Homotopy Quantum Field Theories meets the Crossed Menagerie:
an introduction to HQFTs and their relationship with things
simplicial and with lots of crossed gadgetry.
Notes prepared for the Workshop and School on Higher Gauge
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Introduction

WARNING: These notes for the mini-course have been constructed from the main body of the much larger Menagerie notes. The method used has been to delete sections that were not more or less necessary for this course. (There will be some loose ends therefore, and missing links. These will be given with a ? as the Latex refers to the original cross reference.)

If you want to follow up some of the ideas that lead out of these notes, just look at the version of the Menagerie available on the nLab, [134], and if that does not have the relevant chapter, just ask me! (Beware, the full present version is already 800 pages in length, so don't print too many copies!!!)

There are several points to make. As in the full Menagerie notes, there are no exercises as such, but at various points if a proof could be expanded, or is **left to the reader**, then, yes, bold face will be used to suggest that that is a useful place for more input from the reader. In lots of places, reading the details is not that efficient a way of getting to grips with the calculations and ideas, and there is no substitute for **doing it yourself**. That being said guidance as to how to approach the subject will often be given.

Almost needless to say, there are things that have not been discussed here (or in the Menagerie itself), and suggestions for additional material are welcome. Better still would be for the suggestions to materialise into new entries on the nLab.

Tim Porter,   Anglesey, 2011.

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Chapter 1

Crossed modules - definitions, examples and applications

We will give these for groups, although there are analogues for many other algebraic settings.

1.1 Crossed modules

Definition: A *crossed module*, (C, G, δ), consists of groups C and G with a left action of G on C, written $(g, c) \to {}^{g}c$ for $g \in G$, $c \in C$, and a group homomorphism $\delta : C \to G$ satisfying the following conditions:

CM1) for all $c \in C$ and $g \in G$,
$$\delta({}^{g}c) = g\delta(c)g^{-1},$$

CM2) for all $c_1, c_2 \in C$,
$$^{\delta(c_2)}c_1 = c_2 c_1 c_2^{-1}.$$

(CM2 is called the *Peiffer identity*.)

If (C, G, δ) and (C', G', δ') are crossed modules, a *morphism*, $(\mu, \eta) : (C, G, \delta) \to (C', G', \delta')$, *of crossed modules* consists of group homomorphisms $\mu : C \to C'$ and $\eta : G \to G'$ such that

(i) $\delta'\mu = \eta\delta$ and (ii) $\mu({}^{g}c) = {}^{\eta(g)}\mu(c)$ for all $c \in C$, $g \in G$.

Crossed modules and their morphisms form a category, of course. It will usually be denoted $CMod$.

There is, for a fixed group G, a subcategory $CMod_G$ of $CMod$, which has, as objects, those crossed modules with G as the "base", i.e., all (C, G, δ) for this fixed G, and having as morphisms from (C, G, δ) to (C', G, δ') just those (μ, η) in $CMod$ in which $\eta : G \to G$ is the identity homomorphism on G.

Several well known situations give rise to crossed modules. The verification will be left to you.

1.1.1 Algebraic examples of crossed modules

(i) Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion, then we will say (H, G, i) is a *normal subgroup pair*. In this case, of course, G acts on the left of H by

conjugation and the inclusion homomorphism i makes (H, G, i) into a crossed module, an 'inclusion crossed modules'. Conversely it is an easy exercise to prove

Lemma 1 *If (C, G, ∂) is a crossed module, ∂C is a normal subgroup of G.* ∎

(ii) Suppose G is a group and M is a left G-module; let $0 : M \to G$ be the trivial map sending everything in M to the identity element of G, then $(M, G, 0)$ is a crossed module.

Again conversely:

Lemma 2 *If (C, G, ∂) is a crossed module, $K = Ker\,\partial$ is central in C and inherits a natural G-module structure from the G-action on C. Moreover, $N = \partial C$ acts trivially on K, so K has a natural G/N-module structure.* ∎

Again the proof is left as an exercise.

As these two examples suggest, general crossed modules lie between the two extremes of normal subgroups and modules, in some sense, just as groupoids lay between equivalence relations and G-sets. Their structure bears a certain resemblance to both - they are "external" normal subgroups, but also are "twisted" modules.

(iii) Let G be a group, then, as usual, let $Aut(G)$, denote the group of automorphisms of G. Conjugation gives a homomorphism

$$\iota : G \to Aut(G).$$

Of course, $Aut(G)$ acts on G in the obvious way and ι is a crossed module. We will need this later so will give it its own name, the *automorphism crossed module of the group, G* and its own notation: $\mathsf{Aut}(G)$.

More generally if L is some type of algebra then $U(L) \to Aut(L)$ will be a crossed module, where $U(L)$ denotes the units of L and the morphism send a unit to the automorphism given by conjugation by it.

This class of example has a very nice property with respect to general crossed modules. For a general crossed module, (C, P, ∂), we have an action of P on C, hence a morphism, $\alpha : P \to Aut(C)$, so that $\alpha(p)(c) = {}^p c$. There is clearly a square

$$
\begin{array}{ccc}
C & \xrightarrow{\ =\ } & C \\
{\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \iota} \\
P & \xrightarrow{\ \alpha\ } & Aut(C)
\end{array}
$$

and we can ask if this gives a morphism of crossed modules. 'Clearly' it should. The requirements are that the square commutes and that the actions are compatible in the obvious sense, (recall page 11). To see that the square commutes, we just note that, given $c \in C$, ∂c acts on an $x \in C$, by conjugation by c: ${}^{\partial c}x = c.x.c^{-1} = \iota(c)(x)$, whilst to check that the actions match correctly remember that $\alpha(p)(c) = {}^p x$ by definition, so we do have a morphism of crossed modules as expected.

(iv) We suppose given a morphism

$$\theta : M \to N$$

of left G-modules and form the semi-direct product $N \rtimes G$. This group we make act on M via the projection from $N \rtimes G$ to G.

We define a morphism

$$\partial : M \to N \rtimes G$$

by $\partial(m) = (\theta(m), 1)$, where 1 denotes the identity element of G, then $(M, N \rtimes G, \partial)$ is a crossed module. In particular, if A and B are Abelian groups, and B is considered to act trivially on A, then any homomorphism, $A \to B$ is a crossed module.

(v) Suppose that we have a crossed module, $\mathsf{C} = (C, G, \delta)$, and a group homomorphism $\varphi : H \to G$, then we can form the 'pullback group' $H \times_G C = \{(h, c) \mid \varphi(h) = \delta c\}$, which is a subgroup of the product $H \times C$. There is a group homomorphism, $\delta' : H \times_G C \to H$, namely the restriction of the first projection morphism of the product, (so $\delta'(h, c) = h$). **You are left** to construct an action of H on this group, $H \times_G C$ such that $\varphi^*(\mathsf{C}) := (H \times_G C, H, \delta')$ is a crossed module, and also such that the pair of maps φ and the second projection $H \times_G C \to C$ give a morphism of crossed modules.

Definition: The crossed module, $\varphi^*(\mathsf{C})$, thus defined, is called the *pullback crossed module of* C *along* φ

(vi) As a last algebraic example for the moment, let

$$1 \to K \xrightarrow{a} E \xrightarrow{b} G \to 1$$

be an extension of groups with K a central subgroup of E, i.e. a *central extension* of G by K. For each $g \in G$, pick an element $s(g) \in b^{-1}(g) \subseteq E$. Define an action of G on E by: if $x \in E$, $g \in G$, then

$$^g x = s(g) x s(g)^{-1}.$$

This is well defined, since if $s(g)$, $s'(g)$ are two choices, $s(g) = k s'(g)$ for some $k \in K$, and K is central. (This also shows that this *is* an action.) The structure (E, G, b) is a crossed module.

A particular important case is: for R a ring, let $E(R)$ be, as before, the group of elementary matrices of R, $E(R) \subseteq Gl(R)$ and $St(R)$, the corresponding Steinberg group with $b : St(R) \to E(R)$, the natural morphism, (see later or [123], for the definition). Then this gives a central extension

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

and thus a crossed module. In fact, more generally,

$$b : St(R) \to Gl(R)$$

is a crossed module. The group $Gl(R)/Im(b)$ is $K_1(R)$, the first algebraic K-group of the ring.

1.1.2   Topological Examples

In topology there are several examples that deserve looking at in detail as they do relate to aspects of the above algebraic cases. They require slightly more topological knowledge than has been assumed so far.

(vii) Let X be a pointed space, with $x_0 \in X$ as its base point, and A a subspace with $x_0 \in A$. Recall that the second relative homotopy group, $\pi_2(X, A, x_0)$, consists of relative homotopy classes of continuous maps

$$f : (I^2, \partial I^2, J) \to (X, A, x_0)$$

where ∂I^2 is the boundary of I^2, the square, $[0, 1] \times [0, 1]$, and $J = \{0, 1\} \times [0, 1] \cup [0, 1] \times \{0\}$. Schematically f maps the square as:
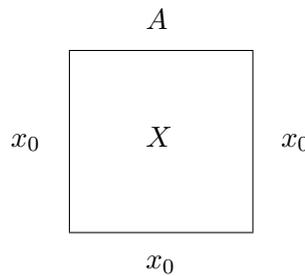
so the top of the boundary goes to A, the rest to x_0 and the whole thing to X. The relative homotopies considered then deform the maps in such a way as to preserve such structure, so intermediate mappings also send J to x_0, etc. Restriction of such an f to the top of the boundary clearly gives a homomorphism

$$\partial : \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

to the fundamental group of A, based at x_0. There is also an action of $\pi_1(A, x_0)$ on $\pi_2(X, A, x_0)$ given by rescaling the 'square' given by
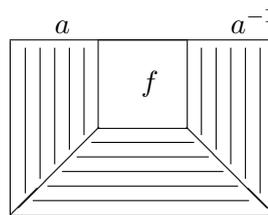
where f is partially 'enveloped' in a region on which the mapping is behaving like a.

Of course, this gives a crossed module

$$\pi_2(X, A, x_0) \to \pi_1(A, x_0).$$

A direct proof is quite easy to give. One can be found in Hilton's book, [94] or in Brown-Higgins-Sivera, [37]. Alternatively one can use the argument in the next example.

(viii) Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence of pointed spaces. Thus p is a fibration, $F = p^{-1}(b_0)$, where b_0 is the basepoint of B. The fibre F is pointed at f_0, say, and f_0 is taken as the basepoint of E as well.

There is an induced map on fundamental groups

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E)$$

and if a is a loop in E based at f_0, and b a loop in F based at f_0, then the composite path corresponding to aba^{-1} is homotopic to one wholly within F. To see this, note that $p(aba^{-1})$ is null homotopic. Pick a homotopy in B between it and the constant map, then lift that homotopy back up to E to one starting at aba^{-1}. This homotopy is the required one and its other end gives a well defined element ${}^a b \in \pi_1(F)$ (abusing notation by confusing paths and their homotopy classes). With this action $(\pi_1(F), \pi(E), \pi_1(i))$ is a crossed module. This will not be proved here, but is not that difficult. Links with previous examples are strong.

If we are in the context of the above example, consider the inclusion map, f of a subspace A into a space X (both pointed at $x_0 \in A \subset X$). Form the corresponding fibration,

$$i^f : M^f \to X,$$

by forming the pullback

$$
\begin{array}{ccc}
M^f & \xrightarrow{\pi^f} & X^I \\
{\scriptstyle j^f}\downarrow & & \downarrow{\scriptstyle e_0} \\
A & \xrightarrow{f} & X
\end{array}
$$

so M^f consists of pairs, (a, λ), where $a \in A$ and λ is a path from $f(a)$ to some point $\lambda(1)$. Set $i^f = e_1 \pi^f$, so $i^f(a, \lambda) = \lambda(1)$. It is standard that i^f is a fibration and its fibre is the subspace $F_h(f) = \{(a, \lambda) \mid \lambda(1) = x_0\}$, often called the *homotopy fibre* of f. The base point of $F_h(f)$ is taken to be the constant path at x_0, (x_0, c_{x_0}).

If we note that

$$\pi_1(F_h(f)) \cong \pi_2(X, A, x_0)$$
$$\pi_1(M^f) \cong \pi_1(A, x_0)$$

(even down to the descriptions of the actions, etc.), the link with the previous example becomes clear, and thus furnishes another proof of the statement there.

(ix) The link between fibrations and crossed modules can also be seen in the category of simplicial groups. A morphism $f : G \to H$ of simplicial groups is a fibration if and only if each f_n is an epimorphism. This means that a fibration is determined by the fibre over the identity which is, of course, the kernel of f. The $(G, \overline{W})$-links between simplicial groups and simplicial sets mean that the analogue of π_1 is π_0. Thus the fibration f corresponds to

$$Ker\, f \xrightarrow{\triangleleft} G$$

and each level of this is a crossed module by our earlier observations. Taking π_0, it is easy to check that

$$\pi_0(Ker\, f) \to \pi_0(G)$$

is a crossed module. In fact any crossed module is isomorphic to one of this form. (Proof left to the reader.)

If $\mathsf{M} = (C, G, \partial)$ is a crossed module, then we sometimes write $\pi_0(\mathsf{M}) := G/\partial C$, $\pi_1(\mathsf{M}) := Ker\, \partial$, and then have a 4-term exact sequence:

$$0 \to \pi_1(\mathsf{M}) \to C \xrightarrow{\partial} G \to \pi_0(\mathsf{M}) \to 1.$$

In topological situations when M provides a model for (part of) the homotopy type of a space X or a pair (X, A), then typically $\pi_1(\mathsf{M}) \cong \pi_2(X)$, $\pi_0(\mathsf{M}) \cong \pi_1(X)$.

MacLane and Whitehead, [117], showed that crossed modules give algebraic models for all homotopy 2-types of connected spaces. We will visit this result in more detail later, but loosely a 2-equivalence between spaces is a continuous map that induces isomorphisms on π_1 and π_2, the first two homotopy groups. Two spaces have the same 2-type if there is a zig-zag of 2-equivalences joining them.

1.1.3   Restriction along a homomorphism φ/ 'Change of base'

Given a crossed module (C, H, ∂) over H and a homomorphism $\varphi : G \to H$, we can form the pullback:

$$
\begin{array}{ccc}
D & \xrightarrow{\psi} & C \\
{\scriptstyle \partial'}\downarrow & & \downarrow{\scriptstyle \partial} \\
G & \xrightarrow[\varphi]{} & H
\end{array}
$$

in $Grps$. Clearly the universal property of pullbacks gives a good universal property for this, namely that any morphism $(\varphi', \varphi) : (C', G, \delta) \to (C, H, \partial)$ factors uniquely through (ψ, φ) and a morphism in $CMod_G$ from (C', G, δ) to (D, G, ∂'). Of course this statement depends on verification that (D, G, ∂') is a crossed module and that the resulting maps are morphisms of crossed modules, but this is routine, and will be **left as an exercise**. (You may need to recall that D can be realised, up to isomorphism, as $G \times_H C = \{(g, c) \mid \varphi(g) = \partial c\}$. It is for you to see what the action is.)

This construction also behaves nicely on morphisms of crossed modules over H and yields a functor,

$$\varphi^* : CMod/H \to CMod_G,$$

which will be called *restriction along φ*.

We next turn to the use of crossed modules in combinatorial group theory.

1.2   Group presentations, identities and 2-syzygies

1.2.1   Presentations and Identities

(cf. Brown-Huebschmann, [38]) We consider a presentation, $\mathcal{P} = (X : R)$, of a group G. The elements of X are called *generators* and those of R *relators*. We then have a short exact sequence,

$$1 \to N \to F \to G \to 1,$$

where $F = F(X)$, the free group on the set X, R is a subset of F and $N = N(R)$ is the normal closure in F of the set R. The group F acts on N by conjugation: ${}^u c = ucu^{-1}$, $c \in N, u \in F$ and the elements of N are words in the conjugates of the elements of R:

$$c = {}^{u_1}(r_1^{\varepsilon_1})\,{}^{u_2}(r_2^{\varepsilon_2}) \ldots {}^{u_n}(r_n^{\varepsilon_n})$$

where each ε_i is $+1$ or -1. One also says such elements are *consequences* of R. Heuristically an *identity among the relations* of $\mathcal{P}$ is such an element c which equals 1. The problem of what this means is analogous to that of working with a relation in R. For example, in the presentation $(a : a^3)$ of C_3, the cyclic group of order 3, if a is thought of as being an element of C_3, then $a^3 = 1$, so why is this different from the situation with the 'presentation', $(a : a = 1)$? To get around that difficulty the free group on the generators $F(X)$ was introduced and, of course, in $F(\{a\})$, a^3 is not 1. A similar device, namely *free crossed modules* on the presentation will be introduced in a moment to handle the identities. Before that consider some examples which indicate that identities exist even in some quite common-or-garden cases.

Example 1: Suppose $r \in R$, but it is a power of some element $s \in F$, i.e. $r = s^m$. Of course, $rs = sr$ and

$$ {}^s r r^{-1} = 1$$

so ${}^s r . r^{-1}$ is an identity. In fact, there will be a unique $z \in F$ with $r = z^q$, q maximal with this property. This z is called the *root of* r and if $q > 1$, r is called a *proper power*.

Example 2: Consider one of the standard presentations of S_3, $(a, b : a^3, b^2, (ab)^2)$. Write $r = a^3$, $s = b^2$, $t = (ab)^2$. Here the presentation leads to F, free of rank 2, but $N(R) \subset F$, so it must be free as well, by the Nielsen-Schreier theorem. Its rank will be 7, given by the Schreier index formula or, geometrically, it will be the fundamental group of the Cayley graph of the presentation. This group is free on generators corresponding to edges outside a maximal tree as in the following diagram:

The Cayley graph of S_3 and a maximal tree in it.
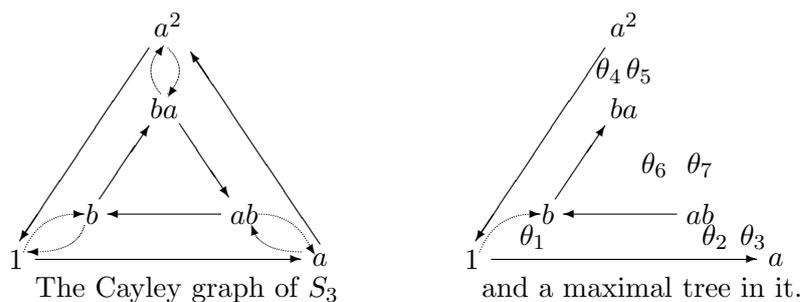
The set of normal generators of $N(R)$ has 3 elements; $N(R)$ is free on 7 elements (corresponding to the edges not in the tree), but is specified as consisting of products of conjugates of r, s and t, and there are infinitely many of these. Clearly there must be some slight redundancy, i.e., there must be some identities among the relations!

A path around the outer triangle corresponds to the relation r; each other region corresponds to a conjugate of one of r, s or t. (It may help in what follows to think of the graph being embedded on a 2-sphere, so 'outer' and 'outside' mean 'round the back face.) Consider a loop around a region.

Pick a path to a start vertex of the loop, starting at 1. For instance the path that leaves 1 and goes along a, b and then goes around aaa before returning by $b^{-1}a^{-1}$ gives $abrb^{-1}a^{-1}$. Now the path around the outside can be written as a product of paths around the inner parts of the graph, e.g. $(abab)b^{-1}a^{-1}b^{-1}(bb)(b^{-1}a^{-1}b^{-1}a^{-1})\ldots$ and so on. Thus r can be written in a non-trivial way as a product of conjugates of r, s and t. (An explicit identity constructed like this is given in [38].)

Example 3: In a presentation of the free Abelian group on 3 generators, one would expect the commutators, $[x,y]$, $[x,z]$ and $[y,z]$. The well-known identity, usually called the Jacobi identity, expands out to give an identity among these relations (again see [38], p.154 or Loday, [111].)

1.2.2   Free crossed modules and identities

The idea that an identity is an equation in conjugates of relations leads one to consider formal conjugates of symbols that label relations. Abstracting this a bit, suppose G is a group and $f : Y \to G$, a function 'labelling' the elements of some subset of G. To form a conjugate, you need a thing being conjugated and an element 'doing' the conjugating, so form pairs (p, y), $p \in G$, $y \in Y$, to be thought of as ${}^p y$, the *formal conjugate* of y by p. Consequences are words in conjugates of relations, *formal consequences* are elements of $F(G \times Y)$. There is a function extending f from $G \times Y$ to G given by

$$\bar{f}(p,y) = pf(y)p^{-1},$$

converting a formal conjugate to an actual one and this extends further to a group homomorphism

$$\varphi : F(G \times Y) \to G$$

defined to be $\bar{f}$ on the generators. The group G acts on the left on $G \times Y$ by multiplication: $p.(p', y) = (pp', y)$. This extends to a group action of G on $F(G \times Y)$. For this action, φ is G-equivariant if G is given its usual G-group structure by conjugations / inner automorphisms. Naively identities are the elements in the kernel of this, but there are some elements in that kernel that are there regardless of the form of function f. In particular, suppose that $g_1, g_2 \in G$ and $y_1, y_2 \in Y$ and look at

$$(g_1, y_1)(g_2, y_2)(g_1, y_1)^{-1}((g_1 f(y_1) g_1^{-1}) g_2, y_2)^{-1}.$$

Such an element is always annihilated by φ. The normal subgroup generated by such elements is called the Peiffer subgroup. We divide out by it to obtain a quotient group. This is the construction of the free crossed module on the function f. If f is, as in our initial motivation, the inclusion of a set of relators into the free group on the generators we call the result the *free crossed module on the presentation* $\mathcal{P}$ and denote it by $C(\mathcal{P})$.

We can now formally define the module of identities of a presentation $\mathcal{P} = (X : R)$. We form the free crossed module on $R \to F(X)$, which we will denote by $\partial : C(\mathcal{P}) \to F(X)$. The *module of identities* of $\mathcal{P}$ is $Ker\,\partial$. By construction, the group presented by $\mathcal{P}$ is $G \cong F(X)/Im\,\partial$, where $Im\,\partial$ is just the normal closure of the set, R, of relations and we know that $Ker\,\partial$ is a G-module. We will usually denote the module of identities by $\pi_{\mathcal{P}}$.

We can get to $C(\mathcal{P})$ in another way. Construct a space from the combinatorial information in $C(\mathcal{P})$ as follows. Take a bunch of circles labelled by the elements of X; call it $K(\mathcal{P})_1$, it is the 1-skeleton of the space we want. We have $\pi_1(K(\mathcal{P})_1) \cong F(X)$. Each relator $r \in R$ is a word

in X so gives us a loop in $K(\mathcal{P})_1$, following around the circles labelled by the various generators making up r. This loop gives a map $S^1 \xrightarrow{f_r} K(\mathcal{P})_1$. For each such r we use f_r to glue a 2-dimensional disc e_r^2 to $K(\mathcal{P})_1$ yielding the space $K(\mathcal{P})$. The crossed module $C(\mathcal{P})$ is isomorphic to $\pi_2(K(\mathcal{P}), K(\mathcal{P})_1) \xrightarrow{\partial} \pi_1(K(\mathcal{P})_1$.

The main problem is how to calculate $\pi_{\mathcal{P}}$ or equivalently $\pi_2(K(\mathcal{P}))$. One approach is via an associated chain complex. This can be viewed as the chains on the universal cover of $K(\mathcal{P})$, but can also be defined purely algebraically, for which see Brown-Huebschmann, [38], or Loday, [111]. That algebraic - homological approach leads to 'homological syzygies'. For the moment we will concentrate on:

1.3 Cohomology, crossed extensions and algebraic 2-types

1.3.1 Cohomology and extensions, continued

Suppose we have any group extension

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

with K Abelian, but not necessarily central. We can look at various possibilities.

If we can split p, by a homomorphism $s : G \to E$, with $ps = Id_G$, then, of course, $E \cong K \rtimes G$ by the isomorphisms,

$$e \longrightarrow (esp(e)^{-1}, p(e)),$$
$$ks(g) \longleftarrow (k, g),$$

which are compatible with the projections etc., so there is an equivalence of extensions

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & K & \longrightarrow & K \rtimes G & \longrightarrow & G & \longrightarrow & 1.
\end{array}
$$

Our convention for multiplication in $K \rtimes G$ will be

$$(k, g)(k', g') = (k^g k', gg').$$

But what if p does not split. We can build a (small) category of extensions $\mathcal{E}xt(G, K)$ with objects such as $\mathcal{E}$ above and in which a morphism from $\mathcal{E}$ to $\mathcal{E}'$ is a diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1.
\end{array}
$$

By the 5-lemma, α will be an isomorphism, so $\mathcal{E}xt(G, K)$ is a groupoid.

In $\mathcal{E}$, the epimorphism p is usually not splittable, but as a function between sets, it is onto so we can pick an element in each $p^{-1}(g)$ to get a transversal (or set of coset representatives), $s : G \to E$. We get a comparison pairing / obstruction map or 'factor set' :

$$f : G \times G \to E$$

$$f(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1},$$

which will be trivial, (i.e., $f(g_1, g_2) = 1$ for all $g_1, g_2 \in G$) exactly if s splits p, i.e., if s is a homomorphism. This construction assumes that we know the multiplication in E, otherwise we cannot form this product! On the other hand given this 'f', we can work out the multiplication. As a set, E will be the product $K \times G$, identified with it by the same formulae as in the split case, noting that $pf(g_1, g_2) = 1$, so 'really' we should think of f as ending up in the subgroup K, then we have

$$(k_1, g_1)(k_2, g_2) = (k_1{}^{s(g_1)}k_2 f(g_1, g_2), g_1g_2).$$

The product is *twisted* by the pairing f. Of course, we need this multiplication to be associative and, to ensure that, f must satisfy a cocycle condition:

$$^{s(g_1)}f(g_2, g_3)f(g_1, g_2g_3) = f(g_1, g_2)f(g_1g_2, g_3).$$

This is a well known formula from group cohomology, more so if written additively:

$$s(g_1)f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0.$$

Here we actually have various parts of the nerve of G involved in the formula. The group G 'is' a small category (groupoid with one object), which we will, for the moment, denote $\mathcal{G}$. The triple $\sigma = (g_1, g_2, g_3)$ is a 3-simplex in $Ner(\mathcal{G})$ and its faces are

$$
\begin{aligned}
d_0\sigma &= (g_2, g_3), \\
d_1\sigma &= (g_1g_2, g_3), \\
d_2\sigma &= (g_1, g_2g_3), \\
d_3\sigma &= (g_1, g_2).
\end{aligned}
$$

This is all very classical. We can use it in the usual way to link $\pi_0(\mathcal{E}xt(G, K))$ with $H^2(G, K)$ and so is the 'modern' version of Schreier's theory of group extensions, at least in the case that K is Abelian.

For a long time there was no obvious way to look at the elements of $H^3(G, K)$ in a similar way. In MacLane's homology book, [114], you can find a discussion from the classical viewpoint. In Brown's [28], the link with crossed modules is sketched although no references for the details are given, for which see MacLane's [116].

If we have a crossed module $C \xrightarrow{\partial} P$, then we saw that $Ker\, \partial$ is central in C and is a $P/\partial C$-module. We thus have a 'crossed 2-fold extension':

$$K \xrightarrow{i} C \xrightarrow{\partial} P \xrightarrow{p} G,$$

where $K = Ker\, \partial$ and $G = P/\partial C$. (We will write $N = \partial C$.)

Repeat the same process as before for the extension

$$N \to P \to G,$$

but take extra care as N is usually not Abelian. Pick a transversal $s : G \to P$ giving $f : G \times G \to N$ as before (even with the same formula). Next look at

$$K \xrightarrow{i} C \to N,$$

and lift f to C via a choice of $F(g_1, g_2) \in C$ with image $f(g_1, g_2)$ in N.

The pairing f satisfied the cocycle condition, but we have no means of ensuring that F will do so, i.e. there will be, for each triple (g_1, g_2, g_3), an element $c(g_1, g_2, g_3) \in C$ such that

$$^{s(g_1)}F(g_2, g_3)F(g_1, g_2g_3) = i(c(g_1, g_2, g_3))F(g_1, g_2)F(g_1g_2, g_3),$$

and some of these $c(g_1, g_2, g_3)$ may be non-trivial. The $c(g_1, g_2, g_3)$ will satisfy a cocycle condition correspond to a 4-simplex in $Ner(\mathcal{G})$, and one can reconstruct the crossed 2-fold extension up to equivalence from F and c. Here 'equivalence' is generated by maps of 'crossed' exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & P & \longrightarrow & G & \longrightarrow & 1 \\
& & \Big\| = & & \Big\downarrow & & \Big\downarrow & & \Big\| = & & \\
1 & \longrightarrow & K & \longrightarrow & C' & \longrightarrow & P' & \longrightarrow & G & \longrightarrow & 1,
\end{array}
$$

but these morphisms need not be isomorphisms. Of course, this identifies $H^3(G, K)$ with π_0 of the resulting category.

What about $H^4(G, K)$? Yes, something similar works, but we do not have the machinery to do it here, yet.

1.3.2   Not really an aside!

Suppose we start with a crossed module $\mathsf{C} = (C, P, \partial)$. We can build an internal category, $\mathcal{X}(\mathsf{C})$, in $Grps$ from it. The group of objects of $\mathcal{X}(\mathsf{C})$ will be P and the group of arrows $C \rtimes P$. The source map

$$s : C \rtimes P \to P \quad \text{is} \quad s(c, p) = p,$$

the target

$$t : C \rtimes P \to P \quad \text{is} \quad t(c, p) = \partial c.p.$$

(That looks a bit strange. That sort of construction usually does not work, multiplying two homomorphisms together is a recipe for trouble! - but it does work here:

$$
\begin{aligned}
t((c_1, p_1).(c_2, p_2)) &= t(c_1{}^{p_1}c_2, p_1p_2) \\
&= \partial(c_1{}^{p_1}c_2).p_1p_2,
\end{aligned}
$$

whilst $t(c_1, p_1).t(c_2, p_2) = \partial c_1.p_1.\partial c_2.p_2$, but remember $\partial(c_1{}^{p_1}c_2) = \partial c_1.p_1.\partial c_2.p_1^{-1}$, so they are equal.)

The identity morphism is $i(p) = (1, p)$, but what about the composition. Here it helps to draw a diagram. Suppose $(c_1, p_1) \in C \rtimes P$, then it is an arrow

$$p_1 \xrightarrow{(c_1, p_1)} \partial c_1.p_1,$$

and we can only compose it with (c_2, p_2) if $p_2 = \partial c_1.p_1$. This gives

$$p_1 \xrightarrow{(c_1, p_1)} \partial c_1.p_1 \xrightarrow{(c_2, \partial c_1.p_1)} \partial c_2 \partial c_1.p_1.$$

The obvious candidate for the composite arrow is $(c_2 c_1, p_1)$ and it works!

In fact, $\mathcal{X}(\mathsf{C})$ is an internal groupoid as $(c_1^{-1}, \partial c_1.p_1)$ is an inverse for (c_1, p_1).

Now if we started with an internal category

$$G_1 \xrightarrow[\underset{\xleftarrow{i}}{t}]{s} G_0 \ ,$$

etc., then set $P = G_0$ and $C = Ker\, s$ with $\partial = t\mid_C$ to get a crossed module.

Theorem 1 *(Brown-Spencer,[42]) The category of crossed modules is equivalent to that of internal categories in Grps.* ∎

You have, almost, seen the proof. As beginning students of algebra, you learnt that equivalence relations on groups need to be congruence relations for quotients to work well and that congruence relations 'are the same as' normal subgroups. That is the essence of the proof needed here, but we have groupoids rather than equivalence relations and crossed modules rather than normal subgroups.

Of course, any morphism of crossed modules has to induce an internal functor between the corresponding internal categories and *vice versa*. That is a **good exercise** for you to check that you have understood the link that the Brown-Spencer theorem gives.

This is a good place to mention 2-groups. The notion of 2-category is one that should be fairly clear even if you have not met it before. For instance, the category of small categories, functors and natural transformations is a 2-category. Between each pair of objects, we have not just a set of functors as morphisms but a small category of them with the natural transformations between them as the arrows in this second level of structure. The notion of 2-category is abstracted from this. We will not give a formal definition here (but suggest that you look one up if you have not met the idea before). A 2-category thus has objects, arrows or morphisms (or sometimes '1-cells') between them and then some 2-cells (sometimes called '2-arrows' or '2-morphisms') between them.

Definition: A *2-groupoid* is a 2-category in which all 1-cells and 2-cells are invertible.
If the 2-groupoid has just one object then we call it a *2-group*.

Of course, there are also 2-functors between 2-categories and so, in particular, between 2-groups. Again this is for **you to formulate, looking up relevant definitions, etc.**

Internal categories in *Grps* are really exactly the same as 2-groups. The Brown-Spencer theorem thus constructs the *associated 2-group of a crossed module*. The fact that the composition in the internal category must be a group homomorphism implies that the '*interchange law*' must hold. This equation is in fact equivalent via the Brown-Spencer result to the Peiffer identity. (It is **left to you** to find out about the interchange law and to check that it is the Peiffer axiom in disguise. We will see it many times later on.)

Here would be a good place to mention that an internal monoid in *Grps* is just an Abelian group. The argument is well known and is usually known by the name of the *Eckmann-Hilton argument*. This starts by looking at the interchange law, which states that the monoid multiplication must be group homomorphism. From this it derives that the monoid identity must also be the group identity and that the two compositions must coincide. It is then easy to show that the group is Abelian.

1.3.3 Perhaps a bit more of an aside ... for the moment!

This is quite a good place to mention the groupoid based theory of all this. The resulting objects look like abstract 2-categories and are 2-groupoids. We have a set of objects, K_0, a set of arrows, K_1, depicted $x \xrightarrow{p} y$, and a set of two cells

$$x \underset{\partial c.p}{\overset{p}{\rightleftarrows}} \Downarrow (c,p) \quad y \; .$$

In our previous diagrams, as all the elements of P started and ended at the same single object, we could shift dimension down one step; our old objects are now arrows and our old arrows are 2-cells. We will return to this later.

The important idea to note here is that a 'higher dimensional category' has a link with an algebraic object. The 2-group(oid) provides a useful way of interpreting the structure of the crossed module *and* indicates possible ways towards similar applications and interpretations elsewhere. For instance, a presentation of a monoid leads more naturally to a 2-category than to any analogue of a crossed module, since kernels are less easy to handle than congruences in Mon.

There are other important interpretations of this. Categories such as that of vector spaces, Abelian groups or modules over a ring, have an additional structure coming from the tensor product, $A \otimes B$. They are *monoidal categories*. One can 'multiply' objects together and this is linked to a related multiplication on morphisms between the objects. In many of the important examples the multiplication is not strictly associative, so for instant, if A, B, C are objects there is an isomorphism between $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, but this isomorphism is most definitely not the identity as the two objects are constructed in different ways. A similar effect happens in the category of sets with ordinary Cartesian product. The isomorphism is there because of universal properties, but it is again not the identity. It satisfies some coherence conditions, (a cocycle condition in disguise), relating to associativity of four fold tensors and the associahedron that we gave earlier, is a corresponding diagram for the five fold tensors. (Yes, there is a strong link, but that is not for these notes!) Our 2-group(oid) is the 'suspension' or 'categorification' of a similar structure. We can multiply objects and 'arrows' and the result is a strict 'gr-groupoid', or 'categorical group', i.e. a strict monoidal category with inverses. This is vague here, but will gradually be explored later on. If you want to explore the ideas further now, look at Baez and Dolan, [12].

(At this point, you do not need to know the definition of a monoidal category, but **remember to look it up in the not too distance future**, if you have not met it before, as later on the insights that an understanding of that notion gives you, will be very useful. It can be found in many places in the literature, and on the internet. The approach that you will get on best with depends on your background and your likes and dislikes mathematically, so we will not give one here.)

Just as associativity in a monoid is replaced by a 'lax' associativity 'up to coherent isomorphisms' in the above, gr-groupoids are 'lax' forms of internal categories in groups and thus indicate the presence of a crossed module-like structure, albeit in a weakened or 'laxified' form. Later we will see naturally occurring gr-groupoid structures associated with some constructions in non-Abelian cohomology. There is also a sense in which the link between fibrations and crossed modules given earlier here, indicates that fibrations are like a related form of lax crossed modules. In the notion

of fibred category and the related Grothendieck construction, this intuition begins to be 'solidified' into a clearer strong relationship.

1.3.4 Automorphisms of a group yield a 2-group

We could also give this section a subtitle:

The automorphisms of a 1-type give a 2-type.

This is really an extended exercise in playing around with the ideas from the previous two sections. It uses a small amount of categorical language, but, hopefully, in a way that should be easy for even a categorical debutant to follow. The treatment will be quite detailed as it is that detail that provides the links between the abstract and the concrete.

We start with a look at 'functor categories', but with groupoids rather than general small categories as input. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are groupoids, then we can form a new groupoid, $\mathcal{H}^{\mathcal{G}}$, whose objects are the functors, $f : \mathcal{G} \to \mathcal{H}$. Of course, functors in this context are just morphisms of groupoids, and, if $\mathcal{G}$, and $\mathcal{H}$ are $G[1]$ and $H[1]$, that is, two groups, G and H, thought of as one object groupoids, then the objects of $\mathcal{H}^{\mathcal{G}}$ are just the homomorphisms from G to H thought of in a slightly different way.

That gives the objects of $\mathcal{H}^{\mathcal{G}}$. For the morphisms from f_0 to f_1, we 'obviously' should think of natural transformations. (As usual, if you are not sufficiently conversant with elementary categorical ideas, pause and look them up in a suitable text of in Wikipedia.) Suppose $\eta : f_0 \to f_1$ is a natural transformation, then, for each x, an object of $\mathcal{G}$, we have an arrow,

$$\eta(x) : f_0(x) \to f_1(x),$$

in $\mathcal{H}$ such that, if $g : x \to y$ in $\mathcal{G}$, then the square

$$
\begin{array}{ccc}
f_0(x) & \xrightarrow{\eta(x)} & f_1(x) \\
\Big\downarrow{\scriptstyle f_0(g)} & & \Big\downarrow{\scriptstyle f_1(g)} \\
f_0(y) & \xrightarrow[\eta(y)]{} & f_1(y)
\end{array}
$$

commutes, so η 'is' the family, $\{\eta(x) \mid x \in Ob(\mathcal{G})\}$. Now assume $\mathcal{G} = G[1]$ and $\mathcal{H} = H[1]$, and that we try to interpret $\eta(x) : f_0(x) \to f_1(x)$ back down at the level of the groups, that is, a bit more 'classically' and group theoretically. There is only one object, which we denote $*$, if we need it, so we have that η corresponds to a single element, $\eta(*)$, in H, which we will write as h for simplicity, but now the condition for commutation of the square just says that, for any element $g \in G$,

$$h f_0(g) = f_1(g) h,$$

i.e., that f_0 and f_1 are *conjugate* homomorphisms, $f_1 = h f_0 h^{-1}$..

It should be clear, (but **check that it is**), that this definition of morphism makes $\mathcal{H}^{\mathcal{G}}$ into a category, in fact into a groupoid, as the morphisms compose correctly and have inverses. (To get the inverse of η take the family $\{\eta(x)^{-1} \mid x \in Ob(\mathcal{G})\}$ and check the relevant squares commute.)

So far we have 'proved':

Lemma 3 *For groupoids, $\mathcal{G}$ and $\mathcal{H}$, the functor category, $\mathcal{H}^{\mathcal{G}}$, is a groupoid.* ∎

We will be a bit sloppy in notation and will write H^G for what should, more precisely, be written $H[1]^{G[1]}$.

We note that it is usual to observe that, for Abelian groups, A, and B, the set of homomorphisms from A to B is itself an Abelian group, but that the set of homomorphisms from one non-Abelian group to another has no such nice structure. Although this is sort of true, the point of the above is that that set forms the set of objects for a very neat algebraic object, namely a groupoid!

If we have a third groupoid, $\mathcal{K}$, then we can also form $\mathcal{K}^{\mathcal{H}}$ and $\mathcal{K}^{\mathcal{G}}$, etc. and, as the objects of $\mathcal{K}^{\mathcal{H}}$ are homomorphisms from $\mathcal{H}$ to $\mathcal{K}$, we might expect to compose with the objects of $\mathcal{H}^{\mathcal{G}}$ to get ones of $\mathcal{K}^{\mathcal{G}}$. We might thus hope for a composition *functor*

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}}.$$

(There are **various things to check**, but we need not worry. We are really working with functors and natural transformations and with the investigation that shows that the category of small categories is 2-category. This means that if you get bogged down in the detail, you can easily find the ideas discussed in many texts on category theory.) This works, so we have that the category, *Grpds* has also a 2-category structure. (It is a '*Grpds*-enriched' category; see later for enriched categories. The formal definition is in section **??**, although the basic idea is used before that.)

We need to recall next that in any category, $\mathcal{C}$, the endomorphisms of any object, X, form a monoid, $End(X) := \mathcal{C}(X, X)$. You just use the composition and identities of $\mathcal{C}$ 'restricted to X'. If we play that game with any groupoid enriched category, C, then for any object, X, we will have a groupoid, $\mathsf{C}(X, X)$, which we might write $\mathsf{End}(X)$, (that is, using the same font to indicate 'enriched') and which also has a monoid structure,

$$\mathsf{C}(X, X) \times \mathsf{C}(X, X) \to \mathsf{C}(X, X).$$

It will be a monoid internal to *Grpds*. In particular, for any groupoid, $\mathcal{G}$, we have such an internal monoid of endomorphisms, $\mathcal{G}^{\mathcal{G}}$, and specialising down even further, for any group, G, such an internal monoid, G^G. Note that this is internal to the category of *groupoids* not of groups, as its monoid of objects is the endomorphism monoid of G, not a single element set. Within G^G, we can restrict attention to the subgroupoid on the automorphisms of G. We thus have this groupoid, $\mathsf{Aut}(G)$, which has as objects the automorphisms of G and, as typical morphism, $\eta : f_0 \to f_1$, a conjugation. It is important to note that as η is specified by an element of G and an automorphism, f_0, of G, the pair, (g, f_0), may then be a good way of thinking of it. (Two points, that may be obvious, but are important even if they are, are that the morphism η is not conjugation itself, but conjugates f_0. One has to specify where this morphism starts, its domain, as well as what it does, namely conjugate by g. Secondly, in (g, f_0), we *do* have the information on the codomain of η, as well. It is $g f_0 g^{-1} = f_1$.)

Using this basic notation for the morphisms, we will look at the various bits of structure this thing has. (Remember, $\eta : f_0 \to f_1$ and $f_1 = g f_0 g^{-1}$, as we will need to use that several times.) We have compositions of these pairs in two ways:

(a) as natural transformations: if

$$\eta : f_0 \to f_1, \quad \eta = (g, f_0),$$
$$\text{and} \quad \eta' : f_1 \to f_2, \quad \eta' = (g', f_1),$$

then the composite is $\eta'\sharp_1\eta = (g'g, f_0)$. (That is easy to check. As, for instance, $f_2 = g'f_1(g')^{-1} = (g'g)f_0(g'g)^{-1}, \ldots$, it all works beautifully). (A word of **warning** here, $(g'g)f_0(g'g)^{-1}$ is the conjugate of the automorphism f_0 by the element $(g'g)$. The bracket does not refer to f_0 applied to the 'thing in the bracket', so, for $x \in G$, $((g'g)f_0(g'g)^{-1})(x)$ is, in fact, $(g'g)f_0(x)(g'g)^{-1}$. This *is* slightly confusing so think about it, so as not to waste time later in avoidable confusion.)

b) using composition, $\sharp_0$, in the monoid structure. To understand this, it is easier to look at that composition as being specialised from the one we singled out earlier,

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}},$$

which is the composition in the 2-category of groupoids. (We really want $\mathcal{G} = \mathcal{H} = \mathcal{K}$, but, by keeping the more general notation, it becomes easier to see the roles of each $\mathcal{G}$.)

We suppose $f_0, f_1 : \mathcal{G} \to \mathcal{H}$, $f_0', f_1' : \mathcal{G} \to \mathcal{H}$, and then $\eta : f_0 \to f_1$, $\eta' : f_0' \to f_1'$. The 2-categorical picture is

with η'' being the desired composite, $\eta'\sharp_0\eta$, but how is it calculated. The important point is the *interchange law*. We can 'whisker' on the left or right, or, since the 'left-right' terminology can get confusing (does 'left' mean 'diagrammatically' or 'algebraically' on the left?), we will often use 'pre-' and 'post-' as alternative prefixes. The terminology may seem slightly strange, but is quite graphic when suitable diagrams are looked at! Whiskering corresponds to an interaction between 1-cell and 2-cells in a 2-category. In 'post-whiskering', the result is the composite of a 2-cell *followed* by a 1-cell:

Post-whiskering:

$$f_0'\sharp_0\eta : f_0'\sharp_0 f_0 \to f_0'\sharp_0 f_1,$$

(It is convenient, here, to write the more formal $f_0'\sharp_0 f_0$, for what we would usually write as $f_0'f_0$.) The natural transformation, η is given by a family of arrows in $\mathcal{H}$, so $f_0'\sharp_0\eta$ is given by mapping that family across to $\mathcal{K}$ using f_0'. (Specialising to $\mathcal{G} = \mathcal{H} = \mathcal{K} = G[1]$, if $\eta = (g, f_0)$, then $f_0'\sharp_0\eta = (f_0'(g), f_0'f_0)$, as is easily checked; similarly for $f_1'\sharp_0\eta$.)

Pre-whiskering:

$$\eta'\sharp_0 f_0 : f_0'\sharp_0 f_0 \to f_1'\sharp_0 f_0,$$

Here the morphism f_0 does not influence the g-part of η' at all. It just alters the domains. In the case that interests us, if $\eta' = (g', f_0')$, then $\eta' \sharp_0 f_0 = (g', f_0' f_0)$.

The way of working out $\eta' \sharp_0 \eta$ is by using $\sharp_1$-composites. First,

$$\eta' \sharp_0 \eta : f_0' f_0 \to f_1' f_1,$$

and we can go

$$\eta' \sharp_0 f_0 : f_0' f_0 \to f_1' f_0,$$

and then, to get to where we want to be, that is, $f_1' f_1$, we use

$$f_1' \sharp_0 \eta : f_1' f_0 \to f_1' f_1.$$

This uses the $\sharp_1$-composition, so

$$
\begin{aligned}
\eta' \sharp_0 \eta &= (f_1' \sharp_0 \eta) \sharp_1 (\eta' \sharp_0 f_0) \\
&= (f_1'(g), f_1' f_0) \sharp_1 (g', f_0' f_0) \\
&= (f_1'(g).g', f_0' f_0),
\end{aligned}
$$

but $f_1'(g) = g' f_0(g)(g')^{-1}$, so the end results simplifies to $(g' f_0(g), f_0' f_0)$. **Hold on!** That looks nice, but we could have also calculated $\eta' \sharp_0 \eta$ using the other form as the composite,

$$
\begin{aligned}
\eta' \sharp_0 \eta &= (\eta' \sharp_0 f_1) \sharp_1 (f_0' \sharp_0 \eta) \\
&= (g', f_0' f_1) \sharp_1 (f_0'(g), f_0' f_0) \\
&= (g' f_0'(g), f_0' f_0),
\end{aligned}
$$

so we did not have any problem. (All the properties of an internal groupoid in *Grps*, or, if you prefer that terminology, 2-group, can be derived from these two compositions. The $\sharp_1$ composition is the 'groupoid' direction, whilst the $\sharp_0$ is the 'group' one.)

We thus have a group of natural transformations made up of pairs, (g, f_0) and whose multiplication is given as above. This is just the semi-direct product group, $G \rtimes Aut(G)$, for the natural and obvious action of $Aut(G)$ on G. This group is sometimes called the *holomorph* of G.

We have two homomorphisms from $G \rtimes Aut(G)$ to $Aut(G)$. One sends (g, f_0) to f_0, so is just the projection, the other sends it to $f_1 = g f_0 g^{-1} = \iota_g \circ f_0$. We can recognise this structure as being the associated 2-group of the crossed module, $(G, Aut(G), \iota)$, as we met on page 12. We call $\mathsf{Aut}(G)$, the *automorphism 2-group* of G..

1.3.5 Back to 2-types

From our crossed module, $\mathsf{C} = (C, P, \partial)$, we can build the internal groupoid, $\mathcal{X}(\mathsf{C})$, as before, then apply the nerve construction internally to the internal groupoid structure to get a simplicial group, $K(\mathsf{C})$.

Definition: Given a crossed module, $\mathsf{C} = (C, P, \partial)$, the nerve (taken internally in *Grps*) of the internal groupoid, $\mathcal{X}(\mathsf{C})$, defined by C, will be called the nerve of C or, if more precision is needed, its *simplicial group nerve* and will be denoted $K(\mathsf{C})$.

The simplicial set, $\overline{W}(K(\mathsf{C}))$, or its geometric realisation, would be called the *classifying space* of C.

We need this in some detail in low dimensions.

$$
\begin{aligned}
K(\mathsf{C})_0 &= P \\
K(\mathsf{C})_1 &= C \rtimes P \qquad\qquad d_0 = t, d_1 = s \\
K(\mathsf{C})_2 &= C \rtimes (C \rtimes P),
\end{aligned}
$$

where $d_0(c_2, c_1, p) = (c_2, \partial c_1.p)$, $d_1(c_2, c_1, p) = (c_2.c_1, p)$ and $d_2(c_2, c_1, p) = (c_1, p)$. The pattern continues with $K(\mathsf{C})_n = C \rtimes (\dots \rtimes (C \rtimes P)\dots)$, having n-copies of C. The d_i, for $0 < i < n$, are given by multiplication in C, d_0 is induced from t and d_n is a projection. The s_i are insertions of identities. (We will examine this in more detail later.)

Remark: A word of caution: for G a group considered as a crossed module, this 'nerve' is not the nerve of G in the sense used earlier. It is just the constant simplicial group corresponding to G. What is often called the nerve of G is what here has been called its classifying space. One way to view this is to note that $\mathcal{X}(\mathsf{C})$ has two independent structures, one a group, the other a category, and *this* nerve is of the category structure. The group, G, considered as a crossed module is like a set considered as a (discrete) category, having only identity arrows.)

The Moore complex of $K(\mathsf{C})$ is easy to calculate and is just $NK(\mathsf{C})_i = 1$ if $i \geq 2$; $NK(\mathsf{C})_1 \cong C$; $NK(\mathsf{C})_0 \cong P$ with the $\partial : NK(\mathsf{C})_1 \to NK(\mathsf{C})_0$ being exactly the given ∂ of C. (This is left as an exercise. It is a useful one to do in detail.)

Proposition 1 *(Loday, [110]) The category $CMod$ of crossed modules is equivalent to the subcategory of $Simp.Grps$, consisting of those simplicial groups, G, having Moore complexes of length 1, i.e. $NG_i = 1$ if $i \geq 2$.* ∎

This raises the interesting question as to whether it is possible to find alternative algebraic descriptions of the structures corresponding to Moore complexes of length n.

Is there any way of going directly from simplicial groups to crossed modules? Yes. The last two terms of the Moore complex will give us:

$$\partial : NG_1 \to NG_0 = G_0$$

and G_0 acts on NG_1 by conjugation via s_0, i.e. if $g \in G_0$ and $x \in NG_1$, then $s_0(g)xs_0(g)^{-1}$ is also in NG_1. (Of course, we could use multiple degeneracies to make g act on an $x \in NG_n$ just as easily.) As $\partial = d_0$, it respects the G_0 action, so CM1 is satisfied. In general, CM2 will not be satisfied. Suppose $g_1, g_2 \in NG_1$ and examine $^{\partial g_1}g_2 = s_0 d_0 g_1.g_2.s_0 d_0 g_1^{-1}$. This is rarely equal to $g_1 g_2 g_1^{-1}$. We write $\langle g_1, g_2 \rangle = [g_1, g_2][g_2, s_0 d_0 g_1] = g_1 g_2 g_1^{-1}.(^{\partial g_1}g_2)^{-1}$, so it measures the obstruction to CM2 for this pair g_1, g_2. This is often called the *Peiffer commutator* of g_1 and g_2. Noting that $s_0 d_0 = d_0 s_1$, we have an element

$$\{g_1, g_2\} = [s_0 g_1, s_0 g_2][s_0 g_2, s_1 g_1] \in NG_2$$

and $\partial\{g_1, g_2\} = \langle g_1, g_2 \rangle$. This second pairing is called the *Peiffer lifting* (of the Peiffer commutator). Of course, if $NG_2 = 1$, then CM2 is satisfied (as for $K(\mathsf{C})$, above).

We could work with what we will call $M(G, 1)$, namely

$$\overline{\partial} : \frac{NG_1}{\partial NG_2} \to NG_0,$$

with the induced morphism and action. (As $d_0 d_0 = d_0 d_1$, the morphism is well defined.) This is a crossed module, but we could have divided out by less if we had wanted to. We note that $\{g_1, g_2\}$ is a product of degenerate elements, so we form, in general, the subgroup $D_n \subseteq NG_n$, generated by all degenerate elements.

Lemma 4

$$\overline{\partial} : \frac{NG_1}{\partial(NG_2 \cap D_2)} \to NG_0$$

is a crossed module. ∎

This is, in fact, $M(sk_1G, 1)$, where sk_1G is the 1-skeleton of G, i.e., the subsimplicial group generated by the k-simplices for $k = 0, 1$.

The kernel of $M(G, 1)$ is $\pi_1(G)$ and the cokernel $\pi_0(G)$ and

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

represents a class $k(G) \in H^3(\pi_0(G), \pi_1(G))$. Up to a notion of 2-equivalence, $M(G, 1)$ represents the 2-type of G completely. This is an algebraic version of the result of MacLane and Whitehead we mentioned earlier. Once we have a bit more on cohomology, we will examine it in detail.

This use of $NG_2 \cap D_2$ and our noting that $\{g_1, g_2\}$ is a product of degenerate elements may remind you of group T-complexes and thin elements. Suppose that G is a group T-complex in the sense of our discussion at the end of the previous chapter (page **??**). In a general simplicial group, the subgroups, $NG_n \cap D_n$, will not be trivial. They give measure of the extent to which homotopical information in dimension n on G depends on 'stuff' from lower dimensions., i.e., comparing G with its $(n-1)$-skeleton. (Remember that in homotopy theory, invariants such as the homotopy groups do not necessarily vanish above the dimension of the space, just recall the sphere S^2 and the subtle structure of its higher homotopy groups.)

The construction here of $M(sk_1G, 1)$ involves 'killing' the images of our possible multiple 'D-fillers' for horns, forcing uniqueness. We will see this again later.

Chapter 2

Crossed complexes

Accurate encoding of homotopy types is tricky. Chain complexes, even of G-modules, can only record certain, more or less Abelian, information. Simplicial groups, at the opposite extreme, can encode all connected homotopy types, but at the expense of such a large repetition of the essential information that makes calculation, at best, tedious and, at worst, virtually impossible. Complete information on truncated homotopy types can be stored in the cat^n-groups of Loday, [110]. We will look at these later. An intermediate model due to Blakers and Whitehead, [164], is that of a crossed complex. The algebraic and homotopy theoretic aspects of the theory of crossed complexes have been developed by Brown and Higgins, (cf. [34, 35], etc., in the bibliography and the forthcoming monograph by Brown, Higgins and Sivera, [37]) and by Baues, [16–18]. We will use them later on in several contexts.

2.1   Crossed complexes: the Definition

We will initially look at reduced crossed complexes, i.e., the group rather than the groupoid based case.

Definition: A *crossed complex*, which will be denoted C, consists of a sequence of groups and morphisms

$$\mathsf{C} : \ldots \to C_n \overset{\delta_n}{\to} C_{n-1} \overset{\delta_{n-1}}{\to} \ldots \to C_3 \overset{\delta_3}{\to} C_2 \overset{\delta_2}{\to} C_1$$

satisfying the following:

CC1) $\delta_2 : C_2 \to C_1$ is a crossed module;

CC2) each C_n, $(n > 2)$, is a left $C_1/\delta_1 C_2$-module and each δ_n, $(n > 2)$ is a morphism of left $C_1/\delta_2 C_2$-modules, (for $n = 3$, this means that δ_3 commutes with the action of C_1 and that $\delta_3(C_3) \subset C_2$ must be a $C_1/\delta_2 C_2$-module);

CC3) $\delta\delta = 0$.

The notion of a morphism of crossed complexes is clear. It is a graded collection of morphisms preserving the various structures. We thus get a category, Crs_{red} of reduced crossed complexes.

As we have that a crossed complex is a particular type of chain complex (of non-Abelian groups near the bottom), it is natural to define its homology groups in the obvious way.

Definition: If C is a crossed complex, its nth *homology group* is

$$H_n(\mathsf{C}) = \frac{Ker\,\delta_n}{Im\,\delta_{n+1}}.$$

These homology groups are, of course, functors from Crs_{red} to the category of Abelian groups.

Definition: A morphism $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$ is called a *weak equivalence* if it induces isomorphisms on all homology groups.

There are good reasons for considering the homology groups of a crossed complex as being its homotopy groups. For example, if the crossed complex comes from a simplicial group then the homotopy groups of the simplicial group are the same as the homology groups of the given crossed complex (possibly shifted in dimension, depending on the notational conventions you are using).

The non-reduced version of the concept is only a bit more difficult to write down. It has C_1 as a groupoid on a set of objects C_0 with each C_k, a family of groups indexed by the elements of C_0. The axioms are very similar; see [37] for instance or many of the papers by Brown and Higgins listed in the bibliography. This gives a category, Crs, of (unrestricted) crossed complexes and morphisms between them. This category is very rich in structure. It has a tensor product structure, denoted $\mathsf{C} \otimes \mathsf{D}$ and a corresponding mapping complex construction, $\mathsf{Crs}(\mathsf{C}, \mathsf{D})$, making it into a monoidal closed category. The details are to be found in the papers and book listed above and will be recalled later when needed.

It is worth noting that this notion restricts to give us a notion of *weak equivalence* applicable to crossed modules as well.

Definition: A morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, between two crossed modules, is called a *weak equivalence* if it induces isomorphisms on π_0 and π_1, that is, on both the kernel and cokernel of the crossed modules.

The relevant reference for π_0 and π_1 is page 16.

2.1.1   Examples: crossed resolutions

As we mentioned earlier, a resolution of a group (or other object) is a model for the homotopy type represented by the group, but which usually is required to have some nice freeness properties. With crossed complexes we have some notion of homotopy around, just as with chain complexes, so we can apply that vague notion of resolution in this context as well. This will give us some neat examples of crossed complexes that are 'tuned' for use in cohomology.

A *crossed resolution* of a group G is a crossed complex, C, such that for each $n > 1$, $Im\,\delta_n = Ker\,\delta_{n-1}$ and there is an isomorphism, $C_1/\delta_2 C_2 \cong G$.

A crossed resolution can be constructed from a presentation $\mathcal{P} = (X : R)$ as follows:

Let $C(P) \to F(X)$ be the free crossed module associated with $\mathcal{P}$. We set $C_2 = C(\mathcal{P})$, $C_1 = F(X)$, $\delta_1 = \partial$. Let $\kappa(\mathcal{P}) = Ker(\partial : C(\mathcal{P}) \to F(X))$. This is the module of identities of the presentation and is a left G-module. As the category $G\text{-}Mod$ has enough projectives, we can form

a free resolution $\mathbb{P}$ of $\kappa(\mathcal{P})$. To obtain a crossed resolution of G, we join $\mathbb{P}$ to the crossed module by setting $C_n = P_{n-2}$ for $n > 3$, $\delta_n = d_{n-2}$ for $n > 3$ and the composite from P_0 to $C(P)$ for $n = 3$.

2.1.2 The standard crossed resolution

We next look at a particular case of the above, namely the standard crossed resolution of G. In this, which we will denote by $\mathsf{C}G$, we have

(i) $C_1G =$ the free group on the underlying set of G. The element corresponding to $u \in G$ will be denoted by $[u]$.

(ii) C_2G is the free crossed module over C_0G on generators, written $[u, v]$, considered as elements of the set $G \times G$, in which the map δ_1 is defined on generators by

$$\delta[u, v] = [uv]^{-1}[u][v].$$

(iii) For $n > 3$, C_nG is the free left G-module on the set G^{n+1}, but in which one has equated to zero any generator $[u_1, \ldots, u_{n+1}]$ in which some u_i is the identity element of G.

If $n > 2$, $\delta : C_{n+1}G \to C_nG$ is given by the usual formula

$$
\begin{aligned}
\delta[u_1, \ldots, u_{n+1}] \quad = \quad & {}^{[u_1]}[u_2, \ldots, u_{n+1}] \\
& + \sum_{i=1}^{n} (-1)^i [u_1, \ldots, u_i u_{i+1}, \ldots, u_{n+1}] + (-1)^{n+1}[u_1, \ldots, u_n].
\end{aligned}
$$

For $n = 2$, $\delta : C_3G \to C_2G$ is given by

$$\delta[u, v, w] = {}^{[u]}[v, w].[u, v]^{-1}.[uv, w]^{-1}[u, vw].$$

This is the crossed analogue of the inhomogeneous bar resolution, $\mathsf{B}G$, of the group G. A groupoid version can be found in Brown-Higgins, [33], and the abstract group version in Huebschmann, [96]. In the first of these two references, it is pointed out that $\mathsf{C}G$, as constructed, is isomorphic to the crossed complex, $\underline{\pi}(BG)$, of the classifying space of G considered with its skeletal filtration.

For any filtered space, $\underline{X} = (X_n)_{n \in \mathbb{N}}$, its *fundamental crossed complex*, $\underline{\pi}(\underline{X})$, is, in general, a non-reduced crossed complex. It is defined to have

$$\underline{\pi}(\underline{X})_n = (\pi_n(X_n, X_{n-1}, a))_{a \in X_0}$$

with $\underline{\pi}(\underline{X})_1$, the fundamental groupoid $\Pi_1 X_1 X_0$, and $\underline{\pi}(\underline{X})_2$, the family, $(\pi_2(X_2, X_1, a))_{a \in X_0}$. It will only be reduced if X_0 consists just of one point.

Most of the time we will only discuss the reduced case in detail, although the non-reduced case will be needed sometimes. Following that, we will often use the notation Crs for the category of *reduced* crossed complexes unless we need the more general case. This may occasionally cause a little confusion, but it is much more convenient for most of the time.

There are two useful, but conflicting, conventions as to indexation in crossed complexes. In the topologically inspired one, the bottom group is C_1, in the simplicial and algebraic one, it is C_0. Both get used and both have good motivation. The natural indexation for the standard crossed resolution would seem to be with C_n being generated by n-tuples, i.e. the topological one. (I am not sure that all instances of the other have been avoided, so please be careful!)

G-augmented crossed complexes.   Crossed resolutions of G are examples of G-augmented crossed complexes. A *G-augmented crossed complex* consists of a pair (C, φ) where C is a crossed complex and where $\varphi : C_1 \to G$ is a group homomorphism satisfying

(i) $\varphi \delta_1$ is the trivial homomorphism;

(ii) $Ker\,\varphi$ acts trivially on C_i for $i \geq 3$ and also on C_2^{Ab}.

A *morphism*

$$(\alpha, Id_G) : (\mathsf{C}, \varphi) \to (\mathsf{C}', \varphi')$$

of G-augmented crossed complexes consists of a morphism

$$\alpha : \mathsf{C} \to \mathsf{C}'$$

of crossed complexes such that $\varphi' \alpha_0 = \varphi$.

This gives a category, Crs_G, which behaves nicely with respect to change of groups, i.e. if $\varphi : G \to H$, then there are induced functors between the corresponding categories.

2.2   Crossed complexes and chain complexes: I

(Some of the proofs here are given in more detail as they are less routine and are not that available elsewhere. A source for much of this material is in the work of Brown and Higgins, [35], where these ideas were explored thoroughly for the first time; see also the treatment in [37].)

We have introduced crossed complexes where normally chain complexes of modules would have been used. We have seen earlier the bar resolution and now we have the standard crossed resolution. What is the connection between them? The answer is approximately that chain complexes form a category equivalent to a reflective subcategory of Crs. In other words, there is a canonical way of building a chain complex from a crossed one akin to the process of Abelianising a group. The resulting reflection functor sends the standard crossed resolution of a group to the bar resolution. The details involve some interesting ideas.

In chapter 2, we saw that, given a morphism $\theta : M \to N$ of modules over a group G, $\partial : M \to N \rtimes G$, given by $\partial(m) = (\theta(m), 1_G)$ is a crossed module, where $N \rtimes G$ acts on M via the projection to G. That example easily extends to a functorial construction which, from a positive chain complex, D, of G-modules, gives us a crossed complex $\Delta_G(\mathsf{D})$ with $\Delta_G(\mathsf{D})_n = D_n$ if $n > 1$ and equal to $D_1 \rtimes G$ for $n = 1$.

Lemma 5 $\Delta_G : Ch(G-Mod) \to Crs_G$ *is an embedding.*

Proof: That Δ_G is a functor is easy to see. It is also easy to check that it is full and faithful, that is it induces bijections,

$$Ch(G-Mod)(\mathsf{A}, \mathsf{B}) \to Crs_G(\Delta_G(\mathsf{A}), \Delta_G(\mathsf{B})).$$

The augmentation of $\Delta_G(A)$ is given by the projection of $A_1 \rtimes G$ onto G.   ■

We can thus turn a positive chain complex into a crossed complex. Does this functor have a left adjoint? i.e. is there a functor $\xi_G : Crs_G \to Ch(G-Mod)$ such that

$$Ch(G-Mod)(\xi_G(\mathsf{C}), \mathsf{D}) \to Crs_G(\mathsf{C}, \Delta_G(\mathsf{D}))?$$

If so it would suggest that chain complexes of G-modules are like G-augmented crossed complexes that satisfy some additional equational axioms. As an example of a similar situation think of 'Abelian groups' within 'groups' for which the inclusion has a left adjoint, namely Abelianisation $(G)^{Ab} = G/[G, G]$. Abelian groups are of course groups that satisfy the additional rule $[x, y] = 1$. Other examples of such situations are nilpotent groups of a given finite rank c. The subcategories of this general form are called *varieties* and, for instance, the study of varieties of groups is a very interesting area of group theory. Incidentally, it is possible to define various forms of cohomology modulo a variety in some sense. We will not explore that here.

We thus need to look at morphisms of crossed complexes from a crossed complex C to one of form $\Delta_G(\mathsf{D})$, and we need therefore to look at morphisms into a semidirect product. These are useful for other things, so are worth looking at in detail.

2.2.1 Semi-direct product and derivations.

Suppose that we have a diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\;\;f\;\;} & K \rtimes G \\
& {}_{\alpha}\searrow & \downarrow{}^{proj} \\
& G &
\end{array}
$$

where K is a G-module (written additively, so we write $g.k$ not ${}^g k$ for the action). This is like the very bottom of the situation for a morphism $f : \mathsf{C} \to \Delta_G(\mathsf{D})$.

As the codomain of f is a semidirect product, we can decompose f, as a function, in the form

$$f(h) = (f_1(h), \alpha(h)),$$

identifying its second component using the diagram. The mapping f_1 is not a homomorphism. As f is one, however, we have

$$(f_1(h_1 h_2), \alpha(h_1 h_2)) = f(h_1)f(h_2) = (f_1(h_1) + \alpha(h_1)f_1(h_2), \alpha(h_1 h_2)),$$

i.e. f_1 satisfies

$$f_1(h_1 h_2) = f_1(h_1) + \alpha(h_1)f_1(h_2)$$

for all $h_1, h_2 \in H$.

2.2.2 Derivations and derived modules.

We will use the identification of G-modules for a group G with modules over the group ring, $\mathbb{Z}[G]$, of G. Recall that this ring is obtained from the free Abelian group on the set G by defining a multiplication extending linearly that of G itself. (Formally if, for the moment, we denote by e_g, the generator corresponding to $g \in G$, then an arbitrary element of $\mathbb{Z}[G]$ can be written as $\sum_{g \in G} n_g e_g$ where the n_g are integers and only finitely many of them are non-zero. The multiplication is by 'convolution' product, that is,

$$\Big(\sum_{g \in G} n_g e_g\Big)\Big(\sum_{g \in G} m_g e_g\Big) = \sum_{g \in G} \Big(\sum_{g_1 \in G} n_{g_1} m_{g_1^{-1} g} e_g\Big).$$

Sometimes, later on, we will need other coefficients that $\mathbb{Z}$ in which case it is appropriate to use the term 'group algebra' of G, over that ring of coefficients.

We will also need the augmentation, $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$, given by $\varepsilon(\sum_{g \in G} n_g e_g) = \sum_{g \in G} n_g$ and its kernel $I(G)$, known as the augmentation ideal.

Definitions: Let $\varphi : G \to H$ be a homomorphism of groups. A φ-*derivation*

$$\partial : G \to M$$

from G to a left $\mathbb{Z}[H]$-module, M, is a mapping from G to M, which satisfies the equation

$$\partial(g_1 g_2) = \partial(g_1) + \varphi(g_1)\partial(g_2)$$

for all g_1, $g_2 \in G$.

Such φ-derivations are really all derived from a universal one.

Definition: A *derived module* for φ consists of a left $\mathbb{Z}[H]$-module, D_φ, and a φ-derivation, $\partial_\varphi : G \to D_\varphi$ with the following universal property:

Given any left $\mathbb{Z}[H]$-module, M, and a φ-derivation $\partial : G \to M$, there is a unique morphism

$$\beta : D_\varphi \to M$$

of $\mathbb{Z}[H]$-modules such that $\beta\partial_\varphi = \partial$.

The derivation ∂_φ is called the *universal φ derivation.*

The set of all φ-derivations from G to M has a natural Abelian group structure. We denote this set by $Der_\varphi(G, M)$. This gives a functor from H-*Mod* to Ab, the category of Abelian groups. If $(D_\varphi, \partial_\varphi)$ exists, then it sets up a natural isomorphism

$$Der_\varphi(G, M) \cong H{-}Mod(D_\varphi, M),$$

i.e., $(D_\varphi, \partial_\varphi)$ represents the φ-derivation functor.

2.2.3   Existence

The treatment of derived modules that is found in Crowell's paper, [54], provides a basis for what follows. In particular it indicates how to prove the existence of $(D_\varphi, \partial_\varphi)$ for any φ.

Form a $\mathbb{Z}[H]$-module, D, by taking the free left $\mathbb{Z}[H]$-module, $\mathbb{Z}[H]^{(X)}$, on a set of generators, $X = \{\partial g : g \in G\}$. Within $\mathbb{Z}[H]^{(X)}$ form the submodule, Y, generated by the elements

$$\partial(g_1 g_2) - \partial(g_1) - \varphi(g_1)\partial(g_2).$$

Let $D = \mathbb{Z}[H]^{(X)}/Y$ and define $d : G \to D$ to be the composite:

$$G \xrightarrow{\eta} \mathbb{Z}[H]^{(X)} \overset{quotient}{\to} D,$$

where η is "inclusion of the generators", $\eta(g) = \partial g$. Thus d by construction, will be a φ-derivation. The universal property is easily checked and hence $(D_\varphi, \partial_\varphi)$ exists.

We will later on construct $(D_\varphi, \partial_\varphi)$ in a different way which provides a more amenable description of D_φ, namely as a tensor product. As a first step towards this description, we shall give a simple description of D_G, that is, the derived module of the identity morphism of G. More precisely we shall identify (D_G, ∂_G) as being $(I(G), \partial)$, where, as above, $I(G)$ is the augmentation ideal of $\mathbb{Z}[G]$ and $\partial : G \to I(G)$ is the usual map, $\partial(g) = g - 1$.

Our earlier observations give us the following useful result:

Lemma 6 *If G is a group and M is a G-module, then there is an isomorphism*

$$Der_G(G, M) \to Hom/G(G, M \rtimes G)$$

where $Hom/G(G, M \rtimes G)$ is the set of homomorphisms from G to $M \rtimes G$ over G, i.e., $\theta : G \to M \rtimes G$ such that for each $g \in G$, $\theta(g) = (g, \theta'(g))$ for some $\theta'(g) \in M$. ∎

2.2.4 Derivation modules and augmentation ideals

Proposition 2 *The derivation module D_G is isomorphic to $I(G) = Ker(\mathbb{Z}[G] \to \mathbb{Z})$. The universal derivation is*

$$d_G : G \to I(G)$$

given by $d_G(g) = g - 1$.

Proof:

We introduce the notation $f_\delta : I(G) \to M$ for the $\mathbb{Z}[G]$-module morphism corresponding to a derivation

$$\delta : G \to M.$$

The factorisation $f_\delta d_G = \delta$ implies that f_δ must be defined by $f_\delta(g - 1) = \delta(g)$. That this works follows from the fact that $I(G)$, as an Abelian group, is free on the set $\{g - 1 : g \in G\}$ and that the relations in $I(G)$ are generated by those of the form

$$g_1(g_2 - 1) = (g_1 g_2 - 1) - (g_1 - 1).$$

∎

We note a result on the augmentation ideal construction that is not commonly found in the literature.

The proof is easy and so will be omitted.

Lemma 7 *Given groups G and H in $\mathcal{C}$ and a commutative diagram*

$$
\begin{array}{ccc}
G & \xrightarrow{\ \delta\ } & M \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \varphi} \\
H & \xrightarrow[\delta']{} & N
\end{array}
\qquad (*)
$$

where δ, δ' are derivations, M is a left $\mathbb{Z}[G]$-module, N is a left $\mathbb{Z}[H]$-module and φ is a module map over ψ, i.e., $\varphi(g.m) = \psi(g)\varphi(m)$ for $g \in G$, $m \in M$. Then the corresponding diagram

$$
\begin{array}{ccc}
I(G) & \xrightarrow{\ f_\delta\ } & M \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \varphi} \\
I(H) & \xrightarrow[\ f_{\delta'}\ ]{} & N
\end{array}
\qquad (**)
$$

is commutative.   ∎

The earlier proposition has the following corollaries:

Corollary 1 *The subset $Im\,d_G = \{g - 1 : g \in G\} \subset I(G)$ generates $I(G)$ as a $\mathbb{Z}[G]$-module. Moreover the relations between these generators are generated by those of the form*

$$(g_1 g_2 - 1) - (g_1 - 1) - g_1(g_2 - 1).$$

∎

It is useful to have also the following reformulation of the above results stated explicitly.

Corollary 2 *There is a natural isomorphism*

$$Der_G(G, M) \cong G{-}Mod(I(G), M).$$

∎

2.2.5 Generation of $I(G)$.

The first of these two corollaries raises the question as to whether, if $X \subset G$ generates G, does the set $G_X = \{x - 1 : x \in X\}$ generate $I(G)$ as a $\mathbb{Z}[G]$-module.

Proposition 3 *If X generates G, then G_X generates $I(G)$.*

Proof: We know $I(G)$ is generated by the $g - 1$s for $g \in G$. If g is expressible as a word of length n in the generators X then we can write $g - 1$ as a $\mathbb{Z}[G]$-linear combination of terms of the form $x - 1$ in an obvious way. (If $g = w.x$ with w of lesser length than that of g, $g - 1 = w - 1 + w(x - 1)$, so use induction on the length of the expression for g in terms of the generators.)   ∎

When G is free: If G is free, say, $G \cong F(X)$, i.e., is free on the set X, we can say more.

Proposition 4 *If $G \cong F(X)$ is the free group on the set X, then the set $\{x - 1 : x \in X\}$ freely generates $I(G)$ as a $\mathbb{Z}[G]$-module.*

Proof: (We will write F for $F(X)$.) The easiest proof would seem to be to check the universal property of derived modules for the function $\delta : F \to \mathbb{Z}[G]^{(X)}$, given on generators by

$$
\delta(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}
$$

then extended using the derivation rule to all of F using induction. This uses essentially that each element of F has a *unique* expression as a reduced word in the generators, X.

Suppose then that we have a derivation $\partial : F \to M$, define $\overline{\partial} : \mathbb{Z}[G]^{(X)} \to M$ by $\overline{\partial}(e_x) = \partial(x)$, extending linearly. Since by construction $\overline{\partial}\delta = \partial$ and is the unique such homomorphism, we are home. ∎

Note: In both these proofs we are thinking of the elements of the free module on X as being functions from X to the group ring, these functions being of 'finite support', i.e. being non-zero on only a finite number of elements of X. This can cause some complications if X is infinite or has some topology as it will in some contexts. The *idea* of the proof will usually go across to that situation but details have to change. (A situation in which this happens is in profinite group theory where the derivations have to be continuous for the profinite topology on the group, see [141].)

2.2.6 (D_φ, d_φ), the general case.

We can now return to the identification of (D_φ, d_φ) in the general case.

Proposition 5 *If $\varphi : G \to H$ is a homomorphism of groups, then $D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G)$, the tensor product of $\mathbb{Z}[H]$ and $I(G)$ over G.*

Proof: If M is a $\mathbb{Z}[H]$-module, we will write $\varphi^*(M)$ for the restricted $\mathbb{Z}[G]$-module, i.e. M with G-action given by $g.m := \varphi(g).m$. Recall that the functor φ^* has a left adjoint given by sending a G-module, N to $\mathbb{Z}[H] \otimes_G N$, i.e. take the tensor of Abelian groups, $\mathbb{Z}[H] \otimes N$ and divide out by $x \otimes g.n \equiv x\varphi(g) \otimes n$.

With this notation we have a chain of natural isomorphisms,

$$
\begin{aligned}
Der_\varphi(G, M) &\cong Der_G(G, \varphi^*(M)) \\
&\cong G-Mod(I(G), \varphi^*(M)) \\
&\cong H-Mod(\mathbb{Z}[H] \otimes_G I(G), M),
\end{aligned}
$$

so by universality,

$$
D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G),
$$

as required. ∎

2.2.7 D_φ for $\varphi : F(X) \to G$.

The above will be particularly useful when φ is the "co-unit" map, $F(X) \to G$, for X a set that generates G. We could, for instance, take $X = G$ as a set, and φ to be the usual natural epimorphism.

In fact we have the following:

Corollary 3 *Let $\varphi : F(X) \to G$ be an epimorphism of groups, then there is an isomorphism*

$$
D_\varphi \cong \mathbb{Z}[G]^{(X)}
$$

of $\mathbb{Z}[G]$-modules. In this isomorphism, the generator ∂_x, of D_φ corresponding to $x \in X$, satisfies

$$
d_\varphi(x) = \partial_x
$$

for all $x \in X$. ∎

(You should check that you see how this follows from our earlier results.)

2.3   Associated module sequences

2.3.1   Homological background

Given an exact sequence

$$1 \to K \to L \to Q \to 1$$

of abstract groups, then it is a standard result from homological algebra that there is an associated exact sequence of modules,

$$0 \to K^{Ab} \to \mathbb{Z}[Q] \otimes_L I(L) \to I(Q) \to 0.$$

There are several different proofs of this. Homological proofs give this as a simple consequence of the Tor^L-sequence corresponding to the exact sequence

$$0 \to I(L) \to \mathbb{Z}[L] \to \mathbb{Z} \to 0$$

together with a calculation of $Tor_1^L(\mathbb{Z}[Q], \mathbb{Z})$, but we are not assuming that much knowledge of standard homological algebra. That homological proof also, to some extent, hides what is happening at the 'elementary' level, in both the sense of 'simple' and also that of 'what happens to the 'elements' of the groups and modules concerned.

The second type of proof is more directly algebraic and has the advantage that it accentuates various universal properties of the sequence. The most thorough treatment of this would seem to be by Crowell, [54], for the discrete case. We outline it below.

2.3.2   The exact sequence.

Before we start on the discussion of the exact sequence, it will be useful to have at our disposal some elementary results on Abelianisation of the groups in a crossed module. Here we actually only need them for normal subgroups but we will need it shortly anyway in the more general form. Suppose that (C, P, ∂) is a crossed module, and we will set $A = Ker\partial$ with its module structure that we looked at before, and $N = \partial C$, so A is a P/N-module.

Lemma 8 *The Abelianisation of C has a natural $\mathbb{Z}[P/N]$-module structure on it.*

Proof: First we should point out that by "Abelianisation" we mean $C^{Ab} = C/[C, C]$, which is, of course, Abelian and it suffices to prove that N acts trivially on C^{Ab}, since P already acts in a natural way. However, if $n \in N$, and $\partial c = n$, then for any $c' \in C$, we have that ${}^n c' = {}^{\partial c} c' = cc'c^{-1}$, hence ${}^n c'(c')^{-1} \in [C, C]$ or equivalently

$$^n(c'[C, C]) = c'[C, C],$$

so N does indeed act trivially on C^{Ab}.   ∎

Of course N^{Ab} also has the structure of a $\mathbb{Z}[P/N]$-module and thus a crossed module gives one three P/N-modules. These three are linked as shown by the following proposition.

Proposition 6 *Let (C, P, ∂) be a crossed module. Then the induced morphisms*

$$A \to C^{Ab} \to N^{Ab} \to 0$$

form an exact sequence of $\mathbb{Z}[P/N]$-modules.

Proof: It is clear that the sequence

$$1 \to A \to C \to N \to 1$$

is exact and that the induced homomorphism from C^{Ab} to N^{Ab} is an epimorphism. Since the composite homomorphism from A to N is trivial, A is mapped into $Ker(C^{Ab} \to N^{Ab})$ by the composite $A \to C \to C^{Ab}$. It is easily checked that this is onto and hence the sequence is exact as claimed. ∎

Now for the main exact sequence result here:

Proposition 7 *Let*

$$1 \to K \xrightarrow{\varphi} L \xrightarrow{\psi} Q \to 1$$

be an exact sequence of groups and homomorphisms. Then there is an exact sequence

$$0 \to K^{Ab} \xrightarrow{\tilde{\varphi}} \mathbb{Z}[Q] \otimes_L I(L) \xrightarrow{\tilde{\psi}} I(Q) \to 0$$

of $\mathbb{Z}[Q]$-modules.

Proof: By the universal property of D_ψ, there is a unique morphism

$$\tilde{\psi} : D_\psi \to I(Q)$$

such that $\tilde{\psi}\partial_\psi = I(\psi)\partial_L$.

Let $\delta : K \to K^{Ab} = K/[K,K]$ be the canonical Abelianising morphism. We note that $\partial_\psi\varphi :$ $K \to D_\psi$ is a homomorphism (since

$$
\begin{aligned}
\partial_\psi\varphi(k_1 k_2) &= \partial_\psi\varphi(k_1) + \psi\varphi(k_1)\partial_\psi\varphi(k_2) \\
&= \partial_\psi\varphi(k_1) + \partial_\psi\varphi(k_2),)
\end{aligned}
$$

so let $\tilde{\varphi} : K^{Ab} \to D_\psi$ be the unique morphism satisfying $\tilde{\varphi}\delta = \partial_\psi\varphi$ with K^{Ab} having its natural $\mathbb{Z}[Q]$-module structure.

That the composite $\tilde{\psi}\tilde{\varphi} = 0$ follows easily from $\psi\varphi = 0$. Since D_ψ is generated by symbols $d\ell$ and $\tilde{\psi}(d\ell) = \psi(\ell) - 1$, it follows that $\tilde{\psi}$ is onto. We next turn to "$Ker\ \tilde{\psi} \subseteq Im\ \tilde{\varphi}$".

If we can prove $\alpha : D_\psi \to I(Q)$ is the cokernel of $\tilde{\varphi}$, then we will have checked this inclusion and incidentally will have reproved that $\tilde{\psi}$ is onto.

Now let $D_\psi \to C$ be any morphism such that $\alpha\tilde{\varphi} = 0$. Consider the diagram

The composite $\alpha\partial_\psi$ vanishes on the image of φ since $\alpha\partial_\psi\varphi = \alpha\tilde{\varphi}\delta$ and $\alpha\tilde{\varphi}$ is assumed zero. Define $d : Q \to C$ by $d(q) = \alpha\partial_\psi(\ell)$ for $\ell \in L$ such that $\psi(\ell) = q$. As $\alpha\partial_\psi$ vanishes on $Im\ \varphi$, this is well defined and

$$
\begin{aligned}
d(q_1 q_2) &= \alpha\partial_\psi(\ell_1 \ell_2) \\
&= \alpha\partial_\psi(\ell_1) + \alpha(\psi(\ell_1)\partial_\psi(\ell_2)) \\
&= d(q_1) + q_1 d(q_2)
\end{aligned}
$$

so d factors as $\bar{\alpha}\partial_Q$ in a unique way with $\bar{\alpha} : I(Q) \to C$. It remains to prove that $\alpha = \tilde{\psi}$, but

$$
\begin{aligned}
\tilde{\psi}\partial_\psi &= I_C(\psi)\partial_L \\
&= \partial_Q \psi
\end{aligned}
$$

by the naturality of ∂. Now finally note that $\bar{\alpha}\partial_Q = d$ and $d\psi = \alpha\partial_\psi$ to conclude that $\tilde{\psi}\partial_\psi$ and $\alpha\partial_\psi$ are equal. Equality of α and $\bar{\alpha}\tilde{\psi}$ then follows by the uniqueness clause of the universal property of (D_ψ, ∂_ψ).

Next we need to check that $K^{Ab} \to D_\psi$ is a monomorphism. To do this we use the fact that there is a transversal, $s : Q \to L$, satisfying $s(1) = 1$. This means that, following Crowell, [54] p. 224, we can for each $\ell \in L$, $q \in Q$, find an element $q \times \ell$ uniquely determined by the equation

$$
\varphi(q \times \ell)) = s(q)\ell s(q\psi(\ell))^{-1},
$$

which, of course, defines a function from $Q \times L$ to K. Crowell's lemma 4.5 then shows

$$
q \times \ell_1 \ell_2 = (q \times \ell_1)(q\psi(\ell_1) \times \ell_2) \quad \text{for} \quad \ell_1, \ell_2 \in L.
$$

Now let $M = \mathbb{Z}[Q]^{(X)}$, with $X = \{\partial\ell : \ell \in L\}$, so that there is an exact sequence

$$
M \to D_\psi \to 0.
$$

The underlying group of $\mathbb{Z}[Q]$ is the free Abelian group on the underlying set of Q. Similarly M, above, has, as underlying group, the free Abelian group on the set $Q \times X$.

Define a map $\tau : M \to K^{Ab}$ of Abelian groups by

$$
\tau(a, \partial\ell) = \delta(q \times \ell).
$$

We check that if $p(m) = 0$, then $\tau(m) = 0$. Since $Ker\ p$ is generated as a $\mathbb{Z}[Q]$-module by elements of the form

$$
\partial(\ell_1 \ell_2) - \partial\ell_1 - \psi(\ell_1)\partial\ell_2,
$$

it follows that as an Abelian group, $Ker\ p$ is generated by the elements

$$
(q, \partial(\ell_1 \ell_2)) - (q, \partial\ell_1) - (q\psi(\ell_1), \partial\ell_2).
$$

We claim that τ is zero on these elements; in fact

$$
\begin{aligned}
\tau(q, \partial(\ell_1 \ell_2)) &= \delta(q \times (\ell_1 \ell_2)) \\
&= \delta(q \times \ell_1) + \delta(q\psi(\ell_1) \times \ell_2) \\
&= \tau(q, \ell_1) + \tau(q\psi(\ell_1), \ell_2).
\end{aligned}
$$

Thus τ induces a map $\eta : D_\psi \to K^{Ab}$ of Abelian groups.

