x$*, i.e. the stabiliser subgroup of* $x \in X$*.*

Another example of groupoids is induced by topological spaces.

Example 3.14. *For every topological space* X *we have a groupoid* $\pi_{\leq 1}(X)$*, whose objects are given by the points* $x \in X$*, and morphisms are homotopy classes of paths* $x \to y$*. We have* $\operatorname{Aut}_{\pi_{\leq 1}(X)}(x) = \pi_1(X, x)$*, by definition. One can show that every groupoid arises as* $\pi_{\leq 1}(X)$ *for a topological space* X*.*[7]

One could arrange groupoids into a category. However, this is often too strict, to capture the higher nature of groupoids. It is infinitely more sensible to consider the 2-category of groupoids instead.

Definition 3.15. *A (strict)* 2*-category* C *consists of the following data:*

- *a class of objects* $\mathbf{Obj}(\mathsf{C})$*,*
- *for every* $X, Y \in \mathbf{Obj}(\mathsf{C})$ *a category* $\mathit{Hom}_{\mathsf{C}}(X, Y)$ *of morphisms,*
- *for* $X, Y, Z \in \mathbf{Obj}(\mathsf{C})$ *a functor* $\circ\colon \mathit{Hom}_{\mathsf{C}}(X, Y) \times \mathit{Hom}_{\mathsf{C}}(Y, Z) \to \mathit{Hom}_{\mathsf{C}}(X, Z)$*,*

[6] Volunteers?
[7] Jack volunteered.

- *for every* $X \in \mathbf{Obj}(\mathsf{C})$ *an object* $\mathrm{id}_X \in \mathit{Hom}_{\mathsf{C}}(X,X)$, *satisfying* $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f$, *for every* $f \in \mathit{Hom}_{\mathsf{C}}(X,Y)$,
- *such that associativity holds, i.e. for* $X, Y, Z, W \in \mathbf{Obj}(\mathsf{C})$ *we want the two natural functors* $\mathit{Hom}_{\mathsf{C}}(X,Y) \times \mathit{Hom}_{\mathsf{C}}(Y,Z) \times \mathit{Hom}_{\mathsf{C}}(Z,W) \to \mathit{Hom}_{\mathsf{C}}(X,W)$ *to agree.*

In other words, a 2-category is a category, where we additionally have 2-morphisms between morphisms.

Example 3.16. *We denote by* Cat *the* 2*-category of (small) categories. It's class of objects is the class of categories whose class of objects is a set (hence, the terminology small).*[8] *The set of morphisms* $\mathit{Hom}_{\mathsf{Cat}}(\mathsf{C},\mathsf{D})$ *is defined to be the set of functors* $\mathsf{Fun}(\mathsf{C},\mathsf{D})$. *We denote by* $[F,G]$ *the set of natural transformations between two functors* $F, G\colon \mathsf{C} \to \mathsf{D}$. *Recall that a natural transformation consists of a morphism* $\eta_X\colon F(X) \to G(X)$ *for every object* $X \in \mathsf{C}$, *such that for every arrow* $\alpha\colon X \to Y$ *the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "F(\alpha)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(\alpha)"] & G(Y) \end{tikzcd}$$

commutes. The full 2*-subcategory of groupoids will be denoted by* Gpd.

Every 2-morphism in the 2-category Gpd is invertible. Such a 2-category is often called a $(2,1)$-category.

3.2.2 Stacks as groupoid-valued sheaves

In order to define prestacks and stacks, we have to clarify the notion of functors between 2-categories. The naive definition, which requires a functor to respect composition on the nose, is too strict to be a good definition by today's understanding of category theory. However, it gets the job done (since every functor can be strictified), and we will therefore stick to it for now.

Definition 3.17. *Let* C *and* D *be* 2*-categories. A* strict functor $F\colon \mathsf{C} \to \mathsf{D}$ *is given by a map between objects* $\mathbf{Obj}(\mathsf{C}) \to \mathbf{Obj}(\mathsf{D})$, *as well as a functor* $\mathit{Hom}_{\mathsf{C}}(X,Y) \to \mathit{Hom}_{\mathsf{C}}(F(X),F(Y))$ *for every pair of objects* X, Y, *which is compatible with composition.*

The definition of sheaves also rested on the notion of equalizers, i.e., limits. Therefore it is necessary to discuss limits in the 2-categorical framework of groupoids. We first give a concrete definition of the limit of a diagram of groupoids.

Definition 3.18. *Let* I *be a category, and* $I \to \mathsf{Gpd}$ *a functor, which sends* $i \in I$ *to the groupoid* C_i. *The limit* $\lim_{i \in I} \mathsf{C}_i$ *is defined to be the following groupoid: its objects are collections* $X_i \in \mathsf{C}_i$, *and for every morphism* $\alpha\colon i \to j$ *in* I *an isomorphism* $\phi_\alpha\colon F(\alpha)(X_i) \xrightarrow{\simeq} X_j$, *such that for two composable arrows* $\alpha\colon i \to j$, *and* $\beta\colon j \to k$, *we have* $\phi_{\beta\circ\alpha} = \phi_\beta \circ \phi_\alpha$.

Limits in 2-categories are often referred to as 2-limits. We drop the prefix, and refer to a limit in the classical sense in a 2-category as strict limit. As one would expect, limits are characterised by a universal property.

[8] This distinction between sets and classes is usually handled by referring to universes. The reader not acquainted with these aspects of set theory can safely ignore all set-theoretic intricacies.

Definition 3.19. *Let* C *be a category. A* prestack[9] *is a strict functor* $\mathsf{C}^{\mathrm{op}} \to \mathsf{Gpd}$*. The* 2*-category of prestacks will be denoted by* $\mathsf{PrSt}(\mathsf{C})$*. If* $(\mathsf{C}, \mathcal{T})$ *is a site, we denote by* $\mathsf{St}_{\mathcal{T}}(\mathsf{C})$ *the full* 2*-subcategory of prestacks* F*, for which for every* $\{U_i \to U\}_{i\in I}$ *the diagram*[10]

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I^2} F(U_i \times_U U_j) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \prod_{(i,j,k)\in I^2} F(U_i \times_U U_j \times_U U_k)$$

induces is a limit *in the* 2*-category of groupoids (i.e., induces an equivalence with the limit defined in Definition 3.18). We call such a prestack a* stack*. The full* 2*-subcategory of stacks will be denoted by* $\mathsf{St}_{\mathcal{T}}(\mathsf{C})$.

In concrete terms, the definition of a stack is tantamount to the following.

Lemma 3.20. *A prestack* F *is a stack if and only if the following two conditions are satisfied.*

(a) For every $\{U_i \to U\}_{i\in I}$*, and a collection of objects* $X_i \in F(U_i)$*, and isomorphisms*

$$\phi_{ij} \colon X_i|_{U_i\times_U U_j} \xrightarrow{\simeq} X_j|_{U_i\times_U U_j},$$

which satisfy the cocycle condition $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ *on* $U_i \times_U U_j \times_U U_k$*, there exists an object* $X \in F(U)$*, together with isomorphisms* $\phi_i \colon X|_{U_i} \xrightarrow{\simeq} X_i$*.*

(b) For every $U \in \mathsf{C}$*, and* $X, Y \in F(U)$*, we have that the functor* $\underline{\mathit{Hom}}(X,Y) \colon \mathsf{C}/U \to \mathsf{Set}$*, which sends* $V \to U$ *to* $\mathit{Hom}(X|_V, Y|_V)$ *is a sheaf.*[11]

Proof. [12] To show that these conditions are sufficient, suppose that we have a prestack F on C satisfying (i) and (ii). Fix a covering $\{U_i \to U\}_{i\in I}$. The commutativity of

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \prod_{i,j,k\in I} F(U_i \times_U U_j \times_U U_k)$$

and the universal property of limits gives us a canonical functor from $F(U)$ to the limit of

$$\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \prod_{i,j,k\in I} F(U_i \times_U U_j \times_U U_k) \ \ (*).$$

It is enough, then, to show that this functor is fully faithful and essentially surjective. Let us denote the two parallel maps

$$\alpha_1, \alpha_2 : \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

and the three parallel maps

$$\beta_1, \beta_2, \beta_3 : \prod_{i,j\in I} F(U_i \times_U U_j) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \prod_{i,j,k\in I} F(U_i \times_U U_j \times_U U_k).$$

[9] Note that our terminology differs from Vistoli's.

[10] parametrised by the category, whose objects are the ordered sets $\{0\}, \{0,1\}, \{0,1,2\} \subset \mathbb{N}$, with order-preserving maps as morphisms.

[11] In Vistoli's convention, a prestack is required to satisfy this property.

[12] This proof was provided by Craig Smith.

Then the limit of $(*)$ has as object collections (C_1, C_2, C_3) where

$$C_1 \in \prod_{i \in I} F(U_i), C_2 \in \prod_{i,j \in I} F(U_i \times_U U_j), C_3 \in \prod_{i,j,k \in I} F(U_i \times_U U_j \times_U U_k)$$

such that there are isomorphisms $\zeta_n : C_2 \xrightarrow{\cong} \alpha_n(C_1)$ and $\xi_m : C_3 \xrightarrow{\cong} \beta_m(C_2)$ that compose to give isomorphisms $C_3 \xrightarrow{\cong} \beta_m \alpha_n(C_1)$. Given (C_1, C_2, C_3) in the limit of $(*)$, we have $C_1 = (X_i)_{i \in I}$, $C_2 = (Y_{i,j})_{i,j \in I}$, $C_3 = (Z_{i,j,k})_{i,j,k \in I}$ for $X_i \in F(U_i), Y_{i,j} \in F(U_i \times_U U_j), Z_{i,j,k} \in F(U_i \times_U U_j \times_U U_k)$. Then the isomorphisms ζ_m are of the form $\zeta_m = (\zeta_m^{i,j})_{i,j \in I}$ giving isomorphisms

$$\phi_{i,j} = \zeta_2^{i,j} (\zeta_1^{i,j})^{-1} : X_i|_{U_i \times_U U_j} \xrightarrow{\cong} X_j|_{U_i \times_U U_j}.$$

Then these satisfy the cocycle condition as $\zeta_2^{i,j}(\zeta_1^{i,j})^{-1}\zeta_2^{j,k}(\zeta_1^{j,k})^{-1}, \zeta_2^{i,k}(\zeta_1^{i,k})^{-1}$ agree when restricted to $F(U_i \times_U U_j \times_U U_k)$. Thus there is an object $X \in F(U)$ with isomorphisms $X|_{U_i} \xrightarrow{\cong} X_i$ by our assumption. So the functor from the limit of $(*)$ to $F(U)$ is essentially surjective.

Morphisms in the limit of $(*)$ are collections of morphisms $(f_1, f_2, f_3) : (C_1, C_2, C_3) \to (C_1', C_2', C_3')$, where $f_i : C_i \to C_i'$, which induce commutative diagrams

$$\begin{tikzcd}
C_2 \arrow[r, "\cong"] \arrow[d] & \alpha_n(C_1) \arrow[d] & C_3 \arrow[r, "\cong"] \arrow[d] & \beta_m(C_2) \arrow[d] \\
C_2' \arrow[r, "\cong"] & \alpha_n(C_1') & C_3' \arrow[r, "\cong"] & \beta_m(C_2').
\end{tikzcd}$$

Thus these morphisms are determined by maps $f_1 : C_1 \to C_1'$ such that $\alpha_1(f_1) = \alpha_2(f_1)$ under the identification $C_2 \cong \alpha_n(C_1)$. That is, morphisms in this limit correspond to morphisms in the equaliser of

$$\mathrm{Hom}(X|_{U_i}, X'|_{U_i}) \rightrightarrows \prod_{i,j \in I} \mathrm{Hom}(X|_{U_i \times_U U_j}, X'|_{U_i \times_U U_j})$$

under our canonical functor. Since we assume that $\mathrm{Hom}(X|_V, X'|_V)$ is a sheaf for every $X, X' \in F(U)$, we have exactly the result that this equaliser is $\mathrm{Hom}(X, X')$ in $F(U)$, and that our functor must be fully faithful. Thus we have that $F(U)$ is isomorphic to the limit of $(*)$, and so F is a stack.

Suppose now that a F is a stack. Then for every $\{U_i \to U\}_{i \in I}$, $F(U)$ is isomorphic to the limit of the diagram

$$\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j) \overset{\textstyle\rightarrow}{\rightrightarrows} \prod_{i,j,k \in I} F(U_i \times_U U_j \times_U U_k) \ (*)$$

So, for a collection of objects $X_i \in F(U_i)$, and isomorphisms $\phi_{ij} : X_i|_{U_i \times_U U_j} \xrightarrow{\cong} X_j|_{U_i \times_U U_j}$ satisfying the cocylce condition, we obtain an object

$$((X_i)_{i \in I}, (X_i|_{U_i \times_U U_j})_{i,j \in I}, (X_i|_{U_i \times_U U_j \times_U U_k})_{i,j,k \in I})$$

of the limit of this diagram. Thus we get an $X \in F(U)$ corresponding to this element. That is, an $X \in F(U)$ together with an isomorphism between the image of X under the map $F(U) \to \prod_{i \in I} F(U_i)$ and the collection $(X_i)_{i \in I}$. So we have isomorphisms $\phi_i : X|_{U_i} \to X_i$.

Suppose now that we have $U \in C$ and $X, Y \in F(U)$. Suppose further that we have a collection of objects $\{U_i \to U\}_{i \in I}$ in C/U. Then for every object $(f_i)_{i\in I}$ in $\prod_{i\in I} \mathrm{Hom}(X|_{U_i}, Y|_{U_i})$ which has the same image under the parallel maps

$$\prod_{i\in I} \mathrm{Hom}(X|_{U_i}, Y|_{U_i}) \rightrightarrows \prod_{i,j\in I} \mathrm{Hom}(X|_{U_i\times_U U_j}, Y|_{U_i\times_U U_j})$$

we obtain a morphism

$$((X_{U_i})_{i\in I}, (X_i|_{U_i x_U U_j})_{i,j\in I}, (X_i|_{U_i\times_U U_j \times_U U_k})_{i,j,k\in I}) \to$$

$$((Y_{U_i})_{i\in I}, (Y_i|_{U_i x_U U_j})_{i,j\in I}, (Y_i|_{U_i\times_U U_j \times_U U_k})_{i,j,k\in I})$$

in the limit of the diagram $(*)$. Since $F(U)$ is isomorphic to the limit of this diagram, we obtain a unique map $f : X \to Y$ that restricts to the maps $f|_{U_i} : X_{U_i} \to Y_{U_i}$. This is precisely the result that

$$\mathrm{Hom}(X, Y) \to \prod_{i\in I} \mathrm{Hom}(X|_{U_i}, Y|_{U_i}) \rightrightarrows \prod_{i,j\in I} \mathrm{Hom}(X|_{U_i\times_U U_j}, Y|_{U_i\times_U U_j})$$

is an equaliser diagram. That is, we have that the functor $\underline{\mathrm{Hom}}(X, Y) : C/U \to \mathrm{Set}$, which sends $V \to U$ to $\mathrm{Hom}(X|_V, Y|_V)$ is a sheaf. Hence we have that these conditions are necessary. $\square$

3.2.3 Examples

The first example is closely related to what we discussed in the Subsection 1.1. We consider a topological space X, and let C be the category $\mathsf{Open}(X)$ of open subsets, with its canonical Grothendieck topology $\mathcal{T}$. We know that it is possible to glue sheaves with respect to open coverings, i.e. given $U = \bigcup_{i\in I} U_i$, and a sheaf G_i on U_i, for each $i \in I$, as well as isomorphisms $\phi_{ij} : G_i|_{U_i\cap U_j} \to G_j|_{U_i\cap U_j}$, satisfying the cocycle condition, there is a sheaf G on U, well-defined up to a unique isomorphism, which restricts to G_i on each U_i.

Example 3.21. *Let $F\colon \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Gpd}$ be the prestack which sends $U \subset X$ to the groupoid*[13] *of set-valued sheaves on U. Then, F is a stack.*

In fact, we could have formulated this example without reference to topological spaces.

Proof of Example 3.21. [14] Suppose we have an open covering $\{U_i \subset U\}_{i\in I}$ and a collection of set-valued sheaves G_i on U_i for $i \in I$ with isomorphisms $\phi_{i,j} : G_i(U_i \cap U_j) \xrightarrow{\cong} G_j(U_i \cap U_j)$ satisfying the cocycle condition $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ on $U_i \cap U_j \cap U_j$. Then, for an open subset $V \subset U$ we have $\{V_i \subset V\}_{i\in I}$ is an open cover for V where $V_i = V \cap U_i$. Then we define

$$G(V) = \{(s_i)_{i\in I} \in \prod_{i\in I} G_i(V_i) \mid \phi_{i,j}(s_i) = s_j \text{ for all } i, j\}.$$

We want to check that this defines a sheaf on U. Suppose we have open subsets $W \subset V \subset U$. Then we have $W_i \subset V_i \subset U_i$ for all $i \in I$ and so we have restriction morphisms $\rho_i : G_i(V_i) \to G_i(W_i)$. Then we can define a map

$$G(V) \to G(W), \;\; (r_i)_{i\in I} \mapsto (\rho_i(r_i))_{i\in I}.$$

[13] This means that we discard all morphisms of sheaves, which are not isomorphisms.
[14] This proof was provided by Craig Smith.

Since $\phi_{i,j}$ are morphisms of sheaves, they commute with the restrictions maps ρ_i, and hence $\phi_{i,j}(\rho_i(r_i)) = \rho_j(\phi_{i,j}(r_i)) = \rho_j(r_j)$ for all $i,j \in I$ for all $(r_i)_{i\in I} \in G(V)$. Hence this map is well defined. It is clear then that G defines a functor $\mathrm{Open}(U)^{\mathrm{op}} \to \mathrm{Set}$, and hence is a presheaf.

Suppose now that we have $V \subset U$ with open covering $\{V^{(j)} \subset V\}_{j\in J}$. Let us again denote $V_i = V \cap U_i$, $V_i^{(j)} = V^{(j)} \cap U_i$. Consider the parallel maps

$$\prod_{j\in J} G(V^{(j)}) \rightrightarrows \prod_{i,k\in J} G(V^{(j)} \cap V^{(k)}).$$

Then we clearly have a map $G(V) \to \prod_{j\in J} G(V_j)$, which we must show is an equaliser of this diagram. But this follows immediately from the fact that

$$G_i(V_i) \to \prod_{j\in J} G_i(V_i^{(j)}) \rightrightarrows \prod_{i,k\in J} G_i(V_i^{(j)} \cap V_i^{(k)})$$

is an equaliser diagram for each $i \in I$.

Now we can define maps $\phi_i : G|_{U_i} \to G_i$, $(s_j)_{j\in I} \mapsto s_i$ which clearly give our required commutative diagram. These maps are clearly isomorphisms as we can define an explicit inverse $s_i \mapsto (\phi_{i,j}(s_i))_{j\in I}$. Thus we have our result. □

The second example is of algebraic nature. We consider the site, introduced in Example 3.11. Recall that its underlying category is $\mathsf{Aff} = \mathsf{Rng}^{\mathrm{op}}$. Coverings are given by morphisms corresponding to faithfully flat ring homomorphisms $R \to S$. We will reformulate the fact that modules satisfy faithfully flat descent, (Theorem 2.25), using the language of stacks. The first formulation below contains the gist of it, although it is strictly speaking incorrect, since the stack of modules, as defined there, isn't even a prestack. We will fix this after having discussed the general idea.

Example 3.22 (Naive/incorrect formulation). *Consider the map $\mathsf{Mod}^{\times}$ which assigns to* $\operatorname{Spec} R \in \mathsf{Aff}^{\mathrm{op}}$ *the groupoid of R-modules (i.e., we discard all non-invertible morphisms of R-modules). The map in induced by a morphism* $\operatorname{Spec} S \to \operatorname{Spec} R$ *(corresponding to $R \to S$) is given by $S \otimes_R -$. As we have seen in Lemma 2.8, we have a natural equivalence of functors $T \otimes_S (S \otimes_R -) \simeq T \otimes_R -$, for every composable chain of ring homomorphisms $R \to S \to T$. But the functors don't strictly agree. We will ignore this subtlety for a moment, and pretend that $\mathsf{Mod}^{\times}$ is a well-defined prestack. Theorem 2.25 implies then that $\mathsf{Mod}^{\times}$ is a stack.*

The problem is that base change isn't strictly transitive, but only up to a natural transformation. In order to fix this, we use a general construction, which could be seen as a strictification strategy. See also [Toe] for a more detailed account.

Definition 3.23. *For a ring R we denote by $\widetilde{\mathsf{Mod}}(R)$ the groupoid, whose objects are S-modules $M(\alpha)$ for every ring homomorphism $\alpha\colon R \to S$, together with isomorphisms $\phi_\gamma\colon M(\alpha) \otimes_S T \xrightarrow{\simeq} M(\beta)$ for every morphism γ of R-algebras:*

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & T \\ S, \arrow[ur, "\beta"'] & \end{tikzcd}$$

such that $\phi_{\mathrm{id}_S}\colon M(\alpha)\otimes_S S \xrightarrow{\simeq} M(\alpha)$ is the canonical map, and for every composable pair of morphisms of R-algebras $\delta\colon S\to T$, and $\epsilon\colon T\to W$, as given by a commutative diagram

$$\begin{tikzcd} & R \arrow[dl, "\alpha"'] \arrow[d, "\beta"] \arrow[dr, "\gamma"] & \\ S \arrow[r, "\delta"] & T \arrow[r, "\epsilon"] & W, \end{tikzcd}$$

we have a commutative square

$$\begin{tikzcd} (M(\alpha)\otimes_S T)\otimes_T W \arrow[r] \arrow[d] & M(\beta)\otimes_T W \arrow[d] \\ M(\alpha)\otimes_S W \arrow[r] & M(\gamma). \end{tikzcd}$$

The essential difference is that the restriction map is obtained by restricting a data to a smaller index set, which is functorial on the nose, on not just up to a natural equivalence. However, we still retain the original information, as the next lemma shows.

Lemma 3.24. *The groupoids $\widetilde{\mathsf{Mod}}(R)$ and $\mathsf{Mod}(R)^\times$ are equivalent.*

Proof. We have a functor $F\colon \widetilde{\mathsf{Mod}}(R)\to\mathsf{Mod}(R)^\times$, which sends $(M(S),\phi^S)$ to $M(R)$. Let

$$G\colon\ \mathsf{Mod}(R)^\times \to \widetilde{\mathsf{Mod}}(R)$$

be the functor, sending $M\in\mathsf{Mod}(R)$ to the tautological collection $(M\otimes_R S, \mathrm{id}_{M\otimes_R S})$.