Finally we check $\eta\tilde{\varphi} =$ identity, so that $\tilde{\varphi}$ is a monomorphism: let $b \in K^{Ab}$, $k \in K$ be such that $\delta(k) = b$, then

$$
\begin{aligned}
\eta\tilde{\varphi}(b) &= \eta\tilde{\varphi}\delta(k) \\
&= \eta\partial_\psi(k) \\
&= \delta(1 \times \varphi(k)),
\end{aligned}
$$

but $1 \times \varphi(k)$ is uniquely determined by

$$
\varphi(1 \times \varphi(k)) = s(1)\varphi(k)s(1\psi\varphi(k))^{-1} = \varphi(k),
$$

since $s(1) = 1$, hence $1 \times \varphi(k) = k$ and $\eta\tilde{\varphi}(b) = \delta(k) = b$ as required. ∎

A discussion of the way in which this result interacts with the theory of covering spaces can be found in Crowell's paper already cited. We will very shortly see the connection of this module sequence with the Jacobian matrix of a group presentation and the Fox free differential calculus. It is this latter connection which suggests that we need more or less explicit formulae for the maps $\tilde{\varphi}$ and $\tilde{\psi}$ and hence requires that Crowell's detailed proof be used, not the slicker homological proof.

2.3.3 Reidemeister-Fox derivatives and Jacobian matrices

At various points, we will refer to Reidemeister-Fox derivatives as developed by Fox in a series of articles, see [81], and also summarised in Crowell and Fox, [55]. We will call these derivatives *Fox derivatives*.

Suppose G is a group and M a G-module and let $\delta : G \to M$ be a derivation, (so $\delta(g_1g_2) = \delta(g_1) + g_1\delta(g_2)$ for all $g_1, g_2 \in G$), then, for calculations, the following lemma is very valuable, although very simple to prove.

Lemma 9 *If $\delta : G \to M$ is a derivation, then*
(i) $\delta(1_G) = 0$;
(ii) $\delta(g^{-1}) = -g^{-1}\delta(g)$ for all $g \in G$;
(iii) for any $g \in G$ and $n \geq 1$,
$$
\delta(g^n) = \left(\sum_{k=0}^{n-1} g^k\right)\delta(g).
$$

Proof: As was said, these are easy to prove.
$\delta(g) = \delta(1g) + 1\delta(g)$, so $\delta(1) = 0$, and hence (i); then

$$
\delta(1) = \delta(g^{-1}g) = \delta(g^{-1}) + g^{-1}\delta(g)
$$

to get (ii), and finally induction to get (iii). ∎

The Fox derivatives are derivations taking values in the group ring as a left module over itself. They are defined for $G = F(X)$, the free group on a set X. (We usually write F for $F(X)$ in what follows.)

Definition: For each $x \in X$, let

$$\frac{\partial}{\partial x} : F \to \mathbb{Z}F$$

be defined by
(i) for $y \in X$,

$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

(ii) for any words, $w_1, w_2 \in F$,

$$\frac{\partial}{\partial x}(w_1 w_2) = \frac{\partial}{\partial x} w_1 + w_1 \frac{\partial}{\partial x} w_2.$$

Of course, a routine proof shows that the derivation property in (ii) defines $\frac{\partial w}{\partial x}$ for any $w \in F$.

This derivation, $\frac{\partial}{\partial x}$, will be called the *Fox derivative with respect to the generator x*.

Example: Let $X = \{u, v\}$, with $r \equiv uvuv^{-1}u^{-1}v^{-1} \in F = F(u, v)$, then

$$\frac{\partial r}{\partial u} = 1 + uv - uvuv^{-1}u^{-1},$$

$$\frac{\partial r}{\partial v} = u - uvuv^{-1} - uvuv^{-1}u^{-1}v^{-1}.$$

This relation is the typical braid group relation, here in Br_3, and we will come back to these simple calculations later.

It is often useful to extend a derivation $\delta : G \to M$ to a linear map from $\mathbb{Z}G$ to M by the simple rule that $\delta(g + h) = \delta(g) + \delta(h)$.

We have

$$Der(F, \mathbb{Z}F) \cong F - Mod(IF, \mathbb{Z}F),$$

and that

$$IF \cong \mathbb{Z}F^{(X)},$$

with the isomorphism matching each generating $x - 1$ with e_x, the basis element labelled by $x \in X$. (The universal derivation then sends x to e_x.)

For each given x, we thus obtain a morphism of F-modules:

$$d_x : \mathbb{Z}F^{(X)} \to \mathbb{Z}F$$

with

$$d_x(e_y) = 1 \quad \text{if } y = x$$
$$d_x(e_y) = 0 \quad \text{if } y \neq x,$$

i.e., the 'projection onto the xth-factor' or 'evaluation at $x \in X$' depending on the viewpoint taken of the elements of the free module, $\mathbb{Z}F^{(X)}$.

Suppose now that we have a group presentation, $\mathcal{P} = (X : R)$, of a group, G. Then we have a short exact sequence of groups

$$1 \to N \xrightarrow{\varphi} F \xrightarrow{\gamma} G \to 1,$$

where $N = N(R)$, $F = F(X)$, i.e., N is the normal closure of R in the free group F. We also have a free crossed module,

$$C \xrightarrow{\partial} F,$$

constructed from the presentation and hence, two short exact sequences of G-modules with $\kappa(\mathcal{P}) = Ker\,\partial$, the module of identities of $\mathcal{P}$,

$$0 \to \kappa(\mathcal{P}) \to C^{Ab} \to N^{Ab} \to 0,$$

and also

$$0 \to N^{Ab} \xrightarrow{\tilde{\varphi}} IF \otimes_F \mathbb{Z}G \to IG \to 0.$$

We note that the first of these is exact because N is a free group, (see Proposition 9, which will be proved shortly), further

$$C^{Ab} \cong \mathbb{Z}G^{(R)},$$

(the proof is left to you to manufacture from earlier results), and the map from this to N^{Ab} in the first sequence sends the generator e_r to $r[N, N]$.

We next revisit the derivation of the associated exact sequence (Proposition 7, page 41) in some detail to see what $\tilde{\varphi}$ does to $r[N, N]$. We have $\tilde{\varphi}(r[N, N]) = \partial_\gamma \varphi(r) = \partial_\gamma(r)$, considering r now as an element of F, and by Corollary 3, on identifying D_γ with $\mathbb{Z}G^{(X)}$ using the isomorphism between IF and $\mathbb{Z}F^{(X)}$, we can identify $\partial_\gamma(x) = e_x$. We are thus left to determine $\partial_\gamma(r)$ in terms of the $\partial_\gamma(x)$, i.e., the e_x. The following lemma does the job for us.

Lemma 10 *Let $\delta : F \to M$ be a derivation and $w \in F$, then*

$$\delta w = \sum_{x \in X} \frac{\partial w}{\partial x} \delta x.$$

Proof: By induction on the length of w. ∎

In particular we thus can calculate

$$\partial_\gamma(r) = \sum \frac{\partial r}{\partial x} e_x.$$

Tensoring with $\mathbb{Z}G$, we get

$$\tilde{\varphi}(r[N, N]) = \sum \frac{\partial r}{\partial x} e_x \otimes 1.$$

There is one final step to get this into a usable form:

From the quotient map $\gamma : F \to G$, we, of course, get an induced ring homomorphism, $\gamma : \mathbb{Z}F \to \mathbb{Z}G$, and hence we have elements $\gamma(\frac{\partial r}{\partial x}) \in \mathbb{Z}G$. Of course,

$$\frac{\partial r}{\partial x} e_x \otimes 1 = e_x \otimes \gamma(\frac{\partial r}{\partial x}),$$

so we have, on tidying up notation just a little:

Proposition 8 *The composite map*

$$\mathbb{Z}G^{(R)} \to N^{Ab} \to \mathbb{Z}G^{(X)}$$

sends e_r to $\sum \gamma(\frac{\partial r}{\partial x}) e_x$ and so has a matrix representation given by $J_\mathcal{P} = \left(\gamma(\frac{\partial r_i}{\partial x_j}) \right)$. ∎

Definition: The *Jacobian matrix* of a group presentation, $\mathcal{P} = (X : R)$ of a group G is

$$J_{\mathcal{P}} = \left(\gamma(\frac{\partial r_i}{\partial x_j}) \right),$$

in the above notation.

The application of γ to the matrix of Fox derivatives simplifies expressions considerable in the matrix. The usual case of this is if a relator has the form rs^{-1}, then we get

$$\frac{\partial rs^{-1}}{\partial x} = \frac{\partial r}{\partial x} - rs^{-1}\frac{\partial s}{\partial x}$$

and if r or s is quite long this looks moderately horrible to work out! However applying γ to the answer, the term rs^{-1} in the second of the two terms becomes 1. We can actually think of this as replacing rs^{-1} by $r - s$ when working out the Jacobian matrix.

Example: Br_3 **revisited.** We have $r \equiv uvuv^{-1}u^{-1}v^{-1}$, which has the form $(uvu)(vuv)^{-1}$. This then gives

$$\gamma(\frac{\partial r}{\partial u}) = 1 + uv - v \quad \text{and} \quad \gamma(\frac{\partial r}{\partial v}) = u - 1 - vu,$$

abusing notation to ignore the difference between u, v in $F(u, v)$ and the generating u, v in Br_3.

Homological 2-syzygies: In general we obtain a truncated chain complex:

$$\mathbb{Z}G^{(R)} \xrightarrow{d_2} \mathbb{Z}G^{(X)} \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{d_0} \mathbb{Z} \to 0,$$

with d_2 given by the Jacobian matrix of the presentation, and d_1 sending generator e_x^1 to $1 - x$, so $Im\, d_1$ is the augmentation ideal of $\mathbb{Z}G$.

Definition: A *homological 2-syzygy* is an element in $Ker\, d_2$..

A homological 2-syzygy is thus an element to be killed when building the third level of a resolution of G. What are the links between homotopical and homological syzygies? Brown and Huebschmann, [38], show they are isomorphic, as $Ker\, d_2$ is isomorphic to the module of identities. We will examine this result in more detail shortly.

Extended example: Homological Syzygies for the braid group presentations: The Artin braid group, Br_{n+1}, defined using $n + 1$ strands is given by

- generators: y_i, $i = 1, \ldots, n$;

- relations: $r_{ij} \equiv y_i y_j y_i^{-1} y_j^{-1}$ for $i + 1 < j$;
  $r_{ii+1} \equiv y_i y_{i+1} y_i y_{i+1}^{-1} y_i^{-1} y_{i+1}^{-1}$ for $1 \leq i < n$.

We will look at such groups only for small values of n.

By default, Br_2 has one generator and no relations, so is infinite cyclic.

The group Br_3: (We will simplify notation writing $u = y_1$, $v = y_2$.)

This then has presentation $\mathcal{P} = (u, v : r \equiv uvuv^{-1}u^{-1}v^{-1})$. It is also the 'trefoil group', i.e., the fundamental group of the complement of a trefoil knot. If we construct $X(2) = K(\mathcal{P})$, this is already a $K(Br_3, 1)$ space, having a trivial π_2. There are no higher syzygies.

We have all the calculation for working with homological syzygies here. The key part of the complex is the Jacobian matrix as that determines d_2:

$$d_2 = \begin{pmatrix} 1 + uv - v & u - 1 - vu \end{pmatrix}.$$

This has trivial kernel, but, in fact, that comes most easily from the identification with homotopical syzygies.

The group Br_4: simplifying notation as before, we have generators u, v, w and relations

$$
\begin{aligned}
r_u &\equiv vwvw^{-1}v^{-1}w^{-1}, \\
r_v &\equiv uwu^{-1}w^{-1}, \\
r_w &\equiv uvuv^{-1}u^{-1}v^{-1}.
\end{aligned}
$$

The 1-syzygies are made up of hexagons for r_u and r_w and a square for r_v. There is a fairly obvious way of fitting together squares and hexagons, namely as a permutohedron, and there is a labelling of such that gives a homotopical 2-syzygy.

The presentation yields a truncated chain complex with d_2

$$\mathbb{Z}G^{(r_u, r_v, r_w)} \xrightarrow{\;d_2\;} \mathbb{Z}G^{(u,v,w)}$$

with

$$d_2 = \begin{pmatrix} 0 & 1 + vw - w & v - 1 - wv \\ 1 - w & 0 & u - 1 \\ 1 + uv - v & u - 1 - vu & 0 \end{pmatrix}$$

and Loday, [111], has calculated that for the permutohedral 2-syzygy, s, one gets another term of the resolution, $\mathbb{Z}G^{(s)}$, and a $d_3 : \mathbb{Z}G^{(s)} \to \mathbb{Z}G^{(r_u, r_v, r_w)}$ given by

$$d_3 = \begin{pmatrix} 1 + vu - u - wuv & v - vwu - 1 - uv - vuwv & 1 + vw - w - uvw \end{pmatrix}.$$

For more on methods of working with these syzygies, have a look at Loday's paper, [111], and some of the references that you will find there.

2.4 Crossed complexes and chain complexes: II

(The source for the material and ideas in this section is once again [35].)

2.4.1 The reflection from Crs to chain complexes

It is now time to return to the construction of a left adjoint for Δ_G.

Theorem 2 (*Brown-Higgins, [35] in a slightly more general form.)* The functor, Δ_G, has a left adjoint.

Proof: We construct the left adjoint explicitly as follows:

Let $f. : (\mathsf{C}, \varphi) \to \Delta_G(M.)$ be a morphism in Crs_G, then we have the following commutative diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_2 & \xrightarrow{\ \delta_2\ } & C_1 & \xrightarrow{\ \delta_1\ } & C_0 & \xrightarrow{\ \varphi\ } & G \\
& & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle Id_G} \\
\cdots & \longrightarrow & M_2 & \xrightarrow{\ \delta_2\ } & M_1 & \xrightarrow{\ \delta_1\ } & M_0 \rtimes G & \xrightarrow{\ pr_G\ } & G
\end{array}
$$

Since the right hand square commutes, f_0 is given by some formula

$$ f_0(c) = (\partial(c), \varphi(c)), $$

where $\partial : C_0 \to M_0$ is a φ-derivation. Thus $\partial = \tilde{f}_0 \partial_\varphi$ for a unique G-module morphism, $\tilde{f}_0 : D_\varphi \to M_0$, and f_0 factors as

$$ C_0 \xrightarrow{\bar{\varphi}} D_\varphi \rtimes G \xrightarrow{\tilde{f}_0 \rtimes G} M_0 \rtimes G, $$

where $\bar{\varphi}(c) = (\partial_\varphi(c), \varphi(c))$.

The map $\partial_\varphi \delta_1 : C_1 \to D_\varphi$ is a homomorphism since

$$
\begin{aligned}
\partial_\varphi \delta_1(c_1 c_2) & = \partial_\varphi \partial_1(c_1) + \varphi \partial_1(c_1) \partial_\varphi \partial_1(c_2) \\
& = \partial_\varphi \partial_1(c_1) + \partial_\varphi \partial_1(c_2),
\end{aligned}
$$

$\varphi \partial_1$ being trivial (because (C, φ) is G-augmented). We thus obtain a map $d : C_1^{Ab} \to D_\varphi$ given by $d(c[C, C]) = \partial_\varphi \partial_1(c)$ for $c \in C_1$. As we observed earlier the Abelian group C_1^{Ab} has a natural $\mathbb{Z}[G]$-module structure making d a G-module morphism.

Similarly there is a unique G-module morphism,

$$ \tilde{f}_1 : C_1^{Ab} \to M_1, $$

satisfying

$$ \tilde{f}_1(c[C, C]) = f_1(c). $$

Since for $c \in C_1$,

$$ (d_1 \tilde{f}_1(c), 1) = f_0(\delta_1 c) = (\tilde{f}_0 \partial_\varphi(\delta_1 c_1), 1), $$

we have that the diagram

$$
\begin{array}{ccc}
C_1^{Ab} & \xrightarrow{\ \tilde{f}_1\ } & M_1 \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_1} \\
D_\varphi & \xrightarrow{\ \tilde{f}_0\ } & M_0
\end{array}
$$

commutes.

We also note that since $\delta_2 : C_2 \to C_1$ maps into $Ker\ \delta_1$, the composite

$$ C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{can} C_1^{Ab} \xrightarrow{d} D_\varphi, $$

being given by $d(\delta_2(c)[C, C] = \partial_\varphi \delta_1 \delta_2(c)$, is trivial and that $\tilde{f}_1 \delta_2(c[C, C]) = f_1 \delta_2(c) = d_2 f_2(c)$, thus we can define $\xi = \xi_G(\mathsf{C}, \varphi)$ by

$$
\begin{aligned}
\xi_n & = C_n \text{ if } n \geq 2 \\
\xi_1 & = C_1^{Ab}, \\
\xi_0 & = D_\varphi,
\end{aligned}
$$

the differentials being as constructed. We note that as $Ker\ \varphi$ acts trivially on all C_n for $n \geq 2$, all the C_n have $\mathbb{Z}[G]$-module structures.

That ξ_G gives a functor

$$Crs \to Ch(G-Mod)$$

is now easy to check using the uniqueness clauses in the universal properties of D_φ and Abelianisation. Again uniqueness guarantees that the process "f goes to $\tilde{f}$" gives a natural isomorphism

$$Ch(G - Mod)(\xi_G(\mathsf{C}, \varphi), \mathsf{M}) \cong Crs_G((\mathsf{C}, \varphi), \Delta_G(\mathsf{M}))$$

as required. ∎

It is relatively easy to extend the above natural isomorphism to handle morphisms of crossed complexes over different groups. For a detailed treatment one needs a discussion of the way that the change of groups functors work with crossed modules or crossed complexes, that is, if we have a morphism of groups $\theta : G \to H$ then we would expect to get functors between Crs_G and Crs_H induced by θ. These do exist and are very nicely behaved, but they will not be discussed here, see [141] for a full treatment in the more general context of profinite groups.

2.4.2 Crossed resolutions and chain resolutions

One of our motivations for introducing crossed complexes was that they enable us to model more of the sort of information encoded in a $K(G,1)$ than does the usual standard algebraic models, e.g. a chain complex such as the bar resolution. In particular, whilst the bar resolution is very good for cohomology with Abelian coefficients for non-Abelian cohomology the crossed version can allow us to push things further, but then comparison on the Abelian theory is very necessary! It is therefore of importance to see how this $K(G,1)$ information that we have encoded changes under the functor $\xi : Crs \to Ch(G-Mod)$.

We start with a crossed resolution determined in low dimensions by a presentation $\mathcal{P} = (X : R)$ of a group, G. Thus, in this case, $C_0 = F(X)$ with $\varphi : F(X) \to G$, the 'usual' epimorphism, and $C_1 \to C_0$ is $C \to F(X)$, the free crossed module on $R \to F(X)$. It is not too hard to show that $C_1^{Ab} \cong \mathbb{Z}[G]^{(R)}$, the free $\mathbb{Z}[G]$-module on R. (The proof is left as an exercise.) This maps down onto $N(R)^{Ab}$, the Abelianisation of the normal closure of R in $F(X)$ via a map

$$\partial_* : \mathbb{Z}[G]^{(R)} \to N(R)^{Ab},$$

given by $\partial_*(e_r) = r[N(R), N(R)]$, where e_r is the generator of $\mathbb{Z}[G]$ corresponding to $r \in R$.

There is also a short exact sequence

$$1 \to N(R) \overset{i}{\to} F(X) \overset{\varphi}{\to} G \to 1$$

and hence by Proposition 7, a short exact sequence

$$0 \to N(R)^{Ab} \overset{\tilde{i}}{\to} \mathbb{Z}[G] \otimes_F I(F) \overset{\tilde{\varphi}}{\to} I(G) \to 0$$

(where we have written $F = F(X)$).

By the Corollary to Proposition 5, we have

$$\mathbb{Z}[G] \otimes_F I(F) \cong \mathbb{Z}[G]^{(X)}.$$

The required map $C_1^{Ab} \to D_\varphi$ is the composite

$$\mathbb{Z}[G]^{(R)} \overset{\partial_*}{\to} N(R)^{Ab} \overset{\tilde{i}}{\to} \mathbb{Z}[G]^{(X)}.$$

We have given an explicit description of ∂_* above, so to complete the description of d, it remains to describe $\tilde{i}$, but $\tilde{i}$ satisfies $\tilde{i}\delta = \partial_\varphi i$, where $\delta : N(R) \to N(R)^{Ab}$, so $\tilde{i}(r[N(R), N(R)]) = d_\varphi(r)$. Thus if r is a relator, i.e., if it is in the image of the subgroup generated by the elements of R, then $\partial(e_r)$ can be written as a finite sum of the form $\sum_x a_x e_x$ and the elements $a_x \in \mathbb{Z}[G]$ are the images of the Fox derivatives.

This operator can best be viewed as the Alexander matrix of a presentation of a group, further study of this operator depends on studying transformations between free modules over group rings, and we will not attempt to study those here.

The rest of the crossed resolution does not change and so, on replacing $I(G)$ by $\mathbb{Z}[G] \to \mathbb{Z}$, we obtain a free pseudocompact $\mathbb{Z}[G]$-resolution of the trivial module $\mathbb{Z}$,

$$\ldots \to \mathbb{Z}[G]^{(R)} \overset{d}{\to} \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}$$

built up from the presentation. This is the complex of chains on the universal cover, $\widetilde{K(G,1)}$, where $K(G,1)$ is constructed starting from a presentation $\mathcal{P}$.

2.4.3   Standard crossed resolutions and bar resolutions

We next turn to the special case of the standard crossed resolution of G discussed briefly earlier. Of course this is a special case of the previous one, but it pays to examine it in detail.

Clearly in $\xi = \xi(\mathsf{C}G, \varphi)$, we have:
$\xi_0 = $ the free $\mathbb{Z}[G]$-module on the underlying set of G, individual generators being written $[u]$, for $u \in G$;
$\xi_1 = $ the free $\mathbb{Z}[G]$ -module on $G \times G$, generators being written $[u, v]$;
$\xi_n = C_n G$, the free $\mathbb{Z}[G]$ -module on G^{n+1}, etc.

The map $d_2 : \xi_2 \to \xi_1$ induced from δ_2 is given by

$$d_2[u, v, w] = u[v, w] - [u, v] - [uv, w] + [u, vw],$$

and the map $d_1 : \xi_1 \to \xi_0$ by

$$\begin{aligned} d_1([u, v]) &= d_\varphi([uv]^{-1}[u][v]) \\ &= v^{-1}u^{-1}(-[uv] + [u] + u[v]), \end{aligned}$$

a unit times the usual bar resolution formula. Thus, as claimed earlier, the standard crossed resolution is the crossed analogue of the bar resolution.

2.4.4   The intersection $A \cap [C, C]$.

We next turn to a comparison of homological and homotopical syzygies. We have almost all the preliminary work already. The next ingredient is a result that will identify the intersection of the kernel of a crossed module, $A = Ker(C \overset{\partial}{\to} P)$ and the commutator subgroup of C.

The kernel of the homomorphism from A to C^{Ab} is, of course, $A \cap [C, C]$ and this need not be trivial. In fact, Brown and Huebschmann ([38], p.160) note that in examples of type $(G, Aut(G), \partial)$,

the kernel of ∂ is, of course, the centre ZG of G and $ZG \cap [G, G]$ can be non-trivial, for instance, if G is dicyclic or dihedral.

We will adopt the same notation as previously with $N = \partial P$ etc.

Proposition 9 *If, in the exact sequence of groups*

$$1 \to A \to C \xrightarrow{p} N \to 1,$$

the epimorphism from C to N is split (the splitting need not respect G-action), then $A \cap [C, C]$ is trivial.

Proof: Given a splitting $s : N \to C$, (so ps is the identity on N), then the group C can be written as $A \rtimes s(N)$. The commutators in C, therefore, all lie in $s(N)$ since A is Abelian, but then, of course, $A \cap [C, C]$ cannot contain any non-trivial elements. ∎

We used this proposition earlier in the case where N is free. We are thus using the fact that subgroups of free groups are free, in that case. Of course, any epimorphism with codomain a free group is split.

Brown and Huebschmann, [38], p. 168, prove that for an group G with presentation $\mathcal{P}$, the module of identities for $\mathcal{P}$ is naturally isomorphic to the second homology group, $H_2(\tilde{K}(\mathcal{P}))$, of the universal cover of $K(\mathcal{P})$, the 2-complex of the presentation. We can approach this via the algebraic constructions we have.

Given a presentation $\mathcal{P} = \langle X : R \rangle$ of a group G, the algebraic analogue of $K(\mathcal{P})$, we have argued above, is the free crossed module $C(\mathcal{P}) \xrightarrow{d} F(X)$ and the chains on the universal cover of $K(\mathcal{P})$ will be given by ξ_G of this, i.e., by the chain complex

$$\mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)}.$$

In general there will be a short exact sequence

$$0 \to \kappa(\mathcal{P}) \cap [C(\mathcal{P}), C(\mathcal{P})] \to \kappa(\mathcal{P}) \to H_2(\xi(C(\mathcal{P})) \to 0.$$

This short exact sequence yields the Brown-Huebschmann result as $N(R)$ will a free group so the epimorphism onto $N(R)$ splits and we can use the above Proposition 9. We thus get

Proposition 10 *If $\mathcal{P} = \langle X : R \rangle$ is a free presentation of G, then there is an isomorphism*

$$\kappa \xrightarrow{\cong} H_2(\xi(C_\mathcal{C}(\mathcal{P})) = Ker(d : \mathbb{Z}[G]^R \to \mathbb{Z}[G]^X).$$

∎

Note: Here we are using something that will not be true in all algebraic settings. A subgroup of a free group is always free, but the analogous statement for free algebras of other types is not true.

2.5 Simplicial groups and crossed complexes

2.5.1 From simplicial groups to crossed complexes

Given any simplicial group G, the formula,

$$\mathsf{C}(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions with, at its 'bottom end', the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0$$

gives a crossed complex with ∂ induced from the boundary in the Moore complex. The detailed proof is too long to indicate here. It just checks the axioms, one by one.

We should have a glance at this formula from various viewpoints, some of which will be revisited later. Once again there is a clear link with the non-uniqueness of fillers for horns in a simplicial group if it is not a group T-complex. We have all those $(NG_n \cap D_n)$ terms involved!

Suppose that we had our simplicial group G and wanted to construct a quotient of it that was a group T-complex. We could do this in a silly way since the trivial simplicial group is clearly a group T-complex, but let us keep the quotient as large as possible. This problem is related to the question of whether the category of group T-complexes forms a reflexive subcategory of $Simp.Grps$. The condition $NG \cap D = 1$ looks like some sort of 'equational specification'. Our question can thus really be posed as follows: Suppose we have a simplicial group morphism $f : G \to H$ and H is a group T-complex. Remember that in group T-complexes, as against the non-algebraic ones, the thin structure is not an added bit of structure. The thin elements are determined by the degeneracies, so whether or not H is or is not a group T-complex is somehow its own affair, and nothing to do with any external factors! Does f factor universally through some 'group T-complexification' of G? Something like

$$G \xrightarrow{\ f\ } H$$

with $G/T(G)$ a group T-complex and $\hat{f}$ uniquely determined by the diagram.

One sensible way to look at such a question is to assume, provisionally, that such a factorisation exists and to see what $T(G)$ would have to be. In general, if $f : G \to H$ is any simplicial group morphism (with no restriction on H for the moment), then with a hopefully obvious notation,

$$f_n(NG_n \cap D(G)_n) \subseteq NH_n \cap D(H)_n,$$

since f sends degenerate elements to degenerate elements and preserves products! Back in our situation in which H is a group T-complex, then $f_n(NG_n \cap D(G)_n) = 1$, for the simple reason that the right hand side of that displayed formula is trivial by assumption. We thus have that if some such $T(G)$ exists, then we must have $NG_n \cap D(G)_n \subseteq T(G)_n$ and our first attempt might be to look at the possibility that they should be equal. This is wrong and for fairly trivial reasons. The subgroup $T(G)_n$ of G_n has to be normal if we are to form the quotient by it, and there is no reason why $NG_n \cap D(G)_n$ should be a normal subgroup in general.

We might then be tempted to take the normal subgroup generated by $NG_n \cap D(G)_n$, but that is 'defeatist' in this situation. We might hope to do detailed calculations with the subgroup and if it is specified as a normal closure, we will lose some of our ability to do that, at least without considerable more effort. (Let's be lazy and see if we can get around that difficulty.) If we look again, we find another thing that 'goes wrong' with any attempt to use $NG_n \cap D(G)_n$ as it is. This subgroup would be within NG_n, of course, and we want to induce a map from the Moore complex of G to that of $G/T(G)$. For that to work, we would need not only $NG_n \cap D(G)_n \subseteq T(G)_n$, but the image of $NG_n \cap D(G)_n$ under d_0 to be in $T(G)_{n-1}$. Going up a dimension, we thus need not only $NG_n \cap D(G)_n$, but $d_0(NG_{n+1} \cap D(G)_{n+1}) \subseteq T(G)_n$. We thus need the product subgroup $(NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$ to be in $T(G)_n$. This looks a bit complicated. Do we need to go any further up the Moore complex? No, because $d_0 d_0$ is trivial. We might thus try

$$T(G)_n = (NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$$

You might now think that this is a bit silly because we would still need this product subgroup to be normal in order to form the quotient ... , but it is! The lack of normality of our earlier attempt is absorbed by the image of the next level up. (That is pretty!)

Of course, there are very good reasons why this works. These involve what are sometimes called *Peiffer pairings*. We will see some of these later.

As a consequence of the above discussion, we more or less have:

Proposition 11 *If G is a group T-complex, then NG is a crossed complex.* ∎

We certainly have a sketch of

Proposition 12 *The full subcategory of Simp.Grps determined by the group T-complexes is a reflective subcategory.* ∎

Of course, the details of the proofs of both of these are left for you to write down. Nearly all of the reasoning for the second result is there for you, but some of the detailed calculations for the first are quite tricky.

The close link between group T-complexes and crossed complexes is evident from these results. You might guess that they form equivalent categories. They do. We will look at the way back from crossed complexes (of groups) to simplicial groups later on, but we need to get back to cohomology.

2.5.2 Simplicial resolutions, a bit of background

We need some such means of going from simplicial groups to crossed complexes so because we can also use simplicial resolutions to 'resolve' a group (and in many other situations). We first sketch in some historical background.

In the 1960s, the connection between simplicial groups and cohomology was examined in detail. The basic idea was that given the adjoint "free-forget" pair of functors between *Groups* and *Sets*, one could generate a free resolution of a group, G, using the resulting comonad (or cotriple) (cf. MacLane, [114]). This resolution was not, however, by a chain complex but by a free simplicial group, F, say. It was then shown (Barr and Beck, [13]) that given any G-module, M, and working in the category of groups over G, one could form the cosimplicial G-module, $Hom_{Gps/G}(F, M)$, and hence, by a dual form of the Dold-Kan theorem, a cochain complex $C(G, M)$, whose homotopy type, and hence whose homology, was independent of the choice of F. This homology was the usual

Eilenberg-MacLane cohomology of G with coefficients in M, but with a shift in dimension (cf. Barr and Beck, [13]).

Other theories of cohomology were developed at about the same time by Grothendieck and Verdier, [8], André, [6, 7], and Quillen, [143, 144]. The first of these was designed for use with "sites", that is, categories together with a Grothendieck topology.

André and Quillen developed, independently, a method of defining cohomology using simplicial resolutions. Their work is best known in commutative algebra, but their methods work in greater generality. Unlike the theory of Barr and Beck (monadic cohomology), they only assume there is enough structure to construct free resolutions; a (co)monad is just one way of doing this. In particular, André, [6, 7], describes a step-by-step, almost combinatorial, process for constructing such resolutions. This ties in well with our earlier comments about using a presentation of a group to construct a crossed resolution and the important link with syzygies. André's method is the simplicial analogue of this.

We will assume for the moment that we have a simplicial resolution, F, of our group, G. Both André and Quillen then consider applying a derived module construction dimensionwise to F, obtaining a simplicial G-module. They then use "Dold-Kan" to give a chain complex of G-modules, which they call the "cotangent complex", denoted L_G or $LAb(G)$, of G (at least in the case of commutative algebras). The homotopy type of $LAb(G)$ does not depend on the choice of resolution and so is a useful invariant of G. We will need to look at this construction in more detail, but will consider a slightly more general situation to start with.

2.5.3 Free simplicial resolutions

Standard theory (cf. Duskin, [64]) shows that if F and F' are free simplicial resolutions of groups, G and H, say, and $f : G \to H$ is a morphism, then f can be lifted to $f' : F \to F'$. The method is the simplicial analogue of lifting a homomorphism of modules to a map of resolutions of those modules, which you should look at first as it is technically simpler. Any two such lifts are homotopic (by a simplicial homotopy).

Of course, f will also lift to a morphism of crossed complexes, $f : \mathsf{C}(F) \to \mathsf{C}(F')$, and any two such lifts will be *homotopic* as crossed complex morphisms. Thus whatever simplicial lift, $f' : F \to F'$, we choose, $\mathsf{C}(f')$ will be a lift in the "crossed" case, and although we do not know at this stage of our discussion of the theory if a homotopy between two simplicial lifts is transferred to a homotopy between the images under C, this does not matter as the *relation* of homotopy is preserved at least in this case of resolutions.

Any group has a free simplicial resolution. There is the obvious adjoint pair of functors

$$U \quad : \quad Groups \to Sets$$
$$F \quad : \quad Sets \to Groups$$

Writing $\eta : Id \to UF$ and $\varepsilon : FU \to Id$ for the unit and counit of this adjunction (cf. MacLane, [114, 115]), we have a comonad (or cotriple) on $Groups$, the free group comonad, $(FU, \varepsilon, F\eta U)$. We write $L = FU$, $\delta = F\eta U$, so that

$$\varepsilon : L \to I$$

is the counit of the comonad whilst

$$\delta : L \to L^2$$

is the comultiplication. (For the reader who has not met monads or comonads before, (L, η, δ) behaves as if it was a monoid in the dual of the category of "endofunctors" on $Groups$, see MacLane, [115] Chapter VI. We will explore them briefly in section **??**, starting on page **??**.)

Now suppose G is a group and set $F(G)_i = L^{i+1}(G)$, so that $F(G)_0$ is the free group on the underlying set of G and so on. The counit (which is just the augmentation morphism from $FU(G)$ to G) gives, in each dimension, face morphisms

$$d_i = L^{n-i} \varepsilon L^i(G) : L^{n+1}(G) \to L^n(G),$$

for $i = 0, \ldots, n$, whilst the comultiplication gives degeneracies

$$s_i : L^n(G) \to L^{n+1}(G)$$

$$s_i = L^{n-1-i} \delta L^i,$$

for $i = 0, \ldots, n - 1$, satisfying the simplicial identities.

Remark: Here we follow the conventions used by Duskin, in his Memoir, [64]. Later we will also need to look at similar resolutions where the labelling of the faces and degeneracies are reversed.

This simplicial group, $F(G)$, satisfies $\pi_0(F(G)) \cong G$ (the isomorphism being induced by $\varepsilon(G) : F_0(G) \to G$) and $\pi_n(F(G))$ is trivial if $n \geq 1$. The reason for this is simple. If we apply U once more to $F(G)$, we get a simplicial set and the unit of the adjunction

$$\eta : 1 \to UF$$

allows one to define for each n

$$\eta U(FU)^n : UL^n \to UL^{n+1},$$

which gives a natural contraction of the augmented simplicial set, $UF(G) \to U(G)$, (cf. Duskin, [64]). We will look at this in detail in our later treatment of augmentations, etc. For the moment, it suffices to accept the fact that we do get a resolution, as we do not need to know the details of why this construction works, at least not yet.

If we denote the constant simplicial group on G by $K(G, 0)$, the augmentation defines a simplicial homomorphism

$$\bar{\varepsilon} : F(G) \to K(G, 0)$$

satisfying $U\bar{\varepsilon}.inc = Id$, where $inc : UK(G, 0) \to UF(G)$ is the 'inclusion' of simplicial sets given by η, and then these extra maps, $(UF)^n \eta U$, in fact, give a homotopy between $inc.U\bar{\varepsilon}$ and the identity map on $UF(G)$, i.e., $\bar{\varepsilon}$ is a weak homotopy equivalence of simplicial groups. Thus $F(G)$ is a free simplicial resolution of G. It is called the *comonadic free simplicial resolution* of G.

This simplicial resolution has the advantage of being functorial, but the disadvantage of being very big. We turn next to a 'step-by-step' method of constructing a simplicial resolution using ideas pioneered by André, [7], although most of his work was directed more towards commutative algebras, cf. [6].

2.5.4   Step-by-Step Constructions

This section is a brief résumé of how to construct simplicial resolutions by hand rather than functorially. This allows a better interpretation of the generators in each level of the resolution. These are the simplicial analogues of higher syzygies. The work depends heavily on a variety of sources, mainly [6], [105] and [124]. André only treats commutative algebras in detail, but Keune [105] does discuss the general case quite clearly. The treatment here is adapted from the paper by Mutlu and Porter, [128].

Recall of notation: We first recall some notation and terminology, which will be used in the construction of a simplicial resolution. Let $[n]$ be the ordered set, $[n] = \{0 < 1 < \cdots < n\}$. Define the following maps: the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \geq i, \end{cases}$$

for $0 \leq i \leq n \neq 0$. The increasing surjective monotone map $\alpha_i^n : [n+1] \to [n]$ is given by

$$\alpha_i^n(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i, \end{cases}$$

for $0 \leq i \leq n$. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$.

2.5.5   Killing Elements in Homotopy Groups

Let G be a simplicial group and let $k \geq 1$ be fixed. Suppose we are given a set, Ω, of elements: $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(\mathsf{G})$, then we can choose a corresponding set of elements $\theta_\lambda \in NG_{k-1}$ so that $x_\lambda = \theta_\lambda \, \partial_k(NG_k)$. (If $k = 1$, then as $NG_0 = G_0$, the condition that $\theta_\lambda \in NG_0$ is immediate.) We want to 'kill' the elements in Ω.

We form a new simplicial group F_n where
1) F_n is the free G_n-group, (i.e., group with G_n-action)

$$F_n = \coprod_{\lambda,t} G_n\{y_{\lambda,t}\} \text{ with } \lambda \in \Lambda \text{ and } t \in \{n, k\},$$

where $G_n\{y\} = G_n * <y>$, the co-product of G_n and a free group generated by y.

2) For $0 \leq i \leq n$, the group homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : G_n \to G_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \quad \text{with} \quad u = t\alpha_i^n, \quad t : [n] \to [k].$$

3) For $0 \leq i \leq n \neq 0$, the group homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : G_n \to G_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map} \quad u = t\delta_i^n \quad \text{is surjective,} \\ t'(\theta_\lambda) & \text{if} \qquad u = \delta_k^k t', \\ 1 & \text{if} \qquad u = \delta_j^k t' \quad \text{with } j \neq k, \end{cases}$$

by extending multiplicatively.

We sometimes denote the F, so constructed by $\mathsf{G}(\Omega)$.

Remark: In a 'step-by-step' construction of a simplicial resolution, (see below), there will thus be the following properties: i) $F_n = G_n$ for $n < k$, ii) F_k = a free G_k-group over a set of non-degenerate indeterminates, all of whose faces are the identity except the kth, and iii) F_n is a free G_n-group on some degenerate elements for $n > k$.

We have immediately the following result, as expected.

Proposition 13 *The inclusion of simplicial groups $G \hookrightarrow F$, where $F = G(\Omega)$, induces a homomorphism*

$$\pi_n(G) \longrightarrow \pi_n(F)$$

for each n, which for $n < k - 1$ is an isomorphism,

$$\pi_n(G) \cong \pi_n(F)$$

and for $n = k - 1$, is an epimorphism with kernel generated by elements of the form $\bar{\theta}_\lambda = \theta_\lambda \partial_k NG_k$, where $\Omega = \{x_\lambda : \lambda \in \Lambda\}$. ∎

2.5.6 Constructing Simplicial Resolutions

The following result is essentially due to André, [6].

Theorem 3 *If G is a group, then it has a free simplicial resolution $\mathbb{F}$.*

Proof: The repetition of the above construction will give us the simplicial resolution of a group. Although 'well known', we sketch the construction so as to establish some notation and terminology.

Let G be a group. The zero step of the construction consists of a choice of a free group F and a surjection $g : F \to G$ which gives an isomorphism $F/Ker\,g \cong G$ as groups. Then we form the constant simplicial group, $F^{(0)}$, for which in every degree n, $F_n = F$ and $d_i^n = \mathrm{id} = s_j^n$ for all i, j. Thus $F^{(0)} = K(F, 0)$ and $\pi_0(F^{(0)}) = F$. Now choose a set, Ω^0, of normal generators of the closed normal subgroup $N = Ker\,(F \xrightarrow{g} G)$, and obtain the simplicial group in which $F_1^{(1)} = F(\Omega^0)$ and for $n > 1$, $F_n^{(1)}$ is a free F_n-group over the degenerate elements as above. This simplicial group will be denoted by $F^{(1)}$ and will be called the *1-skeleton of a simplicial resolution of the group G.*

The subsequent steps depend on the choice of sets, $\Omega^0, \Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$. Let $F^{(k)}$ be the simplicial group constructed after k steps, that is, *the k-skeleton of the resolution.* The set Ω^k is formed by elements a of $F_k^{(k)}$ with $d_i^k(a) = 1$ for $0 \le i \le k$ and whose images $\bar{a}$ in $\pi_k(F^{(k)})$ generate that module over $F_k^{(k)}$ and $F^{(k+1)}$.

Finally we have inclusions of simplicial groups

$$F^{(0)} \subseteq F^{(1)} \subseteq \cdots \subseteq F^{(k-1)} \subseteq F^{(k)} \subseteq \cdots$$

and in passing to the inductive limit (colimit), we obtain an acyclic free simplicial group F with $F_n = F_n^{(k)}$ if $n \le k$. This F, or, more exactly, (F, g), is thus a simplicial resolution of the group G.

The proof of theorem is completed. ∎

Remark: A variant of the 'step-by-step' construction gives: *if G is a simplicial group, then there exists a free simplicial group F and a continuous epimorphism $F \longrightarrow G$ which induces isomorphisms on all homotopy groups.* The details are omitted as they should be reasonably clear.

The key observation, which follows from the universal property of the construction, is a freeness statement:

Proposition 14 *Let $F^{(k)}$ be a k-skeleton of a simplicial resolution of G and $(\Omega^k, g^{(k)})$ k-dimension construction data for $F^{(k+1)}$. Suppose given a simplicial group morphism $\Theta : F^{(k)} \longrightarrow G$ such that $\Theta_*(g^{(k)}) = 0$, then Θ extends over $F^{(k+1)}$.*

This freeness statement does not contain a uniqueness clause. That can be achieved by choosing a lift for $\Theta_k g^{(k)}$ to NG_{k+1}, a lift that must exist since $\Theta_*(\pi_k(F^{(k)}))$ is trivial.

When handling combinatorially defined resolutions, rather than functorially defined ones, this proposition is as often as close to 'left adjointness' as is possible without entering the realm of homotopical algebra to an extent greater than is desirable for us here.

We have not talked here about the homotopy of simplicial group morphisms, and so will not discuss homotopy invariance of this construction for which one adapts the description given by André, [6], or Keune, [105]. Of course, the resolution one builds by any means would be homotopicallly equivalent to any other so, for cohomological purposes, it makes no difference how the resolution is built.

Of course, from any simplicial resolution F of G, you can get an augmented crossed complex $C(F)$ over G using the formula given earlier and this is a crossed resolution.

2.6   Cohomology and crossed extensions

2.6.1   Cochains

Consider a G-module, M, and a non-negative integer n. We can form the chain complex, $K(M, n)$, having M in dimension n and zeroes elsewhere. We can also form a crossed complex, $\mathsf{K}(M, n)$, that plays the role of the nth Eilenberg-MacLane space of M in this setting. We may call it the nth *Eilenberg-MacLane crossed complex* of M:

If $n = 0$, $\mathsf{K}(M, n)_0 = M \rtimes G$, $\mathsf{K}(M, n)_i = 0$, $i > 0$.

If $n \geq 1$, $\mathsf{K}(M, n)_0 = G$, $\mathsf{K}(M, n)_n = M$, $\mathsf{K}(M, n)_i = 0$, $i \neq 0$ or n.

One way to view cochains is as chain complex morphisms. Thus on looking at $Ch(\mathsf{B}G, K(M, n))$, one finds exactly $Z^{n+1}(G, M)$, the $(n+1)$-cocycles of the cochain complex $C(G, M)$. We can also view $Z^{n+1}(G, M)$ as $Crs_G(\mathsf{C}G, \mathsf{K}(M, n))$.

In the category of chain complexes, one has that a homotopy from $\mathsf{B}G$ to $K(M, n)$ between 0 and f, say, is merely a coboundary, so that $H^{n+1}(G, M) \cong [\mathsf{B}G, K(M, n)]$, adopting the usual homotopical notation for the group of homotopy classes of maps from the bar resolution $\mathsf{B}G$ to $K(M, n)$. This description has its analogue in the crossed complex case as we shall see.

2.6.2   Homotopies

Let C, C' be two crossed complexes with Q and Q' respectively as the cokernels of their bottom morphism. Suppose $\lambda, \mu : \mathsf{C} \to \mathsf{C}'$ are two morphisms inducing the same map $\varphi : Q \to Q'$.

A homotopy from λ to μ is a family, $h = \{h_k : k \geq 1\}$, of maps $h_k : C_k \to C'_{k+1}$ satisfying the following conditions:

H1) $h_0 : C_1 \to C'_2$ is a derivation along μ_0 (i.e. for $x, y \in C_0$,

$$h_0(xy) = h_0(x)(^{\mu_0}h_0(y)),)$$

such that

$$\delta_1 h_0(x) = \lambda_0(x)\mu_0(x)^{-1}, \quad x \in C_0.$$

H2) $h_1 : C_1 \to C_2'$ is a C_0-homomorphism with C_0 acting on C_2' via λ_0 (or via μ_0, it makes no difference) such that

$$\delta_2 h_1(x) = \mu_1(x)^{-1}(h_0\delta_1(x)^{-1}\lambda_1(x)) \text{ for } x \in C_1.$$

H3) for $k \geq 2$, h_k is a Q-homomorphism (with Q acting on the C_k' via the induced map $\varphi : Q \to Q'$) such that

$$\delta_{k+1}h_k + h_{k-1}\delta_k = \lambda_k - \mu_k.$$

We note that the condition that λ and μ induce the same map, $\varphi : Q \to Q'$, is, in fact, superfluous as this is implied by $H1$.

The properties of homotopies and the relation of homotopy are as one would expect. One finds $H^{n+1}(G, M) \cong [\mathsf{C}G, \mathsf{K}(M, n)]$. Given that in higher dimensions, this is the same set exactly as $[\mathsf{B}G, K(M, n)]$ means that there is not much to check and so the proof has been omitted.

2.6.3   Huebschmann's description of cohomology classes

The transition from this position to obtaining Huebschmann's descriptions of cohomology classes, [96], is now more or less formal. We will, therefore, only sketch the main points.

If G is a group, M is a G-module and $n \geq 1$, a *crossed n-fold extension* is an exact augmented crossed complex,

$$0 \to M \to C_n \to \ldots \to C_2 \to C_1 \to G \to 1.$$

The notion of similarity of such extensions is analogous to that of n-fold extensions in the Abelian Yoneda theory, (cf. MacLane, [114]), as is the definition of a Baer sum. We leave the details to you. This yields an Abelian group, $Opext^n(G, M)$, of similarity classes of crossed n-fold extensions of G by M.

Given a cohomology class in $H^{n+1}(G, M)$ realisable as a homotopy class of maps, $f : \mathsf{C}G \to \mathsf{K}(M, n)$, one uses f to form an induced crossed complex, much as in the Abelian Yoneda theory:

$$
\begin{array}{ccccccccc}
J_n(G) & \longrightarrow & C_n & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & G \\
\downarrow{\scriptstyle f'} & {\scriptstyle \text{pushout}} & \downarrow & & & & \downarrow & & \| \\
0 & \longrightarrow & M & \longrightarrow & M_n & \longrightarrow \cdots \longrightarrow & M_1 & \longrightarrow & G
\end{array}
$$

where $J_n(G)$ is $Ker(C_nG \to C_{n-1}G)$. (Thus J_nG is also $Im(C_{n+1}G \to C_nG)$ and as the map f satisfies $f\delta = 0$, it is zero on the subgroup $\delta(C_{n+2}G)$ (i.e. is constant on the cosets) and hence passes to $Im(C_{n+1}G \to C_nG)$ in a well defined way.) Arguments using lifting of maps and homotopies show that the assignment of this element of $Opext^n(G, M)$ to $cls(f) \in H^{n+1}(G, M)$ establishes an isomorphism between these groups.

2.6.4   Abstract Kernels.

The importance of having such a description of classes in $H^n(G, M)$ probably resides in low dimensions. To describe classes in $H^3(G, M)$, one has, as before, crossed 2-fold extensions

$$0 \to M \to C_2 \xrightarrow{\partial} C_1 \to G \to 1,$$

where ∂ is a crossed module. One has for any group G, a crossed 2-fold extension

$$0 \to Z(G) \to G \overset{\partial_G}{\to} Aut(G) \to Out(G) \to 1$$

where ∂_G sends $g \in G$ to the corresponding inner automorphism of G. An *abstract kernel* (in the sense of Eilenberg-MacLane, [72]) is a homomorphism $\psi : Q \to Out(G)$ and hence provides, by pulling back, a 2-fold extension of Q by the centre $Z(G)$ of G.

2.7   2-types and cohomology

In classifying homotopy types and in obstruction theory, one frequently has invariants that are elements in cohomology groups of the form $H^m(X, \pi)$, where typically π is the nth homotopy group of some space. When dealing with homotopy types, π will be a group, usually Abelian with a π_1-action, i.e., we are exactly in the situation described earlier, except that X is a homotopy type not a group. Of course, provided that X is connected, we can replace X by a simplicial group, bringing us even nearer to the situation of this section. We shall work within the category of simplicial groups.

2.7.1   2-types

A morphism

$$f : G \to H$$

of simplicial groups is called a *2-equivalence* if it induces isomorphisms

$$\pi_0(f) : \pi_0(G) \to \pi_0(H,)$$

and

$$\pi_1(f) : \pi_1(G) \to \pi_1(H).$$

We can form a quotient category, $Ho_2(Simp.Grps)$, of $Simp.Grps$ by formally inverting the 2-equivalences, then we say two simplicial groups, G and H, have the same *2-type*, (or, more exactly, *homotopy 2-type*), if they are isomorphic in $Ho_2(Simp.Grps)$.

This is, of course, just a special case of the general notion of n-type in which "n-equivalences" are inverted, thus forming the quotient category $Ho_n(Simp.Grps)$.

We recall the following from earlier:

Definition: An *n-equivalence* is a morphism, f, of simplicial groups (or groupoids) inducing isomorphisms, $\pi_i(f)$, for $i = 0, 1, \ldots, n - 1$.

Definition: Two simplicial groups, G and H, have the same *n-type* (or, more exactly, *homotopy n-type* if they are isomorphic in $Ho_n(Simp.Grps)$.

Sometimes it is convenient to say that a simplicial group, G, *is* an n-type. This is taken to mean that it represents an n-equivalence class *and* has zero homotopy groups above dimension $n - 1$.

2.7.2 Example: 1-types

Before examining 2-types in detail, it will pay to think about 1-types. A morphism f as above is a 1-equivalence if it induces an isomorphism on π_0, i.e., $\pi_0(f)$ is an isomorphism. Given any group G, there is a simplicial group, $K(G, 0)$ consisting of G in each dimension with face and degeneracy maps all being identities. Given a simplicial group, H, having $G \cong \pi_0(H)$, the natural quotient map

$$H_0 \to \pi_0(H) \cong G,$$

extends to a natural 1-equivalence between H and $K(\pi_0(H), 0)$.

It is fairly routine to check that

$$\pi_0 : Simp.Grps \to Grps$$

has $K(-, 0)$ as an adjoint and that, as the unit is a natural 1-equivalence, and the counit an isomorphism, this adjoint pair induces an equivalence between the category $Ho_1(Simp.Grps)$ of 1-types and the category, $Grps$, of groups. In other words,

groups are algebraic models for 1-types.

2.7.3 Algebraic models for n-types?

So much for 1-types. Can one provide algebraic models for 2-types or, in general, n-types? We touched on this earlier. The criteria that any such "models" might satisfy are debatable. Perhaps ideally, or even unrealistically, there should be an isomorphism class of algebraic "gadgets" for each 2-type. An alternative weaker solution is to ask that a notion of equivalence between the models is possible, and that only equivalence classes, not isomorphism classes, correspond to 2-types, but, in addition, the notion of equivalence is algebraically defined. It is this weaker possibility that corresponds to our aim here.

2.7.4 Algebraic models for 2-types.

If G is a simplicial group, then we can form a crossed module

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0,$$

where the action of G_0 is via the degeneracy, $s_0 : G_0 \to G_1$, and ∂ is induced by d_0. (As before we will denote this crossed module by $M(G, 1)$.) The kernel of ∂ is

$$\frac{Ker\, d_0 \cap Ker\, d_1}{d_0(NG_2)} \cong \pi_1(G),$$

whilst its cokernel is

$$\frac{G_0}{d_0(NG_1)} \cong \pi_0(G),$$

and so we have a crossed 2-fold extension

$$0 \to \pi_1(G) \to \frac{NG_1}{d_0(NG_2)} \to G_0 \to \pi_0(G) \to 1$$

and hence a cohomology class $k(G) \in H^3(\pi_0(G), \pi_1(G))$.

Suppose now that $f : G \to H$ is a morphism of simplicial groups, then one obtains a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(G) & \longrightarrow & \frac{NG_1}{d_0(NG_2)} & \longrightarrow & G_0 & \longrightarrow & \pi_0(G) & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \pi_1(f)} & & \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \pi_0(f)} & & \\
0 & \longrightarrow & \pi_1(H) & \longrightarrow & \frac{NH_1}{d_0(NH_2)} & \longrightarrow & H_0 & \longrightarrow & \pi_0(H) & \longrightarrow & 1
\end{array}
$$

If, therefore, f is a 2-equivalence, $\pi_0(f)$ and $\pi_1(f)$ will be isomorphisms and the diagram shows that, modulo these isomorphisms, $k(G)$ and $k(H)$ are the same cohomology class, i.e. the 2-type of G determines π_0, π_1 and this cohomology class, k in $H^3(\pi_0, \pi_1)$.

Conversely, suppose we are given a group π, a π-module, M, and a cohomology class $k \in H^3(\pi, M)$, then we can realise k by a 2-fold extension

$$0 \to M \to C \xrightarrow{\partial} G \to \pi \to 1$$

as above.

The crossed module, $\mathsf{C} = (C, G, \partial)$, determines a simplicial group $K(\mathsf{C})$ as follows:

Suppose $\mathsf{C} = (C, P, \partial)$ is any crossed module, we construct a simplicial group, $K(\mathsf{C})$, by

$$K(\mathsf{C})_0 = P, \qquad K(\mathsf{C})_1 = C \rtimes P,$$

$$s_0(p) = (1, p), \; d_0^1(c, p) = \partial c.p, \; d_1^1(c, p) = p.$$

Assuming $K(\mathsf{C})_n$ is defined and that it acts on C via the unique composed face map to $K(\mathsf{C})_0 = P$ followed by the given action of P on C, we set

$$
\begin{aligned}
K(\mathsf{C})_{n+1} & = C \rtimes K(\mathsf{C})_n; \\
d_0^{n+1}(c_{n+1}, \dots, c_1, p) & = (c_{n+1}, \dots, c_2, \partial c_1.p); \\
d_i^{n+1}(c_{n+1}, \dots, c_{i+1}, c_i, \dots, c_1, p) & = (c_{n+1}, \dots, c_{i+1}c_i, \dots c_1, p) \\
& \qquad\qquad\qquad\qquad \text{for } 0 < i < n+1; \\
d_{n+1}^{n+1}(c_{n+1}, \dots, c_1, p) & = (c_n, \dots, c_1, p); \\
s_i^n(c_n, \dots, c_1, p) & = (c_n, \dots, 1, \dots, c_1, p),
\end{aligned}
$$

where the 1 is placed in the ith position.

Clearly $Ker\, d_1^1 = \{(c, p) : p = 1\} \cong C$, whilst $Ker\, d_1^2 \cap Ker\, d_2^2 = \{(c_2, c_1, p) : (c_1, p) = (1,1) \text{ and } (c_2 c_1, p) = (1,1)\} \cong \{1\}$, hence the "top term" of $M(K(\mathsf{C}), 1)$ is isomorphic to C itself, whilst $K(\mathsf{C})_0$ is P itself. The boundary map ∂ in this interpretation is the original ∂, since it maps $(c, 1)$ to $d_0(c)$, i.e., we have

Lemma 11 *There is a natural isomorphism*

$$\mathsf{C} \cong M(K(\mathsf{C}), 1).$$

$\blacksquare$

This construction is the internal nerve of the corresponding internal category in *Grps*, as we noted earlier. All the ideas that go into defining the nerve of a category adapt to handling internal

categories, and they produce simplicial objects in the corresponding ambient category. As we have a simplicial group $K(\mathsf{C})$, we might check if it is a group T-complex, but this is more or less immediate as $NK(\mathsf{C})_n = 1$ for $n \geq 2$, whilst $NK(\mathsf{C})_1$ is $\{(c,p) : p = 1\}$ and $s_0(K(\mathsf{C})_0 = \{(c,p) : c = 1\}$.

Suppose now that we had chosen an equivalent 2-fold extension

$$0 \to M \to C' \xrightarrow{d'} G' \to \pi \to 1$$

The equivalence guarantees that there is a zig-zag of maps of 2-fold extensions joining it to that considered earlier. We need only look at the case of a direct basic equivalence:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C & \xrightarrow{\partial} & G & \longrightarrow & \pi & \longrightarrow & 1 \\
 & & \Big\| & & \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & M & \longrightarrow & C' & \xrightarrow{\partial'} & G' & \longrightarrow & \pi & \longrightarrow & 1
\end{array}
$$

giving a map of crossed modules, $\varphi : \mathsf{C} \to \mathsf{C}'$, where $\mathsf{C}' = (C', G', \partial')$. This induces a morphism of simplicial groups,

$$K(\varphi) : K(\mathsf{C}) \to K(\mathsf{C}'),$$

that is, of course, a 2-equivalence. If there is a longer zig-zag between C and C', then the intermediate crossed modules give intermediate simplicial groups and a zig-zag of 2-equivalences so that $K(\mathsf{C})$ and $K(\mathsf{C}')$ are isomorphic in $Ho_2(Simp.Grps)$, i.e. they have the same 2-type. This argument can, of course, be reversed.

If G and H have the same 2-type, they are isomorphic within the category $Ho_2(Simp.Grps)$, so they are linked in $Simp.Grps$ by a zig-zag of 2-equivalences, hence the corresponding cohomology classes in $H^3(\pi_0(G), \pi_1(G))$ are the same up to identification of $H^3(\pi_0(G), \pi_1(G))$ and $H^3(\pi_0(H), \pi_1(H))$. This proves the simplicial group analogue of the result of MacLane and Whitehead, [117], that we mentioned earlier, giving an algebraic model for 2-types of connected CW-complexes.

Theorem 4 *(MacLane and Whitehead, [117]) 2-types are classified by a group π_0, a π_0-module, π_1 and a class in $H^3(\pi_0, \pi_1)$.* ∎

We have handled this in such a way so as to derive an equivalence of categories:

Proposition 15 *There is an equivalence of categories,*

$$Ho_2(Simp.Grps) \cong Ho(CMod),$$

where $Ho(CMod)$ is formed from $CMod$ by formally inverting those maps of crossed modules that induce isomorphisms on both the kernels and the cokernels. ∎

2.8 Re-examining group cohomology with Abelian coefficients

2.8.1 Interpreting group cohomology

We have had

- A definition of group cohomology via the bar resolution: for a group G and a G-module, M:

$$H^n(G, M) = H^n(C(G, M))$$

together with an identification of $C(G, M)$ with maps from the classifying space / nerve, BG, of G to M, up to shifts in dimension;

- Interpretations

$$
\begin{aligned}
H^0(G, M) &\cong M^G, \text{ the module of invariants} \\
H^1(G, M) &\cong Der(G, M)/Pder(G, M) \\
&- \quad \text{by inspection, where } Pder(G, M) \text{ is the submodule of} \\
&\qquad \text{principal derivations;} \\
H^2(G, M) &\cong Opext(G, M), \text{ i.e. classes of extensions} \\
&\qquad 0 \to M \to H \to G \to 1
\end{aligned}
$$

and we also have

$$
\begin{aligned}
H^n(G, M) &\cong Opext^n(G, M), n \geq 2, \text{ via crossed resolutions} \\
&\cong [\mathsf{C}(G), \mathsf{K}(M, n)]
\end{aligned}
$$

Another interpretation, which will be looked at shortly is as $Ext^n(\mathbb{Z}, M)$, where $\mathbb{Z}$ is given the trivial G-module structure. This leads to

$$H^n(G, M) \cong Ext^{n-1}(I(G), M),$$

via the long exact sequence coming from

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

2.8.2   The Ext long exact sequences

There are several different ways of examining the long exact sequence that we need. We will use fairly elementary methods rather than more 'homologically intensive' one. These latter ones are very elegant and very powerful, but do need a certain amount of development before being used. The more elementary ones have, though, a hidden advantage. The intuitions that they exploit are often related to ones that extend, at least partially, to the non-Abelian case and also to the geometric situations that will be studied later in the notes.

The idea is to explore what happens to an exact sequence of modules

$$\mathcal{E}: \quad 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

over some given ring (we need it for G-modules so there the ring is $\mathbb{Z}[G]$, the group ring of G), when we apply the functor $Hom(-, M)$ for M another module. Of course one gets a sequence

$$Hom(\mathcal{E}, M): 0 \to Hom(C, M) \xrightarrow{\beta^*} Hom(B, M) \xrightarrow{\alpha^*} Hom(A, M)$$

and it is easy to check that this is exact, but there is no reason why α^* should be onto since a morphism $f: A \to M$ may or may not extend to some g defined over the bigger module B. For

instance, if $M = A$, and f is the identity morphism, then f extends if and only if the sequence splits (so $B \cong A \oplus C$). We examine this more closely.

We have

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$
$$\Big\downarrow f$$
$$M$$

and can form a new diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow \overline{f} & & \parallel = & & \\
0 & \longrightarrow & M & \xrightarrow{\overline{\alpha}} & N & \xrightarrow{\overline{\beta}} & C & \longrightarrow & 0
\end{array}
$$

where the left hand square is a pushout. You should check that you see why there is an induced morphism $\overline{\beta} : N \to C$ 'emphusing the universal property of pushouts. (This is important as sometimes one wants this sort of construction, or argument, for sheaves of modules and there working with elements causes some slight difficulties.) The existence of this map is guaranteed by the universal property and does not depend on a particular construction of N. Of course this means that the bottom line is defined only up to isomorphism although we can give a very natural explicit model for N, namely it can be represented as the quotient of $B \oplus M$ by the submodule L of elements of the form $(\alpha(a), -f(a))$ for $a \in A$. Then we have $\overline{\beta}(b, m) = \beta(b)$. (Check it is well defined.) It is also useful to have the corresponding formulae for $\overline{\alpha}(m) = (0, m) + L$ and for $\overline{f}(b) = (b, 0) + L$. This gives an extension of modules

$$f^*(\mathcal{E}): \quad 0 \to M \xrightarrow{\overline{\alpha}} N \xrightarrow{\overline{\beta}} C \to 0.$$

If f extends over B to give g, so $g\alpha = f$, then we have a morphism $g' : N \to M$ given by $g'((m, b) + L) = m + g(b)$. (Check that g' is well defined.)

Lemma 12 *f extends over B if and only if $f^*(\mathcal{E})$ is a split extension.*

Proof: We have done the 'only if'. If $f^*(\mathcal{E})$ is split, there is a projection $g' : N \to M$ such that $g'\overline{\alpha}(m) = m$ for all m. Define $g = g'\overline{f}$ to get the extension. ∎

We thus get a map

$$Hom(A, M) \xrightarrow{\delta} Ext^1(C, M)$$
$$\delta(f) = [f^*(\mathcal{E})]$$

which extends the exact sequence one step to the right.

Here it is convenient to define $Ext^1(C, M)$ to be the set (actually Abelian group) of extensions of form

$$0 \to M \to ? \to C \to 0$$

modulo equivalence (isomorphism of middle terms with the ends fixed). The Abelian group structure is given by Baer sum (see entry in Wikipedia, or many standard texts on homological algebra).

Important aside: 'Recall' the '*snake lemma*: given a commutative diagram of modules with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\
  &   & \mu \downarrow &   & \nu \downarrow &   & \psi \downarrow &   &   \\
0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & 0
\end{array}
$$

there is an exact sequence

$$0 \to Ker\,\mu \to Ker\,\nu \to Ker\,\psi \xrightarrow{\delta} Coker\,\mu \to Coker\,\nu \to Coker\,\psi \to 0$$

This has as a corollary that if μ and ψ are isomorphisms then so is ν. (Do check that you can construct δ and prove exactness, i.e. using a simple diagram chase.)

Back to extensions: It is fairly easy to show that $Hom(\mathcal{E}, M)$ extends even further to 6 terms with

$$\ldots \xrightarrow{\beta^*} Ext^1(B, M) \xrightarrow{\alpha^*} Ext^1(A, M)$$

Here is how α^* is constructed. Suppose $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ gives an element of $Ext^1(B, M)$, then we can form a diagram

$$
\begin{array}{ccccccccc}
\alpha^*(\mathcal{E}_1): & & 0 & \longrightarrow & M & \longrightarrow & \alpha^{-1}(N) & \xrightarrow{p'} & A & \longrightarrow 0 \\
  &   &   &   & = \downarrow &   & \alpha' \downarrow &   & \alpha \downarrow &   \\
\mathcal{E}_1: & & 0 & \longrightarrow & M & \longrightarrow & N & \xrightarrow{p} & B & \longrightarrow 0
\end{array}
$$

by restricting $\mathcal{E}_1$ along α using a pull back in the right hand square. We can give $\alpha^{-1}(N)$ explicitly in the form that the usual construction of pullbacks in categories of modules gives it to us

$$\alpha^{-1}(N) \cong \{(a, n) \mid \alpha(a) = p(n)\}$$

and p' and α' are projections. The construction of β^* is done similarly using pullback along β. It is then easy to check that the obvious extension to $Hom(\mathcal{E}, M)$, mentioned above, is exact, but that there is again no reason why α^* should be onto. (Of course, knowledge of the purely homological way of getting these exact sequence will suggest that there is an $Ext^2(C, M)$ term to come.)

We examine an obstruction to it being so. Suppose given $\mathcal{E}' : 0 \to M \to N_1 \xrightarrow{p'} A \to 0$, giving us an element of $Ext'(A, M)$. If α^* were onto, we would need a $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ such that $\alpha^{-1}(N) \cong N_1$ leaving M fixed and relating to α as above by a pullback. We can splice $\mathcal{E}'$ and $\mathcal{E}_1$ together to get a suitable looking diagram

$$
\mathcal{E}' * \mathcal{E}_1 : \quad 0 \to M \to N' \longrightarrow B \to C \to 0
$$
$$
\phantom{\mathcal{E}' * \mathcal{E}_1 : \quad} p' \searrow \quad \nearrow \alpha
$$
$$
\phantom{\mathcal{E}' * \mathcal{E}_1 : \quad \quad} A
$$

and the row is exact. If we change $\mathcal{E}'$ by an isomorphism than clearly this spliced sequence would react accordingly. If you check up, as suggested, on the Baer sum structure if $Ext^1(A, M)$ and $Ext^2(C, M)$ then you can again check that the above splicing construction yields a homomorphism from the first group to the second. Moreover there is no reason not to extend the splicing construction to a pairing operation on the whole graded family of Ext-groups. This is given in detail in quite a few of the standard books on Homological Algebra, so will not be gone into here.

Two facts we do need to have available are about the structure of $Ext^2(C, M)$. Let $\mathcal{E}xt^2(C, M)$ be the category of 4-term exact sequences

$$0 \to M \to N \to P \to C \to 0$$

and morphisms which are commuting diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\
& & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & {\scriptstyle =}\downarrow \\
0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & C & \longrightarrow & 0
\end{array}
\; ,
$$

then $Ext^2(C, M)$ is the set of connected components of this category. The important thing to note is that the morphisms are not isomorphisms in general, so two 4-term sequences give the same element in $Ext^2(C, M)$ if they are linked by a zig-zag of intermediate terms of this form. The second fact is that the zero for the Baer sum addition is the class of the 4-term extension

$$0 \to M \to M \xrightarrow{0} C \to C \to 0$$

with 'equals' on the unmarked maps.

Suppose now that the top row in

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N_1 & \xrightarrow{\bar{p}} & A & \longrightarrow & 0 \\
& & {\scriptstyle =}\downarrow & & {\scriptstyle \alpha'}\downarrow & & {\scriptstyle \alpha}\downarrow \\
0 & \longrightarrow & M & \longrightarrow & N & \xrightarrow{p} & B & \longrightarrow & 0
\end{array}
$$

is obtained by restriction along α from the bottom row. We now form the spliced sequence

$$0 \to M \to N_1 \xrightarrow{\alpha\bar{p}} B \to C \to 0.$$

We would hope that this 4-term sequence was trivial, i.e. equivalence to the zero one. We clearly must use the given element in $Ext^1(B, M)$ in a constructive way in the proof that it is trivial, so we form the pushout of $\alpha\bar{p}$ along α' getting us a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N_1 & \xrightarrow{\alpha\bar{p}} & B & \longrightarrow & C & \longrightarrow & 0 \\
& & {\scriptstyle =}\downarrow & & {\scriptstyle \alpha'}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0
\end{array}
\; ,
$$

with the middle square a pushout. It is now almost immediate that the morphism from B to B' is split, since we can form a commutative square

$$
\begin{array}{ccc}
N_1 & \xrightarrow{\alpha\bar{p}} & B \\
{\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle =} \\
N & \xrightarrow{p} & B
\end{array}
$$

giving us the required splitting from B' to B. It is now a simple use of the snake lemma, to show that the complementary summand of B in B' is isomorphic to C. We thus have that the bottom row of the diagram above is of the form

$$0 \to M \to N \to B \oplus C \to C.$$

This looks hopeful but to finish off the argument we just produce the morphism:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & M & \xrightarrow{\ 0\ } & C & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow{\scriptstyle incl_2} & & \Big\downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & M & \longrightarrow & N & \xrightarrow[incl_1 p]{} & B \oplus C & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

and we have a sequence of maps joining our spliced sequence to the trivial one. (A similar argument goes through in higher dimensions.) Now you should try to prove that if a spliced sequence is linked to a trivial one then it does come from an induced one. That is quite tricky, so look it up in a standard text. An alternative approach is to use the homological algebra to get the trivialising element (coboundary or homotopy, depending on your viewpoint) and then to construct the extension from that. Another thing to do is to consider how the Ext-groups, $Ext^k(A, M)$, vary in M rather than with A. This will be left to you.

2.8.3   From Ext to group cohomology

If we look briefly at the classical homological algebraic method of defining $Ext^K(A, M)$, we would take a projective resolution $\mathsf{P}.$ of A, apply the functor $Hom(-, M)$, to get a cochain complex $Hom(\mathsf{P}., M)$, then take its (co)homology, with $H^n(Hom(\mathsf{P}., M))$ being isomorphic to $Ext^n(A, M)$, or, if you prefer, $Ext^n(A, M)$ being defined to be $H^n(Hom(\mathsf{P}., M))$. This method can be studied in most books on homological algebra (we cite for instance, MacLane, [114], Hilton and Stammbach, [93] and Weibel, [163]), so is easily accessible to the reader - and we will not devote much space to it here as a result. We will however summarise some points, notation, definitions of terms etc., some of which you probably know.

First the notion of projective module:

Definition: A module P is *projective* if, given any epimorphism, $f : B \to C$, the induced map $Hom(P, f) : Hom(P, B) \to Hom(P, C)$ is onto. In other words any map from P to C can be lifted to one from P to B.

Any free module is projective.

Of the properties of projectives that we will use, we will note that $Ext^n(P, M) = 0$ for P projective and for any M. To see this recall that any n-fold extension of P by M will end with an epimorphism to P, but such things split as their codomain is projective. It is now relatively easy to use this splitting to show the extension is equivalent to the trivial one.

A resolution of a module A is an augmented chain complex

$$\mathsf{P}. : \ldots \to P_1 \to P_0 \to M$$

which is exact, i.e. it has zero homology in all dimensions. This means that the augmentation induces an isomorphism between $P_0/\partial P_1$ and M. The resolution is projective if each P_n is a projective module.

If P. and Q. are both projective resolutions of A, then the cochain complexes $Hom(\mathsf{P}., M)$ and $Hom(\mathsf{Q}., M)$ always have the same homology. (Once again this is standard material from homological algebra so is left to the reader to find in the usual sources.)

An example of a projective resolution is given by the bar resolution, BG., and the construction $C^n(G, M)$ in the first chaper is exactly $Hom(\mathsf{B}G., M)$. This reolution ends with $BG_0 = \mathbb{Z}[G]$ and the resolution resolves the Abelian group $\mathbb{Z}$ with trivial G-module structure. (This can be seen from our discussion of homological syzygies where we had

$$\mathbb{Z}[G]^{(R)} \to \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}.$$

In fact we have

$$H^n(G, M) \cong Ext^n(\mathbb{Z}, M)$$

by the fact that BG. is a projective resolution of $\mathbb{Z}$ and then we can get more information using the short exact sequence

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

As $\mathbb{Z}[G]$ is a free G-module, it is projective and the long exact sequence for $Ext(-, M)$ thus has every third term trivial (at least for $n > 0$), so

$$Ext^n(\mathbb{Z}, M) \cong Ext^{n-1}(I(G), M)$$

giving another useful interpretation of $H^n(G, M)$.

2.8.4   Exact sequences in cohomology

Of course, the identification of $H^n(G, M)$ as $Ext^n(\mathbb{Z}, M)$ means that, if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of G-modules, we will get a long exact sequence in $H^n(G, -)$, just by looking at the long exact sequence for $Ext^n(\mathbb{Z}, -)$.

What is more interesting - but much more difficult - is to study the way that $H^n(G, M)$ varies as G changes. For a start it is not completely clear what this means! If we change the group in a short exact sequence,t

$$1 \to G \to H \to K \to 1$$

say, then what type of modules should be used fro the 'coefficients', that is to say a G-modules or one over H or K. This problem is, of course, related to the change of groups along an arbitrary homomorphism, so we will look at an group homomorphism $\varphi : G \to H$, with no assumptions as to monomorphism, or normal inclusion, at least to start with.

Suppose given such a φ, then the 'restriction functor' is

$$\varphi^* : H-Mod \to G-Mod,$$

where, if N is in $H-Mod$, $\varphi^*(N)$ has the same underlying Abelian group structure as N, but is a G-module via the action, $g.n := \varphi(g).n$. We have already used that φ^* has a left adjoint φ_* given by $\varphi_*(M) = \mathbb{Z}H \otimes_{\mathbb{Z}G} M$. Now we also need a right adjoint for φ^*.

To construct such an adjoint, we use the old device of assuming that it exists, studying it and then extracting a construction from that study. We have M in $G-Mod$ and N in $H-Mod$, and we assume a natural isomorphism

$$G-Mod(\varphi^*(N), M) \cong H-Mod(N, \varphi_\sharp(M)).$$

If we take $N = \mathbb{Z}H$, then, as $H-Mod(\mathbb{Z}H, \varphi_\sharp(M)) \cong \varphi_\sharp(M)$, we have a construction of $\varphi_\sharp(M)$, at least as an Abelian group. In fact this gives

$$\varphi_\sharp(M) \cong G-Mod(\varphi^*(\mathbb{Z}H), M)$$

and as $\mathbb{Z}H$ is also a right G-module, via $h.g := h.\varphi(g)$, we have a left G-module structure of $\varphi_\sharp(M)$ as expected. In fact, this is immediate from the naturality of the adjunction isomorphism using the left hand position of $G-Mod(\varphi^*(\mathbb{Z}H), M)$, as for fixed M, the functor converts the right G-action of $\mathbb{Z}$ to a left one on $\varphi_\sharp(M)$. This allows us to get an explicit elementwise formula for this action as follows: let $m^* : \mathbb{Z}H \to M$ be a left G-module mrphsim This can be specified by what it does to the natural basis of $\mathbb{Z}H$ (as Abelian group), and so is often written $m^* : H \to M$, where the function m^* must satisfy a G-equivariance property: $m^*(\varphi(g).h) = g.m^*(h)$. Any such function can, of course, be extended linearly to a G-module morphism of the earlier form. If $g \in G$, we get a morphism

$$-.\varphi(g) : \varphi^*(\mathbb{Z}H) \to \varphi^*(\mathbb{Z}H)$$

given by 'h goes to $h\varphi(g)$'. This *is* a G-module morphism as the G-module structure is by *left* multiplication, which is independent of this right multiplication. Applying $G-Mod(-, M)$, we get $g.m^*$ is given by

$$g.m^*(h) - m^*(h.\varphi(g)).$$

This is a *left* G-module structure, although at first that may seem strange. That it is linear is easy to check. What take a little bit of work is to check $(g_1 g_2).m^* = g_1(g_2.m^*)$: applying both sides to an element $h \in H$ gives

$$(g_1 g_2).m^*(h) = m^*(h\varphi(g_1)\varphi(g_2)),$$

whilst

$$g_1(g_2.m^*)(h) = (g_2.m^*)(h.\varphi(g_1)) = m^*(h\varphi(g_1)\varphi(g_2)).$$

(The checking that $g_1.m^*$ does satisfy the G-equivariance property is left to the reader.)

Remark: There are great similarities between the above calculations and those needed later when examining bitorsors. This is almost certainly not coincidental.

We built $\varphi_\sharp(M)$ in such a way that it is obviously functorial in M and gives a right adjoint to φ^*. This implies that there is a natural morphism

$$i : N \to \varphi_\sharp\varphi^*(N).$$

We denote this second module by N^*, when the context removes any ambiguity, and especially when φ is the inclusion of a subgroup. The morphism sends n to $n^* : H \to N$, where $n^*(h) = h.n$. (Check that $n^*(\varphi(g).h) = g.n^*(h)$. This reminds us that the codomain of n^* is infact just the *set* N underlying both the H-module N and the G-module $\varphi^*(N)$.)

We examine the cohomology groups $H^n(H, N^*)$. These are the (co)homology groups of the cochain complex $Hom(\mathsf{P}., N^*)$, where $\mathsf{P}.$ is a projective H-module resolution of $\mathbb{Z}$. The adjunction shows that this is isomorphic to $Hom(\varphi^*(\mathsf{P}.), \varphi^*(N))$. If $\varphi^*(\mathsf{P}.)$ is a projective G-module resolution of the trivial G-module $\mathbb{Z}$ then the cohomology of this complex will be $H^n(G, N)$, where N has the structure $\varphi^*(N)$.

The condition that free or projective H modules restrict to free or projective G-modules is satisfied in one important case, namely when G is a subgroup of H, since $\mathbb{Z}H$ is a free Abelian group on the *set H* and H is a disjoint union of right G-cosets, so $\mathbb{Z}H$ splits as a G-module into a direct sum of copies of $\mathbb{Z}G$. This provides part of the proof of Shapiro's lemma

Proposition 16 *If $\varphi : G \to H$ is an inclusion, then for a H-module N, there is a natural isomorphism*

$$H^n(H, N^*) \cong H^n(G, N).$$

$\blacksquare$

Corollary 4 *The morphism $i : N \to N^*$ and the above isomorphism yield the restriction morphism*

$$H^n(H, N) \to H^n(G, N).$$

$\blacksquare$

This suggest other results. Suppose we have an extension

$$1 \to N \to G \to Q \to 1$$

(so here we replace H by G with N in the old role of G, but in addition, being normal in G).

If we look at $\mathsf{B}N$ and $\mathsf{B}G$ in dimension n, these are free modules over the sets N^n and G^n respectively, with the inclusion between them; G is a disjoint union of N-cosets, indexed by elements of Q, so can we use this to derive properties of the cokernel of $\mathbb{Z}G \otimes_{\mathbb{Z}N} \mathsf{B}N \to \mathsf{B}G$, and to tie them into some resolution of Q, or perhaps, of $\mathbb{Z}$ as a trivial Q-module. The answer must clearly be positive, perhaps with some restrictions such as finiteness, but there are several possible ways of getting to an answer having slightly different results. (You have in the (φ_*, φ^*) and $(\varphi^*, \varphi_\sharp)$ adjunctions, enough of the tools needed to read detailed accounts in the literature, so we will not give them here.)

This also leads to relative cohomology groups and their relationship with the cohomology of the quotient Q. We can also consider the crossed resolutions of the various groups in the extension and work, say, with the induced maps

$$\mathsf{C}(N) \to \mathsf{C}(C)$$

looking at its cokernel or better what should be called its homotopy cokernel.

Another possibility is to examine $\mathsf{C}(N)$ and $\mathsf{C}(Q)$ and the cocycle information needed to specify the extension, and to use all this to try to construct a crossed resolution of G. (We will see something related to this in our examination of non-Abelian cohomology a little later.) A simple case of this is when the extension is split, $G \cong N \rtimes Q$ and using a twisted tensor product for crossed complexes, one can produce a suitable $\mathsf{C}(N) \otimes_\tau \mathsf{C}(Q)$ resolving G, (see Tonks, [155]).

Chapter 3

Beyond 2-types

The title of this chapter promises to go beyond 2-types and in particular, we want to model them algebraically. We have so far only done this with the crossed complexes. These do give all the homotopy groups of a simplicial group, but the homotopy types they represent are of a fairly simple type, as they have vanishing Whitehead products.

We will return to crossed complexes later on, but will first look at the general idea of n-types, going into what was said earlier in more detail.

3.1 n-types and decompositions of homotopy types

We will start with a fairly classical treatment of the ideas behind the idea of n-types of topological spaces.

3.1.1 n-types of spaces

We earlier (starting in section 2.7.1) discussed 'n-equivalences' and 'n-types'. As homotopy types are enormously complex in structure, we can try to study them by 'filtering' that information in various ways, thus attempting to see how the information at the nth-level depends on that at lower levels. The informational filtration by n-type is very algebraic and very natural. It has two very satisfying interacting aspects. It gives complete models for a subclass of homotopy types, namely those whose homotopy groups vanish for all high enough n, but, at the same time, gives a set of approximating notions of equivalence that, on all 'spaces', give useful information on weak equivalences.

We start with one version of the topological notion:

Definition: Given a cellular mapping, $f : (X, x_0) \to (Y, y_0)$, between connected pointed spaces, f is said to be an *n-equivalence* if the induced homomorphisms, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, y_0)$, for $1 \le k \le n$, are all isomorphisms. More generally, on relaxing the connectedness requirements on the spaces, a cellular mapping, $f : X \to Y$, is an *n-equivalence* if it induces a bijection on π_0, that is, $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection, and for each $x_0 \in X$ and $1 \le k \le n$, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

Remark: It is important to note that here the mappings are cellular, not just continuous. We will see consequences of this later.

There are alternative descriptions and these can be useful. We recall them next, emphasising certain facts and viewpoints that perhaps have not yet been stressed enough in our earlier treatments, but can be useful for our use of these ideas here.

We start by recalling some standard notions of classical homotopy theory. We let CW be the category of all CW-complexes and *cellular* maps, and CW_{c*} be the corresponding category of pointed connected complexes, again with cellular maps. (The notions below generalise easily to the non-connected multi-pointed case.) If X is such a CW-complex, then we will write X^n for its *n-skeleton*, that is, the union of all the cells in X of dimension at most n. We say that X has *dimension n* if $X = X^n$.

It is important to remember that the homotopy type of X^n is *not* an invariant of the homotopy type of X. (Just think about subdivision if you are in doubt about this.) It was partially to handle this that Henry Whitehead introduced the notion of N-type, as this does give such invariants. The two ways of viewing n-types, which we have already mentioned, are both important. We recall that in one, they are certain equivalence classes of CW-complexes, whilst in the other, they are homotopy types of certain spaces with special characteristics. (Useful sources for this topic include Baues' Handbook article on 'Homotopy Types', [18].)

Let CW_{c*}^{n+1} be the full subcategory of CW_{c*} consisting of complexes of dimension $\leq n+1$. (To emphasise where we are working, we will sometimes write X^{n+1}, Y^{n+1}, etc. for objects here.) Let $f, g : X^{n+1} \to Y^{n+1}$ be two maps in CW_{c*}^{n+1} and $f|_{X^n}, g|_{X^n} : X^n \to Y^{n+1}$ their restrictions to the n-skeleton of X. (Note that the codomain is still the $n+1$-skeleton of Y.)

Definition: We say f, and g, as above, are *n-homotopic* if $f|_{X^n} \simeq g|_{X^n}$ (that is, within Y^{n+1}). We write $f \simeq_n g$ in this case.

It can be useful to remember that f and g, in this, need only be defined on the $(n+1)$-skeleton of X. (This statement is true, but is deliberately silly. We, in fact, assumed that X had dimension $\leq n+1$, but what we said is still useful, since if we have any complex, X, we can restrict to its $(n+1)$-skeleton, X^{n+1}, yet do not need f or g to be defined on *all* of X, merely on X^{n+1}.)

Our first version of (connected) n-types, in this approach, is obtained by taking $CW_{c*}^{n+1}/\simeq_n$, that is, taking the complexes of dimension $\leq n+1$ and the cellular maps between them, and then dividing out the hom-sets by the equivalence relation, $\simeq_n$. From this perspective, we have:

Definition: (à là Whitehead.) A connected *n-type* is an isomorphism class in the category, $CW_{c*}^{n+1}/\simeq_n$.

That sets up, a bit more formally, the first type of description of n-types. If we have a connected CW-complex, X, then we assign to it the isomorphism class of X^{n+1} in $CW_{c*}^{n+1}/\simeq_n$ (for any choice of base point) to get its n-type. From this viewpoint, we get a notion of n-equivalence from the notion of n-homotopy:

Definition: A cellular map, $f : X \to Y$, between CW-complexes is an *n-equivalence* if $f^{n+1} : X^{n+1} \to Y^{n+1}$ gives an isomorphism in $CW_{c*}^{n+1}/\simeq_n$.

This is also called *n-homotopy equivalence*, with the earlier version, that based on the homotopy groups, then called *n-weak equivalence*. It amounts to f^{n+1} having a *n-homotopy inverse*, g^{n+1} : $Y^{n+1} \to X^{n+1}$, so $f^{n+1}g^{n+1} \simeq_n 1_{Y^{n+1}} \; g^{n+1}f^{n+1} \simeq_n 1_{X^{n+1}}$. Here it is *not* claimed that there is some $g : Y \to X$ that extends g^{n+1} to the whole of Y, merely there is a map, g, defined on the $(n + 1)$-skeleton.

(These are stated for connected spaces, but as usual the extension to non-connected complexes is easy to do.)

Let us take these ideas apart one stage more. Suppose that P is a CW-complex of dimension $\leq n$, and $f : X \to Y$ is a n-equivalence in the above sense. We note that, as we are looking at cellular maps and cellular homotopies, the inclusion $i^{n+1} : X^{n+1} \to X$ induces a bijection

$$[P, i^{n+1}] : [P, X^{n+1}] \to [P, X],$$

but then it is clear that

$$[P, f] : [P, X] \to [P, Y]$$

is also a bijection. (Note that if we had required P to have dimension $n + 1$, then $[P, i^{n+1}]$: $[P, X^{n+1}] \to [P, X]$ might not be *injective* as two non-homotopic maps with image in X^{n+1} may be homotopic within the whole of X. That being so $[P, i^{n+1}]$ will be surjective, but just not a bijection. The same would be true for $[P, f]$.)

So much for the first viewpoint, i.e., as equivalence classes of objects in CW_{c*}. For the second approach, that is, n-types as homotopy types of certain spaces delineated by conditions, we work in the bigger category of (pointed connected) CW-complexes and *all continuous maps*, i.e., not just the cellular ones (although, remember, the classical cellular approximation theorem tells us that any (general continuous) map is homotopic to a cellular one). We will temporarily call this category 'spaces', (following the treatment in Baues' Handbook article, [18]). We form spaces/ $\simeq$, the quotient category of 'spaces' and homotopy classes of maps.

Definition: The subcategory, n−types, of spaces/ $\simeq$, is the full subcategory consisting of spaces, X, with $\pi_i(X) = 0$ for $i > n$. Such spaces, or their homotopy types, may also be called *n-types*. The generalisation to the non-connected case should be clear.

We now have two different definitions of n-type of CW-complexes (and that is without mentioning n-types of simplicial sets, simplicial groups $\mathcal{S}$-groupoids, etc.). We need to check on the relationship between them. For this, we introduce *Postnikov functors* and in a later section will study the related *Postnikov tower* that decomposes a homotopy type. Note the Postnikov functors are usually defined so as to be functorial at the level of the homotopy categories, not at the level of the spaces and maps, although this is possible. We will comment on this a bit more later on, but let us describe the main ideas first as these directly relate to the comparison of the two ways of approaching n-types.

Definition: The nth *Postnikov functor*,

$$P_n : CW_{c*}/ \simeq \to \text{n−types}$$

is defined by killing homotopy groups above dimension n, that is, we *choose* a CW-complex, P_nX, with

$$(P_nX)^{n+1} = X^{n+1},$$

and, by attaching cells to X in dimensions $> n$, with $\pi_i(P_nX) = 0$ for $i > n$. If $f : X \to Y$ is a cellular map, we choose a map $P_nf : P_nX \to P_nY$, so that $(P_nf)^{n+1} = f^{n+1}$. The functor P_n takes the homotopy class, $[f]$, to $[P_nf]$.

The first point to note is that the choices are absorbed by the homotopy. To examine this more deeply we make several:

Remarks: (i) First a word about 'killing homotopy groups'. (This is very like the construction of resolutions of a group.)

Suppose that we have a space, X, and a set of representatives, $\varphi_g : S^{n+1} \to X$, of generators, g, of the homotopy group, $\pi_{n+1}(X)$, then we form

$$X(1) := X \sqcup_{\{\varphi_g\}} \bigsqcup_g D^{n+2},$$

i.e., we glue $(n + 2)$-dimensional discs to X, along their boundaries, using the representing maps. We now take $\pi_{n+2}(X(1))$ and a generating set for that, form $X(2)$ by the same sort of construction, and continue to higher dimensions.

If $f : X \to Y$, then each $f(\varphi_g) : S^{n+1} \to Y$ defines an element of $\pi_{n+1}(Y)$, and this will be 'killed' within $\pi_{n+1}(Y(1))$. There is thus a null homotopy for that map within $Y(1)$. We choose one such and use it to extend f over the disc attached by φ_g. Doing this for each generator, we extend f to $f(1) : X(1) \to Y(1)$, and so on.

This is unbelievably non-canonical and non-functorial at the level of spaces, but the different choices can fairly easily be shown to yield homotopy equivalent spaces and homotopic maps. This is discussed in many of the standard algebraic topology textbooks, see, for instance, Hatcher, [92].

The basis of these constructions is a simple extension lemma, (cf. Hatcher, [92], lemma 4.7, p.350, for instance).

Lemma 13 *Given a CW pair, (X, A), and a map, $f : A \to Y$, with Y path connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells in dimension n.* ∎

(ii) Things are clearer when working with simplicial sets as we will see shortly. In that case, there is a good functorial 'Postnikov tower' of Postnikov functors, defined at the level of simplicial sets, and morphisms and not merely at the homotopy level. That works beautifully for what we need, but at the slight cost of moving from 'spaces' to simplicial sets, there using Kan complexes (which is no real bother, as singular complexes are Kan), and finally taking geometric realisations to get back to the spaces. As we said, we will look at this shortly.

There are inclusion maps, $P_n(X) : X \to P_nX$, whose homotopy classes give a natural transformation from the identity to P_n. (This is defined on the homotopy categories of course.) For

$f : X \to Y$ in CW_{c*}, then $P_n f$ can be chosen to make the square

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle p_n(X)}\big\downarrow & & \big\downarrow{\scriptstyle p_n(Y)} \\
P_n X & \xrightarrow[P_n f]{} & P_n Y
\end{array}
$$

commutative 'on the nose'. We note that these maps make each $(P_{n+1}X, X)$ into a CW-pair and, as $P_{n+1}X - X$ has only cells of dimension $n+3$ or greater, and $\pi_i(P_nX) = 0$ in those dimensions, we can apply the extension lemma to the map, $p_n(X) : X \to P_nX$ and thus extend it to $P_{n+1}X$, giving $p_n^{n+1}(X) : P_{n+1}X \to P_nX$, and this satisfies $p_n^{n+1}(X) \cdot p_{n+1}(X) = p_n(X)$. These map, $p_n^{n+1}(X)$ fit into a tower diagram with a 'cone' of maps from X:

The limit of the tower is isomorphic to X itself. This is known as a *Postnikov tower* for X. We will return to such towers in section 3.1.3.

It is useful to refer to $X \to P_nX$, or more loosely to P_nX as a *Postnikov section* of X, or as the nth-*Postnikov section* of X, even though it is only determined up to homotopy equivalence.

We return to the nth Postnikov functor, P_n, and can use it to define n-equivalences in a different way.

Definition: A map, $f : X \to Y$, is called a P_n-equivalence if the induced morphism, $[P_n f]$, in $n-$types is an isomorphism.

Of course, we expect these P_n-equivalences to just be n-equivalences under another name. To examine this, we look again at P_n.

We had the Postnikov functor:

$$
P_n : CW_{c*}/ \simeq \to n-\text{types}.
$$

If we look at $CW_{c*}^{n+1}/ \simeq_n$, we need to see that a P_n construction adapts to give a functor

$$
P_n : CW_{c*}^{n+1}/ \simeq_n \to n-\text{types},
$$

as this does not follow trivially from the previous case. Suppose X and Y are $(n+1)$-dimensional connected pointed CW complexes and $f \simeq_n g : X \to Y$, then $f|_{X^n} \simeq g|_{X^n}$. We have to check that $P_n f \simeq P_n g$.

We have some $h : f|_{X^n} \simeq g|_{X^n} : X_n \times I \to Y^{n+1} \hookrightarrow P_n Y$, and also have the map from $P_n X \times \{0,1\}$ to $P_n Y$ given by $P_n f$ and $P_n g$. These are compatible so define a map from the subcomplex, $X_n \times I \cup P_n X \times \{0,1\}$ of $P_n X \times I$, to $P_n Y$. The cells in $P_n X \times I$ that are not in that subcomplex, all have dimension $n+3$ or greater, since P_n is obtained from X^{n+1} by adding cells. We have $\pi_i(P_n Y) = 0$ for $i > n$, so an application of the extension lemma gives us an extension ver $P_n X \times I$ giving a homotopy between $P_n f$ and $P_n g$, as required. This proves

Lemma 14 P_n *give a functor from* $CW_{c*}^{n+1} / \simeq_n$ *to* $\mathsf{n-types}$. ∎

We claim that this functor is an equivalence of categories, which will show, after a bit more checking, that the two notions of n-equivalence coincide and will relate the main notions of (topological) homotopy n-types.

To prove that P_n is an equivalence of categories, it is, perhaps, easiest to look for a functor in the opposite sense that might serve as a 'quasi-inverse'. If we have that X is a (connected, pointed) CW-complex with $\pi_i(X) = 0$ for $i > n$, then we can take its $(n+1)$-skeleton, X^{n+1} to get something in CW_{c*}^{n+1}. This is not quite a functor, since not all the morphisms in spaces are cellular. Each continuous map between such complexes is *homotopic* to a cellular map, but, whilst taking the $(n+1)$-skeleton *is* a functor with respect to cellular maps, we have to verify that if we choose two cellular approximations for some $f : X \to Y$, then their $(n+1)$-skeletons are, at least, n-homotopic.

Suppose that $f_0, f_1 : X \to Y$ are two cellular maps between n-types (to be thought of, in the first instance, as two 'rival' cellular approximations to some $f : X \to Y$). We assume they are homotopic by a homotopy $h : f_0 \simeq f_1$, which again using cellular approximation, can be assumed to be a cellular homotopy. We take f_0^{n+1} and f_1^{n+1} and see if they are n-homotopic.- Yes they are. They may not be homotopic, since h may use $n + 2$-cells in the process of 'homotoping' between f_0^{n+1} and f_1^{n+1} within Y, but $F_0|_{X^n}$ and $f_1|_{X^n}$ are homotopic via h restricted to $X_n \times I$, i.e., exactly what is needed.

We have checked not only that our idea of taking $(n+1)$-skeletons is compatible with the cellular approximations, but also that that assignment induces a functor from $\mathsf{n-types}$ to $CW_{c*}^{n+1} / \simeq_n$. (Of course, in fact, this is the restriction of a functor from spaces to $CW_{c*} / \simeq_n$, as we nowhere use that X and Y were n-types.)

Theorem 5 *The nth Postnikov functor, P_n, gives an equivalence of categories between $CW_{c*} / \simeq_n$ and* $\mathsf{n-types}$. *A quasi inverse is given by the $(n+1)$-skeleton functor.*

Proof: We examine the two composite functors.

If X is in CW_{c*}, then $(P_n X)^{n+1} = X^{n+1}$, by definition. The inclusion of X^{n+1} into X gives an isomorphism in $CW_{c*} / \simeq_n$, since $\simeq_n$ uses nothing in X above dimension $n+1$.

The other composite starts with an n-type, Y, say, takes Y^{n+1}, then forms $P_n(Y^{n+1})$. The inclusion of Y^{n+1} into Y extends by the extension lemma, to a map $P_n(Y^{n+1}) \to Y$, which induces

isomorphisms on all homotopy groups, so is a weak homotopy equivalence, and thus, as we are handling CW-complexes, is a homotopy equivalence, i.e., an isomorphism in $\mathsf{n-types}$, which completes the proof. ∎

Remark: It is worth noting that, in the above, we have 'naturally' defined maps from X *to* $(P_nX)^{n+1}$ and from $P_n(Y^{n+1})$ *to* Y, which suggests an adjointness behind the equivalence. In fact, we actually did *not* assume that X was in $CW_{c*}^{n+1}/\simeq_n$, so, in some sense, proved that $\mathsf{n-types}$ was equivalent to a homotopically reflective subcategory of CW_{c*}. (Of course, connectedness has nothing to do with the picture and was for convenience only.)

We thus have a fairly complete picture of homotopy n-types and n-equivalence in the topological case. If $f : X \to Y$ is such that $[P_nf]$ is an isomorphism in $\mathsf{n-types}$, then $[f^{n+1}]$ is an isomorphism in $CW_{c*}^{n+1}/\simeq_n$, hence an n-equivalence l Whitehead.

If X and Y are (connected, pointed) $(n+1)$-dimensional CW-complexes, and $f : X \to Y$ is cellular, then f is an n-equivalence if, and only if, it induces isomorphisms on all π_i for $i \le n$. In general, i.e., with no dimensional constraint, as we have defined it, f is an n-equivalence if, and only if f^{n+1} is an n-equivalence in this more restricted sense.

We write $Ho_n(Top)$ for the category of CW-complexes (or more generally, topological spaces, after inverting the n-equivalences. If we are just considering the CW-complexes, this is just the same as $\mathsf{n-types}$ up to equivalence and n-types are just isomorphism classes of objects in this category. (If considering spaces other than those having the homotopy types of CW-complexes, then this is better thought of as the singular n-types, but we will not usually need this level of generality in our development.) It seems that, in his original thoughts on algebraic homotopy theory, Whitehead hoped to find algebraic models for n-types, that is, to find algebraic descriptions of isomorphism classes of spaces within $Ho_n(Top)$. Classifying 1-types is 'easy' as they have models that are just groups, so classification reduces to classifying groups up to isomorphism. This is still not an easy task, but there are a wide range of tools available for it. As was previously mentioned, Mac Lane and Whitehead, [117], gave a complete algebraic model for 2-types. (Note: their 3-types are modern terminology's 2-types.) The model they proposed was the crossed module and we have seen the extension of their result to n-types given by Loday.

It should be pointed out that, although n-equivalence is defined in terms of the π_k, $0 \le k \le n$, the interactions between the various π_ks mean that not every sequence $\{\varphi_k : \pi_k(X) \to \pi_k(Y)\}_{0 \le k \le n}$ can be realised as the induced morphisms coming from some $f : X \to Y$, even if the φ_k are all isomorphisms.

One approach that we will be looking at in our exploration of the basics of Whitehead's idea of Algebraic Homotopy and its implications and developments, is to convert the problems to ones in the study simplicial groups or, more generally, in $\mathcal{S}$-groupoids. For this we will need a knowledge of the corresponding theory for n-types of simplicial sets. This is very elegant, so would, in any case, be worth looking at in some detail.

3.1.2 n-types of simplicial sets and the coskeleton functors

(Sources for this section include, at a fairly introductory level, the description of the coskeleton functors in Duskin's Memoir, [64], his paper, [66], and Beke's paper, [19]. There is also a description of the skeleton and coskeleton constructions in the nLab, [134], (search on 'simplicial skeleton'). The original introduction of this construction would seem to be by Verdier in SGA4, [8], with an early use being in Artin and Mazur's *Étale homotopy*, Lecture Notes, [10].)

First let us summarise some basic ideas. For simplicial sets and simplicially enriched group(oid)s, the definitions of n-equivalence are analogous, and we give them now for convenience:

Definition: For $f : G \to H$ a morphism of $\mathcal{S}$-groupoids, f is an *n-equivalence* if $\pi_0 f : \pi_0 G \to \pi_0 H$ is an equivalence of the fundamental groupoids of G and H and for each object $x \in Ob(G)$ and each k, $1 \le k \le n$,
$$\pi_k f : \pi_k(G\{x\}) \to \pi_k(H\{f(x)\})$$
is an isomorphism.

We write $Ho_n(\mathcal{S}-Grpd)$ for the corresponding category of n-types, i.e., $\mathcal{S}-Grpd(\Sigma_n^{-1})$, where Σ_n is the class of all n-equivalences of $\mathcal{S}$-groupoids. An *n-type of $\mathcal{S}$-groupoids* is atreatment of coskeletons was by verdier,n isomorphism class within $Ho_n(\mathcal{S}-Grpd)$.

Cautionary note: If K is a simplicial set, then as $\pi_k(K) \cong \pi_{k-1}(GK)$, the n-type of K corresponds to the $(n-1)$-type of GK.

We need to look at simplicial n-types, in general, and in some more detail, and will start by the theory for simplicial sets. On a first reading the above summary may suffice.

The theory sketched out in the previous section uses the $(n + 1)$- and n-skeletons of a CW-complex in a neat way. If we go over to simplicial sets as models for homotopy types then skeletons are easy to define, but some points do need making about them.

The n-skeleton of a CW-complex is the union of all cells of dimension less than or equal to n, so the set of higher dimensional cells in an n-skeleton is, clearly, empty. On the other hand, a simplicial set, K, has in addition to the simplices in each dimension, the face and degeneracy operators, i.e., the various $d_i : K_n \to K_{n-1}$ and $s_j : K_n \to K_{n+1}$, so to get the n-skeleton of K, we cannot *just* take the k-simplices for $k \le n$, throwing away everything in higher dimensions, and hope to get a simplicial set. If $\sigma \in K_n$, then the $s_j\sigma$ are in K_{n+1}, so K_{n+1} cannot be empty. The point is rather that, in the n-skeleton, *all* simplices in dimensions greater than n will be degenerate.

Our first task, therefore, is to set this up more abstractly and categorically. A simplicial set, K is a functor, $K : \mathbf{\Delta}^{op} \to Sets$ and we want to restrict attention to those parts of K in dimensions less than or equal to n, discarding, initially, all higher dimensional simplices, before reinstating those that we need.

(We will introduce the ideas for simplicial sets, but we can, and will later, extend the discussion to simplicial groups, and, in general, to simplicial objects in a category, $\mathcal{A}$. The latter situation will require some conditions on the existence of various limits and colimits in $\mathcal{A}$, but we will introduce these as we go along. The ability to use more general categories is a great simplification for later developments.)

Recall that the category, $\mathbf{\Delta}$, consists of all finite ordinals and all order preserving maps between them. Given any natural number n, we can form a full subcategory, $\mathbf{\Delta}[0, n]$, with objects the ordinals $[0], \ldots, [n]$, and all order preserving maps between *them*. As the category of simplicial sets is $\mathcal{S} = Sets^{\mathbf{\Delta}^{op}}$, there is a restriction functor, call *n-truncation* or, more fully, *simplicial n-truncation*,

$$tr^n : \mathcal{S} \to Sets^{\mathbf{\Delta}[0,n]^{op}},$$

which, to a simplicial set, K, assigns the n-truncated simplicial set, $tr^n(K)$, with the same data in dimensions less than $n + 1$, but which forgets all information on higher dimensions. A functor, $K : \mathbf{\Delta}[0, n]^{op} \to Sets$ is equivalent to a system, $K = \{(K_k)_{0 \le k \le n}, d_i, s_j\}$, of sets and functions, (or more generally of objects and arrows of $\mathcal{A}$). These are to be such that the d_i and s_j verify the simplicial identities wherever they make sense.

Remark: Setting up notation and terminology for the more general case, we have a category $Tr^n Simp.\mathcal{A} = \mathcal{A}^{\mathbf{\Delta}[0,n]^{op}}$ of n-truncated simplicial objects in $\mathcal{A}$. The category of n-truncated simplicial sets is then $Tr^n Simp.Sets = Tr^n \mathcal{S} = Sets^{\mathbf{\Delta}[0,n]^{op}}$. Back in the general case, the analogue of the above restriction functor gives us a restriction functor:

$$tr^n : Simp.\mathcal{A} \to Tr^n Simp.\mathcal{A}.$$

If the category $\mathcal{A}$ has finite colimits, then this functor, tr^n has a left adjoint, which we will denote sk^n, and which is called the *n-skeleton* of the truncated simplicial object. The proof that this left adjoint exists is most neatly seen by using the theory of Kan extensions, for which see Mac Lane, [115], here with a discussion starting in section **??**, or the nLab, [134], (search on 'Kan extension'.)

The *idea* of the construction of that left adjoint is, however, quite simple and is just an encoding of the intuitive idea that we sketched out above. We first look at it in the case of a simplicial set. We have K in $Tr^n \mathcal{S}$, and want $(sk^n K)_{n+1}$, that is the first missing level, (after that we can presumably repeat the idea to get the higher levels of $sk^n K$). We clearly need degenerate copise of all simplices in K_n and that suggests, (slightly incorrectly), that we take this $(sk^n K)_{n+1}$ to be the disjoint union of sets, $s_i(K_n) = \{s_i(x) \mid x \in K_n\}$. (The elements $s_i(x)$ are just copies of x indexed by the degeneracy mapping. If you prefer another notation, use pairs (x, s_i) as this corresponds more to one of the usual models of disjoint unions.) This is not right, since, these $s_i(x)$ are not independent of each other. If x is already a degenerate element , say $x = s_j y$ then $s_i x = s_i s_j y$ and, as we will need the simplicial identities to hold in the end result, this must be *the same element* as $s_{j+1} s_i y$, (this is if $i \le j$). In other words, we should not use a disjoint union of these sets, $s_i(K_n)$, but will have to identify elements according to the simplicial identities, that is, we must form some sort of colimit. In fact, one forms a diagram consisting of copies of K_n and K_{n-1}, and then forms its colimit to get $(sk^n K)_{n+1}$. the next task is to define the face and degeneracy maps linking the new level with the old ones, so as to get an $(n + 1)$-truncated simplicial sets. (It is a **good idea** to try this out in some simple cases such as for $n = 1$ and 2 and then to **look up** a 'slick' version, as then you will, more easily, see what makes the slick version work.)

Of course, the use of simplicial *sets* here is not crucial, but if working with simplicial objects in some $\mathcal{A}$, then we will need, as we mentioned earlier, that $\mathcal{A}$ has finite colimits so as to be able to form $(sk^n K)_{n+1}$. The process is then repeated as we now have a $(n + 1)$-truncated object.

Remark: Shortly we will be using skeletons (and coskeletons) of simplicial *groups*. In such a context, it should be noted that not all elements in $(sk^n G)_m$, for $m > n$, need be, themselves, degenerate. For instance, we might have g, and g', in G_n, so have for two different indices, i, j, elements $s_i g$ and $s_j g'$ in $(sk^n G)_{n+1}$, but, more often than not, their product $s_i g.s_j g'$, will not be a degenerate element. This fact is crucial and is one reason why, in homotopy theory, it is possible to have non-trivial homotopy groups above the dimension of a space.

If we are considering simplicial sets, or, more generally, simplicial objects in $\mathcal{A}$, where $\mathcal{A}$ has finite *limits*, the truncation functor, tr^n, has a *right* adjoint, which will be denoted $cosk^n$. This is called the *n-coskeleton functor*. (WARNING: this term will also be used for the *composite $cosk^n \circ tr^n$*, from $Simp.\mathcal{A}$ to itself as it is too useful to 'waste' on the more restrictive situation! Usually no confusion will arise, especially as we will use a slightly different notation.)

The fact that $cosk^n$ is right adjoint to tr^n means that, at least in the case of simplicial sets, $cosk^n$ has a very simple description. If K is a simplicial set and L is an n-truncated simplicial set, then we have
$$Tr^n \mathcal{S}(tr^n(K), L) \cong \mathcal{S}(K, cosk^n L).$$
Taking $K = \Delta[m]$, the simplicial m-dimensional simplex, we get

$$(cosk^n L)_m = \mathcal{S}(\Delta[m], cosk^n L) \cong Tr^n \mathcal{S}(tr^n(\Delta[m]), L),$$

giving us a recipe for the simplices of $cosk^n L$ in all dimensions. As $tr^n \Delta[m]$ is an n-dimensional shell of a m-dimensional simplex, we can think of it intuitively as being a family of n-simplices stuck together along lower dimensional bits in some neat way (governed by the simplicial identities). We thus would expect $cosk^L_m$ to be made up of compatible families of n-simplices of L, and this suggests a 'limit' - which makes sense as $sk^n L$ was thought of as a colimit.

As with the left adjoint of tr^n, the right adjoint can be described as a Kan extension, which would give an explicit 'end' formula and also a limit formula that we could take apart. At this stage in the notes, it is not being assumed that those parts of categorical toolbag are available to us. (They *are* discussed later with Kan extension starting on page **??** and with ends (and coends) discussed in section **??**.) Because of this it seems better to adopt a fairly 'barehands' approach, which is more elementary and nearer the initial intuition of what is needed, but the way to go beyond the limitations of this approach is to understand Kan extensions fully. (The approach that we will use will be adapted from Duskin's memoir, [64].)

For a category, $\mathcal{A}$, with finite limits, we suppose given an n-truncated simplicial object, $L \in Tr^n Simp.\mathcal{A}$ and we consider all the face maps at level n

$$d_0, \ldots, d_n : L_n \to L_{n-1}.$$

Definition: An object, K_{n+1}, together with morphisms $p_0, \ldots, p_{n+1} : K_{n+1} \to L_n$ is said to be the *simplicial kernel* of $(d_0, \ldots, d_n)$ if the family $(p_0, \ldots, p_{n+1})$ satisfies the simplicial identities with respect to the d_is and, moreover, has the following universal property: given any family, $x_0, \ldots, x_{n+1}$ of morphisms from some object, T, to L_n, which satisfy the simplicial identities with

respect to the face morphisms, $d_0, \ldots, d_n$ (so that for $0 \leq i < j \leq n + 1$, $d_i x_j = d_{j+1} x_i$), there is a unique morphism $x = \langle x_0, \ldots, x_{n+1} \rangle : T \to K_{n+1}$ such that for each i, $p_i x = x_i$.

This is clearly just a special type of limit. We would expect to get this K_{n+1}, together wiht the projections, p_i, as some sort of multiple pullback, corresponding to the 'naive' description we gave above. (To gain intuition on this oint, **look at** the case $n = 1$, so we have $d_0, d_1 : L_1 \to L_0$ and want K_2 with maps $p_0, p_1, p_2 : K_2 \to L_1$, and these must satisfy the simplicial identities. It is **worth your while**, if you have not seen this before, to draw a diagram, consisting of some copies of L_1 and L_0, and the face maps built from $d_0, d_1 : L_1 \to L_0$, so that the limit of the diagram is K_2.)

If the simplicial kernel is to do the job, we should be able to use it to take $(cosk^n L)_{n+1} = K_{n=1}$, that is to form a $(n = 1)$-truncated simplicial objects from it having the right properties. We, first, need face and degeneracy morphism defined in a natural way. As the p_i were to satisfy the face simplicial identities, they are the obvious candidates for the face morphisms. We will, then, need to define for each j between 0 and n, a morphism $s_j : L_n \to K_{k+1}$. The universal property of K_{n+1} gives that such a morphism will be of the form

$$s_j = \langle s_{j,0}, \ldots, s_{j,n+1} \rangle,$$

for $s_{j,k} : L_n \to L_n$, and, of course, in this notation $d_i : K_{n+1} \to L_n$ will be the ith projection, p_i. This gives us the recipe for determining the $s_{j,k}$ as we must have, for instance, if $k < j$,

$$s_{j,k} = d_k s_j = s_{j-1} d_k,$$

so as to make sure that the s_j satisfy the simplicial identities. (**It is useful** to list the various cases yourself.) It is now clear that the following holds:

Lemma 15 *The data $((cosk^n L)_k, (d_i), (s_j))$, where $(cosk^n L)_k$ is equal L_k for $k \leq n$ and $(cosk^n L)_{n+1} = K_{n+1}$, the simplicial kernel (as above), the d_i are the structural limit cone projections and the s_j are defined by the universal property and the simplicial identities, defines an $(n+1)$-truncated simplicial object.* ∎

We denote this by $tr^{n+1} cosk^n L$, as it is the next step in the construction of $cosk^n L$.

We have as a consequence the following:

Proposition 17 *Suppose given a simplicial object, T, and a morphism, $f : tr^n T \to L$, then there is a unique morphism,*

$$\tilde{(f)} : tr^{n+1} T \to tr^{n+1} cosk^n L,$$

that extends f in the obvious sense. ∎

We may now construct $cosk^n L$ by successive simplicial kernels in the obvious way, and, generalising the above proposition to each successive dimension, prove that the result gives a right adjoint to tr^n.

Remarks: (i) The n-skeleton functor, that we saw earlier, can be given by an analogous simplicial cokernel construction using the degeneracy operators instead of the faces to give a universal object, and then applying the universal property to obtain the face morphisms. The object $sk^n(L)$

is then obtained by iterating that construction. (This is a **good exercise to follow up on** as it sheds useful light on what the skeleton will be in other situations where our intuitions are less strong than for simplicial sets.)

(ii) We are often, in fact, usually, interested more bby the composites

$$sk_n := sk^n \circ tr^n,$$

and

$$cosk_n := cosk_n \circ tr^n,$$

which will be called the n-skeleton and n-coskeleton functors on $Simp.\mathcal{A}$ (The superfix / siffix notation is just to distinguish them and no special significance should be read into it.)

Proposition 18 *(i) If $p \geq q$, then $cosk_p cosk_q = cosk_q$.*
(ii) If $p \leq q$, then $cosk_p cosk_q = cosk_p$.

Proof: This is a simple **exercise** in the definition, or, alternatively, in the constructions, so is **left to the reader** to work out or check up on in the literature. ∎

A similar result holds for skeletons, and this is, again, **left to you** to investigate.

So far in this section we have just looked at the skeleton and coskeleton functors, but we are wanting these for a discussion of simplicial n-types. If we adopt the view that an n-type is a homotopy type with vanishing homotopy groups above dimension n, this goes across without pain to the context of simplicial sets, and, in fact, to many other situations such as simplicial sheaves on a space or simplicial objects in a (Grothendieck) topos, $\mathcal{E}$.

Aside: A good reason for briefly looking at this is that it introduces several useful concepts and the linked terminology. These in the main are due to Jack Duskin, who developed them for the study of simplicial objects in a topos. We will give the definitions and subsequent discussion within the classical setting of *Sets*, but this is really only because we have not given a thorough and detailed treatment of toposes earlier. The basic point is that if the arguments used in the development are 'constructive' then, usually with some minor changes, the theory will generalise from a category of sets, to one of sheaves, and eventually to any Grothendieck topos. To make that statement more precise would require quite a lot more discussion, and would take us away from our main themes, so investigation is **left to you**.

We start with a slight variant of the Kan fibration definition that we met earlier, (see page **??**). We recall that $\Lambda^i[n]$ is the (n,i)-*horn* or (n,i)-*box*, obtained by discarding the top dimensional n-simplex and its ith face and all the degeneracies of those simplices.

Definition: A simplicial map $p : E \to B$ is a *Kan fibration*, or *satisfies the Kan lifting condition, in dimension n* if, in every commutative square (of solid arrows) of form

$$
\begin{array}{ccc}
\Lambda^i[n] & \xrightarrow{f_1} & E \\
{\scriptstyle inc}\downarrow & {\scriptstyle f}\nearrow & \downarrow{\scriptstyle p} \\
\Delta[n] & \xrightarrow[f_0]{} & B
\end{array}
$$

a diagonal map (indicated by the dashed arrow) exists, i.e., there is an $f : \Delta[n] \to E$ such that $pf = f_0$, $f.inc = f_1$, so f lifts f_0 and extends f_1.

We thus have that p is a Kan fibration if it is one in *all dimensions.* We can refine the above (following Duskin, [65]).

Definition: A simplicial map $p : E \to B$ satisfies the exact Kan lifting condition in dimension n if, in every commutative square (as above), precisely one diagonal map f exists.

Starting with the Kan fibration condition, we singled out the Kan complexes as being those simplicial sets for which the unique map $K \to \Delta[0]$ was a Kan fibration. We clearly can do a similar thing here.

Definition: A simplicial set K is an *exact n-type*, or *n-hypergroupoid*, if $K \to \Delta[0]$ is a Kan fibration that is exact in dimensions greater than n.

The definition of n-hypergroupoid used by Glenn, [84], is slightly different from this as it only requires the (exact) Kan condition in dimensions greater than n, so not requiring K to 'be' a Kan complex in lower dimensions. The n-hypergroupoid terminology is due to Duskin, [65], whilst 'exact n-type' is Beke's, [19].

If we need a version of these ideas in $Simp(\mathcal{E})$ or $Simp.\mathcal{A}$, then we can easily adapt our earlier discussion of horns and Kan objects in that context. For instance:

Proposition 19 *If $\mathcal{A}$ is a finite limit category, a morphism, $p : E \to B$, in $Simp.\mathcal{A}$ is an exact Kan fibration in dimension n if, and only if, the natural maps $E_n \to \Lambda^k[n](E) \times_{\Lambda^k[n](B)} B_n$ are all isomorphisms in $\mathcal{A}$.* ∎

Corollary 5 *In $Simp.\mathcal{A}$, an object, K, is an exact n-type (or n-hypergroupoid) if, and only if, the natural map, $K_k \to \Lambda^j[k](K)$, is an epimorphism for $k \leq n$ and an isomorphism for $k > n$.* ∎

To begin to take 'exact n-types' apart, we will need to look again at look at the coskeleton functors. It is very useful for our purposes to have a description of when a simplicial set, K, is isomorphic to its own n-coskeleton. The following summary is actually adapted from Beke's paper, [19], but is quite well known and moderately easy to prove, so the proof will be **left as an exercise**.

Proposition 20 *For a simplicial set, K, the following are equivalent:*

1) K is isomorphic to an object in the image of $cosk_n$.

2) The natural morphism $K \to cosk_n(K)$ is an isomorphism.

3) Writing $\partial \Delta_k(K)$ for the set

$$\partial \Delta_k(K) = \{(x_0, \ldots, x_k) \mid x_i \in K_{k-1} \text{ and, whenever } i < j, \ d_i x_j = d_{j-1} x_i\},$$

(so $\partial \Delta_k(K) \cong \mathcal{S}(\partial \Delta[k], K)$), the natural 'boundary' map $b_k(x) = (d_0 x, \ldots, d_k x)$, from K_k to $\partial \Delta_k(K)$ is a bijection for all $k > n$.

4) The natural map, $K_k \to Sets^{\mathbf{\Delta}[0,n]^{op}}(tr^n\Delta[k], tr^n(K))$, which sends a k-simplex x of K, considered as its 'name', $\ulcorner x \urcorner : \Delta[k] \to K$, to the n-truncation, of $\ulcorner x \urcorner$, is a bijection for all $k > n$.

5) For any $k > n$, and any pair of (solid) arrows

$$\begin{array}{ccc} \partial\Delta[k] & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta[k] & & \end{array}$$

   there is precisely one (dotted) arrow making the diagram commute.   ∎

As we said, the proof is **left to you**, as it is just a question of translating between different viewpoints.

   Definition: If K satisfies any, and hence all, of the above conditions, it is called *n-coskeletal*.

   The first two conditions can be transferred verbatim for simplicial objects in any category with finite limits, and thus for simplicial objects in a topos. Condition 3 can also be handled in those contexts, using iterated pullbacks to construct $\partial\Delta_k(K)$. Condition 4) can also be used if the category of simplicial objects has finite cotensors (see the discussion of tensors and cotensors in simplicially enriched categories in section **??**, page **??**). A similar comment may be made about 5), since using cotensors allows one to 'internalise' the condition - but it ends up then being 3) in an enriched form. The details will not be needed in our later discussion, so are **left to you if you need them**.
   We use this notion of *n*-coskeletal object in the following way

Proposition 21 *(cf. Beke, [19], proposition 1.3) (i) If K satisfies the exact Kan condition above dimension n, then K must be $(n + 1)$-coskeletal.*
(ii) If K is n-coskeletal, then it satisfies the exact Kan condition above dimension $n + 1$.
(iii) If K is an n-coskeletal Kan complex, then it has vanishing homotopy groups in dimensions n and above.
(iv) An exact n-type has vanishing homotopy groups above dimension n.

Before we prove this, it needs noting that there is an internal version in $Simp(\mathcal{E})$ for $\mathcal{E}$ a topos, see [19]. We have refrained from giving it only to avoid the need to define the homotopy groups of such an object internally.

   Proof: (i) Suppose we are given a map $b : \partial\Delta[k] \to K$ for $k > n + 1$, then we can omit $d_0 b$ to get a $(k, 0)$-horn in K. By assumption, this horn has a filler, $f : \Delta[k] \to K$, so we consider both $d_0 f$ and $d_0 b$. As they have the same boundary and since K satisfies the exact Kan condition above dimension n, they must coincide. We have thus that f is a filler for b. By exactness, we have that it is unique.
   (ii) If $m > n + 1$, $tr_n(\Lambda^k[m]) \to tr_n(\Delta[m])$ is fairly obviously an isomorphism. Now $cosk_n(K)$ satisfies the exact Kan condition in dimension m if, and only if, for any horn, $\underline{x} : \Lambda^k[m] \to K$, there

is a diagram

$$
\begin{array}{ccc}
\Lambda^k[m] & \xrightarrow{\ x\ } & cosk_nK \\
{\scriptstyle inc}\downarrow & {\scriptstyle \exists!}\nearrow & \downarrow \\
\Delta[k] & \longrightarrow & 1
\end{array}
$$

with unique diagonal. Using the adjunction, this gives a diagram

$$
\begin{array}{ccc}
tr_n\Lambda^k[m] & \xrightarrow{\ \overline{x}\ } & K \\
{\scriptstyle inc}\downarrow & {\scriptstyle ?}\nearrow & \downarrow \\
tr_n\Delta[k] & \longrightarrow & 1
\end{array}
$$

and we have noted that the left hand side is an isomorphism if $m > n + 1$.

(iii) If K is Kan, the topological description of homotopy groups goes over to K, i.e., as the group of homotopy classes of maps from $\partial\Delta[n]$ to K mapping a vertex to chosen basepoint. Such a map will fill in dimensions $k \geq n$, so all the $\pi_k(K)$ will be trivial for any base point. (**You should fill in the details of this argument.**)

(iv) This just combines (i) and (iii). ■

We note that (iv) above says that exact n-types are n-types!

3.1.3 Postnikov towers

In the topological case, we saw above that given any (connected) CW-complex, X, we could construct a sequence of Postnikov sections, P_nX, and maps between them, $P_{n+1}X \to P_nX$. We referred to this as a *Postnikov tower* for X. In the simplicial case, we found that the coskeletons gave us a corresponding construction, (and we will shortly see an alternative, if related, one). It is often useful to demand a bit more structure in the tower, structure that is always potentially there but which is usually not in its 'optimal form'. To make them more 'useful', we first review the definition of Postnikov towers and some of their properties. (We refer the reader, who wants a slightly more detailed introduction, to Hatcher's book, [92], p. 410.) First a redefinition,(adapted to our needs from [92])

Definition: A *Postnikov tower* for a (connected) space X is a commutative diagram:

$$
\begin{array}{c}
\vdots \\
\uparrow \ \downarrow \\
X_3 \\
\nearrow \ \uparrow \downarrow {\scriptstyle p_2^3(X)} \\
{\scriptstyle p_3(X)}\ X_2 \\
\nearrow \ \uparrow \downarrow {\scriptstyle p_1^2(X)} \\
{\scriptstyle p_2(X)} \\
X \xrightarrow[p_1(X)]{} X_1
\end{array}
$$

such that

(i) the map $X \to X_n$ induces an isomorphism on π_i for $i \leq n$;

(ii) $\pi_i(X_i) = 0$ for $i > n$.

Remark: A Postnikov tower for X always exists by our discussion in section 3.1.1 and, hidden in that discussion is the information that shows that the tower is unique up to a form of homotopy equivalence for towers.

If we convert each maps $X_n \to X_{n-1}$ into a fibration (in the usual way be pulling back the pathspace fibration on X_{n-1} along this map, see the discussion of the corresponding construction for chain complexes, in section 6.2.1, where the term *mapping cocone* is used), then its fibre (which is, then, the *homotopy fibre* of the original map), will be an Eilenberg-Mac Lane space, $K(\pi_n X, n)$, as the difference between the homotopy groups of X_n and X_{n-1} is exactly $\pi_n(X)$ in dimension n. (More exactly, we should look at the long exact homotopy sequence for this fibration, but we do not have this available within the notes so far so if you need more precision on this refer to Hatcher, [92], or other texts on homotopy theory.)

Definition: A *fibrant Postnikov tower* for X is a Postnikov tower (as above) in which each $X_n \to X_{n-1}$ is a fibration.

The discussion above shows that any Postnikov tower can be replaced, up to homotopy equivalence, by a fibrant one. There is here a technical remark that is worth making, but requires that the reader has met the theory of model categories. (It can safely be ignored if you have not yet met this.) On the category of towers of spaces (or or simplicial sets, etc.) there is a model category structure in which these fibrant towers are exactly the fibrant objects.

Moving over to the simplicial case, we restrict attention to Kan complexes, as they are much better behaved, homotoically, than arbitrary ones. We have the nth coskeleton, $cosk_n K$ of a Kan complex, K, and the first query is whether it is a kan complex itself. Certainly in dimensions lower than n, as it agrees with K there, any k-horn will have a filler. We thus look at an $(n+1)$-horn, $x_0, \ldots, \hat{x}_i, \ldots, x_{n+1}$, corresponding to the map, $\underline{x} : \Lambda^i[n+1] \to cosk_n K$, (using the usual convention with a 'hat' indicating the missing face). All the faces, x_k, are in $(cosk_n K)_n = K_n$, so all toegther they form a $(n+1)$-horn in K, which, of course, can be filled by some $y \in K_{n+1}$ We have its naming map $\ulcorner y \urcorner : \Delta[n+1] \to K$, which we restrict to $sk_n \Delta[n+1]$ to get a filler for our original $\underline{x}$. We thus do have that $cosk_n K$ satisfies the Kan filler condition in dimension $n+1$.

We look, next, at dimension $n = 2$ (expecting, of course, that the situation there will tell us how to handle the general case in higher dimensions). In fact, we have already seen the argument that we will use above.

Suppose $\underline{x} : \Lambda^i[n+2] \to cosk_n K$, then $\underline{x}$ corresponds, under the adjunction to a map, $\overline{x} : sk_n \Lambda^i[n+2] \to K$, but, and this is the neat argument we saw before, $sk_n \Lambda^i[n+2] = sk_n \Delta[n+2]$ (or, if you want to be precise, the inclusion of $\Lambda^i[n+2]$ into $\Delta[n+2]$ restricts to the 'identity' isomorphism on the n-skeletons). This means that $\overline{x}$ is already in $(cosk_n K)_{n+2}$. (Of course, dotting i's and crossing t's, that statement is also not true, but means $\Lambda^i[\ell] \to \Delta[\ell]$ induces a bijection

$$\mathcal{S}(\Delta[\ell], cosk_n K) \xrightarrow{\cong} \mathcal{S}(\Lambda^i[\ell], cosk_n K)$$

for $\ell = n + 2$, and, in fact, for all $\ell \geq n + 2$, so $sk_n \Lambda^i[n+2] \xrightarrow{\cong} sk_n \Delta[n+2]$ for all $\ell \geq n+2$.) We summarise this in a proposition for possible later use.

Proposition 22 *If K is a Kan complex, then so is $cosk_n K$.* ∎

We next glance at the canonical map

$$p_n^{n+1} : cosk_{n+1} K \to cosk_n K.$$

This does not seem to be a fibration, but that is not too worrying since (i) we can replace is by a fibration as in the topological case, and (ii) we will see there is a subtower of this *cosk* tower which is fibrant and very neat and we turn to it next. Its beauty is that it adapts well to many other simplicial settings, such as that of simplicial groups, without much adjustment, and it is functorial.

The canonical map, $p_n = \eta(K) : K \to cosk_n K$, which is the unit of the adjunction, can be very easily described in combinatorial terms, since $(cosk_n K)_m = \mathcal{S}(sk_n \delta[m], K)$. If x is a m-simplex in K, then its 'name' $\ulcorner x \urcorner : \Delta[m] \to K$ determines it precisely and conversely, (by the Yoneda lemma and the equation $\ulcorner x \urcorner \iota_n = x$). There is an inclusion, $i_m : sk_n \Delta[m] \to \Delta[m]$, and $\ulcorner x \urcorner \circ i_m$ is an m-simplex in $cosk_n K$. This is $eta(x)$.

In $(cosk_n K)_m$, there can be simplices that are not restrictions of m-simplices in K and these are, for instance, simplices that, together, 'kill' the homotopy groups (above dimension n, that is.) As K is Kan, $\pi_m(K) \cong [S^m, K]$, the set of pointed homotopy classes of pointed maps from $S^m = \partial \Delta[m+1]$ or alternatively, $S^m = \Delta[m]/\partial \Delta[m]$. (Both identifications are useful and we can go from one to the other since they are weakly homotopy equivalent.) We note that, for instance, $sk_{m-1} S^m = sk_{m-1} \Delta[m]$, so any m-sphere in K has a canonical filler in $cosk_{m-1} K$. Other cases are slightly more tricky, but can be **left to you**, as, in any case, when we consider these more formally slightly later on we will use a slightly different argument.

The image of $\eta(K)$ is, in each dimension m, obtained by dividing K_m by the equivalence relation determined by $\eta(K)_m$, i.e., define $\sim_n$ on K_m by $x \sim_n y$ if, and only if, the representing maps, $x, y : \Delta[m] \to K$ agree on $sk_n \Delta[m]$. (We will dispense with the 'name' notation, $\ulcorner x \urcorner$, here, as it tends to clutter the notation and is not needed, if no confusion is likely to occur. We are thus pretending that $K_m = \mathcal{S}(\Delta[m], K)$, rather tha merely being naturally isomorphic.)

We write $[x]_n$ for the $\sim_n$-equivalence class of x. We note that if $m \leq n$ then $\sim_n$ is simply equality as the n-skeleton of $\Delta[m]$ is all of $\Delta[m]$.

Definition: The simplicial set, $K(n) := K/\sim_n$ is called the nth *Postnikov section* of K.

That $\sim_m$ is compatible with the face and degeneracy maps is **easy to check**, so $K(n)$ is a simplicial set and , equally simply, the natural quotient, $q_n : K \to K(n)$, so $q_n(x) = [x]_n$, is simplicial. (It is the codomain restriction of $p_n = \eta(K)$.) This is best seen using the fact that is is induced from the *cosimplicial* inclusions $sk_n \Delta[m] \to \Delta[m]$. The cosimplicial viewpoint also gives that the inclusions $sk_n \Delta[m] \to sk_{n+1} \Delta[m]$ induce the quotient maps, $q_n^{n+1} : K(n+1) \to K(n)$, (which are the restrictions of the p_n^{n+1}), and that $q_n^{n+1} q_{n+1} = q_n$.

Lemma 16 *For a (connected) Kan complex, K, and for each n:*
  (i) The map $q_n : K \to K(n)$ is a Kan fibration, and $K(n)$ is a Kan complex.
  (ii) The map, $q_n^{n+1} : K(n+1) \to K(n)$, is a Kan fibration.
  (iii) The map, q_n, induces an isomorphism on π_i for $0 \leq i \leq n$.
  (iv) The homotopy groups of $K(n)$ are trivial above dimension n, $K(n)$ is an n-type.

Proof: (i) Suppose we have a commutative diagram

$$
\begin{array}{ccc}
\Lambda^i[m] & \xrightarrow{(x_0,\ldots,\hat{x_i},\ldots,x_m)} & K \\
\downarrow & & \downarrow q_n \\
\Delta[m] & \xrightarrow[{[y]_n}]{} & K(n)
\end{array}
$$

where we have written the i-horn as an $(m+1)$-tuple of $(m-1)$-simplices, with a gap at the 'hat'. We need to lift $[y]_n$ to some y agreeing with the x_ks, i.e., $d_k y = x_k$.

If $m \le n$, there is no problem as q_n the identity in those dimensions.

For $m = n + 1$, we have if y is a representative of $[y]_n$, then as $\sim_n$ is the identity relation in dimension n, $d_k y = x_k$ for $k \ne i$, so y is a suitable lift.

For $m > n+1$, we use that K is Kan to find a filler $x \in K_{m+1}$ for the (m,i)-horn, so $d_k x = x - k$ for $k \ne i$. Now $sk_n \Lambda^i[m] = sk_n \Delta[m]$, as we have used before, and so $q_n(x) = [x]_n = [y]_n$.

In general, if $p : K \to L$ is a surjective Kan fibration and K is a Kan complex, then L is Kan, so the last part of (i) follows.

(ii) Look at $K(n+1)$ and form $K(n+1)(n)$, i.e. divide it out by $\sim_n$. This gives $K(n)$ with the quotient being just q_n^{n+1}. By (i), this will be a fibration.

We next pick a base vertex, $v \in K_0$ and look at the various $\pi_m(K,v)$ and $\pi_m(K(n),[v]_n)$. Clearly, as q_n 'is the identity' in dimensions $m \le n$, the induced morphisms $\pi_m(q_n)$ 'is the identity' in dimensions $m < n$. For (iii), we have, thus, only to examine $\pi_n(q_n)$. Suppose $f : \Delta[n] \to K$ sends $\partial\Delta[n]$ to $\{v\}$, i.e., represents an element of $\pi_n(K)$, and that $q_n f$ is null-homotopic, then $q_n f$ extends to a map, $\overline{F} : \Delta[n+1] \to K(n)$ such that $q_n f = d_0 \overline{F}$, and $d_i \overline{F} = v$ for $i \ne 0$. We can lift $\overline{F}$ to a map $F : \Delta[n+1] \to K$, since q_n is surjective *and* the n-dimensional faces are mapped by the identity. We thus have that f itself was null-homotopic, so $\pi_n(q_n)$ is a monomorphism. As $\pi_n(q_n)$ is cearly an epimorphism, this handles (iii).

(iv) Any map $f : \Delta[m] \to K(n)$ is determined by its restriction, $f| : sk_n \Delta[m] \to K$, but

$$
sk_n \partial\Delta[m] \to sk_n \Delta[m]
$$

is the identity if $m > n$, and $f|_{\partial\Delta[m]}$ is constant with value v, so $\pi_m(K(n)) = 0$ if $m > n$. ∎

We thus have proved the connected case of the following:

Theorem 6 *If K is a Kan complex, $(K(n), q_n^{n+1}, q_n)$, forms a f(functorial) fibrant Postnikov tower for K.* ∎

The non-connected case is a simple extension of this connected one involving disjoint unions, so ...
.

Of course, the inclusion of $K(n)$ into $cosk_n K$ is a weak equivalence.

Remarks: (i) A note of caution seems in order. Some sources tend to confuse $K(n)$ and $cosk_n K$, and whilst, for many homotopical purposes, this is not critical, for certain purposes the use of one is prefereable to that of the other, so it seems better to keep the restriction.

(ii) The study of Postnikov complexes, which abstract the properties of the $K(n)$, is important in the study of coskeletal simplicial sets and nerves of higher categories, for which see the important paper of Duskin, [66].

(iii) Putting a naturally defined model category structure on the category of n-types (and on the corresponding simplicial presheaves and sheaves) has been done using these Postnikov sections, see Biedermann, [20]. He notes that his construction depends on using the Postnikov section approach that we have just outlined, rather than the coskeleton, as that latter one disturbs some of the necessary structure.

(iv) If you need more on Postnikov towers in simplicial sets, a good source is Goerss and Jardine, [85], Chapter 6, whilst Duskin's paper, [66], mentioned above, gives some powerful tools for manipulating them and also coskeletons.

3.1.4 Whitehead towers

Postnikov towers approximate a homotopy type by its tower of n-types, that is, by 'n-co-connected' spaces. The Whitehead tower of a homotopy type produces a sequence of n-connected approximations to it. Before we look at this in detail, let us consider what this should mean. (As a source, we will initially use Hatcher, [92], p. 356 in the topological case, before looking at the simplicial case. Another useful source is the nLab page on 'Whitehead towers', ([134], and search on 'Whitehead tower').)

What we would expect from a naive dualisation of Postnikov tower for a pointed space, X, would be a diagram,

$$\cdots \longrightarrow Z_2 \longrightarrow Z_1 \longrightarrow Z_0$$
$$\downarrow$$
$$X$$
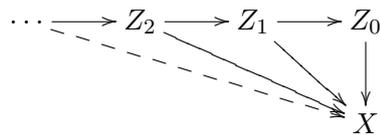
with Z_n an n connected space, (so $\pi_i(Z_n) = 0$ for $i \leq n$), and the composite map $Z_n \to X$ inducing an isomorphism on all homotopy groups, π_i for $i > n$. The space Z_0 would be path connected and homotopy equivalent to the component of X containing the base point. The next space, Z_1 would be simply connected and would have the homotopy properties of the universal cover of Z_0. We would then think of $Z_n \to X$ as an 'n-connected cover' of the (pointed connected component, Z_0, of the)space, X.

Definition: The *Whitehead tower* of a pointed space, (X, x) is a sequence of fibrations

$$\ldots \to X\langle n \rangle \to \ldots \to X\langle 1 \rangle \to X\langle 0 \rangle \to X$$

where each $X\langle n \rangle \to X\langle n-1 \rangle$ induces isomorphisms on the homotopy groups, π_i, for $i > n$ and such that $X\langle n \rangle$ is n-connected, so $\pi_k(X\langle n \rangle)$ is trivial for all $k \leq n$.

The problem of constructing such a tower was posed by Hurewicz and solved by George Whitehead in 1952. We will assume that we have chosen a Postnikov tower for a CW-complex, X, so giving a map $p_n : X \to P_n X$.

We next to form the *homotopy fibre* or *mapping cocone* of this map, over the basepoint, x_0, of $P_n X$. We have already seen this idea, page 15, so will just briefly review how it is constructed. We first form the pullback

$$
\begin{array}{ccc}
M^{p_n} & \xrightarrow{\pi^{p_n}} & (P_n X)^I \\
\downarrow{\scriptstyle j^{p_n}} & & \downarrow{\scriptstyle e_0} \\
X & \xrightarrow{p_n} & P_n X
\end{array}
$$

so M^{p_n} consists of pairs, (x, λ), where $x \in X$ and $\lambda : I \to P_n X$ is a path with $\lambda(0) = p_n(x)$. We set $i^{p_n} = e_1 \circ \pi^{p_n}$, so that $i^{p_n}(x, \lambda) = \lambda(1)$. The fact that $i^{p_n} : M^{p_n} \to P_n X$ is a fibration is standard, as is that $j^{p_n} : M^{p_n} \to X$ is a homotopy equivalence. (If you want a proof of these, **after trying to give one yourself**, there are proofs in many standard textbooks, such as that of Hatcher, and the abstract setting of such results is discussed in Kamps and Porter, [103]. This all fits well into a 'homotopical' context, and that is explored more on the nLab, [134], search under 'mapping cocone' and follow the links.) For brevity, we will write $\overline{X}$ for M^{p_n}, $\overline{p_n} : \overline{X} \to P_n X$ for i^{p_n}. The homotopy fibre of p_n is then the fibre of $\overline{p_n}$ over the base point of $P_n X$. It is $F^h(p_n) = \{(x, \lambda) \mid \lambda(1) = x_0\}$.

We thus have a fibration sequence,

$$F^h(p_n) \to \overline{X} \to P_n X,$$

and, hence, by standard homotopy theory, a long exact sequence of homotopy groups,

$$\ldots \to \pi_k(F^h(p_n)) \to \pi_k(\overline{X}) \to \pi_k(P_n X) \to \pi_{k-1}(F^h(p_n)) \to \ldots$$

Note that $\pi_k(\overline{X}) \cong \pi_k(X)$, since j^{p_n} is a homotopy equivalence. (If you have not met this long fibration exact sequence before, **check it up, briefly** in any standard book on homotopy theory. We will look at it, and also the dual situation in cohomology, in more detail later on, starting in section 6.2.)

If we look at this long exact sequence, below the value $k = n$, the homomorphism $\pi_k(\overline{X}) \to \pi_k(P_n X)$ is an isomorphism, so $\pi_k(F^h(p_n)) = 0$ in that range, whilst as $\pi_k(P_n X) = 0$ if $k > n$, there $\pi_k(F^h(p_n)) \to \pi_k(\overline{X})$ is an isomorphism. Thus the homotopy fibre, $F^h(p_n)$ is n-connected.

This looks good, as this is a functorial construction (or, more exactly, any lack of functoriality is due to a lack of functoriality of the Postnikov tower). We have a composite map $F^h(p_n) \to \overline{X} \to X$. This sends (x, λ) to x, of course. We will write $X\langle n \rangle := F^h(p_n)$, in the expectation that it will form part of a 'Whitehead tower'.

The next ingredient that we need will be a map

$$X\langle n+1 \rangle \to X\langle n \rangle.$$

We do have a (chosen) map $p_n^{n+1} : P_{n+1} X \to P_n X$, which is compatible with the 'projections' $p_n : X \to P_n X$, so $p_n^{n+1} p_{n+1} = p_n$. This induces a map from the homotopy fibre of p_{n+1} to that of p_n. (This is **left to you to check**. The usual proof uses the functoriality of $(-)^I$ and the naturality of the various mappings, and then the universal property of pullbacks. Everything is being 'chosen up to homotopy' so there are subtleties that **do** need thinking about, and it is a good idea to try to get a reasonably homotopy 'coherent' argument going on behind the proof. The construction is a 'homotopy pullback' and the property you a looking for is the analogue of the universal property of pullbacks to this more structured setting. It is, in the long term, important to get used to this sort of situation as well as to the sort of geometric / higher categorical picture that it corresponds to, as this is needed for generalisations.)

We note that the fibre of $X\langle n+1 \rangle \to X\langle n \rangle$ is a $K(\pi_n(X), n)$.

Remarks: (i) The above slightly hides the fact that the construction of a Whitehead tower is only really 'natural' up to homotopy as that was already the case for the Postnikov tower in the topological case.

(ii) For the simplicial case, we can use either the coskeleton based tower or, better, the Postnikov section one, as that is already fibrant as we saw. As the p_n and p_n^{n+1} are fibrations in that case, we

can replace the homotopy pullbacks by pullbacks, and the homotopy fibres by fibres, thus gaining more insight into the relationship of the objects in the corresponding Whitehead tower to the Kan complex being 'resolved'. (The detailed description is **left to you**.)

(iii) The theory and constructions adapt well to other simplicial contexts such as that of simplicial groups, where, as fibrations are simply degreewise epimorphisms, many of the constructions take on a much simpler algebraic aspect.

The case of a topological group, G: In this case, one can find a topological model for each $G\langle n\rangle$ which is a topological group, and, as there is a topological Abelian group model for the $K(\pi, n)$s occurring as the fibres in the tower, there is a short exact sequence

$$1 \to K(\pi_n(G), n) \to G\langle n + 1\rangle \to G\langle n\rangle \to 1.$$

Example: The Whitehead tower of the orthogonal group, $O(n)$.

For large n, the orthogonal group, $O(n)$, has the following homotopy groups:

i	0	1	2	3	4	5	6	7
$\pi_i(O(n))$	C_2	C_2	0	C_∞	0	0	0	C_∞

There are then periodicity results for higher dimensions giving $\pi_{k+8}(O(n)) \cong \pi_k(O(n))$. The first space of the Whitehead tower of $O(n)$ is, of course, $O(n)\langle 0\rangle = SO(n)$, as it is the (0-)connected component of the identity element.

The next space is the group, $O(n)\langle 1\rangle = Spin(n)$, (which we will look at in more detail later; see section 8.1.3). There is a short exact sequence:

$$1 \to C_2 \to Spin(n) \to SO(n) \to 1.$$

The next homotopy group is trivial and $O(n)\langle 2\rangle = O(n)\langle 3\rangle = String(n)$. This is a very interesting group, but we have not yet the machinery to do it justice. (For more on it in our sort of setting, see, for instance, Jurco, [102], Schommer-Pries, [149]. We will return to it later.)

3.2 Crossed squares

We next turn back to algebraic models of these n-types that we have now introduced more formally. We have already seen models for 2-types, namely the crossed modules that we looked at earlier, now we turn to 3-types. There are several different types of model here. We start with one that is relatively simple in its apparent structure.

3.2.1 An introduction to crossed squares

We saw earlier that crossed modules were like normal subgroups except that the inclusion map is replaced by a homomorphism that need not be a monomorphism. We even noted that all crossed modules are, up to isomorphism, obtainable by applying π_0 to a simplicial "inclusion crossed module".

Given a pair of normal subgroups M, N of a group G, we can form a square

$$
\begin{array}{ccc}
M \cap N & \longrightarrow & N \\
\downarrow & & \downarrow \\
M & \longrightarrow & G
\end{array}
$$

in which each morphism is an inclusion crossed module and there is a commutator map

$$h : M \times N \to M \cap N$$

$$h(m,n) = [m,n].$$

This forms a *crossed square* of groups, in fact, it is a special type of such that we will call an *inclusion crossed square*. Later we will be dealing with crossed squares as crossed n-cubes, for $n = 2$. Here we will give an interim definition of crossed squares. The notion is due to Guin-Walery and Loday, [89], and this slightly shortened form of the definition is adapted from Brown-Loday, [40].

3.2.2   Crossed squares, definition and examples

Definition: (First version) A *crossed square* (more correctly *crossed square of groups*) is a commutative square of groups and homomorphisms

$$
\begin{array}{ccc}
L & \stackrel{\lambda}{\longrightarrow} & M \\
\lambda' \downarrow & & \downarrow \mu \\
N & \stackrel{\nu}{\longrightarrow} & P
\end{array}
$$

together with actions of the group P on L, M and N (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h : M \times N \to L$. This structure is to satisfy the following axioms:
(i) the maps λ , λ' preserve the actions of P, furthermore with the given actions, the maps μ, ν and $\kappa = \mu\lambda = \mu'\lambda'$ are crossed modules;
(ii) $\lambda h(m,n) = m^n m^{-1}$, $\lambda' h(m,n) = {}^m n n^{-1}$;
(iii) $h(\lambda\ell, n) = \ell^n \ell^{-1}$, $h(m, \lambda'\ell) = {}^m \ell \ell^{-1}$;
(iv) $h(mm', n) = {}^m h(m', n) h(m, n)$, $h(m, nn') = h(m, n)^n h(m, n')$;
(v) $h({}^p m, {}^p n) = {}^p h(m, n)$;
for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

There is an evident notion of morphism of crossed squares, just preserve all the structure, and we obtain a category Crs^2, the *category of crossed squares*.

Examples
In addition to the above class of examples, we have the following:
(a) Given any simplicial group, G, and two simplicial normal subgroups, M and N, the square

$$
\begin{array}{ccc}
M \cap N & \longrightarrow & N \\
\downarrow & & \downarrow \\
M & \longrightarrow & G
\end{array}
$$

with inclusions and with $h = [\ , \ ] : M \times N \to G$ is a simplicial "inclusion crossed square" of simplicial groups. Applying π_0 to the diagram gives a crossed square and, in fact, all crossed squares arise in this way (up to isomorphism).

b) Any simplicial group, G, yields a crossed square, $M(G, 2)$, defined by

$$
\begin{array}{ccc}
\dfrac{NG_2}{d_0(NG_3)} & \longrightarrow & Ker\, d_1 \\
\downarrow & & \downarrow \\
Ker d_2 & \longrightarrow & G_1
\end{array}
$$

for suitable maps. This is, in fact, part of the construction that shows that all connected 3-types are modelled by crossed squares.

Another way of encoding 3-types is using the truncated simplicial group and Conduché's notion of 2-crossed module.

3.3 2-crossed modules and related ideas

3.3.1 Truncations.

Definition: Given a chain complex, (X, ∂), and an integer n, the *truncation of X at level n* is the complex $t_{n]}X$ defined by

$$
(t_{n]}X)_i = \left\{
\begin{array}{ll}
0 & \text{for } i > n \\
X_n / Im\, \partial_n & \\
X_i & \text{for } i < n.
\end{array}
\right.
$$

For $i < n$, the differential of $t_{n]}X$ is the same as that of X, whilst the nth-differential is induced by ∂.

(For more on truncations see Illusie [98, 99]). Truncation is, of course, functorial.

Remark on terminology: There are several schools of thought on the terminology here. The problem is whether this should be 'truncation' or 'co-truncation'. To some extent both are 'wrong' as n-truncated chain complexes 'should' not have any information available in dimensions greater than n, if the model of simplicial sets was to be followed. This would then be expected to have right and left adjoints, which would correspond, approximately to the coskeleton and skeleton functors of simplicial set theory that we have already seen. At the moment the 'jury' seems to be out and the terminological conventions fairly lax. (We may thus decide to change this later on if convincing arguments are presented.)

This construction will work for chain complexes of groups provided each $Im\, \partial$ is a *normal* subgroup of the corresponding X, i.e., provided X is a *normal chain complex of groups*.

Proposition 23 *There is a truncation functor $t_{n]} : Simp.Grps \to Simp.Grps$ such that there is a natural isomorphism*

$$
t_{n]}NG \cong Nt_{n]}G,
$$

where N is the Moore complex functor from $Simp.Grps$ to the category of normal chain complexes of groups.

Proof: We first note that $d_0(NG_{n+1})$ is contained in G_n as a normal subgroup and that all face maps of G vanish on it. We can thus take

$$
\begin{aligned}
(t_{n]}G)_i &= G_i \text{ for all } i < n \\
(t_{n]}G)_n &= G_n/d_0(NG_{n+1})
\end{aligned}
$$

and for $i > n$, we take the semidirect decomposition of G_i, which we will see shortly, given by Proposition 33, delete all occurrences of NG_k for $k > n$ and replace any NG_n by $NG_n/d_0(NG_{n+1})$. The definition of face and degeneracy is easy as is the verification that $t_{n]}N$ and $Nt_{n]}$ are the same and that the various actions are compatible. ∎

This truncation functor has nice properties. (In the chain complex case, these are discussed in Illusie, [98].)

Proposition 24 *Let $T_{n]}$ be the full subcategory of Simp.Grps defined by the simplicial groups whose Moore complex is trivial in dimensions greater than n and let $i_n : T_{n]} \to Simp.Grps$ be the inclusion functor.*

a) The functor $t_{n]}$ is left adjoint to i_n. (We will usually drop the i_n and so also write $t_{n]}$ for the composite functor.)

b) The natural transformation, η, co-unit of the adjunction, is a natural epimorphism which induces an isomorphism on π_i for $i \leq n$. The unit of the adjunction is isomorphic to the identity transformation, so $T_{n]}$ is a reflective subcategory of Simp.Grps.

c) For any simplicial group G, $\pi_i(t_{n]}G) = 0$ if $i > n$.

d) To the inclusion, $T_{n]} \to T_{n+1]}$, there corresponds a natural epimorphism η_n from $t_{n+1]}$ to $t_{n]}$. If G is a simplicial group, the kernel of $\eta_n(G)$ is a $K(\pi_{n+1}(G), n+1)$, i.e., has a single non-zero homotopy group in dimension $n+1$, that being $\pi_{n+1}(G)$, i.e., is an 'Eilenberg-Mac Lane space' of type $(\pi_{n+1}(G), n+1)$. ∎

As each statement is readily verified using the Moore complex and the semidirect product decomposition, the proof of the above will be left to you, however you will need Proposition 33, page 127.

Definition: We will say that a simplicial group, G, is *n-truncated* if $NG_k = 1$ for all $k > n$.

Of course, $T_{n]}$ is the category of n-truncated simplicial groups.

A comparison of these properties with those of the *coskeleton functors* (cf., above, section 3.1.2, page 80, or for an 'original' source, Artin and Mazur, [10]) is worth making. We will not look at this in detail here, but will just summarise the results. We have met them before and will meet them again later on; see page **??**.

Given any integer $k \geq 0$, there is a functor, $cosk_k$, defined on the category of simplicial sets, which is the composite of a truncation functor (differently defined) and its right adjoint. The n-simplices of $cosk_k X$ are given by $Hom(sk_k\Delta[n], X)$, the set of simplicial maps from the k-skeleton of the n-simplex, $\Delta[n]$, to the simplicial set, X. There is a canonical map from X to $cosk_k X$, whose homotopy fibre is $(k-1)$-connected. The canonical map from $cosk_k X$ to $cosk_{k-1} X$ thus has homotopy fibre an Eilenberg-Mac Lane 'space' of type $(\pi_k(X), k)$.

This k-coskeleton is constructed using finite limits and there is an analogue in any category of simplicial objects in a category, $\mathcal{D}$, provided only that $\mathcal{D}$ has finite limits, thus in particular in *Simp.Grps*. Conduché, [52], has calculated the Moore complex of $cosk_{k+1} G$ for a simplicial group, G, using a construction described in Duskin's Memoir, [64]. His result gives

$$
\begin{aligned}
N(cosk_{k+1}G)_r &= 0 \qquad \text{if } r > k+2 \\
N(cosk_{k+1}G)_{k+2} &= Ker(\partial_{k+1} : NG_{k+1} \to NG_k),
\end{aligned}
$$

and

$$
N(cosk_{k+1}G)_r = NG_r \quad \text{if } r \leq k+1.
$$

There is an epimorphism from $cosk_{n+1}G$ to $t_{n]}G$, which, on passing to Moore complexes, gives

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Ker\,\partial_{k+1} & \longrightarrow & NG_{k+1} & \longrightarrow & NG_k & \xrightarrow{\ \partial_{k+1}\ } & NG_{k-1} \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & NG_k/Im/\,\partial_{m+1} & \longrightarrow & NG_{k-1}
\end{array}
$$

This epimorphism of chain complexes thus has a kernel with trivial homology. The epimorphism therefore induces an isomorphism on all homotopy groups and hence is a weak homotopy equivalence. We may thus use either $t_{n]}G$ or $cosk_{n+1}G$ as a model of the n-type of G.

3.3.2 Truncated simplicial groups and the Brown-Loday lemma

The theory of crossed n-cubes that we have hinted at above is not the only way of encoding higher n-types. Another method would be to use these truncated simplicial groups as suggested above. A detailed study of this is complicated in high dimension, but feasible for 3-types and, in fact, reveals some interesting insights into crossed squares in the process.

As a first step to understanding truncated simplicial groups a bit more, we will give a variant of an argument that we have already seen. We will look at a 1-truncated simplicial group. The analysis is really a simple use of the sort of insights given by the Brown-Loday lemma.

Proposition 25 *(The Brown-Loday lemma) Let N_2 be the (closed) normal subgroup of G_2 generated by elements of the form*

$$
F_{(1),(0)}(x,y) = [s_1 x, s_0 y][s_0 y, s_0 x]
$$

for $x, y \in NG_1 = Ker\,d_1$. Then $NG_2 \cap D_2 = N_2$ and consequently

$$
\partial(NG_2 \cap D_2) = [Ker\,d_0, Ker\,d_1].
$$

$\blacksquare$

Note the link with group T-complex type conditions through the intersection, $NG_2 \cap D_2$.

The form of this element, $F_{(1),(0)}(x,y)$, is obtained by taking the two elements, x and y, of degree 1 in the Moore complex of a simplicial group, G, mapping them up to degree 2 by complementary degeneracies, and then looking at the component of the result that is in the Moore complex term, NG_2. (It is easy to show that G_2 is a semidirect product of NG_2 and degenerate copies of lower

degree Moore complex terms.) The idea behind this pairing can be extended to higher dimensions. It gives the *Peiffer pairings*

$$F_{\alpha,\beta} : NG_p \times NG_q \to NG_{p+q}.$$

In general, these take $x \in NG_p$ and $y \in NG_q$ and (α, β) a complimentary pair of index strings (of suitable lengths), and sends (x, y) to the component in NG_{p+q} of $[s_\alpha x, s_\beta y]$; see the series of papers [126–130]. This again uses the Conduché decomposition lemma, [52], that we will see later on, cf. page 127. It is also worth noting that the Peiffer pairing ends up in $NG_{p+q} \cap D_{p+q}$, so would all be zero in a group T-complex.

A very closely related notion is that of hypercrossed complex as in Carrasco and Cegarra, [49, 50]. There one uses the component of $s_\alpha x.s_\beta y$ in NG_{p+q} to give a pairing and adds cohomological information to the result to get a reconstruction technique for G from NG, i.e., an *ultimate Dold-Kan theorem*, thus hypercrossed complexes generalise 2-crossed modules and 2-crossed complexes to all dimensions.

3.3.3   1- and 2-truncated simplicial groups

Suppose that G is a simplicial group and that $NG_i = 1$ for $i \geq 2$. This leaves us just with

$$\partial : NG_1 \to NG_0.$$

We make $NG_0 = G_0$ act on NG_1 by conjugation as before

$$^g c = s_0(g)cs_0(g)^{-1} \text{ for } g \in G_0, c \in NG_1,$$

and, of course, $\partial(^g c) = g.\partial c.g^{-1}$. Thus the first crossed module axiom is satisfied. For the other one, we note that $F_{(1),(0)}(c_1, c_2) \in NG_2$, which is trivial, so

$$
\begin{aligned}
1 &= d_0([s_1c_1, s_0c_2][s_0c_2, s_0c_1]) \\
&= [s_0d_0c_1, c_2][c_2, c_1] = (^{\partial c_1}c_2)(c_1c_2c_1^{-1})^{-1},
\end{aligned}
$$

so the Peiffer identity holds as well. Thus $\partial : NG_1 \to NG_0$ is a crossed module. As we have already seen that the functor $\mathcal{G}$ provides a way to construct a simplicial group from a crossed module and that the result has Moore complex of length 1, we have the following slight reformulation of earlier results:

Proposition 26 *The category of crossed modules is equivalent to the subcategory $T_{1]}$ of 1-truncated simplicial groups.*   ∎

The main reason for restating and proving this result in this form is that we can glean more information from the proof for examining the next level, 2-truncated simplicial groups.

If we replace our 1-truncated simplicial group by an arbitrary one, then we have already introduced the idea of a Peiffer commutator of two elements, and there we used the term 'Peiffer lifting' without specifying what particular interest the construction had. We recall that here: Given a simplicial group, G, and two elements $c_1, c_2 \in NG_1$ as above, then the *Peiffer commutator* of c_1 and c_2 is defined by

$$\langle c_1, c_2 \rangle = (^{\partial c_1}c_2)(c_1c_2c_1^{-1})^{-1}.$$

We met earlier, $F_{(1),(0)}$, which gives the *Peiffer lifting* denoted

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

where

$$\{c_1, c_2\} = [s_1c_1, s_0c_2][s_0c_2, s_0c_1]$$

and we noted

$$\partial\{c_1, c_2\} = \langle c_1, c_2 \rangle.$$

These structures come into their own for a 2-truncated simplicial group. Suppose that G is now a simplicial group, which is 2-truncated, so its Moore complex looks like:

$$\ldots 1 \to NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0.$$

For the moment, we will concentrate our attention on the morphism ∂_2.

The group NG_1 acts on NG_2 via conjugation using s_0 or s_1. We will use s_0 for the moment, so that if $g \in NG_1$ and $c \in NG_2$,

$$^gc = s_0(g)cs_0(g)^{-1}.$$

It is once again clear that $\partial_2(^gc) = g.\partial_2(c).g^{-1}$ and, as before, we consider, for $c_1, c_2 \in NG_2$ this time, the Peiffer pairing given by

$$[s_1c_1, s_0c_2][s_0c_2, s_0c_1],$$

which is, this time, the component of $[s_1c_1, s_0c_2]$ in NG_3. However that latter group is trivial, so this element is trivial, and hence, so is its image in NG_2. The same calculation as before shows that, with this s_0-based action of NG_1 on NG_2, (NG_2, NG_1, ∂_2) is a crossed module.

We also know that there is a Peiffer lifting

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

which measures the obstruction to $NG_1 \to NG_0$ being a crossed module, since $\partial\{-,-\}$ is the Peiffer commutator, whose vanishing is equivalent to $NG_1 \to NG_0$ being a crossed module. We do not have yet in our investigation a detailed knowledge of how the two structures interact, nor any other distinguishing properties of $\{-,-\}$. We will not give such a detailed derivation here, but from it we can obtain the following:

Proposition 27 *Let G be a 2-truncated simplicial group. The Peiffer lifting*

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

has the following properties:
(i) it is a map such that if $m_0, m_1 \in NG_1$,

$$\partial\{m_0, m_1\} = {}^{\partial m_0}m_1.(m_0m_1m_0^{-1})^{-1};$$

(ii) if $\ell_0, \ell_1 \in NG_2$,

$$\{\partial\ell_0, \partial\ell_1\} = [\ell_0, \ell_1];$$

(iii) if $\ell \in NG_2$ and $m \in NG_1$, then

$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(iv) if $m_0, m_1, m_2 \in NG_1$, then
a) $\{m_0, m_1 m_2\} = \{m_0, m_1\}^{\,(m_0 m_1 m_0^{-1})}\{m_0, m_2\},$
b) $\{m_0 m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1 m_2 m_1^{-1}\};$
(v) if $n \in NG_0$ and $m_0, m_1 \in NG_1$, then

$$^n\{m_0, m_1\} = \{\,^n m_0, \,^n m_1\}.$$

■

The above can be encoded in the definition of a 2-crossed module.

3.3.4   2-crossed modules, the definition

Definition: A *2-crossed module* is a normal complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

together with an action of N on all three groups and a mapping

$$\{-, -\} : M \times M \to L$$

such that

(i) the action of N on itself is by conjugation, and ∂_2 and ∂_1 are N-equivariant;

(ii) for all $m_0, m_1 \in M$,
$$\partial_2\{m_0, m_1\} = {}^{\partial_1 m_0} m_1 . m_0 m_1^{-1} m_0^{-1};$$

(iii) if $\ell_0, \ell_0 \in L$, then
$$\{\partial_2 \ell_0, \partial_2 \ell\} = [\ell_1, \ell_0];$$

(iv) if $\ell \in L$ and $m \in M$, then
$$\{m, \partial \ell\}\{\partial \ell, m\} = {}^{\partial m} \ell . \ell^{-1};$$

(v) for all $m_0, m_1, m_2 \in M$,

(a) $\{m_0, m_1 m_2\} = \{m_0, m_1\}\{\partial\{m_0, m_2\}, (m_0 m_1 m_0^{-1})\}\{m_0, m_2\};$
(b) $\{m_0 m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1 m_2 m_1^{-1}\};$

(vi) if $n \in N$ and $m_0, m_1 \in M$, then

$$^n\{m_0, m_1\} = \{\,^n m_0, \,^n m_1\}.$$

The pairing $\{-, -\} : M \times M \to L$ is often called the *Peiffer lifting* of the 2-crossed module.

The only one of these axioms that looks 'daunting' is (v)a. Note that we have not specified that M acts on L. We could have done that as follows: if $m \in M$ and $\ell \in L$, define

$$^m \ell = \{\partial \ell, m\}\ell.$$

Now (v)a) simplifies to the expression

$$\{m_0, m_1 m_2\} = \{m_0, m_1\}^{(m_0 m_1 m_0^{-1})} \{m_0, m_2\}.$$

We denote such a 2-crossed module by $\{L, M, N, \partial_2, \partial_1\}$, or similar, only adding in notation for the actions and the pairing if explicitly needed for the context. A *morphism of 2-crossed modules* is, fairly obviously, given by a diagram

$$
\begin{array}{ccccc}
L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & N \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
L' & \xrightarrow{\partial_2'} & M' & \xrightarrow{\partial_1'} & N'
\end{array} \ ,
$$

where $f_0 \partial_1 = \partial_1' f_1, f_1 \partial_2 = \partial_2' f_2$,

$$f_1(^n m) = {}^{f_0(n)} f_1(m), \quad f_2(^n \ell) = {}^{f_0(n)} f_2(\ell),$$

and

$$\{-, -\}(f_1 \times f_1) = f_2\{-, -\},$$

for all $\ell \in L$, $m \in M$, $n \in N$.

These compose in an obvious way giving a category which we will denote by $2{-}CMod$.

The following should be clear.

Theorem 7 *The Moore complex of a 2-truncated simplicial group is a 2-crossed module. The assignment is functorial.* ∎

We will denote this functor by $\mathsf{C}^{(2)} : T_{2]} \to 2{-}CMod$. It is an equivalence of categories.

3.3.5 Examples of 2-crossed modules

Of course, the construction of 2-crossed modules from simplicial groups gives a generic family of examples, but we can do better than that and show how these new crossed gadgets link in with others that we have met earlier.

Example 1: Any crossed module gives a 2-crossed module, since if (M, N, ∂) is a crossed module, we need only add a trivial $L = 1$, and the resulting sequence

$$L \to M \to N$$

with the 'obvious actions' is a 2-crossed module! This is, of course, functorial and $CMod$ can be considered to be a full subcategory of $2{-}CMod$ in this way. It is a reflective subcategory since there is a reflection functor obtained as follows:

If

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

is a 2-crossed module, then $Im\, \partial_2$ is a normal subgroup of M and we have (with a small abuse of notation):

Proposition 28 *If $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ is a 2-crossed module then there is an induced crossed module structure on*

$$\partial_1 : \frac{M}{Im\,\partial_2} \to N.$$

$\blacksquare$

But we can do better than this:

Example 2: Any crossed complex of length 2, that is one of form

$$\ldots \to 1 \to 1 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

gives us a 2-crossed complex on taking $L = C_2$, $M = C_1$ and $N = C_0$, with $\{m, m'\} = 1$ for all $m, m' \in M$. We will check this in a moment, but note that this gives a functor from $Crs_{2]}$ to $2-CMod$ extending the one we gave in Example 1.

Of course, (i) crossed complexes of length 2 are the same as 2-truncated crossed complexes.

3.3.6   Exploration of trivial Peiffer lifting

Suppose we have a 2-crossed module

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

with the extra condition that $\{m_0, m_1\} = 1$ for all $m_0, m_1 \in M$. The obvious thing to do is to see what each of the defining properties of a 2-crossed module give in this case.

(i) There is an action of N on L and M and the ∂s are N-equivariant. (This gives nothing new in our special case.)

(ii) $\{-, -\}$ is a lifting of the Peiffer commutator - so if $\{m_0, m_1\} = 1$, the Peiffer identity holds for (M, N, ∂_1), i.e. that is a crossed module;

(iii) if $\ell_0, \ell_1 \in L$, then $1 = \{\partial_2\ell_0, \partial_2\ell_1\} = [\ell_1, \ell_0]$, so L is Abelian
and,

(iv) as $\{-, -\}$ is trivial $^{\partial m}\ell = \ell$, so ∂M has trivial action on L.

Axioms (v) and (vi) vanish.

We leave the reader, if they so wish, to structure this into a formal proof that the 2-crossed module is precisely a 2-truncated crossed complex.

Our earlier discussion should suggest:

Proposition 29 *The category $Crs_{2]}$ of crossed complexes of length 2 is equivalent to the full sub-category of $2-CMod$ given by those 2-crossed complexes with trivial Peiffer lifting.* $\blacksquare$

We leave the proof of this to the reader.

A final comment is that in a 2-truncated simplicial group, G, one obviously has that it satisfies the thin filler condition (cf. page **??**) in dimensions greater than 2, since $NG_k = 1$ for all $k > 2$ and if the Peiffer lifting is trivial in the corresponding 2-crossed module, G satisfies it in dimensions 2 as well. (As D_1 is $s_0(G_0)$, any simplicial group satisfies the thin filler condition in dimension 1.)

In the next section we will give other examples of 2-crossed modules, those coming from crossed squares.

3.3.7   2-crossed modules and crossed squares

We now have several 'competing' models for homotopy 3-types. Since we can go from simplicial groups to both crossed square and 2-crossed modules, there should be some link between the latter two situations. In his work on homotopy n-types, Loday gave a construction of what he called a 'mapping cone' for a crossed square. Conduché later noticed that this naturally had the structure of a 2-crossed module. This is looked at in detail in a paper by Conduché, [53].

Suppose that

$$\begin{array}{ccc} L & \xrightarrow{\ \lambda\ } & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow[\ \mu'\ ]{} & P \end{array}$$

is a crossed square, then its mapping cone complex is

$$L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} P,$$

where $\partial_2 \ell = (\lambda \ell^{-1}, \lambda' \ell)$ and $\partial_1(m, n) = \mu(m)\nu(n)$.

We first note that the semi-direct product $M \rtimes N$ is formed by making N act on M *via* P, i.e.

$$^n m = {}^{\nu(n)}m,$$

where the P-action is the given one. The fact that (λ^{-1}, λ') and $\mu\nu$ are homomorphisms is an interesting and instructive, but easy, exercise:

i) $(m, n)(m', n') = (m^{\nu(n)}m', nn')$, so

$$\begin{aligned} \partial_1((m, n)(m', n')) &= \mu(m^{\nu(n)}m').\nu(nn') \\ &= \mu(m)\nu(n)\mu(m')\nu(n)^{-1}\nu(n)\nu(n') \\ &= (\mu(m)\nu(n))(\mu(m')\nu(n')); \end{aligned}$$

(ii) if $\ell, \ell' \in L$, then, of course,

$$\begin{aligned} \partial_1(\ell\ell') &= (\lambda(\ell\ell')^{-1}, \lambda'(\ell\ell')) \\ &= (\lambda(\ell')^{-1}\lambda(\ell)^{-1}, \lambda'(\ell)\lambda'(\ell')). \end{aligned}$$

whilst

$$\begin{aligned} \partial_1(\ell)\partial_1(\ell') &= (\lambda(\ell)^{-1}, \lambda'(\ell))(\lambda(\ell')^{-1}, \lambda'(\ell')) \\ &= (\lambda(\ell)^{-1}.^{\nu\lambda'(\ell^{-1})}\lambda(\ell')^{-1}, \lambda'(\ell\ell')), \end{aligned}$$

thus the second coordinates are the same, but, as $\nu\lambda' = \mu\lambda$, the first coordinates are also equal.

These elementary calculations are useful as they pave the way for the calculation of the Peiffer commutator of $x = (m, n)$ and $y = (c, a)$ in the above complex:

$$\begin{aligned} \langle x, y \rangle &= {}^{\partial x}y.xy^{-1}x^{-1} \\ &= {}^{\mu m.\nu n}(c, a).(m, n)(^{a^{-1}}c^{-1}, a^{-1})(^{n^{-1}}m^{-1}, n^{-1}) \\ &= (^{\mu m \nu n}c, {}^{\mu m \nu n}a)(m^{\nu(na^{-1})}c^{-1}.^{\nu(na^{-1}n^{-1})}m^{-1}, na^{-1}n^{-1}), \end{aligned}$$

which on multiplying out and simplifying is

$$(^{\nu(na^{-1}n^{-1})}m.m^{-1}, {}^{\mu m}(nan^{-1}).(na^{-1}n^{-1})).$$

(Note that any dependence on c vanishes!)

    Conduché defined the Peiffer lifting in this situation by

$$\{x, y\} = h(m, nan^{-1}).$$

It is immediate to check that this works

$$
\begin{aligned}
\partial_2\{x, y\} &= (\lambda h(m, nan^{-1}), \lambda' h(m, nan^{-1})) \\
&= (^{\nu(na^{-1}n^{-1})}m.m^{-1}, {}^{\mu m}(nan^{-1}).(na^{-1}n^{-1})),
\end{aligned}
$$

by the axioms of a crossed square.

    We will not check all the axioms for a 2-crossed module for this structure, but will note the proofs for one or two of them as they illustrate the connection between the properties of the h-map and those of the Peiffer lifting.

    2CM(iii) : $\{\partial \ell_0, \partial \ell_1\} = [\ell_1, \ell_0]$. As $\partial \ell = (\lambda \ell^{-1}, \lambda' \ell)$, this needs the calculation of

$$h(\lambda \ell_0^{-1}, \lambda'(\ell_0 \ell_1 \ell_0^{-1})),$$

but the crossed square axiom :

$$h(\lambda \ell, n) = \ell.^n \ell^{-1}, \text{ and } h(m, \lambda' \ell) = {}^m \ell.\ell^{-1},$$

together with the fact that the map $\lambda : L \to M$ is a crossed module, give

$$
\begin{aligned}
h(\lambda \ell_0^{-1}, \lambda'(\ell_0 \ell_1 \ell_0^{-1})) &= {}^{\mu \lambda(\ell_0^{-1}}(\ell_0 \ell_1 \ell_0^{-1}).\ell_0 \ell_1^{-1} \ell_0^{-1}) \\
&= [\ell_1, \ell_0].
\end{aligned}
$$

We need $\{(m, n), (\lambda \ell^{-1}, \lambda' \ell)\}\{(\lambda \ell^{-1}, \lambda' \ell), (m, n)\}$ to equal ${}^{\mu(m)\nu(n)}\ell.\ell^{-1}$, but evaluating the initial expression gives

$$
\begin{aligned}
h(m, n.\lambda' \ell.n^{-1})h(\lambda \ell^{-1}, \lambda' \ell.n.\lambda' \ell^{-1}) &= h(m, \lambda'(^n \ell))h(\lambda \ell^{-1}, \lambda' \ell.n.\lambda' \ell^{-1}) \\
&= {}^{\mu(m)\nu(n)}\ell.^{\nu(n)}\ell^{-1}.\ell^{-1}.^{\nu \lambda'(\ell).\nu(n).\nu \lambda' \ell^{-1}}\ell,
\end{aligned}
$$

and this does simplify as expected to give the correct results.

    We thus have two ways of going from a simplicial group, G, to a 2-crossed module:
    (a) directly to get

$$\frac{NG_2}{\partial NG_3} \to NG_1 \to NG_0;$$

    (b) indirectly via $M(G, 2)$ and then by the above construction to get

$$\frac{NG_2}{\partial NG_3} \to Ker\, d_0 \rtimes Ker\, d_1 \to G_1$$

and they clearly give the same homotopy type. More precisely G_1 decomposes as $Ker\, d_0 \rtimes s_0 G_0$ and the $Ker\, d_0$ factor in the middle term of (b) maps down to that in this decomposition by the identity map, thus d_0 induces a quotient map from (b) to (a) with kernel isomorphic to

$$1 \to Ker\, d_0 \overset{=}{\to} Ker\, d_0,$$

which is acyclic/contractible.