We certainly have $(F\circ G)(M)$ agrees with M, by definition. Vice versa, given $(M(S),\phi^S)$, the collection $G(F(M(S),\phi^S))$ is given by $(M(R)\otimes_R S,\mathrm{id}_{M(R)\otimes_R S})$, which is naturally equivalent to $(M(S),\phi^S)$ by means of the system of maps ϕ^S. $\square$

We can now (correctly) formulate the faithfully flat descent as the statement:

Example 3.25. *The prestack $\widetilde{\mathsf{Mod}}$ is a stack over Aff with respect to the Grothendieck topology given by faithfully flat morphisms.*

3.2.4 Fibre products in 2-categories

In the next section we will define the notion of algebraic stacks. The definition of algebraicity uses fibre products of stacks. The universal property of fibre products can be formulated in an arbitrary 2-category.

Definition 3.26. *Let C be a 2-category, an object $X\in\mathsf{C}$ is called* final*, if for every other object $Y\in\mathsf{C}$, there exists a morphism $\phi\colon Y\to X$, and for every two morphisms $\phi,\psi\colon Y\to X$, there exists a unique invertible 2-morphism between ϕ and ψ.*

Final objects in 2-categories provide a convenient way to introduce fibre products.

Definition 3.27. *For a co-span*

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ X \arrow[r, "k"'] & Z, \end{tikzcd}$$

we consider the 2*-category, whose objects are* 2*-commutative diagrams*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X \arrow[r, "k"'] & Z, \end{tikzcd}$$

i.e., we have an invertible 2*-morphism* $\alpha\colon k \circ f \xrightarrow{\simeq} h \circ g$*. A final object in this category of diagrams will be referred to as fibre product, and the object in the top left corner* W *will be denoted by* $X \times_Z Y$*.*

In the 2-category of groupoids Gpd, we have an explicit model for fibre products, which is closely related to the one of Definition 3.18.

Remark 3.28. *Let the co-span of Definition 3.27 be a diagram in the* 2*-category of groupoids. We then have an explicit model for* $X \times_Z Y$*, given by the groupoid, whose objects are triples* (x, y, α)*, with* $x \in X$*,* $y \in Y$*, and* $k(x) \xrightarrow{\alpha} h(y)$ *being an isomorphism between* $k(x)$ *and* $h(y)$ *in the groupoid* Z*.*

In order to illustrate the concept of limits in 2-categories, we give a sample computation of a 2-limit.

Example 3.29. *Let* $\bullet$ *be the set with one element, endowed with the trivial action of an abstract group* G *acting on it. The fibre product* $\bullet \times_{[\bullet/G]} \bullet$ *is equivalent to the set underlying the group* G*.*

Proof. This follows directly from Remark 3.28. As we observe there, a model for the fibre product is given by the set of triples (x, y, α) with $x, y \in \bullet$, and α an automorphism of the object $\bullet \in [\bullet/G]$ (i.e., an element of G). $\square$

4 Algebraicity

In this section we add geometric meaning to the abstract functorial framework we have previously built. Just like manifolds are geometric objects obtained by glueing the local model of Euclidean space along smooth diffeomorphisms; schemes, algebraic spaces, and eventually also algebraic stacks, are modelled in a similar style on affine schemes. The latter will be introduced by means of category theory. We define the category of affine schemes to be the dual of the category of commutative rings. This approach does not "waste time" on constructing a geometric object whose ring of functions agrees with a given commutative ring. Instead, we take the category of rings as it is, and assert dogmatically its geometric content.

4.1 Schemes

4.1.1 Affine schemes

Let us recall one more time the definition of affine schemes.

Definition 4.1. *The category* Aff *of affine schemes is defined to be* $\mathsf{Rng}^{\mathrm{op}}$*, i.e., the opposite category of the category of (commutative and unital) rings.*

We have already constructed a natural Grothendieck topology $\mathcal{T}$ on Aff, using faithfully flat ring homomorphisms. There is a weaker topology, called the *Zariski topology.*

Definition 4.2. *(a) Let* R *be a ring, and* $f \in R$ *an arbitrary element. We denote by* R_f *the ring* $R[t]/(tf-1)$*, obtained by adjoining an inverse for* f*. We say that* $\operatorname{Spec} R_f \to \operatorname{Spec} R$ *is a standard open subscheme of* $\operatorname{Spec} R$*.*

(b) We denote by **Zar** *the Grothendieck topology on* Aff*, given by all collections* $\{\operatorname{Spec} S_i \to \operatorname{Spec} R\}_{i\in I}$*, which are isomorphic to*

$$\{\operatorname{Spec} R_{f_i} \to \operatorname{Spec} R\}_{i\in I},$$

where I *is a finite set, and* $f_i \in R$ *is a collection of elements, satisfying* $\sum_{i\in I} f_i = 1$*.*

The next lemma shows that faithfully flat descent theory is applicable to Zariski open coverings.

Lemma 4.3. *Let* $\{\operatorname{Spec} S_i \to \operatorname{Spec} R\} \in \mathbf{Zar}$*, then* $R \to \prod_{i\in I} S_i$ *is a faithfully flat map of rings.*

Proof. If I is a finite set, $R \to S_i$ a collection of ring homomorphisms, and M an R-module, then

$$M \otimes_R (\prod_{i\in I} S_i) \cong \bigoplus_{i\in I} (M \otimes_R S_i).$$

In particular, we see that $R \to \prod_{i\in I} S_i$ is flat, if and only if each homomorphism $R \to S_i$ is. Recall that a localization $R \to R_f$ is a flat map of rings. Indeed, for an R-module M, the tensor product $M \otimes_R R_f$ can be described as a localization of the module: $M_f = M[t]/M(tf-1)$. Note that we have a canonical map $M \to M_f$, the kernel of which consists precisely of elements $m \in M$, for which there exists a power f^n of f with $f^n \cdot m = 0$. This implies that for an injective map of R-modules $M \hookrightarrow N$, the induced map $M_f \to N_f$ is injective as well.

To conclude the proof we have to show that $\bigoplus_{i\in I} M_{f_i} \cong 0$, if and only if $M = 0$. The canonical map

$$M \to \bigoplus_{i\in I} M_{f_i}$$

would then be zero. In particular, for every $m \in M$, and every $i \in I$, there would exist an $n_i \in \mathbb{N}$, such that $f_i^{n_i} \cdot m = 0$. Let $n = \max_{i\in I}(n_i)$ be the maximum of these positive integers. The relation

$$(\sum_{i\in I} f_i)^{2n} = 1$$

implies that $m = (\sum_{i\in I} f_i)^{2n} \cdot m = 0$. This implies that M is the zero module. □

Corollary 4.4. *The Yoneda embedding* $\mathsf{Aff} \hookrightarrow \mathrm{Pr}(\mathsf{Aff})$ *factorises through the full subcategory* $\mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$ *of Zariski sheaves:*

$$\begin{tikzcd} \mathsf{Aff} \arrow[r] \arrow[dr] & \mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff}) \arrow[d] \\ & \mathrm{Pr}(\mathsf{Aff}). \end{tikzcd}$$

We identify the category Aff with its essential image inside $\mathrm{Pr}(\mathsf{Aff})$, respectively $\mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$. The sheaf corresponding to an object $\operatorname{Spec} R \in \mathsf{Aff}$ is the actual geometric object, associated to the ring R.

Definition 4.5. *A morphism of affine schemes* $\operatorname{Spec} S \cong Y \to X \cong \operatorname{Spec} R$ *is called a* closed immersion*, if it corresponds to a surjective ring homomorphism* $R \twoheadrightarrow S$*.*

This is tantamount to the classical viewpoint that closed subschemes of $\operatorname{Spec} R$ (or a variety) are characterised by ideals in the ring of regular functions R.

Sometimes, the presheaf $\operatorname{Spec} R$ is easier to describe than the ring R (see example (c) below, and Example 4.7).

Example 4.6. *(a) We denote by* $\mathbb{A}^1_{\mathbb{Z}} \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ *the functor, sending a ring* R *to its set of elements. We have a canonical equivalence* $\mathbb{A}^1_{\mathbb{Z}} \cong \operatorname{Spec} \mathbb{Z}[t]$*.*

(b) We denote by $\mathbb{G}_m \in \mathrm{Pr}(\mathsf{Aff})$ *the presheaf sending a ring* R *to the set of units* $R^\times$*. We have a canonical equivalence* $\mathbb{G}_m \cong \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$*.*

(c) Let $\mathrm{GL}_n \in \mathrm{Pr}(\mathsf{Aff})$ *be the functor, which sends a ring* R *to the set of invertible* $(n \times n)$*-matrices over* R*. We have a canonical equivalence* $\mathrm{GL}_n \cong \operatorname{Spec} \mathbb{Z}[t_{11}, \ldots, t_{nn}][\det(t_{ij})^{-1}]$*.*

Proof. We know that the set of ring homomorphisms $\mathbb{Z}[t] \to R$ is in bijection with the set of elements of R (universal property of polynomial rings). In particular $\mathbb{A}^1_{\mathbb{Z}} \simeq \mathsf{Hom}_{\mathsf{Rng}}(\mathbb{Z}[t], -) \simeq \mathsf{Hom}_{\mathsf{Aff}}(-, \operatorname{Spec} \mathbb{Z}[t])$.

Similarly, $\mathbb{Z}[t, t^{-1}] \to R$ is uniquely characterised by the image of t, which has to be a unit. We have $\mathbb{G}_m(R) \simeq \mathsf{Hom}_{\mathsf{Rng}}(\mathbb{Z}[t, t^{-1}], -) \simeq \mathsf{Hom}_{\mathsf{Aff}}(-, \operatorname{Spec} \mathbb{Z}[t, t^{-1}])$.

The last example follows by similar reasoning: the data of an invertible $(n \times n)$-matrix (a_{ij}) over R is given by choosing n^2 coefficients, such that the determinant is an invertible element of R. This functor is corepresented by the ring $\mathbb{Z}[t_{11}, \ldots, t_{nn}][\det(t_{ij})^{-1}]$. $\square$

Example 4.7. *Let* R *be a ring. A* formal group law *over* R *is given by a formal power series* $f(x, y) \in R[[x, y]]$*, satisfying the axioms*

(a) (Unit) $f(x, 0) = f(0, y) = 0$*,*

(b) (Commutativity) $f(x, y) = f(y, x)$*,*

(c) (Associativity) $f(x, f(y, z)) = f(f(x, y), z)$ *in* $R[[x, y, z]]$*.*

The functor FGL*, which sends a ring* R *to the set of formal group laws, is represented by an affine scheme* $\operatorname{Spec} L \in \mathsf{Aff}$*.*

Proof. A power series $f(x,y)$ can be presented in terms of its coefficients $f(x,y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$. The unit axiom translates into the equations $a_{i0} = a_{0j} = 0$ for all i,j. The commutativity condition can be stated as $a_{ij} = a_{ji}$, while the associativity condition can also be presented as an equation the coefficients have to satisfy. Let L be the ring $\mathbb{Z}[(a_{ij})_{i,j=0,\dots,\infty}]/I$, where I is the ideal generated by these equations. The universal property of polynomial algebras guarantees that $\mathsf{Hom}(L,R)$ is canonically equivalent to the set of formal group laws over R. In particular, $\mathsf{FGL} \cong \operatorname{Spec} L$. $\square$

The following theorem is only mentioned as a side remark. It should be understood as a classification result for formal group laws.

Theorem 4.8 (Lazard)**.** *The ring L is isomorphic to a polynomial ring in countably many generators.*[15]

4.1.2 Open subfunctors of affine schemes

The definition of open subfunctors relies on the notion of injectivity and surjectivity for maps of sheaves. While injectivity is straightforward to define, surjectivity requires a little more care.

Definition 4.9. *For a site $(\mathsf{C}, \mathcal{T})$ we say that a map of sheaves $F \to G$ is* injective, *if for every $U \in \mathsf{C}$, the map $F(U) \to G(U)$ is an injective map of sets. It is called* surjective, *if for every $U \in \mathsf{C}$, and every $s \in G(U)$, there exists a covering $\{U_i \to U\}_{i \in I}$, such that each $s|_{U_i} \in G(U_i)$ lies in the image of $F(U_i) \to G(U_i)$.*

We can now define an open subscheme of an affine scheme $\operatorname{Spec} R$. They are given by the sheaf-theoretic union of standard open subfunctors $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$.

Definition 4.10. *Let $\operatorname{Spec} R \in \mathsf{Aff}$, and $F \in \mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$ be a Zariski sheaf. Given an injective map $F \hookrightarrow \operatorname{Spec} R$ in $\mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$, we say that F is an open subfunctor of $\operatorname{Spec} R$, if there exists a (not necessarily finite collection of elements $(f_i)_{i \in I} \in R^I$ of elements of R), such that we have a surjective map of sheaves $\coprod_{i \in I} \operatorname{Spec} R_i \twoheadrightarrow F$, fitting into a commutative triangle*

$$\begin{tikzcd} \coprod_{i \in I} \operatorname{Spec} R_i \arrow[r, two heads] \arrow[dr] & F \arrow[d, hook] \\ & \operatorname{Spec} R. \end{tikzcd}$$

The case where I is a finite set deserves particular attention.

Definition 4.11. *A sheaf $F \in \mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$ is called a* quasi-affine scheme *if there exists a ring R, such that F is an open subfunctor of $\operatorname{Spec} R$, for which the index set I in Definition 4.10 can be chosen to be finite.*

Open subfunctors of schemes, particularly also quasi-affine schemes, are in general not affine. We will prove this for the following example after having developed a criterion for affineness.

[15] See `http://www.math.harvard.edu/~lurie/252xnotes/Lecture2.pdf` for a proof.

Example 4.12. *Consider the sheaf $X = \mathbb{A}^2_{\mathbb{Z}} \setminus 0 \in \mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$, given by the pushout*

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}} \arrow[d] \\ \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m \arrow[r] & X \end{tikzcd} \tag{10}$$

in the category $\mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$. The Zariski sheaf X is a quasi-affine scheme.

Proof. By construction we have a surjection of sheaves $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m \coprod \mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}} \twoheadrightarrow X$. By construction, the map $X \subset \mathbb{A}^2_{\mathbb{Z}}$ is injective. Hence, X is a quasi-affine scheme. □

4.1.3 Schemes

The following definition should be seen as a template to relativise absolute geometric notions. The strategy is to use fibre products with affine schemes.

Definition 4.13. *Let $f : F \to G$ be a morphism in the category $\mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$. We say that f is an* open immersion*, if for every affine scheme* $\operatorname{Spec} R$*, the base change $F \times_G \operatorname{Spec} R \to \operatorname{Spec} R$ is isomorphic to the inclusion of an open subscheme of* $\operatorname{Spec} R$*.*

We can now give a functorial definition of schemes.

Definition 4.14. *A sheaf $F \in \mathsf{Sh}_{\mathbf{Zar}}(\mathsf{Aff})$ is called a* scheme*, if there exists a (not necessarily finite) collection of affine schemes $\{\operatorname{Spec} R_i\}_{i \in I}$, together with morphisms $\operatorname{Spec} R_i \to F$, which are* open immersions*, such that the map $\coprod_{i \in I} \operatorname{Spec} R_i \to F$ is a surjection of sheaves. The category of schemes will be denote by Sch.*

Schemes are obtained by patching affine schemes along open subfunctors. We have already seen a non-trivial example of such an object.

Example 4.15. *Recall the definition of the functor $\mathbb{P}^1_{\mathbb{Z}}$ from Definition 1.11. According to Proposition 1.12 we have a surjective map $\mathbb{A}^1_{\mathbb{Z}} \coprod \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ of Zariski sheaves. Each one of the maps $\mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ is an open immersion. In particular, $\mathbb{P}^1_{\mathbb{Z}}$ is a scheme.*

Proof. Exercise.[16] □

The presheaf or functor corresponding to a scheme is the actual geometric object. It could be thought of as a generalised system of equations which can be solved over any ring.

Definition 4.16. *(a) A presheaf $F \in \mathrm{Pr}(\mathsf{Aff})$ is called affine, if there exists a ring R, and an equivalence $X \cong \operatorname{Spec} R$.*

(b) A map of presheaves $F \to G$ is called affine, if for every affine scheme U, and every morphism $U \to G$, the fibre product $F \times_G U$ is an affine scheme.

(c) We say that $F \to G$ is a closed immersion, if for every affine scheme U, and every morphism $U \to G$, the base change $U \times_G F \to F$ is a closed immersion.

[16] Claudio has volunteered.

The next lemma is a useful tautology.

Lemma 4.17. *Let U be an affine scheme, and $F \to U$ an affine morphism of presheaves, then F is affine.*

Proof. Consider the identity map $U \xrightarrow{\mathrm{id}_U} U$. By assumption, the base change $F \times_U U \cong F$ is affine. ☐

The following observation yields a useful criterion to prove that a scheme is not affine.

Lemma 4.18. *(a) Consider the affine scheme $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t]$. The morphisms $+\colon \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$, and $\cdot\colon \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$, corresponding to the ring homomorphisms $\mathbb{Z}[t] \to \mathbb{Z}[t_1, t_2]$ (mapping $t \mapsto t_1 + t_2$), and $\mathbb{Z}[t] \to \mathbb{Z}[t_1, t_2]$ (mapping $t \mapsto t_1 t_2$), endows $\mathbb{A}^1_{\mathbb{Z}}$ with the structure of a ring object in $\mathrm{Pr}(\mathsf{Aff})$.*

(b) For every $F \in \mathrm{Pr}(\mathsf{Aff})$, we define the ring of regular functions $\Gamma(F)$ on F to be the ring of morphisms $\mathsf{Hom}_{\mathrm{Pr}(\mathsf{Aff})}(F, \mathbb{A}^1_{\mathbb{Z}})$.

(c) For a ring R, we have a canonical equivalence $R \cong \Gamma(\operatorname{Spec} R)$. It is given by the equivalence $R \xrightarrow{\simeq} \Gamma(\operatorname{Spec} R)$, corresponding to the map of sets

$$R \to \mathsf{Hom}_{\mathsf{Rng}}(\mathbb{Z}[t], R) = \mathsf{Hom}_{\mathsf{Aff}}(\operatorname{Spec} R, \mathbb{A}^1_{\mathbb{Z}}),$$

which sends $r \in R$ to the ring homomorphisms mapping t to r.

Proof. [17]

(a) We need to check the commutativity of a lot of diagrams. The important remarks are that $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t]$ is affine and that morphisms between (powers of) $\mathbb{A}^1_{\mathbb{Z}}$ are related to ring homomorphisms between (powers of) $\mathbb{Z}[t]$, thanks to Yoneda's Lemma.

Before we start, let's define the morphism $T : \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \operatorname{Spec}\mathbb{Z}$ as the natural morphism to the terminal object in the category of affine schemes and the diagonal morphism $\Delta : \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$, given by $\mathbb{Z}[t_1, t_2] \longrightarrow \mathbb{Z}[t]$ which sends t_1 and t_2 to t.

i) *Associativity of* $+$. The following diagram

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[r, "id\times +"] \arrow[d, "+\times id"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "+"] \\
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[r, "+"] & \mathbb{A}^1_{\mathbb{Z}}
\end{tikzcd}$$

is commutative if and only if the diagram of rings

$$\begin{tikzcd}
\mathbb{Z}[t_1, t_2, t_3] & \mathbb{Z}[t_1, t_2] \arrow[l] \\
\mathbb{Z}[t_2, t_3] \arrow[u] & \mathbb{Z}[t] \arrow[l] \arrow[u]
\end{tikzcd}$$

[17]This proof was provided by Claudio Onorati.

is. On one hand we get $t \mapsto t_1 + t_2 \mapsto t_1 + t_2 + t_3$; on the other hand we get $t \mapsto t_2 + t_3 \mapsto t_1 + t_2 + t_3$. From now on we systematically avoid to draw the diagram of rings, but we will always check the commutativity on it.

ii) *Existence of neutral element for* $+$. Let's define $0 : \text{Spec } \mathbb{Z} \longrightarrow \mathbb{Z}[t]$ given by sending t to 0. Then,

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \times \text{Spec } \mathbb{Z} \arrow[r, "id\times 0"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "+"] & \text{Spec } \mathbb{Z} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "0\times id"'] \\
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[u, "id\times T"] & \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}. \arrow[u, "T\times id"']
\end{tikzcd}$$

Let's check only the first square, the other one is analogous. Starting from the bottom right object we get $t \mapsto t_1 + t_2 \mapsto t_1 \mapsto t$, that is what we were looking for.

iii) *Existence of the inverse element for* $+$. Let's define $r : \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{A}^1_{\mathbb{Z}}$ sending t to $-t$. Then,

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[r, "+"] & \mathbb{A}^1_{\mathbb{Z}} & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "+"'] \\
 & \text{Spec } \mathbb{Z} \arrow[u, "0"] & \\
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[uu, "r\times id"] & \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[u, "T"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}. \arrow[uu, "id\times r"']
\end{tikzcd}$$

Again, let's look only at the first square, starting from the right-up object: on one hand we get $t \mapsto 0 \mapsto 0$; on the other hand, $t \mapsto t_1 + t_2 \mapsto -t_1 + t_2 \mapsto -t + t = 0$.

iv) *Commutativity of* $+$. Let's define $I : \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ by exchanging the coordinates t_1 and t_2. Then the commutativity of

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[rr, "I"] \arrow[dr, "+"] & & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[dl, "+"'] \\
 & \mathbb{A}^1_{\mathbb{Z}} &
\end{tikzcd}$$

is trivial.

v) *Associativity of* $\cdot$. The commutativity of the following diagram is equivalent to that of the point *i)*,

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[r, "id\times\cdot"] \arrow[d, "\cdot\times id"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "\cdot"] \\
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[r, "\cdot"] & \mathbb{A}^1_{\mathbb{Z}}.
\end{tikzcd}$$

vi) *Existence of unit element for* $\cdot$. Let's define $1 : \text{Spec } \mathbb{Z} \longrightarrow \mathbb{A}^1_{\mathbb{Z}}$ by sending t to 1. Then the commutativity of the following diagram is equivalent to that of the point *ii)*:

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \times \text{Spec } \mathbb{Z} \arrow[r, "id\times 1"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "\cdot"] & \text{Spec } \mathbb{Z} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "1\times id"'] \\
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[u, "id\times T"] & \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}. \arrow[u, "T\times id"']
\end{tikzcd}$$

vii) *Commutativity of* $\cdot$. Again, it is equivalent to point *iv)*:

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[rr, "I"] \arrow[dr, "\cdot"] & & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[dl, "\cdot"'] \\
& \mathbb{A}^1_{\mathbb{Z}} & .
\end{tikzcd}$$

viii) *Left and right distributions.* Finally, let $L : \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ be the left distribution, given by sending t_1 to t_1t_2 and t_2 to t_1t_3, and let $R : \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ be the right distribution, given by sending t_1 to t_1t_3 and t_2 to t_2t_3. Then,

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}} & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "\cdot"'] & \\
\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[u, "+"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[l, "L"'] \arrow[r, "R"] \arrow[u, "id\times +"] \arrow[d, "+\times id"] & \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "+"] \\
& \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}} \arrow[r, "\cdot"] & \mathbb{A}^1_{\mathbb{Z}}.
\end{tikzcd}$$

Let's start by checking the left-up square: on one hand we get $t \mapsto t_1t_2 \mapsto t_1(t_2+t_3)$; on the other hand, $t \mapsto t_1+t_2 \mapsto t_1t_2+t_1t_3$. Of course, they coincide. In the same way we get, for the last square, $t \mapsto t_1t_2 \mapsto (t_1+t_2)t_3$ that is tantamount to $t \mapsto t_1+t_2 \mapsto t_1t_3+t_2t_3$.