3.3.8 2-crossed complexes

(These were not discussed in the lectures in Buenos Aires due to lack of time.) Crossed complexes are a useful extension of crossed modules allowing not only the encoding of an algebraic model for the 2-type, but also information on the 'chains on the universal cover', e.g. if G is a simplicial group, earlier, in section 2.5.1, we had $\mathsf{C}(G)$, the crossed complex constructed from the Moore complex of G, given by

$$C(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions and having at its 'bottom end' the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0.$$

For a crossed complex, $\pi(X)$, coming from a CW-complex (as a filtered space, filtered by its skeleta), these groups in dimensions ≥ 3 coincide with the corresponding groups of the complex of chains on the universal cover of X. In general, the analogue of that chain complex can be extracted functorially from a general crossed complex; see [37] or [141]. The tail on a crossed complex allows extra dimensions, not available just with crossed modules, in which homotopies can be constructed. The category Crs is very much better structured than is $CMod$ itself and so 'adding a tail' would seem to be a 'good thing to do', so with 2-crossed modules, we can try and do something similar, adding a similar 'tail'.

We have an obvious normal chain complex of groups that ends

$$\ldots \to C(G)_3 \to \frac{NG_2}{d_0(NG_3 \cap D_3)} \to NG_1 \to NG_0.$$

Here there are more of the structural Peiffer pairings of the Moore complex NG that survive to the quotient, but it should be clear that, as they take values in the $NG_n \cap D_n$, in general these will again be almost all trivial if the receiving dimension, n, is greater than 2. For $n \leq 2$, these pairings are those that we have been using earlier in this chapter. The one exceptional case that is important here, as in the crossed complex case, is that which gives the action of NG_0 on $C_n(G)$ for $n \geq 3$, which, just as before, gives $C_n(G)$ the structure of a $\pi_0 G$-module. Abstracting from this gives the definition of a 2-crossed complex.

Definition: A *2-crossed complex* is a normal complex of groups

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_0,$$

together with a 2-crossed module structure given on $C_2 \to C_1 \to C_0$ by a Peiffer lifting function $\{-, -\} : C_1 \times C_1 \to C_2$, such that, on writing $\pi = Coker(C_1 \to C_0)$,

(i) each C_n, $n \geq 3$ and $Ker\, \partial_2$ are π-modules and the ∂_n for $n \geq 4$, together with the codomain restriction of ∂_3, are π-module homomorphisms;

(ii) the π-module structure on $Ker\, \partial_2$ is the action induced from the C_0-action on C_2 for which the action of $\partial_1 C_1$ is trivial.

A 2-crossed complex morphism is defined in the obvious way, being compatible with all the actions, the pairings and Peiffer liftings. We will denote by $2 - Crs$, the corresponding category.

There are reduced and unreduced versions of this definition. In the discussion and in the notation we use, we will quietly ignore the groupoid based non-reduced version, but it is easy to give simply by replacing simplicial groups by simplicially enriched groupoids, and making fairly obvious changes to the definitions.

Proposition 30 *The construction above defines a functor, $\mathsf{C}^{(2)}$, from Simp.Grps to $2 - Crs$.* ■

There are no prizes for guessing that the simplicial groups whose homotopy types are accurately encoded in $2 - Crs$ by this functor are those that satisfy the thin condition in dimensions greater than 3. In fact, the construction of the functor $\mathsf{C}^{(2)}$ explicitly kills off the intersection $NG_k \cap D_k$ for $k \geq 3$.

We have noted above that any 2-crossed module,

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

gives us a short crossed complex by dividing L by the subgroup $\{M, M\}$, the image of the Peiffer lifting. (We do not need this, but $\{M, M\}$ is easily checked to be a normal subgroup of L.) We also discussed those 2-crossed complexes that had trivial Peiffer lifting. They were just the length 2 crossed complexes. This allows one to show that crossed complexes form a reflexive subcategory of $2 - Crs$ and to give a simple description of the reflector:

Proposition 31 *There is an embedding*

$$Crs \to 2{-}Crs,$$

which has a left adjoint, L say, compatible with the functors defined from Simp.Grps to $2{-}Crs$ and to Crs, i.e. $\mathsf{C}(G) \cong \mathsf{L}\mathsf{C}^{(2)}(G)$. ■

3.4   Cat^n-groups and crossed n-cubes

3.4.1   Cat^2-groups and crossed squares

In the simplest examples of crossed squares, μ and μ' are normal subgroup inclusions and $L = M \cap N$, with h being the conjugation map. Moreover this type of example is almost 'generic' since, if

$$
\begin{array}{ccc}
M \cap N & \longrightarrow & M \\
\downarrow & & \downarrow \\
N & \longrightarrow & G
\end{array}
$$

is a simplicial crossed square constructed from a simplicial group, G, and two simplicial normal subgroups, M and N, then applying π_0, the square gives a crossed square and, up to isomorphism, all crossed squares arise in this way.

Although when first defined by D. Guin-Walery and J.-L. Loday, [89], the notion of crossed squares was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n-groups (see [110] and below).

Definition: A *cat^1-group* is a triple, (G, s, t), where G is a group and s, t are endomorphisms of G satisfying conditions
(i) $st = t$ and $ts = s$.
(ii) $[Ker\, s, Ker\, t] = 1$.

A cat^1-group is a reformulation of an internal groupoid in *Grps*. (The interchange law is given by the $[Ker, Ker]$ condition; left for you to check) As these latter objects are equivalent to crossed modules, we expect to be able to go between cat^1-groups and crossed modules without hindrance, and we can:

Setting $M = Ker\, s$, $N = Im\, s$ and $\partial = t|M$, then the action of N on M by conjugation within G makes $\partial : M \to N$ into a crossed module. Conversely if $\partial : M \to N$ is a crossed module, then setting $G = M \rtimes N$ and letting s, t be defined by

$$s(m, n) = (1, n)$$

and

$$t(m, n) = (1, \partial(m)n)$$

for $m \in M$, $n \in N$, we have that (G, s, t) is a cat^1-group. Again this is one of those simple, but key calculations that are well worth doing yourself.

For a cat^2-group, we again have a group, G, but this time with two independent cat^1-group structures on it. Explicitly:

Definition: A *cat^2-group* is a 5-tuple (G, s_1, t_1, s_2, t_2), where (G, s_i, t_i), $i = 1, 2$, are cat^1-groups and

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$$

for $i, j = 1, 2, \quad i \neq j$.

There is an obvious notion of morphism between cat^2-groups and with this we obtain a category, $Cat^2(Grps)$.

Theorem 8 *[110] There is an equivalence of categories between the category of cat^2-groups and that of crossed squares.*

Proof: The cat^1-group (G, s_1, t_1) will give us a crossed module with $M = Ker\, s_1$, $N = Im\, s_1$, and $\partial = t|M$, but, as the two cat^1-group structures are independent, (G, s_2, t_2) restricts to give cat^1-group structures on both M and N and makes ∂ a morphism of cat^1-groups as is easily checked. We thus get a morphism of crossed modules

$$
\begin{array}{ccc}
Ker\, s_1 \cap Ker\, s_2 & \longrightarrow & Im\, s_1 \cap Ker\, s_2 \\
\downarrow & & \downarrow \\
Ker\, s_2 \cap Im\, s_1 & \longrightarrow & Im\, s_1 \cap Im\, s_2,
\end{array}
$$

where each morphism is a crossed module for the natural action, i.e., conjugation in G. It remains to produce an h-map, but this is given by the commutator within G, since, if $x \in Ker\, s_2 \cap Im\, s_1$

and $y \in Im\, s_2 \cap Ker\, s_1$, then $[x, y] \in Ker\, s_1 \cap Ker\, s_2$. It is easy to check the axioms for a crossed square. The converse is left as an exercise.   ∎

3.4.2   Interpretation of crossed squares and cat^2-groups

We have said that crossed squares and cat^2-groups give equivalent categories and we will see that, similarly, for the crossed n-cubes and cat^n-groups, which will be introduced shortly. The simplest case of that general situation is one that we have already already met namely that of crossed modules and cat^1-groups, and there we earlier saw how to interpret a crossed modules as being the essential data for a 2-group(oid).

   We thus have, you may recall (combining ideas from pages 21 and 107), that a crossed module, (C, P, ∂), gives us a cat^1-group / 2-group, $(C \rtimes P, s, t)$, with $s(c, p) = p$ being the source of an element (c, p) and $t(c, p) = \partial c.p$ being its target. The definition of cat^2-group does not explicitly use the language of 'internal categories', we mentioned that the $[Ker\, s, Ker\, t] = 1$ condition is a version of the interchange law, and that a cat^{1}group can be interpreted as an internal category in $Grps$. This leads to pictures such as

$$p_1 \xrightarrow{(c_1, p_1)} \partial c_1.p_1,$$

(cf. section 1.3.2, page 21) indicating that (c, p) interprets as an arrow having source and target as indicated. We could equally well use the 2-category or 2-group(oid) style diagram:
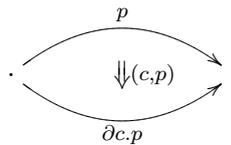
as we discussed earlier in section 1.3.3.

   If we start with a cat^1-group, (G, s, t), then the picture is

$$s(g) \xrightarrow{g} t(g).$$

It thus looks that the source and target are 'objects' of the category structure that we know to be there. Where do they live? Clearly in $Im\, s$ or $Im\, t$, or both. Life is easy on us however. We note that $Im\, s = Im\, t$, since $st = t$ implies that $Im\, t \subseteq Im\, s$, whilst we also have $ts = s$, giving the other inclusion. The subgroup $Im\, s$, corresponds to the group P of the crossed module, considered as a subgroup of the 'big group' $C \rtimes P$.

   It is sometimes more convenient to write an internal category in the form

$$G_1 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} \overset{}{\underset{\iota}{\leftarrow}} G_0 \; ,$$

so that G_1 is an object of arrows and G_0 the object of objects, in our case, the 'group of objects'. The cat^1-group notation replaces the source, target and identity maps by the composites $s = \iota\sigma$ and $t = \iota\tau$. This, of course, gives endomorphisms of G_1, which are simpler to handle than having a 'many sorted' picture with two separate groups. The downside of that simplicity is that the object of objects is slightly hidden. Of course, it is this subgroup, $Im\, s$, and the inclusion of that subgroup

into $G = G_1$ is the morphism denoted ι. It is therefore reasonable to draw the 'objects' as blobs or points rather than as elements of G, e.g., as loops on the single real object of the group thought of as a single object groupoid. The resulting pictures *are* easier to draw! and to interprete.

A cat^2-group is similarly a category-like structure, internal to cat^1-groups, so is a double category internal to the category of groups, as the two category structures are independent of each other. This is emphasised if we look at the elements of a cat^2-group in an analogous way to the above. First suppose that (G, s_1, t_1, s_2, t_2) is a cat^2-group, then we might draw, for each $g \in G$, a square diagram:

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\ t_2 g\ } & \cdot \\
{\scriptstyle s_1 g}\Big\uparrow & g & \Big\uparrow{\scriptstyle t_1 g} \\
\cdot & \xrightarrow[\ s_2 g\ ]{} & \cdot
\end{array}
$$

Now the left vertical arrow is in the subgroup, $Im\, s_1 = Im\, t_1$. (We can refer to $s_1 g$ as the 1-source, and $t_1 g$ as the 1-target, of g, and similarly for 2-source, and so on.) The square is a schema consistent the the equations: $s_1 t_2 = t_2 s_1$, and the three other similar ones. The element $s_1 t_2 g$ is the 1-source of the 2-target of g, so is the vertex at the top left of the square. It is also the 2-target of the 1-source of g, of course.

Such squares compose horizontally and vertically, provided the relevant sources and targets match, but how does this relate to the group structure on G?

Looking back, once more, to a cat^1-group, (G, s, t) and a resulting composition

$$
s(g) \xrightarrow{\ g\ } t(g) = s(g') \xrightarrow{\ g'\ } t(g'),
$$

it is not immediately clear how the composite is to be studied, but look back to the corresponding crossed module based description and it becomes clearer. We had in section 1.3.2,

$$
p \xrightarrow{(c,p)} \partial c.p \xrightarrow{(c',\partial c.p)} \partial c' \partial c.p,
$$

and the composition was given as $(c', \partial c.p) \star (c, p) = (c'c, p)$. Back in cat^1-group language, this corresponds to $g' \star g = g' s(g')^{-1} g$. (We can check that $s(g' s(g')^{-1} g) = s(g)$ and that $t(g' s(g')^{-1} g) = t(g')$, as we would expect.)

We can extend this to cat^2-groups giving a way of composing the squares that we have in this context. For instance, for horizontal composition, we have

$$
\begin{array}{ccccc}
\cdot \to \cdot \to \cdot \\
\Big\uparrow\ g\ \Big\uparrow\ g'\ \Big\uparrow \\
\cdot \to \cdot \to \cdot
\end{array}
\quad = \quad
\begin{array}{cc}
\cdot \to \cdot \\
\Big\uparrow {\scriptstyle g' s_1(g')^{-1} g}\Big\uparrow \\
\cdot \to \cdot
\end{array}
$$

and similarly for vertical composition, replacing s_1 by s_2.

That gives a double category interpretation for a cat^2-group, but how does this relate to a crossed square,

$$
\begin{array}{ccc}
L & \xrightarrow{\ \lambda\ } & M \\
{\scriptstyle \lambda'}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu} \\
N & \xrightarrow[\ \mu'\ ]{} & P
\end{array}
$$

with h-map $h : M \times N \to L$. The construction hinted at earlier is first to form the cat^1-groups of the two vertical crossed modules, giving

$$\partial : L \rtimes N \to M \rtimes P, \text{ with } \partial(\ell, n) = (\lambda(\ell), \nu(n)),$$

with ∂ the induced map. There is an action of $M \rtimes P$ on $L \rtimes N$ (which will be examined shortly) giving a crossed module structure to the result. This action is non-trivial to define (or discover), so here is a way of thinking of it that may help.

We 'know' that a crossed square is meant to be a crossed module of crossed modules, so, if the above ∂ and action does give a crossed module, we will then be able to form a 'big group', $(L \rtimes N) \rtimes (M \rtimes P)$, with a cat^2-group structure on it. The action of $M \rtimes P$ on $L \rtimes N$ will need to correspond to conjugation within this 'big group' as the idea of semi-direct products is, amongst other things, to realise an action: if G acts on H, $H \rtimes G$ has multiplication given by $(h_1, g_1)(h_2, g_2) = (h_1 {}^{g_1} h_2, g_1 g_2)$. In particular, it is easy to **work out**

$$(h, g)^{-1} = ({}^{g^{-1}} h^{-1}, g^{-1}),$$

so

$$(1, g)(h, 1)(1, g)^{-1} = ({}^g h, 1).$$

In our situation, we thus can work out the conjugation,

$$((1, 1), (m, p))((\ell, n), (1, 1))((1, 1), ({}^{p^{-1}} m^{-1}, p^{-1})) = ({}^{(m,p)}(\ell, n), (1, 1)).$$

Now this looks as if we are getting nowhere, but let us remember that any crossed square is isomorphic to the π_0 of an 'inclusion crossed square' of simplicial groups, (this was mentioned on page 95). This suggests that we first look at a group G, and a pair of normal subgroups M, N, and the inclusion crossed square

$$
\begin{array}{ccc}
M \cap N & \longrightarrow & N \\
\downarrow & & \downarrow \\
M & \longrightarrow & G
\end{array}
$$

with $h(m, n) = [m, n]$. If we track the above discussion of the action and the definition of ∂ in this example, we get the induced map, ∂, is the inclusion of $(M \cap N) \rtimes N$ into $M \rtimes G$. Here, therefore, there is, '$gratis$', an action of $M \rtimes G$ on $(M \cap N) \rtimes N$, namely by inner automorphisms / conjugation:

$$
\begin{aligned}
(m, g)(\ell, n)({}^{g^{-1}} m^{-1}, g^{-1})) &= (m, g)(\ell . n . {}^{g^{-1}} m . n^{-1}, ng) \\
&= (m . {}^g \ell . {}^g n . m . {}^g n^{-1}, gmg^{-1}),
\end{aligned}
$$

which can conveniently be written

$$({}^{mg} \ell . [m, {}^g n], {}^g n).$$

This suggests a formula for an action in the general case

$$
\begin{aligned}
{}^{(m,p)}(\ell, n) &= {}^m({}^p \ell, {}^p n) \\
&= ({}^{\mu(m)p} \ell . h(m, {}^p n), {}^p n).
\end{aligned}
$$

If we start with a simplicial inclusion crossed square, and form its 'big simplicial group' simplicially using the previous formula, then this *will* give the action of $M \rtimes P$ on $L \rtimes N$ in the general case,

so our guess looks as if it is correct. Note that in both the particular case of the inclusion crossed square and this general case, we can derive $h(m, n)$ as a commutator within the 'big group'. (Of course, for the first of these, the h-map was defined as a commutator within G.)

We could go on to play around with other facets of this construction. This would be **well worthwhile** - but is better **left to the reader**. For instance, one obvious query is that $(L \rtimes N) \rtimes (M \rtimes P)$ should not be dependent on thinking of a crossed square as a morphism of (vertical) crossed modules. It is also a morphism of horizontal crossed modules, so this 'big group', if it is to give a useful object, should be isomorphic to $(L \rtimes M) \rtimes (N \rtimes P)$. It is, but what is a specific natural isomorphism doing the job. As somehow M has to 'pass through' N, we should expect to have to use the h-map.

There are other 'games to play'. Central extensions gave an instance of crossed modules, so what is their analogue for crossed squares. Double central extensions have been introduced by Janelidze in [100] and have been further studied by others, [76, 86, 147]. They provide a related idea. It is **left to you** to explore any connections that there are.

If we start with a crossed square, as above, what is the analogue of the picture

$$p_1 \overset{(c_1, p_1)}{\longrightarrow} \partial c_1.p_1,$$

representing an element of the 'big group' of a crossed module. Suppose (ℓ, n, m, p) is such an element, then it is easy to see the 2-cell that corresponds to it must be:

$$
\begin{array}{ccc}
\nu(n)p & \xrightarrow{\lambda(\ell).^{\nu(n)}m, \nu(n)p)} & \mu(\lambda(\ell))\nu(n)\mu(m)p \\
\uparrow{\scriptstyle (n,p)} & {\scriptstyle (\ell,n,m,p)} & \uparrow{\scriptstyle \lambda'(\ell)n, \mu(m)p)} \\
p & \xrightarrow[(m,p)]{} & \mu(m)p
\end{array}
$$

The details of how to compose, etc. are again **left to you**. It is, however, worth just checking the way in which the two edges on the top and on the right do match up. The right hand edge will clearly end at $\nu(\lambda'(\ell))\nu(n)\mu(m)p$, which, as $\nu\lambda' = \mu\lambda$, gives the expression on the top right vertex. Of more fun is the top edge. This ends at

$$\mu(\lambda(\ell)).\mu(^{\nu(n)}m).\nu(n).p = \mu(\lambda(\ell)).\nu(n)\mu(m)\nu(n)^{-1}\nu(n)p,$$

so is as required, using the fact that μ is a crossed module.

In such a square 2-cell, the square itself is in the 'big group', the edges are in the cat^1-groups corresponding to vertical and horizontal crossed modules of the crossed square, and the vertices are in P.

Particularly interesting is the case of two crossed modules, $\mu : M \to P$ and $\nu : N \to P$, together with the corresponding $L = M \otimes N$, the Brown-Loday tensor product of the two, (cf. [39, 40]). Approximately, $M \otimes N$ is the universal codomain for an h-map based on the two given sides of the resulting crossed square. (A treatment of this construction has been included in the notes, [141], - please ignore the profinite conditions if using it 'discretely'.)

3.4.3   Cat^n-groups and crossed n-cubes, the general case

Of the two notions named in the title of this section, the first is easier to define.

Definition: A *cat^n-group* is a group G together with $2n$ endomorphisms $s_i, t_i, (1 \leq i \leq n)$ such that

$$s_i t_i = t_i, \text{ and } t_i s_i = s_i \text{ for all } i,$$

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \text{ for } i \neq j$$

and, for all i,

$$[Ker\, s_i, Ker\, t_i] = 1.$$

A cat^n-group is thus a group with n independent cat^1-group structures on it.

As a cat^1-group can also be reformulated as an internal groupoid in the category of groups, a cat^n-group, not surprisingly, leads to an internal n-fold groupoid in the same setting.

The definition of crossed n-cube as an n-fold crossed module was initially suggested by Ellis in his thesis. The only problem was to determine the sense in which one crossed module should act on another. Since the number of axioms controlling the structure increased from crossed modules to crossed squares, one might fear that the number and complexity of the axioms would increase drastically in passing to higher 'dimensions'. The formulation that resulted from the joint work, [75], of Ellis and Steiner showed how that could be avoided by encoding the actions and the h-maps in the same structure.

We write $\langle n \rangle$ for the set $\{1, \ldots, n\}$.

Definition: A *crossed n-cube*, M, is a family of groups, $\{M_A : A \subseteq \langle n \rangle\}$, together with homomorphisms, $\mu_i : M_A \to M_{A-\{i\}}$, for $i \in \langle n \rangle, A \subseteq \langle n \rangle$, and functions, $h : M_A \times M_B \to M_{A \cup B}$, for $A, B \subseteq \langle n \rangle$, such that if $^a b$ denotes $h(a,b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \langle n \rangle$, the following axioms hold:

(1) $\mu_i a = a$ if $a \notin A$

(2) $\mu_i \mu_j a = \mu_j \mu_i a$

(3) $\mu_i h(a,b) = h(\mu_i a, \mu_i b)$

(4) $h(a,b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$

(5) $h(a,a') = [a,a']$

(6) $h(a,b) = h(b,a)^{-1}$

(7) $h(a,b) = 1$ if $a = 1$ or $b = 1$

(8) $h(aa',b) = {}^a h(a',b) h(a,b)$

(9) $h(a,bb') = h(a,b)^b h(a,b')$

(10) $^a h(h(a^{-1},b),c)^c h(h(c^{-1},a),b)^b h(h(b^{-1},c),a) = 1$

(11) $^a h(b,c) = h(^a b, {}^a c)$ if $A \subseteq B \cap C$.

A morphism of crossed n-cubes

$$\{M_A\} \to \{M_A'\}$$

is a family of homomorphisms, $\{f_A : M_A \to M_A' \,|\, A \subseteq \langle n \rangle\}$, which commute with the maps, μ_i, and the functions, h. This gives us a category, Crs^n, equivalent to that of cat^n-groups.

Remarks: 1. In the correspondence between cat^n-groups and crossed n-cubes (see Ellis and Steiner, [75]), the cat^n-group corresponding to a crossed n-cube, (M_A), is constructed as a repeated semidirect product of the various M_A. Within the resulting "big group", the h-functions interpret as being commutators. This partially explains the structure of the h-function axioms.

2. For $n = 1$, these eleven axioms reduce to the usual crossed module axioms. For $n = 2$, they give a crossed square:

$$
\begin{array}{ccc}
M_{\langle 2 \rangle} & \xrightarrow{\;\mu_2\;} & M_{\{1\}} \\
{\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu_1} \\
M_{\{2\}} & \xrightarrow[\;\mu_2\;]{} & M_{\emptyset}
\end{array} \ ,
$$

with the h-map, that was previously specified, being $h : M_{\{1\}} \times M_{\{2\}} \to M_{\langle 2 \rangle}$. The other h-maps in the above definition correspond to the various actions as explained in the definition itself.

Theorem 9 *[75] There are equivalences of categories*

$$
Crs^n \simeq Cat^n(Grps),
$$

∎

3.5 Loday's Theorem and its extensions

In 1982, Loday proved a generalisation of the Mac Lane-Whitehead result that stated that connected homotopy 2-types (they called them 3-types) were modelled by crossed modules. The extension used cat^n-groups, and, as cat^1-groups 'are' crossed modules, we should expect cat^n-groups to model connected $(n + 1)$-types (if the Mac Lane-Whitehead result is to be the $n = 1$ case, see page 95).

We have mentioned that 'simplicial groupoids' model all homotopy types and had a construction of both a crossed module $M(G, 1)$ and a crossed square, $M(G, 2)$ from a simplicial group, G. These are the $n = 1$ and $n = 2$ cases of a general construction of a crossed n-cube from G that we will give in a moment First we note a rather neat result.

We saw early on in these notes, (Lemma 1, page 12), that if $\partial : C \to P$ was a crossed module, then $\partial C \triangleleft P$, i.e. is a normal subgroup of P. A crossed square

$$
\begin{array}{ccc}
L & \xrightarrow{\;\lambda\;} & M \\
{\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\
N & \xrightarrow[\;\mu'\;]{} & P
\end{array}
$$

can be thought of as a (horizontal or vertical,) crossed module of crossed modules:

$$
\begin{array}{ccc}
L & & M \\
\downarrow & \longrightarrow & \downarrow \\
N & & P
\end{array}
$$

(λ, ν) gives such a crossed module with domain (L, N, λ') and codomain (M, P, μ) and so on. (Working out the precise meaning of 'crossed module of crossed modules' and, in particular, what

it should mean to have an action of one crossed module on another, is a very useful exercise; try it!) The image of (λ, ν) is a normal sub-crossed module of (M, P, μ), so we can form a quotient

$$\overline{\mu} : M/\lambda L \to P/\nu N,$$

and this is a crossed module. (This is not hard to check. There are lots of different ways of checking it, but perhaps the best way is just to show how $P/\nu N$ acts on $M/\lambda L$, in an obvious way, and then to check the induced map, $\overline{\mu}$, has the right properties - just by checking them. This gives one a feeling for how the various parts of the definition of a crossed square are used here.)

Another result from near the start of these notes, (Lemma 2), is that $Ker\, \partial$ is a central subgroup of C and ∂C acts trivially on it, so $Ker\, \partial$ has a natural $P/\partial C$-module structure. Is there an analogue of this for a crossed square? Of course, referring again to our crossed square, above, the kernel of (λ, ν) would be $\lambda' : Ker\, \lambda \to Ker\, \nu$ (omitting any indication of restriction of λ' for convenience). Both $Ker\, \lambda$ and $Ker\, \nu$ are Abelian, as they themselves are kernels of crossed modules, so $Ker\, \lambda$ is a $M/\lambda L$-module and $Ker\, \nu$ is a $P/\nu N$-module. (It is left to the diligent reader to work out the detailed structure here and to explore crossed modules that are modules over other ones.)

We had, for a given simplicial group, G, the crossed square

$$
\begin{array}{ccc}
\dfrac{NG_2}{d_0(NG_3)} & \longrightarrow & Ker\, d_1 \\[2ex]
\downarrow & & \downarrow \\[2ex]
Ker d_2 & \longrightarrow & G_1
\end{array}
$$

which was denoted $M(G, 2)$. (The top horizontal and left vertical maps are induced by d_0.) Let us examine the horizontal quotient and kernel.

First the quotient, this has $NG_1/d_0 NG_2$ as its 'top' group and $G_1/Ker\, d_0 \cong G_0$, as its bottom one. Checking all the induced maps shows quite quickly that the quotient crossed module is $M(G, 1)$, up to isomorphism.

What about the kernel? Well, the bottom horizontal map is an inclusion, so has trivial kernel, whilst the top is induced by d_0, and so the kernel here can be calculated to be $Ker\, d_0 \cap NG_2$, divided by $d_0(NG_3)$, but that is $Ker\, \partial / Im\, \partial$ in the Moore complex, so is $H_2(NG)$ and thus is $\pi_2(G)$. We thus have, from previous calculations, that for $M(G, 1)$, there is a crossed 2-fold extension

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

and for $M(G, 2)$, a similar object, a crossed 2-fold extension of crossed modules:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_2(G) & \longrightarrow & Ker\, d_1 & \longrightarrow & NG_2/d_0(NG_3) & \longrightarrow & Ker\, d_1 NG_1/d_0(NG_2) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & 1 & \longrightarrow & Ker\, d_0 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & 1
\end{array}
$$

'Obviously' this should give an element of '$H^3(M(G, 2), (\pi_2(G) \to 1))$', but we have not given any description of what that cohomology group should be. It can be done, but we will not go in that direction for the moment. Rather we will use the route via simplicial groups.

3.5.1 Simplicial groups and crossed n-cubes, the main ideas

We have that simplicial groups yield crossed squares by the $M(G, 2)$ construction, and that, from $M(G, 2)$, we can calculate $\pi_0(G)$, $\pi_1(G)$, and $\pi_2(G)$. If G represents a 3-type of a space (or the 2-type of a simplicial group), then we would expect these homotopy groups to be the only non-trivial ones. (Any simplicial group can be truncated to give one with these π_i as the only non-trivial ones.) This suggests that going from 3-types to crossed squares in a nice way should be just a question of combining the functorial constructions

$$\text{Spaces} \quad \overset{Sing}{\longrightarrow} \quad \text{Simplicial Sets}$$

$$\text{Simplicial Sets} \quad \overset{G(\ )}{\longrightarrow} \quad \mathcal{S}-\text{Groupoids}$$

$$\mathcal{S}-\text{Groupoids} \quad \overset{M(\ ,2)}{\longrightarrow} \quad \text{Crossed squares.}$$

Of course, we would need to see if, for $f : X \to Y$ a 3-equivalence (so f induces isomorphisms on π_i for $i = 0, 1, 2, 3$), what would be the relationship between the corresponding crossed squares. We would also need to know that each crossed square was in sense 'equivalent' to one of the form $M(G, 2)$ for some G constructed from it, in other words to reverse, in part, the last construction. (The other constructions have well known inverses at the homotopy level.)

We will use a 'multinerve' construction, generalising the nerve that we have already met. We will denote this by $E^{(n)}(\mathsf{M})$ for M a crossed n-cube.

For $n = 1$, $E^{(1)}$ is just the nerve of the crossed module, so if $\mathsf{M} = (C, P, \partial)$, we have $E^{(1)}(\mathsf{M}) = K(\mathsf{M})$ as given already on page 28.

For $n = 2$, i.e., for a crossed square, M, we form the 'double nerve' of the associated cat^2-group of M. From M, we first form the 'crossed module of cat^1-groups'

$$L \rtimes N \overset{(\lambda, \nu)}{\longrightarrow} M \rtimes P,$$

where, for instance, in $M \rtimes P$ the source endomorphism is $s(m, p) = (1, p)$ and the target is $t(m, p) = (1, \partial m.p)$. (We could repeat in the horizontal direction to form $(L \rtimes N) \rtimes (M \rtimes P)$, which is the 'big group' of the cat^2-group associated to M, but, in fact, will not do this except implicitly, as it is easier to form a simplicial crossed module in this situation. This,

$$E^{(1)}(L \overset{\lambda'}{\to} N) \longrightarrow E^{(1)}(M \overset{\mu}{\to} P),$$

is obtained by applying the $E^{(1)}$ construction to the vertical crossed modules. The two parts are linked by a morphism of simplicial groups induced from (λ, ν) and which is compatible with the action of the right hand simplicial group on the left hand one. (This action is not that obvious to write down - unless you have already done the previously suggested 'exercises'. It uses the h-maps from $M \times N$ to L, etc. in an essential way, and is, in some ways, best viewed within $(L \rtimes N) \rtimes (M \rtimes P)$ as being derived from conjugation. Details are, for instance, in Porter, [141] or [136] as well as in the discussion of the equivalence between cat^n-groups and crossed n-cubes in the original, [75].)

With this simplicial crossed module, we apply the nerve in the second horizontal direction to get a bisimplicial group, $\mathcal{E}^{(2)}(\mathsf{M})$. (Of course, if we started with a crossed n-cube, we could repeat the application of the nerve functor n-times, one in each direction to get an n-simplicial group $\mathcal{E}^{(n)}(\mathsf{M})$.)

There are two ways of getting from a bisimplicial set or group to a simplicial one. One is the diagonal, so if $\{G_{p,q}\}$ is a bisimplicial group, $\mathsf{diag}(G_{\bullet,\bullet})_n = G_{n,n}$ with fairly obvious face and degeneracy maps. The other is the *codiagonal* (also sometimes called the 'bar construction'). This was introduced by Artin and Mazur, [9]. It picks up related terms in the various $G_{p,q}$ for $p+q = n$. (An example is for any simplicial group, G, on taking the nerve in each dimension. You get a bisimplicial set whose codiagonal is $\overline{W}(G)$, with the formula given later in these notes.) We will consider the codiagonal in some detail later on, (starting on page **??**). The two constructions give homotopically equivalent simplicial groups. Proofs of this can be found in several places in the literature, for instance, in the paper by Cegarra and Remedios, [51]. Here we will set $E^{(n)}(\mathsf{M}) = \mathsf{diag}\mathcal{E}^{(n)}(\mathsf{M})$.

At this stage, for the reader trying to understand what is going on here, it is worth calculating the Moore complex of these simplicial groups. This is technically quite tricky as it is easy to make a slip, but it is not hard to see that they are 'closely related' to the 2-crossed module / mapping cone complex:

$$L \to M \rtimes N \to P$$

that we met earlier, (page 103), that is due to Loday and Conduché, see [53]. Of course, such detailed calculations are much harder to generalise to crossed n-cubes and other techniques are used, see [136] or the alternative version based on the technology of cat^n-groups due to Bullejos, Cegarra and Duskin, [44].

In any of these approaches from a crossed n-cube or cat^n-group, you either extract a n-simplicial group and then a simplicial group, by diagonal or codiagonal, or going one stage further, applying the nerve functor to the n-simplicial group to get a $(n+1)$-simplicial set, which is then 'attacked' using the diagonal or codiagonal functors to get out a simplicial set. This end result is the simplicial model for the crossed n-cube and has the same homotopy groups as M. It is known as the *classifying space of the crossed n-cube or cat^n-group*. (That term is usual, but it actually gives rise to an interesting obvious question, which has a simple answer in some ways but not if one looks at it thoroughly. That question is : *what does this classifying space classify?* That question will to some extent return to haunt us later one. The simple answer would be certain types of simplicial fibre bundles with fibre a $n+1$-type, but that throws away all the hard work to get the crossed n-cube itself, so ... .

Returning to the simplicial group approach, one applies the $M(-, n)$-functor, that we have so far seen only for $n = 1$ and 2, to get back a new crossed n-cube. This is not M itself in general, but is 'quasi-isomorphic' to it.

Definition: A morphism, $f : \mathsf{M} \to \mathsf{N}$, of crossed n-cubes will be called a *trivial epimorphism* if $\mathcal{E}^{(n)}(f) : \mathcal{E}^{(n)}(\mathsf{M}) \to \mathcal{E}^{(n)}(\mathsf{N})$ is an epimorphism (and thus a fibration) of simplicial groups having contractible kernel.

Starting with the category, Crs^n, of crossed n-cubes, inverting the trivial epimorphisms gives a category, $Ho(Crs^n)$, and f will be called a *quasi-isomorphism* if it gives an isomorphism in this category.

Remark: Any trivial epimorphism of crossed modules is a *weak equivalence* in the sense of section 2.1, page 32. This follows from the long exact fibration sequence. Conversely any such

weak equivalence is a quasi-isomorphism.

We can now state Loday's result in the form given in [136]:

Theorem 10 *The functor*

$$M(-, n) : Simp.Grps \to Crs^n$$

induces an equivalence of categories

$$Ho_n(Simp.Grps) \overset{\simeq}{\Rightarrow} Ho(Crs^n).$$

∎

As yet we have not actually given the definition of $M(G, n)$ for $n > 2$ so here it is:

Definition Given a simplicial group, G, the crossed n-cube, $M(G, n)$, is given by:
(a) for $A \subseteq \langle n \rangle$,

$$M(G, n)_A = \frac{\bigcap \{Ker\, d_j^n \ : \ j \in A\}}{d_0(Ker\, d_1^{n+1} \cap \bigcap \{Ker\, d_{j+1}^{n+1} \ : \ j \in A\})};$$

(b) if $i \in \langle n \rangle$, the homomorphism $\mu_i : M(G, n)_A \to M(G, n)_{A \setminus \{i\}}$ is induced from the inclusion of $\bigcap \{Ker\, d_j^n \ : \ j \in A\}$ into $\bigcap \{Ker\, d_j^n \ : \ j \in A \setminus \{i\}\}$;
(c) representing an element in $M(G, n)_A$ by $\overline{x}$, where $x \in \bigcap \{Ker\, d_j^n \ : \ j \in A\}$, (so the overbar denotes a coset), and, for $A, B \subseteq \langle n \rangle$, $\overline{x} \in M(G, n)_A$, $\overline{y} \in M(G, n)_B$,

$$h(\overline{x}, \overline{y}) = \overline{[x, y]} \in M(G, n)_{A \cup B}.$$

Where this definition 'comes from' and why it works is a bit to lengthy to include here, so we refer the interested reader to [141]. From its many properties, we will mention just the following one, linking $M(G, n)$ with $M(G, n-1)$ in a similar way to that we have examined for $n = 2$.

We will use the following notation: $M(G, n)_1$ will denote the crossed $(n-1)$-cube obtained by restricting to those $A \subseteq \langle n \rangle$ with $1 \in A$ and $M(G, n)_0$ that obtained from the terms with $A \subseteq \langle n \rangle$ with $1 \notin A$.

Proposition 32 *Given a simplicial group G and $n \geq 1$, there is an exact sequence of crossed $(n-1)$-cubes:*

$$1 \to K \to M(G, n)_1 \overset{\mu_1}{\to} M(G, n)_0 \to M(G, n-1) \to 1,$$

where, if $B \subseteq \langle n-1 \rangle$ and $B \neq \langle n-1 \rangle$, then $K_B = \{1\}$, whilst $K_{\langle n-1 \rangle} \cong \pi_n(G)$. ∎

There are some special cases of crossed n-cubes, or the associated cat^n-groups that are worth looking at. For instance in [135], Paoli gives a new perspective on cat^n groups. It identifies a full subcategory of them (which are called *weakly globular*) which is sufficient to model connected $n + 1$-types, but which has much better homotopical properties than the general ones. This, in fact, gives a more transparent algebraic description of the Postnikov decomposition and of the homotopy groups of the classifying space, and it also gives a kind of minimality property. Using weakly globular cat^n groups one can also describe a comparison functor to the Tamsamani model of $n + 1$-types (cf. Tamsamani, [154]) which preserves the homotopy type.

3.5.2 Squared complexes

We have met crossed squares and 2-crossed modules and the different ways they encode the homotopy 3-type. We have extended 2-crossed modules to 2-crossed complexes, so it is natural curiosity to try to extend crossed squares to a 'cube' formulation. We will see this is just the start of another hierachy which is in some ways simpler than that suggested by the hypercrossed complexes, and their variants, etc. The first step is the following which was introduced by Ellis, [73].

Definition: A *squared complex* consists of a diagram of group homomorphisms

$$\cdots \longrightarrow C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} L \begin{array}{c} \xrightarrow{\lambda'} N \xrightarrow{\mu} \\ \\ \xrightarrow{\lambda} M \xrightarrow{\mu'} \end{array} P$$

together with actions of P on L, N, M and C_i for $i \geq 3$, and a function $h : M \times N \longrightarrow L$. The following axioms need to be satisfied.

(i) The square $\begin{pmatrix} L \xrightarrow{\lambda} N \\ \lambda' \downarrow \quad \downarrow \mu \\ M \xrightarrow[\mu']{} P \end{pmatrix}$ is a crossed square;

(ii) The group C_n is Abelian for $n \geq 3$

(iii) The boundary homomorphisms satisfy $\partial_n \partial_{n+1} = 1$ for $n \geq 3$, and $\partial_3(C_3)$ lies in the intersection $Ker\,\lambda \cap Ker\,\lambda'$;

(iv) The action of P on C_n for $n \geq 3$ is such that μM and $\mu' N$ act trivially. Thus each C_n is a π_0-module with $\pi_0 = P/\mu M \mu' N$.

(v) The homomorphisms ∂_n are π_0-module homomorphisms for $n \geq 3$.

This last condition does make sense since the axioms for crossed squares imply that $Ker\,\mu' \cap Ker\mu$ is a π_0-module.

Definition: A *morphism of squared complexes*,

$$\varphi : \left(C_*, \begin{pmatrix} L \xrightarrow{\lambda} N \\ \lambda' \downarrow \quad \downarrow \mu \\ M \xrightarrow[\mu']{} P \end{pmatrix} \right) \longrightarrow \left(C'_*, \begin{pmatrix} L' \xrightarrow{\lambda} N' \\ \lambda' \downarrow \quad \downarrow \mu \\ M' \xrightarrow[\mu']{} P' \end{pmatrix} \right)$$

consists of a morphism of crossed squares $(\varphi_L, \varphi_N, \varphi_M, \varphi_P)$, together with a family of equivariant homomorphisms φ_n for $n \geq 3$ satisfying $\varphi_L \partial_3 = \partial'_3 \varphi_L$ and $\varphi_{n-1} \partial_n = \partial'_n \varphi_n$ for $n \geq 4$. There is clearly a category $SqComp$ of squared complexes.

A squared complex is thus a crossed square with a 'tail' attached.

Any simplicial group will give us such a gadget by taking the crossed square to be $M(sk_2G, 2)$,

that is,

$$
\begin{array}{ccc}
\dfrac{NG_2}{d_0(NG_3 \cap D_3)} & \longrightarrow & Ker\, d_1 \\
\downarrow & & \downarrow \\
Ker d_2 & \longrightarrow & G_1
\end{array}
$$

and then, for $n \geq 3$,

$$
C_n(G) = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})}.
$$

The above complex contains not only the information for the crossed square $M(G,2)$ that represents the 3-type, but also the whole of $\mathsf{C}^{(2)}(G)$, the 2-crossed complex of G and thus the crossed complex and the 'chains on the universal cover' of G.

The advantage of working with crossed squares or squared complexes rather than the more linearly displayed models is that they can more easily encode 'non-symmetric' information. We will show this in low dimensions here but will later indicate how to extend it to higher ones. For instance, one gets a building process for homotopy types that reflects more the algebra. In examples, given two crossed modules, $\mu : M \to P$ and $\nu : N \to P$, there is a universal crossed square defining a 'tensor product' of the two crossed modules. We have

$$
\begin{array}{ccc}
M \otimes N & \xrightarrow{\ \lambda\ } & M \\
{\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\
N & \xrightarrow{\ \nu\ } & P
\end{array}
$$

is a crossed square and hence represents a 3-type. It is universal with regard to crossed squares having the same right-hand and bottom crossed modules, (see [39, 40] for the original theory and [141] for its connections with other material).

Equivalently we could represent its 3-type as a 2-crossed module

$$
M \otimes N \longrightarrow M \rtimes N \xrightarrow{\ \mu\nu\ } P
$$

or

$$
M \otimes N \longrightarrow \frac{(M \rtimes N)}{\sim} \longrightarrow \frac{P}{\mu M},
$$

where $\sim$ corresponds to dividing out by the μM action. However, of these, the crossed square lays out the information in a clearer format and so can often have some advantages.

Chapter 4

Classifying spaces, and extensions

We will first look in detail at the construction of classifying spaces and their applications for the non-Abelian cohomology of *groups*. This will use things we have already met. Later on we will need to transfer some of this to a sheaf theoretic context to handle 'gerbes' and to look at other forms of non-Abelian cohomology.

4.1   Non-Abelian extensions revisited

We again start with an extension of groups:

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1.$$

From a section, s, we constructed a factor set, f, but this *is* a bit messy. What do we mean by that? We are working in the category of groups, but neither s nor f are group morphisms. For s, there is an obvious thing to do. The function s induces a homomorphism, k_1, from $C_1(G)$, the free group on the set, G, to E and

$$
\begin{array}{ccc}
C_1(G) & \longrightarrow & G \\
\downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle =} \\
E & \xrightarrow{\ p\ } & G
\end{array}
$$

commutes. One might be tempted to do the same for f, but f is partially controlled by s, so we try something else. When we were discussing identities among relations (page **??**), we looked at the example of taking $X = \{\langle g \rangle \mid g \neq 1, g \in G\}$ and a relation $r_{g,g'} := \langle g \rangle \langle g' \rangle \langle gg' \rangle^{-1}$ for each pair (g, g') of elements of G. (Here we will write $\langle g_1, g_2 \rangle$ for r_{g_1,g_2}.)

We can use this presentation $\mathcal{P}$ to build a free crossed module

$$C(\mathcal{P}) := C_2(G) \to C_1(G).$$

We noted earlier that the identities were going to correspond to tetrahedra, and that, in fact, we could continue the construction by taking $C_n(G) =$ the free G-module on $\langle g_1, \ldots, g_n \rangle$, $g_i \neq 1$, i.e. the normalised bar resolution. This is very nearly the usual bar resolution coming from the nerve of G, but we have a crossed module at the base, not just some more modules.

We met this structure earlier when we were looking at syzygies, and later on with crossed n-fold extensions, but is it of any use to us here?

We know $pf(g_1, g_2) = 1$, so $f(g_1, g_2) \in K$, and $C_2(G)$ is a free crossed module ... . Also, $K \to E$ is a normal inclusion, so is a crossed module ... . Thinking along these lines, we try

$$k_2 : C_2(G) \to K$$

defined on generators by f, i.e., $i(k_2(\langle g_1, g_2 \rangle)) = f(g_1, g_2)$. It is fairly easy to check this works, that

$$\partial k_2(\langle g_1, g_2 \rangle) = k_1 \partial(\langle g_1, g_2 \rangle),$$

and that the actions are compatible, i.e., $\mathbf{k} : C(\mathcal{P}) \to \mathcal{E}$, where will write $\mathcal{E}$ also for the crossed module (K, E, i).

In other words, it seems that the section and the resulting factor set give us a morphism of crossed modules, $\mathbf{k}$. We note however that f satisfies a cocycle condition, so what does that look like here? To answer this we make the boundary, $\partial_3 : C_3(G) \to C_2(G)$, precise.

$$\partial_3 \langle g_1, g_2, g_3 \rangle = {}^{\langle g_1 \rangle}\!\langle g_2, g_3 \rangle \langle g_1, g_2 g_3 \rangle \langle g_1 g_2, g_3 \rangle^{-1} \langle g_1, g_2 \rangle^{-1}$$

and, of course, the cocycle condition just says that $k_2 \partial_3$ is trivial.

We can use the idea of a crossed complex as being a crossed module with a tail which is a chain complex, to point out that $\mathbf{k}$ gives a morphism of crossed complexes:

$$
\begin{array}{ccccccccc}
\mathsf{C}(G) : \ldots & \longrightarrow & C_3(G) & \longrightarrow & C_2(G) & \longrightarrow & C_1(G) & \longrightarrow & G \\
& & \downarrow & & \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle k_1} & & \downarrow \\
\mathcal{E} : \ldots & \longrightarrow & 1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G
\end{array}
$$

where the crossed module $\mathcal{E}$ is thought of as a crossed complex with trivial tail.

Back to our general extension,

$$\mathcal{E} : \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

we note that the choice of a section, s, does not allow the use of an action of G on K. Of course, there is an action of E on K by conjugation and hence s does give us an action of $C_1(G)$ on K. If we translate 'action of G on a group, K', to being a functor from the *groupoid*, $G[1]$, to *Grps* sending the single object of $G[1]$ to the object K, then we can consider the 2-category structure of *Grps* with 2-cells given by conjugation, (so that if K and L are groups, and $f_1, f_2 : K \to L$ homomorphisms, a 2-cell $\alpha : f_1 \Longrightarrow f_2$ will be given by an element $\ell \in L$ such that

$$f_2(x) = \ell f_1(x) \ell^{-1}$$

for all $x \in K$). With this categorical perspective, s *does* give a lax functor from $G[1]$ to *Grps*. We essentially replace the action $G \to Aut(K)$, when s is a splitting, by a lax action (see Blanco, Bullejos and Faro, [21]);

$$
\begin{array}{ccc}
\longrightarrow C_2(G) & \longrightarrow & C_1(G) \\
\downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle k_1} \\
K & \longrightarrow & E \\
\downarrow{\scriptstyle =} & & \downarrow \\
K & \longrightarrow & Aut(K).
\end{array}
$$

Using this lax action and **k**, we can reinterpret the classical reconstruction method of Schreier as forming the semidirect product $K \rtimes C_1(G)$, then dividing out by all pairs,

$$(k_2(\langle g_1, g_2 \rangle), \partial_2(\langle g_1, g_2 \rangle)^{-1}).$$

(We give Brown and Porter's article, [41], as a reference for a discussion of this construction.)

By itself this reinterpretation does not give us much. It just gives a slightly different viewpoint, however two points need making. This formulation is nearer the sort of approach that we will need to handle the classification of gerbes and the use of $K \to Aut(K)$ to handle the lax action of G reveals a problem and also a power in this formulation.

Dedecker, [62], noted that any theory of non-Abelian cohomology of groups must take account of the variation with K. Suppose we have two groups, K and L, and lax actions of G on them. What should it mean to say that some homomorphism $\alpha : K \to L$ is compatible with the lax actions?

A lax action of G on K can be given by a morphism of crossed modules / complexes, $Act_{G,K} : \mathsf{C}(G) \to \mathsf{Aut}(K)$, but $\mathsf{Aut}(K)$ is not functorial in K, so we do not automatically get a morphism of crossed modules, $\mathsf{Aut}(\alpha) : \mathsf{Aut}(K) \to \mathsf{Aut}(L)$. Perhaps the problem is slightly wrongly stated. One might say α is compatible with the lax G-actions if such a morphism of crossed modules existed and such that $Act_{G,L} = \mathsf{Aut}(\alpha)Act_{G,K}$. It is then just one final step to try to classify extensions with a finer notion of equivalence.

Definition: Suppose we have a crossed module, $\mathsf{Q} = (K, Q, q)$. An extension of K by G of the type of Q is a diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
& & {\scriptstyle =}\big\downarrow & & \big\downarrow {\scriptstyle \omega} & & & & \\
& & K & \xrightarrow{\ q\ } & Q & & & &
\end{array}
$$

where ω gives a morphism of crossed modules.

There is an obvious notion of equivalence of two such extensions, where the isomorphism on the middle terms must commute with the structural maps ω and ω'. The special case when $Q = Aut(K)$ gives one the standard notion. In general, one gets a set of equivalence classes of such extensions $Ext_{K \to Q}(G, K)$ and this can be related to the cohomology set $H^2(G, K \to Q)$. This can also be stated in terms of a category $\mathcal{E}xt_\mathsf{Q}(G)$ of extensions of type Q, then the cohomology set is the set of components of this category.

This latter object can be defined using any free crossed resolution of G as there is a notion of homotopy for morphisms of crossed complexes such that this set is $[\mathsf{C}(G), \mathsf{Q}]$. Any other free crossed resolution of G has the same homotopy as $\mathsf{C}(G)$ and so will do just as well. Finding a complete set of syzygies for a presentation of G will do.

Example:

$$G = (x, y \mid x^2 = y^3)$$

This is the trefoil group. It is a one relator presentation and has no identities, so $C(\mathcal{P})$ is already a crossed resolution. A morphism of crossed modules, **k** $: C(\mathcal{P}) \to \mathsf{Q}$, is specified by elements

$q_x, q_y \in Q$, and $a_r \in K$ such that $\mathbf{k}(a_r) = (q_x)^2(q_y)^{-3}$. Using this one can give a presentation of the E that results.

Remark: Extensions correspond to 'bitorsors' as we will see. These in higher dimensions then yields gerbes with action of a gr-stack and a corresponding cohomology. In the case of gerbes, as against extensions, a related notion was introduced by Debremaeker, [58–61]. This has recently been revisited by Milne, [121], and Aldrovandi, [3], who consider the special case where both K and Q are Abelian and the action of Q is trivial. This links with various important structures on gerbes and also with Abelian motives and hypercohomology. In all these cases, Q is being viewed as the coefficients of the cohomology and the gerbes / extensions have interpretations accordingly. Another very closely related approach is given in Breen, [23, 25]. We explore these ideas later in these notes.

We can think of the canonical case $K \to Aut(K)$ as being a 'natural' choice for extensions by K of a group, G. It is the structural crossed module of the 'fibre'. The crossed modules case says we can restrict or, alternatively, lift this structural crossed module to Q. This may, perhaps, be thought of as analogous to the situation that we will examine shortly where geometric structure corresponds to the restriction or the lifting of the natural structural group of a bundle. Both restricting to a subgroup and lifting to a covering group are useful and perhaps the same is true here.

4.2   Classifying spaces

The classifying spaces of crossed modules are never far from the surface in this approach to cohomology and related areas. They will play a very important role in the discussion of gerbes, as, for instance, in Larry Breen's work, [23–25] and later on here.

Classifying spaces of (discrete) groups are well known. One method of construction is to form the nerve, $Ner(G)$, of the group, G, (considered as a small groupoid, $\mathcal{G}$ or $G[1]$, as usual). The classifying space is obtained by taking the geometric realisation, $BG = |Ner(G)|$.

To explore this notion, and how it relates to crossed modules, we need to take a short excursion into some simplicially based notions.

A classifying space of a group classifies principal G-bundles (G-torsors) over a space, X, in terms of homotopy classes of maps from X to BG, using a universal principal G-bundle $EG \to BG$.

This is very topological! If possible, it is useful to avoid the use of geometric realisations, since (i) this restricts one to groups and groupoids and makes handling more general 'algebras' difficult and (ii) for algebraic geometry, the topology involved is not the right kind as a sheaf-theoretic, topos based construction would be more appropriate. Thus the classifying space is often replaced by the nerve, as in Breen, [25].

How about classifying spaces for crossed modules? Given a crossed module, $\mathsf{M} = (C, G, \theta)$, say, we can form the associated 2-group, $\mathcal{X}(\mathsf{M})$. This gives a simplicial group by taking the nerve of the groupoid structure, then we can form $\overline{W}$ of that to get a simplicial set, $Ner(\mathsf{M})$. To reassure ourselves that this *is* a good generalisation of $Ner(G)$, we observe that if C is the trivial group, then $Ner(\mathsf{M}) = Ner(G)$. But this raises the question:

What does this 'classifying space' classify?

To answer that we must digress to provide more details on the functors G and $\overline{W}$, we mentioned earlier.

4.2.1 Simplicially enriched groupoids

We denote the category of simplicial sets by $\mathcal{S}$ and that of simplicially enriched groupoids by $\mathcal{S} - Grpds$. This latter category includes that of simplicial groups, but it must be remembered that a simplicial object in the category of groupoids will, in general, have a non-trivial simplicial set as its 'object of objects', whilst in $\mathcal{S} - Grpds$, the corresponding simplicial object of objects will be constant. This corresponds to a groupoid in which each collection of 'arrows' between objects is a simplicial set, not just a set, and composition is a simplicial morphism, hence the term 'simplicially enriched'. We will often abbreviate the term 'simplicially enriched groupoid' to '$\mathcal{S}$-groupoid', but the reader should note that in some of the sources on this material the looser term 'simplicial groupoid' is used to describe these objects, usually with a note to the effect that this is not a completely accurate term to use.

Remark: Later, in section **??**, we will need to work with $\mathcal{S}$-categories, i.e., simplicially enriched categories. Some brief introduction can be found in [103], in the notes, [139] and the references cited there. We *will* give a fairly detailed discussion of the main parts of the elementary theory of $\mathcal{S}$-categories later.

The loop groupoid functor of Dwyer and Kan, [67], is a functor

$$G : \mathcal{S} \longrightarrow \mathcal{S} - Grpds,$$

which takes the simplicial set K to the simplicially enriched groupoid GK, where $(GK)_n$ is the free groupoid on the directed graph

$$K_{n+1} \underset{t}{\overset{s}{\rightrightarrows}} K_0 \ ,$$

where the two functions, s, source, and t, target, are $s = (d_1)^{n+1}$ and $t = d_0(d_2)^n$ with relations $s_0 x = id$ for $x \in K_n$. The face and degeneracy maps are given on generators by

$$
\begin{aligned}
s_i^{GK}(x) &= s_{i+1}^K(x), \\
d_i^{GK}(x) &= d_{i+1}^K(x), \text{for} \ \ x \in K_{n+1}, 1 < i \leq n
\end{aligned}
$$

and

$$
d_0^{GK}(x) = (d_0^K(x))^{-1}(d_1^K(x)).
$$

This loop groupoid functor has a right adjoint, $\overline{W}$, called the *classifying space* functor. The details as to its construction will be given shortly. It is important to note that if K is reduced, i.e. has just one vertex, then GK will be a simplicial group, so is a well known type of object. This helps when studying these gadgets as we can often use simplicial group constructions, suitable adapted, in the $\mathcal{S}$-groupoid context. The first we will see is the Moore complex.

Definition: Given any $\mathcal{S}$-groupoid, G, its Moore complex, NG, is given by

$$NG_n = \bigcap_{i=1}^{n} Ker(d_i : G_n \longrightarrow G_{n-1})$$

with differential $\partial : NG_n \longrightarrow NG_{n-1}$ being the restriction of d_0. If $n \geq 1$, this is just a disjoint union of groups, one for each object in the object set, O, of G. If we write $G\{x\}$ for the simplicial

group of elements that start and end at $x \in O$, then at object x, one has

$$NG\{x\}_n = (NG_n)\{x\}.$$

In dimension 0, one has $NG_0 = G_0$, so the $NG_n\{x\}$, for different objects x, are linked by the actions of the 0-simplices, acting by conjugation via repeated degeneracies.

The quotient $NG_0/\partial(NG_1)$ is a groupoid, which is the *fundamental groupoid* of the simplicially enriched groupoid, G. We can also view this quotient as being obtained from the $\mathcal{S}$-enriched category G by applying the 'connected components' functor π_0 to each simplicial hom-set $G(x, y)$. If $G = G(K)$, the loop groupoid of a simplicial set K, then this fundamental groupoid is exactly the fundamental groupoid, ΠK, of K and we can take this as defining that groupoid if we need to be more precise later. This means that ΠK is obtained by taking the free groupoid on the 1-skeleton of K and then dividing out by relations corresponding to the 2-simplices: if $\sigma \in K_2$, we have a relation

$$d_2(\sigma).d_0(\sigma) \equiv d_1(\sigma).$$

(**You are left** to explore this a bit more, justifying the claims we have made. You may also like to review the treatment in the book by Gabriel and Zisman, [83].)

For simplicity in the description below, we will often assume that the $\mathcal{S}$-groupoid is *reduced*, that is, its set O, of objects is just a singleton set $\{*\}$, so G is just a simplicial group.

Suppose that NG_m is trivial for $m > n$.

If $n = 0$, then NG_0 is just the group G_0 and the simplicial group (or groupoid) represents an Eilenberg-MacLane space, $K(G_0, 1)$.

If $n = 1$, then $\partial : NG_1 \longrightarrow NG_0$ has a natural crossed module structure.

Returning to the discussion of the Moore complex, if $n = 2$, then

$$NG_2 \xrightarrow{\partial} NG_1 \xrightarrow{\partial} NG_0$$

has a 2-crossed module structure in the sense of Conduché, [52] and above section 3.3. (These statements are for groups and hence for connected homotopy types. The non-connected case, handled by working with simplicially enriched groupoids, is an easy extension.)

In all cases, the simplicial group will have non-trivial homotopy groups only in the range covered by the non-trivial part of the Moore complex.

Now relaxing the restriction on G, for each $n > 1$, let D_n denote the subgroupoid of G_n generated by the degenerate elements. Instead of asking that NG_n be trivial, we can ask that $NG_n \cap D_n$ be. The importance of this is that the structural information on the homotopy type represented by G includes structure such as the Whitehead products and these all lie in the subgroupoids $NG_n \cap D_n$. If these are all trivial then the algebraic structure of the Moore complex is simpler, being that of a crossed complex, and $\overline{W}G$ is a simplicial set whose realisation is the *classifying space of that crossed complex*, cf. [36]. The simplicial set, $\overline{W}G$, is isomorphic to the *nerve* of the crossed complex.

Notational warning. As was mentioned before, the indexing of levels in constructions with crossed complexes may cause some confusion. The Dwyer-Kan construction is essentially a 'loop' construction, whilst $\overline{W}$ is a 'suspension'. They are like 'shift' operators for chain complexes. For example G decreases dimension, as an old 1-simplex x yields a generator in dimension 0, and so

on. Our usual notation for crossed complexes has C_0 as the set of objects, C_1 corresponding to a relative fundamental groupoid, and C_n abstracting its properties from $\pi_n(X_n, X_{n-1}, p)$, hence the natural topological indexing *has* been used. For the $\mathcal{S}$-groupoid $G(K)$, the set of objects is separated out and $G(K)_0$ is a groupoid on the 1-simplices of K, a dimension shift. Because of this, in the notation being used here, the crossed complex $\mathsf{C}(G)$ associated to an $\mathcal{S}$-groupoid, G, will have a dimension shift as well: explicitly

$$C(G)_n = \frac{NG_{n-1}}{(NG_{n-1} \cap D_{n-1})d_0(NG_n \cap D_n)} \quad \text{for } n \geq 2,$$

$C(G)_1 = NG_0$, and, of course, C_0 is the common set of objects of G. In some papers where only the algebraic constructions are being treated, this convention is not used and C is given without this dimension shift relative to the Moore complex. Because of this, care is sometimes needed when comparing formulae from different sources.

4.2.2   Conduché's decomposition and the Dold-Kan Theorem

The category of crossed complexes (of groupoids) is equivalent to a reflexive subcategory of the category $\mathcal{S} - Grpds$ and the reflection is defined by the obvious functor : take the Moore complex of the $\mathcal{S}$-groupoid and divide out by the $NG_n \cap D_n$, see [68, 69]. We will denote by $C : \mathcal{S} - Grpds \longrightarrow Crs$ the resulting composite functor, Moore complex followed by reflection. Of course, we have the formula, more or less as before, (cf. page 52),

$$C(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n) \, d_0(NG_{n+1} \cap D_{n+1})}.$$

The Moore complex functor itself is part of an adjoint (Dold-Kan) equivalence between the category $\mathcal{S} - Grpds$ and the category of hypercrossed complexes, [50], and this restricts to the Ashley-Conduché version of the Dold-Kan theorem of [11].

In order to justify the description of the nerve, and thus the related classifying space, of a crossed complex C, we will specify the functors involved, namely the Dold-Kan inverse construction and the $\overline{W}$. (We will leave **the reader** to chase up the detailed proof of this crossed complex form of the Dold-Kan theorem. The functors will be here, but the detailed proofs that they do give an equivalence will be left to you to give or find in the literature.)This will also give us extra tools for later use. We will first need the Conduché decomposition lemma, [52].

Proposition 33 *If G is a simplicial group(oid), then G_n decomposes as a multiple semidirect product:*

$$G_n \cong NG_n \rtimes s_0 NG_{n-1} \rtimes s_1 NG_{n-1} \rtimes s_1 s_0 NG_{n-2} \rtimes s_2 NG_{n-1} \rtimes \dots s_{n-1} s_{n-2} \dots s_0 NG_0$$

∎

The order of the terms corresponds to a lexicographic ordering of the indices $\emptyset$; 0; 1; 1,0; 2; 2,0; 2,1; 2,1,0; 3; 3,0; $\dots$ and so on, the term corresponding to $i_1 > \dots > i_p$ being $s_{i_1} \dots s_{i_p} NG_{n-p}$.

The proof of this result is based on a simple lemma, which is easy to prove.

Lemma 17 *If G is a simplicial group(oid), then G_n decomposes as a semidirect product:*

$$G_n \cong Ker \ d_n^n \rtimes s_{n-1}^{n-1}(G_{n-1}).$$

∎

We next note that in the classical (Abelian) Dold-Kan theorem, (cf. [56]), the equivalence of categories is constructed using the Moore complex and a functor K constructed via the original direct sum / Abelian version of Conduché's decomposition, cf. for instance, [56].

For each non-negatively graded chain complex, $\mathsf{D} = (D_n, \partial)$. in Ab, $K\mathsf{D}$ is the simplicial Abelian group with

$$(K\mathsf{D})_n = \oplus_a (D_{n-\sharp(a)}, s_a),$$

the sum being indexed by all descending sequences, $a = \{n > i_p \geq \dots \geq i_1 \geq 0\}$, where $s_a = s_{i_p} \dots s_{i_1}$, and where $\sharp(a) = p$, the summand D_n corresponding to the empty sequence.

The face and degeneracy operators in $K\mathsf{D}$ are given by the rules:

(1) if $d_i s_a = s_b$, then d_i will map (D_{n-p}, s_a) to $(D_{(n-1)-(p-1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero;

(2) if $d_i s_a = s_b d_0$, then d_i will map (D_{n-p}, s_a) to (D_{n-p-1}, s_b) as the homomorphism $\partial_{n-p} : D_{n-p} \to D_{n-p-1}$; its components into other direct summands will be zero;

(3) if $d_i s_a = s_b d_j$, $j > 0$, then $d_i(D_{n-p}, s_a) = 0$;

(4) if $s_i s_a = s_b$, then s_i maps (D_{n-p}, s_a) to $(D_{(n+1)-(p+1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero.

This suggests that we form a functor

$$K : Crs \to \mathcal{S} - Grpds$$

using a semidirect product, but we have to take care as there will be a dimension shift, our lowest dimension being C_1:

if C is in Crs, set

$$K(\mathsf{C})_n = C_{n+1} \rtimes s_0 C_n \rtimes s_1 C_n \rtimes s_1 s_0 C_{n-1} \rtimes \cdots \rtimes s_{n-1} s_{n-2} \dots s_0 C_1.$$

The order of terms is to be that of the proposition given above. The formation of the semidirect product is as in the proof we hinted at of that proposition, that is the bracketing is inductively given by

$$(C_{n+1} \dots \rtimes s_{n-2} \dots s_0 C_2) \rtimes (s_{n-1} C_n \rtimes \dots \rtimes s_{n-1} \dots s_0 C_1);$$

each $s_\alpha(C_{n+1-\sharp(\alpha)})$ is an indexed copy of $C_{n+1-\sharp(\alpha)}$; the action of

$$s_{n-1} C_{n-1} \rtimes \dots \rtimes s_{n-1} \dots s_0 C_0 \quad (\cong s_{n-1} K(\mathsf{C})_{n-1})$$

on $C_{n+1} \rtimes \dots s_{n-2} \dots s_0 C_1$, is given componentwise by the actions of each C_i and as C is a crossed complex, these are all via C_0. This implies, of course, that the majority of the components of these actions are trivial.

To see how this looks in low dimensions, it is simple to give the first few terms of the simplicial group(oid). As we are taking a reduced crossed complex as illustration, the result is a simplicial group, $K(\mathsf{C})$, having

- $K(\mathsf{C})_0 = C_1$

- $K(\mathsf{C})_1 = C_2 \rtimes s_0(C_1)$

- $K(\mathsf{C})_2 = (C_3 \rtimes s_0 C_2) \rtimes (s_1 C_2 \rtimes s_1 s_0 C_1)$

- $K(\mathsf{C})_3 = (C_4 \rtimes s_0 C_3 \rtimes s_1 C_3 \rtimes s_1 s_0 C_2) \rtimes (s_2 C_3 \rtimes s_2 s_0 C_2 \rtimes s_2 s_1 C_2 \rtimes s_2 s_1 s_0 C_1).$

and so on.

The face and degeneracy maps are determined by the obvious rules adapting those in the Abelian case, so that if $c \in C_k$, the corresponding copy of c in $s_\alpha C_k$ will be denoted $s_\alpha c$ and a face or degeneracy operator will usually act just on the index. The exception to this is if, when renormalised to the form $s_\beta d_\gamma$ using the simplicial identities, γ is non-empty. If $d_\gamma = d_0$ then $d_\gamma c$ becomes $\delta_k c \in C_{k-1}$, otherwise $d_\gamma c$ will be trivial.

Lemma 18 *The above defines a functor*

$$K : Crs \to \mathcal{S} - Grpds$$

such that $\mathsf{C} K \cong Id.$ ∎

This extends the functor $K : CMod \to Simp.Grps$, given earlier, to crossed complexes as there $C_k = 1$ for $k > 2$.

One obvious question, given our earlier discussion of group T complexes, and its fairly obvious adaptation to groupoid T-complexes, is if we start with a crossed complex C and construct this simplicially enriched groupoid $K(\mathsf{C})$, is this a groupoid T-complex? As the thin filler condition for groupoid T-complexes involves the Moore complex, it is enough to look at the single object simplicial group case. We have the following:

Proposition 34 *If* C *is a crossed complex, then* $K\mathsf{C}$ *is a group T-complex.*

Proof: We have to check that $NK(\mathsf{C})_n \cap D_n = 1$. We suppose $g \in NK(\mathsf{C})_n$ is a product of degenerate elements, then, using the semidirect decomposition, we can write g in the form

$$g = s_1(g_1) \dots s_{n-1}(g_{n-1}). \qquad (*)$$

The only problem in doing this is handling any element that comes from C_0, but this can be done via the action of C_0 on the C_i.

As $g \in Ker\, d_n$, we have

$$1 = d_n g = s_1 d_{n-1}(g_1) \dots s_{n-2} d_{n-1}(g_{n-2}).g_{n-1},$$

so we can replace g_{n-1} by a product of degenerate elements and use $s_{n-1} s_i = s_i s_{n-2}$ and rewriting to obtain a new expression for g in the form (*), but with no s_{n-1} term. Repeating using d_{n-1} on this new expression yields that the new g_{n-2} is also in D_{n-1} and so on until we obtain

$$g = s_0(g^{(1)})$$

where $g^{(1)} \in D_{n-1}$, writing $g^{(1)}$ in the form (*) gives

$$g = s_0 s_0(g_1^{(1)} \dots s_0 s_{n-2}(g_{n-2}^{(1)}),$$

but $d_1 d_n g = 1$, so $g_{n-2}^{(1)} \in D_{n-2}$. Repeating we eventually get $g = s_0 s_0(g^{(n)})$ with $g^{(2)} \in D_{n-2}$. This process continues until we get $g = s_0^{(n)}(g^{(n)})$ with $g^{(n)} \in K(\mathsf{C})_0$, but $d_1 \dots d_n g = g^{(n)}$ and $d_1 \dots d_n g = 1$, so $g = 1$ as required. ∎

Note that this proof, which is based on Ashley's proof that simplicial Abelian groups are group T-complexes (cf., [11]), depends in a strong way on being able to write g in the form $(*)$, i.e., on the triviality of almost all the actions together with the explicit nature of the action of C_0.

Collecting up the pieces we have all the main points in the proof of the following Dold-Kan theorem for crossed complexes.

Theorem 11 *There is an equivalence of categories*

$$Grpd.T-comp. \xleftrightarrow{\simeq} Crs.$$

∎

Checking that we do have all the parts necessary and providing any missing pieces is a good exercise, so will be **left to you**. A treatment more or less consistent with the conventions here can be found in [141].

4.2.3 $\overline{W}$ and the nerve of a crossed complex

We next need to make explicit the $\overline{W}$ construction. The simplicial / algebraic description of the nerve of a crossed complex, C is then as $\overline{W}(K(\mathsf{C}))$. We first give this description for a general simplicially enriched groupoid.

Let H be an $\mathcal{S}$-groupoid, then $\overline{W}H$ is the simplicial set described by

- $(\overline{W}H)_0 = ob(H_0)$, the set of objects of the groupoid of 0-simplices (and hence of the groupoid at each level);

- $(\overline{W}H)_1 = arr(H_0)$, the set of arrows of the groupoid H_0:

and for $n \geq 2$,

- $(\overline{W}H)_n = \{(h_{n-1}, \ldots, h_0) \mid h_i \in arr(H_i) \text{ and } s(h_{i-1}) = t(h_i), 0 < i < n\}$.

Here s and t are generic symbols for the domain and codomain mappings of all the groupoids involved. The face and degeneracy mappings between $\overline{W}(H)_1$ and $\overline{W}(H)_0$ are the source and target maps and the identity maps of H_0, respectively; whilst the face and degeneracy maps at higher levels are given as follows:

The face and degeneracy maps are given by

- $d_0(h_{n-1}, \ldots, h_0) = (h_{n-2}, \ldots, h_0)$;

- for $0 < i < n$, $d_i(h_{n-1}, \ldots, h_0) = (d_{i-1}h_{n-1}, d_{i-2}h_{n-2}, \ldots, d_0 h_{n-i} h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$;

and

- $d_n(h_{n-1}, \ldots, h_0) = (d_{n-1}h_{n-1}, d_{n-2}h_{n-2}, \ldots, d_1 h_1)$;

whilst

- $s_0(h_{n-1}, \ldots, h_0) = (id_{dom(h_{n-1})}, h_{n-1}, \ldots, h_0)$;

and,

- for $0 < i \leq n$, $s_i(h_{n-1}, \ldots, h_0) = (s_{i-1}h_{n-1}, \ldots, s_0 h_{n-i}, id_{cod(h_{n-i})}, h_{n-i-1}, \ldots, h_0)$.

Remark: We note that if H is a constant simplicial groupoid, $\overline{W}(H)$ is the same as the nerve of that groupoid for the algebraic composition order. Later on, when re-examining the classifying space construction, we may need to rework the above definition in a form using the functional composition order.

To help understand the structure of the nerve of a (reduced) crossed complex, C, we will calculate $Ner(\mathsf{C}) = \overline{W}(K(\mathsf{C}))$ in low dimensions. This will enable comparison with formulae given earlier. The calculations are just the result of careful application of the formulae for $\overline{W}$ to $H = K(\mathsf{C})$:

- $Ner(\mathsf{C})_0 = *$, as we are considering a *reduced* crossed complex - in the general case, this is C_0;

- $Ner(\mathsf{C})_1 = C_1$, as a *set* of 'directed edges' or arrows - we will avoid using a special notation for 'underlying set of a group(oid)';

- $Ner(\mathsf{C})_2 = \{(h_0, h_1) \mid h_1 = (c_2, s_0(c_1)), h_0 = c_1', \text{ with } c_2 \in C_2, c_1, c_1' \in C_1\}$, and such a 2-simplex has faces given as in the diagram

$$c_1 \qquad c_1'$$
$$(h_1, h_0)$$
$$\delta c_2.c_1.c_1'$$
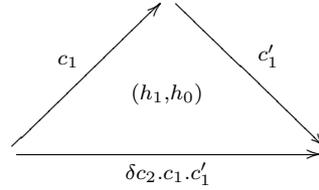
Note that $h_1 : c_1 \longrightarrow \delta c_2.c_1$ in the internal category corresponding to the crossed module, (C_2, C_1, δ), so the formation of this 2-simplex corresponds to a right whiskering of that 2-cell (in the corresponding 2-groupoid) by the arrow c_1';

- $Ner(\mathsf{C})_3 = \{(h_2, h_1, h_0) \mid h_1 = (c_3, s_0 c_2^0, s_1 c_2^1, s_1 s_0 c_1), h_1 = (c_2', s_0(c_1')), h_0 = c_1''\}$ in the evident notation. Here the faces of the 3-simplex (h_2, h_1, h_0) are as in the diagrams, (in each of which the label for the 2-simplex itself has been abbreviated):

d_3 :

$$c_1 \qquad c_1'$$
$$c_2^1$$
$$\delta c_2^1.c_1.c_1'$$

; d_2 :

$$c_1 \qquad \delta c_2'.c_1'.c_1''$$
$$c_2^0 c_2^1$$
$$\delta(c_2^0 c_2^1).c_1.\delta c_2'.c_1'.c_1''$$

d_1 :

$$\delta c_2^1.c_1.c_1' \qquad c_1''$$
$$\delta c_3.c_2^0.{}^{\delta c_2^1.c_1} c_2'$$
$$\delta(c_2^0 c_2^1).c_1.\delta c_2'.c_1'.c_1''$$

; d_0 :
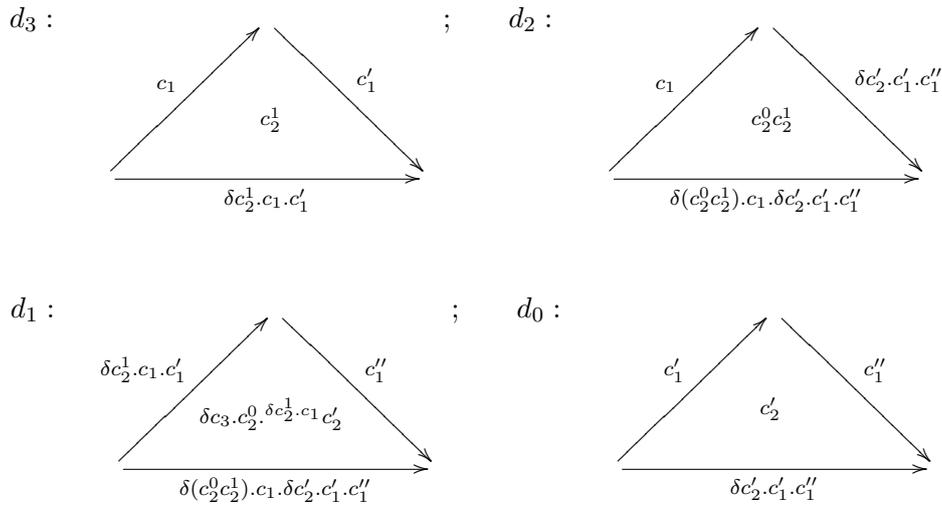
$$c_1' \qquad c_1''$$
$$c_2'$$
$$\delta c_2'.c_1'.c_1''$$

The only face where any real thought has to be used is d_1. In this the d_1 face has to be checked to be consistent with the others. The calculation goes like this:

$$\delta(\delta c_3.c_2^{0.\,\delta c_2^1.c_1}c_2').(\delta c_2^1.c_1.c_1').c_1'' \;=\; \delta c_2^0.(\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}).\delta c_2^1.c_1.c_1'.c_1''$$
$$=\; \delta(c_2^0 c_2^1).c_1.\delta c_2'.c_1'.c_1''$$

This uses (i) $\delta\delta c_3$ is trivial, being a boundary of a boundary, and (ii) the second crossed module rule for expanding $\delta(^{\delta c_2^1.c_1}c_2')$ as $\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}$.

This diagrammatic representation, although useful, is limited. A recursive approach can be used as well as the simplicial / algebraic one given above. In this, $Ner(\mathsf{C})$ is built up via its skeletons, specifying a simplex in $Ner(\mathsf{C})_n$ as an element of C_n, together with the empty simplex that it 'fills', i.e. the set of compatible $(n-1)$-simplices. This description is used by Ashley, ([11], p.37). More on nerves of crossed complexes can be found in Nan Tie, [131, 132]. There is also a very neat 'singular complex' description, $Ner(\mathsf{C})_n = Crs(\pi(n), \mathsf{C})$, where $\pi(n)$ is the free crossed complex on the n-simplex, $\Delta[n]$. We will have occasion to see this in more detail later.

This singular complex description shows another important feature. If we have an n-simplex $f : \pi(n) \to \mathsf{C}$, we will say it is *thin* if the image $f(\iota_n)$ of the top dimensional generator in $\pi(n)$ is trivial. The nerve together with the filtered set of thin elements forms a T-complex in the sense of section **??**. This is discussed in Ashley, [11], and Brown-Higgins, [36].

4.3   Simplicial Automorphisms and Regular Representations

The usual enrichment of the category of simplicial sets is given by :
for each $n \geq 0$, the set of n-simplices is

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L),$$

together with obvious face and degeneracy maps.
Composition : for $f \in \underline{\mathcal{S}}(K, L)_n$, $g \in \underline{\mathcal{S}}(L, M)_n$, so $f : \Delta[n] \times K \to L$, $g : \Delta[n] \times L \to M$,

$$g \circ f := (\Delta[n] \times K \overset{diag \times K}{\longrightarrow} \Delta[n] \times \Delta[n] \times K \overset{\Delta[n] \times f}{\longrightarrow} \Delta[n] \times L \overset{g}{\to} M);$$

Identity : $id_K : \Delta[0] \times K \overset{\cong}{\to} K$.

Definition: The simplicial set, $\underline{\mathcal{S}}(K, L)$, defined above, is called the *simplicial mapping space of maps from K to L.*

This **clearly** is functorial in both K and L. (Of course, with differing 'variance'. It is 'contra-variant' in K, so that $\underline{\mathcal{S}}(-, L)$ is a functor from $\mathcal{S}^{op}$ to $\mathcal{S}$, but $\underline{\mathcal{S}}(K, -) : \mathcal{S} \to \mathcal{S}$. In the category, $\mathcal{S}$, each of the functors 'product with K' for K a simplicial set, has a right adjoint, namely this $\underline{\mathcal{S}}(K, -)$. Technically $\mathcal{S}$ is a *Cartesian closed category* a notion we will explore briefly in the next section. In any such setting we can restrict to looking at endomorphisms of an object, and, here we can go further and get a simplicial group of automorphisms of a simplicial set, K, analogously to our construction of the automorphism 2-group of a group (recall from section 1.3.4).

Explicitly, for fixed K, $\underline{\mathcal{S}}(K, K)$ is a simplicial monoid, called the *simplicial endomorphism monoid of K* and $\mathsf{aut}(K)$ will be the corresponding simplicial group of invertible elements, that is the *simplicial automorphism group of K*.

If $f : K \times \Delta[n] \longrightarrow L$ is an n-simplex, then we can form a diagram

$$
\begin{array}{ccc}
K \times \Delta[n] & \xrightarrow{\ (f,p)\ } & L \times \Delta[n] \\
& \searrow \quad \swarrow & \\
& \Delta[n] &
\end{array}
$$

in which the two slanting arrow are the obvious projections, (so $(f,p)(k,\sigma) = (f(k,\sigma),\sigma)$). Taking $K = L$, $f \in \mathsf{aut}(K)$ if and only if (f,p) is an isomorphism of simplicial sets.

Given a simplicial set K, and an n-simplex, x, in K, there is a representing map,

$$\mathbf{x} : \Delta[n] \longrightarrow K,$$

that sends the top dimensional generating simplex of $\Delta[n]$ to x.

As was just said, the mapping space construction, above, is part of an adjunction,

$$\mathcal{S}(K \times L, M) \cong \mathcal{S}(L, \underline{\mathcal{S}}(K, M)),$$

in which, given $\theta : K \times L \longrightarrow M$ and $y \in L_n$, the corresponding simplicial map

$$\bar{\theta} : L \longrightarrow \underline{\mathcal{S}}(K, M)$$

sends y to the composite

$$K \times \Delta[n] \xrightarrow{K \times \mathbf{y}} K \times L \xrightarrow{\ \theta\ } M \ .$$

In a simplicial group G, the multiplication is a simplicial map, $\#_0 : G \times G \longrightarrow G$, and so, by the adjunction, we get a simplicial map

$$G \longrightarrow \underline{\mathcal{S}}(G, G)$$

and this is a simplicial monoid morphism. This gives the *right regular representation* of G,

$$\rho = \rho_G : G \longrightarrow \mathsf{aut}(G).$$

We will look at this idea of *representations* in more detail later.

This morphism, ρ, needs careful interpretation. In dimension n, an element $g \in G_n$ acts by multiplication on the right on G, but even in dimension 0, this action is not as simple as one might think. (NB. Here $\mathsf{aut}(G)$ is the simplicial group of 'simplicial automorphisms of the underlying simplicial set of G' as, of course, multiplication by an element does not give a mapping that respects the group structure.) Simple examples are called for:

In general, 0-simplices give simplicial maps corresponding to multiplication by that element, so that for $g \in G_0$, and $x \in G_n$,

$$\rho(g)(x) = x \#_0 s_0^{(n)}(g).$$

Suppose, now, $g \in G_1$, then $\rho(g) \in \mathsf{aut}(G)_1 \subset \underline{\mathcal{S}}(G, G)_1 = \mathcal{S}(G \times \Delta[1], G)$. In other words, $\rho(g)$ is a homotopy between $\rho(d_1 g)$ and $\rho(d_0 g)$. Of course, it is an invertible element of $\underline{\mathcal{S}}(G, G)_1$ and this will have implications for its properties as a homotopy, and, to use a geometric term, we will loosely refer to it as an *isotopy*.

In dimension 1, we, thus, have that elements give isotopies, and in higher dimensions, we have 'isotopies of isotopies', and so on.

Of course, the existence of these automorphism simplicial groups, $\mathsf{aut}(K)$, leads to a notion of a *(permutation) representation for a simplicial group*, G, as being a simplicial group morphism from G to $aut(K)$ for some simplicial set K. Likewise, if we have a simplical vector space, V, then we can construct a group of its automorphisms and thus consider linear representations as well. We will return to this later so give no details here.

4.4   Simplicial actions and principal fibrations

We saw, back in the first chapter, (page **??**), the idea of a group, G, acting on a set, X. This is clearly linked to what was discussed in the previous section. A group action was given by a map,

$$a : G \times X \to X,$$

(and we may write $g.x$, or simply gx, for the image $a(g,x)$), satisfying obvious conditions such as an 'associativity' rule $(g_2.g_1).x = g_2.(g_1.x)$ and an 'identity' rule $1_G.x = x$, both for all possible gs and xs. Of course, this 'action by g' gives a permutation of X, that is, a bijection form X to itself.

4.4.1   More on 'actions' and Cartesian closed categories

We know that the behaviour we have just been using for simplicial sets is also 'there' in the much simpler case of *Sets*, i.e., given sets X, Y and Z, there is a natural isomorphism

$$Sets(X \times Y, Z) \cong Sets(X, Sets(Y, Z)),$$

given by sending a 'function of two variables', $f : X \times Y \to Z$, to $\tilde{f} : X \to Sets(Y, Z)$, where $\tilde{f}(x) : Y \to Z$ sends y to $f(x, y)$. (We often write Z^Y for $Sets(Y, Z)$, since, for instance, if $Y = \{1, 2\}$, a two element set, $Sets(Y, Z) \cong Z \times Z = Z^2$, in the usual sense.) Technically, this is saying that $- \times Y$ has an adjoint given by $Sets(Y, -)$.

Definition: A category, $\mathcal{C}$, is *Cartesian closed* or a *ccc*, if it has all finite products and for any two objects, Y and Z, there is an *exponential*, Z^Y, in $\mathcal{C}$, so that $(-)^Y$ is right adjoint to $- \times Y$.

Recall or note: To say that $\mathcal{C}$ has all products says that, for any two objects X and Y in $\mathcal{C}$, their product $X \times Y$ is also there, and that there is a *terminal object*, and conversely. If you have not really met 'terminal objects' explicitly before an object T is *terminal* if, for any X in $\mathcal{C}$, there is a *unique* morphism from X to T. The simplest examples to think about are (i) any one element (singleton) set is terminal in *Sets*, (ii) the trivial group is terminal in *Groups*, and so on. The dual notion is *initial object*. An object, I, is *initial* if there is a unique morphism from I to X, again for all X in $\mathcal{C}$. The empty set is initial in *Sets*; the trivial group is initial in *Groups*.

If you have not formally met these, now is a good time to check up in texts that give an introduction to category theory and categorical ideas. In particular, it is worth thinking about why the terminal object in a category, if it exists, is the 'empty product', i.e., the product of an empty family of objects. This can initially seem strange, but is a *very useful insight* that will come in later, when we discuss sheaves.

We can use this property of *Sets*, and $\mathcal{S}$, or more generally for any ccc, to give a second description of a group action. The function $a : G \times X \to X$ gives, by the adjunction, a function

$$\tilde{a} : G \to Sets(G, G).$$

This set, $Sets(G, G)$, is a monoid under composition, and we can pick out $Perm(X)$ or if you prefer the notations, $Symm(X)$ or $Aut(X)$, the subgroup of self bijections or permutations of G. In this guise, an action of G on X is a group homomorphism from G to $Perm(X)$. (You might like to **consider how this selection of the invertibles** in the 'internal' monoid, $\mathcal{C}(X, X)$, could be done in a general ccc.)