(*b*) Let's define the following operations on $\Gamma(F)$:

$$\begin{array}{rccc} + : & \Gamma(F) \times \Gamma(F) & \longrightarrow & \Gamma(F) \\ & (f,g) & \longmapsto & f+g \end{array}$$

and

$$\begin{array}{rccc} \cdot : & \Gamma(F) \times \Gamma(F) & \longrightarrow & \Gamma(F) \\ & (f,g) & \longmapsto & f\cdot g, \end{array}$$

defined locally by

$$\begin{array}{rccc} (f+g)(\text{Spec } S) : & F(\text{Spec } S) & \longrightarrow & \mathbb{A}^1_{\mathbb{Z}}(\text{Spec } S) \\ & \sigma & \longmapsto & f(S)(\sigma)+g(S)(\sigma) \end{array}$$

and

$$\begin{array}{rccc} (f\cdot g)(\text{Spec } S) : & F(\text{Spec } S) & \longrightarrow & \mathbb{A}^1_{\mathbb{Z}}(\text{Spec } S) \\ & \sigma & \longmapsto & f(S)(\sigma)\cdot g(S)(\sigma). \end{array}$$

Now we should check the commutativity of the same diagrams of the previous point. However, it is clear that the check is only moved from $\Gamma(F)$ to $\mathbb{A}^1_{\mathbb{Z}}$ and so the thesis follows from the point (a).

(c) Let Spec R be an affine scheme. Then by definition and Example 4.6 (a) we get

$$\Gamma(\text{Spec } R) = Hom_{\mathrm{Pr(Aff)}}(\text{Spec } R, \mathbb{A}^1_{\mathbb{Z}}) =$$
$$= Hom_{\mathrm{Pr(Aff)}}(\text{Spec } R, \text{Spec } \mathbb{Z}[t]) = Hom_{\mathrm{Rng}}(\mathbb{Z}[t], R),$$

where in the last passage we have used the Yoneda's Lemma. Finally, as observed several times in these notes, there is a natural isomorphism between $Hom_{\mathrm{Rng}}(\mathbb{Z}[t], R) \cong R$ given by sending t to r, for every $r \in R$.

□

As a corollary we obtain our first examples of non-affine schemes.

Corollary 4.19. *(a) The quasi-affine scheme $X = \mathbb{A}^2_{\mathbb{Z}} \setminus 0$ of Example 4.12 is non-affine.*

(b) The map $X \to \mathbb{P}^1_{\mathbb{Z}}$ is affine, in particular, $\mathbb{P}^1_{\mathbb{Z}}$ cannot be affine.

Proof. [18]

(a) Recall that X rises as the pushout of the following diagram

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}} \arrow[d] \\ \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m \arrow[r] & X \end{tikzcd}$$

and so, by the Remark **??**, this defines a commutative diagram

$$\begin{tikzcd} \Gamma(X) \arrow[r] \arrow[d] & \Gamma\left(\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m\right) \arrow[d, "\alpha"] \\ \Gamma\left(\mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}}\right) \arrow[r, "\beta"] & \Gamma\left(\mathbb{G}_m \times \mathbb{G}_m\right). \end{tikzcd}$$

that turns out to be cartesian in the category of ring Rng (here α and β are just the inclusion maps). Now, since $\mathbb{G}_m \times \mathbb{G}_m$, $\mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m$ are affine schemes, the previous lemma tells us that

$$\Gamma\left(\mathbb{G}_m \times \mathbb{G}_m\right) = \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}],$$
$$\Gamma\left(\mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}}\right) = \mathbb{Z}[t_1, t_1^{-1}, t_2]$$

and

$$\Gamma\left(\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m\right) = \mathbb{Z}[t_1, t_2, t_2^{-1}].$$

[18] This proof was provided by Claudio Onorati.

Finally, if we want that two polynomials $f(t_1, t_1^{-1}, t_2) \in \mathbb{Z}[t_1, t_1^{-1}, t_2]$ and $g(t_1, t_2, t_2^{-1}) \in \mathbb{Z}[t_1, t_2, t_2^{-1}]$ coincide in $\mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}]$, then we must request at least that they only depend on t_1 and t_2. For this reason we get

$$\Gamma(X) = \mathbb{Z}[t_1, t_2].$$

If X was an affine scheme, then $\operatorname{Spec}\Gamma(X) = X$. But $\operatorname{Spec}\Gamma(X) = \mathbb{A}^2_{\mathbb{Z}} \neq X$, so it cannot be affine.

□

We have the following definition.

Definition 4.20. *A map $F \to G$ is called* schematic*, if for every affine scheme U and every map $U \to G$ the base change $F \times_G U$ is a scheme.*

4.2 Algebraic spaces and stacks

4.2.1 Finite presentation

Definition 4.21. *A ring homomorphism $R \to S$ is called of* finite presentation*, if there exists a factorisation*

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R[t_1, \ldots, t_n] \arrow[d, two heads] \\ & S, \end{tikzcd}$$

such that the kernel of the surjection $R[t_1, \ldots, t_n] \twoheadrightarrow S$ is a finitely generated ideal.

There is a categorical reformulation of this definition, which will be useful in generalising this notion to general morphisms of presheaves.

Lemma 4.22. *We denote by I a directed set, i.e., it is endowed with a partial ordering $\leq$, such that for every pair of indices $(i, j) \in I^2$ there exists a $k \in I$, such that $i \leq k$ and $j \leq k$. We consider diagrams of rings $(T_i)_{i \in I}$, parametrised by I. For a ring homomorphism $R \to S$ to be of* finite presentation *is equivalent to the natural map*

$$\operatorname*{colim}_{i \in I} \mathsf{Hom}_R(S, T_i) \to \mathsf{Hom}_R(S, \operatorname*{colim}_{i \in I} T_i)$$

to be an equivalence*, for every directed system of R-algebras.*

Proof. [19] Suppose we have a ring homomorphism $R \to S$ such that, for every diagram of rings $(T_i)_{i \in I}$ parametrised by a directed set I we have that

$$\varinjlim_{i \in I} \operatorname{Hom}_R(S, T_i) \to \operatorname{Hom}_R(S, \varinjlim_{i \in I} T_i)$$

is an equivalence. Then S is an R-algebra. Let $\mathcal{I}$ be the set of finite subsets $I \subset S$, a directed set ordered by inclusion, and let T_I be the R-subalgebra of S generated by I. Then it is

[19] This proof was provided by Craig Smith.

clear that $\varinjlim_{I\in\mathcal{I}} T_I = S$. So the identity morphism in $\mathrm{Hom}_R(S, \varinjlim_{I\in\mathcal{I}} T_I)$ corresponds to an element of $\varinjlim_{I\in\mathcal{I}} \mathrm{Hom}_R(S, T_I)$ under the equivalence, and hence corresponds to an element of some $\mathrm{Hom}_R(S, T_I)$ for some finite subset $I \subset S$. Thus there is an isomorphism between S and some finitely generated R-subalgebra of S. So S is a finitely generated R-algebra. So there exists $n \in \mathbb{N}$ and an epimorphism $R[t_i, t_2, ..., t_n] \twoheadrightarrow S$ mapping $\{t_1, ..., t_n\}$ bijectively to a finite generating set of S. So the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R[t_1, t_2, ..., t_n] \arrow[d, two heads, "\pi"] \\ & S \end{tikzcd}$$

commutes. Now let $\mathcal{J}$ be the set of finite subsets $J \subset \ker\pi$, a directed set ordered by inclusion, and now let T_J be the quotients $R[t_1, ..., t_n]/\langle J\rangle$. Then whenever $J \subset J'$ we have a projection $T_J \twoheadrightarrow T_{J'}$, giving a directed diagram of rings. It is clear that $\varinjlim_{J\in\mathcal{J}} T_J = R[t_1, ..., t_n]/\ker\pi \cong S$. Indeed, this limit is a coproduct of each T_J under the equivalence relation generated by $x\ y$ if $x - y \in \langle J\rangle \cap \langle J'\rangle$ where $x \in T_J, y \in T_{J'}$. It then follows that $x\tilde{y} \Leftrightarrow x - y \in \ker\pi$. Thus the isomorphism $R[t_1, ..., t_n]/\ker\pi \cong S$ is contained in $\mathrm{Hom}_R(S, \varinjlim_{J\in\mathcal{J}} T_J)$ and so, as before, must come from some $\mathrm{Hom}_R(S, R[t_1, ...t_n]/\langle J\rangle)$. Thus we may conclude that $\ker\pi$ is a finitely generated ideal. Thus $R \to S$ is finitely presented.

Now suppose $R \to S$ is finitely presented. That is, there is $n \in \mathbb{N}$ and an epimorphism $\pi : R[t_1, t_2, ..., t_n] \twoheadrightarrow S$ such that the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R[t_1, t_2, ..., t_n] \arrow[d, two heads, "\pi"] \\ & S \end{tikzcd}$$

commutes and $\ker\pi = \langle r_1, ..., r_m\rangle$ is a finitely generated ideal. Let $s_i = \phi(t_i)$. Suppose we have a directed set I and a diagram of rings $(T_i)_{i\in I}$ indexed by I. Then there is a natural map

$$\varinjlim_{i\in I} \mathrm{Hom}_R(S, T_i) \to \mathrm{Hom}_R(S, \varinjlim_{i\in I} T_i).$$

Suppose $f \in \mathrm{Hom}_R(S, \varinjlim_{i\in I} T_i)$. Then, for each $i = 1, 2, ..., n$, there is $j_i \in I$ with $f(s_i) \in T_{j_i}$. Then, as I is directed, there is $j \in I$ with $j_i \le j$ for all i. Then, as f is a map into $\varinjlim_{i\in I} T_i$ we have that $f(s_i) \in T_j$ for all i. So we may define a map $f' : R[t_1, t_2, ...t_n] \to T_j$, $t_i \to f(s_i)$. Furthermore, for $i = 1, ..., m$ there is $k_i \in I$ with $k_i \ge j$ such that r_i is mapped to 0 under the map $f' : R[t_1, t_2, ...t_n] \to T_j \to T_{k_i}$ since f is a well defined map into $\varinjlim_{i\in I} T_i$. That is, the relation given by r_i is satisfied in some T_{k_i}, and we can take these $k_i \ge j$. Since I is directed, there is $k \in I$ with $k_i \le k$ for all i. Then it is clear that the map $f'' : R[t_1, t_2, ...t_n] \to T_j \to T_k$ descends to a well defined map from S. So f defines a map $\tilde{f} : S \to T_k$ and hence comes from an element of $\varinjlim_{i\in I} \mathrm{Hom}_R(S, T_i)$. Thus the natural map is surjective. Now suppose $f \in \varinjlim_{i\in I} \mathrm{Hom}_R(S, T_i)$ map to the zero morphism of $\mathrm{Hom}_R(S, \varinjlim_{i\in I} T_i)$ under this correspondence. Then f is an equivalence class f_i of morphisms in the coproduct of all $\mathrm{Hom}_R(S, T_i)$ such that $f_i(s)$ are equivalent to 0 for all $s \in S$ and $i \in I$. So each f_i is equivalent to the zero morphisms in this coproduct. Thus f itself is zero in $\varinjlim_{i \in I} \mathrm{Hom}_R(S, T_i)$. Thus the correspondence is also injective. So we have a bijective correspondence. □

4.2.2 Smooth and étale morphisms

For the purpose of illustration we fix a ring $R = \mathbb{Z}[x_1, \ldots, x_m]/(f_1, \ldots, f_n)$, and define $X = \operatorname{Spec} R$. For a field k, the affine scheme $\operatorname{Spec} k$ is often described as an abstract k-point. The reason is that the set of morphisms $\mathsf{Hom}_{\mathsf{Sch}}(\operatorname{Spec} k, X) \cong \mathsf{Hom}_{\mathsf{Rng}}(R, k)$ is canonically equivalent to the solution set of the set system of equations $(f_1, \ldots, f_n)$ over the field k. Let now $k[\epsilon] = k[t]/(t^2)$, be the so-called *ring of dual numbers*. The affine scheme $\operatorname{Spec} k[\epsilon]$ could analogously be referred to as an abstract tangent vector defined over k. Indeed, the relation $\epsilon^2 = 0$ allows us to treat ϵ as the algebraic incarnation of infinitesimals, as used by Newton and Leibniz in their treatment of differentiation.

Lemma 4.23. *Let $J(x_1, \ldots, x_m)$ be the Jacobi matrix of the system of equations $(f_1, \ldots, f_n)$. For a fixed solution $x_i = \lambda_i$ over k, given by* $\operatorname{Spec} k \to X$ *the set of commuting triangles*

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & X \\ \operatorname{Spec} k[\epsilon] \arrow[ur] & \end{tikzcd}$$

corresponds to the kernel of $J(\lambda_1, \ldots, \lambda_m)$, i.e. the tangent space at the chosen solution.

Proof. For $\lambda_i, \mu_i \in k$ we have

$$f_i(\lambda_1 + \mu_1\epsilon, \ldots, \lambda_m + \mu_m\epsilon) = f(\lambda_1, \ldots, \lambda_m) + \epsilon \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \mu_j.$$

So we see that this expression is 0, if and only if $f(\lambda_1, \ldots, \lambda_m) = 0$, and $(\mu_1, \ldots, \mu_m) \in \ker J(\lambda_1, \ldots, \lambda_m)$. □

Inspired by this discussion, we have the following definition.

Definition 4.24. *Let $f \colon F \to G$ be a morphism of presheaves.*

(a) We say that f is formally étale, *if for every ring surjective homomorphism $R \twoheadrightarrow S$, with the kernel I satisfying $I^2 = 0$ (known as a* square-zero extension*), a unique dashed arrow exists, rendering the diagram*

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r] \arrow[d] & F \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & G \end{tikzcd} \tag{11}$$

commutative.

(b) The morphism f is called formally smooth, *if for every square-zero extension, a dashed arrow rendering the diagram* (11) *commutative, exists.*

(c) *We call f* formally unramified*, if for every square-zero extension, there exists at most one dashed arrow as in* (11)*, rendering the diagram commutative.*

By definition, a formally étale morphism is formally smooth and formally unramified.
One can show the following properties (see [Mil80]).

Proposition 4.25. (a) *A morphism of finite presentation of affine scheme* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is smooth (respectively étale), if and only if it corresponds to a ring homomorphism $R \to S$, which is isomorphic to*

$$R \to R[x_1, \dots, x_m]/(f_1, \dots, f_n),$$

with the Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ *being surjective (respectively bijective).*

(b) *Smooth morphisms are flat.*

4.2.3 Various Grothendieck topologies

In this paragraph we define the Grothendieck topologies **fppf**, **et**, and **smth**; corresponding to flat maps coverings which are additionally assumed to be of finite presentation, respectively étale, respectively smooth. In this section only the fppf topology will be needed, but we will refer to the other topologies in later sections.

Definition 4.26. (a) *The Grothendieck topology consisting of coverings isomorphic to* $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$*, where I is a finite set, each $R \to R_i$ étale, and $R \to \prod_{i\in I} R_i$ is fully faithful, will be denoted by* **et***.*

(b) *The Grothendieck topology consisting of coverings isomorphic to* $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$*, where I is a finite set, each $R \to R_i$ smooth, and $R \to \prod_{i\in I} R_i$ is fully faithful, will be denoted by* **smth***.*

(c) *The Grothendieck topology consisting of coverings isomorphic to* $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$*, where I is a finite set, each $R \to R_i$ flat and of finite presentation, and $R \to \prod_{i\in I} R_i$ is fully faithful, will be denoted by* **fppf***.*

The following proposition implies that the Grothendieck topologies **et** and **smth** have equivalent categories of sheaves. We record it for later use.

Proposition 4.27. *Let $V \to U$ be a smooth morphism of affine schemes. There exists a covering* $\{U_i \to U\}_{i\in I} \in \mathbf{et}$*, such that $V \times_U U_i \to U_i$ has a section $s_i \colon U_i \to V \times_U U_i$.*

Proof. The proof relies on Proposition 4.25(a), and is an application of the strategy which is used to deduce the Regular Value Theorem in differential geometry from the Inverse Function Theorem. We leave it as an exercise to the keen reader.[20] □

[20] Alexander has volunteered.

4.2.4 Algebraic spaces

The definition of algebraic spaces is formally very similar to the functorial definition of schemes. The role of open immersions will taken by étale maps. Heuristically speaking, an algebraic space is obtained by glueing affine schemes along étale equivalence relations. We will discuss this interpretation in Paragraph 4.2.6.

Definition 4.28. *An fppf sheaf F is called an* algebraic space*, if there exists a collection of schematic étale morphisms $\{U_i \to F\}_{i\in I}$, where each U_i is affine, such that $p\colon \coprod_{i\in I} U_i \to F$ is a surjection. We call the map $p\colon \coprod_{i\in I} U_i \to F$ an* atlas *for F.*

One can construct examples of algebraic spaces which are not schemes. We defer this to a later section, respectively the exercises.

Definition 4.29. *A morphism of algebraic spaces $F \to G$ is called* smooth*, respectively étale, if there an atlas $\coprod_{i\in I} U_i \to F$ for F, and an atlas $\coprod_{j\in J} V_j \to G$ for G, which fit in a commutative diagram*

$$\begin{array}{ccc} \coprod_{i\in I} U_i & \longrightarrow & \coprod_{j\in J} V_j \\ \downarrow & & \downarrow \\ F & \longrightarrow & G, \end{array}$$

such that the corresponding maps $U_i \to V_j$ are smooth (respectively étale).

Examples of algebraic spaces arise naturally when considering quotients of schemes with respect to free group actions. This way one can construct algebraic spaces which are not schemes, using for example an involution on a non-projective scheme, constructed by Hironaka. Moreover, a theorem of Artin states that for proper smooth algebraic spaces, defined over $\mathbf{C}$, the resulting compact complex manifold is Moishezon, and the converse is true as well. Such a result is not known for schemes.

A morphism of presheaves, whose fibres are algebraic spaces, is called *representable*. The same notion makes sense for prestacks, and it is the first step in the definition of algebraic stacks.

Definition 4.30. *(a) A map of prestacks $F \to G$ is called* representable*, if for every affine scheme U mapping into G, i.e. $U \to G$, the fibre product $F \times_G U$ is an algebraic space.*

(b) A representable map of prestacks $F \to G$ is called smooth (respectively étale), if for every affine scheme U, and morphism $U \to G$ the base change $F \times_G U \to U$ is a smooth (respectively étale) morphism of algebraic spaces.

This definition allows us to define the notion of an atlas for algebraic stacks.

4.2.5 Algebraic stacks

Definition 4.31. *Let $(\mathsf{C}, \mathcal{T})$ be a site. A map of stacks $F \to G$ is called a* surjection*, if for every $U \in \mathsf{C}$, and every object $X \in G(U)$, there exists a covering $\{U_i \to U\}_{i\in I}$, and object $Y_i \in F(U_i)$, such that each $X|_{U_i}$ is isomorphic to Y_i.*

We can now define the notion of algebraic stacks.

Definition 4.32. *(a) An fppf stack F is called an* algebraic stack *(or* Artin stack*), if there exists a collection of representable smooth morphisms $\{U_i \to F\}_{i \in I}$, where each U_i is affine, such that $p \colon \coprod_{i \in I} U_i \to F$ is a surjection of stacks. We call the map $p \colon \coprod_{i \in I} U_i \to F$ an* atlas *for F.*

(b) An fppf stack F is called a Deligne-Mumford stack*, if there exists an atlas $\coprod_{i \in I} U_i \twoheadrightarrow F$, with each map $U_i \to F$ being* étale*.*

We have the following observation, the first part of which follows directly from the definitions.

Remark 4.33. *For an fppf sheaf F being a Deligne-Mumford stack is equivalent to being an algebraic space. Using a result similar to Proposition 4.27, one can show that an fppf sheaf is an algebraic space if and only if it is an algebraic stack (hence, a smooth atlas implies the existence of an étale atlas).*

The following lemma can often be found as one of the axioms an algebraic stack has to satisfy.

Lemma 4.34. *Let F be an algebraic stack, and $f \colon U \to F$ a morphism, where U is an affine. Then, f is* representable*, in the sense that for every affine scheme V, mapping into X, the fibre product $U \times_F V$ is an algebraic space.*

Proof. Let $p \colon X = \coprod_{i \in I} U_i \to F$ be an atlas for F. We begin with the assumption (which will be removed later) that there exists a factorisation

$$\begin{tikzcd} V \arrow[r, "s"] \arrow[dr] & X \arrow[d, "p"] \\ & F. \end{tikzcd}$$

Then we have $U \times_F V \cong (X \times_F U) \times_U V$. By definition of an atlas, the fibre product $X \times_F U$ is an algebraic space. Since the fibre product of algebraic spaces defines an algebraic space, we have won in this case. As a next step we need to remove the assumption that a section s exists. We will do this by means of the observation that such a section exists always locally in the smooth topology **smth**. Indeed, we have a natural map from the fibre product

$$X \times_F V \to X,$$

and the projection $X \times_F V \to V$ is a surjective smooth morphism, since $X \to F$ is an atlas. We conclude therefore that $U \times_F (X \times_F V) \to U \times_F V$ is a surjective smooth morphism from an algebraic space onto $U \times_F V$. Choosing an atlas for $U \times_F (X \times_F V)$ we also obtain an atlas for $U \times_F V$. $\square$

4.2.6 Presentations

Algebraic spaces can be represented as quotients of schemes with respect to étale equivalence relations.

Definition 4.35. *Let U be a scheme, an étale equivalence relation on U is given by a monomorphism $R \hookrightarrow U \times U$, such that each of the composition $R \to U \times U \xrightarrow{p_i} X$ for $i = 1, 2$ is étale, such that for each ring R, the set $X(R) \subset U(R) \times U(R)$ is an equivalence relation.*

Proposition 4.36. *Let F be an algebraic space, and $p\colon \coprod_{i\in I} U_i \to F$ an atlas. We denote the scheme $\coprod_{i\in I} U_i$ by U. The fibre product $R = U \times_F U \hookrightarrow U$ is an étale equivalence relation. Moreover, $R \rightrightarrows U \to F$ is a co-equalizer diagram in $\mathsf{Sh}_{\mathbf{fppf}}(\mathsf{Aff})$.*

Proof. Exercise.[21] □

The converse is true as well, i.e., given an étale equivalence relation R on an algebraic scheme U, the co-equalizer $R \rightrightarrows U \to F$ exists, and F is an algebraic space. This works even if U is an algebraic space. We will prove this result in a later section.