As we mentioned, the category, $\mathcal{S}$, is also Cartesian closed, and we can use the above observation, together with our identification of the simplicial group of automorphisms, $\mathsf{aut}(Y)$, of a simplicial set Y from our earlier discussion, to describe the action of a simplicial group, G, on a simplicial set, Y. A *simplicial action* would thus be, equivalently, a simplicial map,

$$a : G \times Y \to Y,$$

satisfying associativity and identity rules, or a morphism of simplicial groups,

$$\tilde{a} : G \to \mathsf{aut}(Y).$$

We thus have the well known equivalence of 'actions' and 'representations'. This will be another recurring theme throughout these notes with embellishments, variations, etc. in different contexts. it is sometimes the 'aut'-object version that is easiest to give, sometimes not, and for some contexts, although $\mathcal{C}(X, X)$ will always be a monoid internal to some base category, the automorphisms may be hard to 'carve' out of it. (The structure may only be 'monoidal' not 'Cartesian' closed, for instance.) For this reason it pays to have both approaches.

We can identify various properties of group actions for a special mention. Here G may be a group or a simplicial group (or often more generally, but we do not need that yet) and X will be a set respectively a simplicial set, etc. (We choose a slightly different form of condition, than we will be using later on. The links between them can be **left to you**.)

Definition: (i) A left group action

$$a : G \times X \to X,$$

is said to be *effective* (or *faithful*) if $gx = x$ for all $x \in X$ implies that $g = 1_G$.

(ii) The G-action is said to be *free* (or sometimes, *principal*, cf. May, [119]) if $gx = x$ for some $x \in X$ implies $g = 1_G$.

(iii) If $x \in X$, the *orbit* of x is the set $\{g.x \mid g \in G\}$.

Clearly (i) can be, more or less equivalently, stated as, if $g \neq 1_G$, then there is an $x \in X$ such that $gx \neq x$. This is a form sometimes given in the literature. Whether or not you consider it equivalent depends on your logic. The use of negation means that in some context this formulation of the condition is less easy to use than the former.

For future use, it will be convenient to also have slightly different, but equivalent, ways of viewing these simplicial actions. For these we need to go back again to the simplicial mapping space, $\underline{\mathcal{S}}(K, L)$ and the composition, (see page 132). Suppose we have, as there, three simplicial sets, K, L and M, and the composition:

$$\underline{\mathcal{S}}(K, L) \times \underline{\mathcal{S}}(L, M) \to \underline{\mathcal{S}}(K, M).$$

(The product is symmetric so this is equivalent to

$$\underline{\mathcal{S}}(L, M) \times \underline{\mathcal{S}}(K, L) \to \underline{\mathcal{S}}(K, M).$$

The former is the viewpoint of the 'algebraic' concatenation composition order, the latter is the 'analytic' and 'topological' one. Of course, which you choose is up to you. We will tend to use the second, but sometimes .... .)

We want to look at the situation where $K = \Delta[0]$. As $\Delta[0]$ is the terminal object in $\mathcal{S}$, $\Delta[0] \times \Delta[n] \cong \Delta[n]$, so $\underline{\mathcal{S}}(\Delta[0], L) \cong L$. If we substitute from this back into the previous composition, we get

$$eval : L \times \underline{\mathcal{S}}(L, M) \to M.$$

(It is equally valid, to write the product around the other way, giving

$$eval : \underline{\mathcal{S}}(L, M) \times L \to M,$$

which correspond better to the 'analytic' Leibniz composition order. We will often use this form as well.) In either notational form, this is the simplicially enriched evaluation map, the analogue of $eval(x, f) = f(x)$ in the set theoretic case. (We will usually write *eval* for this sort of map.) Of course, if $L = M$, this situation is exactly that of the simplicial action of the simplicial monoid of self maps of L on L itself.

We can take the simplicial version apart quite easily, to see what makes it work.

Going back one stage, if $g \in \underline{\mathcal{S}}(K, L)_n$ and $f \in \underline{\mathcal{S}}(L, M)_n$, we can form their composite using the trick we saw earlier, in the discussion in section 4.3, page 132. We can replace $g : K \times \Delta[n] \to L$, by a map over $\Delta[n]$, given by $\overline{g} = (g, p_2) : K \times \Delta[n] \to L \times \Delta[n]$, and then compose with $f : L \times \Delta[n] \to M$ to get the composite $f \circ g \in \underline{\mathcal{S}}(K, M)_n$, or use the 'over $\Delta[n]$ version to get $\overline{f \circ g} = \overline{f}\overline{g} : K \times \Delta[n] \to M \times \Delta[n]$. We note

$$\overline{f \circ g}(k, \sigma) = (f(g(k, \sigma), \sigma), \sigma),$$

(yes, we do need all those σs!).

Next we try the formulae with $K = \Delta[0]$ and '$g = \ulcorner x \urcorner$', the 'naming' map for an n-simplex, x, in L. That is not quite right, and to make things 'crystal clear', we had better be precise. The naming map for x has domain $\Delta[n]$ and we need the corresponding map, g, defined on $\Delta[0] \times \Delta[n]$. (Here the notation is getting almost 'silly', but to track things through it is probably necessary to do this, at least once! It shows how the details are there and can be taken out from the abstract packaging if and when we need them.) This map g is defined by $g(s_0^{(m)}\iota_0, \sigma) = \ulcorner x \urcorner(\sigma)$, and this is 'really' given by $g(s_0^{(n)}(\iota_0), \iota_n)$ as that special case determines the others by the simplicial identities, so that, for $\sigma \in \Delta[n]_m$, so $\sigma : [m] \to [n]$, $g(s_0^{(m)}\iota_0, \sigma) = L_\sigma g(s_0^{(n)}(\iota_0), \iota_n)$. (It may help here to think of σ as one of the usual face inclusions or degeneracies, at least to start with.) We have not yet

used what g is, but $g(s_0^{(n)}(\iota_0), \iota_n) = x$, that is all! We can now work out (with all the identifications taken into account),

$$eval(x, f) = \overline{f \circ g}(s_0^{(n)}\iota_0, \iota_n) = f(x, \iota_n).$$

We might have guessed that this was the formula, ... what else could it be? This derivation, however, obtains it consistently with the natural 'action' formula, without having to check any complicated simplicial identities.

We will use this formula in the next chapter when discussing the structure of fibre bundles in the simplicial context.

4.4.2 G-principal fibrations

Specialising down to the simplicial case for now, suppose that G is a simplicial group acting on a simplicial set, E, then we can form a quotient complex, B, by identifying x with $g.x$, $x \in E_q$, $g \in G_q$. In other words the q-simplices of B are the orbits of the q-simplices of E, under the action of G_q. We note that this works (for **you to check**).

Lemma 19 *(i) The graded set, $\{B_q\}_{q \geq 0}$ forms a simplicial set with induced face and degeneracy maps, so that, if $[x]_G$ denotes the orbit of x under the action of G_q, then $d_i^B[x]_G = [d_i^E x]_G$, and similarly $s_i^B[x]_G = [s_i^E x]_G$.*
(ii) The graded function, $p : E \to B$, $p(x) = [x]_G$, is a simplicial map. ∎

Definition: A map of the form $p : E \to B$, as above, is called a *principal fibration*, or, more exactly, *G-principal fibration* if we need to emphasise the simplicial group being used.

A morphism between two such objects will be a simplicial map over B, which is G-equivariant for the given G-actions.

(Any such morphism will be an isomorphism; **for you to check**.)

We will denote the set of isomorphism classes of G-principal fibrations on B by $Princ_G(B)$.

This definition really only makes sense if such a p is a fibration. Luckily we have:

Proposition 35 *Any map $p : E \to B$, as above, is a Kan fibration.*

Proof: Suppose $p : E \to B$ is a principal fibration. We assume that we have (cf. page **??**) a commutative diagram

$$
\begin{array}{ccc}
\Lambda^i[n] & \xrightarrow{f_1} & E \\
{\scriptstyle inc}\downarrow & & \downarrow{\scriptstyle p} \\
\Delta[n] & \xrightarrow{f_0} & B
\end{array}
$$

and will write $b = f_0(\iota_n)$ for the corresponding n-simplex in B, and $(x_0, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n)$ a compatible set of $(n-1)$-simplices up in E, in other words, a (n, i)-horn in E and a filler, b, for its image down in B.

Pick a $x \in E_n$ such that $p(x) = b$, then as $d_j p(x) = p(x_j)$, we have there are unique elements $g_j \in G_{n-1}$ such that $d_j x = g_j x_j$. ('Uniqueness' comes from the assumed properties of the action.)

It is easy to **check** (again using 'uniqueness') that the g_js give a (n,i)-horn in G, which, since G is a 'Kan complex', has a filler (use the algorithm in section **??**). Let g be the filler and set $y = g^{-1}x$. It is now **easy to check** that $d_k y = x_k$ for all $k \neq i$, i.e., that y is a suitable filler.   ∎

We need to investigate the class of these principal fibrations (for some fixed G). (We will tend to omit specific mention of the simplicial group G being used if, within a context, it is 'fixed', so, for instance, if we are not concerned with a 'change of groups' context.)

Let us suppose that $p : E \to B$ is a principal fibration and that $f : X \to B$ is any simplicial map. We can form a pullback fibration

$$
\begin{array}{ccc}
E_f & \xrightarrow{\ f'\ } & E \\
{\scriptstyle f^*(p)}\downarrow & & \downarrow{\scriptstyle p} \\
X & \xrightarrow[\ f\ ]{} & B.
\end{array}
$$

Is this pullback a G-principal fibration? Or to use terminology that we introduced earlier (section **??**), is the class of principal fibrations pullbacks stable?

There are several proofs of the result that it is, some of which are very neat, but here we will use the trusted method of 'brute force and ignorance', using as little extra machinery as possible. We have a reasonable model for E_f, so we should expect to be able to give it an explicit G-action in a fairly obvious natural way. We then can see what the orbits look like. That sounds a simple plan and it in fact works nicely.

We will model E_f as $E \times_B X$. (Previously, we had it around the other way as $X \times_B E$, but the two are isomorphic and this way is marginally easier notationally.) Recall the n-simplices in $E \times_B X$ are pairs (e,x) with $e \in E_n$, $x \in X_n$ and $p(e) = f(x)$. The G-action is staring at us. It surely must be

$$g \cdot (e,x) = (g \cdot e, x),$$

but does this work? We note $p(e) = [e]_G$, the G-orbit of e, so $p(g \cdot e) = p(e) = f(x)$, so we end up in the correct object. (**You are left** to check that this *is* a G-action and that it is free and effective.) What are the orbits?

We have (e,x) and $e',y)$ will be in the same orbit provided that there is a g such that $(g \cdot e, x) = (e', y)$, but that means that $x = y$ and that e and e' are in the same G-orbit within E. This has various consequences, which you are **left to explore**, but it is clear that, up to isomorphism, the map $f^*(p)$, which is projection onto the x component, *is* the quotient by the action. We have verified (except for the bits **left to you**:

Proposition 36 *If $p : E \to B$ is a G-principal fibration, and $f : X \to B$ is a simplicial map, then $(E_f, X, f^*(p))$ is a G-principal fibration.*   ∎

Of particular interest is the case when $X = \Delta[n]$, so that f is a 'naming' map, (cf. page **??**), $\ulcorner b \urcorner$, for some n-simplex, $b \in B_n$. We can, in this case, E_f as being the 'fibre' over b, although b is in dimension n.

This is very useful because of the following:

Lemma 20 *If $p : E \to \Delta[n]$ is a G-principal fibration, then $E \cong \Delta[n] \times G$, with p corresponding to the first projection.*

Before launching into the proof, it should be pointed out that here $\Delta[n] \times G$, should really be written $\Delta[n] \times U(G)$, where $U(G)$ is the underlying simplicial set of G. Of course there is a natural free and effective G-action on $U(G)$, with exactly one orbit. We have suppressed the U as this is a common 'abuse' of notation.

Proof: We have a single non-degenerate n-simplex in $\Delta[n]$, namely ι_n, which corresponds to the identity map in $\Delta[n]_n = \mathbf{\Delta}([n], [n])$. We pick any $e_n \in p^{-1}(\iota_n)$ and get a map, $\ulcorner e_n \urcorner : \Delta[n] \to E$, naming e_n. Of course, the composite, $p \circ \ulcorner e_n \urcorner$, is the identity on $\Delta[n]$. (This means that the fibration is 'split', in a sense we will see several times later on.)

Suppose $e \in E_m$, then $p(e) = \mu \in \Delta[n]_m = \mathbf{\Delta}([m], [n])$. We have another possibly different element in $p^{-1}(\mu)$, since $\mu : [m] \to [n]$ induces $E(\mu) : E_n \to E_m$, and so we have an element $E(\mu)(e_n)$. (**You can easily check** that, as p is a simplicial map, $p(E(\mu)(e_n)) = \mu$, i.e. $E(\mu)(e_n) \in p^{-1}(\mu)$, but therefore there is a unique element $g_m \in G_m$ such that $g_m \cdot E(\mu)(e_n) = e$. Starting with e, we got a unique pair $(\mu, g_m) \in (\Delta[n] \times G)_m$ and, from that pair, we can retrieve e by the formula. (You are **left to check** that this yields a simplicial isomorphism over $\Delta[n]$.) ∎

We will see this sort of argument several times later. We have a 'global section,' here $\ulcorner e_n \urcorner$, of some G-principal 'thing' (fibration, bundle, torsor, whatever) and the conclusion is that the 'thing' is trivial' that is, a product thing.

4.4.3 Homotopy and induced fibrations

A key result that we will see later is that, if you use homotopic maps to pullback something like a fibration, or its more structured version, a fibre bundle, then you get 'related' pullbacks. Here we will look at the simplest, least structured, case, where we are forming pullbacks of *fibrations*. As this is a very important result, we will include quite a lot of detail.

As $\Delta[1]_0 = \Delta([0], [1])$, it has two elements, which we will write as e_0 and e_1, where $e_i(0) = i$, for $i = 0, 1$. (We will use this simplified notation several times later in the notes and should point out that e_0 corresponds to δ_1, and so induces d_1 if passing to simplicial notation, whilst e_1 is δ_0, corresponding to d_1, which is the 'face opposite 1', hence is 0. This is slightly confusing, but the added intuition of $K \times \Delta[1]$ being a cylinder with $K \times \ulcorner e_0 \urcorner : K \cong K \times \Delta[0] \to K \times \Delta[1]$ being inclusion at the bottom end is too good to pass by!)

In what follows, we will quietly write e_i instead of $\ulcorner e_i \urcorner$, as it is a lot more convenient.

Proposition 37 *Let $p : E \to B$ be a Kan fibration and let $f, g : A \to B$ be homotopic simplicial maps, with $F : f \simeq g$, a specific homotopy, then there is a homotopy equivalence over A between $f^*(p) : E_f \to A$ and $g^*(p) : E_g \to A$.*

Proof: We first write $f = F \circ (A \times e_0)$, then we form E_f in two stages, by forming two pullbacks:

$$
\begin{array}{ccccc}
E_f & \xrightarrow{\ i_f\ } & E_F & \longrightarrow & E \\
{\scriptstyle f^*(p)}\downarrow & & \downarrow{\scriptstyle F^*(p)} & & \downarrow{\scriptstyle p} \\
A & \xrightarrow[A \times e_0]{} & A \times \Delta[1] & \xrightarrow[F]{} & B
\end{array}
$$

A similar construction works, of course, for E_g using $A \times e_1$.

We have, from Lemma **??**, that, as $F^*(p)$ is a Kan fibration, so is $q_f := \underline{\mathcal{S}}(E_f, F^*(p))$, and so also is $q_g := \underline{\mathcal{S}}(E_g, F^*(p))$. These maps just compose with $F^*(p)$, so

$$q_f(i_f) = f^*(p) \times e_0.$$

Next we note that $f^*(p) \times \Delta[1] : E_f \times \Delta[1] \to A \times \Delta[1]$, so is in $\underline{\mathcal{S}}(E_f, A \times \Delta[1])_1$ and $f^*(p) \times e_0 = d_1(f^*(p) \times \Delta[1])$. We now have a $(1,1)0$-horn, $(-, i_f)$ in $\underline{\mathcal{S}}(E_f, E_F)$, whose image $(-, q - f(i_f))$ in $\underline{\mathcal{S}}(E_f, A \times \Delta[1])$ has a filler, namely $f^*(p) \times \Delta[1]$. We can thus lift that filler to one y_f, say, in $\underline{\mathcal{S}}(E_f, E_F)_1$, with $d_1(y_f) = i_f$, and, of course, $q_f(y_f) = f^*(p) \times \Delta[1]$. What is the other end, $d_0(y_f)$?

This is also in $\underline{\mathcal{S}}(E_f, E_F)_0$, so is a simplicial map from E_f to E_F. This suggests it might be a map of fibrations. Does

$$
\begin{array}{ccc}
E_f & \xrightarrow{\ d_0(y_f)\ } & E_F^{F^*(p)} \\[2pt]
{\scriptstyle f^*(p)}\big\downarrow & & \\[6pt]
A & \xrightarrow[\ A\times e_1\ ]{} & A \times \Delta[1]
\end{array}
$$

commute? We calculate,

$$
\begin{aligned}
F^*(p)d_0(y_f) &= q_F(d_0(y_f)) \\
&= d_0(q_f(y_f)) \\
&= d_0(f^*(p) \times \Delta[1]) \\
&= (A \times e_1) \circ f^*(p),
\end{aligned}
$$

so it is, but this means that, as bottom 'right-hand corner' of the square, had E_g as its pullback, we get a map, $\alpha : E_f \to E_g$, over A, so that $f^*(p) = g^*(p)\alpha$, and $d_0(y_f) = i_g\alpha$. This gives us the first part of our homotopy equivalence.

Reversing the roles of f and g, we get a y_g in $\underline{\mathcal{S}}(E_g, E_F)_1$ with $d_0(y_g) = i_g$, then $q_g(y_g) = g^*(p) \times \Delta[1]$, and we get a $\beta : E_g \to E_f$ such that $f^*(p)\beta = g^*(p)$ and $i_f\beta = d_1(y_g)$.

We now have to look at the composites $\alpha\beta$ and $\beta\alpha$, and to show they are homotopic (over A) to the identities. Of course, we need only produce one of these as the other will follow 'similarly', on reversing the roles of f and g.

Considering $s_0(\alpha) \in \underline{\mathcal{S}}(E_f, E_g)_1$ and $y_g \in \underline{\mathcal{S}}(E_g, E_F)_1$, we have a composite (really a composite homotopy), that we will denote by $\xi \in \underline{\mathcal{S}}(E_f, E_F)_1$. We can check (**for you to do**) that $d_0(\xi) = d_0(y_f)$ and $d_1(\xi) = d_i(y_g)\alpha = i_f\beta\alpha$. We thus have a horn

$$
\begin{array}{ccc}
& i_f\beta\alpha & \\
& & \\
h = & \quad\searrow{\scriptstyle \xi} & \\
i_f & \xrightarrow[\ \ y_f\ \ ]{} & d_0(y_f)
\end{array}
$$

in $\underline{\mathcal{S}}(E_f, E_F)$. We look at its image in $\underline{\mathcal{S}}(E_f, A \times \Delta[1])$, and **check** it cna be filled by $s_0(f^*(p) \times \Delta[1])$, that means that, as $F^*(p)$ is a Kan fibration, we can find a filler, z, for h, so set $w := d_2(z)$. (This is a composite homotopy, as if it was topologically 'y_f followed by the reverse of ξ.') this homotopy, w, is in $\underline{\mathcal{S}}(E_f, E_F)$, not in $\underline{\mathcal{S}}(E_f, E_f)$, but otherwise does the right sort of thing.

To get a homotopy with E_f as codomain, we use the lft hand pullback square of the above double pullback diagram, so have to work out $F^*(p)(w)$. This is just our $q_f(w)$ and that, by the description of z as a filler is $d_2 s_0(f^*(p) \times \Delta[1]) = s_0 d_1(f^*(p) \times \Delta[1]) = f^*(p).pr_{E_f}.(A \times e_0)$, so we have a map $w' : E_f \times \Delta[1] \to E_f$, as in the diagram

$$
\begin{array}{ccc}
E_f \times \Delta[1] & & \\
& w' \searrow & w \\
& E_f \xrightarrow{\quad i_f \quad} E_F & \\
f^*(p).pr_{E_f} & \Big\downarrow f^*(p) \qquad \Big\downarrow F^*(p) \\
& A \xrightarrow[A \times e_0]{} A \times \Delta[1],
\end{array}
$$

where $pr_{E_f} : E_f \times \Delta[1] \to E_f$ is the projection. Note that w' is a homotopy *over* A, so is 'in the fibres'.

This w' certainly goes between the right objects, but is it the required homotopy. We check

$$i_f.w'.e_1 = w.e_1 = i_f \beta \alpha,$$

but i_f is the induced map from $A \times e_0$, which is a (split) monomorphism, so i_f is itself a monomorphism, and so $w'.e_1 = \beta \alpha$. Similarly $w'.e_0 = id_{E_f}$, so w' does what was hoped for.

We reverse the roles of α and β, and of f and g, to get the last part of the proof. ∎

4.5 $\overline{W}$, W and twisted Cartesian products

Suppose we have simplicial sets, Y, a potential 'fibre' and B, a potential 'base', which will be assumed to be pointed by a vertex, $*$. Inspired by the sort of construction that works for the construction of group extensions, we are going to try to construct a fibration sequence,

$$Y \longrightarrow E \longrightarrow B.$$

Clearly the product $E = B \times Y$ will give such a sequence, but can we somehow *twist* this Cartesian product to get a more general construction? We will try setting $E_n = B_n \times Y_n$ and will change as little as possible in the data specifying faces and degeneracies. In fact we will take all the degeneracy maps to be exactly those of the Cartesian product, and all but d_0 of the face maps likewise. This leaves just the zeroth face map.

In, say, a covering space considered as a fibration with discrete fibre, the fundamental group(oid) of the base acts by automorphisms / permutations on the fibre, and the fundamental group(oid) is generated by the edges, hence by elements of dimension one greater than that of the fibre, so we try a formula for d_0 of form

$$d_0(b, y) = (d_0 b, t(b)(d_0 y)),$$

where $t(b)$ is an automorphism of Y, determined by b in some way, hence giving a function $t : B_n \longrightarrow \mathsf{aut}(Y)_{n-1}$. Note here Y is an arbitrary simplicial set, not the underlying simplicial set of a simplicial group as was previously the case when we considered aut, but this makes no difference to the definition.

Of course, with these tentative definitions, we must still have that the simplicial identities hold, but it is easy to check that these will hold exactly if t satisfies the following equations

$$
\begin{aligned}
d_i t(b) &= t(d_{i-1}b) \quad \text{for} \quad i > 0, \\
d_0 t(b) &= t(d_1 b) \#_0 t(d_0 b)^{-1}, \\
s_i t(b) &= t(s_{i+1}b) \quad \text{for} \quad i \geq 0, \\
t(s_0 b) &= *.
\end{aligned}
$$

A function, t, satisfying these equations will be called a *twisting function*, and the simplicial set E, thus constructed, will be called a *regular twisted Cartesian product* or *T.C.P.* We write $E = B \times_t Y$.

It is often useful to assume that the twisting function is 'normalised' so that $t(*)$ is the identity automorphism. We usually will tacitly make this assumption if the base is pointed.

If this construction is to make sense, then we really need also a 'projection' from E to B and Y should be isomorphic to its fibre over the base point, $*$. The obvious simplicial map works, sending (b, y) to b. It *is* simplicial and clearly has a copy of Y as its fibre.

Of course, a twisting function is not a simplicial map, but the formulae it satisfies look closely linked to those of the Dwyer-Kan loop group(oid) construction, given earlier, page 125. In fact:

Proposition 38 *A twisting function, $t : B \longrightarrow \mathsf{aut}(Y)$, determines a unique homomorphism of simplicial groupoids $t : GB \to \mathsf{aut}(Y)$, and conversely.* ∎

Of course, since G is left adjoint to $\overline{W}$, we could equally well note that t gave a simplicial morphism $t : B \longrightarrow \overline{W}(\mathsf{aut}(Y))$, and conversely.

Of course, we could restrict attention to a particular class of simplicially enriched groupoids such as those coming from groups (constant simplicial groups), or nerves of crossed modules, or of crossed complexes, etc. We will see some aspects of this in the following chapter, but we will be generalising it as well.

This adjointness gives us a 'universal' twisting function for any simplicial group, H. We have the general natural isomorphism,

$$
\mathcal{S}(B, \overline{W}H) \cong Simp.Grpds(G(B), H),
$$

so, as usual in these situations, it is very tempting to look at the special case where $B = \overline{W}H$ itself and hence to get the counit of the adjunction from $G\overline{W}(H)$ to H corresponding to the identity simplicial map from $\overline{W}H$ to itself. By the general properties of adjointness, this map 'generates' the natural isomorphism in the general case.

From our point of view, the two natural isomorphic sets are much better viewed as being $\mathsf{Tw}(B, H)$, the set of twisting functions $\tau : B \to H$, so the key case will be a 'universal' twisting function, $\tau_H : \overline{W}H \to H$ and hence a universal twisted Cartesian product $\overline{W}H \times_{\tau_H} H$. (Notational point: the context tells us that the fibre H is the underlying simplicial set of the simplicial group, H, but no special notation will be used for this here.) This universal twisted Cartesian product is called the *classifying bundle for H* and is denoted WH. We can unpack its definition from its construction, but will not give the detailed derivation (which is suggested as a **useful exercise**). Clearly

$$
(WH)_n = H_n \times_t \overline{W}(H)_n,
$$

so from our earlier description of $\overline{W}(H)$, we have

$$WH_n = H_n \times H_{n-1} \times \ldots \times H_0.$$

The face maps are given by

$$d_i(h_n, \ldots, h_0) = (d_i h_n, \ldots, d_0 h_{n-i}.h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$$

for all i, $0 \le i \le n$, whilst

$$s_i(h_n, \ldots, h_0) = (s_i h_n, \ldots s_0 h_{n-i}, 1, h_{n-i-1}, \ldots, h_0).$$

(It is noteworthy that $d_0(h_n, \ldots, h_0) = (d_0 h_n.h_{n-1}, h_{n-2}, \ldots, h_0)$ so the universal twist, τ_H, must somehow be built in to this. In fact τ_H is an 'obvious' map as one would hope. We have $\overline{W}(H)_n = H_{n-1} \times \ldots \times H_0$ and we need $(\tau_H)_n : \overline{W}(H)_n \to H_{n-1}$, since it is to be a twisting map and so has degree -1. The obvious formula to try is that τ_H is the projection map - and it works. The details are left to you. A glance back at the formula for the general d_0 in a twisted Cartesian product will help.)

We start by showing that $p : W(H) \to \overline{W}(H)$ is a principal fibration. This simplicial map just is the projection onto the second factor in the T.C.P. To prove this is such a principal fibration, we first examine $W(H)$ more closely and then at an obvious action. The simplicial set, $W(H)$, contains a copy of (the underlying simplicial set of) H as the fibre over the element $(1, 1, \ldots, 1) \in \overline{W}(H)$. There is then a fairly obvious action of H on $W(H)$, given by, in dimensions n,

$$h'.(h_n, \ldots, h_0) = (h' h_n, \ldots, h_0).$$

In other words, just using multiplication on the first factor. As multiplication is a simplicial map, $H \times H \to H$, or simply glancing at the formulae, we have that this *is* a simplicial actions.

That action is *free*, since the regular representation is free as an action. (After all, this is just saying that, if $gx = x$ for some $x \in H$, then $g = 1$, so is obvious!) The action is also faithful / effective, for similar reasons. What are the orbits? As the action only changes the first coordinate, and does that freely and faithfully, the orbits coincide with the fibres of the projection map from $W(H)$ to $\overline{W}(H)$, so that p is also the quotient map coming from the action. It follows that

Lemma 21 *The simplicial map*

$$W(H) \to \overline{W}(H),$$

is a principal fibration.   ∎

The following observations now are either corollaries of this, simple to check or should be looked up in 'the literature'.

1). The simplicial set, $W(H)$, is a Kan complex.

2). $W(H)$ is contractible, i.e., is homotopy equivalent to $\Delta[0]$.

3). The simplicial map,

$$W(H) \to \overline{W}(H),$$

is a Kan fibration with fibre the underlying simplicial set of H, (so the long exact sequence of homotopy groups together with point 2) shows that $\pi_n(\overline{W}H) \cong \pi_{n-1}(H)$).

4). If $p : E \to B$ is a principal H-bundle, that is, E is $H \times_t B$ for some twisting function, $t : B \to H$, then we have a simplicial map

$$f_t : B \to \overline{W}(H)$$

given by $f_t(b) = (t(b), t(d_0 b), \ldots, t(d_0^{n-1}b))$, and we can pull back $(W(H) \to \overline{W}(H))$ along f_t to get a principal H-bundle over B

$$
\begin{array}{ccc}
E' & \longrightarrow & W(H) \\
{\scriptstyle p'}\downarrow & & \downarrow \\
B & \underset{f_t}{\longrightarrow} & \overline{W}(H).
\end{array}
$$

We can, of course, calculate E' and p' precisely:

$$
\begin{aligned}
E' & \cong \ \{((h_n, h_{n-1}, \ldots, h_0), b) \mid h_{n-1} = t(b), \ldots h_0 = t(d_0^{n-1}b)\} \\
& \cong \ \{(h_n, b) \mid h_n \in H_n, b \in B_n\} \\
& = \ H_n \times B_n.
\end{aligned}
$$

It should come as no surprise to find that $E' \cong H \times_t B$, so is E itself up to isomorphism, and that p' is p in disguise.

The assignment of f_t to t gives a one-one correspondence between the set, $Princ_H(B)$, of H-equivalence classes of principal H-bundles with base B, and the set, $[B, \overline{W}(H)]$, of homotopy classes of simplicial maps from B to $\overline{W}(H)$.

An important thing to remember is that not all T.C.Ps are principal fibrations. To get a T.C.P., we just need a fibre Y, a base, B, and a simplicial group, G, acting on Y, together with our twisting function, $t : B \to \overline{W}(G)$. From B and t, we can build a principal fibration which is, of course, a T.C.P. but has fibre the underlying simplicial set of G. To build the T.C.P., $B \times_t Y$, we need the *additional* information about the *representation* $G \to \mathsf{aut}(Y)$, that is, the action of G on the fibre, and, of course, that representation need not be an isomorphism. In general, we have: 'fibre bundle = principal fibration plus representation', as a rule of thumb. This is not just in the simplicial case. (We will consider fibre bundles and similar other structures in a lot more detail in the next chapter.)

A good introduction to simplicial bundle theory can be found in Curtis' classical survey article, [56] section 6, or, for a thorough treatment, May's book, [119]. For full details, you are invited to look there, at least to know what is there. We have not gone into all the detail here. We will revisit the overall theory several times later on, drawing parallels and comparisons that will, it is hoped, shed light both on it and on geometrically related theories elsewhere in the area.

4.6   More examples of Simplicial Groups

We have already seen several general constructions of simplicial groups, for instance, the simplicial resolutions of a group, the loop group on a reduced simplicial set, the internal nerve of a crossed module / cat^1-group, and so on. The previous few sections give some ideas for other construction leading to simplicial groups. We will concentrate on two such.

Let G be a topological (or Lie) group (so a group internal to 'the' category of topological spaces - whichever one is most appropriate for the situation). The singular complex functor, $Sing : Top \to \mathcal{S}$, preserves products,

$$Sing(X \times Y) \cong Sing(X) \times Sing(Y),$$

so it follows that, as the multiplication on G is continuous, there is an induced simplicial map,

$$Sing(G) \times Sing(G) \to Sing(G).$$

With the map induced from the maps that picks out the identity element and that give the inverse, this makes $Sing(G)$ into a simplicial group. This gives a large number of interesting simplicial groups, corresponding to general linear, orthogonal, and other topological (or Lie) groups of various dimension. Of course, the homotopy groups of these simplicial groups correspond to those of the groups themselves.

A closely related construction involves a similar idea to the $\mathsf{aut}(K)$ simplicial group, that we used when discussing simplicial bundles, twisted Cartesian products, etc., a few sections ago. We had a simplicial set, K, and hence a simplicial monoid, $\underline{\mathcal{S}}(K, K)$, of endomorphisms of K. The simplicial group, $\mathsf{aut}(K)$, was the corresponding simplicial group of simplicial automorphisms of K. We had a representation of such an $f : K \times \Delta[k] \to K$ as $(f, p) : K \times \Delta[k] \to K \times \Delta[k]$ and this was an automorphism *over* $\Delta[k]$, (look back to page 133).

This sort of construction will work in any situations where the basic category being studied is 'simplicially enriched', i.e. the usual hom-sets of the category form the vertices of simplicial hom-sets and the composition maps between these are simplicial. We will formally introduce this idea later, (see Chapter **??**, and in particular section **??**, page **??**). Here we will give some examples of this type of idea in situations that are useful in geometric and topological contexts.

We will assume that X is a (locally finite) simplicial complex. In applications X is often $\mathbb{R}^n$, or S^n or similar. We think of the product, $\Delta^k \times X$, as a 'bundle over the k-simplex, Δ^k, or, if working in the piecewise linear (PL) setting, a PL bundle over Δ^k. The simplicial group, $\mathcal{H}(X)$, is then the simplicial group having $\mathcal{H}(X)_k$ being the set of homeomorphisms of $\Delta^k \times X$ over Δ^k, or, alternatively, the (PL) bundle isomorphisms of $\Delta^k \times X$. As a variant, if $A \subset X$ is a subcomplex, one can restrict to those bundle isomorphisms that fix $\Delta^k \times A$ pointwise.

Various examples of this were used to study the problem of the existence and classification of triangulations and smoothings for manifolds. The construction occurs, for instance, in Kuiper and Lashof, [108, 109]. Later on starting in section **??**, we will look at another variant of these examples concerning microbundle theory, (see Buoncristiano, [46, 47]), as it gives a nice interpretation of some simplicial bundles in a geometric setting.

Chapter 5

Non-Abelian Cohomology: some ideas

One of the problems to be faced when presenting the applications of crossed modules, etc., is that such is the breadth of these applications that they may safely be assumed to be potentially of interest to mathematicians of very differing backgrounds, algebraists of many different hues, geometers both algebraic and differential, theoretical physicists and, of course, algebraic topologists. To make these notes as useful as possible, some part of the more basic 'intuitions' from the background material from some of these areas has been included at various points. This cannot be 'all inclusive' nor 'universal' as different groups of potential readers have different needs. The real problems are those of transfer of 'technology' between the areas and of explanation of the differing terminology used for the same concept in different contexts. Often, essentially the same idea or result will appear in several places. This repetition is not just laziness on the authors behalf. The introduction of a concept bit-by-bit from various angles almost necessitates such a treatment.

For the background on bundle-like constructions (sheaves, torsors, stacks, gerbes, 2-stacks, etc.), the geometric intuition of 'things over X' or X-parametrised 'things' of various forms, does permeate much of the theory, so we will start with some fairly basic ideas, and so will, no doubt, for some of the time, be 'preaching to the converted', however that 'bundle' intuition is so important for this and later sections that something more than a superficial treatment is required.

(In the original lectures at Buenos Aires, I did assume that that intuition was understood, but in any case concentrated on the 'group extension' case rather than on 'gerbes' and their kin. By this means I avoided the need to rely too heavily on material that could not be treated to the required depth in the time available. However I cannot escape the need to cover some of that material here!)

Initially crossed modules, etc., will not be that much in evidence, *but* it is important to see how they do enter in 'geometrically' or their later introduction can seem rather artificial.

We start by looking at descent, i.e., the problem of putting 'local' bits of structure into a global whole.

5.1   Descent: Bundles, and Covering Spaces

(Remember, if you have met 'descent' or 'bundles', then you should 'skim' this section only / anyway.)

We will look at these structures via some 'case studies' to start with.

5.1.1   Case study 1: Topological Interpretations of Descent.

Suppose A and B are topological spaces and $\alpha : A \to B$ a continuous map (sometimes called a 'space over B' or loosely speaking a 'bundle over B', although that can also have a more specialised meaning later). The space, B, will usually be called the *base*, whilst A is the *total space* of the bundle, α.

An obvious and important example is a product, $A = B \times F$, with α being the projection. We call this a *trivial bundle* on B.

If $U \subset B$ is an open set, then we get a restriction $\alpha_U : \alpha^{-1}(U) \to U$. If $V \subset B$ is another open set, we, of course, have $\alpha_V : \alpha^{-1}(V) \to V$ and over $U \cap V$ the two restrictions 'coincide', i.e., if we form the pullbacks

$$
\begin{array}{ccc}
? & \longrightarrow & \alpha^{-1}(U) \\
\downarrow & & \downarrow \\
U \cap V & \longrightarrow & U
\end{array}
\qquad\qquad
\begin{array}{ccc}
? & \longrightarrow & \alpha^{-1}(V) \\
\downarrow & & \downarrow \\
U \cap V & \longrightarrow & V
\end{array}
$$

the resulting spaces over $U \cap V$ are 'the same'. (We have to be a bit careful since we formed them by pullbacks so they are determined only 'up to isomorphism' and we should take care to interpret 'the same' as meaning 'being isomorphic' as spaces over $U \cap V$. This care will be important later.) Now assume that for each $b \in B$, we choose an open neighbourhood $U_b \subset B$ of b. We then have a family

$$\alpha_b : A_b \to U_b \qquad\qquad b \in B,$$

where we have written A_b for $\alpha^{-1}(U_b)$, and we know information about the behaviour over intersections.

Can we reverse this process? More precisely, can we start with a family $\{\alpha_b : A_b \to U_b : b \in B\}$ of maps (with A_b now standing for an arbitrary space) and add in, say, information on the 'compatibility' over the intersections of the cover $\{U_b : b \in B\}$ so as to rebuild a space over B, $\alpha : A \to B$, which will restrict to the given family.

We will need to be more precise about that 'compatibility', but will leave it aside until a bit later. Clearly, indexing the cover by the elements of B is a bit impractical as usually we just need, or are given, some (open) cover, $\mathcal{U}$, of B, and then can choose, for each $b \in B$, some set of the cover containing b. This way we do not repeat sets unless we expressly need to. Thinking like this we have a cover $\mathcal{U}$ and for each U in $\mathcal{U}$, a space over U, $\alpha_U : A_U \to U$. To encode the condition on compatibility on intersections, we need some (temporary) notation: If $U, U' \in \mathcal{U}$, write $(A_U)_{U'}$ for the restriction of A_U over the intersection $U \cap U'$, similarly $(\alpha_U)_{U'}$ for the restriction of α_U to $U \cap U'$. This is given by the further pullback of α_U along the inclusion of $U \cap U'$ into U, so we also get a map

$$(\alpha_U)_{U'} : (A_U)_{U'} \to U \cap U'.$$

We noted that if the family $\{\alpha_U \mid U \in \mathcal{U}\}$ did come from a single $\alpha : A \to B$, then the α_Us agreed up to isomorphism on the intersections, i.e., we needed homeomorphisms

$$\xi_{U,U'} : (A_U)_{U'} \xrightarrow{\cong} (A_{U'})_U$$

over $U \cap U'$ if we were going to give an adequate description. (These are sometimes called the *transition functions* or gluing *cocycles*.) This, of course, means that

$$(\alpha_{U'})_U \circ \xi_{U,U'} = (\alpha_U)_{U'}.$$

Clearly we should require

1. $\xi_{U,U}$ = identity,

but also if U'' is another set in the cover, we would need

2. $\xi_{U',U''} \circ \xi_{U,U'} = \xi_{U,U''}$

over the triple intersection $U \cap U' \cap U''$.

(This condition 2. is a *cocycle condition*, similar in many ways to ones we have met earlier in apparently very different contexts.)

These two conditions are inspired by observation on decomposing an original bundle. They give us 'descent data', but are our 'descent data' enough to construct and, in general, to classify such spaces over B? The obvious way to attempt construction of an α from the data $\{\alpha_U; \xi_{U,U'}\}$ is to 'glue' the spaces A_U together using the $\xi_{U,U'}$. 'Gluing' is almost always a colimiting process, but as that can be realised using coproducts (disjoint union) and coequalisers (quotients by an equivalence relation), we will follow a two step construction

Step 1: Let $C = \sqcup_{U \in \mathcal{U}} A_U$ and $\gamma : C \to \sqcup_{U \in \mathcal{U}} U$, the induced map. Thus if we consider a specific U in $\mathcal{U}$, we will have inclusions of A_U into C and U into $\sqcup U$ and a diagram

$$
\begin{array}{ccc}
A_U & \hookrightarrow & C = \sqcup A_U \ . \\
\alpha_U \downarrow & & \downarrow \gamma \\
U & \hookrightarrow & \sqcup U
\end{array}
$$

Remember that a useful notation for elements in a disjoint union is a pair, (element, index), where the index is the index of the set in which the element is. We write (a, U) for an element of C, then $\gamma(a, U) = (\alpha_U(a), U)$, since $a \in A_U$.

Step 2: We relate elements of C to each other by the rule:

$$(a, U) \sim (a', U')$$

if and only if

(i) $\alpha_U(a) = \alpha_{U'}(a')$,

and

(ii) we want to glue corresponding elements in fibres over the same point of B so need something like $\xi_{U,U'}(a) = a'$. Although intuitively correct, as it says that if a and a' are over the same point of $U \cap U'$ then they are to be 'related' or 'linked' by the homeomorphism, $\xi_{U,U'}$, a close look at the formula shows it does not quite make sense. Before we can apply $\xi_{U,U'}$ to a, we have to restrict a to be in $(A_U)_{U'}$ and the result will be in $(A_{U'})_U$. Perhaps the neatest way to present this is to look at another disjoint union, this time $\sqcup_{U,U'}(A_U)_{U'}$, and to map this to $C = \sqcup_{U \in \mathcal{U}} A_U$ in two ways. The first of these, $\mathbf{a}$, say, takes the component $(A_U)_{U'}$ and injects it into C via the injection of A_U. The second map, $\mathbf{b}$, first sends $(A_U)_{U'}$ to $(A_{U'})_U)$ using $\xi_{U,U'}$ then sends that second component to $(A_{U'})$ and thus into C. We thus get the correct version of the formula for (ii) to be:

there is an $x \in \sqcup_{U,U'}(A_U)_{U'}$ such that $\mathbf{a}(x) = a$ and $\mathbf{b}(x) = a'$.

The two conditions on the homeomorphisms ξ readily imply that this is an equivalence relation and that the α_U together define a map

$$\alpha : A = C/_\sim \ \to B$$

given by

$$\alpha[(a, U)] = \alpha_U(a),$$

on the equivalence class, $[(a, U)]$ of (a, U). For this to be the case, we only needed $\alpha_U(a) = \alpha_{U'}(a')$ to hold. Why did we impose the second condition, i.e., the cocycle condition? Simply, if we had not, we would risked having an equivalence relation that crushed C down to B. Each fibre $\alpha^{-1}(b)$ might have been a single point since each $\alpha_U^{-1}(a)$ could have been in a single equivalence class.

We now have a space over B, $\alpha : A \to B$ (with A having the quotient topology, which ensures that α will be continuous).

If we had started with such a space, decomposed over $\mathcal{U}$, then had constructed a 'new space' from that data, would we have got back where we started? Yes, up to isomorphism (i.e., homeomorphism over B). To discuss this, it helps to introduce the category, Top/B, of spaces over B. This has continuous maps $\alpha : A \to B$ (often written (A, α)) as its objects, whilst a map from (A, α) to $\alpha' : A' \to B$ will be a continuous map $f : A \to A'$ making the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ f\ \ } & A' \\
 & {\scriptstyle \alpha}\searrow \quad \swarrow {\scriptstyle \alpha'} & \\
 & B &
\end{array}
$$

commutative. This, however, raises another question.

If we have such an f and an (open) cover $\mathcal{U}$ of B, we restrict f to $\alpha^{-1}(U)$ to get

$$
f_U : A_U \to A'_U
$$

which, of course, is in Top/U. If we have data,

$$
\{\alpha_U : A_U \to U, \{\xi_{U,U'}\}\}
$$

for (A, α) and similarly for (A', α'), and morphisms

$$
\{f_U : A_U \to A'_U\},
$$

when can we 'rebuild' $f : A \to A'$? We would expect that we would need a compatibility between the various f_U and the $\xi_{U,U'}$ and $\xi'_{U,U'}$. The obvious condition would be that whenever we had U, U' in $\mathcal{U}$, the diagram

$$
\begin{array}{ccc}
(A_U)_{U'} & \xrightarrow{(f_U)_{U'}} & (A'_U)_{U'} \\
{\scriptstyle \xi_{U,U'}}\Big\downarrow & & \Big\downarrow{\scriptstyle \xi'_{U,U'}} \\
(A_{U'})_U & \xrightarrow{(f_{U'})_U} & (A'_{U'})_U
\end{array}
$$

should commute, where we have extended our notation to use $(f_U)_{U'}$ for the restriction of f_U to $\alpha^{-1}(U \cap U')$. To codify this neatly we can form each category, Top/U, for $U \in \mathcal{U}$, then form the category, D, consisting of families of objects, $\{\alpha_U : U \in \mathcal{U}\}$, of $\prod Top/U$ together with the extra structure of the $\xi_{U,U'}$. Morphisms in D are families $\{f_U\}$ as above, compatible with the structural isomorphisms $\xi_{U,U'}$.

Remark: For any specific pair consisting of a family, $\mathcal{A} = \{(A_U, \alpha_U) : U \in \mathcal{U}\}$ and the extra $\xi_{U,U'}$s is a *set of descent data for* $\mathcal{A}$. We will look at both this construction and its higher dimensional relatives in quite a lot of detail and generality later on. The category of these things

and the corresponding morphisms can be called the *category of descent data relative to the cover*, $\mathcal{U}$.

The reason for the use of the word 'descent' is that, in many geometric situations, structure is easily encoded on some basic 'patches'. This structure, that is locally defined, 'descends' to the space giving it a similar structure. In many cases, the A_U have the fairly trivial form $U \times F$ for some fibre F. This fibre often has extra structure and the $\xi_{U,U'}$ have then to be structure preserving automorphisms of the space, F. The term 'bundle' is often used in general, but some authors restrict its use to this *locally trivial* case. The classic case of a locally trivial bundle is a Möbius band as a bundle over the circle. Locally, on the circle, the band is of form $U \times [-1, 1]$, but globally one has a twist. A bit more formally, and for use later, we will define:

Definition: A bundle $\alpha : A \to B$ is said to be *locally trivial* if there is an open cover $\mathcal{U}$ of B, such that, for each U in $\mathcal{U}$, A_U is homeomorphic to $U \times F$, for some fibre F, compatibly with the projections, α_U and $p_U : U \times F \to U$.

We will gradually build up more precise intuitions about what 'compatibly' means, and as we do so, the above definition will gain in precision and strength.

5.1.2 Case Study 2: Covering Spaces

This is a classic case of a class of 'spaces over' another space. It is also of central importance for the development of possible generalisations to higher 'dimensions', (cf. Grothendieck's *Pursuit of Stacks*, [87].) We have a continuous map

$$\alpha : A \to B$$

and for any point $b \in B$, there is an open neighbourhood U of b such that $\alpha^{-1}(U)$ is the disjoint union of open subsets of A, each of which is mapped homeomorphically onto U by α. The map α is then called a *covering projection*. On such a U, $\alpha^{-1}(U)$ is $\sqcup U_i$ over some index set which can be taken to be $\alpha^{-1}(b) = F_b$, the fibre over b. Then we may identify $\alpha^{-1}(U)$ with $U \times F_b$ for any $b \in U$. This F_b is 'the same' up to isomorphism for all $b \in U$. If B is connected then for any $b, b' \in B$, we can link them by a chain of pairwise intersecting open sets of the above form and hence show that $F_b \cong F_{b'}$. We can thus take each $\alpha^{-1}(U) \cong U \times F$ and F will be a discrete space provided B is nice enough. The descent data in this situation will be the local covering projections

$$\alpha_U : U \times F \to U$$

together with the homeomorphisms

$$\xi_{U,U'} : (U \cap U') \times F \to (U \cap U') \times F$$

over $(U \cap U')$. Provided that $(U \cap U')$ is connected, this $\xi_{U,U'}$ will be determined by a permutation of F.

We often, however, want to allow for non-connected $(U \cap U')$. For instance, take B to be the unit circle S^1, $F = \{-1, 1\}$,

$$U_1 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x > -0.1\}$$

$$U_2 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x < 0.1\}.$$

The intersection, $U_1 \cap U_2$, is not connected, so we specify ξ_{U_1, U_2} separately on the two connected components of $U_1 \cap U_2$. We have

$$U_1 \cap U_2 = \{(x, y) \in S^1 \mid |x| < 0.1, y > 0\} \cup \{(x, y) \mid |x| < 0.1, y < 0\}.$$

Let $\xi_{U_1, U_2}((x, y), t) = \begin{cases} ((x, y), t) & \text{if } y > 0 \\ ((x, y), -t) & \text{if } y < 0, \end{cases}$
so on the part with negative y, ξ exchanges the two leaves. The resulting glued space is either viewed as the edge of the Möbius band or as the map,

$$S^1 \to S^1$$

$$e^{i\theta} \mapsto e^{i2\theta}.$$

Remark: The well known link between covering spaces and actions of the fundamental group $\pi_1(B)$ on Sets is at the heart of this example.

A neat way to picture the n-fold covering spaces of S^1 for $n \geq 2$ is to consider a knot on the surface of a torus, $S^1 \times S^1$, for instance the trefoil. The projection to the first factor of $S^1 \times S^1$ gives a covering, as does the second projection. It is **also instructive** to consider the covering space $\mathbb{R}^2 \to S^1 \times S^1$, working out what the various transitions are for a cover. We note the way that quotients of $\mathbb{R}^n$ by certain geometrically defined group actions, yields neat examples of coverings (although some may be 'ramified', an area we will not stray into here.)

In general, when we have a local product structure, so $\alpha^{-1}(U) \cong U \times F$, the homeomorphisms $\xi_{U,U'}$ have a nicer description than the general one, since being 'over' the intersection, they have to have the form that interprets at the product levels as being $\xi_{U,U'}(x, y) = (x, \xi'_{U,U'}(x)(y))$ where $\xi'_{U,U'} : U \cap U' \to Aut(F)$. In the case of covering spaces F is discrete, so $\xi'_{U,U'}(x)$ will give a permutation of F.

5.1.3   Case Study 3: Fibre bundles

The examples here are to introduce / recall how torsors / principal fibre bundles are defined topologically and also to give some explicit instances of how fibre bundles arise in geometry.

(Often in this context, the terminology 'total space' is used for the source of the bundle projection.)

First some naturally occurring examples.

(i) Let S^n denote the usual n-sphere represented as a subspace of $\mathbb{R}^{n+1}$,

$$S^n = \{\underline{x} \in \mathbb{R}^{n+1} \big| \, \|\underline{x}\| = 1\},$$

where $\|\underline{x}\| = \sqrt{\langle \underline{x} \mid \underline{x} \rangle}$ for $\langle \underline{x} \mid \underline{y} \rangle$, the usual Euclidean inner product on $\mathbb{R}^{n+1}$. The *tangent bundle* of S^n, τS^n is the 'bundle' with total space,

$$TS^n = \{(\underline{b}, \underline{x}) \mid \langle \underline{b} \mid \underline{x} \rangle = 0\} \subset S^n \times \mathbb{R}^{n+1}.$$

We thus have a projection

$$p : TS^n \to S^n$$

given by $p(\underline{b}, \underline{x}) = \underline{b}$, as a space over S^n.

Similarly the *normal bundle*, νS^n, of S^n is given with total space,

$$NS^n = \{(\underline{b}, \underline{x}) \mid \underline{x} = k\underline{b} \text{ for some } k \in \mathbb{R}\} \subset S^n \times \mathbb{R}^{n+1}.$$

The projection map $q : NS^n \to S^n$ gives, as before, a space over S^n, $\nu S^n = (NS^n, q, S^n)$.

Another example extends this to a geometric context of great richness.

(ii) First we need to introduce generalisations, the Grassmann varieties, of projective spaces and in order to see what topology it is to have, we look at a related space first. The *Stiefel variety* of k-frames in $\mathbb{R}^n$, denoted $V_k(\mathbb{R}^n)$, is the subspace of $(S^{n-1})^k$ such that $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^n)$ if and only if each $\langle v_i \mid v_j \rangle = \delta_{i,j}$, so that it is 1 if $i = j$ and is zero otherwise. Note $V_1(\mathbb{R}^n) = S^{n-1}$.

The *Grassmann variety* of k-dimensional subspaces of $\mathbb{R}^n$, denoted $G_k(\mathbb{R}^n)$, is the set of k-dimensional subspaces of $\mathbb{R}^n$. There is an obvious function,

$$\alpha : V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n),$$

mapping $(v_1, \ldots, v_k)$ to $span_{\mathbb{R}}\langle v_1, \ldots, v_k \rangle \subseteq \mathbb{R}^n$, that is, the subspace with $(v_1, \ldots, v_k)$ as basis. We give $G_k(\mathbb{R}^n)$ the quotient topology defined by α. (For $k = 1$, we have $G_1(\mathbb{R}^n)$ is the real projective space of dimension $n - 1$.)

This geometric setting also produces further important examples of 'bundles', this time on these Grassmann varieties.

Consider the subspace of $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ given by those (V, x) with $x \in V$. Using the projection $p(V, x) = V$ gives the bundle,

$$\gamma_k^n = (\gamma_k^n, p, G_k(\mathbb{R}^n)).$$

This is *canonical k-dimensional vector bundle on $G_k(\mathbb{R}^n)$.*

Similarly the *orthogonal complement bundle*, $^*\gamma_k^n$, has total space consisting of those (V, x) with $\langle V \mid x \rangle = 0$, i.e., x is orthogonal to V.

All of these 'bundles' have vector space structures on their fibres. They are all *locally trivial* (so in each case $\alpha^{-1}(U) \cong U \times F$ for suitable open subsets U of the base), and the resulting $\xi_{U,U'}$ have form

$$\xi_{U,U'}(x, t) = (x, \xi'_{U,U'}(x))(t)$$

where $\xi'_{U,U'} : U \cap U' \to G\ell_M(\mathbb{R})$ for suitable M. (As usual, $G\ell_M(\mathbb{R})$, which may sometimes also be denoted $G\ell(M, \mathbb{R})$, is the *general linear group of non-singular $M \times M$ matrices over* $\mathbb{R}$. Here it is considered as a topological group. It also has a smooth structure and is an important example of a *Lie group*.) Such *vector bundles* are prime examples of the situation in which the fibres have extra structure.

We will see, use and study vector bundles in more detail later on, for the moment, we introduce the example of a *trivial vector bundle* in addition to those geometrically occurring ones above. We will work over the real numbers as our basic field, but could equally well use $\mathbb{C}$ or more generally.

Definition: A *trivial (real) vector bundle* of dimension m, on a space B is one of the form $\mathbb{R}^m \times B \to B$, the mapping being, naturally, the projection. We will denote this by ε_B^m.

Even more structure can be encoded, for instance, by giving each fibre an inner product structure with the requirement that the $\xi'_{U,U'}$ take values in $O_M(\mathbb{R})$, or $O(M,\mathbb{R})$, the *orthogonal group*, hence that they preserve that extra structure. Abstracting from this we have a group, G, which acts by automorphisms on the space, F, and have our descent data isomorphisms $\xi_{U,U'}$ of the form $\xi_{U,U'}(x,t) = (x, \xi'_{U,U'}(x))(t)$ for some continuous $\xi'_{U,U'} : U \cap U' \to G$.

As usual, if G is a (topological) group, by a *G-space*, we mean a space X with an action (left action):

$$G \times X \to X,$$

$$(g,x) \to g.x.$$

The action is *free* if $g.x = x$ implies $g = 1$. The action is *transitive* if given any x and y in X there is a $g \in G$ with $g.x = y$. Let X^* be the subspace

$$X^* = \{(x, g.x) : x \in X, g \in G\} \subseteq X \times X,$$

(cf. our earlier discussion of action groupoids on page **??**).

There is a function (called the *translation function*)

$$\tau : X^* \to G$$

such that $\tau(x, x')x = x'$ for all $(x, x') \in X^*$. We note

(i) $\tau(x, x) = 1$,

(ii) $\tau(x', x'')\tau(x, x') = \tau(x, x'')$,

(iii) $\tau(x', x) = \tau(x, x')^{-1}$

for all $x, x', x'' \in X$.

A G-space, X, is called *principal* provided X is a free, transitive G-space with *continuous* translation function $\tau : X^* \to G$.

Proposition 39 *Suppose X is a principal G-space, then the mapping*

$$G \times X \to X \times X$$

$$(g,x) \to (x, g.x)$$

is a homeomorphism.

Proof: The mapping is continuous by its construction. Its inverse is (τ, pr_1), which is also continuous. ∎

This is often taken as the definition of a principal G-space, so you could try to prove the converse. We, in fact, need a fibrewise version of this.

Given any G-space, X, we can form a quotient X/G with a continuous map $\alpha : X \to X/G$. A bundle $\mathsf{X} = (X, \alpha, B)$ is called a *G-bundle* if X has a G-action, so that B is homeomorphic to X/G compatibly with the projections from X. The bundle is a *principal G-bundle* if X is a principal G-space *over* B. What does this mean? In a G-bundle, as above, the fibres of α are orbits of the G-action, so the action is 'fibrewise'. We can replace G by $\underline{G} = G \times B$ and, thinking of it as a

space over B, perhaps rather oddly, write the action within the category Top/B. We replace the product in Top by that in Top/B, which is just the pullback along projections in Top. The action is thus

$$\underline{G} \times_B X \to X$$

over B, or just $\underline{G} \times \mathsf{X} \to \mathsf{X}$ in the notation valid in Top/B. Now 'principalness' will say that the action is free and transitive, and that the translation function is a continuous map *over B*. A neater way to handle this is to use the above proposition and to define X to be a principal G-bundle if the corresponding morphism over B,

$$\underline{G} \times \mathsf{X} \to \mathsf{X} \times \mathsf{X}$$

is an isomorphism in Top/B. We will not explore this more here as that *is*, more or less, the way we will define G-torsors later on, except that we will be using a bundle or sheaf of groups rather than simply $\underline{G}$.

We note that if $\xi = (X, p, B)$ is a principal G-bundle then the fibre $p^{-1}(b)$ is homeomorphic to G for any point $b \in B$. It is usual in topological situations to require that the bundle be locally trivial. For the moment, we can summarise the idea of principal G-bundle as follows:

A principal G-bundle is a fibre bundle $p : X \to B$ together with a continuous left action $G \times X \to X$ by a topological group G such that G preserves the fibers of p and acts freely and transitively on them.

Later we will see other more categorical views of principal G-bundles. As we have mentioned, they will reappear as 'G-torsors' in various settings. For the moment we need them to provide the link to the general notion of fibre bundle.

For F, a (right) G-space with action $G \times F \to F$, we can form a quotient, X_F, of $F \times X$ by identifying (f, gx) with (fg, x). The composite

$$F \times X \overset{pr_2}{\to} X \to X/G$$

factors via X_F to give $\beta : X_F \to X/G$, where $\beta(f, x)$ is the orbit of x, i.e., the image of x in X/G. The earlier examples of 'bundles' were all examples of this construction. The resulting (X_F, β, B) is called a *fibre bundle* over B ($= X/G$).

Note: The theory of fibre bundles was developed by Cartan and later by Ehresmann and others from the 1930s onwards. Their study arose out of questions on the topology and geometry of manifolds. In 1950, Steenrod's book, [151], gave what was to become the first reasonably full treatment of the theory. Atiyah, Hirzebruch and then, in book form, Husemoller, [97] in 1966 linked this theory up with K-theory, which had come from algebraic geometry. The books contain much of the basic theory including the local coordinate description of fibre bundles which is most relevant for the understanding of the descent theory aspects of this area (cf. Chapter 5 of Husemoller, [97]). The restriction of looking at the local properties relative to an open cover makes this treatment slightly too restrictive for our purposes. It *is* sufficient, it seems, for many of the applications in algebraic topology, differential geometry and topology and related areas of mathematical physics, however as Grothendieck points out (SGA1, [88], p.146), in algebraic geometry *localisation of properties*, although still linked to certain types of "base change" (as here with base change along the map

$$\sqcup \, \mathcal{U} \to B$$

for $\mathcal{U}$ an open cover of B), needs to consider other families of base change. These are linked with some problems of commutative algebra that are interesting in their own right and reveal other aspects of the descent problem, see [22]. For these geometric applications, we need to replace a purely topological viewpoint by one in which *sheaves* take a front seat role.

(The Wikipedia entries for principal G-space, principal bundle and 'fiber' bundle are good places to start seeing how these concepts get applied to problems in geometry. For a picture of how to build a fibre bundle out of wood, see http://www.popmath.org.uk/sculpmath/pagesm/fibundle.html.)

5.1.4   Change of Base

This is a theme that we will revisit several times. Suppose that we have a good knowledge of 'bundles' over some space, B', but want bundles over another space, B. We have a continuous map, $f : B \to B'$, and hope to glean information on bundles on B by comparing them with those on B', using f in some way. (We could be looking to transfer the information the other way as well, but this way will suffice for the moment!)

What we have used when restricting to open subsets of a base space was pullback and that works here as well. Suppose $p' : A' \to B'$ is a principal G-bundle over B', then we form the pullback

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

Categorically the pullback, as it is characterised by a universal property, is only determined up to isomorphism, but we can pick a definite model for A in the form

$$A' \times_{B'} B = \{(a, b) \mid p'(a) = f(b)\},$$

with $a \in A'$ and $b \in B$. The projection of A onto B is given by sending (a, b) to b and the map from A to A' by the obvious other projection. As we have an action of G on the left of A' it is tempting to see if there is one on A and the obvious thing to attempt is $g.(a, b) = (g.a, b)$. Does this make sense? Yes, because $p'(g.a) = p'(a)$, since B' is the space of orbits of the action of G on A'. Is $A \to B$ then a principal G-bundle? Again the answer is yes. To gain some idea why look at the fibres. We know the fibres of a principal G bundle are copies of the *space* G, and fibres of the pullback are the same as fibres of the original. The action is concentrated in the fibres as the orbit space of the action *is* the base.

The one question is whether the map

$$\underline{G} \times_B A \to A \times_B A$$

is an isomorphism. You can see that it is in two ways. The elements of A are pairs (a, b), as above. The map is $((g, b), (a, b)) \mapsto ((a, b), (g.a, b))$ and this is clearly in the fibres as the second component in each pair is the same. It has an inverse surely, (since an element in $A \times BA$, has the form $((a_1, b), (a_2, b))$ and since A' is a principal bundle we can continuously find g such that $a_2 = g.a_1$). The alternative approach is to note that the map fits into a diagram with lots of pull back squares and to note that is is induced from the corresponding map for (A', B', p').

We thus have, it would seem, that $f : B \to B'$ induces a 'functor' from the category of principal G-bundles over B' to the corresponding one over B. (The word 'functor' is given between inverted

commas since we have not discussed morphisms between bundles of this form. That is left to you both to formulate the notion and to check that the inverted commas can be removed. In any case we will be considering this in the more general setting of G-torsors slightly later in this chapter.)

We thus have induced bundles, $f^*(\mathsf{A}')$, but different maps, f, can lead to isomorphic bundles. More precisely, suppose f and g are two maps from B to B', then if f and g are homotopic (under mild compactness conditions on the spaces) it is fairly easy to prove that for any (principal) bundle A' on B', the two bundles $f^*(\mathsf{A}')$, and $g^*(\mathsf{A}')$, are isomorphic. We will not give the details here as they are in most text books on the area, (see, for instance, [97], or [104]), but the idea is that if $H : B \times I \to B'$ is a homotopy between f and g, we get a bundle $H^*(\mathsf{A})$ with base $B \times I$. You now use local triviality of the bundle to cover $B \times I$ by open sets over which this bundle trivialises. Using compactness of B, we get a sequence of points t_i in I and an open cover of $B \times I$ made up of open sets of the form $U \times (t_i, t_{i+2})$. Now we work our way up the cylinder showing that the bundle over each slice $B \times \{t_i\}$ is isomorphic to that on the previous slice. (There are lots of details left vague here and you should look them up if you have not seen the result before.)

This result shows that categories of principal bundles over homotopically equivalent spaces will be equivalent, and, in particular, that over any contractible space, all principal bundles are isomorphic to each other and hence are all isomorphic to the product principal bundle. It also shows that if we can cover B with an open cover made up of contractible open sets that all bundles trivialise over that cover.

Remarks: In many different theories of bundle-like objects there is an *induced bundle* construction given by pullback along a continuous map on the 'bases'. In *most* of those cases, it seems, homotopic maps induce isomorphic 'bundles', again with possibly a compactness requirement of some sort on the bases.. This happens with vector bundles, (as follows from the result on principal bundles mentioned above.) In these cases, the only bundles of that type on a contractible space will be product bundles. (We will keep this vague directing the reader to the literature as before.)

5.2 Descent: simplicial fibre bundles

To understand topological descent, as in the theory of fibre bundles as sketched out above, it is useful to see the somewhat simpler simplicial theory. This has aspects that are not so immediately obvious as in the topological case, yet some of these will be very useful when we get further in our study handling sheaves and later on stacks.

The basics of simplicial fibre bundle theory were developed in the 1950s and early 1960s, the start being in a paper by Barratt, Gugenheim and Moore, [14]. We have already discussed several of the features of this theory. A useful survey is given by Curtis, [56], and a full description of the theory are available in May's book, [119], with many aspects also treated in Goerss and Jardine, [85].

5.2.1 Fibre bundles, the simplicial viewpoint

We earlier saw how, in the simplicial setting, the G-principal fibrations, when pulled back over any simplex of their base, gave a trivial product fibration. It is this feature that we abstract to get a working notion of simplicial fibre bundle.

Definition: A *(simplicial) fibre bundle* with fibre, Y, over a simplicial set, B, is a simplicial map, $f : E \to B$ such that for any n-simplex, $b \in B_n$, (for any n), the pullback over the representing ('naming') map, $\ulcorner b \urcorner : \Delta[n] \to B$, is a trivial bundle, that is, isomorphic to a product of Y with $\Delta[n]$ together with its projection onto $\Delta[n]$.

We thus have a diagram

$$
\begin{array}{ccc}
Y \times \Delta[n] & \longrightarrow & E \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\
\Delta[n] & \xrightarrow[\ulcorner b \urcorner]{} & B,
\end{array}
$$

which is a pullback.

It is worthwhile just thinking about the comparison between this and what we have been looking at for topological bundles. The role played there by the open covering is taken by the family of *all* simplices of the base. (From this one can build a neat category, and in a very similar way from a plain classical open cover you can form all finite (non-empty) intersections, add them into the cover and build a category from these and the inclusions between them. It will pay to retain that thought for when we launch into discussion of sheaves, and, in particular, stacks, etc.)

It is, thus, important to note that in any simplicial fibre bundle, we have fibres over all simplices, not just the 'vertices'. The 'fibre' over an n-simplex, b, of the base, is given by the pullback

$$
\begin{array}{ccc}
E \times_B \Delta[n] & \longrightarrow & E \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\
\Delta[n] & \xrightarrow[\ulcorner b \urcorner]{} & B,
\end{array}
$$

The usual notion of 'fibre' then corresponds to the case where $n = 0$. We will sometimes write $E(b) = E \times_B \Delta[n]$, since $E \times_B \Delta[n]$ as a notation, does not actually record the b being considered. For instance, given $e \in E_n$, we have the fibre through e will be $E(p(e))$.

Examples of fibre bundles: (i) **Trivial product bundles**:

Lemma 22 *The trivial product bundle, $p_B : Y \times B \to B$, is a fibre bundle in this sense.*

Proof: To see this, we pick an arbitrary, $\ulcorner b \urcorner : \Delta[n] \to B$, and embed it in the commutative diagram:

$$
\begin{array}{ccccc}
Y \times \Delta[n] & \xrightarrow{Y \times \ulcorner b \urcorner} & Y \times B & \longrightarrow & Y \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_B} & & \downarrow \\
\Delta[n] & \xrightarrow[\ulcorner b \urcorner]{} & B & \longrightarrow & \Delta[0],
\end{array}
$$

where the two arrows with codomain $\Delta[0]$ are the unique such maps, (since $\Delta[0]$ is terminal in $\mathcal{S}$). This means that both the right-hand square and the outer rectangle are pullbacks, and then it is an elementary (standard) exercise of category theory to show that the left hand square is also a pullback, which completes the proof. ∎

(ii) Any G-principal fibration is a fibre bundle, since we saw, Lemma 20, that the fibre bundle condition was satisfied. The fibre in this case is the underlying simplicial set of the simplicial group, G.

5.2.2 Atlases of a simplicial fibre bundle

The idea of atlases originally emerged in the theory of manifolds. manifolds are specified by local 'charts' and, of course, a collection of charts makes, yes you guessed, ... here we will see how that idea can be adapted to a simplicial setting.

Let (E, B, p) be a fibre bundle with fibre, Y, then we see that, for any $b \in B_n$, there is an isomorphism

$$\alpha(b) : Y \times \Delta[n] \to E \times_B \Delta[n],$$

given by the diagram:

$$
\begin{array}{ccccc}
Y \times \Delta[n] & \xrightarrow[\cong]{\alpha(b)} & E \times_B \Delta[n] & \xrightarrow{p_1} & E \\
& \searrow{\scriptstyle p_2} & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p} \\
& & \Delta[n] & \xrightarrow{\ulcorner b \urcorner} & B
\end{array}
$$

using the universal property of pullbacks. Set $a(b) : Y \times \Delta[n] \to E$ to be the composite $p_1\alpha(b)$.

Remark: If we think of b as a 'patch' over which (E, B, p) trivialises, then $\alpha(b)$ is the trivialising isomorphism identifying E 'restricted to the patch b' with a product. A face of b may be shared with another n-simplex, so we can expect interactions / transitions between the different descriptions / trivialisations.

Definition: The family $\alpha = \{\alpha(b) \mid b \in B\}$ (or, equivalently, $\mathbf{a} = \{a(b) \mid b \in B\}$) will be called an *atlas* for (E, B, p).

That α determines $\mathbf{a}$ is obvious, but we have also $\alpha(b)(y, \sigma) = (a(b)(y, \sigma), \sigma)$, so $\mathbf{a}$ also determines α. We should also point out that in the definition, we are using $b \in B$ as a convenient shorthand for $b \in \bigsqcup_n B_n$.

It is often useful to think of $\alpha(b)$ as an element of $\underline{\mathcal{S}}(Y, E \times_B \Delta[n])_n$ and $a(b) \in \underline{\mathcal{S}}(Y, E)_n$, since this makes the following idea very clear.

Suppose we consider the automorphism simplicial group, $\mathsf{aut}(Y)$, (cf. page 132) and a subsimplicial group, G, of it. Pick a family $\mathbf{g} = \{g(b) \mid b \in B\}$, of elements of G, where, if $b \in B_n$, $g(b) \in G_n$. There is a new atlas $\alpha \cdot \mathbf{g} = \{\alpha(b)g(b) \mid b \in B\}$ obtained by 'precomposing' with $\mathbf{g}$. (We can also use $\mathbf{a} \cdot \mathbf{g}$ with the obvious definition.)

Definition: Two atlases, α and α', are said to be *G-equivalent* is $\alpha' = \alpha \cdot \mathbf{g}$ for some family, $\mathbf{g}$, of elements from G.

So far, there has been no requirement on the atlas α to respect faces and degeneracies in any way. In fact, we do not really *want* to match faces, since, even in such a simple case as the Möbius

band, strict preservation of faces (something like $a(d_i b) = d_i(a(b))$, perhaps) would not allow the 'twisting' that we would need.) On the other hand, if we have $a(b)$ defined for a non-degenerate simplex, b, then we already have a suitable $a(s_i b)$ around, namely $s_i a(b)$, so why not take that! (You may like to **investigate** this with regard to theuniversal property that we used to define the $\alpha(b)$s.)

Definition: An atlas, **a**, is *normalised* if, for each $b \in B$, $a(s_i b) = s_i a(b)$ in $\underline{\mathcal{S}}(Y, E)$.

Lemma 23 *Given any atlas, **a**, there is a normalised atlas, **a′**, that agrees with **a** on the non-degenerate simplices of B.*   ∎

The proof, which is simply a question of making a definition, then verifying that it works is **left to you**.

Turning to the face maps, as we said, we do not necessarily have $a(d_i b) = d_i a(b)$, but we might expect the two sided to be linked by an automorphism of the fibre, of some type. We know

$$d_i(\alpha(b)) = (Y \times \Delta[n-1] \overset{Y \times \delta_i}{\to} Y \times \Delta[n] \overset{\alpha(b)}{\to} E \times_B \Delta[n],$$

is an isomorphism onto its image. The ith face inclusion $\delta : \Delta[n-1] \to \Delta[n]$ also induces

$$E \times \delta_i : E \times_B \Delta[n-1] \to E \times_B \Delta[n],$$

which we will call θ, and which element-wise is given by $\theta(e, \sigma) = (e, \delta_i \circ \sigma)$, and the image of $\theta \circ \alpha(d_i b)$ is the same as that of $d_i(\alpha(b))$, namely elements of the form $(e, \delta_i \circ \sigma)$. We thus obtain an automorphism, $t_i(b)$, of $Y \times \Delta[n-1]$ with

$$\alpha(d_i b) \circ t_i(b) = d_i(\alpha(b)).$$

('Corestricting' $\alpha(d_i b)$ and $d_i(\alpha(b))$ to that image, we have $t_i(b) = \alpha(d_i b)^{-1} \circ d_i(\alpha(b))$, so $t_i(b)$ is uniquely determined.)

This 'corestriction' argument is reasonably clear as an element based level, but it leaves a lot to check. It is useful to give an equivalent more categorical construction of t, which gets around the verification, for instance, that $t_i(b)$ is a simplicial map - which was 'swept under the carpet' in the above - and is more 'universally valid' as it shows what categorical and simplicial properties are being used.

Let us go back a stage, therefore, and take things apart as 'pullbacks' and in quite some detail. This is initially a bit tedious perhaps, but it is worth doing.

- $\ulcorner d_i b \urcorner$ is the composite

$$\Delta[n-1] \overset{\delta_i}{\to} \Delta[n] \overset{\ulcorner b \urcorner}{\to} B,$$

and so $\alpha(d_i b)$ fits in a diagram:

- We have $\alpha(b) : Y \times \Delta[n] \to E \times_B \Delta[n]$ and want to obtain a restriction of it to the ith face, i.e., to $Y \times \Delta[n-1]$ along $Y \times \delta_i$, *and*, at the same time, that 'corestriction' to $E \times_B \Delta[n-1]$. We want to form the square diagram

$$
\begin{array}{ccc}
Y \times \Delta[n-1] & \xrightarrow{\widetilde{d_i(\alpha(b))}} & E \times_B \Delta[n-1] \\
{\scriptstyle Y \times \delta_i}\Big\downarrow & & \Big\downarrow{\scriptstyle E \times_B \delta_i} \\
Y \times \Delta[n] & \xrightarrow[\cong]{\alpha(b)} & E \times_B \Delta[n],
\end{array}
$$

where the top horizontal arrow, $\widetilde{d_i(\alpha(b))}$, is 'induced from' $\alpha(b)$. We should check how exactly it is built. As it is goinginto an object specified by a pullback, we need only specify its two components, that is, the projections onto E and $\Delta[n-1]$. (Of course, this is exactly what we did in in the element-wise description.) The component going to E is just found by going the other way around the square and folowing that composite by p_1 down to E. The component to $\Delta[n-1]$ is just the projection, p_2. (To see what is going on **draw a diagram yourself**.) We have to verify that the square commutes. This uses the pullback 'uniqueness' clause for $E \times_B \Delta[n]$.

- We note that the corestriction, $\widetilde{d_i(\alpha(b))}$, is a monomorphism, as its composite with $E \times_B \delta_i$ is one. We claim it is an isomorphism. It remains to show, for instance, that it is a split epimorphism. (That is relative easy to try, so is a good place to attack what is needed.)

First note that

$$
\begin{array}{ccc}
Y \times \Delta[n-1] & \xrightarrow{Y \times \delta_i} & Y \times \Delta[n] \\
\Big\downarrow & & \Big\downarrow \\
\Delta[n-1] & \xrightarrow[\delta_i]{} & \Delta[n]
\end{array}
$$

is a pullback, as is also

$$
\begin{array}{ccc}
E \times_B \Delta[n-1] & \xrightarrow{E \times_B \delta_i} & E \times_B \Delta[n] \\
\Big\downarrow & & \Big\downarrow \\
\Delta[n-1] & \xrightarrow[\delta_i]{} & \Delta[n].
\end{array}
$$

(In each case, you can put an obvious pullback square to the right, so that the composite 'rectangle' is again a pullback - that same argument again.) We build the inverse to $\tilde{d} :=$ $\widetilde{d_i(\alpha(b))}$, using the first of these two squares. The component of that inverse going to $\Delta[n-1]$ is the obvious one, whilst to $Y \times \Delta[n]$, we use $\alpha(b)$. (You are **left to check commutativity**.) To check then that this map we have constructed, does split $\tilde{d}$, we use the uniqueness clause for the second of these pullbacks.

The final step in proving that $\tilde{d}$ is an isomorphism is the 'usual' proof that if a morphism is both a monomorphism and a split epimorphism then the splitting is, in fact, the inverse for the original monomorphism (which is thus an isomorphism). (**If you have not seen this before**, first **check** the categorical meaning of monomorphism, then **work out a proof** of the fact.)

We, therefore, have

$$Y \times \Delta[n-1] \xrightarrow[\cong]{\alpha(d_ib)} E \times_B \Delta[n-1]$$

and

$$Y \times \Delta[n-1] \xrightarrow[\cong]{\tilde{d}} E \times_B \Delta[n-1] \ ,$$

both over $\Delta[n-1]$, as you **easily check** from the above. We thus get

$$t_i(b) = \alpha(d_ib)^{-1}.\tilde{d},$$

and this is in $\mathsf{aut}(Y)_{n-1}$. We note that these elements are completely determined by the normalised atlas.

   Definition: The automorphisms, $t_i(b)$, for $b \in B$ are called the *transition elements* of the atlas, α.
   If the transition elements all lie in a subgroup, G of $\mathsf{aut}(Y)$, then we say α, (or, equivalently, $\mathbf{a}$), is a *G-atlas*.
   An atlas, α, is *regular* if, for $i > 0$, its transition elements, $t_i(b)$, are all identities.

   We thus have that, in a regular normalised atlas, we just need to specify the $t_0(b)$, as these may be non-trivial. (To see where this theory is going at this point, you may find it helps to think $t =$ 'twisting', as well as, $t =$ 'transition', and to look back at our discussion of T.C.P.s (section 4.5, page 141).)

Lemma 24 *Every (normalised) G-atlas is G-equivalent to a (normalised) regular G-atlas.*

Proof: We start with a G-atlas, which we will assume normalised. (The unnormalised case is more or less identical.) We will use it in the form $\mathbf{a}$, rather than α, but, of course, this really makes no difference. We will build, by induction, a G-equivalent regular one, $\mathbf{a}'$.
   On vertices, we take $a'(b) = a(b)$. That gets us going, so we now assume $a'(b)$ is defined for all simplices of dimension less than n, and that $\mathbf{a}'$ is regular and G-equivalent to $\mathbf{a}$, to the extent that this makes sense. We next want to define $a'(b)$ for b, a (non-degenerate) n-simplex. (The degenerate ones are handled by the normalisation condition.)
   We look at the $(n,0)$-horn in B corresponding to b, i.e., made up of all the d_ib for $i \neq 0$. We have elements $g_i(b)$ such that

$$a'(d_ib) = a(d_ib)g_i(b),$$

since $\mathbf{a}'$ is G-equivalent to $\mathbf{a}$ in this dimension, then, using

$$a(d_ib)t_i(b) = d_i(a(b)),$$

we get $a'(d_ib) = d_i(a(b)).t_i(b)^{-1}.g_i(b) = d_i(a(b)).h_i$, where we have set $h_i = t_i(b)^{-1}.g_i(b)$. Since $\mathbf{a}'$, so far defined is regular, we have, for $0 < i \leq j$, after a bit of simplicial identity work (**for you**), that

$$d_id_j(a(b))d_ih_j = d_id_j(a(b))d_{j-1}h_i,$$

which implies that $d_i h_j = d_{j-1} h_i$, the the hs form a $(n, 0)$-horn in G. we now wheel out our method for filling horns in G to get a $h \in G_n$ with $d_i h = h_i$, for $i > 0$, and we set $a'(b) = a(b)h$. we heck

$$
\begin{aligned}
d_i a'(b) &= d_i a(b))d_i h \\
&= d_i a(b)h_i \\
&= a'(d_i b).
\end{aligned}
$$

The resulting $\mathbf{a}'$, is now defined up to and including dimension n, is normalised and regular, and G-equivalent to $\mathbf{a}$. We get this in all dimensions by induction. ∎

5.2.3 Fibre bundles are T.C.P.s

We saw earlier that G-principal fibrations were locally trivial and hence are fibre bundles, and that twisted Cartesian products (T.C.Ps) are principal fibrations. We now have regular atlases, yielding structures that look like twisting functions. This suggests that the various ideas are really 'the same'. We will not comlete all the details that show that they are, since that theory is in various texts (for instance, May's book, [119]), but will more-or-less complete our *sketch* of the interrelationships.

There remains, for our sketch, an investigation of the transition elements for simplicial fibre bundles and a 'sketch proof' that fibre bundles are just T.C.Ps.

Suppose we have some simplicial fibre bundle and a normalised regular G-atlas, $\mathbf{a} = \{a(b) \mid b \in B\}$, giving as the only possibly non-trivia transition elements, the $t(b) := t_0(b)$. We thus have

$$
d_0 a(b) = a(d_0 b).t(b).
$$

(To avoid looking back all the time to the definition of twisting function, we repeat it here for convenience and also to adjust conventions. We had:

a function, t, satisfying the following equations will be called a *twisting function*:

$$
\begin{aligned}
d_i t(b) &= t(d_{i-1} b) \quad \text{for} \quad i > 0, \\
d_0 t(b) &= t(d_0 b)^{-1} t(d_1 b), \\
s_i t(b) &= t(s_{i+1} b) \quad \text{for} \quad i \geq 0, \\
t(s_0 b) &= *.
\end{aligned}
$$

(**Warning:** The version on page 142 corresponded to the 'algebraic' diagrammatic composition order, and here we have used the 'Leibniz' composition order so we have adjusted the second equation accordingly.)

Lemma 25 *The transition elements, $t(b)$, above, define a twisting function.*

Proof: We use the defining equation (above) for the $t(b)$ and, in particular, the uniqueness of these elements with this property, (together with the 'regular' and 'normalised' conditions for $\mathbf{a}$). We leave the majority of the cases **to you**, since conce you have seen one or two of these, the others are easy.

(We wil do a very easy one as a 'warm up', then the important, and more tricky, one relating toe d_0 and d_1, i.e., the twist.)

Applying the equation above to $s_0 b$, we get

$$d_0 a(s_0 b) = a(d_0 s_0 b).t(s_0 b) = a(b).t(s_0 b),$$

but **a** is normalised, so $a(s_0 b) = s_0(b)$ and the left hand side is thus just $a(b)$. we can thus conclude that $t(s_0 b)$ is the identity. (That was easy!)

We now turn to the relation involving $t(d_0 b)$ and $t(d_1 b)$, etc.:

$$d_0 a(d_1 b) = a(d_0 d_1 b).t(d_1 b),$$

but we also have

$$d_0 a(d_0 b) = a(d_0 d_0 b).t(d_0 b),$$

and, of course, $d_0 d_1 b) = d_0 d_0 b$.

We next apply d_0 to the 'master equation', simply giving

$$d_0 d_0 a(b) = d_0 a(d_0 b).d_0 t(b),$$

and to $d_1 a(b) = a(d_1 b)$ to get

$$d_0 d_1 a(b) = d_1 a(d_1 b).$$

Again using the simplicial identity $d_0 d_1 = d_0 d_0$, we rearrange terms algebraically to get

$$d_0 t(b) = t(d_0 b)^{-1} t(d_1 b),$$

as expected.

The other equations are **left to you**. (You just mix applying a d_i or s_i to the 'master equation' inside (i.e, on b) and outside, then use normalisation, regularity and the simplicial identities.) ∎

It is thus possible to use E to find **a** and thus t, and thence to form $B \times_t Y$. We need now to compare $B \times_t Y$ with E.

To start with we will do something that looks as if it is 'cheating'. We have, for $b \in B_n$ that $a(b) \in \underline{\mathcal{S}}(Y, E)$, so *do* have a *graded* map

$$\mathbf{a} : B \to \underline{\mathcal{S}}(Y, E).$$

Our assumptions about **a** being regular, normalised, etc., imply that this is very nearly a simplicial map. (The only thing that goes wrong is the d_0-face compatibility.)

If **a** was simplicial, we could 'fli[it' through the adjunction to get $\xi : B \times Y \to E$. We know how to do this. We form the composite

$$B \times Y \xrightarrow{\mathbf{a} \times Y} \underline{\mathcal{S}}(Y, E) \times Y \xrightarrow{eval} E,$$

where *eval* is the map we met earlier (page 137), and which, as you will recall, we worked hard to get a complete description of. For $y \in Y_n$, and $f : Y \times \Delta[n] \to E \in \underline{\mathcal{S}}(Y, E)_n$, we had that

$$eval(f, y) = f(y, \iota_n),$$

where, as always, ι_n is the unique non-degenerate n-simplex in $\Delta[n]$, corresponding to the identity map on $[n]$ in the description $\Delta[n] = \mathbf{\Delta}(-, [n])$. We can pretend that **a** is simplicial, see what ξ is given by and then see how much it is or is not simplicial. We can read off, if $y \in Y_n$ and $b \in B_n$,

$$\xi(b, y) = a(b)(y, \iota_n).$$

This map ξ is 'as simplicial as is $\mathbf{a}$'. We will check this, or part of it, by hand, but although it follows from generalities on the adjunction process, verifying the conditions needs care.

First we note that if $f : Y \times \Delta[n] \to E$, then $d_i f = f \circ (Y \times \Delta[\delta - i])$, where $\delta_i : [n-1] \to [n]$ is the ith face inclusion (so we get $\Delta[\delta_i] : \Delta[n-1] \to \Delta[n]$). We examine the evaluation map in detail as it is the key to the calculation. By its construction, it is bound to be simplicial, but we need also to see what that means at this 'elementary' level. We have

$$\begin{array}{ccc}
\underline{\mathcal{S}}(Y,E)_n \times Y_n & \xrightarrow{\;eval\;} & E_n \\
{\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i} \\
\underline{\mathcal{S}}(Y,E)_{n-1} \times Y_{n-1} & \xrightarrow{\;eval\;} & E_{n-1}
\end{array}$$

and, for $i > 0$,

$$\begin{aligned}
d_i \xi(b,y) &= d_i(a(b)(y,\iota_n) = eval(d_i a(b), d_i y) \\
&= eval(a(d_i b), d_i y) = \xi(d_i b, d_i y) = \xi d_i(b, y).
\end{aligned}$$

Similarly, we have, for s_i that $s_i \xi = \xi s_i$. That just leaves $d_0 \xi$ and, of course

$$d_0 \xi(b, y) = eval(a(d_0 b).t(b), d_0 y),$$

by the same sort of argument, and then this is $a(d_0(b))(t(b)d_0 y, \iota_{n-1}) = \xi(d_0 b, t(b) d_0 y)$. (You may want to check this last bit for yourself. You need to translate to-and-fro between a G-actions on Y as being $a : G \times Y \to Y$ and the adjoint $a : G \to \mathsf{aut}(Y)$, again using $eval$.)