Definition 4.37. *A* smooth groupoid *in algebraic spaces is given by two algebraic spaces X_0, and X_1, together with two smooth representable maps $s, t\colon X_1 \rightrightarrows X_0$, as well as a composition law $X_1 \times_{t,X_0,s} X_1 \to X_1$, such that for each ring R the resulting object in sets $X_1(R) \rightrightarrows X_0(R)$, with induced composition, corresponds to a groupoid.*

Proposition 4.38. *Let F be an algebraic stack, and $U \to F$ an atlas. Then, $U \times_F U \rightrightarrows U$, with composition given by the natural morphism*

$$(U \times_F U) \times_{p_2,U,p_1} (U \times_F U) \to (U \times_F U)$$

is a smooth groupoid in algebraic spaces.

Proof. Exercise.[22] □

4.2.7 Base change

So far we have been working with stacks over $\mathsf{Aff} = \mathsf{Rng}^{\mathrm{op}}$. As we alluded to in the introduction, we could also fix a base ring R, and replace the category Rng by Alg_R, the category of R-algebras. We could then denote the opposite category by Aff_R, and refer to its objects as affine schemes defined over R, and consistently replace every appearance of Aff by Aff_R. It turns out that this exercise in notation is subsumed entirely in category theory.

Definition 4.39. *For a 2-category C and an object $X \in \mathsf{C}$, we denote by C/X the so-called* over-2-category*, whose objects are morphisms $Y \to X$, and whose 1-morphisms are strictly commuting triangles*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[dl] \\ X, & \end{tikzcd}$$

and the set of 2-morphisms between two 1-morphisms agrees with the set of 2-morphisms in C.

In Subsection 1.2, the category of R-algebras was defined in a similar way, but with the direction of the arrows inverted. In particular we see that the following is true, essentially by definition.

Remark 4.40. *For a ring R, the category $\mathsf{Aff}_R = \mathsf{Alg}_R^{\mathrm{op}}$ is equivalent to $\mathsf{Aff}/(\operatorname{Spec} R)$.*

This tautology extends to the 2-category of (algebraic) stacks defined over R.

[21] Volunteers?
[22] Volunteers?

Proposition 4.41. *The* 2*-category of prestacks* $\mathsf{PrSt}(\mathsf{Aff}_R)$*, defined over a ring* R*, is equivalent to the* 2*-category* $\mathsf{PrSt}(\mathsf{Aff})/(\operatorname{Spec} R)$*. Moreover, this equivalence respects the notion of stacks and algebraicity.*

Proof. Exercise.[23] □

For every ring homomorphism $R \to S$ we obtain a natural 2-functor $\mathsf{St}(\mathsf{Aff}_R) \to \mathsf{St}(\mathsf{Aff}_S)$, which preserves the notion of algebraicity. This base change functor is given by the fibre product along the map $\operatorname{Spec} S \to \operatorname{Spec} R$.

5 Torsors and quotient stacks

One of the simplest examples of an algebraic stack is the stack of torsors for an algebraic group. It can be described as the quotient stack of a point by the trivial action of a group G. We will discuss the notion of algebraic groups, torsors, their relation with cohomology, and show algebraicity of the stack of torsors and more general quotient stacks.

5.1 Group objects

A group object in a category C can be defined as follows.

Definition 5.1. *Let* C *be a category, the structure of a group object on* $X \in \mathsf{C}$ *is equivalent to a factorisation of functors*

$$\begin{tikzcd} \mathsf{C}^{\mathrm{op}} \arrow[r] \arrow[dr, "\mathit{Hom}_{\mathsf{C}}(-,X)"'] & \mathsf{Grp} \arrow[d] \\ & \mathsf{Set}, \end{tikzcd}$$

where $\mathsf{Grp} \to \mathsf{Set}$ *is the forgetful functor from groups to sets.*

By Yoneda, this is equivalent to the standard definition of group objects (if C has finite products), as one often finds it in the literature, but the functorial point of view will be of advantage to us.

Lemma 5.2. *If* C *has finite products, the structure of a group object on* $X \in \mathsf{C}$ *is equivalent to the following collection of morphisms:*

(a) $e\colon \bullet \to X$*, where* $\bullet \in \mathsf{C}$ *is a final object,*

(b) $m\colon X \times X \to X$*,*

(c) $\iota\colon X \to X$*,*

satisfying the group axioms, which are stipulated in form of the commutativity of the diagrams

(a) (unit)

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[dr, "\mathrm{id}_X"'] & X \times \bullet \arrow[d, "m\circ(\mathrm{id}_X \times e)"] \\ & X, \end{tikzcd}$$

[23] Volunteers?

(b) (associativity)

$$\begin{tikzcd} X \times X \times X \arrow[r, "m\times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_X \times m"'] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"] & X, \end{tikzcd}$$

(c) (inverses)

$$\begin{tikzcd} X \times X \arrow[r, "\mathrm{id}_X \times \iota"] \arrow[d] & X \times X \arrow[d, "m"] \\ \bullet \arrow[r, "e"] & X. \end{tikzcd}$$

We have already seen a few examples of group schemes (i.e., group objects in the category of schemes).

Example 5.3. *Fix a base ring R. A group R-scheme is a group object in the category Sch_R.*

(a) Let $\mathbb{A}^1_R\colon \mathsf{Alg}_R \to \mathsf{Set}$ be the functor sending an R-algebra $R \to S$ to the set underlying S. We have $\mathbb{A}^1_R \cong \mathbb{A}^1_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$, in particular we see that $\mathbb{A}^1_R$ is a scheme. It carries a natural structure of a group object, given by the group-valued functor, sending $R \to S$ to the abelian group $(S, +)$.

(b) Similarly, we may define $\mathbb{G}_{m,R}$ as the group-valued functor, sending an R-algebra $R \to S$ to the set of invertible elements $S^\times$. It is a commutative group R-scheme.

(c) We have a group scheme $\mathrm{GL}_{n,R}$, sending an R-algebra S to the group of invertible $(n \times n)$-matrices.

If G is an affine group R-scheme, then Lemma 5.2 implies that R-algebra $\Gamma(G)$ of regular functions on G is endowed with the structure of a Hopf R-algebra.

Definition 5.4. *A Hopf R-algebra consists of an R-algebra S, together with R-algebra maps*

(a) $e^\sharp\colon S \to R$, the co-unit*,*

(b) $m^\sharp\colon S \to S \otimes_R S$, called comultiplication*,*

(c) $\iota^\sharp\colon S \to S$, the co-inverse*,*

such that $(S, e^\sharp, m^\sharp, \iota^\sharp)$ satisfies the axiom of a group object (see Lemma 5.2) in $\mathsf{Alg}_R^{\mathrm{op}} \cong \mathsf{Aff}_R$.

All of the examples of 5.3 are affine group schemes, therefore correspond to Hopf algebras.

One can show that a group object in algebraic spaces, which is of finite type over a field k, is in fact a group scheme. This follows from the general principle that group objects in geometric categories "don't have corners". Heuristically speaking, if U is a neighbourhood of the unit element of G, and $g \in G$ an arbitrary element, then the translate gU defines a neighbourhood of the element g. Hence, we see that a group object G has a uniform local geometry. This implies that if P is a geometric property which holds for an open dense subset of points in G, then it has to hold for all points of G. An example of such a property for algebraic spaces is, lying an open *subscheme* $U \subset G$. In [The, 0ADC] it is shown that every algebraic space of finite type over a field k has an open dense subset which is a scheme. Hence, every group object in algebraic spaces of finite type over a field is actually a scheme.

5.2 Torsors

5.2.1 Definition and basic properties

We fix a site $(\mathsf{C}, \mathcal{T})$, and consider a group-valued sheaf $G^{\mathrm{op}}\colon \mathsf{C} \to \mathsf{Grp}$ (i.e., a group object in $\mathsf{Sh}(\mathsf{C})$). We assume that the Yoneda embedding factorises through the category $\mathsf{Sh}_{\mathcal{T}}(\mathsf{C})$ of sheaves, i.e., for every $U \in \mathsf{C}$, the representable functor $\mathsf{Hom}_{\mathsf{C}}(-, U)$ is a sheaf.

Definition 5.5. *Let $Y \in \mathsf{Sh}_{\mathcal{T}}(\mathsf{C})$ be endowed with a G-action, and $\pi\colon Y \to X$ be a morphism G-invariant morphism in $\mathsf{Sh}_{\mathcal{T}}(\mathsf{C})$. Then we say that π is a G-torsor, if for every $U \to X$, where $U \in \mathsf{C}$ the G-action on $Y \times_X U$ is $\mathcal{T}$-locally trivial, i.e., there exists $\{U_i \to U\}_{i \in I}$, such that $Y \times_X U_i \cong U_i \times G$ as a G-space.*

Paraphrasing the definition above, one could also refer to a G-torsor as a locally trivial G-bundle.

Lemma 5.6. *We say that $\pi\colon Y \to X$ is a* trivial torsor, *if we have an isomorphism $Y \cong X \times G$, respecting the projection π, and the G-action. A G-torsor is trivial if and only if there exists a section $s\colon X \to Y$, $\pi \circ s = \mathrm{id}_X$.*

Proof. A section s induces a map of G-torsors $X \times G \xrightarrow{(s \times \mathrm{id}_G)} Y \times G \to Y$. Lemma 5.7 below implies that it is an isomorphism.

Vice versa, given an isomorphism $X \times G \cong Y$, as torsors, it induces a section

$$X \xrightarrow{\simeq} X \times \bullet \xrightarrow{\mathrm{id}_X \times e} X \times G \xrightarrow{a} Y.$$

This concludes the proof. □

Lemma 5.7. *A map of G-torsors over X is a commutative diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X, \end{tikzcd}$$

where $Z \to X$ and $Y \to X$ are G-torsors. Then, every map of G-torsors is an isomorphism.

Proof. By Yoneda's lemma, it suffices to show that for every $U \in \mathsf{C}$, and $U \to X$, the map $Z \times_X U \to Y \times_X U$, of G-torsors over U is an isomorphism. Moreover, since X, Y, Z are sheaves, and a map of sheaves is an isomorphism, if and only it locally is an isomorphism, it suffices to prove the assertion for trivial G-torsors.

Let $\phi\colon U \times G \to U \times G$ a G-equivariant map of G-torsors. For every $V \in \mathsf{C}$, and every morphism $V \to U$, we obtain a G-equivariant map of $G(V)$-torsors $U(V) \times G(V) \to U(V) \times G(V)$. Every map of torsors in sets is an equivalence, since the transitivity and freeness of the action guarantee bijectivity of the map. Yoneda's lemma implies now that ϕ is an isomorphism. □

Corollary 5.8. *Every commutative diagram of sheaves, and G-equivariant maps*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ W \arrow[r] & X, \end{tikzcd}$$

where $Z \to W$ and $Y \to X$ are G-torsors, is cartesian.

Proof. The commutative diagram induces a map of G-torsors $Z \to W \times_X Y$. According to Lemma 5.7, it is an isomorphism. $\square$

We now switch to the category $\mathsf{C} = \mathsf{Aff}_R$, and show that the notion of torsors is often surprisingly well-behaved with respect to changing the topology.

Definition 5.9. *For a group-valued sheaf G on Aff_R, we denote by $BG\colon \mathsf{Aff}_R^{\mathrm{op}} \to \mathsf{Gpd}$ the prestack, sending an R-algebra S to the groupoid of collections $(\pi_T\colon Y_T \to \operatorname{Spec} T, \phi_T)$, where π_T is a G-torsor on $\operatorname{Spec} T$ for each S-algebra T, and $\phi_T\colon Y_T \cong Y_S \times_{\operatorname{Spec} R} G$ is an equivalence of G-torsors. Moreover, this collection is assumed to satisfy the conditions: $\phi_{\mathrm{id}_S}\colon Y_S \times_{\operatorname{Spec} S} S \xrightarrow{\simeq} Y_S$ is the canonical map, and for every composable pair of morphisms of R-algebras $\delta\colon S \to T$, and $\epsilon\colon T \to W$, as given by a commutative diagram*

$$\begin{tikzcd} & R \arrow[dl, "\alpha"'] \arrow[d, "\beta"] \arrow[dr, "\gamma"] & \\ S \arrow[r, "\delta"] & T \arrow[r, "\epsilon"] & W, \end{tikzcd}$$

we have a commutative square

$$\begin{tikzcd} (Y_S \times_{\operatorname{Spec} S} \operatorname{Spec} T) \times_{\operatorname{Spec} T} \operatorname{Spec} W \arrow[r] \arrow[d] & Y_T \times_{\operatorname{Spec} T} \operatorname{Spec} W \arrow[d] \\ Y_S \times_{\operatorname{Spec} S} \operatorname{Spec} W \arrow[r] & Y_W. \end{tikzcd}$$

The prestack BG is actually a stack. This is tautological, using descent for sheaves.

Lemma 5.10. *The prestack BG of* **fppf***-torsors is a stack with respect to the* **fppf** *topology.*

The inclusions of Grothendieck topologies **et** $\subset$ **smth** $\subset$ **fppf** imply that every torsor with respect to the étale topology is also a torsor with respect to the smooth and **fppf** topology. However, G is a smooth affine group scheme, these are actually equivalences, i.e, a torsor with respect to the **fppf** topology is also a torsor with respect to the coarser topology **et**. The proof stretches over the following two paragraphs.

Proposition 5.11. *Let G be a smooth affine[24] group R-scheme. Then, a G-torsor $Y \to X$, with respect to the* **fppf** *is also a G-torsor in the étale (and hence also smooth) topology.*

Since every smooth morphism has a section in the étale topology, it suffices to construct a covering of $\{U_i \to X\}$ of X in the smooth topology, such that $Y \times_X U_i \cong U_i \times_{\operatorname{Spec} R} G$ as a G-torsor. This follows directly from Proposition 4.27, which states that every smooth morphism has a section in the étale topology. It is a worthwhile exercise to think through the details of the proof.

Lemma 5.12. *Let G be a smooth affine group R-scheme. Then every G-torsor with respect to the smooth topology is also a G-torsor with respect to the étale topology.*

Proof. Exercise.[25] $\square$

[24] We could relax this condition.
[25] Alex has volunteered.

Another example is that every GL_n-torsor with respect to the **fppf** or **et** topology, is actually a torsor with respect to the much coarser Zariski topology.

Proposition 5.13. *For every affine scheme $U \in \mathsf{Aff}_R$, the groupoids of $\mathrm{GL}_{n,R}$-torsors and rank n vector bundles (i.e., finite projective modules, locally of rank n) are equivalent. Since every finite projective module is Zariski locally free, we see that every GL_n-torsor, defined with respect to the* **fppf** *or étale topology, is also a GL_n-torsor with respect to the topology* **Zar**.

Proof. We fix an affine scheme U, in order to establish an equivalence of groupoids of GL_n-torsors (denoted by $B\,\mathrm{GL}_n(U)$), and rank n vector bundles ($\mathsf{VB}_n(U)$), a priori we have to define mutually inverse functors $F\colon B\,\mathrm{GL}_n(U) \to \mathsf{VB}_n(U)$, and $G\colon \mathsf{VB}_n(U) \to B\,\mathrm{GL}_n(U)$.

We will only construct the functor G, and verify locally that it is an equivalence. Since VB_n and BG are stacks, this is sufficient to conclude the proof.

The functor $G\colon \mathsf{VB}_n(U) \to B\,\mathrm{GL}_n(U)$ sends a vector bundle E to the torsor whose total space is the sheaf on Aff/U, sending $V \xrightarrow{f} U$ to the set of bases of f^*E. Since the set of bases of a vector space is freely and transitively acted on by $\mathrm{GL}_n(V)$, we obtain that G is a fully faithful functor. It is also easily seen to be locally essentially surjective (hence globally essentially surjective, by the stack property), since every GL_n-torsor is locally trivial, thus the image of the trivial vector bundle. □

We can also give a direct description of the functor $F\colon B\,\mathrm{GL}_n(U) \to \mathsf{VB}_n(U)$. It takes a GL_n-torsor $Y \to X$, and first extracts a cocycle, by choosing an **fppf** covering $\{U_i \to U\}_{i\in I}$, over which Y becomes trivial (the cocycle is obtained by comparing trivialisations over the fibre products $U_i \times_U U_j$). Since automorphisms of a vector bundle are GL_n-valued regular functions, we can also view this as a cocycle for the glueing of vector bundles. Hence, obtain a well-defined vector bundle E_Y associated to the torsor Y.

A more conceptual description of the total space of E_Y is given by the following general construction:

$$E_Y \cong \mathbb{A}^n \times^{\mathrm{GL}_n} Y = (\mathbb{A}^n \times Y)/\,\mathrm{GL}_n\,.$$

This comparison result for torsors is far from being a purely abstract statement. It generalises a classical result in number theory, concerning the vanishing of a certain Galois cohomology set. In order to explain the connection with Galois cohomology, we have to comment on the connection between torsors and the first cohomology set.

Remark 5.14. *For a site $(\mathsf{C}, \mathcal{T})$, and a group sheaf G on C we define the first cohomology set $H^1_{\mathcal{T}}(U,G)$, given by a colimit (ranging over coverings $\mathfrak{U}$ of U in $\mathcal{T}$) of quotients $Z^1_{\mathfrak{U}}/C^0_{\mathfrak{U}}$, where C^0 is a group acting on a set Z^1, both of which will be defined subsequently. For every $\mathfrak{U} = \{U_i \to U\} \in \mathcal{T}$, we let $C^0_{\mathfrak{U}}$ be the group of cochains $(\psi_i)_{i\in I} \in \prod_{i\in I} G(U_i)$. We denote by $Z^1_{\mathfrak{U}}$ the set consisting of cocycles $(\phi_{ij})_{(i,j)\in I^2} \in \prod_{(i,j)\in I^2} G(U_i \times_U U_j)$, satisfying the cocycle condition $\phi_{ij}\phi_{jk} = \phi_{ik}$ on $U_i \times_U U_j \times_U U_k$. A cochain (ψ_i) acts on (ϕ_{ij}) by the formula*

$$\psi_j \phi_{ij} \psi_i^{-1}.$$

One sees that $H^1_{\mathcal{T}}(U,G)$ is in bijection with the set of isomorphism classes of G-torsors on U.

Corollary 5.15 (Hilbert's Theorem 90)**.** *Let K be a field, with separable closure L. Then, the Galois cohomology group $H^1_{\mathrm{Gal}}(K, \mathrm{GL}_n(L))$ vanishes.*

Proof. It is a general fact that $H^1_{\mathrm{Gal}}(K, L^\times) \cong H^1_{\mathbf{et}}(\operatorname{Spec} K, \mathrm{GL}_n)$, which relies on a morphism of field $K \to L$ being étale, if and only if it is separable (see Example 1.7(a) in [Mil80]).

The group $H^1_{\mathbf{et}}(\operatorname{Spec} K, \mathrm{GL}_n)$ is equivalent to the set of isomorphism classes of rank n vector bundles on $\operatorname{Spec} K$, according to Proposition 5.13, i.e., rank n vector spaces over K. Therefore we obtain the vanishing result from the fact that, up to isomorphism, there is precisely one vector space of dimension n. $\square$

5.2.2 Faithfully flat descent revisited

In this paragraph we discuss descent for smooth morphisms. We will use the classification of smooth maps of affine schemes in terms of Jacobi matrix, which was stated in Proposition 4.25(a). At first we have to verify that being of finite presentation descends along faithfully flat maps.

Lemma 5.16. *For ring homomorphisms $R \to S$, and $R \to R'$ faithfully flat, the co-base change $R' \to S \otimes_R R'$ is of finite presentation if and only if $R \to S$ is.*

Proof. Being of finite presentation is certainly invariant under tensor products. Hence, we focus on establishing the descent result. By Lemma 4.22, it suffices to show that for every directed system $(T_i)_{i\in I}$ of R-algebras, the natural map

$$\operatorname*{colim}_{i\in I} \mathsf{Hom}_R(S, T_i) \to \mathsf{Hom}_R(S, \operatorname*{colim}_{i\in I} T_i)$$

is an equivalence. Faithfully flat descent for ring homomorphisms (see Proposition 2.27) implies that we have a commutative diagram, with the columns being equalizer diagrams

$$\begin{tikzcd}
\operatorname{colim} \mathsf{Hom}_R(S, T_i) \arrow[r] \arrow[d] & \mathsf{Hom}_R(S, \operatorname{colim} T_i) \arrow[d] \\
\operatorname{colim} \mathsf{Hom}_{R'}(S \otimes_R R', T_i \otimes_R R') \arrow[r, "\simeq"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{Hom}_{R'}(S \otimes_R R', \operatorname{colim} T_i \otimes_R R') \arrow[d, shift left] \arrow[d, shift right] \\
\operatorname{colim} \mathsf{Hom}_{R'\otimes_R R'}(S \otimes_R R' \otimes_R R', T_i \otimes_R R' \otimes_R R') \arrow[r, "\simeq"] & \mathsf{Hom}_{R'\otimes_R R'}(S \otimes_R R' \otimes_R R', \operatorname{colim} T_i \otimes_R R' \otimes_R R').
\end{tikzcd}$$

Since the middle and bottom horizontal arrows are isomorphisms, the equalizer property implies that also the top horizontal map is an isomorphism. $\square$

We can now turn to the case of smooth and étale morphisms.

Lemma 5.17. *Let $V \to U$ be a morphism of affine schemes, and $U' \to U$ a faithfully flat morphism of affine schemes. Then, the base change $V' = V \times_U U' \to U'$ is smooth, respectively étale, if and only if the map $V \to U$ is smooth, respectively étale.*

Proof. Choose a presentation $U = \operatorname{Spec} R$, $V = \operatorname{Spec} R[t_1, \dots, t_m]/(f_1, \dots, f_n)$. We have to show that the Jacobi matrix $J = (\frac{\partial f_i}{\partial t_j})$ is surjective (respectively bijective), if and only if the Jacobi matrix J', for $V' \to U'$ is surjective (respectively bijective). But since we have a presentation $V' = \operatorname{Spec} R'[t_1, \dots, t_m]/(f_1, \dots, f_n)$, we obtain for J' the matrix associated to J with respect to the natural map from R-matrices to R'-matrices.

The Jacobi matrix J can be seen as an R-linear map from the R-module R^m to R^n. Since $R \to R'$ is faithfully flat, we obtain from Lemma 2.21(a) that J is surjective (respectively bijective) if and only if the Jacobi matrix for the map $V' \to U'$ is surjective (respectively bijective). $\square$

Corollary 5.18. *Let $F \to G$ a representable morphism of prestacks over R. Then for a faithfully flat ring homomorphism $R \to S$, the base change $F \times_{\operatorname{Spec} R} \operatorname{Spec} R \to G \times_{\operatorname{Spec} R} \operatorname{Spec} S$ is smooth (respectively étale) if and only if $F \to G$ is smooth (respectively étale).*

We also have to show that affineness descends.

Lemma 5.19. *Let $F \to U$ be a map of* **fppf** *sheaves, and $V \to U$ an fppf map of affine schemes, such that $F \times_U V$ is affine, then F is an affine scheme.*

Proof. We use the notation $U = \operatorname{Spec} R$, and $V = \operatorname{Spec} S$. We will show that the ring $T' = \Gamma(F \times_U V)$ is naturally endowed with a descent datum, hence defines an R-algebra T by faithfully flat descent (Theorem 2.25). We have a natural isomorphism $\operatorname{Spec} T \times_U V \to F \times_U V$, which satisfies the descent condition, hence an isomorphism of stacks $F \cong \operatorname{Spec} T$.

We have a commutative diagram with the columns being colimit diagram (or "co-equalizers") by descent theory

$$\begin{tikzcd}
F \times_U (V \times_U V \times_U V) \arrow[r, "\simeq"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \operatorname{Spec} T' \otimes_S (S \otimes_R \otimes_R S \otimes_R S) \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
F \times_U (V \times_U V) \arrow[r, "\simeq"] \arrow[d, shift left=1] \arrow[d, shift right=1] & \operatorname{Spec} T' \otimes_S (S \otimes_R S) \arrow[d, shift left=1] \arrow[d, shift right=1] \\
F \times_U V \arrow[r, "\simeq"] \arrow[d] & \operatorname{Spec} T' \arrow[d] \\
F \arrow[r, dashed, "\simeq"] & \operatorname{Spec} T,
\end{tikzcd}$$

We therefore obtain from faithfully flat descent, and the condition that F is a sheaf that F is equivalent to $\operatorname{Spec} T$. ☐

5.2.3 The tautological trivialisation

Every torsor is trivial when pulled back to itself.

Proposition 5.20. *Let $\pi \colon Y \to X$ be a G-torsor over an affine R-scheme X, where G is a smooth affine group R-scheme. Then, π is a covering in the smooth topology, and we have a canonical equivalence of G-torsors*

$$Y \times_X Y \cong Y \times G.$$

Proof. We obtain from Lemmas 5.17 and 5.19 that π is a smooth affine map. Since faithfully flatness descends, one sees that π is indeed a covering in the smooth topology.

In order to prove the second assertion, it suffices to produce a map of G-torsors $Y \times G \to Y \times_X Y$ over Y (every map of G-torsors is automatically an isomorphism, since the G-action is free and locally transitive. The required map is given by $\mathrm{id}_Y \times a$, i.e. the identity in the first, and the G-action in the second component. ☐

This concludes the proof of Proposition 5.11.

5.3 Algebraicity of quotient stacks

5.3.1 Algebraicity of the stack of torsors

We have already seen that BG is a stack (Lemma 5.10). This allows us to check algebraicity of BG.

Theorem 5.21. *For a smooth affine*[26] *group R-scheme G, the stack BG is algebraic. An atlas is given by the map $p\colon \operatorname{Spec} R \to BG$, corresponding to the trivial G-torsor on $\operatorname{Spec} R$.*

Proof. We will show that p is a surjective, smooth, affine morphism. By the Definition 4.32 of algebraic stacks, this implies the assertion.

Surjectivity of p amounts to the statement that for every G-torsor Y on an affine scheme U, there exists an fppf covering $\{U_i \to U\}_{i\in I}$, such that $Y \times_U U_i$ is equivalent to the trivial G-torsor on U_i. However, this local triviality property is the defining property of torsors (see Definition 5.5).

We now have to check that p is representable and smooth. Let $U \to BG$ be an arbitrary map, classified by a G-torsor Y on U, where U is an affine R-scheme. We claim that the fibre product $\operatorname{Spec} R \times_B GU$ is equivalent to Y. The first part of Proposition 5.20 states that Y is affine, and $Y \to U$ is a smooth map. This shows that $U \to BG$ is affine and smooth.

In order to see that $\operatorname{Spec} R \times_B GU$ is equivalent to Y, we observe that for every affine R-scheme V we have that the fibre product of groupoids $\bullet \times_{BG(V)} U(V)$ agrees with the set of isomorphisms from the trivial $G(V)$-torsor to the $G(V)$-torsor $Y(V)$. This set is canonically equivalent to the set $Y(V)$. □

5.3.2 Quotient stacks

Many examples of algebraic stacks are actually quotient stacks. As we will explain in the next subsection, they provide a geometric counterpart for equivariant constructions in geometry.

In order to get started, we consider a set X with a group action, and describe maps into the quotient groupoid $[X/G]$ in terms of torsors.

Example 5.22. *In Example 3.13 we defined a groupoid $[X/G]$, whose object are the elements of X, and morphisms from x to y correspond to $g \in G$ with $g \cdot x = y$. If U is a set, then the groupoid of morphisms $U \to [X/G]$ is equivalent to the groupoid of pairs $(\pi\colon Y \to U, f : Y \to X)$, where π is a G-torsor (i.e. a G-set with a free and transitive action), and f is a G-equivariant map.*

Proof. We will be content with exhibiting functors in both directions, and leave the verification that they are mutually inverse to the reader. Given a pair $(Y \to U, Y \to X)$ as above, one applies the $[-/G]$ construction, to obtain a map $U \to [X/G]$. Vice versa, given $U \to [X/G]$, we form the pullback square

$$\begin{tikzcd} U \times_{[X/G]} X \arrow[r] \arrow[d] & X \arrow[d] \\ U \arrow[r] & {[X/G]}. \end{tikzcd}$$

One sees that $U \times_{[X/G]} X \to U$ is a G-torsor, and the top horizontal map $U \times_{[X/G]} X \to X$ is G-equivariant. □

[26] We could relax this assumption.

The example discussed above motivates the definition of quotient stacks, which we record in heuristic form, i.e. in non-strict form (imitate Definition 5.9 for a rigorous definition).

Definition 5.23. *Let G be a group-valued fppf sheaf acting on an fppf sheaf X. The quotient stack $[X/G]$ is defined to be the prestack, sending an affine R-scheme U to the groupoid of pairs $(\pi\colon Y \to U, f\colon Y \to X)$, where π is a G-torsor, and f is a G-equivariant map.*

A verification similar to the one for BG shows that $[X/G]$ is a stack.

Lemma 5.24. *The prestack $[X/G]$ is a stack.*

Proof. Exercise.[27] □

Theorem 5.25. *If G is a smooth affine[28] group scheme, and X an algebraic R-space, then $[X/G]$ is an algebraic stack. The canonical projection $p\colon X \to X/G$, corresponding to the trivial pair $(X \times G \to X, X \times G \to X)$ is a surjective, smooth, affine morphism.*

Proof. The second statement implies the first: to see this choose an atlas $\coprod_{i\in I} U_i \to X$ for the algebraic space X. The composition with p defines then an atlas for the stack $[X/G]$.

In order to conclude that p is surjective, smooth, and affine, we use base change invariance of these properties, and the fact that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ {[X/G]} \arrow[r] & BG \end{tikzcd}$$

is a cartesian square. To see this, choose an affine R-scheme V, and consider the corresponding square of sets

$$\begin{tikzcd} X(V) \arrow[r] \arrow[d] & \bullet \arrow[d] \\ {[X/G](V)} \arrow[r] & BG(V). \end{tikzcd}$$

For each object of $[X/G](V)$ we can find an fppf covering, such that the corresponding G-torsor is trivial, hence the square is evidently cartesian. This proves the assertion. □

5.4 Philosophy: quotient stacks and equivariant objects

In this subsection we give a heuristic account of the prevalent point of view, that object on a quotient stack $[X/G]$ correspond to G-equivariant objects on X. We will therefore refrain from giving proofs in this subsection.

Lemma 5.26. *Let Z be an fppf sheaf, with trivial G-action. A G-invariant map $X \to Z$ corresponds to a morphism $[X/G] \to Z$.*

[27] Volunteers?

[28] We could relax this condition

Similarly, there's the notion of a G-equivariant vector bundle. Rank n vector bundles are equivalent to GL_n-torsors, which yields a convenient way to stating the next result without having to define vector bundles on stacks first. We begin with the definition of G-equivariance.

Definition 5.27. *Let $\pi\colon Y \to X$ be an H-torsor, and X a space acted on by G. A G-equivariant structure on Y amounts to a an action of G on Y, which commutes with the action of H, such that the map π is G-equivariant.*

One can then show the following result:

Proposition 5.28. *The datum of a G-equivariant H-torsor on X is equivalent to an H-torsor on $[X/G]$, i.e. a morphism $[X/G] \to BH$.*

It is often easier, and conceptually preferable, to replace equivariant constructions by analogous objects for stacks. For example, the study of equivariant cohomology is subsumed by the cohomology of stacks.

6 The stack of G-bundles on a curve

This section is devoted to establishing algebraicity of the stack of G-bundles on a curve, where G is a smooth affine group scheme. A good source for this material is Gaitsgory's geometric Langlands seminar[29], we will follow their presentation closely.

6.1 Preliminaries

6.1.1 Curves and vector bundles on curves

We begin by recalling a few basic facts about algebraic curves and vector bundles on algebraic curves.

Definition 6.1. *A curve over a field k is a proper, smooth k-scheme of dimension* 1*.*

We also record a relative version of Serre duality for cohomology groups of vector bundles.

Theorem 6.2 (Serre duality)**.** *Let X be a k-curve, and R a k. We denote by ω_X the line bundle of* 1*-forms on X. For every vector bundle E on $X \times \operatorname{Spec} R$ we have a canonical equivalence of R-modules*

$$H^0(X \times \operatorname{Spec} R, E) \cong H^1(X \times \operatorname{Spec} R, E^\vee \otimes \omega_X)^\vee.$$

The assumption of being of dimension 1 implies an important cohomology vanishing result.

Lemma 6.3. *Let X be a k-curve, and R a k. For every quasi-coherent sheaf M on $X \times \operatorname{Spec} R$ we have $H^i(X \times \operatorname{Spec} R, M) = 0$ for $i \geq 2$.*

6.1.2 Recollection of cohomology of coherent sheaves

Theorem 6.4 (Semicontinuity Theorem)**.** *For a projective morphism for noetherian schemes $Y \to X$, and a coherent sheaf M, which is X-flat, we have that $\dim H^i(Y_x, M_x)$ is an upper semicontiuous function on X, i.e. the sets $\{x \in X \,|\, \dim H^i(Y_x, M_x) \leq n\}$ are open.*

[29] `http://www.math.harvard.edu/~gaitsgde/grad_2009/`

6.1.3 Stacks which are locally of finite presentation

We have cited a few results on the cohomology of coherent sheaves which (at least in the way cited) required the assumption that the schemes were of finite type. We will be interested in applying these results to families of vector bundles, i.e., a vector bundle on a product $X \times S$, where S is an affine scheme, and X a curve over a field k. The definition of stacks requires S to vary through all affine schemes, so stipulating the condition of being of finite type a priori seems to seriously hinder the application of cohomology. We will circumvent this problem by showing that the stack of bundles $\mathsf{Bun}(X)$ is *locally of finite presentation.* This condition stipulates that the stack can be recovered by restricting it to the category $(\mathsf{Aff}_R^{\mathrm{fp}})^{\mathrm{op}}$, i.e. affine R-schemes which are of finite presentation.

Definition 6.5. *A prestack $F\colon \mathsf{Aff}_R^{\mathrm{op}} \to \mathsf{Gpd}$ is said to be* locally of finite presentation*, if the map*

$$\operatorname*{colim}_{i\in I} F(T_i) \to F(\operatorname*{colim}_{i\in I} T_i)$$

is an equivalence for every filtered inverse system of affine R-schemes T_i.

As a formal consequence we obtain the following straightforward assertion.

Lemma 6.6. *Let R be a ring, then every prestack $F\colon \mathsf{Aff}_R^{\mathrm{op}}$ which is locally of finite presentation, can be recovered from its restriction to the category $(\mathsf{Aff}_R^{\mathrm{fp}})^{\mathrm{op}}$.*

Proof. We will show that ring can be written as a directed colimit of finitely presented rings. In fact, we can express a ring R as an ascending union of subrings R', which are finitely generated over $\mathbb{Z}$: every ring R contains either a copy of $\mathbb{Z}$ or $\mathbb{F}_p$ as subrings; for $x \in R$ we can consider the subring $\mathbb{Z}[x]$, respectively $\mathbb{F}_p[x]$. This shows that every $x \in R$ lies in a finitely generated subring R'. Since the set of finitely generated subrings is clearly directed with respect to inclusion, we obtain the statement. □

One can show that all stacks in this section are locally of finite presentation.[30]

6.1.4 A criterion for surjectivity

In this paragraph we record a short and useful criterion, which allows to check surjectivity of a map of prestacks $F \to G$, by evaluating at fields. (...)

Lemma 6.7. *Let $f\colon F \to G$ be a morphism of stacks which is smooth and representable. We then have that f is surjective if and only if for every field-valued point $x \in G(k')$ there exists a finite field extension k''/k (can be chosen to be separable), such that the object in $G(k'')$ induced by x lies in the image of the map of groupoids $F(k'') \to G(k'')$.*

Proof. This is an interesting exercise.[31] □

6.1.5 Quot schemes

(...)

[30] This will be added to the notes at a later point.
[31] Volunteers?

6.2 The stack of vector bundles on a curve

6.2.1 Families of vector bundles

Recall that we fix a field k and consider all schemes, spaces, and stacks as objects over k. Consequently we drop the subscript k in a lot of places, hence write $\times$, where we should write $\times_{\operatorname{Spec} k}$.

Definition 6.8. *Let X be a curve over a field k, and X an affine scheme. A* family of rank n vector bundles *parametrised by S is a vector bundle of rank n, denoted by E, on the fibre product $X \times S$.*

Describing the moduli problems of vector bundles has been a well-researched topic in the last century, and this quest is continuing to the present day. In the language of this course we can think of this as the study of the stack $\mathsf{Bun}_n \colon \mathsf{Aff}_k^{\mathrm{op}} \to \mathsf{Gpd}$, sending $S \in \mathsf{Aff}_k$ to the groupoid of S-families of rank n vector bundles. Since pullback of vector bundles is not a strict functor, we have to strictify the functor. We omit this process and leave it to the reader to copy Definition 3.23.

Definition 6.9. *We define $\mathsf{Bun}_n \colon \mathsf{Aff}_k^{\mathrm{op}} \to \mathsf{Gpd}$ to be the prestack sending an affine k-scheme S to the groupoid of vector bundles of rank n on $X \times S$. The connected component of rank n and degree d vector bundles will be denoted by $\mathsf{Bun}_n^d(X)$.*

It is central to the theory of vector bundles that this is actually an algebraic stack.

Theorem 6.10. *The prestack Bun_n is an algebraic stack.*

The proof needs to verify to assertions: that Bun_n is a stack, which follows directly from descent theory, and the existence of an atlas. We will deal with the latter in the following paragraph, and conclude the present one by remarking on the details of the proof that Bun_n is a stack. (...)

6.2.2 The atlas

We will produce an atlas for $\mathsf{Bun}_n(X)$, using an open subscheme of $\mathsf{Quot}_X(\mathcal{O}_X(m)^\ell)$, where $\mathcal{O}_X(1)$ denotes an ample line bundle on X with tensor powers $\mathcal{O}_X(m)$, and m, and ℓ are appropriately chosen integers. In order to achieve this, we remark first that every vector bundle of sufficiently high degree can be obtained as a quotient of $\mathcal{O}_X(m)^\ell$.

Lemma 6.11. *Let E be a vector bundle on the curve X, with a chosen ample line bundle $\mathcal{O}_X(1)$ (i.e., of positive degree). For an integer m, we denote by $E(m)$ the tensor product $E \otimes \mathcal{O}_X(1)^{\otimes m}$. There exists an integer m_0, such that for all $m \geq m_0$ we have that $H^1(X, E(m)) = 0$. Under this assumption, $E(m)$ is generated by global sections, i.e., the map*

$$H^0(X, E(m)) \otimes \mathcal{O}_X \to E(m)$$

is surjective. The dimension ℓ of $H^0(X, E(m))$ is locally constant in S-families.

Proof. This is just a general fact for ample line bundles. See [**?**] for a proof in a far more general context. (...) □

The Semicontinuity Theorem 6.4 implies that the open subscheme U is well-defined.

Definition 6.12. *Let $U^{(m)}$ be the open subscheme of $\mathsf{Quot}_n^d(\mathcal{O}_X(-m)^\ell)$, consisting of quotients*

$$(\mathcal{O}_X(m)^\ell) \twoheadrightarrow E,$$

such that E satisfies the vanishing condition $H^1(X, E(m)) = 0$, and $d(E) = d$, and $n = \mathrm{rk}(E)$.

This definition allows us to state a corollary of Lemma 6.11.

Corollary 6.13. *The map $p\colon \coprod_{m\geq 0} U^{(m)} \to \mathsf{Bun}_n^d$ is a surjection.*

Proof. We have seen in Lemma 6.7 that it suffices to check that every field-valued point $E \in \mathsf{Bun}_n^d(X)(k')$, can be lifted to $\mathsf{Quot}_n^d(\mathcal{O}_X(-m)^\ell)(k'')$, after passing to a field extension k''/k'. We show that p is smooth and representable in Corollary 6.15. Lemma 6.11 implies right away that every vector bundle E on $X \times \operatorname{Spec} k'$ we can express $E(m)$ as a quotient of $\mathcal{O}_X^\ell \cong H^0(X, E(m)) \otimes \mathcal{O}_X$, hence E as a quotient of $\mathcal{O}_X(-m)^\ell$. As mentioned above, Lemma 6.7, this concludes the argument. □

We will show that p is an atlas. In order to verify representability and smoothness, we will check that p is a GL_ℓ-torsor. This will be a corollary of the following geometric statement.

Lemma 6.14. *Let $q\colon X \times S \to S$ be the canonical projection. For every rank n vector bundle E on $X \times S$, such that the fibrewise degree is equal to d, and satisfying the assumptions of Definition 6.12, we have a bijection between surjections $\phi\colon\ \mathcal{O}_{X\times S}^\ell \twoheadrightarrow E(m)$ and isomorphisms $\phi^\sharp\colon\ \mathcal{O}_S^\ell \xrightarrow{\simeq} q_*(E(m))$.*

Proof. The pullback functor q^* is left adjoint to the pushforward functor q_*. In particular, we obtain a bijection between maps $\phi\colon q^*\,\mathcal{O}_X^\ell \cong \mathcal{O}_{X\times S}^\ell \to E(m)$, and maps $\phi^\sharp\,\mathcal{O}_S \to q_*(E(m))$. It remains to show that surjections on one side correspond to isomorphisms on the other.

We begin by showing that $\phi^\sharp$ being an isomorphism implies that ϕ is a surjection. This is immediate, since the data of an isomorphism $\mathcal{O}_S^\ell \xrightarrow{\simeq} q_*(E(m))$ is equivalent to choosing an $\Gamma(S)$-linear basis for the space of global sections of E on $X \times S$. Since E is generated by its global sections by assumption, we obtain that the induced map $\phi\colon\ \mathcal{O}_{X\times S}^\ell \twoheadrightarrow E(m)$ is a surjection.

To show that $\phi^\sharp$ is an isomorphism if ϕ is a surjection, we begin by observing that $\phi^\sharp = q_*(\phi)$. Indeed, we have that the natural map $\mathcal{O}_S \to q_*(\mathcal{O}_{X\times S})$, corresponding to $q^*\,\mathcal{O}_S \xrightarrow{\simeq} \mathcal{O}_{X\times S}$, is an equivalence, because X has only constant global sections. Moreover, since E is generated by global sections, the map ϕ is a surjection. To conclude the proof we show that $q_*(E(m))$ is a vector bundle. This follows as an application of Theorem III.12.11 in [Har77]. Hence, we have a surjection between vector bundles of the same rank. Such a map is always an isomorphism. □

Corollary 6.15. *The map $p\colon U^{(m)} \to \mathsf{Bun}_n^d(X)$ is a GL_ℓ-torsor over its image, i.e., for every affine scheme V, and morphism $V \to \mathsf{Bun}_n$, the base change $V \times_{\mathsf{Bun}_n^d} U \to V$ is a GL_ℓ-torsor over an open subscheme of V.*

Proof. A map $V \to \mathsf{Bun}_n^d(X)$ corresponds to a rank n, degree d vector bundle E on $X \times S$. Consider the cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & V \arrow[d] \\ \mathsf{Quot}_n^d(\mathcal{O}_X(-m)^\ell) \arrow[r] & \mathsf{Bun}_n^d(X). \end{tikzcd}$$

By definition of fibre products of groupoids, for every affine scheme W, the groupoid $P(W)$ is the set of surjections $\mathcal{O}_{X\times W}(-m)^\ell \twoheadrightarrow E$. By Lemma 6.14, this set is equivalent to the set of isomorphisms $\mathcal{O}_W^\ell = q_* \mathcal{O}_{X\times W}^\ell \xrightarrow{\cong} q_* E(m)$. This set of isomorphisms (if non-empty) is freely transitively acted on by $\mathrm{GL}(W)$. Since p is surjective, P is non-empty, after replacing V by an **fppf** covering. This shows that $P \to V$ is indeed a GL_ℓ-torsor. □

Corollary 6.16. *The stack* $\mathsf{Bun}_n^d(X)$ *is algebraic, with atlas* $p\colon \coprod_{m\geq 0} U^m \to \mathsf{Bun}_n^d(X)$.

Proof. We have seen in Corollary 6.13 that p is a surjection. Corollary 6.15 showed that p is a GL_ℓ-torsor. Since GL_ℓ is a smooth and affine group scheme, we obtain from Proposition 5.20 that p is a smooth and affine morphism. Therefore, p satisfies all conditions required to be an atlas for $\mathsf{Bun}_n^d(X)$. □

6.3 The stack of G-bundles on a curve

6.3.1 Stacks of G-bundles as mapping stacks

Let F and Y be stacks. The mapping stack $\mathsf{Map}(Y, F)$ is defined as the prestack

$$\mathsf{Map}(Y,F)(U) = \mathsf{Hom}(Y \times U, F),$$

where $\mathsf{Hom}(Y\times U, F)$ is a groupoid, because stacks form a 2-category. The stack property for F and Y implies that $\mathsf{Map}(Y, F)$ is a stack as well. In general, it is unreasonable to expect mapping stacks to be algebraic, if however Y is a projective scheme, the chances for this to happen are much better. In fact, we have just established algebraicity of the mapping stack $\mathsf{Bun}_n(X) \cong \mathsf{Map}(X, \mathsf{VB}_n)$.

We fix a smooth affine group k-scheme G, and as before a curve X, defined over k.

Definition 6.17. *The stack* $\mathsf{Bun}_G(X)$ *of* G*-bundles on* X*, is defined (as the strictification, similar to Definition 3.23), of the groupoid-valued functor, sending an affine* k*-scheme* S *to the groupoid of* G*-torsors on* $X \times S$.

Since rank n vector bundles correspond to GL_n-torsors (see Proposition 5.13), we have $\mathsf{Bun}_{\mathrm{GL}_n}(X) \cong \mathsf{Bun}_n(X)$.