This gives us that, if we *define* a new d_0 on this product by *twisting* it using t (and, of course, this is just giving us $B \times_t Y$ as we have already seen it, on page 141) with, explicitly,

$$d_0(b, y) = (d_0 b, t(b)(d_0 y)),$$

then we actually obtain

$$\xi : B \times_t Y \to E$$

as a simplicial map. We note that $p\xi = p_B$, the projection onto B of the T.C.P., so ξ is 'over B'.

Proposition 40 *This map ξ is an isomorphism (over B).*

Proof: We start by constructing, for each $b \in B_n$, a map $\nu(b) : E(b) \to Y$, where, as before, $E(b) = E \times_B \Delta[n]$, the pullback of E along $\ulcorner b \urcorner$, so is the 'fibre over b'. We have $\alpha(b) : Y \times \Delta[n] \to E(b)$ is an isomorphism, and so we can form $\nu(b) := pr_Y \alpha(b)^{-1} : E(b) \to Y$. Using this we send and n-simplex e to $(p(e), \nu(p(e))(e, \iota_n))$, where $(e, \iota_n) \in E(p(e))$ This gives us something in $B \times_t Y$ and ξ is then easily seen to send that n-simplex back to e. That the other composite is the identity ia also easy (for **you to check**). ∎

We thus have a pretty full picture of how principal fibrations are principal fibre bundles, given by twisted Cartesian products of a particular type, that principal H-fibre bundles are classified by $\overline{W}(H)$, since $Princ_H(B) \cong [B, \overline{W}(H)]$, that general fibre bundles in the simplicial context are T.C.P.s and so correspond to a principal bundle and a representation of the corresponding group, and probably some other things as well. As these have been spread over different chapters, since we wanted to make use of the ideas as we went along, **you may find it helpful** to now read one of the texts, such as [119] or the survey, [56], that give the whole theory in one go. We will periodically be recalling part of this, making comparisons with other ideas and methods, and possibly pushing this theory on new directions (as this is 'classical').

5.2.4   . . . and descent in all that?

In earlier sections, we looked at descent in a topological context. There we used an open cover, $\mathcal{U}$, of the base space and had transitions, $\xi_{U,U'}$, on intersections of these open patches, with a condition on triple intersections. The idea was to take the A_U for the various open sets, U, of the cover $\mathcal{U}$, and to glue them together, using the $\xi_{U,U'}$ to get the right amount of 'twisting' from patch to patch, with the cocycle condition to ensure the different gluings are compatible.

That somehow looks initially very different from what we have been doing in our discussion of simplicial fibre bundles. We would not expect to have 'open sets', but what takes their place in the simplicial context. We will look at this only briefly, but from several directions. The ideas that we would use for a full treatment will be studied in more depth in the following chapters. This therefore is a 'once over lightly' treatment of just a few of the ideas and insights. The ideas will be recalled, and treated in some depth in later chapters, but not always from the same perspective.

We start by looking at the open cover from a simplicial viewpoint.

Definition: The *Čech complex*, *Čech nerve* or simply, *nerve*, of the open covering, $\mathcal{U}$, is the simplicial complex, $N(\mathcal{U})$, specified by:

- Vertex set : the collection of open sets in $\mathcal{U} = \{U_a \mid a \in A\}$ (alternatively, the set, A, of labels or indices of $\mathcal{U}$);

- Simplices : the set of vertices, $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p \rangle$, belongs to $N(\mathcal{U})$ if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection.

As usual, if we choose an order on the indexing set, i.e., the set of vertices of $N(\mathcal{U})$, then we can construct a neat simplicial set out of this, so the $\langle U_0, U_1 \rangle \in N(\mathcal{U})_1$ means $U_0 \cap U_1 \neq \emptyset$ and U_0 is listed before U_1 in the chosen order. (We could, of course, not bother about the order and just consider all possible simplices. For instance, $\langle U_0, U_0, U_1 \rangle$ woud be $s_0 \langle U_0, U_1 \rangle$, but apparently the same simplex, $\langle U_1, U_0, U_0 \rangle = s_1 \langle U_1, U_0 \rangle$, will also be there. This gives a larger simplicial set, but does have the advantage of being constructed without involving an order. **You are left to investigate** if this second construction gives something really different from the other. It is larger, but does it retract to the other form, for instance.)

(For simplicity of exposition, we will assume local triviality, so $A_U = U \times F$, for some 'fibre' F.) Looking at our transition functions, $\xi_{U,U'}$, they assign elements of the group, G, which acts on F, to these 1-simplices, $\langle U, U' \rangle$. (We assume G is a discrete group, not one of the more complex topological groups that also occur in this context.) Taking the group, G, we can form the constant simplicial group $K(G, 0)$, which has G in all dimensions and identity maps for all face and degeneracy morphism. This, then, gives a simplicial map from $N(\mathcal{U})$ to $\overline{W}K(G, 0)$. (You can check this if you wish, *but* we will be looking at it in great detail later on anyway.) We thus get a twisted Cartesian product $N(\mathcal{U}) \times_t K(G, 0)$. That gives us one way of seeing simplicial fibre bundles as being generalisations of the topological ones. They replace a very simple constant simplicial group by an arbitrary one, so have 'higher order transitions' acting as well. Untangling the complex intuitions and interpretations of this simple idea will be one of the themes from now on, not constantly 'up front', but quietly increasing in importance as we go further.

Another way of thinking of descent data is as 'building plans' for the fibre bundle given the bits, $A_U \cong U \times F$. We took the disjoint union, $\sqcup_U A_U$, then 'quotiented' by the gluing instructions encoded in the descent data, (see section 5.1.1). This is a fairly typical simple example of a colimit construction. We will study the categorical notion of colimit (and limit) later in some detail and

will use it, and generalisations, many times. (These notes are intended to be reasonably accessible to people who have not had much formal contact with the theory of categories, although some basic knowledge of terminology is assumed as has been mentioned several times already. If you have not met 'colimits' formally, then **do** look up the definition. It may initially not 'mean' much to you, but it will help if you have some intuition. Something like: colimits are 'gluing' processes. You form a 'disjoint union' (coproduct), putting pieces out ready for use in the construction, then 'divide out' by an equivalence relation given, or at least, generated, by some maps between the different pieces.) We will see, more formally, the way that topological descent fits into this colimit / gluing intuition later on, but it is clearly also here in this simplicial context.

We have our basic pieces, $Y \times \Delta[n]$, and we glue them together using the 'combinatorial' information encoded in the simplicial set B. One way to view that is by using a neat construction of a category from a simplicial set.

Suppose we have a simplicial set, B. then we can form a small category $Cat(B)$ (also denoted (Yon, B), as it is an example of a *comma category*). This has as its set of objects the simplices, b, of B, or, more usefully, their representing maps, such as $\ulcorner b \urcorner : \Delta[n] \to B$. If $\ulcorner b \urcorner$ and $\ulcorner c \urcorner : \Delta[m] \to B$ are two such, not necessarily of the same dimension, then a morphism in $Cat(B)$ from $\ulcorner b \urcorner$ to $\ulcorner c \urcorner$ 'is' a diagram:

$$\Delta[n] \xrightarrow{\Delta[\mu]} \Delta[m]$$
$$\ulcorner b \urcorner \searrow \quad \swarrow \ulcorner c \urcorner$$
$$B$$

i.e., $\mu : [n] \to [m]$ is a morphism in $\mathbf{\Delta}$, so is a 'monotone map' which induces $\Delta[\mu]$ as shown. Saying that the diagram commutes says, of course, that $\ulcorner b \urcorner = \ulcorner c \urcorner \circ \Delta[\mu]$. Again, of course, $b \in B_n$ and $c \in B_m$ and μ induces a map $B_\mu : B_m \to B_n$. The obvious relationship corresponding to 'commutative' is that $B_\mu(c) = b$ and this holds. (You can take this, in the definition of morphism, to replace commutativity of the triangle as it is equivalent, then it comes out as saying 'a morphism $\mu : \ulcorner b \urcorner \to \ulcorner c \urcorner$ is a $\mu : [n] \to [m]$ such that $B_\mu(c) = b$, but it is very worth while checking through the above at a categorical level as well.)

If now you look back at our discussion of the reconstruction of (E, B, p) from the various patches, $Y \times \Delta[n]$, which corresponded to an n-simplex b in B, the process of gluing these together is completely analogous to our earlier discussion. It is again a 'colimit'. (You may, quite rightly ask, 'how come we get a twisted Cartesian *product* from a disjoint union type construction?' This is neat - and, of course, you may have seen it before. Looking just at sets A and B, if we form $A \times B$, then $A \times B = \coprod\{\{a\} \times B \mid a \in A\}$, so we can write a product as a disjoint union of (identical) labelled copies of the second set, each indexed by an element of the first one. (First and second here are really interchangeable of course.) We will see this type of construction several times later on. For instance if G is a simplicial groupoid and K is a simplicial set, we can form a new simplicial groupoid $K \otimes G$ with $(K \otimes G)_n$ being a disjoint union (coproduct) of copies of G_n indexed by the n-simplices of K. We will see this in detail later on, so this mention is 'in passing', but it is hopefully suggestive as to the sort of viewpoint we can use and adapt later.

The structure of simplicial fibre bundles is thus closely linked to the same intuitions and techniques used in the topological case. We now turn to sheaves, and will see those same ideas coming out again, with of course, their own flavour in the new context.

Chapter 6

Hypercohomology and exact sequences

6.1 Hyper-cohomology

6.1.1 Classical Hyper-cohomology.

We have several times mentioned this subject and so should provide some slight introduction to the basic ideas. We will go right back to basics, even though we have already used some of the ideas previously, usually without comment. Most of this first part may be well known to you.

The basic idea is that of a graded, or more precisely $\mathbb{Z}$-graded, group and variants such as graded vector spaces, or graded modules, or sheaves of such on some space, B or in some topos $\mathcal{E}$.

Definition (First form): A $\mathbb{Z}$-*graded vector space* (gvs) is vector space together with a direct sum decomposition, $\mathsf{V} = \bigoplus_{p \in \mathbb{Z}} V_p$. The elements of V_p are said to be *homogeneous of degree p*. If $x \in V_p$, write $|x| = p$.

A graded vector space could equally well be defined as a family $\{V_i\}_{i \in \mathbb{Z}}$ of vector spaces, since we could then form their direct sum and obtain the first version.

Definition (Second form): A $\mathbb{Z}$-*graded vector space* (gvs) is a $\mathbb{Z}$-indexed family, $\{V_i\}_{i \in \mathbb{Z}}$, of vector spaces.

(The definitions are, pedantically, not completely equivalent as one can have a constant family with all V_i equal, but that is really a smokescreen and causes no problem.)

Both versions are useful. For example, if K is a simplicial set, we can define a graded vector space using the second version by taking V_n to be the vector space with basis indexed by the elements of K_n if $n \geq 0$ and to be the trivial vector space if $n < 0$. From our treatment of simplicial sets, it would be somewhat artificial to define $\mathsf{V} = \bigoplus_{i \in \mathbb{Z}} V_i$. For another example, the other description fits better. The polynomial ring, $\mathbb{R}[x]$, is a graded vector space with V_n having basis $\{x^n\}$, i.e., V_n is the subspace of degree n monomials over $\mathbb{R}$. The whole space, $\mathbb{R}[x]$, is here by far the more natural object.

For graded groups, etc., just substitute 'group' etc. for 'vector space' and correspondingly, 'direct product' for 'direct sum'.

Definition: A morphism $f : \mathsf{V} \to \mathsf{W}$ of graded vector spaces is *homogeneous* if $f(V_p) \subseteq W_{p+q}$ for all p and some common q, called the *degree of f*. The set of such morphisms of given degree is $Hom(\mathsf{V}, \mathsf{W})_q = \prod_p Hom(V_p, W_{p+q})$.

An endomorphism, $d : \mathsf{V} \to \mathsf{V}$, of degree -1 is called a *differential* or *boundary* (which depending largely on the context) if $d \circ d = 0$.

A gvs with a differential is really just a chain complex, where $d_n : V_n \to V_{n-1}$ and $d_{n-1}d_n = 0$.

Definition: A graded vector space together with a differential is variously called a *differential graded vector space* (dgvs), or a *chain complex*. Some authors reserve that latter term for a positively graded differential vector space, or module, or .... . The elements of V_n are called *n-chains*, those of $Ker\, d_n$, *n-cycles*, and those of $Im\, d_{n+1}$, *n-boundaries*.

A graded vector space V is *positively graded* if $V_i = 0$ for all $i < 0$. It is, on the other hand, *negatively graded* if $V_i = 0$ for $i > 0$.

The classical convention is to write V^{-n} instead of V_n for all n in the negatively graded case. This, of course, has the effect that if (V, d) is a differential graded vector space which is negatively graded, then d has apparent degree $+ 1$, $d^n : V^n \to V^{n+1}$. In the usual terminology that will give a *cochain complex*. For some purposes, it is usual to adapt the terminology somewhat, for instance to use chain complex as a synonym for dgvs without mention of positive or negative, but then also to use cochain complex for what is essentially the same type of object, but with 'upper index' notation, so $\mathsf{V} = (V^n, d^n)$ with $d^n : V^n \to V^{n+1}$. Terms such as 'bounded above', 'bounded below' or simply 'bounded' are also current where they correspond respectively to $V_n = 0$ for large positive n, or large negative n or both. We will make little use, if any, of these in the context of these notes, but it is a good thing to be aware of the existence of the various conventions and to check before assuming that a given source uses exactly the same one as that which you are used to!

For simplicity of exposition, we will initially concentrate our attention on general dgvs, which we will often call *chain complexes* and will attempt to be reasonably consistent - although that is virtually impossible! We will extend that terminology to dg-modules and dg-groups if and when needed.

- The elements of a chain complex are called *chains*. If $c \in C_n$, it is an *n-chain*. If $dc_n = 0$, it is called an *n-cycle* and, if $c \in Im\, d_{n+1}$, an *n-boundary*. If 'n' is not important, or is understood, it may be omitted.

- A *chain map* $f : \mathsf{V} \to \mathsf{W}$ of chain complexes is a graded map of degree 0, $\{f_n : V_n \to W_n\}$ compatible with the differentials, so, for all n,

$$d_n^W f_n = f_{n-1} d_n^V,$$

and, of course, we will drop the V and W superfixes whenever possible. The category of differential vector spaces and chain maps will be variously denoted dgvs, or Ch_k with variants $\mathsf{dgk - mod}$, $\mathsf{dgk - mod}_{\geq 0}$, Ch_k^+ and so on, denoting the k-module version, a positively graded

variant, and an alternative notation. (These, and other, notations are all used in the literature with the precise convention usually evident from the context. To some extent the choice, say of dgvs as against Ch is determined by the use intended, but this is not completely consistent.)

- A *chain homotopy* between two chain maps $f, g : \mathsf{V} \to \mathsf{W}$ is a graded map of degree 1, $s : \mathsf{V} \to \mathsf{W}$ such that
$$f_n - g_n = d_{n+1}s_n + s_{n-1}d_n.$$

- The *homology* of a chain complex, $\mathsf{V} = (V, d)$, is the graded object
$$H_n(\mathsf{V}) = \frac{Ker\, d_n}{Im\, d_{n+1}}.$$

If we are using upper indices, for whatever reason, the more usual term will be 'cohomology',
$$H^n(\mathsf{V}^*) = \frac{Ker(d^n : V^n \to V^{n+1})}{Im(d^{n-1} : V^{n-1} \to V^n)}.$$

This most often occurs in the situation where C is a chain complex and A is a vector space / module or similar, then we form $Hom(\mathsf{C}, A)$, by applying the functor $Hom(-, A)$ to C. Of course, $d_n : C_n \to C_{n-1}$ induces a differential
$$Hom(C_{n-1}, A) \to Hom(C_n, A)$$

and the elements of $Hom(C_n, A)$ are called *cochains*, with *cocycles*, and *coboundaries* as the corresponding elements of kernels and images. The notation $Hom(\mathsf{C}, A)^n$ is used for the object $Hom(C_{-n}, A)$, so this 'dual' has negative grading if C has positive grading, and is given upper indexing. The homology of $Hom(\mathsf{C}, A)$ is then called the *cohomology of C with coefficients in A*. (We will try to follow usual terminology as given in standard homological algebra texts, e.g. the classic [114].)

- More generally, if C and D are both chain complexes (of modules), then we can form the graded Abelian group, $Hom(\mathsf{C}, \mathsf{D})$, with $Hom(\mathsf{C}, \mathsf{D})_n$ being the Abelian group of graded maps of degree n from C to D. This means, of course,
$$Hom(\mathsf{C}, \mathsf{D})_n = \prod_{p=-\infty}^{\infty} Hom(C_p, D_{p+n}),$$

as before.

We make this into a chain complex by specifying, for $f \in Hom(\mathsf{C}, \mathsf{D})_n$, its 'boundary' ∂f by, if $c \in C_p$,
$$(\partial f)_p c = \partial^{\mathsf{D}}(f_p c) + (-1)^{n+1} f_{p-1}(\partial^{\mathsf{C}} c).$$

(In the event that you have not seen this before, check that (i) $\partial\partial = 0$, (ii) if f is of degree 0, then it is a chain map if and only if $\partial f = 0$ and (iii) a chain homotopy, s, between two chain maps, $f, g \in Hom(\mathsf{C}, \mathsf{D})_0$, is precisely an $s \in Hom(\mathsf{C}, \mathsf{D})_1$ with $\partial s = f - g$.)

The homology of $Hom(\mathsf{C}, \mathsf{D})$ is called the *hyper-cohomology of C with coefficients in D*. The case where $D_0 = A$ and $D_n = 0$ if $n \neq 0$ is the cohomology we saw earlier. In general, $H^0(Hom(\mathsf{C}, \mathsf{D}))$, i.e., chain maps modulo coboundaries, is just the group of chain homotopy

classes of chain maps by (ii) and (iii) above. As is usual in homological (and homotopical) algebra, we usually need good conditions on C and D to get really good invariants from this construction - typically C needs to be 'projective' or D 'injective', or C needs to be 'fibrant' or D 'cofibrant'. Our use of this will be somewhat hidden by the situations we will be considering.

6.1.2   Čech hyper-cohomology

The main type of application for us will be the 'hyper'-version of Čech cohomology. In this, or at least in its simplest form, we have a space, X, and we form the colimit over the open covers, $\mathcal{U}$, of X of the hyper-cohomology groups $H^n(C(\mathcal{U}), \mathsf{D})$. In more detail:

The classical Čech cohomology of X with coefficients in a sheaf of R-modules, A, is defined via open covers $\mathcal{U}$ of X. If $\mathcal{U}$ is an open cover of X, then we form the chain complex, $C(\mathcal{U})$, by taking $N(\mathcal{U})$, the nerve of $\mathcal{U}$, and letting $C(\mathcal{U})_n$ be the sheaf of free R-modules generated by $N(\mathcal{U})_n$ with $\partial = \sum_{k=0}^n (-1)^k d_k$ being the differential. This can either be thought of as a complex of (sheaves of) R-modules or in the straight forward module version. We take coefficients in another sheaf of R-modules, A, and form $H^n(C(\mathcal{U}), A)$.

If $\mathcal{V}$ is a finer cover than $\mathcal{U}$, there is a chain map from $C(\mathcal{V})$ to $C(\mathcal{U})$. Recall if $\mathcal{V} < \mathcal{U}$, for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ with $V \subseteq U$, and $(x, V_0, \ldots, V_n) \in N(\mathcal{V})_n$, we can map it to a corresponding $(x, U_0, \ldots, U_n) \in N(\mathcal{U})_n$ with each $V_i \subseteq U_i$. This is not well defined as several Us might work for a particular V, so the construction of the chain map involves a choice, however it does induce, firstly, a chain map from $C(\mathcal{V})$ to $C(\mathcal{U})$, which is determined up to (coherent) homotopy and thus a *well defined* map on cohomology, $H^*(C(\mathcal{U}), A) \to H^*(C(\mathcal{V}), A)$.

The Čech cohomology, $\check{H}^*(X, A) = colim_{\mathcal{U}} H^*(C(\mathcal{U}), A)$, was the first, historically, of the sheaf type cohomologies. Others apply to a topos rather than merely a space. The obvious hyper-variant of this replaces A by a sheaf of chain complexes (of whatever variety you like, provided they are 'Abelian'), so $H^n(C(\mathcal{U}), \mathsf{D}) = H^n(Hom(C(\mathcal{U}), \mathsf{D}))$ and then $\check{H}^*(X, \mathsf{D}) = colim_{\mathcal{U}} H^*(C(\mathcal{U}), \mathsf{D})$.

We should 'deconstruct' this a bit to see why it is relevant to us.

To simplify our lives no end, we will assume D is a presheaf of chain complexes of R-modules which is positive, ($D_n = 0$ if $n < 0$). By the method of construction of colimits of modules, etc., we can find for any element of $\check{H}^*(X, \mathsf{D})$, an open cover $\mathcal{U}$ of X and a representing element in $H^*(C(\mathcal{U}), \mathsf{D})$. We can thus, further, find a representing n-cocycle from $C(\mathcal{U})$ to D, i.e., an element in $\prod_p Hom(C(\mathcal{U})_p, D_{n+p})$.

To simplify still further, we look at low values of n:

- for $n = 0$, we have some $\mathbf{f} = \{f_p : C(\mathcal{U})_p \to D_p\}$, which satisfies $\partial \mathbf{f} = 0$, so $\mathbf{f}$ forms a chain map. In some of our most interesting cases, D is usually very short, e.g. $D_n = 0$ if $n > 1$, so $\mathsf{D} = (D_1 \to D_0)$ with zeroes elsewhere in other dimensions. Then the only f_ps that contribute to $\mathbf{f}$ are f_0 and f_1. Over an open set, U_i, of the cover, f_0 will be a local section, $f_{0,i}$, of D_0, since 0-simplices of $N(\mathcal{U})$ have form (x, U_i) over $x \in U_i$. Similiarly 1-simplices are, of course, represented by (x, U_i, U_j) with $x \in U_{ij}$, so f_1 corresponds to local sections $f_{1,ij} : U_{ij} \to D_1$. The boundary in $C(\mathcal{U})$ of (x, U_i, U_j) is $(x, U_j) - (x, U_i)$, so

$$d^{\mathsf{D}} f_{1,ij} = f_{0,j}(x) - f_{0,i}(x),$$

  or

$$f_{0,j}(x) = d^{\mathsf{D}} f_{1,ij} + f_{0,i}(x).$$

If we look at the non-Abelian analogue of this, it gives

$$f_{0,j}(x) = d^{\mathsf{D}} f_{1,ij}.f_{0,i}(x),$$

which 'is' the equation $p_j = \partial(c_{ij})p_i$. (You could explore the cases where D is slightly longer, or what about a non-Abelian version?)

- for $n = 1$, we expect to find a formula corresponding to the coboundaries that we met on 'changing the local sections' for M-torsors. If h, (yes, 'h' as in 'homotopy') is a degree 1 map in $Hom(C(\mathcal{U}), \mathsf{D})$ and D has length 1 as above, the only case that contributes is $h_0 : C(\mathcal{U})_0 \to D_1$ and hence $h_{0,i} : U_i \to D_1$. You are **left to check** that this does give (the Abelian version of) the coboundary / chain homotopy formula.

6.1.3 Non-Abelian Čech hyper-cohomology.

The idea should be fairly obvious in its general form. We replace our overall structural viewpoint of chain complexes or sheaves of such, by our favorite non-Abelian analogue. For instance, we could take D to be a sheaf of simplicial groups, or crossed complexes, or n-truncated simplicial groups or . . . . These would really include sheaves of 2-crossed modules and clearly we might try sheaves of 2-crossed complexes, and so on. Some of these classes of coefficient are very likely to turn out to be useful in the future if recent developments in algebraic and differential geometry are any indication. We cannot consider all of them here. The first is the easiest to deal with and to some extent includes the others. It is not structurally the neatest, but ... .

If D is a sheaf of simplicial groups, then we might be tempted to replace $C(\mathcal{U})$ by the free simplicial group sheaf on $N(\mathcal{U})$. It is very important to note that this is NOT the same as $\mathcal{G}(N(\mathcal{U}))$ and we should pause to consider this point.

Let K be a simplicial set and G a simplicial group. The set of simplicial maps from K to the underlying simplicial set of G is isomorphic to $Simp.Grps(FK, G)$ by the standard adjunction between the free group functor, F, and the forgetful functor, U from $Grps$ to $Sets$. Complications might seem to arise if one tries to work with $\underline{\mathcal{S}}(K, UG)$ and $\underline{Simp.Grps}(FK, G)$, as initially it needs to be noted that $\underline{\mathcal{S}}(K, UG) = \mathcal{S}(K \times \Delta[n], UG)$ and one has to think of the relationship between $F(K \times \Delta[n])$ and $F(K) \otimes \Delta[n]$, the latter in the sense of our earlier discussion of tensoring in simplicially enriched categories, page **??**. (This problem is, in fact, not really there, as although F does not preserve products, the product $K \times \Delta[n]$ is actually being thought of, and constructed, as a *colimit* and F, as a left adjoint, behaves nicely with respect to such.) We will not explore that further here and will, in fact, stick with $\underline{\mathcal{S}}(N(\mathcal{U}), \mathsf{D})$ rather than use F. (Note that by a useful result of Milnor, FK and $\mathcal{G}SK$ are isomorphic for a reduced simplicial set K, where S is the reduced suspension; see [56] and the paper, [122], which can be found in Adams, [2].) The relationship between $\underline{\mathcal{S}}(K, UG)$ and other related constructions such as $\underline{\mathcal{S}}(K, \overline{W}G) \cong \underline{\mathcal{S}-Grpds}(\mathcal{G}(K), G)$, is given by the induced fibration sequence,

$$\underline{\mathcal{S}}(K, UG) \to \underline{\mathcal{S}}(K, WG) \to \underline{\mathcal{S}}(K, \overline{W}G),$$

coming from the fibration,

$$UG \to WG \to \overline{W}G.$$

If we work within our favourite topos $\mathcal{E}$, or with bundles over B, this still holds true. It is also the case that WG is (naturally) contractible.

Back with hyper-cohomology, let D be a sheaf of simplicial groups and form $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(D))$. We put forward the homotopy groups of this simplicial group as being one analogue of $H^*(C(\mathcal{U}), D)$ in this context. (If D is Abelian, it will be KD for some sheaf of chain complexes, D, and the Dold-Kan theorem, plus the freeness of $C(\mathcal{U})$, give a correspondence between the elements in the two cases. Since we have $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(D))$ is a simplicial Abelian group in that case, its homotopy is its homology and the detailed correspondence passes down to homology without any pain. We thus do have a generalisation of the Abelian situation with our formula.)

We have $\pi_n(\mathcal{U}, D) := \pi_n(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(D)))$ is thus a candidate for a 'non-Abelian' Čech cohomology relative to $\mathcal{U}$ with coefficients in D. (If $n > 1$, it is an Abelian group, which makes it suspiciously well behaved - in fact too well behaved! We really need not these π_n, but rather the various algebraic models for the various k-types of the homotopy type $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(D))$, i.e., we could do with examining $M(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(D)), k)$, the crossed k-cube of that simplicial group. (For those of you who hanker for the simple life, it should be pointed out that when discussing extensions, we already had that there was a groupoid of extensions $\mathcal{E}xt(G, K)$, and although we could extract information from that groupoid to get cohomology groups, the natural invariant is really that groupoid, not the cohomology group as such. We can extract information from such an invariant, just as we can extract homotopy information from a homotopy type. To keep the information tractable we often truncate, or kill off, some of the structure to make the extraction process more amenable to calculation.)

We are, however, running before we can walk here! The case we have met earlier is for $n = 0$, i.e., $[N(\mathcal{U}), D]$, and we could pass to the colimit over covers to get $\check{H}^0(B, D)$. This is without restriction on the sheaf of simplicial groups, D. Our earlier example was with $D = K(M)$ for $M = (C, P, \partial)$, a sheaf of crossed modules. (Breen in [23] calls this the *zeroth cohomology of the crossed module*, M, but as it varies covariantly in M perhaps his later terminology, [26], as the *zeroth Čech non-Abelian cohomology of B with coefficients in* M, is more appropriate.)

What about $\check{H}^1(B, M)$?

This will be $colim_{\mathcal{U}} H^1(N(\mathcal{U}), M)$, which is $colim_{\mathcal{U}} \pi_1(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), K(M)))$. From the long exact fibration sequence, this will be isomorphic to $colim_{\mathcal{U}}[N(\mathcal{U}), \overline{W}K(M)]$ and so should classify some sort of simplicial $K(M)$-bundles on B. It does, but we need to wait until a later chapter for the details.

The set $[N(\mathcal{U}), \overline{W}K(M)]$ has elements which are homotopy classes of maps from $N(\mathcal{U})$ to $\overline{W}K(M)$ and by the properties of the loop groupoid construction, $\mathcal{G}$ of section 4.2.1, page 125, each such is adjoint to a morphism of sheaves of $\mathcal{S}$-groupoids from $\mathcal{G}(N(\mathcal{U}))$ to $K(M)$. The category of crossed modules is equivalent, via K and $M(-, 2)$, to a full reflective subcategory / variety of $\mathcal{S}-Grpds$, and this extends to sheaves, so the elements of $[N(\mathcal{U}), \overline{W}K(M)]$ correspond to homotopy classes of crossed module morphisms from $M(\mathcal{G}N(\mathcal{U}), 2)$ to M. In particular, for nice spaces, B, one would expect there to be 'nice' covers $\mathcal{U}$, such that $N(\mathcal{U})$ corresponded, via geometric realisation, to B itself, then taking $M = M(\mathcal{G}N(\mathcal{U}), 2)$ itself, one would have a sort of universal element in $\check{H}^1(B, M)$, corresponding in this level, to a universal simplicial sheaf over B, extending in part the construction and properties of the universal covering space. This argument is one form of the 'evidence' for believing Grothendieck's intuition in 'En Poursuite des Champs / Pursuing Stacks', [87]. There seems no good reason why, for any nice class of simplicial groups that form a variety, $\mathcal{V}$, with perhaps having some stability with respect to homotopy types, there should not be a 'universal $\mathcal{V}$-stack' over B. The above corresponds to the case of crossed modules, but crossed complexes and many of the other types of crossed objects that we have met earlier would seem to

be relevant here. The main hole in our understanding of this is not really how to do it, rather it is how to interpret the theory once it is there. This form of crossed homotopical algebra would extend Galois theory to higher 'levels', but what do the invariants tell us algebraically?

That provides some overview of this general case, but in our earlier situation, with extensions of groups, we used a crossed resolution of a group, G, not a simplicial one. We have also mentioned once or twice that the category, Crs, of crossed complexes is monoidal closed. This would suggest (i) that given a topos $\mathcal{E}$, and, in particular, given a space B and $\mathcal{E} = Sh(B)$, the category of crossed complexes *in* $\mathcal{E}$, denoted $Crs_\mathcal{E}$, would be monoidal closed, (ii) there would be a free crossed complex on a cover / hypercover in $\mathcal{E}$, i.e., if we have a simplicial object K in $\mathcal{E}$, we would get a crossed complex object, $\pi(K)$, and if $K \to 1$ is a 'weak equivalence' then there would be a local contracting homotopy on $\pi(K)$, i.e., $\pi(K) \to 1$ would be a 'weak equivalence' of crossed complex bundles (recall 1 is the terminal object of $\mathcal{E}$, so in the case of $\mathcal{E} = Sh(B)$ is the singleton sheaf), then (iii) if $\mathrm{CRS}_\mathcal{E}$ denotes the internal 'hom' of crossed complex bundles, we would be looking at the model $\mathrm{CRS}_\mathcal{E}(\pi(K), \mathsf{D})$ for a crossed complex, D, in $\mathcal{E}$ and would want the homotopy colimit of these over (hyper-)covers, K, so as to get a well-structured model. Of course, if $\mathcal{E} = Sh(B)$ and we have 'nice' (hyper-)covers K, then we would expect the homotopy type of this to stabilise, up to homotopy, so $\mathrm{CRS}_\mathcal{E}(\pi(K), \mathsf{D})$ would be the same, up to homotopy, as that homotopy colimit. This plan almost certainly works, but has not been followed through as yet, at least, in all its gory detail. The first part looks very feasible given the construction of $\mathrm{CRS}(\mathsf{C}, \mathsf{D})$ for (set based) crossed complexes, C and D. (A source for this is Brown and Higgins, [34] and it is discussed with some detail in Kamps and Porter, [103], p. 222-227.) We will not give the details here. The other parts also look to work as the set based originals are given by explicit constructions, all of which generalise to $Sh(B)$. If that does all work then one has a Crs-based 'hyper-cohomology' crossed complex, $\check{\mathrm{CRS}}(B, \mathsf{D}) = hocolim_K \mathsf{Crs}(\pi(K), \mathsf{D})$, whose homotopy groups represent the analogue of hyper-cohomology.

If you are wary of not having a group or groupoid as an 'answer' for what is this 'hypercohomology', think of various analogous situations. For instance, for total derived functor theory, in homological and homotopical algebra, from a functor you get a complex, but it is the homotopy type of that complex which is used, not just its homotopy groups. In algebraic K-theory, it is quite usual to refer to the algebraic K-theory of a ring as being the (homotopy type of) a simplicial set or space. The algebraic K-groups are then the homotopy invariants of that simplicial set. In other words, in 'categorifying', one naturally ends up with an object whose homotopy type encapsulates the invariants that you are mostly used to, but that object is the thing to work with, not just the invariants themselves.

6.2 Mapping cocones and Puppe sequences

Exact sequences in cohomology can be constructed in various ways. One of these is related to the fibration and cofibration seqences of homotopy theory. If one has a fibration of spaces, then it leads to a long exact sequence of homotopy groups. Of course, not all maps are fibrations, but any map, $f : X \to Y$, can be replaced, up to homotopy, by a fibration, and its fibre Γ_f, then codes up homotopy information about f. This fibre is usually called the *homotopy fibre* of f and we have already met it in our list of common examples leading to crossed modules; see page 15. Later on we will need to use the construction to extend our simplicial interpretations of non-Abelian cohomology,

but, by way of introduction, to start with both that construction (mapping cocylinders and mapping cocones/homotopy fibres) and the resulting homotopy exact sequences (Puppe sequences) will be looked at in a much simpler setting, namely that of chain complexes. Initially we will concentrate on the dual situation as that is slightly easier to understand geometrically.

(A very useful concise introduction to this theory can be found in May's book, [120], starting about page 55, and, for results on chain complexes, page 90.)

6.2.1   Mapping Cylinders, Mapping Cones, Homotopy Pushouts, Homotopy Cokernels, and their cousins!

We need various 'homotopy kernels', 'homotopy fibres' and more general 'homotopy limits' for our discussion. We have also already mentioned 'homotopy colimits' in passing several times, and so it seems a good idea to examine this general area from an elementary point of view.

We will work with a chain map $f : C \to D$ of chain complexes of modules over some ring R. We will use a *cylinder* $C \otimes I$. This is given by tensoring C with the chain complex, I,

$$0 \longrightarrow R \overset{\partial}{\longrightarrow} R \oplus R \longrightarrow 0,$$

$$\partial(e_1^1) = e_1^0 - e_0^0.$$

There is one generator, e_1^1, in dimension 1, and two in dimension zero, corresponding to the interval $I = [0, 1]$ or $\Delta[1]$ having one 1-cell and two 0-cells, e_1^0 and e_1^0, the superfix denoting the dimension of that generator . We should give a formal definition of a tensor product of chain complexes, even though you may have met this before.

Definition: If C and D are chain complexes, their tensor product $C \otimes D$ has

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and boundary / differential given on generators by

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^{|c|} c \otimes (\partial' d),$$

where $|c|$ is the degree of c, (that is, $c \in C_{|c|}$).

We note the connection between $\otimes$ and Hom, namely that, given chain complexes, C, D, and E, there are natural isomorphisms

$$Hom(C \otimes D, E) \cong Hom(C, Hom(D, E)),$$

so $- \otimes D$ and $Hom(D, -)$ are adjoint.

Example:

$$\begin{aligned}
(C \otimes I)_n &\cong C_n \otimes I_0 \oplus C_{n-1} \oplus I_1 \\
&\cong C_n \oplus C_n \oplus C_{n-1}
\end{aligned}$$

(We will denote elements in this direct sum as column vectors, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, but will usually write $(x, y, z)^t$, or even (x, y, z) if we are being lazy!)

The isomorphism matches $c_n \otimes e_0^0$ with $(c_n, 0, 0)^t$, $c_n \otimes e_1^0$ with $(0, c, 0)^t$ and $c_{n-1} \otimes e_1^1$ with $(0, 0, c_{n-1})^t$. We can therefore calculate $\partial(x, y, z)^t$ explicitly for $(x, y, z)^t \in C_n \oplus C_n \oplus C_{n-1}$.

$$\begin{aligned} \partial(x, 0, 0)^t &= (\partial x, 0, 0)^t \\ \partial(0, y, 0)^t &= (0, \partial y, 0,)^t \end{aligned}$$

and, as $(0, 0, z)^t$ corresponds to a "$c_{n-1} \otimes e_1^1$", its boundary is

$$\begin{aligned} \partial(c_{n-1} \otimes e_1^1) &= \partial(c_{n-1}) \otimes e_1^1 + (-1)^{n-1} c_{n-1} \otimes \partial(e_1^1) \\ &= \partial(c_{n-1}) \otimes e_1^1 + (-1)^{n-1} c_{n-1} \otimes e_1^0 + (-1)^n c_{n-1} \otimes e_0^0 \end{aligned}$$

i.e. $\partial(0, 0, z)^t = ((-1)^n z, (-1)^{n+1} z, \partial z)^t$. This allows us to use, if we want to, a matrix representation of the boundary in $\mathsf{C} \otimes \mathsf{I}$ as

$$\begin{pmatrix} \partial & 0 & (-1)^{n-1} \\ 0 & \partial & (-1)^n \\ 0 & 0 & \partial \end{pmatrix}$$

and thus would allow us to use such a description to *define* a cylinder $\mathsf{C} \otimes \mathsf{I}$ for C, a chain complex in a more abstract setting such as that of an arbitrary Abelian category.

There are obvious chain maps,

$$\mathsf{e}_i : \mathsf{C} \to \mathsf{C} \otimes \mathsf{I},$$

$i = 0, 1$, corresponding to the ends of the cylinder, and a projection,

$$\sigma : \mathsf{C} \otimes \mathsf{I} \to \mathsf{C},$$

corresponding to 'squashing' the cylinder onto the base.

This, of course, leads to a notion of homotopy between chain maps.

Definition: A *(cylindrical) homotopy*, h, between two chain maps, $\mathsf{f}, \mathsf{g} : \mathsf{C} \to \mathsf{D}$, is a chain map,

$$\mathsf{h} : \mathsf{C} \otimes \mathsf{I} \to \mathsf{D},$$

with $\mathsf{he}_0 = \mathsf{f}$ and $\mathsf{he}_1 = \mathsf{g}$.

This notion of a 'cylindrical' homotopy, h, between two chain maps is easy to analyse. We have $h_n : C_n \oplus C_n \oplus C_{n-1} \to D_n$ and the conditions $\mathsf{he}_0 = \mathsf{f}$ and $\mathsf{he}_1 = \mathsf{g}$ become, in terms of coordinates, $h_n(x, 0, 0) = f_n(x)$, and $h_n(0, y, 0) = g_n(y)$, thus the 'free' or 'unbound' information for h is contained in $h_n(0, 0, z)$. This map, h, restricted to the C_{n-1}-summand gives a degree one map $\mathsf{h}' = \{h'_{n-1} : C_{n-1} \to D_n\}$. We have assumed that h is a chain map, so with our convention for the boundary on $\mathsf{C} \otimes \mathsf{I}$, we get:

$$\begin{aligned} \partial h'_{n-1}(z) = \partial h_n(0, 0, z) &= h \partial(0, 0, z) \\ &= h((-1)^{n-1} z, (-1)^n z, \partial z) \\ &= (-1)^{n-1}(f_{n-1}(z) - g_{n-1}(z)) + h'(\partial z). \end{aligned}$$

We thus have that, if we put $s_n = (-1)^n h'_n$, we will get a chain homotopy $\mathsf{s} : \mathsf{C} \to \mathsf{D}$, from f to g. Conversely any chain homotopy will yield a cylindrical homotopy.

Notational comment: The convention on signs that we have adopted is not the only on $\mathsf{C} \otimes \mathsf{I}$ and, as you can easily check, this will determine a different boundary on the chain complex, although the individual terms of the complex are still isomorphic to $C_n \oplus C_n \oplus C_{n-1}$.

Later we will consider the suspension $\mathsf{C}[1]$ of C and this has $C[1]_n = C_{n-1}$. Different sources on differential graded objects may adopt different conventions as to the form of the boundary for $\mathsf{C}[1]$. Quite often the convention chosen is $\partial_n^{\mathsf{C}[1]} = (-1)^n \partial_{n-1}^{\mathsf{C}}$, as this absorption of the $(-1)^n$ makes certain graded maps that naturally occur into chain maps and thus greatly simplifies the formulae and to some extent the theory.

These sign conventions are extremely useful in the study of differential graded algebras as in rational homotopy theory, cf. [80]. We are using chain complexes here mainly as an illustrative example, so will not need to adopt those conventions here. The reader is, however, advised that if working with differential graded (dg) structures, attention to the compatibility between the simplicial and 'dg' conventions is essential if your calculations are not going to look wrong! There is no essential difference in the geometric intuitions between the approaches, but confusion can easily arise if this is not recognised early on in work at this interface.

Given our chain map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, we can form a *mapping cylinder* on f by the pushout

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\;\mathsf{f}\;} & \mathsf{D} \\
{\scriptstyle \mathsf{e}_0}\downarrow & & \downarrow{\scriptstyle \mathsf{j}_\mathsf{f}} \\
\mathsf{C} \otimes \mathsf{I} & \xrightarrow{\;\pi_\mathsf{f}\;} & \mathsf{M}_\mathsf{f}
\end{array}
$$

and we can set $\mathsf{i}_\mathsf{f} = \pi_\mathsf{f}\mathsf{e}_1$. The fact that the e_i are split by $\mathsf{s} : \mathsf{C} \otimes \mathsf{I} \to \mathsf{C}$ means that we can form a commutative square

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\;\mathsf{f}\;} & \mathsf{D} \\
{\scriptstyle \mathsf{e}_0}\downarrow & & \downarrow{\scriptstyle =} \\
\mathsf{C} \otimes \mathsf{I} & \xrightarrow{\;\mathsf{fs}\;} & \mathsf{D}
\end{array}
$$

and obtain an induced map $\mathsf{p}_\mathsf{f} : \mathsf{M}_\mathsf{f} \to \mathsf{D}$ satisfying $\mathsf{p}_\mathsf{f}\mathsf{j}_\mathsf{f} = id_\mathsf{D}$ and $\mathsf{p}_\mathsf{f}\pi_\mathsf{f} = \mathsf{fs}$. The second equation then gives $\mathsf{p}_\mathsf{f}\mathsf{i}_\mathsf{f} = \mathsf{f}$, as an easy consequence.
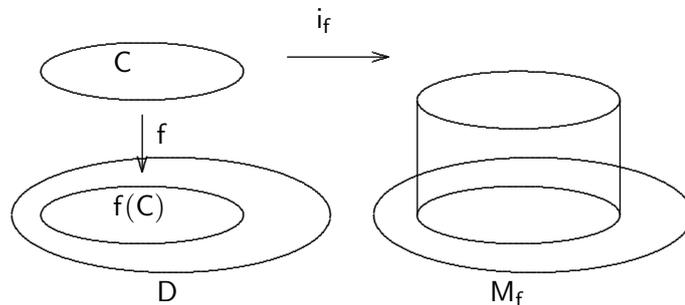
In addition, $\mathsf{j}_\mathsf{f}\mathsf{p}_\mathsf{f} : \mathsf{M}_\mathsf{f} \to \mathsf{M}_\mathsf{f}$ is homotopic to the identity by a homotopy that is constant on composition with j_f, i.e., D is a strong deformation retract of M_f.

Note that we have not shown this last fact. That is **left for you to do**. We should also note that most of this does not use any specific properties of chain complexes nor of the cylinder that we have been using. The same arguments would work for any 'reasonable' cylinder functor on a category with pushouts. The construction of a homotopy from $j_f p_f$ to the identity *does* use a few more properties. (**Try to investigate what is needed.** A quite detailed discussion of this from one point of view can be found in Kamps and Porter, [103], in a form fairly compatible with that used here.) We will need to use this mapping cylinder construction several times more in different contexts, so abstraction is useful.

Aside: In [103], you will also find a proof that i_f satisfies a *homotopy extension property*, i.e., it is a *cofibration*. The description above shows that any f can be factored as a cofibration composed with a strong deformation retraction.

Before we leave mapping cylinder-type constructions as such, we also need to comment on the dual situation, as that is really what we need for our immediate task. In many situation we can form a cocylinder, D^I, either instead of, or as well as, a cylinder. For instance, in the setting of chain complexes, we can set $D^I = Hom(I, D)$ and then, as is easily checked, $D^I{}_n \cong D_n \oplus D_n \oplus D_{n+1}$. The boundary is left to you to write down. The adjointness isomorphism gives the connection with the cylinder and also with chain homotopies. We can form a *mapping cocylinder* by a pullback:

$$
\begin{array}{ccc}
M^f & \xrightarrow{\ \pi^f\ } & D^I \\
{\scriptstyle j^f}\downarrow & & \downarrow{\scriptstyle e^0} \\
C & \xrightarrow{\ f\ } & D.
\end{array}
$$

There is a morphism $p^f : C \to M^f$ splitting j^f, so $j^f p^f = id$, and also $p^f j^f \simeq id$. Writing $i^f = e_1 \pi^f$, we have $i^f p^f = f$. This map i^f is a fibration, even in the abstract case under reasonable conditions on the context and the properties of the cocylinder functor, and we find, for instance in the topological setting, the method we used to replace an arbitrary map into a fibration, up to homotopy, (look back to page 15).

Returning now to mapping cylinders, we have $i_f : C \to M_f$ inserting C as the 'top' of the cylinder part of M_f. The *mapping cone*, C_f, (or, sometimes, $C(f)$) of f is obtained by quotienting out by the image of i_f. (This is usually visualised by imagining C_f as a copy of D together with a cone, $C(C)$ on C glued to it using f.)
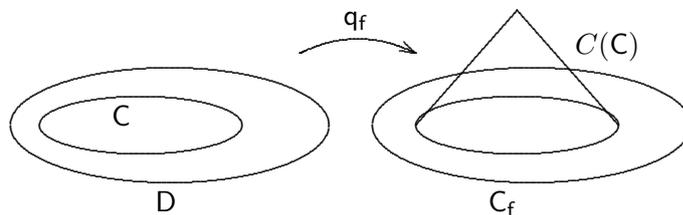
We note that the map $j_f : D \to M_f$ composed with the quotient $q : M_f \to C_f$ gives a map, $q_f : D \to C_f$ and that the cone structure provides a homotopy between the composite, $C \to D \to C_f$, and the trivial map, $C \to C_f$. We should look at this more closely.

If we compose the cylindrical homotopy given by the identity on $\mathsf{C} \otimes \mathsf{I}$ with π_{f}, we get a homotopy between $\pi_{\mathsf{f}}\mathsf{e}_0$ and $\pi_{\mathsf{f}}\mathsf{e}_1$, but $\pi_{\mathsf{f}}\mathsf{e}_0 = j_{\mathsf{f}}\mathsf{f}$ and $\pi_{\mathsf{f}}\mathsf{e}_1 = \mathsf{i}_{\mathsf{f}}$. Finally composing everything with $\mathsf{q} : \mathsf{M}_{\mathsf{f}} \to \mathsf{C}_{\mathsf{f}}$, we have a homotopy between $\mathsf{q}j_{\mathsf{f}}\mathsf{f} = \mathsf{q}_{\mathsf{f}}\mathsf{f}$ and $q\mathsf{i}_{\mathsf{f}}$, which latter map is trivial.

Dually we can get a *homotopy (mapping) cocone*: we take the homotopy cocylinder M^{f} and the map $\mathsf{i}^{\mathsf{f}} : \mathsf{M}^{\mathsf{f}} \to \mathsf{D}$ and form its fibre over the 'basepoint', that is the zero, of D. Of course that 'fibre' is just the kernel of i^{f} in our chain complex case study.

Aside on homotopy cokernels, etc.

In discussion on kernels and cokernels in Abelian and additive categories, it is quite often noted that the cokernel of a map, $\varphi : A \to B$, say in an Abelian category, gives a pushout

$$
\begin{array}{ccc}
A & \xrightarrow{\ \varphi\ } & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Coker\,\varphi
\end{array}
$$

and that the pushout square property is exactly the universal property defining cokernels. The construction of the mapping cone gives a similar square:

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\ \mathsf{f}\ } & \mathsf{D} \\
\downarrow & \nearrow\!\!\!\!\!\diagup & \downarrow{\scriptstyle \mathsf{q}_{\mathsf{f}}} \\
0 & \longrightarrow & C_{\mathsf{f}}
\end{array}
$$

but it is only homotopy commutative (or rather homotopy coherent as there is the natural *explicit* homotopy, $\mathsf{h}_{\mathsf{f}} : \mathsf{q}_{\mathsf{f}}\mathsf{f} \Rightarrow 0$). This homotopy coherent square has a universal property with respect to homotopy coherent squares based on $0 \leftarrow \mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D}$. This makes it reasonable to call the result a *homotopy pushout* and then to say that C_{f} is the *homotopy cokernel* or sometimes the *homotopy cofibre* of f. It is, of course, an example of a homotopy colimit, but note that it is necessary to give not only C_{f} plus q_{f} to get the full universal property (as would be the case for an ordinary colimit), but also h_{f}.

Exercise: The construction of the mapping cylinder is also a homotopy pushout. Try to formulate a good notion of homotopy pushout and identify that construction as an example of one such. The main idea is to start with two maps

$$
B \xleftarrow{b} A \xrightarrow{c} C
$$

with common domain and to form a homotopy coherent square

$$
\begin{array}{ccc}
A & \xrightarrow{\ c\ } & C \\
{\scriptstyle b}\downarrow & \nearrow\!\!\!\!\!\diagup & \downarrow{\scriptstyle b'} \\
B & \xrightarrow{\ c'\ } & D,
\end{array}
$$

where h is a homotopy $A \times I \to D$ between $b'c$ and $c'b$. For instance, use a repeated pushout

operation on the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ c\ } & C \\
\ \downarrow{\scriptstyle e_0} & & \vdots \\
A \xrightarrow{\ e_1\ } A \times I & & \vdots \\
\ \downarrow{\scriptstyle b} & & \vdots \\
B & \dashrightarrow & D
\end{array}
$$

to construct its colimit, which will be a *double mapping cylinder*. The homotopy h is then clear. Specialise down to the case of b being the identity to complete. Note that homotopy pushouts are determined 'up to homotopy', not 'up to isomorphism', so you may not quite get what you expect and different construction may give different, but homotopic, models for it!

This discussion of homotopy cokernels is almost 'general'. It works, more or less, in any setting where there is a null object, corresponding to 0, having a nice cylinder that preserves pushouts, and, of course, enough pushouts. In our well behaved case study of chain complexes, we can track the construction in the direct sum decomposition if we so wish.

Homotopy commutative v. homotopy coherent: It is quite important to note a sort of theme that occurs both here and earlier in our discussion of bitorsors and M-torsors. An M-torsor was a C-torsor, E together with a definite choice of global section for $\partial_*(E)$. We did not just say the $\partial_*(E)$ is trivialisable, we specified a trivialisation as part of the structure.

Here with homotopy pushouts, we do not just have a homotopy commutative square, but specify a definite choice of homotopy linking the two composite maps around the square, i.e., we give a 'homotopy coherent square'. This passage from 'there is a homotopy such that ...' to specifying one is of prime importance in interpreting non-Abelian cohomology.

We have concentrated, so far, on the case of chain complexes. We do need to caste a glance at the topological case. The above description in terms of homotopy cokernels goes through for pointed spaces.

Suppose $f : X \to Y$ is a map of pointed spaces, we can form M_f and factorise f as $p_f i_f = f$, where i_f is a cofibration and p_f is the retraction part of a strong deformation retraction, so in particular is a homotopy equivalence.

Using the cofibration $i_f : X \to M_f$, we divide out, identifying its image to a point, to get C_f as a quotient space, or directly as a homotopy pushout

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow & \swarrow & \downarrow{\scriptstyle q_f} \\
* & \longrightarrow & C_f,
\end{array}
$$

where $q_f = q j_f$ with $q : M_f \to C_f$ the quotient map.

6.2.2 Puppe exact sequences

The map q_f is a cofibration, under reasonable conditions on the spaces involved, and we can form the quotient of C_f by identifying the image of this map to a point: $SX \cong C_f/Y$, giving the

(reduced) suspension, SX, on X. This can be defined directly as $(X \times I)/(X \times \{0,1\} \cup * \times I)$, where $*$ is the base point of X. It is also the homotopy pushout

$$
\begin{array}{ccc}
X & \longrightarrow & * \\
\downarrow & \swarrow & \downarrow \\
* & \longrightarrow & SX,
\end{array}
$$

where the homotopy is the quotient map from $X \times I$ to SX.

This gives us a sequence of maps

$$X \xrightarrow{f} Y \to C_f \to SX \xrightarrow{Sf} SY \to SC_f \to S^2 X \to \dots,$$

where we have extended the bit that we have actually constructed by applying S to it and grafting it to the old part. This sequence is known, variously, as the *long cofibre sequence* of f, the *Puppe sequence* of f or the *cofibre Puppe sequence*. It is 'homotopy exact' - what does that mean?

Recall that in an exact sequence, say, of Abelian groups, the kernel of one map is the image of the previous one, so in particular, the composition of pairs of maps in the sequence is always trivial. In the above sequence of pointed spaces, there is an *explicit* null-homotopy from each composition of pairs of adjacent maps to the corresponding trivial map that send the domain to the base point of the codomain. This is clear for the first composable pair $X \xrightarrow{f} Y \to C_f$ as that is exactly what C_f was designed to do! (Some treatments of these sequences in fact construct them by repeating that basic construction of C_f from f for subsequent maps starting with $Y \to C_f$, and then showing that the resulting terms match, up to homotopy, with those of the above sequence. We do not adopt that approach here, although it has some very good points to it.)

The next pair $Y \to C_f \to SX$ is trivial anyway. The checking that $C_f \to SX \xrightarrow{Sf} SY$ is homotopy exact is omitted. It can be found in the literature or you can attempt it yourself. This is thus the analogue of the composites being trivial in an exact sequence. The arguments used for these also show that an analogue of the other part of 'exactness' also holds. For this it seems advisable to indicate a more precise statement. (The temptation to use the words 'exact statement' here must be resisted!) That statement is the usual one here, and goes as follows. (It will need a certain amount of commentary, which will be given shortly.)

For any pointed space, Z, applying the functor $[-, Z]$ to the above sequence yields a long exact sequence of groups and pointed sets,

$$\dots \to [S^2 X, Z] \to [SC_f, Z] \to [SY, Z] \to [SX, Z] \to [C_f, Z] \to [Y, Z] \to [X, Z].$$

We have already recalled the meaning of exactness for sequences of groups. The extension of that to pointed sets should be clear: we replace 'kernel' by 'preimage of the base point' whilst 'image' has the same meaning. If we examine the exactness at $[Y, Z]$, this says that if $g : Y \to Z$ is such that gf is null homotopic, (that is, there is some $h : gf \simeq *$), then there is some $\overline{g} : C_f \to Z$ such that $g = \overline{g} q_f$, and conversely. But that is just what the construction of C_f does, as the nullhomotopy extends the map on Y to the cone on the X part of C_f. In fact, of course, different nullhomotopies will extend to different maps on C_f and you are left to think about the way in which these different null homotopies are, or are not, 'observed' by the sequence. To start you thinking, if $h, h' : gf \simeq *$, then we have a self homotopy of $*$, intuitively, '$hh'^{(-1)}$'. The map

$hh'^{(-1)} : X \times I \to Z$ sends both ends of the cylinder to the basepoint and as it is constructed from pointed homotopies, it also sends $* \times I$ there. It thus induces a map from SX to Z, giving a possible link back to $[SX, Z]$. Again the theme of homotopy coherence v. homotopy commutativity is nearby as if we record the possible null homotopies then we get other information cropping up elsewhere in the sequence.

In this discussion of 'homotopy exact sequences', we have still to complete our discussion of the cofibre sequence of a chain map and also we will have need not so much of this cofibre form of the Puppe sequence, but rather the Puppe 'fibre' long exact sequence of a map. We start with the chain cofibre sequence.

So far we have

$$\mathsf{C} \to \mathsf{D} \to \mathsf{C_f}$$

and, in elementary terms,

$$(\mathsf{C_f})_n \cong D_n \oplus C_{n-1},$$

i.e., the pushout of D and a cone on C. (The differential / boundary is **left to you**.) There is an inclusion of D into $\mathsf{C_f}$, and, surprise surprise, the quotient is $\mathsf{C}[1]$, it has C_{n-1} in dimension n, so is the chain complex analogue of the suspension. (Here we must repeat the warning about sign conventions. The suspension is often considered to have boundary $(-1)^n \partial_n$, corresponding to the needs for the 'suspension map' to be a chain map. This is just due to a different convention on the boundary map of the cylinder. As we need this as a step to understanding the *simplicial* situation, our convention is slightly more appropriate.)

Of course, if E is another chain complex, then applying $[-, \mathsf{E}]$ should give us a long exact sequence. (All is not really as simple as that here as it is usually better to work in what is called the *derived category* of chain complexes rather than just dividing out by homotopy. Initially you should try this for chain complexes of free modules as you cannot always create the maps you want in more general contexts. This general situation *is* important and will be needed in certain aspects later on, but we will ignore the complication here. It is a very useful exercise to show the long exactness for chain complexes of free (or projective) modules, before trying to understand the complication if the freeness condition is removed.)

Now we turn to 'fibre Puppe sequences' in the topological case: we have our $f : X \to Y$ and form the *mapping cocylinder*, M^f, with $i^f : M^f \to Y$ being a fibration and $M^f \simeq X$ in a controlled way, (homotopy coherence again - and, yes, M^f *is* given by a homotopy pullback.) We form the fibre of i^f, and this is $C^f = F_h(f)$, the homotopy fibre of f that we have met before (cf. page 15). This is also a homotopy pullback:

$$
\begin{array}{ccc}
C^f & \longrightarrow & * \\
{\scriptstyle f^f}\downarrow & {\scriptstyle \nearrow\!\!\!\!\!\nearrow} & \downarrow \\
X & \underset{f}{\longrightarrow} & Y,
\end{array}
$$

wher q^f is the composite $C^f \to M^f \to X$. We can realise this very neatly by first using the pullback

$$
\begin{array}{ccc}
\Gamma Y & \longrightarrow & Y^I \\
\downarrow & & \downarrow{\scriptstyle e_0} \\
* & \longrightarrow & Y
\end{array}
$$

giving the object of paths that start at $*$. This has a second map to Y induced by e_1, giving $\Gamma Y \to Y$, which is a fibration. This is the dual analogue of the cone on X in this dual context.

(The notation ΓY is 'traditional', but is also traditional for the set of global sections of a bundle! No confusion should arise!) This space ΓY is contractible in a geometrically pleasing way - the homotopy reduces the 'active' part of each path until it does nothing: if $\alpha : I \to Y$ with $\alpha(0) = *$, then $\alpha_t(s) = *$ if $s \le t$ and is $\alpha(s - t)$ if $t \le s \le 1$. The α_t form a homotopy, essentially a path, from α to the constant path at $*$. We can realise C^f as the pullback:

$$
\begin{array}{ccc}
C^f & \longrightarrow & \Gamma Y^I \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;f\;} & Y.
\end{array}
$$

(A useful observation here is that this pullback absorbs the homotopy of the homotopy pullback by replacing the $*$ by a contractible space. That *is* an example of a general process, a 'rectification' or 'rigidification' process, but this will not be explored until much later in these notes.)

Example 1: The neat example that illustrates the importance of this homotopy fibre construction is to take Y to be an arcwise connected space, X a proper subspace (so the inclusion f is very far from being a fibration). The fibre of f over a point $y \in Y$ is either a single point, if $y \in X$, or empty, if it is not. We think of y as being a map $y : * \to Y$, picking out that element, and change y along a path y_t, from being in X, say y_0, to not being in X, at y_1. That path is a homotopy between the maps y_0 and y_1, so although y_0 and y_1 are homotopic maps, the fibre over y_t changes homotopy type as t varies. On the other hand, the homotopy fibre has the same homotopy type along the whole of y_t. (We saw earlier (page 15) that the fundamental group of $F_h(f)$ was $\pi_2(Y, X)$ and does not change, up to specified isomorphisms, as one varies t between 0 and 1.)

Example 2: This first example was with f far from being a fibration. What if f *is* a fibration? (We, as usual, want to concentrate on the intuitions behind the facts here so will not explore this in depth, but it will be useful to have some picture of what is happening, leaving details either **to the reader** to provide or to find, as the results are fairly easy to find in the literature.)

First note the obvious
$$ f^{-1}(*) = \{x \mid f(x) = *\}, $$
whilst
$$ C^f = \{(x, \lambda) \mid \lambda \in \Gamma Y, \lambda(0) = *, \lambda(1) = f(x)\}, $$
so, in particular, there is a map from $f^{-1}(*)$ to C^f, mapping x to (x, c), where c is the constant path at $*$. We would like to see when this map is a homotopy equivalence. We have that underlying it, in some sense, is the map sending $*$ to $c \in \Gamma Y$, which is a homotopy equivalence, in fact a strong deformation retraction. If you try to see if this will induce in some way a retraction from C^f to $f^{-1}(*)$, then you hit the problem of what path an element (x, λ) should trace out in order to get to some $(x', c) \in f^{-1}(*)$. This would have to project down onto a path in X and in general there will not be one. If we assume that f is a fibration however, we can see more clearly what to do. (Recall that a fibration has a homotopy lifting property and it is that we will use.)

Examine the following diagram:

$$
\begin{array}{ccc}
C^f & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle f} \\
C^f \times I & \longrightarrow & Y.
\end{array}
$$

The bottom horizontal map here is the composite $C^f \times I \to \Gamma Y \to Y$. The first of these is the inclusion, then the second is the homotopy retracting ΓY to a point, composed with the projection onto Y. The top horizontal map is q^f, so the diagram commutes. As f is assumed to be a fibration, there is a lift of the bottom map to a homotopy $C^f \times I \to X$, extending q^f on its 'zero' end. Its other end gives a map which has image in the fibre of f, so we have what we want - except for **checking details**!

This is very useful as it says: if f is a fibration, we do not need to turn it into one before taking its fibre! Why is that useful? Look at the fibre Puppe sequence so far

$$C^f \to X \to Y.$$

We said that ΓY is a fibration, so $q^f : C^f \to X$ is also a fibration. We can take its homotopy fibre, which will look messy to say the least, or its fibre, which is a lot easier to calculate!

$$\begin{aligned}(q^f)^{-1}(*_X) &= \{(\lambda, x) \mid \lambda(0) = *_Y, \lambda(1) = f(x), x = *_X\} \\ &= \{\lambda \mid \lambda(0) = \lambda(1) = *_Y\},\end{aligned}$$

so $(q^f)^{-1}(*_X) \cong \Omega Y$, the space of loops, at the base point, of Y. (This is neat, of course, as Ω is a functor, which is adjoint to S, the reduced suspension. Whether it is **right or left adjoint is left to you**! Thus we have a linkage between the right and left Puppe sequence constructions.) That fact gives us the tool to open up the whole of the sequence. It goes

$$\ldots \to \Omega^2 Y \to \Omega C^f \to \Omega X \overset{\Omega f}{\to} \Omega Y \to C^f \to X \overset{f}{\to} Y.$$

Given a pointed space Z, we can apply $[Z, -]$ to this sequence to get our long exact sequence

$$\ldots \to [Z, \Omega^2 Y] \to [Z, \Omega C^f] \to [Z, \Omega X] \overset{[Z, \Omega f]}{\to} [Z, \Omega Y] \to [Z, C^f] \to [Z, X] \overset{[Z, f]}{\to} [Z, Y],$$

(and once **you have sorted out** right or left adjunctions, you will find many terms you recognise from the other type of Puppe sequence).

Our treatment here has been deliberately informal. The importance of these sequences for cohomology cannot be over emphasised and we **suggest that you look** at some formal treatments, both for the algebraic case (via derived and triangulated categories, e.g. Neeman, [133]) and via the topological case consulting, say, May, [120] in the first instance before looking into the theory in other sources. There are abstract versions in homotopical algebra, see, for instance, in Hovey, [95], and a neat categorical treatment in Gabriel and Zisman, [83].

One final point before passing from descriptions of Puppe sequences to using them is the interpretation of exactness at the various points in the sequence. For instance, at $[Z, C^f]$, an element is represented by a map, g say, to C^f, and as C^f is given by a pullback, g decomposes via the two projections into a pair (g_X, g_Γ) with $g_X : Z \to X$ and $g_\Gamma : Z \to \Gamma Y$ such that $f g_X = e_1 g_\Gamma$. Going one step further, $\Gamma Y \subset Y^I$, so g_Γ gives a homotopy between $*$, the constant map to the basepoint, and $f g_X$. Now suppose $[Z, f] : [Z, X] \to [Z, Y]$ sends a homotopy class $[k]$ to the basepoint, then fk is homotopic to $*$ and we can build a $g : Z \to C^f$ from k and that homotopy. The more difficult part of the exactness at $[Z, X]$ follows. Back to $[Z, C^f]$, suppose our $g = (g_X, g_\Gamma)$ gets sent to the 'point' of $[Z, X]$, then $q^f g_X$ must be null homotopic. Pick such a null homootpy $h : Z \times I \to X$ and use the fact that q^f is a fibration to lift h to $\overline{h} : Z \times I \to C^f$. The 'other end ' of $\overline{h}$, i.e., $\overline{h} e_1$ is such that $q^f \overline{h} e_1$ is $*$, so $\overline{h} e_1$ is into the fibre of q^f, but that is ΩY. It remains to put the various pieces

together. The details can be found in many sources, but what is important to retain is the way of constructing a corresponding element in the previous stage. A trivialisation of an element yields a class in another stage. This should remind you of M-torsors, of categorisation and of homotopy cohenrence.

6.3   Puppe sequences and classifying spaces

6.3.1   Fibrations and classifying spaces

In his discussion of bitorsors, etc., in [23], Breen makes use of Puppe sequences of maps between classifying spaces. Suppose $v : H \to G$ is a morphism of simplicial groups, then we get an induced map of classifying spaces $Bv : BH \to BG$. We can take BG to be $\overline{W}G$ as being the neatest construction from our simplicial viewpoint. (Detailed calculations with $\overline{W}G$, etc., are quite easy in the simple case that we will need, but do get complicated if G has lots of non-trivial terms in its Moore complex. Another point worth making is that the detailed formulae for $\overline{W}G$ given earlier, page 130, use the algebraic composition order and therefore sometimes seem to reflect 'right actions'. This can be got around in either of two ways. The formulae for both $\overline{W}$ and G, the Dwyer-Kan $\mathcal{S}$-groupoid functor, can easily be reversed to get equivalent ones using the other composition order. This may be needed later when considering cocycles, etc., however the second argument uses that $\overline{W}G$ determines a Kan complex that is determined up to homotopy type - so either method will lead to the same $[-, \overline{W}G]$ and thus *most* of the time we can ignore the composition order. To ignore it, or forget it, completely is not a good idea, but we can face the problem, if and when it is needed.)

We thus are looking at $Bv : BH \to BG$. If v is not surjective, then we can use the mapping cocylinder construction, suitably adapted, to replace it by a fibration and fibrations of simplicial groups are exactly the surjective morphisms. We can thus study, without loss of generality, the surjective case and, of course, that means using the exact sequence

$$K \xrightarrow{u} H \xrightarrow{v} G$$

of simplicial groups and studying the effect of the functor B on it.

We 'clearly' get a long Puppe sequence, ending with

$$\ldots \to \Omega BH \to \Omega BG \to C^{Bv} \to BH \to BG.$$

Such a Puppe sequence can be constructed from the 'obvious' cocylinder functor, $\mathcal{S}_*(\Delta[1], -)$, but only works really well if applied to Kan complexes. Luckily these simplicial sets *are* Kan, so we can proceed accordingly. We note that as v is a fibration of simplicial groups, Bv is a fibration of simplicial sets, so we can hope that C^{Bv} can be more easily calculated than would be the case in general.

To see why Bv is a fibration, imagine we have a $\underline{g} \in BG_n$ and thus $\underline{g}$ has the form $(g_{n-1}, \ldots, g_0)$ with $g_i \in G_i$. We can find $h'_i \in H_i$ such that $v(\underline{h'_i}) = g_i$, $i = 0, \ldots, n-1$. If we are given a (n, k)-horn, $\overline{h}$, in BH that maps down to the (n, k)-horn, $(d_n \underline{g}, \ldots, \widehat{d_k \underline{g}}, \ldots, d_0 \underline{g})$, of $\underline{g}$ (using the traditional $\widehat{}$ notation for an omitted element), then $\underline{h}^{-1}.\overline{h}'$ gives a horn over the trivial (n, k)-horn of BG, that is, we can *translate* the filling problem to the identity, where it is essentially that of proving that $\overline{W}G$ is a Kan complex, which is easier to handle and we will do so in a moment. Note

this argument uses a transversal in each dimension, although we did not explicitly label it as being one, namely $g_i \mapsto h_i'$, which is suggestive of other uses of transversals in these notes.

An indirect, but neat, proof that $\overline{W}$ preserves fibrations and weak equivalences is to be found on p. 303 of the book, [85], by Goerss and Jardine. They note that this implies that $\mathcal{G}$ preserves cofibrations and weak equivalences, which is also very useful.

Postponing the proof that classifying spaces are Kan for the moment, the last thing to identify is the fibre of Bv, but this is easy, since we have an explicit description of Bv. It sends $\underline{h} = (h_{n-1}, \ldots, h_0)$ to $(v(h_{n-1}), \ldots, v(h_0))$, so its fibre is exactly the image by Bu of BK. We can thus use that, for fibrations, the fibre and homotopy fibre coincide up to equivalence, to conclude $C^{Bv} \simeq BK$ and our Puppe sequence now looks like

$$\ldots \to \Omega BH \to \Omega BG \to BK \to BH \to BG.$$

6.3.2 $\overline{W}G$ is a Kan complex

We have left this aside because we want to examine it in some detail, and those details were not needed at that point in our discussion. As an example of what might be done, suppose that G satisfies some extra condition such as the vanishing of its Moore complex in certain dimensions or that it satisfies the thin filler condition above some dimension, then the constructive description of $\overline{W}G$ suggests that it might be feasible to analyse $\overline{W}G$ to see if it satisfies some similar condition.

We will give the verification for a simplicial group, however, in many of the applications, we will need the construction for a simplicial group object in a topos, $\mathcal{E}$. This will allow us to talk of the classifying space of a sheaf of simplicial groups without worrying about the context. All the structure, however, is specified in a constructive way, and so goes across without any pain to a general topos. It also goes across without difficulty to an $\mathcal{S}$-groupoid. (I learnt this via Phil Ehlers' MSc thesis, [70], in which he did all the calculations explicitly.)

For convenience, we repeat the formulae for $\overline{W}G$, from page 130, making small adjustments, since we will not be looking at the groupoid case here, so let G be a simplicial group.

The simplicial set, $\overline{W}G$, is described by

- $(\overline{W}G)_0$ is a single point, so $\overline{W}(G)$ is a reduced simplicial set;

- $(\overline{W}G)_n = G_{n-1} \times \ldots G_0$, as sets, for $n \geq 1$.

The face and degeneracy mappings between $\overline{W}(G)_1$ and $\overline{W}(G)_0$ are the source and target maps and the identity maps of G_0, respectively; whilst the face and degeneracy maps at higher levels are given as follows:

The face and degeneracy maps are given by

- $d_0(g_{n-1}, \ldots, g_0) = (g_{n-2}, \ldots, g_0)$;

- for $0 < i < n$, $d_i(g_{n-1}, \ldots, g_0) = (d_{i-1}g_{n-1}, d_{i-2}g_{n-2}, \ldots, d_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \ldots, g_0)$;

and

- $d_n(g_{n-1}, \ldots, g_0) = (d_{n-1}g_{n-1}, d_{n-2}g_{n-2}, \ldots, d_1 g_1)$,

whilst

- $s_0(g_{n-1}, \ldots, g_0) = (1, g_{n-1}, \ldots, g_0)$;

and,

- for $0 < i \leq n$, $s_i(g_{n-1}, \ldots, g_0) = (s_{i-1}g_{n-1}, \ldots, s_0g_{n-i}, 1, g_{n-i-1}, \ldots, g_0)$.

Let us start in a low dimension to see what problems there may be. For $n = 2$, suppose we had a $(2, 2)$ box in $\overline{W}G$, so we have a pair, (x_0, x_1), of elements of $\overline{W}G_1$, which fit together, so $d_0 x_0 = d_0 x_1$. (We think of this as $(x_0, x_1, -)$, a list of possible faces, with a gap in the d_2-position.) We want some $y \in \overline{W}G_2$ such that $d_0 y = x_0$ and $d_1 y = x_1$.

Expanding things (in fact this is purely formal here, but lays down notation for later), we thus have $x_0 = (x_{0,0})$, $x_1 = (x_{1,0})$. (The condition on the faces happens to be trivial here since $\overline{W}G_0$ is a single point.) These $x_{i,0}$ are in G_0, for $i = 0, 1$. Similarly y will be of form (y_1, y_0), and we can examine what the desired conditions imply

$$
\begin{aligned}
x_{0,0} &= d_0 y = y_0 \\
x_{1,0} &= d_1 y = d_0 y_1 . y_0 .
\end{aligned}
$$

We thus already know y_0 and need to find a y_1 with $d_0 y_1 = x_{1,0} x_{0,0}^{-1}$. Clearly, we can find one, for instance, $s_0(x_{1,0} x_{0,0}^{-1})$ will do and we can even find *all* such, since any other suitable y_1 will have form $k s_0(x_{1,0} x_{0,0}^{-1})$ for some $k \in Ker\, d_o$. In other words, we really do know a lot about the possible fillers for our horn, even being able to count them if G is a finite simplicial group!

Next in line, we suppose that we have $(x_0, -, x_2)$ and want y such that $d_0 y = x_0$, $d_2 y = x_2$. Expanding these, using the same notation as before, we have, once again, that $x_{0,0} = d_0 y = y_0$, but now

$$
x_{2,0} = d_2 y = d_1 y_1 .
$$

Again we have y_0 and can solve $d_1 y_1 = x_{2,0}$, using $y_1 = s_0(x_{2,0})$, and, to get all fillers, $k s_0(x_{2,0})$ with $k \in Ker\, d_1$.

That was easy! What about $(2, 0)$-horns? These *are* slightly harder, as the other types did give us $d_0 y$ and thus handed us y_0 'on a plate', but it is only '*slightly*'.

We have $(-, x_1, x_2)$, $x_i = (x_{i,0})$ and want $y = (y_1, y_0)$. We thus know

$$
\begin{aligned}
x_{1,0} &= d_1 y = d_0 y_1 . y_0 \\
x_{2,0} &= d_2 y = d_1 y_1 .
\end{aligned}
$$

We do not know y_0, but do know $d_1 y_1$ and can solve to get $y_1 = k s_0(x_{2,0})$ with $k \in Ker\, d_1$ as before. We then have $y_0 = (d_0(k) x_{2,0})^{-1} x_{1,0}$ for the general filler.

Although that is simple, it is also easy to see that it can be extended, with modifications, to higher dimensions.

If we have a (n, n)-horn in $\overline{W}G$, then we have $(x_0, \ldots, x_{n-1}, -)$ with $x_i = (x_{i,n-2}, \ldots, x_{i,0}) \in \overline{W}G_{n-1}$. for $i = 0, 1, \ldots, n-1$. The compatibility condition is non-trivial here, so we note that

$$
d_i x_j = d_{j-1} x_i
$$

if $i < j$.

We need to find all $y = (y_{n-1}, \ldots, y_0)$ with $d_i y = x_i$ for all $i < n$. We thus have

$$x_0 = d_0 y = (y_{n-2}, \ldots, y_0),$$

but this means that we know all but the top dimensional element of the string that is y. Next

$$x_1 = d_1 y = (d_0 y_{n-1}.y_{n-2}, \ldots, y_0),$$

so we glean the information that

$$d_0 y_{n-2} = x_{1,n-2}.x_{0,n-2}^{-1}.$$

Continuing, we get, for $k > 1$ and in the range $k < n$, that

$$x_k = d_k y = (d_{k-1}y_{n-1}, d_{k-2}y_{n-2}, \ldots, d_0 y_{n-k}.y_{n-k-1}, \ldots, y_0),$$

and here the only new information is that which we get on $d_{k-1}y_{n-1}$, which can be read off as being $x_{k,n-2}$.

We should note that the compatibility condition tells us that there will be no inconsistencies in the rest of this string. For instance, we seem to have

$$x_{k,n-k-1} = d_0 y_{n-k}.y_{n-k-1}.$$

As we know y_{n-k-1} and y_{n-k}, we can check that we do not have a conflict:

$$
\begin{aligned}
y_{n-k} &= x_{0,n-k} \\
y_{n-k-1} &= x_{0,n-k-1},
\end{aligned}
$$

but then $x_{k,n-k-1}$ needs to be $d_0 x_{0,n-k}.x_{0,n-k-1}$, which is the $(n-k-1)$-component of $d_k x_0$. The compatibility condition tells us

$$d_0 x_k = d_{k-1}x_0,$$

and we leave the reader to check that the $(n-k-1)$-component of this equation is exactly

$$x_{k,n-k-1} = d_0 x_{0,n-k}.x_{0,n-k-1},$$

as hoped for.

Collecting things up, we know $d_\ell y_{m-1}$ for $\ell = 0, \ldots, n-2$, i.e., we have a $(n-1, n-1)$-horn in G itself. We know not only that G is a Kan complex, but how to fill horns algorithmically, so can find a suitable y_{n-1} and hence a filler, y for the original (n,n)-horn in $\overline{W}G$.

The intermediate cases of (n,i)-horns in $\overline{W}G$ for $0 < i < n$ are very similar and are **left to you**. In each case, as we have $d_0 y = x_0$, we just have to work on the first element, y_{n-1} in the string giving us y. The other parts give us a horn in G, which encodes the available information on the faces of y_{n-1}. We fill this horn to get y_{n-1}, and hence to fill the original horn in $\overline{W}G$. In each case, we can fill because we know that the underlying simplicial set of G is a Kan complex. We have the algorithm for fillers and so can analyse the set of fillers for a given horn, the algorithm giving a definite coset representative. For instance, in the (n,n)-horn, above, we found y exactly except in the first, highest dimensional position, y_{n-1}. We use the algorithm to find *one* filler / solution

for y_{n-1}, then know any other will differ from it by an element of $\bigcap_{i=0}^{n-2} Ker\, d_i$. This latter group is essentially a 'translate' of NG_{n-1} using the argument that Carrasco used to simplify Ashley's group T-complex condition (see the comment in the discussion of group T-complexes, page **??**).

We still have to handle the $(n,0)$-horn case, so should not be too pleased with ourselves yet! That was the slightly awkward case for the $n = 2$ situation that we studied earlier, as we do not have y_{n-2} given us initially.

Suppose $(-, x_1, \ldots, x_n)$ is the horn and we have to find a $y \in \overline{W}G_n$ satisfying $d_i y = x_i$ for $i = 1, \ldots, n$. Using the same notation, we have

$$x_1 = d_1 y = (d_0 y_{n-1} . y_{n-2} . y_{n-3}, \ldots, y_0)$$

and we get all the y_i except y_{n-1} and y_{n-2}. We then have

$$x_i = d_i y = (d_{i-1} y_{n-1}, \ldots, y_0)$$

and so get all the faces of y_{n-1}, except that zeroth one. We can thus fill the resulting $(n-1, 0)$-box in G (using the algorithm) to find a suitable y_{n-1}. We still do not have y_{n-2}, but as we now have y_{n-1}, we can read off $d_0 y_{n-1}$ from our solution to get

$$y_{n-2} = (d_0 y_{n-1})^{-1} . x_{1,n-1}.$$

We thus do get a filler for our $(n,0)$-horn *and* can analyse the set of fillers / solutions if we need to.

Theorem 12 *For any simplicial group, G, the classifying space, $\overline{W}G$, is a Kan complex*   ∎

Perhaps it occurs to you that it should be possible to adapt this constructive proof to give a proof that, if $f : G \to H$ is a surjection of simplicial groups, and thus a fibration, then $\overline{W}f$ will be a Kan fibration. We know already that $\overline{W}f$ *is* a fibration, as we saw this earlier, quoting some results in Goerss and Jardine, [85], but it should not be too difficult to construct a proof which took transversals in the necessary dimensions and *found* lifts for horns accordingly. This is left as a bit of a **challenge to the reader**. It is not just an exercise for amusement, however, as the analysis of fillers could give some interesting results in some cases.

We mentioned that most of this went across 'without pain' to the case of simplicial objects in a topos, $\mathcal{E}$, and hence to simplicial sheaves on a space. Perhaps a few words are needed, however, to show how this can be done. We start by thinking about how to talk about the Kan fibrations in $\mathcal{E}$, or more generally in any category with finite limits. For any object K in $Simp(\mathcal{E})$, we can form an object of $\mathcal{E}$ corresponding to the 'set of (n,k)-horns' in K. To see how to think about this, we look at $(2,1)$-horns. These correspond, in the set based case, to pairs of 1-simplices, (x_0, x_2), with $d_0 x_2 = d_1 x_0$, so are elements of the pull back:

$$
\begin{array}{ccc}
\Lambda^0[2](K) & \longrightarrow & K_1 \\
\downarrow & & \downarrow {\scriptstyle d_1} \\
K_1 & \xrightarrow{\ d_0\ } & K_0
\end{array}
$$

More generally, for a simplicial set K, $\Lambda^k[n](K)$, the set of (n,k)-horns in K is given by an iterated pullback or limit of a diagram. (If you have not seen this before, or ever handled it yourself, do try to formulate the diagram in as neat a way as possible - 'neat' is a question of taste! It is technically quite easy, but gives good practice in converting concepts across to diagrams and hence to finite limit categories.)

We thus can mimic this to get an object, $\Lambda^k[n](K)$, and an induced map, $K_n \to \Lambda^k[n](K)$, which maps an n-simplex to the (n,k)-horn of its faces other than the kth one.

Definition: If $\mathcal{E}$ is a finite limit category, a morphism, $p : E \to B$, in $Simp(\mathcal{E})$ is a *Kan fibration* if the natural maps $E_n \to \Lambda^k[n](E) \times_{\Lambda^k[n](B)} B_n$ are all epimorphisms in $\mathcal{E}$.

We can equally obtain the meaning of a *Kan object* in $Simp(\mathcal{E})$.

Beke, [19], uses the term *local Kan fibration* for what has been called a Kan fibration in $\mathcal{E}$ above. That 'local' terminology is especially good when talking about the topos case, but with, later on in these notes, a use of 'locally Kan' enriched category, it did seem a bit risky to over use 'local Kan'!

We now return to the case of simplicial groups in the usual sense.

Corollary 6 *Suppose that $NG_{n-1} = 1$, then, for any i, with $0 \leq i \leq n$, any (n,i)-horn in $\overline{W}G$ has a unique filler.*

Proof: We noted that different fillers for an (n,i)-horn differed by elements of NG_{n-1}, or its translates, thus if that group is trivial, ... ∎

Of course, we expect $\overline{W}G$ to have the same homotopy groups as G, displaced by one dimension, since there is the fibration sequence

$$G \to WG \to \overline{W}G$$

with WG contractible, so this corollary comes as no surprise. What is interesting is the detail that it gives us. If $NG_k = 1$, then clearly $\pi_k(G) = 1$ and hence $\pi_{k+1}(\overline{W}G)$ is trivial as well, but that there are unique fullers in the structure is perhaps a bit surprising, at least until one sees why.

Suppose that, as usual, G is a simplicial group and $D = (D_n)_{n \geq 1}$ is the graded subgroup of products of degeneracies. Within $\overline{W}G_n$, let

$$T_n = D_{n-1} \times G_{n-2} \times \ldots \times G_0,$$

be the subset of those elements whose first component is a product of degenerate elements, yielding a graded subset of $\overline{W}G$.

Corollary 7 *If G is a group T-complex, then $(\overline{W}G, T)$ is a simplicial T-complex.*

Proof: There is not that much to check. We know, by the proof of the theorem, that every horn has a filler in T. Uniqueness follows from the fact that G is a group T-complex. The other conditions are as easy to check as well, so are **left to you**. ∎

Corollary 8 *If G is thin in dimensions greater than n, then $\overline{W}G$ has a unique T-filler for all horns above dimension $n + 1$.* ∎

The property of being a T-complex involves all dimensions and here we are meeting some sort of weaker 'filtered' condition. This condition was studied extensively by Duskin, and used in various forms in [64, 65] and in later work. It was also used by his students Glenn, [84], and Nan Tie, [131, 132], who looked at some of the links with T-complexes. They are also used, more recently, by Beke, [19], and we, in fact, studied his approach earlier when discussing the coskeleton functors, (in particular, in our brief discussion of exact n-types and n-hypergroupoids, cf. page 85).

6.3.3 Loop spaces and loop groups

We now turn to ΩBG. Although not strictly necessary, it will help to shift our perspective slightly and talk a bit more on some generalities. Let S^0 be the pointed simplicial set with two vertices and only degenerate simplices in dimensions higher than 1. In other words, it is the 0-sphere. The reduced suspension SS^0 is S^1, the circle, which can also be realised as $\Delta[1]/\partial\Delta[1]$, the circle realised as the interval with the ends identified to a single point. The loop space, ΩK, on a pointed connected simplicial set, K, is then $\underline{\mathcal{S}}_*(S^1, K)$, or more briefly, K^{S^1}, the simplicial set of pointed maps from S^1 to K. (It will be a Kan complex if K is one.) As in the topological case, ΩK has the structure of an 'H-space'. This refers to a compositional structure *up to homotopy*, so we have

$$\mu : \Omega K \times \Omega K \to \Omega K,$$

given by composition of loops. Topologically this is just that: first do one loop, then the other, then rescale to get a map from $[0, 1]$ again. The rescaling means that this μ is not associative, but is associative up to a homotopy. There are also 'reverses', which are inverses up to homotopy, and it all fits together to make ΩK a 'group up to homotopy'. (Again the homotopies can be linked together to make a homotopy coherent version of a group.) The same can be done in the simplicial case provided that K is Kan. (This is a **good exercise to attempt**, to see once more the use of 'fillers' as a form of algebraic structure.)

If K is not reduced, we can replace it by a homotopy equivalent reduced simplicial set. (In fact we want $K = \overline{W}G$ and that *is* reduced.) For such a K, the simplicial group GK is often called the *loop group* of K. (Look back to page 125, if you need to review the construction of GK.) What is the connection between ΩK and GK?