Lemma 6.18. *We have an equivalence of prestacks* $\mathsf{Bun}_G(X) \cong \mathsf{Map}(X, BG)$*. In particular,* $\mathsf{Bun}_G(X)$ *is a stack.*

Proof. By definition, $\mathsf{Map}(X, BG)(S) = \mathsf{Hom}(X \times S, BG)$, but the latter is equivalent to the groupoid of G-torsors on $X \times S$, hence to $\mathsf{Bun}_G(X)(S)$. □

Theorem 6.19. *Let* $G \hookrightarrow \mathrm{GL}_n$ *an embedding of* G *into the general linear group. The induced map* $\mathsf{Bun}_G(X) \to \mathsf{Bun}_n(X)$ *is representable. In particular,* $\mathsf{Bun}_G(X)$ *is an algebraic stack.*

We will prove this assertion in the next paragraph, after having established that the mapping space $\mathsf{Map}(X, Y)$ is a scheme, if X is projective, and Y quasi-projective.

6.3.2 Presentability statements

We will need the following generalisation of mapping stacks.

Definition 6.20. *Let $Y \to X \times S$ be a morphism of sheaves, where $S = \operatorname{Spec} R$ is an affine k-scheme, and X a curve. We denote by $\mathsf{Sect}_{X\times S}(Y)$ the space, sending an affine R-scheme U to the set of commutative diagrams*

$$\begin{tikzcd} & Y \arrow[d] \\ X \times U \arrow[ur] \arrow[r] & X \times S. \end{tikzcd}$$

The reason for introducing these stacks of sections is their appearance in the proof of Theorem 6.19. It will be important to us to understand representability of these spaces of sections for quasi-projective morphisms. We begin by establishing this in the special case that Y is a vector bundle over $X \times S$.

Lemma 6.21. *Let E be a vector bundle on $X \times S$. Denote by $\mathsf{Sect}_{X\times S}(E)$ the stack which sends an affine R-scheme U to the set of sections of the pullback π^*E on $X \times U$. Then we have*

$$\mathsf{Sect}_{X\times S}(E) \cong \operatorname{Spec}\operatorname{Sym}(H^1(X \times S, E^\vee \otimes \omega_X)).$$

Proof. We have to show that affine R-scheme $\operatorname{Spec}\operatorname{Sym}(H^0(X\times S, E^\vee \otimes \omega_X))$ represents the functor $\mathsf{Sect}_{X\times S}(E)$. This is equivalent to the R-algebra $\operatorname{Sym}(H^0(X \times S, E^\vee \otimes \omega_X))$ corepresenting the functor $T \mapsto \mathsf{Sect}_{X\times S}(E)(T)$. Hence, we have to show that for every R-algebra T there's a natural bijection between maps of S-modules

$$\mathsf{Hom}_R(H^1(X \times S, E^\vee \otimes \omega_X), T) \cong H^1(X \times S, E^\vee \otimes \omega_X)^\vee \otimes T$$

and the set of sections

$$H^0(X \times T, E \otimes_S T) \cong H^0(X \times S, E) \otimes_S T.$$

However, according to Serre duality (see Theorem 6.2), we have a canonical equivalence

$$H^0(X \times S, E) \cong H^1(X \times S, E^\vee \otimes \omega_X)^\vee.$$

This implies the claim. □

The main result on spaces of sections is the following.

Proposition 6.22. *Let S be an affine scheme, and $Y \to X \times S$ a quasi-projective morphism, i.e., a morphism of schemes, which can be factored as*

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[drr] & Z \arrow[r, "j"] \arrow[dr] & \mathbb{P}(E) \arrow[d] \\ & & X \times S, \end{tikzcd}$$

where $\mathbb{P}(E)$ is the projective space bundle, associated to a vector bundle E over $X \times S$, and i is an open immersion, and j a closed immersion. Under these assumptions we have that $\mathsf{Sect}_{X\times S}(Y)$ is a scheme.

Proof. Lemmas 6.23 and 6.24 below allow us to reduce the proof of the Proposition to showing that $\mathsf{Sect}_{X\times S}(\mathbb{P}(E))$ is a scheme. As we will see, this is the content of Grothendieck's celebrated representability result for Quot schemes.

Let $U \to S$ be a morphism of affine schemes. A section $s\colon X \times U \to \mathbb{P}(E)$, i.e., a U-point of $\mathsf{Sect}_{X\times S}(\mathbb{P}(E))$, corresponds to a surjection $\pi_U^* E \twoheadrightarrow M$, where M is a line bundle on $X \times U$.

This should be compared to the U-points of $\mathsf{Quot}_{X\times S/S}(E)$. By definition, $\mathsf{Quot}_{X\times S/S}(E)$ consists of quotients $\pi^* E \twoheadrightarrow M$, such that M is U-flat.

We claim that the natural morphism $\mathsf{Sect}_{X\times S}(\mathbb{P}(E)) \to \mathsf{Quot}_{X\times S/S}(E)$ is an open immersion. This implies in particular that $\mathsf{Sect}_{X\times S}(\mathbb{P}(E))$ is a scheme.

We have to show that for every $U \to \mathsf{Quot}_{X\times S/S}(E)$ the base change

$$\mathsf{Sect}_{X\times S}(\mathbb{P}(E)) \times_{\mathsf{Quot}_{X\times S/S}(E)} U \hookrightarrow U$$

is an open immersion. To see this, we observe that for quotient $\pi_U^* E \twoheadrightarrow M$ on $X \times U$, there exists an open subscheme V of $X \times U$, such that M is locally free (we say that being locally free is an open condition for coherent sheaves). A proof of this can be found in [**?**]. We denote the closed complement by $Z = X \times U \setminus V$. Recall that the canonical projection is denoted by $q\colon X \times U \to U$. The image $q(Z) \subset U$ is closed, since X is proper. We have an open subscheme $U' = U \setminus q(Z)$. By construction, the restriction $M|_{X\times U'}$ is locally free. Since the rank of a vector bundle is locally constant we may replace U' be the union of connected components where $\mathcal{M}$ is of rank 1. The resulting open subscheme $U' \subset U$ is equivalent to the fibre product $\mathsf{Sect}_{X\times S}(\mathbb{P}(E)) \times_{\mathsf{Quot}_{X\times S/S}(E)} U \subset U$. □

We have to show that open immersions of targets induce open immersions of spaces of sections. This is a consequence of properness of X.

Lemma 6.23. *Let $Y \hookrightarrow Z$ be an open immersion of schemes over $X \times S$, then the induced map $\mathsf{Sect}_{X\times S}(Y) \to \mathsf{Sect}_{X\times S}(Z)$ is also an open immersion.*

Proof. It suffices to prove that for every $U \to S$ and a map $U \to \mathsf{Sect}_{X\times S}(Z)$ the base change $\mathsf{Sect}_{X\times S}(Y) \times_{\mathsf{Sect}_{X\times S}(Z)} U \hookrightarrow U$ is an open immersion. A map $U \to \mathsf{Sect}_{X\times S}(Z)$ corresponds to a section $X \times U \to Z \times_S U$. Since X is proper, the subset $U' = U \setminus (q(Z \setminus Y))$ is the maximal open subscheme, such that we have a factorisation $X \times U' \to Y \times_S U$. This concludes the proof. □

The case of closed immersions is slightly more subtle, and ultimately relies on Serre duality for X.

Lemma 6.24. *Let $Y \hookrightarrow Z$ be a closed immersion of finite presentation of sheaves over $X \times S$, then the induced map $\mathsf{Sect}_{X\times S}(Y) \to \mathsf{Sect}_{X\times S}(Z)$ is also a closed immersion.*

Proof. We have to show that for every cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \arrow[d] \\ \mathsf{Sect}_{X\times S}(Y) \arrow[r] & \mathsf{Sect}_{X\times S}(Z), \end{tikzcd}$$

where U is an affine R-scheme, the base change $P \to U$ is a closed immersion. The map $U \to \mathsf{Sect}_{X\times S}(Z)$ corresponds to a section $s\colon X \times U \to Z$. Let W be the base change $Y \times_Z (X \times U)$.

We have a closed immersion of finite presentation $W \hookrightarrow X \times U$. For every affine R-scheme U' with a morphism $U' \to U$, we have that $P(U')$ is the subset of $\mathsf{Sect}_{X\times S}(Z)$, consisting of morphisms factoring through Y. Hence we have $P(U') = \mathsf{Sect}_{X\times U}(W)(U')$, i.e., $\mathsf{Sect}_{X\times U}(W) \cong P$. Since $W \hookrightarrow X \times U$ is a closed subscheme of finite presentation, there is a morphism of vector bundles $f\colon E \to E'$, such that $W = f^{-1}(0)$. Particularly, we see that we have

$$\mathsf{Sect}_{X\times U}(W) \cong \mathsf{Sect}_{X\times U}(E) \times_{\mathsf{Sect}_{X\times U}(E')} U,$$

where $U \times \mathsf{Sect}_{X\times U}(E')$ corresponds to the section given by the zero section

$$0_{E'}\colon X \times U \to E'.$$

The map $\mathsf{Sect}_{X\times U}(E) \to \mathsf{Sect}_{X\times U}(E')$ is certainly a closed immersion, hence the base change $\mathsf{Sect}_{X\times U}(W) \to U$ is a closed immersion too. Technically we're using that a closed immersion of vector bundles $E \hookrightarrow E'$ induces a closed immersion of spaces of sections. However, inspecting the proof of Lemma 6.21, it is easy to show that this holds, since an inclusion of sheaves $E_1 \hookrightarrow E_2$ induces a surjection $E'^{\vee} \otimes \omega_X \twoheadrightarrow E^{\vee} \otimes \omega_X$. The induced map of first cohomology groups is also a surjection, since a coherent sheaf on a curve has vanishing second cohomology groups. This yields a surjection of algebras

$$\operatorname{Sym} H^1(X \times U, E'^{\vee} \otimes \omega_X) \twoheadrightarrow \operatorname{Sym} H^1(X \times U, E^{\vee} \otimes \omega_X),$$

thus a closed immersion of affine schemes. $\square$

We will now explain the link with proving representability of the map $\mathsf{Bun}_G(X) \to \mathsf{Bun}_n(X)$. This requires one more definition.

Definition 6.25. *For a G-torsor $Y \to U$, and a sheaf Z with a G-action, we denote by $Y \times^G Z$ the sheaf $(Y \times Z)/G \to U$ over U.*

One checks easily that $Y \times^G Z \to U$ is étale locally equivalent to $U \times Z$ (since G-torsors are étale locally trivial).

Lemma 6.26. *Let $G \hookrightarrow H$ be an embedding of smooth affine group k-schemes. For an affine k-scheme S we have a cartesian diagram*

$$\begin{tikzcd}
\mathsf{Sect}_{X\times S}(E \times^H H/G) \arrow[r] \arrow[d] & S \arrow[d] \\
\mathsf{Bun}_G(X) \arrow[r] & \mathsf{Bun}_H(X).
\end{tikzcd}$$

Sketch. [32] The embedding $G \hookrightarrow H$ allows us to describe the map $\mathsf{Bun}_G(X) \to \mathsf{Bun}_H(X)$ as sending a G-torsor E to $E^H = E \times^G H$. In particular we have have a commutative diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[dr] & E^H \arrow[d] \\
& X \times S,
\end{tikzcd}$$

[32] Volunteers for turning this into a proof as an exercise?

where the map $E \hookrightarrow E^H$ is actually a closed immersion. For each fibre it specifies a element in H/G (after choosing a trivialization of the H-torsor). Hence, the data of the above diagram is equivalent to a map $X \times S \to E^H \times^H H/G$. □

Using this lemma, the Proposition 6.22 implies Theorem 6.19.

Proof of Theorem 6.19. The quotient H/G is quasi-projective.[33] One can show that for any H-bundle E on an affine scheme, $E \times^H H/G$ is quasi-projective too, since the embedding into projective space is given by a G-equivariant ample line bundle. In this case, we obtain by virtue of Proposition 6.22 that $\mathsf{Sect}_{X\times S}(E \times^H H/G)$ is a quasi-projective scheme. □

7 Smooth algebraic stacks

A smooth variety is intuitively speaking very close to a complex manifold. The tangent space of a smooth variety reflects the local geometry. This section is devoted to the smooth algebraic stacks. We will see that the natural analogue of the tangent space at a point is actually a complex. Similar, the sheaf of tangent vector fields for varieties, has to be substituted by the tangent complex. After having studied the basic properties of smooth algebraic stacks, we will attempt to give a definition of this object. On the way there we encounter higher descent conditions, which will make a reappearance in the study of gerbes.

7.1 Smooth morphisms

The following definition can often be used as a template to define properties of algebraic stacks.

Definition 7.1. *An algebraic R-stack $F\colon \mathsf{Aff}_R^{\mathrm{op}} \to \mathsf{Gpd}$ is called* smooth*, if there exists an atlas $p\colon U \to F$, such that U is a smooth R-scheme.*

An important example is given by classifying stacks of G-torsors.

Example 7.2. *Let G be a smooth affine R-scheme. The algebraic stack BG of Definition 5.9 is smooth.*

Proof. We have seen in Theorem 5.21 that an atlas for BG is given by $\operatorname{Spec} R \to BG$, given by the trivial G-torsor on $\operatorname{Spec} R$. Since the identity map $\operatorname{Spec} R \to \operatorname{Spec} R$ is smooth, we see that BG is a smooth algebraic R-stack. □

It turns out that if F is a smooth algebraic stack, then every atlas of F is smooth.

Lemma 7.3. *Let F be a smooth algebraic stack and $q\colon V \to F$ an atlas. Then, V is smooth.*

Proof. Consider the cartesian square

$$\begin{tikzcd} U \times_F V \arrow[r, "p'"] \arrow[d, "q'"'] & V \arrow[d, "q"] \\ U \arrow[r, "p"] & F. \end{tikzcd}$$

[33] See `http://math.uchicago.edu/~jpwang/writings/Quotient.pdf` for a proof of this classical result, using sheaf-theoretic language.

By assumption, the maps p and q are smooth, representable, and surjective. Since this is a base change invariant notion, p' and q' are smooth and representable as well. In particular, $U \times_F V$ is an algebraic space, and the map $p' \colon U \times_F V \to V$ is smooth and representable. By Definition 4.29, we see that there exists an atlas $W \to U \times_F V$, such that each of the affine schemes W_i is smooth. Without loss of generality we may assume that W and V are affine. We then see that we have a surjective smooth map of affine schemes $W \to V$, where W is R-smooth, since there is a smooth morphism $W \to U$, and U is R-smooth. Representing V as $\operatorname{Spec} R[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$, and W as $\operatorname{Spec} R[t_1, \ldots, t_n; s_1, \ldots, s_{n'}]/(f_1, \ldots, f_m; g_1, \ldots, g_{m'})$, and applying the chain rule to

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "h"'] & V \arrow[d, "g"] \\ & \operatorname{Spec} R, \end{tikzcd}$$

we see that the Jacobi matrices satisfy $\partial h = \partial g \cdot \partial f$. Therefore, ∂h being surjective implies that ∂g is surjective too. $\square$

The lemma shows that the following statements are equivalent.

(a) The algebraic R-stack F is smooth.

(b) There exists an atlas $U \to F$, such that U is smooth over R.

(c) For every atlas $U \to F$, U is smooth over R.

Instead of smoothness, we could have worked with a different property, for example being locally of finite presentation, a locally complete intersection, or Cohen-Macaulay, etc.

Remark 7.4. *Let P be a property of affine schemes, such that for a surjective smooth morphism $V \to U$ of affine R-schemes, the scheme V has property P, if and only if U has property P. Then, we say that an algebraic R-stack F has property P, if and only if there exists an atlas $\coprod_{i \in I} U_i \to F$, such that each U_i has property P. Equivalently, one could demand that for every atlas $\coprod_{i \in I} U_i \to F$, the scheme U has property P.*

Theorem 7.5. *Let F be an algebraic R-stack, which is locally of finite presentation. It is smooth, if and only is for every square-zero extension of affine schemes $U \to V$ (see Definition 4.24), and commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & F \arrow[d] \\ V \arrow[r] & G \end{tikzcd}$$

there exists an étale covering $V' \to V$, such that we have a dashed arrow

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & F \arrow[d] \\ V' \arrow[r] \arrow[ur, dashed] & G, \end{tikzcd}$$

where $U' = V' \times_V U$.

We devote the rest of this subsection to the proof of this theorem. For future reference it will be useful to have the notion of formally smooth morphisms for stacks (for the case of affine schemes see Definition 4.24).

Definition 7.6. *We say that a morphism of stacks $F \xrightarrow{f} G$ we say that f is* formally smooth, *if for every square-zero extension of affine schemes $U \to V$, and every commutative square*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & F \arrow[d] \\ V \arrow[r] \arrow[ur] & G, \end{tikzcd}$$

there exists an étale covering $V' \to V$, we have a dashed arrow rendering the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & F \arrow[d] \\ V' \arrow[r] \arrow[ur, dashed] & G, \end{tikzcd}$$

commutative, where $U' = U \times_V V'$.

We can now give the proof of Theorem 7.5.

Proof of Theorem 7.5. The proof relies on the so-called *topological invariance property* of étale morphisms, which is proven in [Mil80, Thm. 3.23]. It states that for a square-zero extension $U \to V$, the functor

$$(V' \to V) \mapsto (V' \times_V U \to U)$$

provides an equivalence between étale V-schemes and étale U-schemes. The intuitive content is that étale morphisms correspond to local diffeomorphims for manifolds, hence should not be sensitive to nilpotent thickenings.

Let $X \to F$ be an atlas for F, we also choose an atlas Y for the algebraic space $X \times_F U$. We have a commutative diagram

$$\begin{tikzcd} X \times_F U \arrow[r] \arrow[d] & X \arrow[d] \\ U \arrow[r] \arrow[d] & F \arrow[d] \\ V \arrow[r] & \operatorname{Spec} R. \end{tikzcd}$$

The map $X \times_F U \to U$ is a smooth morphism of schemes. In particular, there exists an étale covering $U' \to U$, such that we have a section (Proposition 4.27)

$$\begin{tikzcd} & X \times_F U \arrow[d] \\ U' \arrow[r] \arrow[ur] & U. \end{tikzcd}$$

By the topological invariance of étale morphisms, we know that there is an étale covering $V' \to V$, and an isomorphism $U' \cong V' \times_V U$. This allows us to draw the commutative diagram

$$\begin{tikzcd}
& Y \arrow[d] & \\
& X \times_F U \arrow[r] \arrow[d] & X \arrow[d] \\
U' \arrow[r] \arrow[d] \arrow[ur] \arrow[uur, bend left] & U \arrow[r] \arrow[d] & F \arrow[d] \\
V' \arrow[r] & V \arrow[r] & \operatorname{Spec} R.
\end{tikzcd}$$

Composing a few arrows, we obtain

$$\begin{tikzcd}
U' \arrow[r] \arrow[dd] & X \arrow[d] \\
& F \arrow[d] \\
V' \arrow[r] \arrow[uur, dashed] \arrow[ur, dashed] & \operatorname{Spec} R
\end{tikzcd}$$

by applying the lifting criterion for the smooth morphism of schemes $X \to \operatorname{Spec} R$. $\square$

Let us demonstrate the usefulness of this criterion for smoothness, by applying it to the stack $\mathsf{Bun}_1(X) = \mathsf{Pic}(X)$.

Example 7.7. *The stack of line bundles on a curve* $\mathsf{Pic}(X) = \mathsf{Bun}_1(X)$ *is smooth.*

Proof. Let $i\colon U \to V$ be a square-zero extension of affine k-schemes. We let $U = \operatorname{Spec} B$, and $V = \operatorname{Spec} A$, and $A \to B$ the corresponding ring homomorphism. Its kernel will be denoted by I. The set of isomorphism classes of line bundles on $X \times V$ is given by $H^1(X \times S, \mathcal{O}_{X\times V}^{\times})$. We have a short exact sequence of sheaves

$$0 \to \pi^* I \xrightarrow{1+?} \mathcal{O}_{X\times V}^{\times} \to i_* \mathcal{O}_{X\times U}^{\times} \to 0,$$

where we use that $I^2 = 0$, in order to get a well-defined map $x \mapsto 1 + x$, from $\pi^* I$ to $\mathcal{O}_{X\times V}$. The corresponding long exact sequence of sheaf cohomology groups contains the portion

$$H^1(X \times V, \mathcal{O}_{X\times V}^{\times}) \to H^1(X \times U, \mathcal{O}_{X\times U}^{\times}) \to H^2(X \times V, \pi^* I) = 0,$$

where we use that X is a scheme of dimension 1, and V is affine, to conclude that the second cohomology group above is zero. Therefore, we see that every line bundle on $X \times U$ can be extended to a line bundle on $X \times V$. $\square$

This result (and in a way also its proof) also hold for Bun_n, and even Bun_G. One can show by similar means that the deformation theory of bundles on a curve is always unobstructed. The exact sequence of abelian groups, used in the proof for $\mathsf{Pic}(X)$ has to be replaced by an exact sequence of pointed, non-abelian cohomology sets

$$H^1(X \times V, G(\mathcal{O}_{X\times V})) \to H^1(X \times U, G(\mathcal{O}_{X\times U})) \to H^2(X \times V, \mathsf{g} \otimes_k \pi^* I) = 0,$$

where g denotes the Lie algebra of G.

7.2 The cotangent complex

7.2.1 Dual numbers and the tangent space

This paragraph should be labelled as being very heuristic. We focus on conveying ideas and won't be bothered with making everything completely rigorous, although it certainly could be done.

For a field k, we define the algebra of dual numbers $k[\epsilon]$ to be the k-algebra $k[t]/(t^2)$. We have seen in Lemma 4.23 that for a variety X, the set $X(k[\epsilon])$ corresponds to the the disjoint union of tangent spaces T_xX at the k-points $x \in X(k)$. The existence of this vector space structure can be explained in purely formal terms.

Lemma 7.8. *Let $k[\epsilon_1, \epsilon_2]$ be the k-algebra $k[t_1, t_2]/(t_1^2, t_2^2, t_1t_2)$. We have a natural k-algebra homomorphism*

$$+^\sharp \colon k[\epsilon] \to k[\epsilon_1, \epsilon_2],$$

characterised by $\epsilon \mapsto \epsilon_1 + \epsilon_2$, and for each $\lambda \in k$ a map $m_\lambda^\sharp \colon k[\epsilon] \to k[\epsilon]$, sending ϵ to $\lambda\epsilon$. With respect to the identification of the fibre $X(k[\epsilon]) \times_{X(k)} \{x\}$ with the tangent space T_xX of Lemma 4.23, addition and multiplication by $\lambda \in k$ are given by the maps

$$X(k[\epsilon_1]) \times_{X(k)} X(k[\epsilon_2]) \xrightarrow{\simeq} X(k[\epsilon_1, \epsilon_2]) \xrightarrow{+^\sharp} X(k[\epsilon]),$$

and $X(k[\epsilon]) \xrightarrow{m_\lambda^\sharp} X(k[\epsilon])$.

The proof of this lemma is an easy exercise, provided that, we clarify the isomorphism $X(k[\epsilon_1]) \times_{X(k)} X(k[\epsilon_2]) \xrightarrow{\simeq} X(k[\epsilon_1, \epsilon_2])$. This amounts to the assertion that

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] \arrow[d] & \operatorname{Spec} k[\epsilon_2] \arrow[d] \\
\operatorname{Spec} k[\epsilon_1] \arrow[r] & \operatorname{Spec} k[\epsilon_1, \epsilon_2]
\end{tikzcd}$$

is a pushout diagram. More concretely, we have to identify the set $X(k[\epsilon_1, \epsilon_2])$ with the set of pairs of tangent vectors, based at the same k-point $X(k[\epsilon]) \times_{X(k)} X(k[\epsilon])$.

Indeed, choosing a presentation of X as affine variety $\operatorname{Spec} \mathbb{Z}[t_1, \dots, t_n]/(f_1, \dots, f_m)$, as in Lemma 4.23, we see by the same means as in the proof of Lemma 4.23 that a ring homomorphism $\mathbb{Z}[t_1, \dots, t_n]/(f_1, \dots, f_m) \to k[\epsilon_1, \epsilon_2]$, corresponds to the choice of two elements in the kernel of the Jacobi matrix $\left(\frac{\partial f_i}{\partial t_j}\right)(\mathrm{x})$, evaluated at the corresponding k-point.

Remark 7.9. *The object* $\operatorname{Spec} k[\epsilon] \to \operatorname{Spec} k$ *is a* co-k-vector space *object in the category of morphisms in* $\mathsf{Sh}(\mathsf{Aff})$*. In particular, for every* $F \in \mathsf{Sh}(\mathsf{Aff})$*, and fixed morphism* $\operatorname{Spec} k \to F$*, the set of commutative diagrams*

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & F \\ \operatorname{Spec} k[\epsilon] \arrow[ur] & \end{tikzcd}$$

is naturally endowed with the structure of a vector space.

At least from the point of view of this course it is natural to ask what happens if we allow F to be a stack. Evidently, the collection of commutative diagrams as above forms a groupoid in this case. The structure on $k[\epsilon]$ induces on this groupoid a "vector space structure". The right notion in this case is given by *strict Picard groupoids.*

Definition 7.10. *A k-vector space object* P *in the strict* 1*-category (obtained by discarding* 2*-morphisms) of groupoids is called a* strict Picard groupoid.

For the sake of clarity we unravel the definition of a Picard groupoid a bit more. We have an addition map

$$+\colon P \times P \to P,$$

and for each $\lambda \in k$ a multiplication by λ map

$$m_\lambda\colon P \to P,$$

such that the following diagrams are strictly commutative:

$$\begin{tikzcd} P \times P \times P \arrow[r, "(+,\mathrm{id}_P)"] \arrow[d, "(\mathrm{id}_P,+)"'] & P \times P \arrow[d, "+"] \\ P \times P \arrow[r, "+"] & P, \end{tikzcd}$$

encoding associativity,

$$\begin{tikzcd} P \times P \arrow[r, "(p_2,p_1)"] \arrow[dr, "+"'] & P \times P \arrow[d, "+"] \\ & P, \end{tikzcd}$$

corresponding to (strict) commutativity,

$$\begin{tikzcd} P \arrow[r, "(m_\lambda,m_\mu)"] \arrow[dr, "m_{\lambda+\mu}"'] & P \times P \arrow[d, "+"] \\ & P, \end{tikzcd}$$

as well as

$$\begin{tikzcd}
P \times P \arrow[r, "(m_\lambda, m_\lambda)"] \arrow[d, "\mathrm{id}_{P\times P}"'] & P \times P \arrow[d, "+"] \\
P \arrow[r, "m_\lambda"] & P,
\end{tikzcd}$$

tantamount to distributivity, and we also need the condition $m_1 = \mathrm{id}_P$; moreover we require an object $\mathbf{0} \in P$, such that $x \mapsto x + \mathbf{0}$ is the identity functor.

Example 7.11. *Let $C^\bullet = [V_{-1} \xrightarrow{d} V_0]$ be a length 2 cochain complex of vector spaces. There exists a canonical Picard groupoid $P_{C^\bullet}$, with set of objects V_0, and $\mathsf{Hom}_{P_{C^\bullet}}(x, y)$ being equivalent to the set of $v \in V_{-1}$, such that $y = x + f(v)$.*

This is the linear analogue of the quotient groupoid construction of Example 3.13. Indeed, we may think of the vector space V_{-1} as acting on the vector space V_0, by means of the linear map $f : V_{-1} \to V_0$.

Lemma 7.12. *Every strict Picard groupoid P is equivalent to $P_{C^\bullet}$, for some cochain complex $C^\bullet$. A morphism of complexes $C^\bullet \xrightarrow{f} D^\bullet$ induces an equivalence of groupoids $P_{C^\bullet} \to P_{D^\bullet}$, if and only if f is a quasi-isomorphism, i.e., if $H^i(f)$ is an isomorphism for all i.*

Proof. The strict nature of the definition of a strict Picard groupoid, implies that the set of objects $\mathbf{Obj}\, P = V_0$ is endowed with the structure of a vector space. Similarly, one checks that $V_{-1} = \mathrm{Aut}_P(\mathbf{0})$ inherits the structure of a vector space. Let $C^\bullet = [V_{-1} \xrightarrow{0} V_0]$, we then have a natural functor $P_{C^\bullet} \to P$, which is the identity on objects. It is essentially surjective by construction, and easily shown to be fully faithful, thus it is an equivalence of groupoids. This proves the first assertion.

Similarly, given a cochain complex $C^\bullet = [V_{-1} \xrightarrow{f} V_0]$, we easily see by definition by $\mathrm{Aut}_P(\mathbf{0}) = \ker f = H^{-1}(C^\bullet)$, while the set of isomorphism classes is given by $H^0(C^\bullet) = \operatorname{coker} f$. This shows that $P_{C^\bullet} \to D^\bullet$ is fully faithful and essentially surjective (i.e., an equivalence) if and only if it induces an isomorphism on H^{-1} and H^0. Since all other cohomology groups of a length 2 cochain complex, concentrated in degrees $[-1, 0]$, are zero, this is the case if and only if $C^\bullet \to D^\bullet$ is a quasi-isomorphism. $\square$

One way of interpreting this lemma is as saying that the 2-category of Picard groupoids is equivalent to the 2-category of cochain complexes, supported in degrees $[-1, 0]$, localised in a 2-categorical sense at quasi-isomorphisms. Recall that localising in the 1-categorical sense truncates the crucial structures given by 2-morphisms in the 2-category of Picard groupoids. This is one of the many reasons why the derived category is sometimes too crude for applications which is sensitive to this higher data.

The nomenclature suggests that there is also a notion of Picard groupoids, without any strictness assumptions. We will not go into the details of their definition, which only requires the axioms of an abelian group to hold up to a coherent system of invertible 2-morphisms. However, we remark that Lemma 7.12 can be generalised to the non-strict setting. The place of cochain complexes concentrated in degrees $[-1, 0]$ is in this case taken by spectra E with homotopy groups $\pi_i(E)$ concentrated in degrees $[0, 1]$. The ostensible difference in the support of the degrees corresponds to the fact that spectra are really analogous to chain complexes, and not cochain complexes, and is therefore just a consequence of the distinction between homological and cohomological gradings.

Definition 7.13. *Let F be an algebraic stack, and $\operatorname{Spec} k \xrightarrow{x} F$ a k-point. The cochain complex (well-defined up to quasi-isomorphism) corresponding to the Picard groupoid of commutative diagrams*

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & F \\ \operatorname{Spec} k[\epsilon] \arrow[ur] & \end{tikzcd}$$

is called the tangent complex *of F at x, and will be denoted by $(T_x X)^\bullet$.*

We would like to construct a sheafy version of the tangent complex, i.e. a complex of sheaves on F, which could be understood as the natural analogue of the sheaf of tangent vectors on a variety. We have not yet discussed the how to define quasi-coherent sheaves on algebraic stacks. We will therefore return to this interesting question at a later point.

Example 7.14. *Let G be a smooth affine group k-scheme. We denote by BG the classifying stacks of Definition 5.9. We denote the Lie algebra of G by g. Let X be a smooth scheme with an action by G.*

(a) For the k-point $\operatorname{Spec} k \xrightarrow{x} BG$ given by the standard atlas of BG, we have that $T_x^\bullet BG$ is quasi-isomorphic to the complex $[\mathsf{g} \to 0]$.

(b) More generally, if $x\colon \operatorname{Spec} k \to X$ is a k-point of X, then we have the infinitesimal action $\mathsf{g} \to T_x X$, given by deriving the action of G, at $x \in X$. The complex $T_x^\bullet[X/G]$ is quasi-isomorphic to $[\mathsf{g} \to T_x X]$.

Proof. Exercise.[34] □

The description of the tangent complex $T_x^\bullet BG$ as $[\mathsf{g} \to 0]$, allows us to take a glimpse at the exciting and recent area of shifted symplectic structures [PTVV11].

Remark 7.15. *Let G be a reductive group scheme, k of characteristic zero, for example $G = \mathrm{GL}_n$. There exists a non-degenerate symmetric pairing $b\colon \mathsf{g} \times \mathsf{g} \to k$, which induces an equivalence $\mathsf{g}^\vee \cong \mathsf{g}$. In particular, we obtain an equivalence*

$$[\mathsf{g} \to 0]^\vee[-2] \cong [\mathsf{g} \to 0].$$

The dual of the tangent complex at x should be the cotangent at x. This equivalence between tangent complex and cotangent complex is actually the shadow of a much finer structure, which is a so-called 2-shifted symplectic structure *on BG. See [PTVV11, p. 26] for details.*

7.2.2 Kähler differentials and basic deformation theory

In this section we attempt to obtain a more conceptual understanding of formal smoothness for a morphism of affine schemes. We will use this to obtain a cue for how the case of algebraic stacks should be treated.

Let $A \twoheadrightarrow B$ be a square-zero extension, corresponding to the ideal $I \in A$. Since we have $I^2 = 0$, and $B \cong A/I$, there is a canonical B-module structure on I. Vice versa, given a B-module M, we

[34] Volunteers?

have a square-zero extension $B[M] \twoheadrightarrow B$, where $B[M] = B \oplus M$, where the ring structure on $B[M]$ is defined as $(b, m) \cdot (b', m') = (bb', b'm + bm')$. Not every square-zero extension is of this shape, consider for instance $\mathbb{Z}/p^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$. The following lemma amounts to an easy computation.

Lemma 7.16. *Let $R \to S$ be a ring homomorphism. Two dashed arrows as in the commutative diagram*

$$\begin{tikzcd} B & S \arrow[l] \arrow[dl, dashed] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

differ by an R-linear derivation $S \to M = \ker(A \twoheadrightarrow B)$.

There is an S-module $\Omega^1_{S/R}$, receiving the universal R-linear derivation. I.e., there is a derivation

$$d \colon S \to \Omega^1_X,$$

which is R-linear, and satisfies that for every R-linear derivation $S \to M$, there exists a unique map of S-modules $\Omega^1_X \to M$, such that the diagram

$$\begin{tikzcd} S \arrow[r, "d"] \arrow[dr] & \Omega^1_X \arrow[d] \\ & M \end{tikzcd}$$

commutes. We call Ω^1_X the sheaf of *Kähler differentials*. It is an easy exercise to construct the S-module Ω^1_X as a quotient of S, by dividing by the submodule constructed by imposing the conditions of an R-linear derivation.

Lemma 7.16 can therefore be restated as saying that the set of these dashed arrows (if non-empty) is a torsor over $\mathsf{Hom}_B(\Omega^1_X \otimes_S B, M)$. Hence, it implies directly that $\operatorname{Spec} S \to \operatorname{Spec} R$ is *formally unramified*, if and only if $\mathsf{Hom}(\Omega^1_{S/R} \otimes_S B, M) = 0$ for all B-modules M. Particularly, choosing $B = S$, we see that this implies $\Omega^1_{S/R} = 0$ by Yoneda's lemma.

This connects to our previous discussion of tangent spaces in the following way. The space of dashed arrows

$$\begin{tikzcd} k & S \arrow[l] \arrow[dl, dashed] \\ k[\epsilon] \arrow[u] & k, \arrow[l] \arrow[u] \end{tikzcd}$$

which corresponds to the fibre of $(\operatorname{Spec} S)(k[\epsilon]) \to (\operatorname{Spec} S)(k)$, at a chosen k-point $x \in (\operatorname{Spec} S)(k)$. By the discussion above we see that it is also equivalent to $\mathsf{Hom}_k(\Omega_{S/k} \otimes_S k, k) = (\Omega^1_{S/k})^\vee$, which agrees with the expected picture that $\Omega_{S/k} \otimes_S k$ is the fibre of the cotangent space at x, and its k-linear dual is the tangent space at x.

7.2.3 The cotangent sheaf for smooth Deligne-Mumford stacks

In this subsection we fix a smooth Deligne-Mumford R-stack F, and define for every affine scheme mapping into it $\operatorname{Spec} B = U \xrightarrow{f} F$, a module, formally denoted by $f^*\Omega^1_{F/R}$. The constructed module will have the property that the set of dashed arrows in

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & F \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} R \end{tikzcd}$$

is equivalent to $\mathsf{Hom}_B(f^*\Omega_{F/R}, M)$, where $M = \ker(A \twoheadrightarrow B)$.

At this point it is not yet possible for us to define of $\Omega^1_{F/R}$ as a quasi-coherent sheaf on the Deligne-Mumford stack F itself, but we will see in the last section that the above collection of pullbacks $f^*\Omega^1_{F/R}$ amounts to exactly such an object.[35]

We choose an atlas $p\colon X \to F$, such that p is an étale representable morphism, and X is a scheme, given by a disjoint union of smooth affine R-schemes.

We assume that there exists an extension to a commutative diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \operatorname{Spec} B \arrow[ur, "g"] \arrow[r, "f"] \arrow[d] & F \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dash dot] & \operatorname{Spec} R, \end{tikzcd}$$

which is étale locally on $\operatorname{Spec} B$ always the case. Since p is étale, we have a bijection between dashed and dotted arrows in the above diagram. In particular, we may define $f^*\Omega^1_{F/R}$ in this case as $g^*\Omega^1_{X/R}$. One can then show that the latter is independent of the choice of g, and faithfully flat descent theory yields a well-defined sheaf $f^*\Omega^1_{F/R}$.

7.2.4 Towards the cotangent complex for smooth algebraic stacks

In this last paragraph we remark on the ingredients of defining the cotangent complex for smooth stacks. An excellent source to learn about are the notes of Sam Raskin in the aforementioned seminar by Gaitsgory[36].

Deligne-Mumford stacks only have zero-dimensional stabilizer groups. Therefore we don't expect the cotangent sheaf to exhibit stacky phenomena. This is also visible from Example 7.14, where we have seen that the tangent complex of BG was given by $[\mathfrak{g} \to 0]$. Hence, a zero-dimensional smooth group G will yield a zero tangent complex.

As in Example 7.14, we expect the cotangent complex to be a length 2 complex, concentrated in degrees 0 and 1. For every atlas $p\colon U \to F$ (hence p is a smooth map) of a *smooth algebraic stack* F, we will construct a complex $p^*L^\bullet_{F/R}$, called the *pullback of the cotangent complex*. As in the previous section, one could use a descent construction, to construct $f^*L^\bullet_{F/R}$ for every map $U \to F$, where U is affine. However, this is a finical procedure. The complexes will only be independent

[35]In line with the theme of this course this hints at the fact that we will resolve the problem of not having an obvious notion of quasi-coherent sheaves on a stack, by turning the question into its answer. The above picture will then turn into a complete tautology.

[36]`http://www.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Sept22(Dmodstack1).pdf`

of the resulting choices up to quasi-isomorphism. In general it is impossible to glue objects in the derived category of chain complexes, defined by crude localisation. Instead one needs to work with a higher categorical enhancement. For length 2 complexes it suffices to work with 2-categories, in fact the 2-category of Picard groupoids does the trick. Just like the glueing of vector bundles involves checking cocycle conditions on triple fibre products $X \times_Y X \times_Y X$, glueing Picard groupoids makes use of higher cocycle conditions on $X \times_Y X \times_Y X \times_Y X$.

Definition 7.17. *(a) For an atlas $p: X \to F$ we define a quasi-coherent sheaf $\Omega^1_{X/F}$ on X as follows. Let $\Omega^1_{X \times_F X/X}$ be the cokernel of the natural map $p'^*\Omega^1_X \to \Omega^1_{X \times_F X}$. One could then show that this object descends along $X \times_F X \to X$, but there's a cheap trick to avoid this verification: the diagonal $\Delta_{X/F}: X \to X \times_F X$ is a section of the natural projection, we simply define $\Omega^1_{X/F}$ as $\Delta^*_{X/F}\Omega^1_{X \times_F X/X}$.*

(b) We define a length 2 complex (supported in degrees $[0,1]$) on X as follows

$$p^*L_{F/R} = [\Omega^1_X \to \Omega^1_{X/F}].$$

The motivation behind this definition is that for F a scheme, the above complex is quasi-isomorphic to $\ker(\Omega^1_X \to \Omega^1_{X/F}) = \Omega^1_F$.

The following statement is an interesting exercise.[37]

Lemma 7.18. *For a commutative diagram of atlases*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & F \end{tikzcd}$$

we have a quasi-isomorphism $f^*q^*L_{F/R} \simeq p^*L_{F/R}$.

8 Quasi-coherent sheaves

8.1 Quasi-coherent sheaves on prestacks

We have already seen the cotangent sheaf Ω^1_X of a Deligne-Mumford stack F as a natural example of a quasi-coherent sheaf on a stack. Since we were lacking the concept of a sheaf on a stack we couldn't define Ω^1_F itself, but only the pullbacks $f^*\Omega^1_X$ for all morphisms $f: U \to F$, where U is an affine scheme. In this section we introduce quasi-coherent sheaves on an algebraic stack F as a compatible system of R-modules M, for each $\operatorname{Spec} R \to F$. Hence, in retrospect we obtain a well-defined cotangent sheaf Ω^1_F for any Deligne-Mumford stack F. This definition makes sense for an arbitrary prestack F, and this extra generality turns out to be more than just an exercise in abstraction, as demonstrated by the fact that the category quasi-coherent sheaves on so-called de Rham spaces, which are not algebraic, are equivalent to categories of D-modules on varieties.[38]

[37] Volunteers?

[38] See http://www.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Nov17-19(Crystals).pdf for a proof of this statement.

8.1.1 A stacky definition of quasi-coherent sheaves

A quasi-coherent sheaf on an affine scheme $\operatorname{Spec} R$ is an R-module. In Definition 3.23 we introduced a stack $\widetilde{\mathsf{Mod}}\colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Gpd}$, which sends a ring R to a groupoid equivalent to the groupoid of R-modules. In particular, we have that the groupoid of R-modules is up to equivalence given by the groupoid of morphisms $\operatorname{Spec} R \to \widetilde{\mathsf{Mod}}$.

Definition 8.1 (Provisional definition)**.** *Let F be a prestack, we define a quasi-coherent sheaf on F to be a morphism $F \to \widetilde{\mathsf{Mod}}$. The groupoid of such morphisms will be denoted by $\mathsf{QCoh}(F)^{\times}$.*

By virtue of the Yoneda Lemma a quasi-coherent sheaf on an affine scheme $U \cong \operatorname{Spec} R$ corresponds indeed to an R-module. How should we imagine a quasi-coherent sheaf on a general prestack F? The following lemma gives a tautological, but useful answer to this question.

Lemma 8.2. *If $F \in \mathsf{PrSt}(\mathsf{Aff})$ is equivalent to a colimit of affine schemes $\operatorname{colim}_{i \in I} \operatorname{Spec} R_i$, where I is a category indexing the colimit, then*

$$\mathsf{QCoh}(F)^{\times} \cong \lim_{i \in I} \widetilde{\mathit{Mod}}(R_i).$$

In particular, a quasi-coherent sheaf on F consists of a collection of R_i-modules M_i for each $i \in I$, as well as isomorphisms

$$\phi_{ij}\colon M_j \xrightarrow{\simeq} M_i \otimes_{R_i} R_j,$$

for each morphism $j \to i$ in I, such that we have that $\phi_{i \xrightarrow{\mathrm{id}} i}\colon M_i \xrightarrow{\simeq} M_i \otimes_{R_i} R_i$ is the canonical isomorphism, and for each commuting triangle

$$\begin{tikzcd} k \arrow[r] \arrow[dr] & j \arrow[d] \\ & i. \end{tikzcd}$$

a cocycle identity expressed by the commutative square

$$\begin{tikzcd} M_j \otimes_{R_j} R_k \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & (M_i \otimes_{R_i} R_j) \otimes_{R_j} R_k \arrow[d, "\simeq"] \\ M_k \arrow[r, "\simeq"] & M_i \otimes_{R_i} R_k. \end{tikzcd}$$

Proof. The first assertion follows from the universal property of colimits. For any prestack F we have that

$$\mathsf{Hom}_{\mathsf{PrSt}(\mathsf{Aff})}(\operatorname*{colim}_{i \in I} U_i, F) \cong \lim_{i \in I} \mathsf{Hom}_{\mathsf{PrSt}(\mathsf{Aff})}(U_i, F).$$

The second assertion is a restatement of the explicit model for limits in 2-categories, which we discussed in ??. $\square$

As a direct consequence, one obtains the following statement.

Corollary 8.3. *Let X be a scheme, and $(\operatorname{Spec} R_i)_{i\in I}$ a Zariski open covering of X be affine schemes, for each tuple of indices $(i_0, \ldots, i_{k-1})$, we let $(\operatorname{Spec} R_{i_0,\ldots,i_{k-1},i})_{i\in I_{i_0,\ldots,i_{k-1}}}$ be a Zariski open covering of affine schemes of the intersection $\operatorname{Spec} R_{i_0} \cap \cdots \cap \operatorname{Spec} R_{k-1}$. Then the datum of a quasi-coherent sheaf on X is equivalent to giving a compatible system of $R_{i_0\ldots i_k}$-modules for each tuple of indices.*

We demonstrate the computational value of this principle with an example of a prestack which we will take up again in Paragraph 8.2, in a more general context. At first we have to introduce some notation.

Definition 8.4. *An abstract group G induces a group-valued sheaf $\coprod_{g\in G} \operatorname{Spec} R$ on Aff_R, which we denote by the same letter. We denote by BG^{triv} the prestack which assigns to every $U \in \mathsf{Aff}_R$ the full subgroupoid of* trivial *G-torsors of $BG(U)$.*

It turns out that the natural map $BG^{triv} \to BG$ realises BG as the *stackification* of BG^{triv}, as we will discuss in Paragraph 8.2. We will now use Lemma 8.2 to compute $\mathsf{QCoh}(BG^{triv})$.

Lemma 8.5. *For an abstract group G, and a ring R we denote by $BG^{triv}\colon \mathsf{Aff}_R^{\mathrm{op}} \to \mathsf{Gpd}$ the prestack of Definition 8.4. Let $\mathsf{Rep}_G(R)$ be the category of G-representations on R-modules. Then we have an equivalence $\mathsf{QCoh}(BG^{triv})^\times \cong \mathsf{Rep}_G(R)^\times$.*

We will deduce this from the following lemma.