It is clear there should be one as the free group construction involved in the definition of GK uses concatenation of strings of simplices and that is the algebraic analogue of composition of paths, however it is associative, has inverses, etc., as it gives a group. It looks like an abstract algebraic model of ΩK, which replaces the homotopy coherent multiplication by an algebraic one, but, as a result, gets a much bigger structure. Even in dimension 0, $\Omega K_0 \cong K_1$, whilst GK_0 is the free group on K_1. (This is again a **useful place** to see what the two structures look like, in low dimensions, and to see if there is a 'natural' map between them.) If we could replace Ω by G, our life would simplify as G is left adjoint to $\overline{W}$ and so, for any simplicial group, H, there is a natural map

$$G\overline{W}H \to H,$$

which is a weak equivalence, i.e., it induces isomorphisms on all homotopy groups, then we would be able to identify three more terms of the Puppe sequence. In fact for any reduced K, GK and ΩK *are* weakly equivalent. We will not give the proof, referring instead to the discussion in Goerss and Jardine, [85], in particular the proof on p. 285. (This is very neat for us as it uses both ΓK, there called PK, and induced fibrations in a very similar way to our earlier treatment of the Puppe

sequence.) If G is more interesting and is not reduced, then GK is equivalent to a disjoint union, indexed by $\pi_0(G)$, of simplicial sets that 'look like' copies of ΩG, namely loops, not at the identity element, but at some representative of a connected component of G. This will shortly be linked up with the décalage construction.

Putting all this together, we get that if

$$K \xrightarrow{u} H \xrightarrow{v} G$$

is a short exact sequence of simplicial groups, then the Puppe sequence of Bv ends:

$$\Omega G \to K \xrightarrow{u} H \xrightarrow{v} G \to BK \xrightarrow{Bu} BH \xrightarrow{Bv} BG.$$

We need to add what might be considered a cautionary note. To emphasise the *ideas* behind this sequence, we have handled the case of simplicial groups. For many of the applications, we have to work with sheaves of simplicial groups or, more generally, simplicial group objects in some topos, $\mathcal{E}$. In those cases the meaning of such terms as 'fibration' or 'weak equivalence' needs refining, much as the notion of 'equivalence' between categories needs adjusting before it can be used to its full potential with the 'stacks' that we will meet in the next chapter. The category in which one 'does' one's homotopy is then naturally to be considered with a Quillen model category structure and $[-,-]$ is replaced by $Ho(Simp(\mathcal{E}))(-,-)$, the 'hom-set' in the category obtained from that of simplicial objects in $\mathcal{E}$ by inverting the weak equivalences. These technicalities *do* complicate things to quite a large amount and are very non-trivial to describe in detail, however the idea is the same and the technicalities are there just to bring that idea to its most rigorous form. We have left out these technicalities to concentrate on the intuition, but they cannot be completely ignored. (Some idea of the possible detailed approaches to this can be found in Illusie's thesis, [98, 99], Jardine's paper, [101] and various more recent works on simplicial sheaves.)

6.3.4 Applications: Extensions of groups

Suppose we have our old situation, namely an extension of groups, or rather of sheaves of groups,

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to 1$$

(as in section **??**). We can replace each by a constant simplicial group, L by $K(L,0)$, etc. (To simplify notation we will, in fact, abbreviate $K(L,0)$ back to L, whenever this is feasible.) We now apply the classifying space construction and take the corresponding Puppe sequence. The result will be

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to BL \to BM \to BN.$$

(Here we are abusing notation even more, as the first three terms are the underlying simplicial sheaves of the corresponding sheaves of simplicial groups, which are , ... and so on, but writing $U(K(L,0))$ seems silly and it would get worse, so ... .)

Note that in this sequence, we have that $\Omega^2 BN$ is equivalent to ΩN, which is contractible, which explains the 1 on the left hand end.. The classifying spaces are the nerves of the corresponding groupoids, $BL = Ner(L[1])$, etc.

All this is happening in $Sh(B)$ (or, more generally, in a topos, $\mathcal{E}$). Given an open cover $\mathcal{U}$ of B, with nerve $N(\mathcal{U})$, we get a long exact sequence of groups and pointed sets:

$$1 \to [N(\mathcal{U}), L] \to [N(\mathcal{U}), M] \to [N(\mathcal{U}), N] \to [N(\mathcal{U}), BL] \to [N(\mathcal{U}), BM] \to [N(\mathcal{U}), BN],$$

and passing to the colimit over coverings, this gives

$$1 \to L(B) \to M(B) \to N(B) \to \check{H}^1(B, L) \to \check{H}^1(B, M) \to \check{H}^1(B, N).$$

This is exactly the exact sequence that we discussed earlier, again in section **??**. Note that we have not yet got our hands on any substitute for the $\check{H}^2(B, L)$, that exists in the Abelian case.

6.3.5   Applications: Crossed modules and bitorsors

Suppose $\mathsf{M} = (C, P, \partial)$ is a sheaf of crossed modules. It would be good to examine the simplicial view of relative M-torsors in a similar way. We have a sheaf of simplicial groups given by $K(\mathsf{M})$ and have identified $colim[N(\mathcal{U}), K(\mathsf{M})] = colim H^0(N(\mathcal{U}), \mathsf{M})$ with $\pi_0(\mathsf{M}-Tors)$, which is a group. We also showed that any M-torsor, (E, t), had that E is a C-torsor with t a trivialisation of $\partial_*(E)$. This suggests some sort of exact sequence:

$$\pi_0(\mathsf{M}-Tors) \to \pi_0(Tors(C)) \overset{\partial_*}{\to} \pi_0(Tors(P)),$$

i.e., anything in $Tors(C)$ that is sent to the base point (that is, the class of the trivial torsor) in $Tors(P)$, comes from an M-torsor. We can see this geometrically as we saw earlier. What is neat is that if (E, t) and (E', t') are M-torsors, with E and E' equivalent as C-torsors, then we can assume $E = E'$ and can use the trivialisations t and t' to obtain a global section, p, of P such that $t' = p.t$. The implication is that

$$P(B) \to \pi_0(\mathsf{M}-Tors) \to \pi_0(Tors(C))$$

is also exact. This can also be seen from the Puppe sequence.

First a very useful bit of the simplicial toolkit. We form the décalage of $K(\mathsf{M})$. (Recall $K(\mathsf{M})$ is the simplicial group associated to M, that is, it is formed as the internal nerve of the internal category corresponding to M, that it has P in dimension 0, $C \rtimes P$ in dimension 1, etc. It also has a Moore complex which is of length 1 and is exactly $C \overset{\partial}{\to} P$.)

What is the décalage?

Definition: The *décalage* of an arbitrary simplicial set, Y, is the simplicial set, $Dec\,Y$, defined by shifting every dimension down by one, 'forgetting' the last face and degeneracy of Y in each dimension. More precisely

- $(Dec\,Y)_n = Y_{n+1}$;
- $d_k^{n, Dec\,Y} = d_k^{n+1, Y}$;
- $s_k^{n, Dec\,Y} = s_k^{n+1, Y}$.

This comes with a natural projection, $d_{last} : Dec\,Y \to Y$, given by the 'left over' face map. (Check it is a simplicial map.) We will denote this by p, for 'projection'. Moreover this map gives a homotopy equivalence

$$Dec\,Y \simeq K(Y_0, 0),$$

between $Dec\,Y$ and the constant simplicial set on Y_0. The homotopy can be constructed from the 'left-over' degeneracy, s_{last}^Y. (A full discussion of the décalage can be found in Illusie's thesis,

[98, 99] and Duskin's memoir, [64]. Be aware, however, some sources may use the alternative form of the construction that forgets the *zeroth* face rather than the *last* one. This works just as well. The translation between the two forms is quite easy, if sometimes a bit time consuming!)

Of course, this same construction works for simplicial objects in any category. We need it mainly for (sheaves of) simplicial groups and, in particular, as hinted at earlier, we need $Dec\,K(\mathsf{M})$. We list some properties of this simplicial group:

(i) $Dec\,K(\mathsf{M})_0 \cong C \rtimes P$, $Dec\,K(\mathsf{M})_1 \cong C \rtimes C \rtimes P$, and in general, $Dec\,K(\mathsf{M})_n \cong C^{(n+1)} \rtimes P$. The face maps are given by

$$\begin{aligned} d_0(c_n, \ldots, c_0, p) &= (c_n, \ldots, c_1, \partial c_0.p) \\ d_i(c_n, \ldots, c_0, p) &= (c_n, \ldots, c_i c_{i-1}, \ldots, c_0, p) \quad 0 < i < n \\ d_0(c_n, \ldots, c_0, p) &= (c_n c_{n-1}, \ldots, c_0, p) \end{aligned}$$

with degeneracies given by suitable insertions of identities.

(ii) $Dec\,K(\mathsf{M})$ has Moore complex isomorphic to one of the form

$$C \to C \rtimes P.$$

Here we clearly have $Ker\,d_1 = \{(c_1, c_0, p) \mid c_1 = c_0^{-1}, p = 1\} \cong C$. We also have a boundary, induced by d_0, so the boundary sends $(c^{-1}, c, 1)$ to $(c^{-1}, \partial c)$. If this looks strange, **just check** that $(c^{-1}, c, 1)((c')^{-1}, c', 1) = ((cc')^{-1}, cc', 1)$. (Don't forget the Peiffer identity!)

(iii) The boundary is a monomorphism and its image is the kernel of the homomorphism from $C \rtimes P$ to P that sends (c, p) to $\partial c.p$. (That makes sense as that is the target / codomain map of the internal category or cat^1-group associated to M.)

(iv) $Dec\,K(\mathsf{M})$ is homotopy equivalent to the constant simplicial group on P. (This can be seen from the Moore complex, but also from the retraction of $Dec\,K(\mathsf{M})$ onto the subsimplicial group given by all $(1, \ldots, 1, p)$. That map is a deformation retraction with the 'extra degeneracy', $s_{\ell ast}$, of the décalage construction giving the homotopy, (**for you to check**). This is neat, because it is explicit and natural and thus can provide a more geometric picture than merely stating that there is a weak equivalence of simplicial groups between $Dec\,K(\mathsf{M})$ and $K(P, 0)$.)

(v) The morphism $\mathsf{p} : Dec\,K(\mathsf{M}) \to K(\mathsf{M})$ is an epimorphism, hence is a fibration. (It is, in fact, split at each level by the last degeneracy map of $K(\mathsf{M})$.) We can give p explicitly by $\mathsf{p}(c_n, \ldots, c_0, p) = (c_{n-1}, \ldots, c_0, p)$, hence:

(vi) The kernel of p is given by $Ker\,\mathsf{p} = \{(c, 1, \ldots, 1, 1) \mid c \in C\}$ with the face and degeneracy maps given by the restrictions of the above, so $Ker\,\mathsf{p}$ is isomorphic to $K(C, 0)$.

(vii) Within the context of our much earlier discussion of crossed modules as being given by fibrations (page 15), we had that if G is a simplicial group and $N \lhd G$ a normal simplicial subgroup, then applying π_0 to the inclusion of N into G gave us a crossed module. The proof that, up to isomorphism, all crossed modules arise in this way was left to the reader! Here it is:

If we take $G = Dec\,K(\mathsf{M})$, and $N = Ker\,\mathsf{p}$, then $\pi_0(N) \to \pi_0(G)$ is $\partial : C \to P$ and the actions agree, (all 'up to isomorphism', of course).

This is at the heart of the algebraic proof of Loday's theorem (see 3.5) that cat^n-groups / crossed n-cubes model all connected homotopy $(n+1)$-types. Its appearance here is not accidental.

We thus have an exact sequence of simplicial groups arising from M:

$$1 \to Ker\,\mathsf{p} \to Dec\,K(\mathsf{M}) \to K(\mathsf{M}) \to 1$$

corresponding to

$$K(C,0) \to K(P,0) \to K(\mathsf{M}),$$

(which is not exact!).

At a crossed module level, we get

$$
\begin{array}{ccccc}
1 & \longrightarrow & 1 & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & P & \longrightarrow & P
\end{array}
$$

is homotopy exact, or, more exactly (pun intended!) that

$$
\begin{array}{ccccc}
1 & \longrightarrow & C & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & C \rtimes P & \longrightarrow & P
\end{array}
$$

is exact.

If we pass to the Puppe sequence, it will end

$$\Omega K(\mathsf{M}) \to C \to P \to K(\mathsf{M}) \to BC \to BP \to BK(\mathsf{M}).$$

Going through the usual process of applying $[N(\mathcal{U}), -]$ for an open cover $\mathcal{U}$ of the base space B, followed by the colimit over such $\mathcal{U}$s, we get

Proposition 41 *For any crossed module, M, there is an exact sequence*

$$1 \to \check{H}^{-1}(B,\mathsf{M}) \to C(B) \to P(B) \to \pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(C)) \to \pi_0(Tors(P)) \to \check{H}^1(B,\mathsf{M}).$$

■

There are two 'mysterious' terms here. The second is the 1st Čech hypercohomology of B with coefficients in M. We have, sort of, met this earlier. It is

$$\check{H}^1(B,\mathsf{M}) = colim_{\mathcal{U}}[N(\mathcal{U}), BK(\mathsf{M})].$$

The treatment we have given it here, and the language we have available, is however not yet rich enough to yield a good geometric interpretation. For that we will need stacks and gerbes, and we will start on them in the next chapter!

The other strange term is $\check{H}^{-1}(B,\mathsf{M})$, which comes from the various $[N(\mathcal{U}), \Omega K(\mathsf{M})]$. We can calculate $\Omega K(\mathsf{M})$ explicitly using its description as the simplicial group of maps from S^1_* to $K(\mathsf{M})$.

Lemma 26 *(i) There are isomorphisms $\Omega K(\mathsf{M}) \cong K(\pi_1(\mathsf{M}),0)$, the constant simplicial group on the kernel $\pi_1(\mathsf{M}) = Ker(\partial : C \to P) \cong \pi_1(K(\mathsf{M}))$.*

(ii) There are isomorphisms $\check{H}^{-1}(B,\mathsf{M}) = \check{H}^0(B,\pi_1(\mathsf{M})) \cong \pi_1(\mathsf{M})(B)$, the group of global sections of $\pi_1(\mathsf{M})$.

Proof: This is just a question of calculation so is left to you the reader. ■

6.3.6 Examples and special cases revisited

We can use the analyses of Puppe sequences and their applications to refine a bit more the information on relative M-torsors for the 'examples and special cases'. We first apply our exact sequence of the previous paragraph.

The first example is when $\mathsf{M} = (1, P, inc)$ and the exact sequence confirms the isomorphism between $P(B)$ and $\pi_0(\mathsf{M}-Tors)$. When M is $A[1] = (A \to 1)$ for Abelian A, the sequence gives, as expected, confirmation that $\pi_0(\mathsf{M}-Tors) \cong \pi_0(Tors(A))$ and that the latter has a group structure.

For an inclusion crossed module / normal subgroup pair, we can compare the exact sequence coming from $1 \to N \to P \to G \to 1$ with that from $\mathsf{M} = (N, P, \partial)$, with ∂ the inclusion. The induced maps give us a map of exact sequences

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & N(B) & \longrightarrow & P(B) & \longrightarrow & \pi_0(\mathsf{M}-Tors) & \longrightarrow & \pi_0(Tors(N)) & \longrightarrow & \pi_0(Tors(P)) & \longrightarrow & \check{H}^1(B, \mathsf{M}) \\
& & \Big\| & & \Big\| & & \Big\downarrow & & \cong\Big\downarrow & & \cong\Big\downarrow & & ?\Big\downarrow \\
1 & \longrightarrow & N(B) & \longrightarrow & P(B) & \longrightarrow & G(B) & \longrightarrow & \check{H}^1(B, N) & \longrightarrow & \check{H}^1(B, P) & \longrightarrow & \check{H}^1(B, G)
\end{array}$$

which again gives $\pi_0(\mathsf{M}-Tors) \cong G(B)$, and suggests that the mysterious $\check{H}^1(B, \mathsf{M})$, in this special case, is our better known $\check{H}^1(B, G)$, i.e., $\pi_0(Tors(G))$.

The last case we looked at was $\mathsf{M} = (M, G, 0)$. The long exact sequence has the induced map, ∂_*, trivial, so gives us

$$1 \to G(B) \to \pi_0(\mathsf{M}-Tors) \to \pi_0(Tors(M)) \to 1.$$

To examine the other situation considered on page **??**, we need to apply our analysis of exact sequences of simplicial groups to another case.

6.3.7 Devissage: analysing $\mathsf{M}-Tors$

We saw that for any (sheaf of) crossed module(s) M, we had a short exact sequence

$$\begin{array}{ccc}
K & \longrightarrow C \longrightarrow & N \\
\downarrow & \downarrow & \downarrow \\
1 & \longrightarrow P \longrightarrow & P,
\end{array}$$

or

$$\pi_1(\mathsf{M})[1] \to \mathsf{M} \to \pi_0(\mathsf{M})$$

if you prefer, (as $\pi_0(\mathsf{M}) = \pi_0(K(\mathsf{M})) = P/N$). (We only saw this for a crossed module, but clearly the argument goes through with only trivial changes in any topos, given suitable definitions!) Applying the associated simplicial group functor, K, this gives that

$$K(\pi_1(\mathsf{M}), 1) \to K(\mathsf{M}) \to K(\pi_0(\mathsf{M}), 0)$$

is an exact sequence of simplicial groups.

Theorem 13 *For any crossed module, M, there is an exact sequence*

$$1 \to \pi_0(Tors(\pi_1(\mathsf{M}))) \to \pi_0(\mathsf{M}-Tors)) \to \pi_0(\mathsf{M})(B) \to$$

$$\check{H}^2(B, \pi_1(\mathsf{M})) \to \check{H}^1(B, \mathsf{M}) \to \pi_0(Tors(\pi_0(\mathsf{M}))).$$

Proof: The proof merely is to identify the various terms from the Puppe sequence. Firstly the general form of such sequences, seen above, gives

$$\to \check{H}^{-1}(B, \pi_0(\mathsf{M})) \to \check{H}^0(B, \pi_1(\mathsf{M})[1]) \to \check{H}^0(B, K(\mathsf{M})) \to \check{H}^0(B, \pi_0(\mathsf{M})) \to \check{H}^1(B, \pi_1(\mathsf{M})[1]) \to \dots$$

The first of these terms is trivial since for a general crossed module, $\Omega K(\mathsf{N})$ is $K(Ker\partial, 0)$, up to equivalence, so in our case in which $\mathsf{N} = (1 \to \pi_0(\mathsf{M}))$, it will be trivial. (Remember $\check{H}^{-1}(B, \mathsf{N}) = colim_{\mathcal{U}}[N(\mathcal{U}), \Omega K(\mathsf{N})]$.)

The next term $\check{H}^0(B, \pi_1(\mathsf{M})[1]) \cong \check{H}^1(B, \pi_1(\mathsf{M})) \cong \pi_0(Tors(\pi_1(\mathsf{M})))$, by our earlier calculations (case (ii) above). The next two terms are routine to handle, whilst that $\check{H}^1(B, \pi_1(\mathsf{M})[1])$ is isomorphic to $\check{H}^2(B, \pi_1(\mathsf{M}))$ is a classical result that is easy to check anyhow. Finally the remaining terms are standard. ∎

Note that this gives some new information on $\mathsf{M}-Tors$, indicating the difference between this category for general M and for the particular special cases considered earlier.

Chapter 7

Topological Quantum Field Theories

(As a basic reference for the initial parts of this chapter, you might look at Joachim Kock's book, [107], or Quinn's introductory lectures, [145].)

7.1 What is a topological quantum field theory?

Topological Quantum Field Theories form a relatively new area of mathematics, somewhere near the 'frontier' , always a fuzzy one, between mathematics, and mathematical physics. To some extent it studies 'space-times', and uses them to look for possibly new invariants and properties of manifolds. It has interesting interactions with cohomology theory and we will look at some of these. (almost universally we will use the abbreviation 'TQFT' for 'topological quantum field theory'.)

7.1.1 What is a TQFT?

In Topological Quantum Field Theory, one studies $(d-1)$-dimensional orientable smooth or piecewise linear manifolds and the d-dimensional (orientable) cobordisms between them, pictured, for $d = 2$, as:
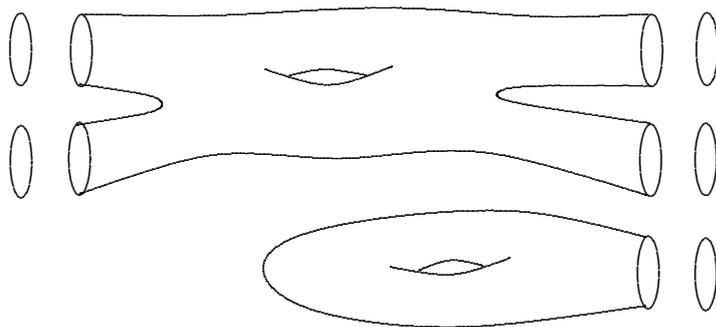
$$X = S^1 \sqcup S^1 \qquad\qquad M : X \longrightarrow Y \qquad\qquad Y = S^1 \sqcup S^1 \sqcup S^1$$
$$\text{but we may write } Y = S^1 \otimes S^1 \otimes S^1,$$

For later use we record the following:

Definition: A *d-cobordism*, $W : X_0 \to X_1$, is a compact oriented d-manifold, W, whose boundary is the disjoint union of pointed closed oriented $(d-1)$-manifolds, X_0 and X_1, such that the orientation of X_1 (resp. X_0) is induced by that on W (resp., is opposite to the one induced from that on W).

After some technical difficulties, one shows these form a category, $d-Cob$, with $(d-1)$-manifolds as its objects and, more-or-less, the d-cobordisms as the morphisms. (The 'more-or-less' is that as far as their being morphisms is concerned, we have to consider two cobordisms to be 'the same' if they are isomorphic relative to the boundary, i.e., there is an isomorphism between them that preserves the boundaries. This is examined in more detail in the sources that have been indicated for the basic theory.) If and when it is necessary, we will add a suffix PL, $Diff$ or Top to distinguish the cases in which the manifolds are to be piecewise linear (PL), smooth ($Diff$) or merely topological.) This category has a monoidal category structure given by disjoint union, $\sqcup$, but which will often be written as a tensor, $\otimes$. The unit of the structure is the empty manifold, $\emptyset$. We thus formally have $d-Cob = (d-Cob, \sqcup, \emptyset)$, with the convention that we used in section **??**, page **??**.

We will often use the case $d = 2$ as an illustrative example. For instance, in the above picture, $M = M_1 \otimes M_2$, where $M_1 : X \to Y_1 = S^1 \otimes S^1$, $M_2 : \emptyset \to Y_2 = S^1$ and $Y = Y_1 \otimes Y_2$.

We will also need the monoidal category, $Vect^\otimes = (Vect_\Bbbk, \otimes, \Bbbk)$, of (finite dimensional) complex vector spaces with the usual tensor product. (We could use fields, $\Bbbk$, other than the usual one, $\mathbb{C}$, of complex numbers and the minimal one for things to be fairly simple, $\mathbb{Q}$. We may even use a commutative ring, R, with some restriction on the characteristic, although characteristic zero will always work.)

Definition: A TQFT is a monoidal functor, $Z : d-Cob \to Vect^\otimes$, or, more generally, to $R - Mod^\otimes$, so Z preserves $\otimes$ and $Z(\emptyset) = \mathbb{C}$, resp. R. For an object, X of $d-Cob$, $Z(X)$ is sometimes called the *state space* or *state module* of X.

Terminology: There is some disagreement as to terminology when considering $d-Cob$, and as we will try to stay relatively close to sources, we will hit this full on! There are two basic conventions. In one the key dimension is that of the manifolds, whilst in the other it is that of the cobordisms. The above uses the second of these. (Later when discussing homotopy quantum field theories, the first convention tends to dominate the literature, so we will need to be careful.) There is also an intermediate situation in which one writes $(n+1)-Cob$, emphasising both dimensions, so $(2+1)-Cob$ is the monoidal category of 2-dimensional manifolds, and cobordisms that are 3 dimensional. In this convention, which is a very useful one, a $(2+1)$ dimensional TQFT is one defined on what we would denote as $3-Cob$.

We mentioned 'technical difficulties'. These relate mostly to composition of cobordisms and identification of identities. We will not go into the details, as this is well discussed in the main sources, using different ways of getting around the difficulties. The simplest way is to *think* of the morphisms as equivalence classes of cobordisms, under isomorphism (so diffeomorphism if in the smooth case), relative to the two ends. That gives sufficient detail to be going on with. We will discuss this some more later on, but the **reader is urged** to look at one or more of the sources to see the means used to get a monoidal category structure.

Definition: A *morphism*, $\varphi : Z \to Z'$, of TQFTs will be a monoidal (natural) transformation between them.

All such morphisms are, in fact, isomorphisms. (**You are left** to try to prove this or to look it up.)

Exploring briefly the simple case of $d = 1$, clearly any 1-manifold is a disjoint union, $X = (S^1)^{\otimes n}$, of n copies of S^1 for some $n \geq 0$, so $Z(X) = Z(S^1)^{\otimes n}$, and much of the structure of Z will be about the vector space $Z(S^1)$. This is not quite right. The point is that we have to take into account an orientation of the circle. Let us fix S^1 to have an anticlockwise orientation and write $-S^1$ for the opposite.

If we put $A = Z(S^1)$, we get an algebra structure on A given by a linear map

$$\mu : A \otimes A \to A,$$

that is, from $Z(S^1 \sqcup S^1)$ to $Z(S^1)$. (We will look at algebras in this way in more detail in the next section.) To get this we use the cobordism:
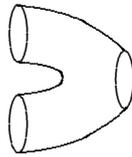
known also as the 'pair of pants'. It has a useful representation as a disc with two holes
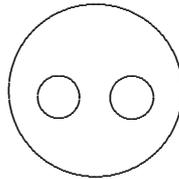
with all three circles given an anti-clockwise orientation. The outer circle corresponds to the right hand 'output' end with two inner circles being 'inputs' at the left side of the previous picture.

The cobordism
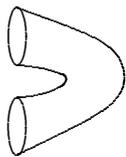
gives a bilinear form $A \otimes A \longrightarrow \mathbb{C}$, which is not hard to show is non-degenerate, so A must be finite dimensional (because the pairing gives an isomorphism between A and its dual). We thus do not really have to *impose* finite dimensionality on the vector spaces as, if they do form a TQFT, they will be finite dimensional. It also shows that $Z(-S^1) \cong A^*$, the dual space of A, since we can picture this cobordism as constructed from a cylinder, which is bent back on itself. This is quite general and does not just apply to this 1+1 dimensional case.

A lot of other structure can be visualised in similar ways. The algebra A has a unit

$$\mathbb{C} \to A$$

corresponding to

which is, of course a cobordism from the empty manifold to the circle.

The proof that this is a unit is simply:
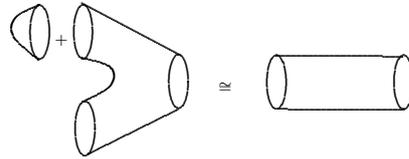
i.e., $\mu(1, a) = a$, and so on.

We can read the diagrams 'in a mirror' to get a coalgebra structure, $A \to A \otimes A$, corresponding to
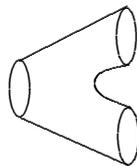
There are similarly a copairing, $\mathbb{C} \to A \otimes A$, and a counit, $A \to \mathbb{C}$, and these all satisfy a bunch of equations including a 'Frobenius equation' linking algebraic and coalgebraic structures that we will see shortly (in the next section).

Verification of axioms such as associativity for the algebra structure can all be done in a diagrammatic form. You compose the corresponding cobordisms, and they are evidently isomorphic / equivalent. As we will see, for the Frobenius equation is easier to understand in diagrammatic form than to write the equations.

What is the sort of algebra involved here. We explore this in the next section.

7.1.2   Frobenius algebras: an algebraic model of some of the structure

(These are discussed in some detail in Kock's book, [107], and occur, under a different name, in Quinn's notes, [145]. They are also discussed in a Wikipedia article, which you can safely be left to find and read. We will give a brief introduction to them strongly influenced by Joachim Kock's lectures at the Almería workshop on TQFTs.)

We will fix a field $\mathbb{k}$. We think of this usually as being $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$, but others can be useful. As usual, certain situations may benefit from using a more general commutative ring, R, as a 'ground ring'.

We first look at algebras, as this introduces a way of thinking about algebraic structures in a monoidal category, here, $Vect_{\mathbb{k}}^{\otimes} = Vect^{\otimes} = (Vect_{\mathbb{k}}, \otimes, \mathbb{k})$, so the objects are vector spaces over $\mathbb{k}$, the 'multiplication' is the usual tensor product for which $\mathbb{k}$ is the unit. (We may sometimes omit the suffix $\mathbb{k}$ if it is not essential for the discussion.)

Definition: A $\mathbb{k}$-*algebra*, A, is a monoid in the monoidal category, $Vect^{\otimes}$.

Taking this apart, a monoid in the usual situation is a set, M, with a multiplication $\mu : M \times M \to M$, and a unit, satisfying associativity and unit axioms. Internalising this into a monoidal category

we replace the product structure of *Set* with the $\otimes$ of the monoidal category, and, making several other slight adjustments,we have that A has a multiplication

$$\mu : A \otimes A \to A,$$

and a unit

$$\eta : \mathbb{k} \to A,$$

satisfying an associativity condition namely that

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu \otimes A} & A \otimes A \\
{\scriptstyle A \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
$$

commutes,
and unit laws (left unit)

$$
\begin{array}{ccc}
\mathbb{k} \otimes A & \xrightarrow{\eta \otimes A} & A \otimes A \\
 & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mu} \\
 & & A
\end{array}
$$

It is best if one requires also a right unit law, and, if we were being very strict with ourselves, we should allow for the fact that the monoidal category, $Vect^{\otimes}$, is not strict, but we will not handle this here.

Definition: A *Frobenius algebra* is a finite dimensional $\mathbb{k}$-algebra, A, equipped with a non-degenerate 'associative' pairing,

$$\beta : A \otimes A \to \mathbb{k}.$$

The pairing is sometimes called the *Frobenius form* of the algebra, A.

There are some terms here that need a bit more detail. First:

Definition: A pairing, as above, is said to be *associative* if for all $x, a, y \in A$, we have

$$\beta(x, ay) = \beta(xa, y).$$

There are two forms of non-degeneracy. Given a pairing, $\beta : V \otimes W \to \mathbb{k}$, we get an induced map

$$V \to W^* = Vect_{\mathbb{k}}(W, \mathbb{k})$$

given by

$$v \mapsto \overline{v}$$

where $\overline{v}(w) = \beta(v \otimes w)$. There is a similarly defined one from W to V^*.

Definition: (i) A pairing, $\beta : V \otimes W \to \mathbb{k}$, is *weakly non-degenerate* if the induced maps, $V \to W^*$ and $W \to V^*$, are both injective.

(ii) A pairing, β, is *non-degenerate* if there is some $\gamma : \Bbbk \longrightarrow W \otimes V$ such that

$$V \xrightarrow{V \otimes \gamma} V \otimes W \otimes V \xrightarrow{\beta \otimes V} V$$

is the identity on V, whilst

$$W \xrightarrow{\gamma \otimes W} W \otimes V \otimes W \xrightarrow{W \otimes \beta} W$$

is that on W.

Here we are, once again, slightly abusing notation, since we should really write $V \xrightarrow{\cong} V \otimes \Bbbk \xrightarrow{V \otimes \gamma}$ $V \otimes W \otimes V$, and so on, and even take note of the associativity isomorphisms $V \otimes (W \otimes V) \cong$ $(V \otimes W) \otimes V$, as $Vect_{\Bbbk}^{\otimes}$ is not a strict monoidal category, however it usual to leave such details aside, unless strictly needed.

The weak form of degeneracy is equivalent to the strong one if V and W are finite dimensional.

Remark: There are two other forms of definition that can be given for Frobenius algebra. One is less categorical, the other is more so. We will look at the second of these in a bit of detail later, but we note that the less categorical one is sometimes very useful when verifying that an example algebra *is* a Frobenius algebra. It is as follows:

Proposition 42 *A $\Bbbk$-algebra, A, is Frobenius if and only if (i) it is finite dimensional and (ii) it has a linear functional, $\varepsilon : A \to \Bbbk$, such that $Null(\varepsilon)$ contains no non-zero left ideal.*

Sketch proof: Given ε, define $\beta(x \otimes y)$ to be $\varepsilon(xy)$, and, conversely, given a β, define $\varepsilon(x) = \beta(1 \otimes c) = \beta(x \otimes 1)$. The conditions can then be safely left as an exercise.   ■

Examples of Frobenius algebras

1. Take $A = \Bbbk$ itself with ε any non-zero map.

2. Any finite field extension of $\Bbbk$ will yield a Frobenius algebra. As an example, consider $\mathbb{C}$ as an $\mathbb{R}$-algebra, taking $\varepsilon(x + iy) = x$.

3. Any matrix ring, $Mat_{n \times n}(\Bbbk)$, gives a Frobenius algebra on taking ε to be the trace.

4. Any finite dimensional semi-simple algebra is Frobenius.

5. For G a finite group, define as before $\Bbbk G$ to be the group algebra of G, and take $\varepsilon : \Bbbk G \to \Bbbk$ to be the usual augmentation (see page 36, adapted to have coefficients in $\Bbbk$).

6. For G again a finite group, and taking $\Bbbk = \mathbb{C}$, let

$$R(G) = \{\varphi : G \to \mathbb{C}^{\times} \mid \varphi \text{ is constant on conjugacy classes}\}$$

be the ring of class functions of G. (Here $\mathbb{C}^{\times}$ is the group of non-zero complex numbers.) We set

$$\beta(\varphi, \psi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \psi(g^{-1}).$$

This again is a Frobenius algebra.

To handle Frobenius algebras, or more generally *Frobenius objects* in a more general monoidal category, it is useful to use a graphical calculus. (The treatment here is again strongly based on Joachim Kock's lectures in Almería.)

Objects are labelled A, B, etc., but maps are labelled by lines with circled function labels, or sometimes, for ease of typing, labelled bullet points, on them:

$$A \quad\longrightarrow\quad \boxed{f} \quad\longrightarrow\quad B$$

Tensors are represented by vertical juxtaposition, so a map $g : A \otimes B \to C$ becomes

$$B \searrow \quad A \nearrow \quad \boxed{g} \longrightarrow C$$
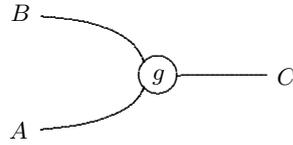
or more complicated, for $g : A \otimes A' \to B \otimes B' \otimes B''$,

$$A' \quad A \quad \boxed{g} \quad B'' \quad B' \quad B$$
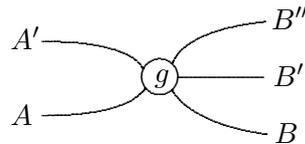
With this, we note the algebra structure: the multiplication

$$\mu : A \otimes A \to A$$

$$A \quad A \quad \boxed{\mu} \longrightarrow A$$

and the unit,

$$\eta : \Bbbk \to A$$
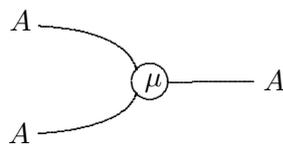
$$\boxed{\eta} \longrightarrow A$$

This perhaps takes a bit more thought. We have $\Bbbk = A^{\otimes 0}$, so, as for tensor powers such as $A^{\otimes n}$, we stack n copies of A, here we stack *no* copies of A!

For the Frobenius form, we have

$$\beta : A \otimes A \to \Bbbk$$

$$A \quad A \quad \boxed{\beta}$$
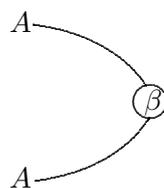
or, if we want to specify the structure in the other form, we can give a counit

$$\varepsilon : A \to \Bbbk$$

$$A \longrightarrow \boxed{\varepsilon}$$
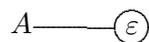
The γ that was used in the definition of 'non-degenerate' becomes:

$$\gamma : \Bbbk \to A \otimes A$$
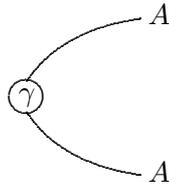
These diagrams are, more or less, the diagrams relating to 0+1 TQFTs and, again more or less, you can construct the 1+1 versions by taking a product of a diagram with S^1. We will return to this point a little later. For the moment, playing with these diagrams gives some neat pictorial versions of the axioms of Frobenius algebra. We will not give them all - they can be found in numerous sources in the literature - but give some as a taster.

Associativity

Unit
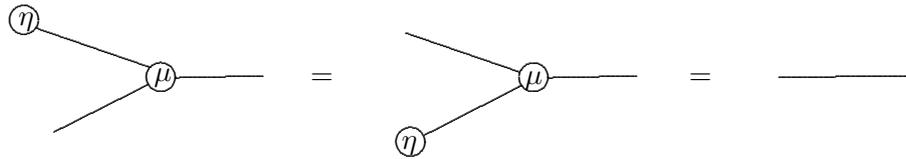
Non-degeneracy
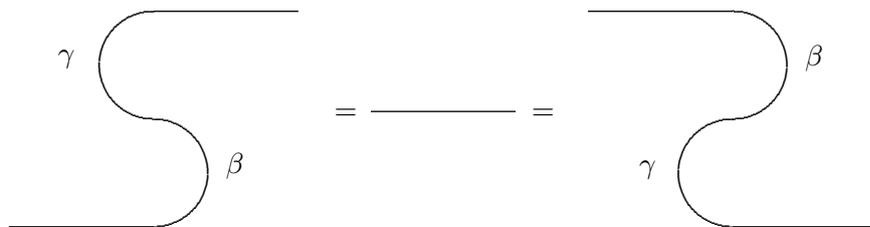
and **you are left** to do the relation between β and ε, etc.

We can now look at the more categorical version of the definition of Frobenius algebra. We state it as a definition, but, of course, it is a *re-definition* really.

Definition: A *Frobenius $\Bbbk$-algebra* is $\Bbbk$-vector space with maps

$$\mu : A \otimes A \to A, \qquad \eta : \Bbbk \to A,$$
$$\delta : A \to A \otimes A, \qquad \varepsilon : \Bbbk \to A,$$

such that (i) unit rule, (ii) counit rule (i.e., mirror of unit), and (iii) the Frobenius rule:
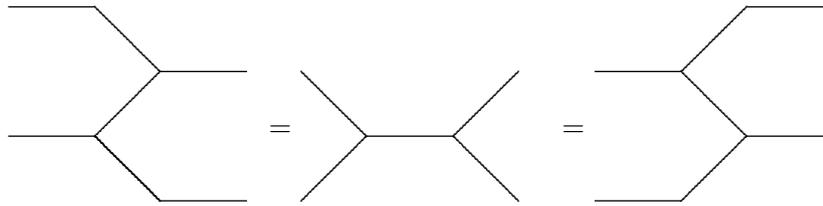
Remark: You can derive associativity from these.

Proposition 43 *This definition is equivalent to the earlier one.*

Sketch proof plan: First define δ using μ and γ, then check the axioms etc. In the other direction, we need β and this can be constructed from μ and ε. ∎

It takes a bit of time to become proficient in manipulating these diagrams, but they are very often used in studies of 'tensor categories'. If you prefer to think of surface diagrams, just take any of the above and take its 'product with' S^1, the circle. The manipulations envisaged above are just homeomorphisms of the resulting cobordisms. As an example, the last one of the sketch proof (constructing β from μ and ε), is
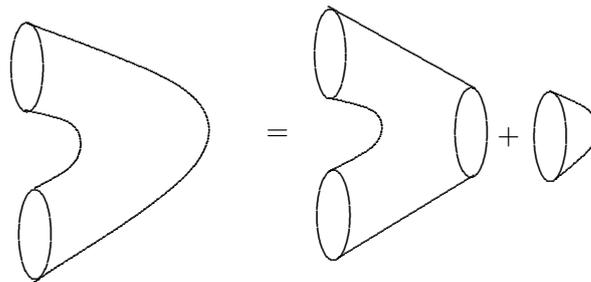
in the surface form, that is, sewing a disc on a pair of pants is isomorphism to a cylinder.

From the description of Frobenius algebras, it becomes more or less easy to prove that

Theorem 14 *The category of 2d TQFTs is equivalent to that of commutative Frobenious algebras.*

Proof: The main idea is clear. We start with a 1+1 TQFT, Z, and then $Z(S^1)$ will be Frobenius algebra. Conversely given a Frobenius algebra, A, we define $Z(S^1) = A$ and then start generating the other structure. (For this, it is simplest to look at the generators and relations for $2-Cob$, and then to check each part in turn. You can find this in Joachim Kock's book, [107], amongst other places.)

There are one or two points that need noting. We have, in the statement of the result, used the term 'commutative Frobenius algebra'. If we place ourselves in $(Vect_{\Bbbk}, \otimes, \Bbbk)$, there is a symmetry $\sigma : A \otimes B \to B \otimes A$, or as a diagram:
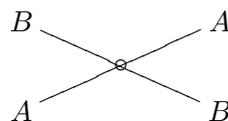
A *commutative Frobenius algebra* is a Frobenius algebra such that
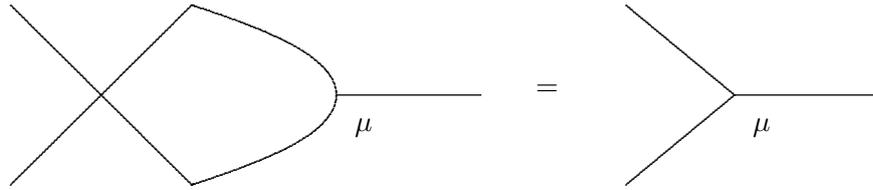
(There is a notion of *symmetric Frobenius algebra*, where one requires β to be a symmetric form:
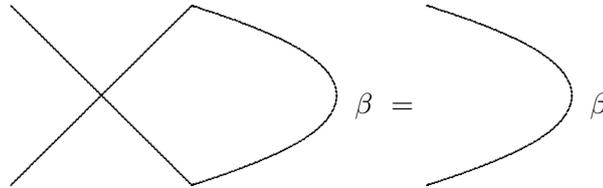
but note $Mat_{n\times n}(\Bbbk)$ is symmetric, but is not commutative.)

The cobordism
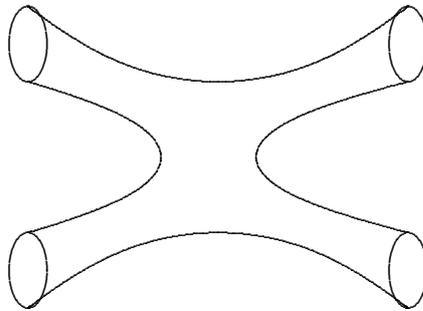
shows that 2 dimensional TQFTs will be commutative.

The reader is left to collect up the pieces, check that functors, etc. work as hoped, and, if all else fails, to look up a neat proof in one of the sources mentioned earlier. (The first published full proof is by Lowell Abrams, [1].)   ∎

The above has a useful interpretation in terms of *Frobenius objects*. If we look at the definition above (page 206), it is easy to see how we can adapt it to give a *Frobenius object* in a suitably structured monoidal category. We will use a description derived from that given in Rodrigues' paper, [148].

7.1.3 Frobenius objects

Let $\mathcal{A}$ be a symmetric monoidal category with monoidal structure, denoted $\otimes$, and with $\Bbbk$ as unit. (That is, we will suspend our convention that $\Bbbk$ is a commutative ring for rest of this discussion! Of course, in one of the main examples i.e. $Vect_{\Bbbk}^{\otimes}$ it still is as that is the unit in this case.)

Defintiion: We say $\mathcal{A}$ has a (left) *duality structure* if for each object A, there is an object A^*, the dual of A, and morphisms

$$b_A : \Bbbk \to A \otimes A^*,$$

$$d_A : A^* \otimes A \to \Bbbk$$

such that
(i)
$$(A \xrightarrow{\cong} \Bbbk \otimes A \xrightarrow{b_A \otimes A} A \otimes A^* \otimes A \xrightarrow{A \otimes d_A} A \otimes \Bbbk \xrightarrow{\cong} A) = Id_A$$
and
(ii)
$$(A^* \xrightarrow{\cong} A^* \otimes \Bbbk \xrightarrow{A^* \otimes b_A} A^* \otimes A \otimes A^* \xrightarrow{d_A \otimes A^*} \Bbbk \otimes A^* \xrightarrow{\cong} A^*) = Id_{A^*},$$
where the unlabelled isomorphisms are the structural isomorphisms of $\mathcal{A}$ corresponding to $\Bbbk$ being a left and right unit for $\otimes$.

The assignment of A^* to A extends to give a functor from $\mathcal{A}$ to $\mathcal{A}^{op}$, the opposite category. If $f : A \to B$ is a morphism in $\mathcal{A}$, its *dual* or *adjoint morphism* $f^* : B^* \to A^*$ is given by the composition

$$B^* \xrightarrow{\cong} B^* \otimes \Bbbk \xrightarrow{B^* \otimes b_A} B^* \otimes A \otimes A^* \xrightarrow{B^* \otimes f \otimes A^*} B^* \otimes B \otimes A^* \xrightarrow{d_B \otimes A^*} \Bbbk \otimes A^* \xrightarrow{\cong} A^*.$$

Definition: If $\mathcal{A}$ has a duality structure as above, a *Frobenius object* in $\mathcal{A}$ consists of

- an object A of $\mathcal{A}$;

- a 'multiplication' morphism, $\mu : A \otimes A \to A$;

- a 'unit' morphism, $\eta : \Bbbk \to A$ such that (A, μ, η) is a monoid in $(\mathcal{A}, \otimes)$;
and

- a symmetric 'inner product' morphism,

$$\rho : A \otimes A \to \Bbbk,$$

such that (i)

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{A \otimes \mu} & A \otimes A \\
{\scriptstyle \mu \otimes A} \downarrow & & \downarrow {\scriptstyle \rho} \\
A \otimes A & \xrightarrow{\quad \rho \quad} & \Bbbk
\end{array}
$$

commutes (so writing $\mu(a, b) = ab$, $\rho(ab, c) = \rho(a, bc)$),
and
(ii) ρ is non-degenerate, i.e., the following two induced maps from A to A^* are isomorphisms:

$$A \xrightarrow{\cong} A \otimes \Bbbk \xrightarrow{A \otimes b_A} A \otimes A \otimes A^* \xrightarrow{\rho \otimes A^*} \Bbbk \otimes A^* \xrightarrow{\cong} A^*$$

and

$$A \xrightarrow{\cong} \Bbbk \otimes A \xrightarrow{\rho^* \otimes A} A^* \otimes A^* \otimes A \xrightarrow{A^* \otimes d_A} A^* \otimes \Bbbk \xrightarrow{\cong} A^*.$$

(This second composite tacitly uses the isomorphisms $(A \otimes A)^* \cong A^* \otimes A^*$, and $\Bbbk^* \cong \Bbbk$ which hold since $\mathcal{A}$ is assumed to be symmetric monoidal.)

Examples: (i) Frobenius objects in the category, $(Vect_\Bbbk, \otimes)$, or, more generally, $(Mod\ R), \otimes)$ are Frobenius algebras in the usual sense.

One of the main reasons for mentioning Frobenius objects is the following result, which is really a very remarkable one.

Theorem 15 *The category* $2-Cob$ *is the free symmetric monoidal category on a commutative Frobenius object.* ∎

We are not going to prove this. (A proof can be found in Joachim Kock's book, [107].) The Frobenius object is the circle. The meaning of the result is that if you have an assignment that sends the circle to a Frobenius object in another category, compatibly with the structural maps of Frobenius objects, then that assignment extends uniquely to a monoidal functor defined on $2-Cob$. We will see other similar results later on. The amazing thing about this is that we have a geometric situation involving cobordisms, etc., and yet have a universal property and a complete categorical characterisation of $2-Cob$.

7.2   How can one construct TQFTs?

7.2.1   Finite total homotopy TQFT (FTH theory)

As a first exercise in constructing a TQFT, we will look at the 'toy' example given by Quinn in his notes, [145]. We will not give all the details, but will sketch some of the main ideas. This not only gives an easily understood method, but in many ways is a precursor for the TQFTs and HQFTs that we will meet later on, when the link with earlier material in these notes will be more explicit.

   The basic category is not as complicated as $d-Cob$, as we can get away with objects and 'cobordisms' merely being CW complexes.

   The idea is that one fixes a space, B, and then the TQFT, Z_B, is to have, for a space, Y, the state module is

$$Z_B(Y) = \Bbbk[Y, B],$$

the $\Bbbk$-vector space with a basis corresponding to the homotopy classes of maps from Y to B. Here $\Bbbk$ needs to be of characteristic zero, so $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ will be enough to be going on with. We need $Z_B(Y)$ to be finite dimensional, so need $[Y, B]$ to be a finite set.

   Definition: A space B is said to have *finite total homotopy, (FTH)* if it has finitely many components, and for any base point $b \in B$, and any i, $\pi_i(B, b)$ is finite, all but finitely many of these groups being trivial.

   Example: (i) For any finite group, G, its classifying space, BG has finite total homotopy.
   (ii) If $\mathsf{C} = (C, P, \partial)$ is a finite crossed module, then $B\mathsf{C}$, its classifying space, being the realisation of $\overline{W}K(\mathsf{C})$ has only π_1 and π_2 non-trivial, and these two groups are finite, being $Coker\,\partial$ and $Ker\,\partial$, respectively.

   The importance of the idea is due to the following lemma:

Lemma 27 *If B has finite total homotopy and Y is a finite CW-complex, then the set, $[Y, B]$, is finite.* ∎

   The proof is by induction on the number of cells in Y, using a long exact sequence argument. From now on in this section, we will assume that B has finite total homotopy.

The 'cobordism' role in this CW-complex context is a CW-triad, $(X; Y_1, Y_2)$, thought of as $X : Y_1 \to Y_2$. The two subcomplexes are *disjoint* subcomplexes of X. For such a $(X; Y_1, Y_2)$, we will need to define

$$Z_X := Z_{(X;Y_1,Y_2)} : Z_B(Y_1) \to Z_B(Y_2).$$

Suppose $[f_1]$ is a basis element of $Z_B(Y_1)$, so $f_1 : Y_1 \to B$ (and, as is customary, $[f_1]$ will denote the corresponding homotopy class). We must have

$$Z_X^B([f_1]) = \sum_{[f_2]} \mu_{X,f_1,f_2}.[f_2],$$

where the sum is over all $[f_2] \in [Y_2, B]$. The μ are just matrices over $\Bbbk$, expressing this linear transformation in terms of the given bases, but we do not know that much about them! Of course, they are constrained by the axioms of TQFTs (i.e., monoidal functoriality) and, thus, respect for the composition of cobordisms. It might be possible to find the most general form that they can take, but rather than that we will follow Quinn in his notes, [145], and give a solution, sketching parts of the verification of the axioms (adapted to the slightly wider context of this theory). We first need some terminology and notation.

Let X be a space with finite total homotopy.

Definition: The *homotopy order* of X is the rational number, $\sharp^\pi(X)$, defined, if X is connected, to be equal to

$$\sharp^\pi(X, x) := \prod_{i=1}^\infty \sharp(\pi_i(X, x))^{(-1)^i} = (\sharp\pi_1)^{-1}(\sharp\pi_2)(\sharp\pi_3)^{-1}...,$$

for any basepoint, $x \in X$, and, if X is not connected, as $\prod \sharp^\pi(X, x)$, the product of the homotopy orders of the connected components of X, based at a representative family of base points, thus one in each component.

Now let $f_1 : Y_1 \to B$, $f_2 : Y_2 \to B$, and set

$$Maps_{f_1}(X, B)_{[f_2]} = \{F : X \to B \mid F|_{Y_1} = f_1, F|_{Y_2} \simeq f_2\},$$

which is a subspace of the space of maps from X to B. An argument similar to that for the lemma above shows that this has finite total homotopy, provided $(X; Y_1, Y_2)$ is a finite CW-triad. (It is interesting that changing f_1 within its homotopy class does not change the homotopy order of this space of maps.)

We can now give Quinn's scaling factor matrix, μ:

$$\mu_{X,f_1,f_2} = \sharp^\pi(Map_{f_1}(X, B)_{[f_2]}).$$

We will also use the space of maps $Map_{f_1}(X, B)$, which is $\{F : X \to B \mid F|_{Y_1} = f_1\}$, then we have:

Lemma 28

$$Z_X^B([f_1]) = \sum_{[F]} \sharp^\pi(Map_{f_1}(X, B), F)[F|_{Y_2}],$$

where the sum is over all homotopy classes, rel Y_2, of maps $F : X \to B$, restricting to f_1 on Y_1. ∎

The proof is by a repackaging of the previous expression, so is **left to you**.

We will look at the composition of 'cobordisms'. First another lemma, which will also be useful later on.

Lemma 29 *Suppose $F \to E \xrightarrow{p} B$ is a fibration of spaces with finite total homotopy and assume that B is connected, then*

$$\sharp^{\pi}(E) = \sharp^{\pi}(F)\sharp^{\pi}(B).$$

Proof: This should be clear from the long exact homotopy sequence of a fibration. ∎

Now assume given triads, $(X_1; Y_1, Y_2)$, $(X_2; Y_2, Y_3)$, then so is $(X_1 \sqcup_{Y_1} X_2, Y_1, Y_2)$ and

$$X_1 \to X_1 \sqcup_{Y_1} X_2$$

is a cofibration, so

$$Map_{f_1}(X_1 \sqcup_{Y_1} X_2, B) \to Map_{f_1}(X_1, B)$$

is a fibration. To use the lemmas, we need to work out the fibre over a map, $F : X_1 \to B$, but we can identify this with $Map_{F|Y_2}(X_2, B)$ by the pushout property of $X_1 \sqcup_{Y_1} \sqcup X_2$. This gives, after a fairly obvious calculation involving the previous lemma:

Lemma 30

$$Z^B_{X_1 \sqcup_{Y_1} X_2} = Z^B_{X_2} Z^B_{X_1}.$$

∎

The other properties are **left for you to investigate**.

The following is a closely related construction in which B is the classifying space of a finite group. This provides a first example of a lattice or triangulation based construction of a TQFT via a 'state sum' model.

7.2.2   How can we construct TQFTs ... from finite groups?

One method of generation of TQFTs which is frequently used is based on simplicial lattices or triangulations and we will use this. Although it is a bit more complicated than some of the other constructions, it generalises nicely to higher dimensions *and* has a nice interpretation.

(The version here, and in the next few sections, is based on constructions of Dave Yetter, [165, 166], see also the papers, [137, 138]. The original idea is discussed quite fully in the first of the two papers by Yetter. It is a version of a construction due to Dijkgraaf and Witten, [63].)

First we work with triangulations of the oriented manifolds and cobordisms. For our immediate use of 'triangulations', we will work with an intuitive idea of triangulation. (You can base that intuition informally on wire-grid models such as are used in computer graphics.) We will, later on, have to look at them in a bit more detail.

We will go into quite a lot of detail on this construction itself as the methods it uses are quite intuitively simple, but are also the basis for those that we will use later on, which are perhaps less so.

Fix a finite group, G, and let X be a space with triangulation, **T**. It will be useful, but initially not essential, to have that **T** is an ordered triangulation, so will consist of a simplicial complex,

T, a homeomorphism between $|T|$ and X, and, in addition, a *total order* on the set of vertices, $V(T) = T_0$, of T. (Sometimes it may be notationally useful to specify the vertices of T as being explicitly indexed by natural numbers in agreement with the ordering, so then $T_0 = \{v_0, v_1, \ldots, v_n\}$ where $\sharp(T_0) = n$.) The choice of the ordering is not crucial in any way. We will initially use it to help the construction along, and later will need it to turn simplicial complexes into simplicial sets using the construction that we saw near the start of these notes, page **??**.

Definition: A *G-colouring* of **T** is a map,

$$\lambda : T_1 \to G,$$

such that given $\sigma \in T_2$, $\lambda(e_1)^{\varepsilon_1}\lambda(e_2)^{\varepsilon_2}\lambda(e_3)^{\varepsilon_3} = 1$, where the boundary $\partial\sigma$ of *sigma* is given by $\partial\sigma = e_1^{\varepsilon_1}e_2^{\varepsilon_2}e_3^{\varepsilon_3}$.

To help understand the formula, we look at a very simple example.

Picture: To simplify, assume the orientation is given and, as above, the vertices of **T** are ordered. If we write $\sigma = (a, b, c)$, then $a < b < c$, and we assume the order is compatible with the orientation:
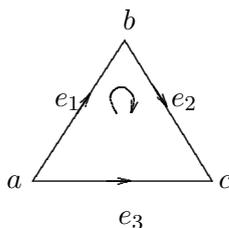
The boundary of σ is $e_1 e_2 e_3^{-1}$, so a coloring, λ gives
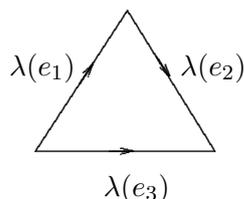
with $\lambda(e_1)\lambda(e_2)\lambda(e_3)^{-1} = 1$.

The intuition is: on looking at G-valued functions on edges, integrating around a triangle is to give you 'nothing', that is the identity element of G.

The G-valued functions concerned are typically those associated with transition functions of a bundle, usually of G-sets, i.e., a G-torsor or principal G-bundle. That intuition then corresponds to situations where a G-bundle on X is being specified by charts, and the elements, g, h, k, etc., are transition automorphisms of the fibre. (Because of this, the triangle condition above is a *cocycle condition.* It is often also termed a 'flatness' condition, as in the differential case, it corresponds to a condition on a 'connection' which say that it is 'flat'.) The construction methods for the TQFT then use manipulations of the pictures as the triangulation is changed by subdivision, etc.

Another closely related view of this is to consider continuous functions, $f : X \to BG$, to the classifying space of G. We can assume that f is a cellular map, using a suitable cellular model of BG, and at the cost of replacing f by a homotopic map, and by subdividing the triangulation of X. From this perspective, the previous model is a combinatorial description of such a continuous *characteristic* map, f. The edges of the triangulation pick up group elements since the end points of each edge get mapped to the base point of BG, and $\pi_1 BG \cong G$, whilst the faces give a realisation of the cocycle condition. Likewise we can, and, later on will, use a labelled decomposition of the objects as regular CW-complexes.

So much for the moment on the G-colourings as such, ..., what do we do with them?

Since G is assumed to be finite, the set, $\Lambda_G(\mathbf{T})$, of all G-colourings of $\mathbf{T}$ is also finite. Let $Z_G(X, \mathbf{T})$ be the vector space having $\Lambda_G(\mathbf{T})$ as basis. (The vector space will usually be over $\mathbb{C}$, but any other field of suitable characteristic will do, provided the constructions used do not involve elements that 'aren't there. For instance, sometimes a formula will have the order of a group raised to a fractional power and, clearly, that is fine if we work over $\mathbb{C}$ or $\mathbb{R}$, but could be problematic over $\mathbb{Q}$. Because of this we will work over $\mathbb{C}$, rather than having to change the ground field or ring each time a new construction needs an extra condition. In general, of course, we could replace 'vector space' by free module over a commutative ring with a short list of conditions on the ring.)

We will need, later, to consider subdivisions of triangulations and their effect on these vector spaces, and, as $\mathbf{T}$ is being ordered and that structure is part of the structure needed for the construction, we need, in considering subdivisions of $\mathbf{T}$, to take the ordering into consideration. (As has been said before, here the ordering could be avoided, but it is great help in the exposition even in this simple case, and will be more or less essential in more complicated cases later on. It is another instant of introducing structure to help with a construction, although once the thing is constructed, we can show that it is independent of the extra structure.)

Definition: A *subdivision* of an ordered triangulation, $\mathbf{T}$, is an ordered triangulation, $\mathbf{T}'$, such that the underlying triangulation is a subdivision of the underlying triangulation of $\mathbf{T}$ and the inclusion of $V(T_0)$ into $V(T_0')$ is a monotone function for the given orderings.

Comment: We are basing this definition on a fairly informal definition of triangulations and subdivisions, but it will suffice for the moment. Shortly we will make this a bit more formal.

Yetter uses a very simple form of subdivision, namely 'edge-stellar' subdivision. Although, in fact, we will also use other means and other types of subdivision, it is worth briefly noting the justification that he gives for his choice. We will need a result of Alexander's from [4]. To be able to state this, we first need the idea of a dimensionally homogeneous polyhedron. A polyhedron is *dimensionally homogeneous* if there is a dimension k such that every point is contained in some closed k-simplex.

Theorem 16 *(Alexander, 1930) If X is a dimensionally homogeneous polyhedron, then any two triangulations of X are related by a series of edge-stellar subdivisions and inverses of such.*   ∎

From this it is not hard to prove:

Corollary 9 *Any two ordered triangulations of an n-manifold are related by a sequence of edge-stellar subdivisions and their inverses.*   ∎

Remark: In fact, Yetter restricts to surfaces, so the full force of these results is not needed. In [165], he considers this case of a finite group, but later uses very similar methods for a finite crossed module / categorical group; see below and [166], but, in both, the case of surfaces, with 3-dimensional cobordisms between them, is what is considered in detail and this is quite reasonable as we will see.

Whichever type of triangulation you use, as it is extra structure beyond the basic manifolds, it is necessary to eliminate dependence on this. We will turn to this shortly, but we also need to consider cobordisms, so how are they studied?

Suppose $(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are two triangulated *oriented* d-manifolds (and, as in [165], let us restrict to surfaces and thus to $d = 2$ for simplicity of exposition, although, for much of the time, this will make no difference). A triangulated cobordism, $(M, \mathcal{T})$, between them will be a cobordism, M, between X and Y, i.e., M will be an oriented manifold of dimension $(d + 1)$, (so here usually 3), with boundary $X \sqcup -Y$, that is, 'X disjoint union with Y, which is oriented with the opposite orientation', and $\mathcal{T}$ will be an (ordered) triangulation of M that restricts to $\mathbf{T}$ on X and to $\mathbf{S}$ on Y. We can consider G-colourings of $(M, \mathcal{T})$ as well, and we can define a linear map,

$$Z_G^!(M, \mathcal{T}) : Z_G(X, \mathbf{T}) \to Z_G(Y, \mathbf{S}),$$

by, for $\lambda \in \Lambda_G(\mathbf{T})$,

$$Z_G^!(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_G(\mathcal{T}) \\ \mu|\mathbf{T} = \lambda}} \mu|\mathbf{S},$$

and extending linearly.

In other words, you have a basis colouring, λ, on the 'input' end and look at those colourings of $(M, \mathcal{T})$ that extend it, then see what they give you at the 'output' end, summing over all the possible answers.

This, at the same time, looks good and also slightly suspect. We started with a basis element of $Z_G(X, \mathbf{T})$ and ended up with lots of basis elements for $Z_G(Y, \mathbf{S})$. That is, somehow, too 'inflationary'. Even for a single basis element, λ', in $Z_G(Y, \mathbf{S})$, there would be triangulations, $\mathcal{T}$, of M, which would give a large number of copies of λ' in this sum. Perhaps a compensating factor, say depending on the size of G, is needed to correspond more to an 'average' value over these μs. We need to see 'how many' there are. For that, we need to look 'inside' the cobordism and how the colourings of $\mathcal{T}$ change with subdivision.

We will call the vertices, edges, etc., in $\mathcal{T}$, *interior* if they are away from the ends. Suppose that we subdivide *one* interior edge, e, of $\mathcal{T}$, giving a new triangulation, $\mathcal{T}'$, of M. For each old colouring, λ, of $\mathcal{T}$, we now have $\sharp(G)$ G-colourings of $\mathcal{T}'$, which are the same on all other edges, since on the subdivided e, we can assign any $g \in G$ to one half of it with $g' = \lambda(e)g^{-1}$ or $g^{-1}\lambda(e)$ given to the other, which one to use depending on the ordering. This does not disturb the face relations in the relevant 2-simplices:
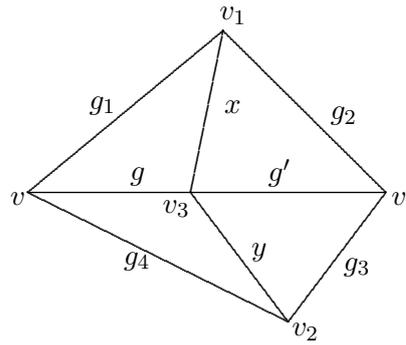
keeping $g_1, \ldots, g_4$ fixed, it is easy to find unique values for x and y giving a new colouring, and not changing anything outside the immediate neighbourhood of the edge, e. There are **several cases to check** depending on the relative position of the new vertex, v_3, in the ordering, but they clearly are all easily handled. (We will in any case look at this in a bit more detail shortly, but it is a good idea to have **tried the calculations yourself**.) We have

$$Z_G^!(M, \mathcal{T}') = \sharp(G) Z_G^!(M, \mathcal{T}).$$

Now, for a general $\mathcal{T}$, set $n_{\mathcal{T}}$ to be the number of vertices in $\mathcal{T}$, but not in the two ends, so $n_{\mathcal{T}} = \sharp(\mathcal{T}_0 - (T_0 \cup S_0)) = \sharp(\mathcal{T}_0) - \sharp(T_0) - \sharp(S_0)$, as S_0 and T_0 are disjoint.

Lemma 31 *If $\mathcal{T}$ and $\mathcal{T}'$ are any two triangulations of M that agree with $\mathbf{T}$ and $\mathbf{S}$ on the two ends, then*

$$\sharp(G)^{-n_{\mathcal{T}}} Z_G^!(M, \mathcal{T}) = \sharp(G)^{-n_{\mathcal{T}'}} Z_G^!(M, \mathcal{T}').$$

Proof: You just consider a mutual subdivision, $\mathcal{T}''$, and compare the two linear maps of the statement of the result with $Z_G^!(M, \mathcal{T}'')$, using the earlier comment on the 'one-extra-vertex' case, and induction. The details are best **left to you**.   ∎

Of course, this means that:

Corollary 10 *This common value is independent of the triangulation, $\mathcal{T}$.*   ∎

This value is still dependent on the triangulations on the two ends, $\mathbf{T}$ and $\mathbf{S}$. We denote this common value, $Z_G^!(M, \mathbf{T}, \mathbf{S})$. We have thus, in this new linear map, included compensatory scaling factors to handle the subdivisions in the cobordism, but even if a big step in the right direction, they still do not give us the final linear map that we want. That has to be compatible with composition and preserve the monoidal structure.

It is easy to see that the above is all compatible with the monoidal structure, since within the cobordism setting $\otimes$ interprets as disjoint union, and so a G-colouring of $X \otimes Y$ will be given precisely by a G-colouring of X together with a G-colouring of Y; that is all. We thus have $Z_G(X \otimes Y, \mathbf{T} \otimes \mathbf{S}) \cong Z_G(X, \mathbf{T}) \otimes Z_G(Y, \mathbf{S})$, with the isomorphism originating on the given bases. (Hopefully, the above notation is more or less self explanatory.)

There remains the question of compatibility of the above with composition. There is an obvious composition of triangulated cobordisms. Suppose $(M, \mathcal{T})$ is a triangulated cobordism from $(X, \mathbf{T})$ to $(Y, \mathbf{S})$, and $(N, \mathcal{S})$ another from $(Y, \mathbf{S})$ to $(Z, \mathbf{R})$. We can form a cobordism, $M +_Y N$, from X to Z by gluing the two given cobordisms along the copies of Y (i.e., by forming a pushout). This

clearly comes with a triangulation, which could, sensibly, be denoted $\mathcal{T} +_\mathbf{S} \mathcal{S}$. This is not, however, an arbitrary triangulation of $M +_Y N$ as, on the copy of Y 'in its middle', the triangulation agrees with $\mathbf{S}$. This affects the compensating factors:

Lemma 32 *In the above situation*

$$Z_G^!(N, \mathbf{S}, \mathbf{R}).Z_G^!(M, \mathbf{T}, \mathbf{S}) = \sharp(G)^{\sharp(S_0)} Z_G^!(M +_Y N, \mathbf{T}, \mathbf{R}).$$

Proof: The compensating factor for the term $Z_G^!(M +_Y N, \mathbf{T}, \mathbf{R})$ uses

$$
\begin{aligned}
n^{\mathcal{T}+\mathbf{s}\mathcal{S}} &= \sharp((\mathcal{T} +_\mathbf{S} \mathcal{S})_0 - \mathbf{T}_0 - \mathbf{R}_0) \\
&= n^{\mathcal{T}} + n^{\mathcal{S}} + \sharp(S_0).
\end{aligned}
$$

Now look at what happens in the composite on the left. ∎

We thus let $Z_G(M, \mathbf{T}, \mathbf{S}) = \sharp(G)^{-\frac{1}{2}(\sharp(T_0)+\sharp(S_0))} Z_G^!(M, \mathbf{T}, \mathbf{S})$, dividing the effects of the two ends between them to compensate for this. (**Think about this**. It is a nice way of getting this to work, although we should also consider, and from several different angles, **why** does it get it to work! It looks like a factor introduced to make things work and, in fact, is, but it *should* have some other more 'elegant' description, but what should that be? One extra point to note is that this does potentially need $\sqrt{\sharp(G)}$ to be an element of the ground field, so, as mentioned before, for convenience we take $\Bbbk$ to be $\mathbb{R}$ or $\mathbb{C}$, although with a bit of attention in any particular case, we do not need all the extra elements that this provides.)

We thus obtain:

Corollary 11 *For M, $\mathbf{T}$, $\mathbf{S}$, etc. as above:*

$$Z_G(N, \mathbf{S}, \mathbf{R}).Z_G(M, \mathbf{T}, \mathbf{S}) = Z_G(M +_Y N, \mathbf{T}, \mathbf{R}).$$

∎

Note: For simplicity, we have assumed that the field for the vector spaces has characteristic 0, but, in fact, for the above, we only need $\sharp(G)$ to be invertible in it. (This may, perhaps, remind you of parts of group representation theory, and that is not just coincidence.)

We thus have a monoidal 'functor' from the 'category' of triangulated surface and cobordisms to that of vector spaces. Although this looks good, we have left 'functor' and 'category' in inverted commas, because the so-called 'functor' is not going to preserve identities, and, worse than that, it is not so clear what the identities should be in this case of triangulated surfaces. Oh dear! But there is another point left outstanding, namely that we have manifolds with *triangulations*, and that $Z_G(X, \mathbf{T})$, etc., depend on the choice of triangulation. On handling *that* point, we will actually end up close to managing the 'identity' problem.

We are thus back with subdivisions. Suppose we are given X and $\mathbf{T}$, as before, and let $\mathbf{T}'$ be obtained by subdividing $\mathbf{T}$ at a single edge, e, divided into two parts, e_1 and e_2, for instance, if $v < v'$,

$$v \xrightarrow{e} v' \quad \text{goes to} \quad v \xrightarrow{e_1} . \xrightarrow{e_2} v',$$

(in the case where the new vertex is between v and v' in the ordering on T_0'). We define a function

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T})$$

by multiplying the values of a colouring on the subdivided bits of the edge, (taking into account signs to handle the vertex ordering), so, in this simple case: for λ, a G-colouring of $\mathbf{T}'$,

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e) = \lambda(e_1)\lambda(e_2),$$

with $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)$ taking the same value as λ on all other edges of $\mathbf{T}$, $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e') = \lambda(e')$ if $e' \in T_1$, $e' \neq e$.

It is really *necessary* to draw some diagrams, as above, to check that, if λ is a colouring of $\mathbf{T}'$, then $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)$ is one of $\mathbf{T}$. We work with the diagram given earlier (page 216) with g corresponding to $\lambda(e_1)$, and g' to $\lambda(e_2)$. (There are several cases to check corresponding to the placement of the 'new vertex' in the order on the vertices of $\mathbf{T}$ and the form of that ordering. We will just treat the case where in $\mathbf{T}$, $v < v_1 < v_2 < v'$ and the new vertex, v_3, lies between v_2 and v' in the order on $\mathbf{T}'$. **You are left to think about the other cases**, but you should quickly realise that reversing the order on a pair of vertices just replaces $\lambda(e)$ by its inverse, so the other cases are easy once one is done. Of course, this is just another instance of the argument we sketched when looking at subdivisions of the triangulations of the cobordisms.

We have four equations, with, from the top two triangles,

$$
\begin{aligned}
g_1 x &= \lambda(e_1) \\
x\lambda(e_2) &= g_2
\end{aligned}
$$

and similarly for the lower two triangles. The value, $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e)$ is $\lambda(e_1)\lambda(e_2)$, and it is immediate that $g_1 g_2 = \mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e)$, as required.

Remark: That was easy, and, of course, is the basic calculation in all (co)homological situations. We 'integrate' along the edge labelled x first in one direction, then later in the opposite one and adding up the contributions cancels that x out. The only slightly subtle point is that as G may be non-commutative, we need to be careful about the multiplication order. (Here we have used 'algebraic' concatenation order as that is what is used 'traditionally' in this area.) The actual formula for $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ that we gave above only works (without adjustment that is) if the new vertex is between v and v' in the order on T_0'. As commented above, if another order occurs, say $v < v' < v_3$, then e_2 may be, as here, reversed in direction and, in that case, we would have $\lambda(e_1)\lambda(e_2)^{-1}$ in the expression. This is easy to do in this case of G being a group, and of being in low dimensions, but still looks as if numerous similar cases would be needed in general. It one replaces the group, G, by a crossed module, as we will shortly, the complications would look to be getting out of hand, so we *do need to keep our eyes open* for a neater way of handling things, that is, other than a proof by exhaustion! Case-by-case analysis is always there as a backup, but begins to look unfeasible for later on. We will continue to use it for the moment as it *does* emphasise the combinatorics of what is going on and thus when we do go on to slicker methods, we will some background intuition of how these methods encode this combinatorial analysis.

Going back to subdivisions, if $\mathbf{T}'$ is obtained by subdividing edges in $\mathbf{T}$, then it can be obtained inductively by doing the above simple case repeatedly. (It is clear the order in which this is done

is immaterial to the end result.) We thus have

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T}),$$

in general, together with a resulting linear extension from $Z_G(X, \mathbf{T}')$ to $Z_G(X, \mathbf{T})$.

What happens to the cobordisms? We have, say, M from X to Y, as before, and hence $Z_G(X, \mathbf{T}', \mathbf{S})$ and $Z_G(X, \mathbf{T}, \mathbf{S})$. If $\eta \in \Lambda_G(\mathbf{T}')$, a similar argument to before shows

$$Z_G(X, \mathbf{T}', \mathbf{S}) = \sharp(G)^{-\frac{1}{2}(\sharp(T_0') - \sharp(T_0))} Z_G(X, \mathbf{T}, \mathbf{S}) \circ \mathrm{res}_{\mathbf{T}',\mathbf{T}},$$

in other words, the obvious diagram does not commute. (Note: if you are cross-referencing, or consulting, the original source, the relevant diagram on page 5 of [166] has an arrow going in the wrong direction.) A similar calculation would apply to a subdivision of $\mathbf{S}$. To obtain better compatibility of the 'res' maps with the cobordism induced ones, we thus scale the restriction map, defining a new

$$res_{\mathbf{T}',\mathbf{T}} : Z_G(X, \mathbf{T}') \to Z_G(X, \mathbf{T}),$$

by $res_{\mathbf{T}',\mathbf{T}} = \sharp(G)^{-\frac{1}{2}(\sharp(T_0') - \sharp(T_0))} \mathrm{res}_{\mathbf{T}',\mathbf{T}}$, and, thus adjusted, we will then have the desired compatibility between the cobordism structure and the restriction maps coming from subdivision.

Before we look at that however, we note another essential feature of the restriction maps, namely that they are epimorphisms. It is clear that, even at the 'basic' non-linear level

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T}),$$

is 'onto', as, given a colouring, λ, of $\mathbf{T}$, we can even work out a colouring of $\mathbf{T}'$ that maps down to it. We just look at a simple case to give the idea. Suppose $\mathbf{T}'$ contains just one more vertex subdividing the edge e as before. We define a colouring of $\mathbf{T}'$ by assigning to most edges the same value as for λ, but, using the same notation as before, to e_1 we assign 1, and, to e_2, assign $\lambda(e)$. The other edges in $\mathbf{T}'$ now can be assigned values so that the result is a colouring. (Just **try it out** on the diagram we saw earlier.)

We thus have a diagram of finite dimensional vector spaces and epimorphisms, indexed by the (directed) category of triangulations of X. To eliminate the dependence of $Z_G(X, \mathbf{T})$ on the triangulations, we just take the colimit of this diagram.

Let $Z_G(X) = colim_{\mathbf{T}} Z_G(X, \mathbf{T})$. This vector space is finite dimensional. Although fairly simple to prove, this fact is important, so we will spend a moment examining it. It is important, since otherwise Z_G would not give us a TQFT, since $Vect$ is the category of *finite* dimensional vector spaces. (The study of infinite dimensional vector spaces, of course, usually uses different tools for their study than does that of their finite dimensional counterparts and that machinery, norms, completeness, etc., is not really what is being used in this theory.) It is also important since we can, in the process of proving this, show that $Z_G(X)$ is a quotient of $Z_G(X, \mathbf{T})$, in fact, the natural linear maps,

$$r_{\mathbf{T}}^X : Z_G(X, \mathbf{T}) \to Z_G(X),$$

are epimorphisms, so it suffices to examine their kernels to obtain a neat and useful representation of the elements of $Z_G(X)$, ..., but we are going too fast and must backtrack.

We note

(i) for any polyhedron, X, the category of triangulations of X is directed. More exactly, for any simplicial complex, K, and subdivisions, K', K'', of K, there is a subdivision, K''', finer than both K' and K''. There are subtleties here, but this works fine for X a piecewise linear (PL) manifold, which is sufficient for us. Some of the subtle points are discussed in [5], but there is also a correction note available, adjusting one or two of the statements. We have already referred to Alexander's paper, [4] and this will suffice for us. The link between triangulations and coverings that we will look at shortly, is also relevant here.

(ii) If $\mathcal{V} : \mathcal{D} \to Vect$ is a functor, where $\mathcal{D}$ is a directed set (thought of as a directed category), and, for each $d' < d$, the corresponding $V_d^{d'} : \mathcal{V}(d') \to \mathcal{V}(d)$ is an epimorphism, then, for any d, the canonical map

$$r_d : \mathcal{V}(d) \to colim\mathcal{V}$$

is an epimorphism. To see this, note how $V = colim\mathcal{V}$ can be constructed. You first form the direct sum $\bigoplus_{d \in \mathcal{D}} \mathcal{V}(d)$, and then divide out by the equivalence relation:

$v_{d_1} \equiv v_{d_2}$ if there is a v_{d_3}, for d_3 such that
(i) $d_3 < d_1$, and (ii) $d_3 < d_2$,
with
(iii) $V_{v_1}^{d_3}(v_{d_3}) = v_{d_1}$ and (iv) $V_{v_3}^{d_3}(v_{d_3}) = v_{d_2}$,

so two elements are to be equivalent if they are both images of some third element 'further back' in the diagram. If we write $[v]$ for the equivalence class determined by v, then $r_d(v_d) = [v_d]$, where, notationally, we do not distinguish between $v_d \in \mathcal{V}(d)$ and its image in the direct sum $\bigoplus \mathcal{V}(d)$. To check the statement that r_d is an epimorphism, we need only show that any $[v]$ has a representative in $\mathcal{V}(d)$, but $[v]$ will be a finite sum of elements of the form $[v_{d'}]$ for a finite family of indices d'. Using that $\mathcal{D}$ is directed, we can find a d'', finer than d and also finer than *all* of the d's, and then using the fact that all the $V_{d'}^{d''}$ are epimorphisms, pick elements $v'' \in \mathcal{V}(d'')$, each mapping down to the corresponding $v_{d'}$, finally replace the $v_{d'}$ in the sum by the equivalent $V_{d'}^{d''}(v'')$ to get an element, equivalent to v, but which is just in the image of r_d. As $[v]$ was arbitrary, this shows that r_d is itself an epimorphism.

This verification is standard, and elementary, but it shows why $Z_G(X)$ is finite dimensional in a very concrete and effective way. It also helps identify the kernel of $r_{\mathbf{T}}^X$, or, in general, r_d, since if $v \in Ker\, r_d$, there must be some d' and d'' with d'' less than both d and d', and an element $v'' \in \mathcal{V}(d'')$ such that $V_d^{d''}(v'') = v$ and $v'' \in Ker\, V_{d'}^{d''}$.

This, thus, gives a good description of the elements of $Z_G(X)$. You just take a $Z_G(X, \mathbf{T})$ and work out the kernel of $r_{\mathbf{T}}^X$. We next return to the cobordisms.

As we know that the linear maps, $Z_G(X, \mathbf{T}), \mathbf{S})$, are compatible with the restriction maps, $res_{\mathbf{T}',\mathbf{T}}$ (and $res_{\mathbf{S}',\mathbf{S}}$), it is now easy to check that a cobordism M from X to Y induces a linear map,

$$Z_G(M) : Z_G(X) \to Z_G(Y).$$

(This does require a bit of care as the domain is a colimit over the category of triangulations of X, whilst the codomain is over that to triangulations of Y. This is, however, quite easy to handle, so the details are **left to you**.)

Theorem 17 *(Yetter, [165]) The assignment, above, defines a monoidal functor, $Z_G : d - Cob \to Vect^{\otimes}$.* ∎

We will not prove this here. We will discuss generalisations of it later on and will indicate why they work.

Of more immediate interest than a straightforward direct proof is an interpretation of the resulting TQFT. The compensatory factors looks mysterious. What are they 'really'? Although the initial idea may be clear, the process of finding those compensatory factors does cloud the view a bit. The theory left like this looks a bit like cohomology when cocycles were the only way of looking at things. Cocycles are useful and have a geometric interpretation provided, for instance, we think of them as transitions between structure over some open cover. (We will turn towards that in the next section linking triangulations and open coverings.)

Yetter's proof of the above result uses an important observation, which also tells us a lot more about this TQFT. First we note that, as X is a triangulable manifold, we can work out the fundamental groupoid, ΠX, of X, up to equivalence, using the classical *edge path groupoid* construction. (For details of the origins of this construction, see below.) We have really met this several times earlier in these notes, but not explicitly, so here it is.

Given a simplicial complex, K, we form the free groupoid, $F_{Grpd}(K^{(1)})$, on the 1-skeleton, $K^{(1)}$ of K, (i.e., the 1-dimensional subcomplex, and hence 'graph', made up of the edges and vertices of K), then we divide by relations corresponding to the 2-simplices. This is worthwhile making explicit, as it make what follows more or less trivial.

We introduce some notation. We will put an order on the vertices of K, for convenience. The 1-simplices of K will be denoted $\langle v_0, v_1 \rangle$, and the elements of $F_{Grpd}(K^{(1)})$ will be composable chains of these such as $\langle v_0, v_1 \rangle \langle v_1, v_2 \rangle$, where the reverse of the given order corresponds to inverting the element so, if $v_1 < v_0$, then $\langle v_1, v_0 \rangle$ will be the same as the 'virtual' element, $\langle v_0, v_1 \rangle^{-1}$. (We have essentially got rid of the imposed order, already, at this step. We can thus assume that we only take those edges, $\langle v_0, v_1 \rangle$, with $v_0 < v_1$.) For each 2-simplex, $\langle v_0, v_1, v_2 \rangle$, of K, we then introduce the relation

$$\langle v_0, v_1 \rangle \langle v_1, v_2 \rangle \equiv \langle v_0, v_2 \rangle.$$

(Here we can again assume $v_0 < v_1 < v_2$.) The resulting quotient groupoid, denoted $\Pi_{edge} K$, is the edge path groupoid of K.

Remark: Of course, this construction is discussed, classically, in Spanier, [150], p.136, and, according to Brown, [30, 31], was already essentially in Reidemeister's book, [146].

We note that, if we order the vertices of K, then we have a simplicial set that corresponds to it, (see page **??**), and hence the simplicially enriched groupoid, $G(K)$, (cf. page 125). We considered taking π_0 of each of the simplicial sets $G(K)(v, v')$ to get the fundamental groupoid of K as a quotient of $G(K)$, (again review page 125, and the discussion on the pages that follow that one). This is, of course, the same construction as the edge path groupoid, but has the advantage of having the higher n-types of K available in the $\mathcal{S}$-groupoid, $G(K)$. We will return to this point in a later section, when we replace the finite group, G, by a finite n-type, or similar.

Suppose that λ is a G-colouring of $(X, \mathbf{T})$, then considering the group, G, as a single object groupoid, $G[1]$, the assignment, λ, extends to a functor (or, if you prefer, a morphism of groupoids) from $F_{Grpd}(T^{(1)})$ to $G[1]$. This would be the case just with 'any-old' assignment of elements of G to edges of T, i.e., even without the cocycle-like condition for the values around each 2-simplex,

$\sigma \in T_2$. That extra family of conditions means that actually λ induces $\lambda : \Pi_{edge} T \to G[1]$.

As $\Pi_{edge} T$ is equivalent to ΠX, this gives a representation, $\lambda : \Pi X \to G[1]$. This will not be uniquely determined by the previous one since there will always be many different equivalences between $\Pi_{edge} T$ and ΠX. We have that $\Pi_{edge} T$ is more or less the same as $\Pi X | T_0 |$, the fundamental groupoid of X, based at the vertices of T_0. The usual retraction of ΠX onto this subgroupoid involves choices of paths and different choices give different, but conjugate, morphisms, λ. This is not conclusive, but may give sufficient intuition to see the feasibility of proving Yetter's main representation theorem:

Theorem 18 *(Yetter, [165], p.7) The vector space, $Z_G(X)$, is isomorphic to the vector space whose base is the set of conjugacy classes of representations from ΠX to $G[1]$.* ∎

Of course, a *representation* in this context is just a groupoid morphism and hence is just a functor. Two groupoid morphisms, $f_0, f_1 : \mathcal{G} \to \mathcal{H}$, are conjugate if and only if they are homotopic and if and only if, as functors, there is a natural transformation between them.

There is another 'take' on G-colourings that is worth mentioning here. It is somehow 'adjacent' to that which we have just given. We made the link between $\Pi_{edge} T$ and $G(T)$ above. This link extends to a very simply defined one between $G(T)$ and G-colourings. This then gives a bit more substance to our hint of link with G-torsors, etc.

Let **T** be an ordered triangulation of a space, X, and as usual, let $K(G, 0)$ denote the constant simplicial group of value G, (i.e., $K(G, 0)_n = G$ for all n, and all the face and degeneracy maps in $K(G, 0)$ being identity isomorphisms). (On a niggling notational point, perhaps we should really write $K(G, 0)[1]$, or $K(G[1], 0)$, to include the information that we are thinking of the group, G, as a one object groupoid. This extra precision needs to be kept in mind, but will not be used except if it turns out to be useful at a particular place in our discussions.)

Proposition 44 *Suppose that λ is a G colouring of T, then λ defines an $\mathcal{S}$-groupoid morphism*

$$\lambda' : G(T) \to K(G, 0)$$

given by

$$\lambda_0 \langle a, b \rangle = \lambda(a, b) \in G = K(G, 0)_0;$$

and, if $\sigma = \langle a_0, \ldots, a_{n+1} \rangle \in T_{n+1}$, $n \geq 1$,

$$\lambda'_n \sigma = s_0^n \lambda(a_0, a_1).$$

Proof: First note that we use the ordering to convert the simplicial complex, T, to a simplicial set. (Here we will not recall this again, but when we have different coefficients than just G, a bit later in this chapter, then we will need to be more precise at the corresponding point in the discussion.)

Remembering that $G(T)$ is free in each dimension, we only have to see what λ' does to non-degenerate simplices, (as the values on degenerate ones will be determined by the fact the morphism is simplicial). We then have to check that the simplicial identities work for this choice of λ'_ns. Most of this is routine and inconsequential, so is left **to the reader**, but it is worth noting what happens in dimension 1 with the d_0-face relation.

We will sometimes use an overbar to give the generating element, $\overline{\sigma}$, of $G(T)$ that corresponds to a simplex σ in T. This provides a bit more precision that is sometimes useful (although it gets awkward if the convention is slavishly followed).