Lemma 8.6. *Consider the category $I = [\bullet/G]$. We claim that*

$$BG^{triv} \cong \operatorname*{colim}_I \operatorname{Spec} R,$$

where each $g \in G$ is sent to the identity $\mathrm{id}_{\operatorname{Spec} R}$.

Proof. As a groupoid we have that $BG^{triv}(\operatorname{Spec} S) = [(\operatorname{Spec} R)(S)/G]$, where G acts trivially. We have to show that BG^{triv} satisfies the universal property of the colimit for the I-indexed constant system $(\operatorname{Spec} R)_{i\in I}$. Indeed, if $(F)_{i\in I}$ is another constant I-indexed system of prestacks, and $(\operatorname{Spec} R)_{i\in I} \xrightarrow{(f_i)_{i\in I}} (F)_{i\in I}$ is a natural transformation between them, this amounts to choosing $x \in F(R)$, as well as an isomorphism $\psi_g \in \operatorname{Aut}_{F(R)}(x)$ for every $g \in G$, such that $\psi_{gh} = \psi_g\psi_h$. This defines a natural map $[\operatorname{Spec} R(S)/G] \to F(S)$, by composing a given $\operatorname{Spec} S \to \operatorname{Spec} R$ with $x\colon \operatorname{Spec} R \to F$, and sending the isomorphism of $[\operatorname{Spec} R(S)/G]$ corresponding to $g \in G$, to the automorphism of the composition $\operatorname{Spec} S \to \operatorname{Spec} R \to F$, induced by ψ_g. ☐

Proof of Lemma 8.5. We have seen in Lemma 8.6 that BG^{triv} can be written as a colimit $\operatorname{colim}_I \operatorname{Spec} R$. According to Lemma 8.2 this implies that $\mathsf{QCoh}(BG^{triv})^\times$ is equivalent to the category of pairs $(M, (\phi_g)_{g\in G})$, where M is an R-module, and for each $g \in G$ we have an isomorphism $\phi_g\colon M \xrightarrow{\simeq} M \otimes_R R$, such that $\phi_1 = \mathrm{id}_M$, and for each identity $gh = k$ we have a commutative square

$$\begin{tikzcd}
M \otimes_R R \arrow[r, "h"] \arrow[d, "g^{-1}"'] & (M \otimes_R R) \otimes_R R \arrow[d] \\
M \arrow[r, "k"] & M \otimes_R R.
\end{tikzcd}$$

Composing the isomorphism corresponding to g with the canonical equivalence of R-modules given by $M \otimes_R R \xrightarrow{\simeq} M$, we obtain a G-action on M. ☐

Every prestack can be expressed as a colimit in a tautological manner.

Lemma 8.7. *For a prestack $F\colon \mathsf{Aff} \to \mathsf{Gpd}$ we denote by $\mathsf{Aff}\,/F$ the category of morphisms $U \to F$, where U is an affine scheme. The Yoneda Lemma, and the universal property of colimits yields a canonical map*

$$\operatorname*{colim}_{U \in \mathsf{Aff}\,/F} U \to F,$$

which is an equivalence of prestacks.

Proof. Exercise.[39] □

Hence we obtain the following corollary of Lemma 8.2, which will be the model for the definition of category of quasi-coherent sheaves on a prestack.

Corollary 8.8. *The groupoid of quasi-coherent sheaves on a prestack F is equivalent to*

$$\mathsf{QCoh}(F)^{\times} \cong \lim_{\operatorname{Spec} R \in (\mathsf{Aff}\,/F)^{\mathrm{op}}} \widetilde{\mathit{Mod}(R)}.$$

8.1.2 The category of quasi-coherent sheaves as a limit

The downside of the treatment of the last section is that it only describes quasi-coherent sheaves, and isomorphisms between quasi-coherent sheaves. In order to include general morphisms, we give the following definition.

Definition 8.9. *The category of quasi-coherent sheaves $\mathsf{QCoh}(F)$ is defined to be the limit*

$$\lim_{\operatorname{Spec} R \in \mathsf{Aff}\,/F} \mathit{Mod}(R),$$

in the 2-category of categories (here we employ the stricitfication procedure of Definition ??, to get a well-defined limit).

Using the explicit model for limits in 2-categories we see that the datum of a quasi-coherent sheaf on F amounts to an R_i-module M_i for every morphism $\operatorname{Spec} R_i \to F$, such that for a diagram

$$\begin{tikzcd} \operatorname{Spec} R_1 \arrow[r] \arrow[dr] & \operatorname{Spec} R_0 \arrow[d] \\ & F, \end{tikzcd}$$

we have an equivalence $\phi_{01}\colon M_0 \otimes_{R_0} R_1 \cong M_1$, such that for $\mathrm{id}_{R_0}\colon R_0 \to R_0$ the isomorphism ϕ_{00} is the tautological one, and for a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R_2 \arrow[r] \arrow[dr] & \operatorname{Spec} R_1 \arrow[r] \arrow[d] & \operatorname{Spec} R_0 \arrow[dl] \\ & F & \end{tikzcd},$$

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we have a commutative square

$$\begin{tikzcd}
M_1 \otimes_{R_1} R_2 \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & (M_0 \otimes_{R_0} R_1) \otimes_{R_1} R_2 \arrow[d, "\simeq"] \\
M_2 \arrow[r, "\simeq"] & M_0 \otimes_{R_0} R_2.
\end{tikzcd}$$

We want the category of quasi-coherent sheaves on $\operatorname{Spec} R$ to be equivalent to the category of R-modules.

Lemma 8.10. *We have an equivalence of categories* $\mathsf{QCoh}(\operatorname{Spec} R) \cong \mathit{Mod}(R)$.

Proof. The indexing category $(\mathsf{Aff} / \operatorname{Spec} R)$ has a final object $\operatorname{Spec} R \to \operatorname{Spec} R$, hence the limit is equivalent to $\mathsf{QCoh}(\operatorname{Spec} R)$. $\square$

8.1.3 Stackification and quasi-coherent sheaves

Let $(\mathsf{C}, \mathcal{T})$ be a site. A morphism of presheaves $F \to F^\sharp$ is called a sheafification, if it is injective, and surjective in the sense of sheaves, i.e., for every $x \in F^\sharp(U)$ there exists an $\{U_i \to U\}_{i \in I}$, such that $x|_{U_i}$ lies in the image of $F(U_i) \to F^\sharp(U_i)$. If this is the case, one has for an arbitrary sheaf G that $\mathsf{Hom}(F^\sharp, G) \cong \mathsf{Hom}(F, G)$. By virtue of the Yoneda Lemma we have that $F^\sharp$ is unique up to a unique isomorphism. Just like presheaves can be sheafified, stacks can be stackified.

Definition 8.11. *A morphism of prestacks $F \to F^\sharp$ on a site $(\mathsf{C}, \mathcal{T})$ is called a stackification, if $F^\sharp$ is a stack, and for each $U \in \mathsf{C}$ the morphism of groupoids $F(U) \to F^\sharp(U)$ is surjective in the sense of stacks, i.e. for each $x \in F^\sharp(U)$, there exists an $\{U_i \to U\}_{i \in I}$, such that $x|_{U_i}$ lies in the essential image of $F(U_i) \to F^\sharp(U_i)$, and for each pair of objects $x, y \in F(U)$ we have that the induced map of presheaves of isomorphisms on C/U $\underline{\mathit{Hom}}_F(x, y) \to \underline{\mathit{Hom}}_{F^\sharp}(x, y)$ is a sheafification.*

An example of a stackification is given by the map $BG^{triv} \to BG$.

Lemma 8.12. *The map $BG^{triv} \to BG$ is a stackification.*

The stackfication satisfies a universal property.

Lemma 8.13. *If $F \to F^\sharp$ is a stackification then for any stack $G \in \mathsf{St}_{\mathcal{T}}(\mathsf{C})$ we have a canonical equivalence $\mathit{Hom}(F^\sharp, G) \cong \mathit{Hom}(F, G)$.*

In particular we see from the Yoneda Lemma that a stackification is unique in a 2-categorical manner. Since we have a stack of quasi-coherent sheaves, i.e. $\widetilde{\mathsf{Mod}}$, we see directly that the groupoid of quasi-coherent sheaves is not affected by stackification.

Corollary 8.14. *If $F \to F^\sharp$ is a stackification then* $\mathsf{QCoh}(F^\sharp)^\times \cong \mathsf{QCoh}(F)^\times$.

Proof. The equivalence $\mathsf{Hom}(F^\sharp, \widetilde{\mathsf{Mod}}) \cong \mathsf{Hom}(F, \widetilde{\mathsf{Mod}})$ implies that $\mathsf{QCoh}(F^\sharp) \cong \mathsf{QCoh}(F)$. $\square$

This suggests that the same statement should be true for categories of quasi-coherent sheaves.

Lemma 8.15. *For a stackification $F \to F^\sharp$ of a prestack F we have an equivalence* $\mathsf{QCoh}(F^\sharp) \cong \mathsf{QCoh}(F)$*, given by pullback along the map $F \to F^\sharp$.*

Proof. Exercise.[40] □

Since the map $BG^{triv} \to BG$ is a stackification, we see obtain an explicit description of the category of quasi-coherent sheaves on the stack BG.

Corollary 8.16. *For an abstract group G, and a ring R we denote by $BG\colon \mathsf{Aff}_R^{\mathrm{op}} \to \mathsf{Gpd}$ the prestack of Definition 5.9. Let $\mathsf{Rep}_G(R)$ be the category of G-representations on R-modules. Then we have an equivalence $\mathsf{QCoh}(BG)^{\times} \cong \mathsf{Rep}_G(R)^{\times}$.*

8.2 BG and G-representations

We will refine the previous example of a category of quasi-coherent sheaves. Recall from the discussion around Definition 5.4 that for an affine group R-scheme G, the ring of regular functions $\Gamma(G)$ has the structure of a Hopf R-algebra, i.e., we have a comultplication map $m^\sharp\colon \Gamma(G) \to \Gamma(G) \otimes_R \Gamma(G)$ satisfying the categorical analogues of the comonoid axioms (also have a co-unit map, and co-inverse map). A regular G-representation is a $\Gamma(G)$-comodule M, i.e. an R-module together with a comultiplication map

$$a^\sharp\colon M \to M \otimes_R \Gamma(G),$$

such that the diagram

$$\begin{tikzcd} M \arrow[r, "a^\sharp"] \arrow[d, "a^\sharp"'] & M \otimes_R \Gamma(G) \arrow[d, "\mathrm{id}_M \otimes m^\sharp"] \\ M \otimes_R \Gamma(G) \arrow[r, "a^\sharp \otimes \mathrm{id}_{\Gamma(G)}"] & M \otimes_R \Gamma(G) \otimes_R \Gamma(G) \end{tikzcd}$$

commutes. There is also the so-called co-unit axiom, corresponding to commutativity of

$$\begin{tikzcd} M \arrow[r, "m^\sharp"] \arrow[dr, "\mathrm{id}_M"'] & M \otimes \Gamma(G) \arrow[d, "e^\sharp"] \\ & M \end{tikzcd}$$

We denote the category of regular G-representations by Rep_G.

Lemma 8.17. *The category $\mathsf{QCoh}(BG)$ is equivalent to the category Rep_G of regular G-representations, i.e., the category of $\Gamma(G)$-comodules.*

We will give a proof in the next paragraph, after having discussed a theorem in category theory.

8.2.1 The Barr-Beck Theorem

We fix a category C, a *comonad* on C is an endofunctor $T\colon \mathsf{C} \to \mathsf{C}$, together with a natural transformation $u^\sharp\colon T \to \mathrm{id}_{\mathsf{C}}$ (co-unit), and $m^\sharp\colon T \to T \circ T$ (comultiplication). Moreover we stipulate

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that the following diagrams commute:

$$\begin{tikzcd}
T \arrow[r, "m^\sharp"] \arrow[d, "m^\sharp"'] & T \circ T \arrow[d, "u^\sharp \circ \mathrm{id}_T"] \\
T \circ T \arrow[r, "\mathrm{id}_T \circ u^\sharp"] & T,
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[r, "m^\sharp"] \arrow[d, "m^\sharp"'] & T \circ T \arrow[d, "m^\sharp \circ \mathrm{id}_T"] \\
T \circ T \arrow[r, "\mathrm{id}_T \circ m^\sharp"] & T \circ T \circ T.
\end{tikzcd}$$

This is equivalent to the following definition.

Definition 8.18. *For a category C we have a monoidal category of endofunctors $\mathsf{Fun}(\mathsf{C}, \mathsf{C})$, with monoidal structure given by composition of functors. A* comonad *in C is a coalgebra object in this category.*

A *comodule* for a comonad consists of an object $M \in \mathsf{C}$, and a map $a^\sharp \colon M \to TM$, such that the diagrams

$$\begin{tikzcd}
M \arrow[r, "a^\sharp"] \arrow[dr, "\mathrm{id}_M"'] & TM \arrow[d, "u^\sharp"] \\
 & M,
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \arrow[r, "a^\sharp"] \arrow[d, "a^\sharp"'] & TM \arrow[d, "m^\sharp"] \\
TM \arrow[r, "a^\sharp"] & T^2M
\end{tikzcd}$$

commute. In the literature, comodules over a comonad T are referred to as T-algebras. We will not apply this nomenclature. Equivalently, we may define comodules for a comonad in categorical jargon as follows.

Definition 8.19. *The category $\mathsf{Fun}(\mathsf{C}, \mathsf{C})$ acts on C (by applying a functor F to an object $X \in \mathsf{C}$). A comodule in C of a co-algebra object T in $\mathsf{Fun}(\mathsf{C}, \mathsf{C})$ (hence a comand) is called a comodule of T. We denote the category of T-comodules by $\mathit{Mod}_T(\mathsf{C})$.*

The definition of comonads and comodules is reminiscent of Hopf algebras and comodules.

Example 8.20. *Let G be an affine group R-scheme, i.e., $\Gamma(G)$ is a Hopf R-algebra. Then the functor $- \otimes_R \Gamma(G) \colon \mathit{Mod}(R) \to \mathit{Mod}(R)$ has the structure of a comonad. We have an equivalence $\mathit{Mod}_{-\otimes_R \Gamma(G)}(\mathit{Mod}(R)) \cong \mathsf{Rep}_G$.*

Coming from the abstract side of things, pairs of adjoint functors are a source for comonads (and the dual concept, so-called *cocomonads*[41]).

[41] Often also abbreviated as monads

Lemma 8.21. *Let $L\colon \mathsf{D} \to \mathsf{C}$ be a functor which has a right adjoint* $\mathrm{R}\colon \mathsf{C} \to \mathsf{D}$*. Then, the composition $LR \in \mathsf{Fun}(\mathsf{C}, \mathsf{C})$ has the structure of a comonad, induced by the co-unit of the adjunction $LR \to \mathrm{id}_{\mathsf{C}}$ (coming from the identity map $R \to R$ by adjunction), and the map $LR = L(\mathrm{id}_{\mathsf{C}})R \to L(RL)R = LRLR$, using the unit $\mathrm{id}_{\mathsf{C}} \to RL$ (derived from the identity map $L \to L$ via the adjunction).*

Proof. Exercise.[42] □

Adjunctions are not only the source of comonads, but also of comodules.

Lemma 8.22. *Let L and R be the functors of Lemma 8.21, then for every $X \in \mathsf{D}$, the image $LX \in \mathsf{C}$ has the structure of an LR-comodule. The map $a^\sharp\colon LX \to LRLX$ is induced by the unit* $\mathrm{id}_{\mathsf{C}} \to RL$*. Hence, we have a natural factorisation*

$$\begin{tikzcd} \mathsf{D} \arrow[r, "L"] \arrow[dr, "\widetilde{L}"'] & \mathsf{C} \\ & \mathit{Mod}_{LR}(\mathsf{C}). \arrow[u] \end{tikzcd}$$

Proof. Exercise.[43] □

The functor $\widetilde{L}\colon \mathsf{D} \to \mathsf{Mod}_{LR}(\mathsf{C})$ can be shown to be an equivalence, if a few technical conditions are satisfied.

Theorem 8.23 (Easy Barr-Beck)**.** *The functor $\widetilde{L}\colon \mathsf{D} \to \mathit{Mod}_{LR}(\mathsf{C})$ is an equivalence if the following conditions hold:*

- *the categories C and D have equalizers,*
- *the functor L preserves equalizers,*
- *the functors L and R are conservative, i.e. a morphism $X \xrightarrow{f} Y$ in D is an isomorphism if and only if $R(f)$ is an isomorphism.*

Proof. Using the assumptions we will define an inverse functor $\widetilde{R}\colon \mathsf{Mod}_T(\mathsf{C}) \to \mathsf{D}$. This relies on an observation, which we will establish in Lemma 8.24 below. Namely that for every $M \in \mathsf{D}$ we have an equalizer diagram

$$M \to RLM \rightrightarrows RLRLM.$$

Taking this statement for granted, we define $\widetilde{R}(M)$ as the equalizer of $RM \rightrightarrows RLRM$. Since L preserves equalizers, and assuming that $M = LN$ we obtain an equalizer diagram

$$\widetilde{R}LN \to RLN \rightrightarrows RLRLN,$$

the universal property of equalizers yields therefore a natural equivalence $\widetilde{R}\widetilde{L}N \cong N$.

Vice versa, we see that $\widetilde{L}\widetilde{R}M$ is equal to the equalizer (since L preserves equalizers)

$$L\widetilde{R}M \to LRM \rightrightarrows LRLRM,$$

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[43]Volunteers?

and it suffices to show that $M \to LRM \rightrightarrows LRLRM$ is also an equalizer diagram. We apply R to this diagram and obtain

$$RM \to RL(RM) \rightrightarrows RLRL(RM),$$

which is an equalizer as mentioned above. Since R is a right adjoint, it preserves limits, hence $M \to LRM \rightrightarrows LRLRM$ is an equalizer diagram too, since we assumed R to be conservative. $\square$

It remains to prove the lemma which was used in the proof of the Barr-Beck Theorem.

Lemma 8.24. *If L and R are functors satisfying the same assumptions as in Theorem 8.23, then we have for every $M \in \mathsf{D}$ that*

$$M \to RLM \rightrightarrows RLRLM$$

is an equalizer diagram.

Proof. Let us denote the equalizer of $RLM \rightrightarrows RLRLM$ by M'. The universal property of equalizers yields a canonical map $M \to M'$, we will verify that it is an isomorphism. In fact, we will show that $LM \to LM'$ is an isomorphism. Since L is conservative, this is sufficient to deduce the assertion.

Applying the functor L to the above equalizer diagram we obtain an equalizer diagram

$$LM' \to LRLM \rightrightarrows LRLRLM,$$

since L preserves limits. We claim that

$$LM \to LRLM \rightrightarrows LRLRLM$$

is also an equalizer diagram, which then shows that the natural map $LM \to LM'$ is an isomorphism, and hence concludes the proof by the discussion above. To see that this assertion holds, we observe that the unit $LR \to \mathrm{id}_{\mathsf{C}}$ gives rise to a splitting of the fork-shaped diagram

$$\begin{tikzcd} LM \arrow[r, shift right] & LRLM \arrow[l, dotted, shift right] \arrow[r, shift left] \arrow[r, shift right] & LRLRLM. \arrow[l, dotted] \end{tikzcd}$$

However, every split fork is an equalizer diagram, since the splitting can be used to verify the universal property. $\square$

A direct comparison of the proofs of Theorem 8.23 with Theorem 2.25 shows that the Barr-Beck Theorem functions as an abstract descent result. We invite the reader to further contemplate this analogy and to deduce faithfully flat descent from the Barr-Beck Theorem.

8.2.2 Quasi-coherent sheaves on BG: the proof

It remains to prove that $\mathsf{QCoh}(BG) \cong \mathsf{Rep}_G$, for a faithfully flat group R-scheme G. In order to do this, we will construct a right adjoint functor p_* for the pullback $p^*\colon \mathsf{QCoh}(BG) \to \mathsf{QCoh}(\operatorname{Spec} R) \cong \mathsf{Mod}(R)$, where $p\colon \operatorname{Spec} R \to BG$ is the canonical map. We will then show that the conditions of the Barr-Beck Theorem 8.23 are satisfied, and conclude the proof by verifying that the comonad given by the adjunction is equivalent to $- \otimes_R \Gamma(G)$, which we studied in Example 8.20. Since $\mathsf{Mod}_{-\otimes_R \Gamma(G)}(\mathsf{Mod}(R))$, this concludes the proof that $\mathsf{QCoh}(BG) \cong \mathsf{Rep}_G$.

Lemma 8.25. *Let $p\colon F \to G$ be an affine and flat morphism of prestacks, then the functor $p^*\colon \mathsf{QCoh}(G) \to \mathsf{QCoh}(F)$ has a right adjoint $p_*\colon \mathsf{QCoh}(F) \to \mathsf{QCoh}(G)$, which is conservative.*

Sketch. We will deduce this from the base change identity for flat morphisms of affine schemes. At first we construct the functor p_*. For every $M \in \mathsf{QCoh}(F)$, corresponding to a compatible system of pullbacks f^*M, given for every $f\colon V \to F$ we have to define g^*p_*M, for every $g\colon U \to G$. Since $p\colon F \to G$ is an affine morphism of prestacks we can form the fibre product

$$\begin{tikzcd} V \arrow[r] \arrow[d, "p_U"'] & F \arrow[d] \\ U \arrow[r, "g"] & G, \end{tikzcd}$$

and define g^*p_*M as $(p_U)_*f^*M$. For $U' \to U$, we obtain two pullback squares

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "p_{U'}"'] & V \arrow[r] \arrow[d, "p_U"'] & F \arrow[d] \\ U' \arrow[r, "g'"] & U \arrow[r, "g"] & G, \end{tikzcd}$$

hence base change for the first square yields a natural isomorphism

$$(g')^*(p_U)_*(f^*M) \cong (p_{U'})_*(f \circ f')^*M.$$

Since pushforward along maps of affine schemes is a conservative functor (it corresponds to restriction of scalars along a ring homomorphism $R \to S$), we see that p_* is conservative as well. $\square$

The next lemma is only a restatement of the definition of a faithfully flat morphism.

Lemma 8.26. *If $p\colon F \to G$ is a representable and faithfully flat morphism of prestacks, then the functor p^* preserves equalizers and is conservative.*

The only assertion left to check is that p^*p_* is equivalent to $- \otimes_R \Gamma(G)$. The definition of the functor p_* reveals that for every $f\colon U \to \operatorname{Spec} R$, we send f^*M to $\pi_*\pi^*M$, where $\pi\colon \widetilde{U} \to U$ is the canonical projection of the fibre product $U \times_{BG} U \to U$. We have seen however that $\widetilde{U} \cong U \times G$, since a map $U \to BG$, factoring through $\operatorname{Spec} R \to BG$ classifies the trivial torsor on U. Hence, $\pi_*\pi^*M$ is equivalent to $M \otimes \Gamma(G)$.

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