Let $\langle a_0, a_1, a_2 \rangle$ be a non-degenerate 2-simplex of T, so $\overline{\langle a_0, a_1, a_2 \rangle}$ will be a generator of $G(T)_1$ (check back for the definition of all the structure of $G(K)$, for K a simplicial set, page 130).

$$\begin{aligned} \lambda_1' \overline{\langle a_0, a_1, a_2 \rangle} &=& s_0 \lambda_0(a_0, a_1) \\ &=& s_0 \lambda(a_0, a_2) s_0 \lambda(a_1, a_2)^{-1}, \end{aligned}$$

since λ satisfies the flatness / cocycle condition on 2-simplices, so $\lambda(a_0, a_1)\lambda(a_1, a_2)\lambda(a_0, a_2)^{-1} = 1$. Now clearly $d_0 \lambda_1' = \lambda_0 d_0'$, as required. ∎

This correspondence is clearly bijective. If $\lambda' : G(T) \to K(G, 0)$ is a $\mathcal{S}$-groupoid morphism, then it gives back a G-colouring of T and everything matches.

Proposition 45 *There is a bijection*

$$\Lambda_G(T) \leftrightarrow \mathcal{S}-Gpds(G(T), G[1]).$$

∎

We have, on passing to Moore complexes, that $N(K(G, 0))_n = G$ if $n = 0$ and is trivial otherwise, so $N(G(T))_n$ is 'killed off' by λ' in all dimensions except 0, and, of course, there we get an induced map from $\pi_0(G(T))$ to G, but $\pi_0(G(T))$ is, as we noted, just $\Pi_{edge}(T)$ again.

The advantage of this simplicial viewpoint will be clearer when we pass to generalisations, but we note also that λ' corresponds to a morphism of simplicial sets,

$$\lambda' : T \to \overline{W}(K(G, 0)) = Ner(G[1]) = BG,$$

by the adjointness of G and $\overline{W}$ that we have used several times. It thus corresponds to an isomorphism class of simplicial principal G-bundles on T (or G-torsors, if you prefer). As G is a finite group, this bundle also can be thought of as a finite covering space on T, or as a twisted Cartesian product, as in section 4.5, starting page 141.

As we said above, we will not give a proof of the result of Yetter here, but will look at generalisations later on. The proof in [165] is interesting and quite neat, so may be worth looking at anyway. It is also useful since aspects of it are used by Yetter in his paper on TQFTs associated to categorical groups, [166]. We will look at this shortly, but in the next section must 'backfill' on some of the ideas on triangulations, etc.

To finish up this section, we note a consequence of Lemma 32 / Corollary 11. Suppose M is a closed $(d+1)$-dimensional manifold and consider it as a cobordism from the empty d-manifold to itself. Next pick any triangulation, $\mathcal{T}$ of M. We have the domain and codomain of M in $(d+1)-Cob$ are empty, so there is only one G-colouring of them, namely the unique empty function from the empty set of edges of the empty triangulation of the empty manifold. (This, of course, corresponds to $Z_G(\emptyset) \cong \mathbb{C}$, which is the unit of the monoidal structure of $Vect^{\otimes}$.)

Our next task is to work out $Z_G^!(M, \mathcal{T})(\lambda)$, for this unique G-colouring λ. The formula gives

$$Z_G^!(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_G(\mathcal{T}) \\ \mu | \mathbf{T} = \lambda}} \mu | \mathbf{S},$$

but λ corresponded to the unique basis element of the vector space, $Z_G(\emptyset)$ and hence is 1, as is each $\mu|_S$, since $\mathbf{S}$ is also empty. This means that we have a contribution of 1 for each $\mu \in \Lambda_G(\mathcal{T})$, that is,

$$Z_G^!(M, \mathcal{T}) = \sharp(\Lambda_G(\mathcal{T})),$$

the number of G-colourings of $\mathcal{T}$.

Now we apply Lemma 32 or Corollary 11. Recall $n_{\mathcal{T}}$ is the number of vertices of $\mathcal{T}$.

Proposition 46 *The number*

$$I_G(M) = (G)^{-n_{\mathcal{T}}} \sharp(\Lambda_G(\mathcal{T})),$$

is independent of the triangulation.   ∎

This is just a special case of Corollary 11.

   Remarks: (i) We note, almost just for fun, that in this situation, the scaling factor to get from $Z_G^!(M, \mathcal{T})$ to $Z_G(M, \mathcal{T})$ has value 1, since both T_0 and S_0 are empty.

   (ii) This invariant, $I_G(M)$ is the simplest case of the *Yetter invariants* of M. When M is 3-dimensional, then it is an 'untwisted' version of the Dijkgraaf-Witten invariant. We will see this construction in other instances when we have generalised the construction of Z_G to having 'target' a finite crossed module.

   This invariant has a homotopy theoretic description, which shows that it is a homotopy invariant of M. This description is

$$I_G(M) = \frac{[M, BG]}{\sharp(G)}.$$

7.2.3   Triangulations and coverings

We mentioned that the intuition behind the finite group case was linked to the transition functions of a G-torsor or principal G-bundle. With such 'transition functions', (cf. page **??**), one has an open cover over which the bundle / torsor is assumed to trivialise. Recall that by this we mean that we have a cover, $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, say, of a space X and a 'bundle', $p : Y \to X$, such that if we restrict to a U_α, $p^{-1}(U_\alpha) \to U_\alpha$ is just the projection of a product $U_\alpha \times F \to U_\alpha$ for some nice 'fibre' F. In other words, the bundle is locally trivial. How does this correspond to the triangulation approach? For this we need to look more closely at triangulations. (For convenience, we will repeat some material on simplicial complexes from earlier in the notes, but sometimes from a slightly different perspective. This may seem slightly strange sometimes, as we have been using the associated definitions, notation, etc., for quite some time on a partially informal basis.)

   The history, and some of complications, of the fact that manifolds can be triangulated, and thus can be represented by simplicial complexes, is well discussed in Stillwell's book, [152], at a fairly non-technical level and in Anderson and Mnev, [5], for more technicalities. (One of the claims mentioned later in that paper has been withdrawn, and a correction made available. The paper

still is well worth consulting, as it is quite short, well written and to the point.) We will, in fact, be using other more technical aspects of that general area within examples later on and will then need to introduce more detail than given in [152].

Here we retain the formal definition of a triangulation.

Definition: A *triangulation*, $\mathbf{T} = (K, f)$, of a space, X, consists of a simplicial complex, K, and a homeomorphism, $f : |K| \to X$.

We will usually confuse $|K|$ with X, and so will call X, itself, a *polyhedron* in this case.

We put a formal definition of ordered triangulation here as well for convenience of reference. We will leave you to adjust the definition of ordered subdivision that we gave above.

Definition: A finite *ordered triangulation* of a space, X, is a triangulation, $\mathbf{T} = (K, f)$, together with a bijection, $K_0 \leftrightarrow \{0, 1, \ldots, n\}$, for some n. (Note that the bijection is part of the structure.)

We will also need a more formal definition of *subdivision*. Firstly it may help to look up the canonical geometric realisation of an (abstract) simplicial complex, (see page **??**). The following definition of subdivision is from Spanier, [150], p. 121. (It is not independent of the use of geometric realisations, so to some extent seems 'external' to the theory of abstract simplicial complexes. This is relevant when considering the observational viewpoint for this area, (see below). It is however a useful definition and *is* the usual one!)

Definition: If K is a simplicial complex, a *subdivision* of K is a simplicial complex, K', such that
a) the vertices of K' are (identified with) points of $|K|$;
b) if s' is a simplex of K', there is a simplex, s, of K such that $s' \subset |s|$; and
c) the mapping from $|K'|$ to $|K|$, that extends the mapping of vertices of K' to the corresponding points of $|K|$, is a homeomorphism (thus continuous with a continuous inverse),

..., and the corresponding notion for triangulations:

Definition: If $\mathbf{T} = (K, f)$ is a triangulation of a space, X, a subdivision of $\mathbf{T}$ is a triangulation $\mathbf{T}' = (K', f')$, where K' is a subdivision of K, and $f' : |K'| \to X$ compatible with f, i.e., f' is equal to $|K'| \to |K| \xrightarrow{f} X$.

An important idea for us will be that of the star of a vertex in a triangulation. (Now is a good time to briefly look back at the construction of the geometric realisation of an abstract simplicial complex given in the first chapter, (page **??**), if you did not do it above. The important point to hold on to is the idea of a point in $|K|$ as being a function from the vertex set, $V(K)$, of K to the unit interval, $[0, 1]$. Recall the condition on such a function, $\alpha : V(K) \to [0, 1]$, to be a 'point' in $|K|$ was that its *support*, $\{v \in V(K) \mid \alpha(v) \neq 0\}$, forms one of the simplices in K.) We have

Definition: If K is a simplicial complex and $\sigma \in S(K)$ is a simplex of K, the subspace,

$$|\sigma| = \{\alpha \in K \mid \alpha(v) \neq 0 \Rightarrow v \in \sigma\}$$

is called the *closed simplex* corresponding to σ.

We can now define the *star of a vertex, v,* by:

Definition: The *star of a vertex, v*, in a simplicial complex K is the open subset of $|K|$ given by

$$st(v) = \bigcup \{ Int|s| \mid v \text{ is a vertex of } s \} \cup v,$$

the union of the interiors of those closed simplices that have v as a vertex together with that vertex itself.

Alternatively, and equivalently, given any vertex v of K, its *star* is defined by

$$st(v) = \{ \alpha \in |K| \mid \alpha(v) \neq 0 \}.$$

Remark: More generally than needed when discussing triangulations, sometimes it is useful to think of a simplicial complex, K, as encoding combinatorial information on an 'observed space', X, with a continuous map, $f : |K| \to X$ or $f : X \to |K|$, giving the translation between the two contexts. We might be observing a physical object (thought of as the space, X). The vertices are observations of 'points' in X. (We will briefly explore this a bit more below.) The case of a triangulation then corresponds to the simplicial complex, K, being, somehow, a correct encoding of the structure as it allows a complete reconstruction of the 'space', X, itself to be made.

In such an 'observational interpretation', for each vertex, $v \in V(K)$, the set, $st(v)$, is an open set in $|K|$ and, if $\alpha : V(K) \to [0, 1]$ is loosely interpreted as a 'fuzzy superposition of observations', then $st(v)$ consists of those such 'observations' that 'observe' the notional point, v.

The other ingredient for our comparison between triangulations and coverings is the formal definition of the nerve of an open covering. We have been using this idea in another more structured form when we have considered an open cover as specifying a simplicial sheaf, but here we will need the older, non-sheaf theoretic version, due to Čech and Alexandrov back in the 1930s, that we briefly mentioned in section 5.2.4, where we introduced the idea in discussion the descent aspect of simplicial fibre bundles. The link between the two ideas is by taking connected components of the spatial part of the simplicial étale space version of the simplicial sheaf. (This is certainly not 'optimal', but makes the connection through to the study of triangulations more clear and 'classical'.)

We assume given a space, X, and an open covering, $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, of X.

Definition: The *Čech complex*, *Čech nerve* or simply, *nerve*, of the open covering, $\mathcal{U}$, is the simplicial complex, $N(\mathcal{U})$, specified by:

- Vertex set : the collection of open sets in $\mathcal{U}$ (alternatively, the set, A, of labels or indices of $\mathcal{U}$);

- Simplices : the set of vertices, $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p \rangle$, belongs to $N(\mathcal{U})$ if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection.

You will probably have remembered that we have already used the notation, $N(\mathcal{U})$, for another thing that (cf. section 6.1.2) we called the nerve of the open cover $\mathcal{U}$. That was the simplicial sheaf version of this. The connection between the two is very close, so any confusion that may arise is not that serious. If, as in the case of manifolds, we can refine covers so that each U_α is a contractible open set, then one can use either the sheaf of the simplicial complex version of the nerve with no great advantage to either. In that sort of situation the Yetter construction we saw earlier can be adapted to give one using simplicial sheaves, and it seems feasible that it can be extended to one with a sheaf of finite groups as the 'background' coefficients. (There are technicalities here that we will not go into for the moment.)

Remark (continued): In various parts of mathematics and mathematical physics, as we said above, it is sometimes useful to think of an open set as the support of an 'observation'. Physically, one cannot make measurements *at a point*, rather one uses the abstract idea of value *at a point* as a convenience for the average measurement 'locally' near the *point* in some space. One can thus replace 'point' by 'observed open set' and then see how overlapping 'observations' fit together. The nerve then serves as the combinatorial gadget that 'organises' the observations.

Similarly, in the relatively new area of *topological data analysis*, the information on a spatial model of some phenomenon is given by a point cloud of sample values, so the sample points are thought of as being small regions, and so are replaced by small discs. A similar idea comes in when considering Voronoi patches and the related Delaunay triangulations, (for which area it is suggested that you use a websearch to find a summary). Variants of the nerve construction are then use to help in the construction of geometric and topological models of the phenomena.

The idea of a triangulation is physically slightly problematic as the observer 'imposes' a triangulation on the space or space-time being observed. If 'points' are suspect then perhaps imposed triangulations are even more so!

If the space, X, is a polyhedron, then we can easily obtain a link between nerves and triangulations, so as to connect up this 'observational' idea with the 'imposition' of a triangulation.

The vertex stars give an open covering of $|K|$ and the following classical result tells us that the nerve of this covering is K itself (up to isomorphism):

Proposition 47 *(cf. Spanier [150], p. 114) Let X be a polyhedron and let $\mathcal{U} = \{st(v) \mid v \in V(K)\}$ be the open cover of X by vertex stars. The vertex map, φ, from K to $N(\mathcal{U})$, defined by*

$$\varphi(v) = \langle st(v) \rangle,$$

is a simplicial isomorphism,

$$\varphi : K \cong N(\mathcal{U}).$$

∎

As an example, suppose a triangle, as simplicial complex, has vertices

$$V(K) = \{v_0, v_1, v_2\}$$

and simplices $\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}$. (This is the *triangle* not the *2-simplex*, so there is no 2-dimensional face.) This obviously provides a triangulation of the circle, S^1, which you are left to 'draw'.

The above result, and the example, illustrate that for polyhedra (and thus for triangulated manifolds), an approach via open coverings is at least as strong as that via triangulations. Triangulations give open coverings that themselves give back the triangulation. If we, on the other hand, start with an open covering of a polyhedron, can we always find a triangulation that is finer than it in the sense that any open star of a vertex is completely within some open set of the covering? The following classical result (for instance, in Spanier, [150], p.125) tells us that we can, and hence that, for polyhedra, the two approaches, triangulations and open coverings are, in fact, of equal strength:

Theorem 19 *Let $\mathcal{U}$ be any open covering of a (compact) polyhedron X, then X has triangulations finer than $\mathcal{U}$.*   ■

We thus have a method of going between a triangulations based approach to an open coverings based one, both theoretically and intuitively. This works extremely well for spaces that are triangulable, however many spaces encountered in diverse areas of mathematics are not manifolds or even polyhedra, and then, evidently, triangulation based ideas cannot be directly used. The open covering based ones have no such restriction, but that advantage will not concern us in this chapter as we mainly deal with polyhedra and manifolds, (which will often be PL ones, and hence have well understood triangulations).

7.2.4   How can we construct TQFTs ... from a finite crossed module?

In Yetter's second construction of a TQFT in [166], he replaced the finite group, G, by a finite crossed module, $\mathsf{C} = (C, P, \partial)$. It should be fairly clear, given the route we have taken so far, how we can treat this from our perspective. We look at C-colourings as being an assignment of elements of P to edges of a triangulation, elements of C to the 2-simplexes with a boundary condition, and with any tetrahedrons giving some cocycle condition.

In pictures:
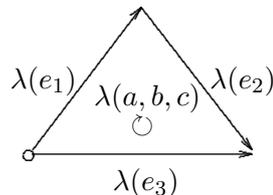
where $e_1 = \langle a, b \rangle$, $e_2 = \langle b, c \rangle$ and $e_3 = \langle a, c \rangle$, $\lambda(a, b, c) \in C$, and the various $\lambda(e_i) \in P$.

The boundary condition, as given by Yetter, is then that

$$\lambda(e_1)\lambda(e_2)\lambda(e_3)^{-1} = \partial\lambda(a, b, c)^{-1}.$$

If we think of the categorical group or 2-group, $\mathcal{X}(\mathsf{C})$, instead of C itself, this means that, for $\sigma = \langle a, b, c \rangle$,

$$\tilde{\lambda}(\sigma) : \lambda(e_1)\lambda(e_2) \Rightarrow \partial\lambda(a, b, c)\lambda(e_1)\lambda(e_2),$$

and, of course, $\lambda(e_3)$ is the expression on the right here. (Here $\tilde{\lambda}(\sigma) = (\lambda(\sigma), \lambda(e_1)\lambda(e_2)) \in \mathcal{X}(\mathsf{C})_1 = C \rtimes P$.)

It will also be necessary to have a 2-flatness / cocycle condition for a tetrahedron. This will say that the diagram in the 2-category / 2-group, $\mathcal{X}(\mathsf{C})$, corresponding to the faces of a tetrahedron,

$\langle a_0, a_1, a_2, a_3 \rangle$, must commute. We will first look at this condition within the 2-group, $\mathcal{X}(\mathsf{C})$, using the two compositions, $\sharp_0$ and $\sharp_1$, then we will convert from 2-categorical to simplicial notation (in fact, in two different ways) to get this condition in more simplicial language.

We assume $\mathbf{T}$ is an ordered simplicial complex and that we have λ as above. The faces of a tetrahedron, $\langle a_0, a_1, a_2, a_3 \rangle$, form a square diagram (as we have seen several times before). In this case, the vertices of that square correspond to the objects of $\mathcal{X}(\mathsf{C})$, or, if you prefer, to elements of P. We have

$$
\begin{array}{ccc}
\lambda(a_0 a_1)\lambda(a_1 a_2)\lambda(a_2 a_3) & \xrightarrow{\;\;①\;\;} & \lambda(a_0 a_2)\lambda(a_2 a_3) \\
{\scriptstyle ②}\downarrow & & \downarrow{\scriptstyle \tilde{\lambda}(a_0, a_2, a_3)} \\
\lambda(a_0 a_1)\lambda(a_1 a_3) & \xrightarrow[\tilde{\lambda}(a_0, a_1, a_3)]{} & \lambda(a_0 a_3)
\end{array}
$$

where $① = \tilde{\lambda}(a_0, a_1, a_2)\sharp_0\lambda(a_2, a_3)$ and $② = \lambda(a_0, a_1)\sharp_0\tilde{\lambda}(a_1, a_2, a_3)$, (cf. section **??**, page **??**). The cocycle condition is that this commutes in $\mathcal{X}(\mathsf{C})$, where the composition used is the category composition in $\mathcal{X}(\mathsf{C})$, that is, $\sharp_1$. (You can easily convert this to a condition in the crossed module, C, if you like. The above references more or less tell you what form things should take, but certain conventions are here different from those back in section **??**. You can also refer to the paper by Faria Martins and Porter, [79], but the best course of action is to **work it out for yourself**.)

As the categorical composition, $\sharp_1$, is determined by the other structure and the group multiplication (which is $\sharp_0$), we could rewrite this condition solely using these. In fact, as T is 'simplicial', it seems better to translate the conditions into simplicial ones. (This may seem a bit arbitrary, but we have seen the efficiency of simplicial methods when handling coherence in earlier chapters, and cocycle conditions are coherence conditions, Never fear, the translation works and is worth it, ...)

To help with this, we replace $\mathcal{X}(\mathsf{C})$ by $K(\mathsf{C})$. Recall that this is the simplicial group obtained by taking the (internal) nerve of the (internal) category structure of $\mathcal{X}(\mathsf{C})$, everything being done 'internally' in the category of groups, cf. page **??**. This functor is also one part of the Dold-Kan equivalence between crossed complexes and simplicial T-complexes, see page 128. If you want yet another glimpse of $K(\mathsf{C})$ in its many and varied manifestations, think of $\mathcal{X}(\mathsf{C})$ as a one-object 2-category, $\mathcal{X}(\mathsf{C})[1]$, and similarly think of the simplicial group, $K(\mathsf{C})$, as an $\mathcal{S}$-groupoid (with one-object); that really should be $K(\mathsf{C})[1]$, of course. In section **??**, (page **??**), we saw that any 2-category gives a simplicially enriched category by using the nerve functor on each hom-category. As the nerve functor embeds Cat in $\mathcal{S}$, the resulting simplicial enriched category is really the same as the 2-category. In our case here, that $\mathcal{S}$-category is $K(\mathsf{C})[1]$.

In the above square diagram, the vertices are vertices of $K(\mathsf{C})$ and the edges are 1-simplices of $K(\mathsf{C})$, so we need to use the simplicial form of the $\sharp_1$-composition so as to work out the diagonal of the square in two different ways. (The results of these calculations must be equal as the square has to be commutative.)

We have actually used this simplicial form several times already, but perhaps not always with an explicit mention! That means we should give it 'for convenience'. Given a pair of 1-simplices, g_0, g_2, in a simplicial group, G, with $d_0 g_2 = d_1 g_0$ (so the picture is

$$
\cdot \xrightarrow{\;g_2\;} \cdot \xrightarrow{\;g_0\;} \cdot
$$

within G), we clearly can form a $(2,1)$-horn (cf. page **??** if you have forgotten what that means). Using the algorithm given in Proposition **??**, we can fill it. This gives an explicit element, $x = s_1 g_2 . s_1 s_0 d_0 (g_2)^{-1} . s_0 (g_0)$, in G_2 and its d_1 face is $g_2 . s_0 d_0 (g_2)^{-1} . g_0$. This can be thought of as the

composite of g_0 and g_2, and, as x is thin, *is* that composite if $NG_2 \cap D_2 = 1$. In our square diagram, this composite will be a diagonal arrow (pointing SE). The formula for the top composed with the right-hand side is

$$\tilde{\lambda}(a_0, a_1, a_2) s_0 \lambda(a_2, a_3).(s_0 \lambda(a_1, a_2) s_0 \lambda(a_2, a_3))^{-1} \tilde{\lambda}(a_0, a_2, a_3)$$
$$= \tilde{\lambda}(a_0, a_1, a_2) s_0 \lambda(a_1, a_2)^{-1} \tilde{\lambda}(a_0, a_2, a_3)$$

and the composite '(left side) $\sharp_1$ (bottom)' is

$$s_0 \lambda(a_0, a_1) \tilde{\lambda}(a_1, a_2, a_3) s_0 \lambda(a_1, a_3)^{-1} s_0 \lambda(a_0, a_1) \tilde{\lambda}(a_0, a_1, a_3),$$

and these two must be equal, since the square is to commute in $\mathcal{X}(\mathsf{C})$.

Although seeming moderately complex, this cocycle condition is very like others that we have seen before, at least in its general form. We will also see how, when encoded simplicially, it becomes just the condition corresponding to the *existence* of a simplicial morphism.

Let us now look at the assignment λ again in a new light:

- λ assigns vertices in $K(\mathsf{C})$ to edges, i.e., to elements in T_1;

- λ assigns edges in $K(\mathsf{C})$ to elements in T_2.

This suggests another construction that we have seen before. We have, for a simplicial set, K, the loop groupoid, $G(K)$, (see page 125). This was an $\mathcal{S}$-groupoid, and the vertices were generated by the edges / 1-simplices of K, the 1-simplices were generated by K_2, and so on. This 'hints' that a C-colouring might correspond to, perhaps, an $\mathcal{S}$-groupoid map from $G(T)$ to $K(\mathsf{C})$, so we should 'check this out' as an idea. (This was the case when C was just a group; see page 222 and Proposition 44.)

First note that, as T is an *ordered* triangulation of X, T can be replaced by its associated simplicial set. This, as was mentioned back in section **??** (page **??**), has as its n-simplices those totally ordered sets, $\langle a_0, a_1, \ldots, a_n \rangle$, of vertices of T (so $a_0 \leq a_1 \leq \ldots \leq a_n$) such that $\{a_0, a_1, \ldots, a_n\}$ is a simplex of T, after deletion of any repeats. The face maps omit the corresponding element, so, for instance, $d_0 \langle a_0, a_1, a_2 \rangle = \langle a_1, a_2 \rangle$, and the degeneracies repeat, in an obvious way, a vertex in the list, so, for example, $s_1 \langle a_0, a_1 \rangle = \langle a_0, a_1, a_1 \rangle$.

If we look at a C-colouring, λ, it is fairly clear that there is possible way to define a simplicial morphism,

$$\lambda' : G(T) \to K(\mathsf{C}),$$

using λ, and encoding the same information, so let us try it.

- the simplicially enriched groupoid, $G(T)$, has T_0 as its set of objects, whilst $K(\mathsf{C})$ is a simplicial group. (We, perhaps, should write $K(\mathsf{C})[1]$ or $K(\mathsf{C}[1])$ here, but will often omit the $[1]$, unless in a situation where that little bit of extra precision seems to be useful or needed.) We thus have a unique 'object map' underlying λ'.

- We want $\lambda'_0 : G(T)_0 \to K(\mathsf{C})_0 = P$, and $G(T)_0$ is the free groupoid on the directed graph given by the 1-skeleton of T, so this suggests $\lambda'_0 \langle a_0, a_1 \rangle = \lambda(a_0, a_1)$, where, as when we discussed $G(K)$, we will tacitly confuse a simplex in K_{n+1} with the corresponding generator in $G(K)_n$. Note that the degenerate 1-simplices of form, $\langle a, a \rangle$ are not covered by this definition, but, since, according to the defining relations in $G(T)$, the elements that they generate have been discarded or, rather, equated to identities, the corresponding value for $\lambda'_0 \langle a, a \rangle$ will be the identity element, $\langle a \rangle$, of the group P.

- For $\lambda_1' : G(T)_1 \to K(\mathsf{C})_1$, we want $\lambda_1'\langle a_0, a_1, a_2 \rangle \in K(\mathsf{C})_1$. Because of the twist in the d_0-face of $G(T)$, (cf. page 125), we need to take a bit of care. (**It may help just to draw a triangle**, label it using λ, as above, and see the difference between the $G(T)$ '2-cell' and that in the given λ.) We take

  - $\lambda_1'\langle a_0, a_1, a_2 \rangle = \tilde{\lambda}(a_0, a_1, a_2).s_0\lambda(a_1, a_2)^{-1}$, provided $a_0 < a_1 < a_2$. (You should check this works for both d_0 and d_1, for instance that $d_1\lambda_1' = \lambda_0'd_1$.)

  - On $\langle a_0, a_0, a_1 \rangle = \langle s_0(a_0, a_1) \rangle = id_{\langle a_0 \rangle}$, (since the relations $\langle s_0^{G(T)}x \rangle = id$, for each $x \in T_n$ hold in $G(T)$), there is no problem as λ' is to preserve identities.

  - As $\langle a_0, a_1, a_1 \rangle = s_1^T\langle a_0, a_1 \rangle = s_0^{G(T)}\langle a_0, a_1 \rangle$, λ_1', on this, must be $s_0\lambda_0'\langle a_0, a_1 \rangle$, since it is a simplicial morphism, so again 'no problem'.

  In fact, this already determines λ'. We investigate why by looking at λ_2':

- For $\lambda_2' : G(T)_2 \to K(\mathsf{C})_2$, we use the fact that the simplicial group, $K(\mathsf{C})$, is a group T-complex to find out what $y = \lambda_2'\langle a_0, a_1, a_2, a_3 \rangle$ must be. We use the face operators, d_1 and d_2, to get values on a $(2,0)$-horn in $K(\mathsf{C})$. Filling that horn 'thinly', we get a unique thin candidate for what y must be and then check that it works for d_0. (It must work, because $K(\mathsf{C})$ has no homotopy above level 1. That was the significance of the 2-flatness / cocycle / tetrahedron condition on λ itself.) This gives a new form of the cocycle condition.

There is, in fact, a small complication here. The earlier form of the cocycle condition was written in terms of 2-commutativity of a tetrahedron in the 2-group, $\mathcal{X}(\mathsf{C})$. It thus uses the geometric or Duskin nerve of $\mathcal{X}(\mathsf{C})$. (More exactly, it uses the lax, rather than the op-lax form of this nerve.) You may recall that we looked at this in section **??**. It was the 2-categorical form of the homotopy coherent nerve of $\mathcal{X}(\mathsf{C})$, considered as the $\mathcal{S}$-enriched category, $\mathcal{X}(\mathsf{C})[1]$. Of course, $\mathcal{X}(\mathsf{C})$ corresponded to the simplicial group, $K(\mathsf{C})$, and is really the 'same thing', viewed from a slightly different angle (and, of course, with care taken on the convention for compositional order).

On the other hand, here we are using the loop-groupoid, $G(T)$, so a morphism from there to $K(\mathsf{C})$ corresponds to a simplicial map from T to $\overline{W}(K(\mathsf{C}))$, and that is the *other* model for the nerve. We had (Proposition **??** in Chapter **??**) that $Ner(\mathcal{X}(\mathsf{C}))$ and $\overline{W}(K(\mathsf{C}))$ were isomorphic. We have that our cocycle condition translated to the commutativity of that square, and thus to the fact that the assignment, λ, extends to a simplicial map from T to $Ner(\mathcal{X}(\mathsf{C}))$, whilst the construction above states that λ' extends to one from $G(T)$ to $K(\mathsf{C})$, i.e., from T to $\overline{W}(K(\mathsf{C}))$. As the two nerves are isomorphic, we have:

Proposition 48 *A C-colouring, λ, of $\mathbf{T}$ corresponds uniquely to*

(i) *a simplicial map $\lambda : T \to Ner(\mathcal{X}(\mathsf{C}))$; also to*

(ii) *a $\mathcal{S}$-groupoid morphism, $\lambda' : G(T) \to K(\mathsf{C})[1]$, and thus to*

(iii) *a simplicial map, $\lambda' : T \to \overline{W}(K(\mathsf{C}))$.* ∎

This is really just a corollary of the Bullejos-Cegarra result, ([43]), that we mentioned in Chapter **??**, and proved in its conjugate form. It will be very useful, because of other results that we have looked at on the coskeletal properties of $Ner(\mathcal{X}(\mathsf{C}))$, and of $\overline{W}(K(\mathsf{C}))$. It means that we can use either monoidal categorical, simplicial set or T-complex / simplicial group methods as convenient,

and can, up to a point, choose which context to work with so as to optimise our chances of obtaining results (and hopefully with fairly intuitive proofs). From an expositional viewpoint, we can simply take as a definition:

Definition: A C-colouring of $\mathbf{T}$ will be defined to be a simplicial map from the associated simplicial set, T, to $\overline{W}(K(\mathsf{C}))$.

This will replace the equivalent earlier one, but either of the other two formulations could also be used. This will enable us shortly to generalise in two distinct but related directions, not only to colourings with values in a more general 'finite' simplicial group, but also to more general monoidal categories, provided that suitable finiteness conditions are satisfied (corresponding to some extent to the fact that C is a finite 'categorical group' here), - but, as usual, we are getting ahead of ourselves! We need to construct Z_C from C.

To do this, we follow the same basic path as we did when considering G-colourings in section 7.2.2. We let $\Lambda_\mathsf{C}(\mathbf{T})$ be the set of C-colourings of $\mathbf{T}$, and then $Z_\mathsf{C}(X, \mathbf{T})$ be the (complex) vector space with basis labelled by $\Lambda_\mathsf{C}(\mathbf{T})$, We then turn to cobordisms. Let $(M, \mathcal{T})$ be an ordered triangulated cobordism from $(X, \mathbf{T})$ to $(Y, \mathbf{S})$ and define, as before

$$Z_\mathsf{C}^!(M, \mathcal{T}) : Z_\mathsf{C}(X, \mathbf{T}) \to Z_\mathsf{C}(Y, \mathbf{S}),$$

by, for $\lambda \in \Lambda_\mathsf{C}(\mathbf{T})$,

$$Z_\mathsf{C}^!(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_\mathsf{C}(\mathcal{T}) \\ \mu|\mathbf{T}=\lambda}} \mu|\mathbf{S},$$

on the basis elements, then extending linearly.

We, as before, need to normalise this with respect to independence from the triangulation $\mathcal{T}$ (keeping, of course, $\mathbf{T}$ and $\mathbf{S}$ fixed), and also for composition. What will the compensating scaling factor be in this case?

To answer this, it will help to re-examine subdivision, from a slightly different point of view. We will use the form of the definition of C-colouring corresponding to a morphism

$$\mu : \mathcal{T} \to \overline{W}(K(\mathsf{C})).$$

We will need to use the proof that $\overline{W}(K(\mathsf{C}))$ is Kan, which uses the explicit algorithms for filling horns in a simplicial group to get explicit algorithms for filling horns in $\overline{W}G$; see section 6.3.2 and the arguments there. This leads to the lemma. (Remember $\Lambda^i[n]$ is obtained from $\Delta[n]$ by omitting the unique non-degenerate n-simplex, and its ith face.)

In the following, by a *finite simplicial group*, we mean one in which each G_n is a finite group and all but a finite number of terms in its Moore complex are trivial, so the Moore complex has finite length. We note that a finite simplicial group represents a homotopy type with finite total homotopy in the sense that we met earlier in section 7.2.1.

Lemma 33 *Suppose $\lambda : \Lambda^0[2] \to \overline{W}G$ is a (2,0)-horn, for a finite simplicial group, G, then the number of fillers of λ is $\sharp(NG_1)$.*

Proof: We know that $\overline{W}G$ is Kan, so we have a filler for λ. Any two such fillers share the same d_1 and d_2-faces, but, then in the first place of their expressions, they must differ only by

multiplication by an element of NG_1. To see this, remember that an element in $\overline{W}G_2$ has form (g_1, g_0) with $g_i \in G_i$, $i = 0, 1$, then

$$
\begin{aligned}
d_1(g_1, g_0) &= d_0 g_1 . g_0 \\
d_2(g_1, g_0) &= d_1 g_1.
\end{aligned}
$$

If (g_1, g_0) and (g_1', g_0') are two fillers for the same $(2,0)$-horn, then $d_1(g_1' g_1^{-1}) = 1$, so $g_1' g_1^{-1} \in NG_1 = Ker\, d_1$. The expressions for the faces then give that the relationship between g_0 and g_0' is determined by that between g_1 and g_1'. The number of such fillers is thus exactly $\sharp(NG_1)$. ■

Generally most of the results that we will see in this section hold either 'as they are' or with slight adaptation if we replace $K(\mathsf{C})$ by a general finite simplicial group, G, and we will often, though not always, state and prove results in that extra generality.

There are higher dimensional versions of this lemma as we will see. Note that similar results hold for $(2,1)$- and $(2,2)$-horns. In each case, there are $\sharp NG_1$ different fillers. **You are left to prove** these two alternative forms. For that you will need results on $\sharp(Ker\, d_0 \cap Ker\, d_2)$ and $\sharp(Ker\, d_0 \cap Ker\, d_1)$, namely that they are the same as $\sharp NG_1$. More generally, for a simplicial group G, some $n \geq 0$ and some $0 \leq r \leq n$, let

$$
NG_n^{(r)} = \bigcap \{ Ker\, d_i \mid i \neq r \},
$$

so, for example, $NG_n^{(0)} = NG_n$, and $Ker\, d_0 \cap Ker\, d_2$, above, is $NG_2^{(1)}$, and so on.

Lemma 34 *There is a bijection*

$$
NG_n^{(r)} \leftrightarrow NG_n.
$$

■

The proof of this is **left to you**. Note that it is not claimed to be an isomorphism, nor to be compatible with face or degeneracies in any way. It is just a bijection, but explicit formulae can be given. (The result is used by Cegarra and Carasco, [50] and [49], to reduce the group T-complex condition, $T3$, of Ashley to the single one, $D_n \cap NG_n = 1$, as we mentioned on page **??**.)

Of course, for our purposes here, we have $\mathsf{C} = (C, P, \partial)$ is a finite crossed module and $G = K(\mathsf{C})$, so $\sharp NG_1 = \sharp(C)$. A similar calculation to the one we gave shows that any $(3, i)$-horn in $\overline{W}K(\mathsf{C})$ has a unique filler, since $\sharp NG_2 = 1$. Although this is a consequence of later more general lemmas, **it is easy to check directly** and is quite fun!

Now let us concentrate on the simplest case. We assume $\mathcal{T}'$ is formed from $\mathcal{T}$ by taking an interior edge, e, and subdividing it. We will look at the case where $e = \langle a_0, a_1 \rangle$, so $a_0 < a_1$, and will subdivide e with a new vertex, v, which is greater than all the vertices of all simplices incident to e. (It is easy to adapt the argument to other relative positions of the new vertex.)

We assume given a C-colouring, $\mu : \mathcal{T} \to \overline{W}K(\mathsf{C})$, which agrees with some given $\lambda : T \to \overline{W}K(\mathsf{C})$ on the 'input end'. We want to see what colourings, $\mu' : \mathcal{T}' \to \overline{W}K(\mathsf{C})$, there are which agree with μ on simplices which are not subdivided in the passage from $\mathcal{T}$ to $\mathcal{T}'$ and which 'sum' to the value of μ on subdivided ones. (The meaning of this last condition will emerge as we go along.)

The edge, e, of $\mathcal{T}$ has been replaced, in $\mathcal{T}'$, by $e_0 = \langle a_0, v \rangle$ and $e_1 = \langle a_1, v \rangle$. (**It is a good idea** to glance back at the way this was handled for G-colourings, page 215.) We need to assign a value to $\mu'(e_0)$ and any value in P will do. We then assign a value to $\mu'(e_1)$, so as to retain the overall value of $\mu(e)$ on the original edge, i.e., $\mu(e) = \mu'(e_0)\mu'(e_1)^{-1}$, with the inverse resulting from the change in direction along e_1, of course. (We, so far, have had a possibility of making one out of a possible $\sharp(P)$ choices.)

Next look at a 2-simplex, σ, incident to e in $\mathcal{T}$. Again there are several cases to consider, but they are all similar, so we assume $\sigma = \langle a_0, a_1, a_2 \rangle$, i.e., $a_2 > a_1$, (and we also have assumed previously that $a_2 < v$). Let $\sigma_0 = \langle a_0, a_2, v \rangle$, $\sigma_1 = \langle a_1, a_2, v \rangle$ and we look at possible values of $\mu'(\sigma_0)$. We know $\mu'\langle a_0, v \rangle$ as this is $\mu'(e_0)$. We also know $\mu'\langle a_0, a_2 \rangle$. We thus have a $(2,0)$-horn and can fill that in exactly $\sharp(C)$ ways by our lemma above.

From any fixed filler, we obtain $\mu'\langle a_2, v \rangle$. We now have no choice for $\mu'\langle a_1, a_2, v \rangle$, since it and $\mu'\langle a_0, a_2, v \rangle$ must combine to give $\mu\langle a_0, a_1, a_2 \rangle$. You may, quite rightly, ask what 'combine' must mean. In the original paper by Yetter, [166], the 12 different possibilities are given corresponding to the relative position of the new vertex amongst the three vertices of $\langle a_0, a_1, a_2 \rangle$, thus an explicit answer can be given. Perhaps, however, a slightly different perspective is worth taking, one that can be pushed further later on, as it is not so much combinatorial as 'geometric' or 'topological'. A purely combinatorial form might become increasingly difficult with increasing dimension, but a suitable geometric construction should mean 'the same' whatever dimension it is in.

We will think of the process of subdivision in a slightly different way and here a certain amount of informal imagery may help. We will make it more formal slightly later on.

Given $\mathcal{T}$ and e, we think of the new vertex, v, as being slightly off e and form the cone, $e * v$, on e with vertex v:
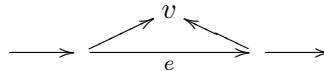
The diagram contains both e and the subdivided e (by going over the top of the triangular 'lump'). More generally, if σ is a simplex of $\mathcal{T}$ incident to e, then the join, $\sigma * v$, of v with σ will contain a copy of the subdivision of σ, resulting from subdividing e. (It may help to look at the classical treatment of subdivision as described, say, in Spanier, [150], p. 123.) Perhaps you can imagine the process of subdividing e as being like the formation of a little 'lump' on the union of the simplices incident to e. The original level corresponds to $\mathcal{T}$, the level going 'over the lump' corresponds to $\mathcal{T}'$, but we have both $\mathcal{T}$ and $\mathcal{T}'$ in the same simplicial set, which can be very useful. If you do not like the 'lump' picture, then instead you can make a cylinder on M, triangulate one end using $\mathcal{T}$ and the other end using $\mathcal{T}'$, then, if a simplex σ is not incident to e, triangulate $\sigma \times I$ in the standard way (as both ends of that part of the cylinder are the same), that is, using the product triangulation, whilst those simplices σ incident to e, (which are therefore subdivided in passing to $\mathcal{T}'$), you use a triangulation generalising the one given below for e itself:
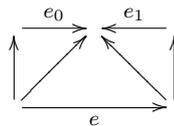
The arguments that we will use can be adapted to either picture. (Of course, the 'lump' is a subcomplex of the cylinder.)

Now, retaining that imagery, look at σ, σ_0, and σ_1, as part of a join $\sigma * v$, which will be a 3-simplex. (Remember we are in the case $a_0 < a_1 < a_2 < v$, and this 3-simplex is $\langle a_0, a_1, a_2, v \rangle$ in this new simplicial set.) We will look at the faces of this and where they are mapped to in $\overline{W}K(\mathsf{C})$, but, before we can really do that, we look at the cone, $v * e$, i.e., the new 2-simplex, $\langle a_0, a_1, v \rangle$. We have

- $d_0\langle a_0, a_1, v \rangle = \langle a_1, v \rangle$, so this can be assigned $\mu'(e_1)$;

- $d_1\langle a_0, a_1, v \rangle = \langle a_0, v \rangle$, so, similarly, corresponds to $\mu'(e_0)$;

- $d_2\langle a_0, a_1, v \rangle = \langle a_0, a_1 \rangle$, and we use $\mu(e)$ on this.

From our discussion of composition above, we have that this gives us a thin filler for the $(2,1)$-horn, corresponding to the d_0 and d_2 values and the d_1 of that filler will be $\mu(e)\mu'(e_1)$. We can see that this gives us *exactly* what we need to be able to say that $\mu'(e_0)\mu'(e_1)^{-1} = \mu(e)$, i.e., the thin filler *is* this equation, or perhaps slightly more exactly, *is the reason* for this equation. (This works perfectly for the case of G-colourings that we looked at earlier, and so is highly relevant here. We take it as the 'definition' of that equality.)

This suggests looking at the faces of $\sigma * v = \langle a_0, a_1, a_2, v \rangle$. We list them with a μ or μ' image where available.

- $d_0(\sigma * v) = \langle a_1, a_2, v \rangle = \sigma_1$, so use $\mu'(\sigma_1)$;

- $d_1(\sigma * v) = \langle a_0, a_2, v \rangle = \sigma_0$, so use $\mu'(\sigma_0)$;

- $d_2(\sigma * v) = \langle a_0, a_1, v \rangle$, use the thin filler above;

- $d_3(\sigma * v) = \langle a_0 a_1, a_2 \rangle = \sigma$, so use $\mu(\sigma)$.

Taking the $(3,1)$-horn, we find a thin filler in $\overline{W}K(\mathsf{C})$, and this will, in fact, be *unique*, since $NG_2 = 1$. We thus get that $\mu'(e_1)$ is *uniquely determined* by the values of $\mu(\sigma)$ and $\mu'(e_0)$, as claimed.

We have to take account of all such contributions, from each simplex, σ, incident to e. We introduce a measure of the complexity of the triangulation that will help us handle this. Let K be an arbitrary (finite) simplicial complex and let $\chi_k(K) = (-1)^k \chi(sk_k K)$, where χ is the Euler characteristic, thus

$$
\begin{aligned}
\chi_0(K) &= \sharp(K_0), \\
\chi_1(K) &= \sharp(K_1) - \chi_0(K), \\
\chi_2(K) &= \sharp(K_2) - \chi_1(K),
\end{aligned}
$$

and so on.

Now, if $\mathcal{T}$ is a triangulation of the cobordism, M, as before, we set

$$
\begin{aligned}
\chi_0^{int}(\mathcal{T}) &= \sharp(\mathcal{T}_0 - T_0 - S_0), \\
\chi_1^{int}(\mathcal{T}) &= \sharp(\mathcal{T}_1 - T_1 - S_1) - \chi_0^{int}(\mathcal{T}), \\
\chi_k^{int}(\mathcal{T}) &= \sharp(\mathcal{T}_k - T_k - S_k) - \chi_{k-1}^{int}(\mathcal{T}).
\end{aligned}
$$

Later on, we will also need, $\chi_0^\partial(\mathcal{T}) = \sharp(T_0 \cup S_0)$, and then inductively,

$$\chi_k^\partial(\mathcal{T}) = \sharp(T_k \cup S_k) - \chi_{k-1}^\partial \mathcal{T},$$

the count of the boundary k-simplices of $(M, \mathcal{T})$.

The usefulness of these modified Euler characteristics is because of the following in which e is an interior 1-simplex of the triangulation, $\mathcal{T}$:

Lemma 35 *(Yetter, [166], for the case $k = 2$.) The number, $s_{k+1}(e)$, of $(k + 1)$-simplices in $M - \partial M$ that are incident to e is*

$$\chi_k^{int}(\mathcal{T}') - \chi_k^{int}(\mathcal{T}),$$

where $\mathcal{T}'$ is obtained from $\mathcal{T}$ by edge stellar subdivision of the edge e.

Proof: The subdivision takes each $(k + 1)$-simplex, σ, that is incident to e and replaces it by two such, incident to σ_0 and σ_1, respectively, (in the notation we used earlier). Typically, if the new vertex, v is considered greater than all other vertices of $\sigma = \langle a_0, \ldots, a_{k+1} \rangle$, the two replacement $(k + 1)$-simplices will be $\langle a_0, a_2 \ldots, a_{k+1}, v \rangle$ and $\langle a_1, a_2 \ldots, a_{k+1}, v \rangle$. These meet in the face $\langle a_2 \ldots, a_{k+1}, v \rangle$ of dimension k. Other relative positions of v in the ordering yield similar results.

The net gain in χ_k^{int} in passing from $\mathcal{T}$ to $\mathcal{T}'$ is thus equal to $s_{k+1}(e)$.   ■

Now let

$$Z_{\mathsf{C}}^!(M, \mathbf{T}, \mathbf{S}) = \sharp(P)^{-\chi_0^{int}(\mathcal{T})} \sharp(C)^{-\chi_1^{int}(\mathcal{T})} Z_{\mathsf{C}}(M, \mathcal{T}).$$

Proposition 49 *The linear mapping, $Z_{\mathsf{C}}^!(M, \mathbf{T}, \mathbf{S})$, is independent of the choice of the triangulation, $\mathcal{T}$, extending $\mathbf{T}$, and $\mathbf{S}$ to the cobordism M.*   ■

The proof has essentially the same form as the earlier case, page 216, in which the place of the crossed module, C, was taken by the group, G, so this will be **left to you**.

We next need to 'fix' the problem of lack of 'compositionality'. The argument will be more or less the same as that for the G-colouring case. We refer you back to page 217 for the notation, etc. We get

Lemma 36

$$Z_G^!(N, \mathbf{S}, \mathbf{R}).Z_G^!(M, \mathbf{T}, \mathbf{S}) = \sharp(P)^{\chi_0(\mathbf{S})} \sharp(C)^{\chi_1(\mathbf{S})} Z_G^!(M +_Y N, \mathbf{T}, \mathbf{R}).$$

Proof: The adjustment $\sharp(P)^{\chi_0(\mathbf{S})} \sharp(C)^{\chi_1(\mathbf{S})}$ corresponds to the different ways of colouring the $\mathbf{S}$, which is common to both cobordisms, the left hand term has a 'compensating factor' without any contribution from $\mathbf{S}$. The compensatory scaling factor on the right hand term has this adjustment factor in the denominator, so the equation balances.   ■

We again distribute this adjustment between the two ends of M to get a new linear map:

$$Z_{\mathsf{C}}(M, \mathbf{T}, \mathbf{S}) = \sharp(P)^{-\frac{1}{2}\chi_0^\partial(\mathcal{T})} \sharp(C)^{-\frac{1}{2}\chi_1^\partial(\mathcal{T})} Z_{\mathsf{C}}^!(M, \mathbf{T}, \mathbf{S}).$$

Note that $\chi_k^\partial(\mathcal{T})$ only depends on $\mathbf{T}$ and $\mathbf{S}$, so is, in fact, independent of $\mathcal{T}$. As an evident corollary, we obtain, with similar notation to before:

Corollary 12 *For M, $\mathbf{T}$, $\mathbf{S}$, etc., as above:*

$$Z_{\mathsf{C}}(N, \mathbf{S}, \mathbf{R}).Z_{\mathsf{C}}(M, \mathbf{T}, \mathbf{S}) = Z_{\mathsf{C}}(M +_Y N, \mathbf{T}, \mathbf{R}).$$

∎

As we are following a parallel trail, more or less, to that that we used for Z_G, the next step is to go for the restriction maps,

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_{\mathsf{C}}(\mathbf{T}') \to \Lambda_{\mathsf{C}}(\mathbf{T}),$$

corresponding to subdividing $\mathbf{T}$. We already have partially met this when discussing the compensating terms for $\mathcal{T}'$ and $\mathcal{T}$, but that 'lump' imagery can, and should, be made more precise. In fact, this means that we will be able to treat arbitrary subdivisions, not just edge-stellar ones, if we want to.

As we mentioned before, one way to view a subdivision of a simplex is to see it as a cone on a subdivision of the boundary of the simplex. Suppose $s = \langle a_0, \ldots, a_n \rangle$ is an n-simplex in T, and take some point $v \in |s|$, the realisation of s. (We have $v \in \langle s' \rangle = \langle \{a \in \{a_0, \ldots, a_n\} \mid v(a) \neq 0\} \rangle$, often called the *carrier* of v . We will assume $s' = s$, so v is in the interor of $|s|$; if this is not the case, then replace s by s'.) Take ∂s to be the boundary of s and form $\partial s * v$, the join of ∂s with the 'new' vertex. The 'lump' comes from thinking of $|s|$ in the canonical realisation (i.e., in $\mathbb{R}^{V(T)}$, cf. page **??**). Subdividing adds a new vertex, which will increase the dimension of the ambient space. The 'new vertex', v, is not in $\mathbb{R}^{V(T)}$. We take s and form the cone $s * v$. This is the 'lump'! If s has dimension n, this has dimension $n + 1$. It retains s itself, so, as a simplicial complex, it has T in it as a subcomplex, but it also has T' as one as well. (Of course, we have simplified things a bit here as, if $dim\, T > n$, then T' may have more than one simplex subdivided as that subdivision will need to be constructed 'up the skeletons'.) If needed, this 'ambient complex' provides a setting for various constructions. Most importantly, it contains both T and T' as deformation retracts, but we can do better than this algebraically.

We again will assume that $dim\, T = n$, so we are just handling one simplex and, again, for simplicity, assume that v, the new vertex is placed after all the a_i in the ordering on $V(T')$. (The other possibilities are not really any more difficult, but are a bit more 'messy'.) Within $G(T')$, we have a (n, n)-horn with i^{th} face the generator corresponding to $\langle a_0, \ldots, \widehat{a_i}, \ldots, a_n, v \rangle$, where the $\widehat{}$, of course, means that this term is omitted. Using the filling algorithm, we can fill this and obtain d_n of the filler, representing the original $s \in G(T)$. This filler, then, gives strong deformation retraction data:

$$r_{\mathbf{T}}^{\mathbf{T}'} : G(\mathbf{T}) \to G(\mathbf{T}'),$$

$$s_{\mathbf{T}'}^{\mathbf{T}} : G(\mathbf{T}') \to G(\mathbf{T}),$$

with $s_{\mathbf{T}'}^{\mathbf{T}} r_{\mathbf{T}}^{\mathbf{T}'} = id_{G(\mathbf{T})}$, and $r_{\mathbf{T}}^{\mathbf{T}'} s_{\mathbf{T}'}^{\mathbf{T}} \simeq id_{G(\mathbf{T}')}$, essentially using the description of the filler. Here $r_{\mathbf{T}}^{\mathbf{T}'}$ assigns to the generator corresponding to s, the algebraic composite obtained from d_n of the filler. For $s_{\mathbf{T}'}^{\mathbf{T}}$, choices have to be made. It can be chosen to be induced by a simplicial approximation to the identity on $X = |T| = |T'|$, or, more explicitly, by mapping all but one of the n-simplices generating the subdivided simplex to the identity, or, rather, to a degeneracy, with the remaining one mapping to s itself, thus collapsing $G(T')$ back to $G(T)$. (In fact, we will not use any explicit form of $s_{\mathbf{T}'}^{\mathbf{T}}$, just that it exists.) There are many different ways of constructing $s_{\mathbf{T}'}^{\mathbf{T}}$, but they are all homotopic. (If constructed simplicially, they are even *contiguous*.)

We note that $r_{\mathbf{T}}^{\mathbf{T}'}$ expresses the generator, s, *algebraically* as a pasting of the subdivided s. If s is not a top dimensional simplex, then the subdivision will need propagating up the skeletons of T, but this again can be done by a filling argument within $G(T')$ to get the $r_{\mathbf{T}}^{\mathbf{T}'}$ expressing s as a composite of its subdivided parts. If a subdivision, T', of T is obtained by subdividing several times, then we just iterate the above construction to get the appropriate $r_{\mathbf{T}}^{\mathbf{T}'}$. In terms of our 'lump', this algebraic construction inserts T as a subcomplex of the lumpy complex, L, and there is an *elementary collapse* from L to T, similarly there is one from L to T'. The maps $r_{\mathbf{T}}^{\mathbf{T}'}$ and $s_{\mathbf{T}'}^{\mathbf{T}}$ are algebraic models of the composites obtained from $T \overset{insert}{\to} L \overset{collapse}{\to} T'$, and the other way around. The construction of L, together with using fillers, gives the homotopy, $r_{\mathbf{T}}^{\mathbf{T}'} s_{\mathbf{T}'}^{\mathbf{T}} \simeq id_{G(\mathbf{T}')}$. (For 'elementary collapses' in general, look up sources on Simple Homotopy Theory.)

We define our restriction map, $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$, by:
if $\lambda : G(T') \to K(\mathsf{C})$ is a C-colouring of $\mathbf{T}'$, then

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_{\mathsf{C}}(\mathbf{T}') \to \Lambda_{\mathsf{C}}(\mathbf{T})$$

sends λ to $\lambda \circ r_{\mathbf{T}}^{\mathbf{T}'}$ and then extend linearly to get it defined on $Z_{\mathsf{C}}(X, \mathbf{T}')$.

We immediately have that $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ is an epimorphism, since it is split by sending λ' to $\lambda' \circ s_{\mathbf{T}'}^{\mathbf{T}}$. We repeat that none of this requires that $\mathbf{T}'$ is obtained by edge-stellar subdivisions from $\mathbf{T}$, so we will usually quietly drop that assumption, except if needed later for some specific argument. We also note for use later that this definition did not require other than a (finite) simplicial group, G as 'coefficients' to make the definition work. No specific properties of $K(\mathsf{C})$ are used as everything important happens back in $G(T)$.

It will pay to examine the strong deformation retraction data for the 'lump' slightly more closely. Suppose that $\mathbf{T}'$ is obtained from $\mathbf{T}$ by forming $\partial s * v$ for some $s \in T_n$ and $v \in \langle s \rangle$, the carrier of s. This gave us a (n, n)-horn in $G(T')$, which we used to get the *algebraic* analogue of the 'lump' and thus the composite corresponding to $s \in G(T)$. That horn's parts all were n-simplices, so gave $(n-1)$-simplices of $G(T')$. The filling algorithm *only* uses these and their faces and so everything happens in $sk_{n-1}G(T')$, including the homotopy $r_{\mathbf{T}}^{\mathbf{T}'} s_{\mathbf{T}'}^{\mathbf{T}} \simeq id_{G(\mathbf{T}')}$. Intuitively, this is a 'thin' homotopy, i.e., if $n \geq 1$, the images of all the $(n-1)$-simplices use only $sk_{n-1}G(T')$. We will return to this slightly later on, formalising things a bit more.

As we had earlier for the G-colourings, we now define:

$$Z_{\mathsf{C}}(X) = colim_{\mathbf{T}} Z_{\mathsf{C}}(X, \mathbf{T}),$$

and can lift from that earlier discussion that the natural linear maps,

$$r_{\mathbf{T}}^X : Z_{\mathsf{C}}(X, \mathbf{T}) \to Z_{\mathsf{C}}(X),$$

are epimorphisms. We therefore have that $Z_{\mathsf{C}}(X)$ is a finite dimensional vector space. One point is that the $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ maps need to be scaled so as to obtain compatibility with the maps coming from the cobordisms. The scaling, in this case, gives

$$res_{\mathbf{T}',\mathbf{T}} : Z_{\mathsf{C}}(X, \mathbf{T}') \to Z_{\mathsf{C}}(X, \mathbf{T}),$$

by

$$res_{\mathbf{T}',\mathbf{T}} = \sharp(P)^{-\frac{1}{2}(\chi_0(T') - \chi_0(T))} \sharp(C)^{-\frac{1}{2}(\chi_1(T') - \chi_1(T))} \mathrm{res}_{\mathbf{T}',\mathbf{T}}.$$

This rescaling of the res$_{\mathbf{T'},\mathbf{T}}$ does not change the end result, $Z_{\mathsf{C}}(X)$, at least, up to isomorphism. It does, however, change the quotient maps $r_{\mathbf{T}}^X$ by a scalar factor, and this *is* important, especially when doing calculations, as we will see slightly later.

We have

$$
\begin{array}{ccc}
Z_{\mathsf{C}}(X,\mathbf{T}) & & \\
\uparrow & \searrow{\scriptstyle r_{\mathbf{T}}^X} & \\
{\scriptstyle \mathrm{res}_{\mathbf{T'},\mathbf{T}}} & & Z_{\mathsf{C}}(X) \\
\downarrow & \nearrow{\scriptstyle r_{\mathbf{T'}}^X} & \\
Z_{\mathsf{C}}(X,\mathbf{T'}) & &
\end{array}
$$

commutes where $\mathrm{res}_{\mathbf{T'},\mathbf{T}}(\lambda) = [\lambda]$, etc., but replacing $\mathrm{res}_{\mathbf{T'},\mathbf{T}}$ by its scaled version, $res_{\mathbf{T'},\mathbf{T}}$ leads to a diagram that does not commute. This can be fixed by scaling the quotienting maps, so replacing each $r_{\mathbf{T}}^X$ by $\rho_{\mathbf{T}}^X := \sharp(P)^{-\frac{1}{2}\chi_0(T))}\sharp(C)^{-\frac{1}{2}\chi_1(T))}r_{\mathbf{T}}^X$ gives us back commutativity, yet does not disturb the universal property of the codomain.

With that the following result should come as no surprise.

Theorem 20 *(Yetter, [166]) For any dimension d, the construction above applied to $(d+1)$–Cob_{PL} gives rise to a $(d+1)$-dimensional TQFT, Z_{C}.*

Proof: The 'proof' sketched above *is* incomplete. It is clear that although we 'thought' $d = 2$, there was no point at which we actually used that. The arguments did use tetrahedra, but these ended up mapped across to $\overline{W}K(\mathsf{C})$, so ended as being filled with thin elements, leading to commutativity and the 2-flatness conditions.

The other more major point we have left out of the discussion is that of identities. We will **leave this to you to check up** in the sources (although they are not all that explicit on this).∎

Remarks: We have skated over the actual construction of $(d+1)-Cob$ as a category. This is well discussed in many convenient sources, so we will not go into it in all its gory detail, however some more comments probably are necessary.

The objects are d-manifolds, the morphisms are ...? What? We think of them as being cobordisms, but cobordisms compose *only up to homeomorphism* and similarly for identities. (It is the same old problem that is at the heart of a lot of what we have been doing.) For instance, $M := X \times [0,1]$ will be a cobordism between X and itself, so may serve as an identity morphism, but if $N : X \to Y$ is another cobordism, $M +_X N$ is not equal to N. (The only 'cobordism' that would work to give us equality would be to think of the d-manifold X as a $(d+1)$-cobordism, and that would open a 'can of worms'!) Of course, $M +_X N$ *is* homeomorphic to N, so really we should use either homeomorphism classes of cobordisms as the morphisms or encode the homeomorphisms somehow into the structure of the 'category'. This latter idea is, in the end, the better one, but means that the use of 'weak categories', or better, some form of quasicategory, is the way out of the difficulty. For more on this, see Lurie's summary, [113]. This involves as well the notion of extended TQFT, but we will not explore that here.

The epimorphism,

$$
r_{\mathbf{T}}^X : Z_{\mathsf{C}}(X,\mathbf{T}) \to Z_{\mathsf{C}}(X),
$$

means that we 'merely' have to understand when two C-colourings yield the same element of $Z_{\mathsf{C}}(X)$ and we will understand $Z_{\mathsf{C}}(X)$. We have analogues of the results of our earlier analysis of the G-colourings.

Proposition 50 *Suppose* $\lambda : T \to \overline{W}K(\mathsf{C})$, *and* $\lambda' : T' \to \overline{W}K(\mathsf{C})$ *are two* C-*colourings such that* $r_{\mathbf{T}}^X(\lambda) = r_{\mathbf{T}'}^X(\lambda')$, *then there is a joint subdivision,* $\mathbf{T}''$, *of* $\mathbf{T}$ *and* $\mathbf{T}'$ *and simplicial approximations to the identity,* $\overline{s} : \mathbf{T}'' \to \mathbf{T}$ *and* $\overline{s'} : \mathbf{T}'' \to \mathbf{T}'$, *such that* $\lambda \circ s$ *and* $\lambda' \circ s'$ *are thinly homotopic. (If* $T_0 \cap T_0'$ *is non-empty, the homotopy can be chosen to be constant on the vertices of this intersection.)*

Proof: We know that $\mathbf{T}''$ and a λ'' exist such that $res_{\mathbf{T}'',\mathbf{T}}(\lambda'') = \lambda$ and $res_{\mathbf{T}'',\mathbf{T}'}(\lambda'') = \lambda'$ from our discussion of the G-colouring case. We thus have

$$\lambda = \lambda'' \circ \overline{r},$$

where $\overline{r} : \mathbf{T} \to \overline{W}G(\mathbf{T}'')$ is adjoint to $r : G(\mathbf{T}) \to G(\mathbf{T}'')$. We noted that $rs : G(\mathbf{T}) \to G(\mathbf{T})$ is thinly homotopic to the identity, where s is part of the strong deformation retraction for $G(\mathbf{T})$ and $G(\mathbf{T}'')$, chosen by collapsing the 'lump' or, equivalently, as a simplicial approximation to the identity. (It can be chosen so as to fix old vertices if that is needed.) We thus have

$$\lambda \circ \overline{s} = \lambda'' \circ \overline{r} \circ \overline{s} \simeq_{thin} \lambda'' \circ \eta_{\mathbf{T}''} : \mathbf{T}'' \to \overline{W}G(T'') \to K(\mathsf{C}),$$

where $\eta_{T''}$ is the unit of the adjunction between $\overline{W}$ and G. To understand this enough to proceed, we will need to make precise the notion of thin homotopy here.

    Definition: Suppose that G and H are two $\mathcal{S}$-groupoids, and $f_0, f_1 : G \to H$ are two morphisms of $\mathcal{S}$, which are homotopic by a homotopy h. We say h is a *thin* homotopy if for each $x \in G_n$, the homotopy restricted to x takes values in $sk_n H$, the n-skeleton of H.

    Of course, we can use the description of homotopies as families of maps, $\{h_i^n : G_n \to H_{n+1}\}$, to be slightly more explicit; (refer back to page **??** if you need that description). We would have that h is thin if for all n and appropriate i, $h_i^n : G_n \to D(H_{n+1})$, the subgroupoid in H_{n+1} generated by the degenerate elements. As we are handling $\mathcal{S}$-groupoids, we have a map from $Ob(G)$ to H_0 related to the f_0 on the objects in the evident way. This means that a thin homotopy can move the objects along a 'path' in the codomain. **You are left to examine** the way in which 'thinness' of homotopies between maps from $G(T)$ to a simplicial group, G, transforms into special homotopies between maps from T to $\overline{W}G$. Again 'thin' seems a good term to use for this type of homotopy, but **you are left to formalise it**.

    In words, a thin homotopy deforms the images of a n-simplex, x, from $f_0(x)$ to $f_1(x)$, within the n-skeleton, never using any non-degenerate parts of H_{n+1}. (We will briefly discuss the terminology after the end of the proof.) It is clear that the homotopy, $h : rs \simeq id_{G(T)}$, is thin in the sense above, as it uses just composites of degenerate elements. It will correspond to a thin homotopy from $\overline{rs}$ to the unit of the adjunction. (Recall that there are *explicit* formulae for the adjointness relationship between G and $\overline{W}$.)

    We thus have thin homotopies

$$\lambda \circ \overline{s} \simeq_{thin} \lambda'' \circ \eta_{T''},$$

and, similarly,

$$\lambda' \circ \overline{s'} = \lambda'' \circ \overline{r} \circ \overline{s'} \simeq_{thin} \lambda'' \circ \eta_{T''},$$

which we can glue to get one as required. ∎

Remark: The notion of 'thin homotopy' goes right back to the heart of the link between simplicial sets and weak infinity categories, and thus to the link between geometric or topological cohomology and infinity category theory.

When forming the fundamental group or groupoid of a space, X, the homotopies used are rather special. Take for instance, when proving that composition of (normalised) path classes is associative. We have paths, $a, b, c, : I \to X$ such that $a * (b * c)$ is defined, (so end points match in the usual way). Here $a * b$ is the usual normalised concatenated path,

$$a * b(t) = \begin{cases} a(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ b(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Its essence is a map from $I_{d_0} \sqcup_{d_1} I \cong [0,2]$ to I sending t to $\frac{1}{2}t$. This composition starts as a co-operation in the 'models', i.e., in the intervals used to 'probe' the space. Proving that $a * (b * c)$ is homotopic to $(a * b) * c$ uses a homotopy defined *in the interval*, so the image of that homotopy in X happens completely within the trace of the composite path. It does not sweep out any 'surface'. Its image is a very 'thin' region of X. The homotopy has domain I^2 of dimension 2, but its image, somehow, has 'dimension 1'.

The singular complex of X gives us another instance of such thinness. The simplicial set, $Sing(X)$ is a Kan complex. The fillers for horns can be chosen to be 'thin', since if $x : \Lambda^i[n] \to Sing(X)$ is a horn, then we can find a filler which just uses the information in the horn. That is clearly of dimension $n - 1$. We can use a retraction of Δ^n to the geometric realisation of $\Lambda^i[n]$, which is the union of all but the ith face of Δ^n. We then have the filler $\overline{x} \in Sing(X)_n$ is the composite

$$\Delta^n \to |\Lambda^i[n]| \xrightarrow{x} X.$$

this, of course, only uses the parts of $Sing(X)$ of dimension $(n - 1)$ or less. It is a 'thin' filler in an intuitive sense (although it does not give a T-complex structure to $Sing(X)$, since that would require uniqueness of such a thin filler; see the discussion back on page **??**.)

When defining the fundamental 2-groupoid of a (Hausdorff) space, X, Hardie, Kamps and Kieboom use thin homotopies to prevent a collapse of the 2-dimensional information on X (see [32, 90] and also [91]), and, of course, in the work of Brown and Higgins, filtered spaces and filtered homotopies are used for the same purpose, [37].

A similar idea was used by John Roberts explicitly for cohomology and, via Street's study, [153], this led to the work of Verity on complicial sets, [160] and their weakened form, [159, 161, 162], that we saw earlier, page **??**.

In [138], the naturally occurring need for thin homotopy was filled by what was, there, called 'filtered homotopy'. Here we used 'thin' rather than 'filtered' as that term really fits the bill better.

Finally. in the differential geometric setting, a related notion of thinness has become quite common, (cf. [15, 48, 118]). Here we need X to be some sort of smooth space such as a smooth manifold, and $a, b : I \to X$ smooth paths with the same end-points (often with the property that they are constant in a neighbourhood of the end points of the interval). A (fixed end-point) homotopy, $h : I^2 \to X$, between them is thin if the rank of the differential, $Dh : T_{\underline{t}}I^2 \to T_{h(\underline{t})}X$, is everywhere smaller than 2. Intuitively that means that there is no 'transversal' deformation of the path and, again, we get the idea that the homotopy does not 'sweep out any surface' in X.

In our situation, the clear connection is with the notion of thin elements in a group T-complex (page **??**). The thin elements there are products of degenerate elements, but to get a T-complex, we needed the $NG \cap D = 1$ condition, which corresponded to uniqueness of thin fillers.

The above result has a partial converse:

Proposition 51 *Suppose that T is an ordered triangulation of X and $\lambda, \lambda' : T \to \overline{W}K(\mathsf{C})$ are C-colourings. If there is a thin homotopy, $h : \lambda \simeq \lambda'$, then $r_T^X(\lambda) = r_T^X(\lambda')$.* ∎

We will not prove this here, so **look at the proof in** [137]. The method is probably clear. You have to look for a subdivision of T and a C-colouring, λ'', which restricts to the two given colourings. The first thing to do is to look at h at the level of $\langle v \rangle \times I$ for the various $v \in T_0$ and to build a subdivision bit-by-bit, together with the C-colouring. (To some extent you can think of this as taking two copies of T, displacing one a bit then forming the joint subdivision in a fairly standard way. That *is* incorrect, but gives a bit of intuition that might help.)

As a result of this, we obtain

Theorem 21 *The vector space, $Z_{\mathsf{C}}(X)$, has a basis which is in bijective correspondence with the set, $[G(T), K(\mathsf{C})]_{thin}$, of thin homotopy classes of morphisms from $G(T)$ to $K(\mathsf{C})$, for any triangulation* **T** *of X.* ∎

Any simplicial map, $\lambda : G(T) \to K(\mathsf{C})$ must kill all the higher dimensional Moore complex terms of $G(T)$, since the Moore complex of $K(\mathsf{C})$ is precisely C. The Moore complex, $N(G(T))$, will be mapped to $M(G(T), 2)$, which is the crossed module (of groupoids over T_0 as its set of objects),

$$\frac{NG(T)_1}{d_0(NG(T)_2)} \xrightarrow{\partial} G(T)_0,$$

and there will be an induced map of crossed modules,

$$\lambda : M(G(T), 2) \to \mathsf{C}.$$

As we have a crossed module of groupoids as the domain and a single object one as codomain, a homotopy between two such C-colourings can assign an element of $C_0 = P$ to each object $\langle a_0 \rangle$ of $G(T)$. (This is the conjugation that is apparent in the simple G-colouring case.) Usually, in an arbitrary homotopy of crossed modules, there would be a derivation from $G(T)_0 = M(G(T), 2)_0$ to $C_1 = C$, but, if that homotopy is thin, this derivation must be trivial. As there is nothing in C above dimension 1, we have that thin homotopies are just conjugations.

It is worth experimenting with what happens if, instead of C being a crossed *module*, we had that it was a crossed *complex*. We would have (after adapting earlier results) the crossed complex $\pi(T)$ and a C-colouring would be a morphism of crossed complexes, $\lambda : \pi(T) \to \mathsf{C}$. **You are left to investigate** what 'thin' means in this case. It may help to consider $\mathrm{CRS}(\pi(T), \mathsf{C})$, the mapping crossed complex. We will look at an even more general setting in the next section. (For the above, it may be useful to look at Faria Martins and Porter, [79] and others of Faria Martins' papers, [77, 78], for some methods. Some of the suggested investigation is not detailed anywhere in the published literature and the outcomes of some of the questions that may occur to you may not be explicitly known.)

We now turn to the version of the Yetter invariant of a closed manifold, M, with target a finite crossed module. This is the value taken by $Z_{\mathsf{C}}(M)$ on the empty C-colouring.

As in our earlier discussion, M is a $(d+1)$ dimensional closed manifold (usually assumed to be PL, since we use triangulations). It is thought of as being a cobordism from the empty d-manifold to itself. We have, almost as before (page 224), that, for $\mathcal{T}$, a triangulation of M,

$$Z_{\mathsf{C}}(M, \mathcal{T}) = \sharp(\Lambda_{\mathsf{C}}(\mathcal{T})),$$

and we set

$$I_{\mathsf{C}}(M) = \sharp(P)^{-\chi_0(\mathcal{T})} \sharp(C)^{-\chi_1(\mathcal{T})} \sharp(\Lambda_{\mathsf{C}}(\mathcal{T})),$$

Corollary 13 *(of Proposition 56, page 255) The quantity, $I_{\mathsf{C}}(M)$, is independent of the choice of triangulation.* ∎

We can do some trivial manipulations of this expression. We have $\chi_1(\mathcal{T}) = \sharp(\mathcal{T}_1) - \sharp(\mathcal{T}_0)$, so, to ease notation, writing $n_i = \sharp(\mathcal{T}_i)$, then

$$I_{\mathsf{C}}(M) = \frac{\sharp(C)^{n_0}}{\sharp(P)^{n_0} \sharp(C)^{n_1}} \sharp(\Lambda_{\mathsf{C}}(\mathcal{T})).$$

If we write $N = \partial C$, $\pi_0 = P/N$ and $\pi_1 = Ker\, \partial$, then clearly

$$\frac{\sharp(C)^{n_0}}{\sharp(P)^{n_0}} = \left(\frac{\sharp(\pi_1)}{\sharp(\pi_0)}\right)^{n_0},$$

and the expression begins to be similar to the scaling factors for Quinn's FTH theory, (see page 211). The precise link is explored in Faria-Martins and Porter, [79], using a bit more of the theory of crossed complexes than we have assumed here, so you are left to look that up. (Remember to check on the conventions as right actions are being used there.) The description uses the space $Map(M, B\mathsf{C})$ and is essentially

$$I_{\mathsf{C}}(M) = \sharp^{\pi}(Map(M, B\mathsf{C}))$$

in the notation that we introduced earlier (page 211).

This quantity, $I_{\mathsf{C}}(M)$, is referred to as the *Yetter Invariant* for M (with target, C).

Example: Although we will investigate this more fully in the next section, it is interesting to note that, if ∂ is a monomorphism, so that C is isomorphic to an inclusion crossed module, $N \overset{\triangleleft}{\to} P$ (with $N = \partial C$), then

$$I_{\mathsf{C}}(M) = I_{\pi_0}(M),$$

as one would expect.

This follows because π_1 is trivial and, in this situation,

$$\sharp(\Lambda_{\mathsf{C}}(\mathcal{T})) = \sharp(N)^{n_1} \sharp(\Lambda_{\pi_0}(\mathcal{T})).$$

7.3   Examples, calculations, etc.

The above example suggests that there should be links between Z_G and Z_C, for C, an inclusion crossed module with $\pi_0 \cong G$. More generally, we might suggest there to be a relationship between Z_C and the two related TQFTs, Z_{π_0} and $Z_{\pi_1[1]}$, where $\pi_1[1]$ is to be the crossed module, $\pi_1 \to 1$. That link is much harder to investigate and we will *not* be giving any deep general results on, just scratching the surface, as little is known about the general situation.

   To start on this open problem in a semi-systematic way, let us look at some examples of Z_C for different types of crossed module, C.

   We retain our previous notation for spaces, (ordered) triangulations, etc.

7.3.1   Example 0: The trivial example

As a first, almost silly, example, we can consider Z_G, when G is the trivial group. **You are left to check** that the resulting TQFT is the trivial one, which is constant with value the ground field.

7.3.2   Example 1: $C = (1, G, inc) = K(G, 0)$

We would expect that, in this case, Z_C would reduce to just Z_G, and, of course, it does, but, as usual, it is useful to examine even this simplest case although it is 'clear'.

   A C-colouring of a triangulation T of a manifold, X, is an assignment of elements of G to the edges on T satisfying the commutativity condition (of page 213), since the only value that we can assign to a triangle is 1, i.e., a C-colouring *is* just a G-colouring. Next glance at each of the scaling factors in turn. In each we set $\sharp(C) = 1$, and, of course, retrieve the corresponding values used in the construction of Z_G.

   That was 'obvious'. We will have to work harder when $C = (N, P, inc)$ with $P/N \cong G$, but that is for later.

7.3.3   Example 2: $C = (A, 1, \partial) = K(A, 1)$

Here we take A to be a finite Abelian group, so C has $P = 1$, the trivial group. A C-colouring, λ of T assigns an element, $\lambda(a, b, c)$, in A to each triangle, $\langle a, b, c \rangle$, in T. These are to satisfy: for $a < b < c < d$, (so $\langle a, b, c, d \rangle$ is a 3-simplex of T),

$$\lambda(a, b, c)\lambda(a, c, d) = \lambda(b, c, d)\lambda(a, b, d).$$

This is just the simplified form of the colouring cocycle condition (page 230) that results since A is Abelian allowing several terms to cancel in the original general formula.

   Writing things additively, this is just

$$\lambda(b, c, d) - \lambda(a, c, d) + \lambda(a, b, d) - \lambda(a, b, c) = 0,$$

so C-colourings are just 2-cocycles in the classical sense of simplicial cohomology theory (cf. section **??**) applied to the simplicial complex, T. We have, in fact, a bijection

$$\Lambda_C(T) \leftrightarrow Z^2(T, A),$$

where we follow the traditional notation, as used, for instance, on page **??**, in denoting by $Z^2(T, A)$, the group of 2-cocycles of T with coefficients in A.

Looking at the restriction maps, we find that $res_{\mathbf{T'},\mathbf{T}}$ sends a 2-cocycle on $\mathbf{T'}$ to, at worst, a multiple of a cocycle on $\mathbf{T}$, never anything more complex in the way of linear combinations. Moreover, it is part of a chain homotopy equivalence between $C(\mathbf{T'}, A)$ and $C(\mathbf{T}, A)$, the simplicial chain complexes on $\mathbf{T'}$ and $\mathbf{T}$ respectively, since it is induced by part of a strong deformation retraction on the corresponding loop $\mathcal{S}$-groupoid level. If we now check through the quotient maps from $Z_{\mathsf{C}}(X, T)$ to $Z_{\mathsf{C}}(X)$, we find they correspond exactly to the quotienting from $\mathbb{C}[Z^2(T, A)]$ to $\mathbb{C}[H^2(X, A)]$. (The argument has a 'classical' feel to it and **you should be able to build it fairly easily**. It can be found in Yetter's paper, [166].) In other words, $Z_{\mathsf{C}}(X)$ is the vector space $\mathbb{C}[H^2(X, A)]$ generated by $H^2(X, A)$.

Turning to the cobordisms, we have $M : X \to Y$ and a triangulation, $\mathcal{T}$, of M extending T on X and S on Y. The analysis of the scaling factors uses simple counting arguments based on the identification of $\sharp(A)^{\sharp(\mathcal{T}_i)}$ as being

$$\frac{\sharp(C^i(\mathcal{T}, A))}{\sharp(C^i(T, A))\sharp(C^i(S, A))},$$

followed by use of the exact sequences

$$0 \to Z^i(\mathcal{T}, A) \to C^i(\mathcal{T}, A) \to B^i(\mathcal{T}, A) \to 0,$$

etc. Again, it is worth playing with these before turning to Yetter's paper for the analysis in the case of surfaces. (His argument crucially uses that each edge in T is incident to two 2-simplices, i.e., that he is working with $3 - Cob$.) The situation for other dimensions than $d = 2$ **does not seem to be known**.

We will not be using Yetter's result, so merely record the value of the Yetter invariant in this case, for M, a 3-manifold. It is

$$I_{\mathsf{C}}(M) = \frac{\sharp(H^0(M, A))\sharp(H^2(M, A))}{\sharp(H^1(M, A))},$$

which certainly suggests a good candidate for it for higher dimensional manifolds.

7.3.4 Example 3: C, a contractible crossed module

The above two examples seem to suggest that Z_{C}, in general, should depend on the homotopy of C, i.e. at least on its π_0 and π_1, (or π_1 and π_2, depending on whether algebraic or topological grading is being used). This idea then suggests that we look at a very simple test case, namely a crossed module for which both π_0 and π_1 are trivial.

For a group, P, we can form $\mathsf{P} = (P, P, =)$. Of course, the obvious morphism, $\mathsf{P} \to \mathbf{1}$, is a weak homotopy equivalence and $K(\mathsf{P})$ is, in fact, contractible as a simplicial group, as is easily verified. We might guess Z_{P} was, therefore, the same as $Z_{\mathbf{1}}$, the trivial TQFT.

To verify that our guess is correct, we can use the plan sketched out earlier for determining $Z_{\mathsf{C}}(X)$, namely first looking at $Z_{\mathsf{C}}(X, T)$, then analyse the kernel of r_T^X using Propositions 50 and 51, or, alternatively, by showing that any $\lambda : G(T) \to K(\mathsf{P})$ is thinly homotopic to the trivial P-colouring, so by Theorem 21, $Z_{\mathsf{P}}(X) \cong \Bbbk$.

We will take this chance to take apart the morphisms from $G(T)$ to this crossed module, P, as this provides a good illustration of various features of colourings in an elementary example, that we, hopefully, can put to good use later.

We first recall that if $a < b$ in T_0, then $\langle a, b \rangle$ will define a 0-simplex in $G(T)(a, b)$. In the next dimension, if $a < b < c$, the 1-simplex, $\langle a, b, c \rangle \in G(T)(a, b)_1$ goes from $\langle a, b \rangle$ to $\langle a, c \rangle . \langle b, c \rangle^{-1}$. Now assume we have a P-colouring,

$$\lambda : G(T) \to K(\mathsf{P}),$$

then $\lambda\langle a, b, c \rangle$ is of form $(p(a, b, c), \lambda(z, b))$ and, as usual, we have $\partial p(a, b, c)\lambda(a, b) = \lambda(a, c)\lambda(b, c)^{-1}$. Let us abbreviate $\lambda(a, b) = x_2$, etc, (so $x_i = \lambda d_i\langle a, b, c \rangle$), and $\lambda\langle a, b, c \rangle = x$, and we translate the condition to give

$$x : x_2 \Rightarrow x_1.x_0^{-1},$$

and, hence, $\partial x = x_1.x_0^{-1}.x_2^{-1}$, (cf. the definition of C-colouring on page 228). That is true for any crossed modules, but in P, ∂ is the identity morphism, so $x = x_1 x_0^{-1} x_2^{-1}$. We have:

Lemma 37 *Any assignment, $\lambda : T_1 \to P$, extends uniquely to a P-coloring of T.* ∎

We have, within $K(\mathsf{P})$, the 1-simplex

$$x_2 \xrightarrow{(x, x_2)} x_1 x_0^{-1},$$

corresponding to λ, and we want to show λ is equivalent to 1, the trivial P-colouring. We can shift the vertices to 1 along paths such as

$$x_2 \xrightarrow{(x_2^{-1}, x_2)} 1,$$

so need next to see how to use thin elements to produce a thin homotopy extending this. On (x, x_2), we can build a 'local homotopy':

in which the diagonal is also (x_2^{-1}, x_2). The top-left 2-simplex is degenerate, being just $s_1(x_2^{-1}, x_2)$. The bottom right one is thin. It is the thin filler of the $(2, 1)$-horn given by the non-diagonal edges. (Recall (page 229) that the composite 'g_2 then g_0' is obtained by forming the filler,

$$s_1 g_2.s_1 s_0 d_0(g_2)^{-1}.s_0(g_0),$$

and then taking its d_1-face to get $g_2.s_0 d_0(g_2)^{-1}.g_0$.) The composite on the diagonal is thus the product $(x, x_2).(1, x_0 x_1^{-1}).(x_0 x_1^{-1}.x_1 x_0^{-1})$, calculated within $K(\mathsf{P})_1 = P \rtimes P$, where the right hand P acts by conjugation on the left hand one. It is **routine to check** that this is (x_2^{-1}, x_2).

As $K(\mathsf{P})$ has no non-thin n-simplexes for $n > 1$, this is automatically going to give a thin homotopy, but it *is* useful to see exactly its form and the manner in which it depends explicitly on λ. The fact that the simplicial group, $K(\mathsf{P})$, has no non-thin elements in higher dimensions also tells us that we do not have to do anything more! Our local thin homotopy extends without problem to a thin homotopy between λ and 1, and hence the P-colouring, λ, is equivalent to the constant colouring, 1, but, as it was arbitrary, we have that $Z_\mathsf{P}(X) \cong \mathbb{C}$, as expected.

The transformations corresponding to the cobordisms also need looking at. To do this, we examine a cobordism, $M : X \to Y$, as usual, and obtain the commutative diagram:

$$
\begin{array}{ccc}
Z_{\mathsf{P}}(X,T) & \xrightarrow{Z_{\mathsf{P}}(M,T,S)} & Z_{\mathsf{P}}(Y,S) \\
\rho_T^X \downarrow & & \downarrow \rho_S^Y \\
Z_{\mathsf{P}}(X) & \xrightarrow[Z_{\mathsf{P}}(M)]{} & Z_{\mathsf{P}}(Y)
\end{array}
$$

We know $Z_{\mathsf{P}}(X)$ and $Z_{\mathsf{P}}(Y)$ are copies of $\mathbb{C}$, so have to work out $Z_{\mathsf{P}}(M)[1]$ as a number. (We really only need to show it to be constant, i.e., independent of M, as that would suffice. Actually we can calculate it exactly, so do not need that safety net!)

As the information that we will need is a bit scattered through the previous section, we will recall each fact that we need and will give it in general (we will reuse some of this slightly later) as well as in the form for P. As usual $\mathsf{C} = (C, P, \partial)$ will be our general crossed module.

- If $\lambda \in \Lambda_{\mathsf{C}}(T)$ and, as before, $\mathcal{T}$ triangulates M, extending T on X and S on Y, then

$$
\begin{aligned}
Z_{\mathsf{C}}^!(M, \mathcal{T})(\lambda) &= \sum \{ \mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T}), \mu|_T = \lambda \} \\
&= \sum \{ \mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T})_\lambda \},
\end{aligned}
$$

  thereby introducing $\Lambda_{\mathsf{C}}(\mathcal{T})_\lambda$ as a shorthand for the set of μ extending λ on T. (This does not change for $\mathsf{C} = \mathsf{P}$.)

-
$$
Z_{\mathsf{C}}^!(M, T, S)(\lambda) = \sharp(P)^{-\chi_0^{int}(\mathcal{T})} \sharp(C)^{-\chi_1^{int}(\mathcal{T})} Z_{\mathsf{C}}^!(M, \mathcal{T})(\lambda),
$$

  which, in the case $\mathsf{C} = \mathsf{P}$, gives

$$
Z_{\mathsf{C}}^!(M, T, S)(\lambda) = \frac{\sharp(P)^{\sharp(T_1)} \sharp(P)^{\sharp(S_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}} \sum \{ \mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T})_\lambda \}
$$

  using the definition of $\chi_1^{int}(\mathcal{T})$.

-
$$
Z_{\mathsf{C}}(M, T, S)(\lambda) = \sharp(P)^{-\frac{1}{2}\chi_0^\partial(\mathcal{T})} \sharp(C)^{-\frac{1}{2}\chi_1^\partial(\mathcal{T})} Z_{\mathsf{C}}^!(M, T, S)(\lambda),
$$

  and, when $\mathsf{C} = \mathsf{P}$,

$$
Z_{\mathsf{C}}(M, T, S)(\lambda) = \frac{\sharp(P)^{\frac{1}{2}\sharp(T_1)} \sharp(P)^{\frac{1}{2}\sharp(S_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}} \sum \{ \mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T})_\lambda \}.
$$

-
$$
\rho_T^X(\lambda) = \sharp(P)^{-\frac{1}{2}\chi_0(T)} \sharp(C)^{-\frac{1}{2}\chi_1(T)} [\lambda],
$$

  so, for the case, $\mathsf{C} = \mathsf{P}$, in which $[\lambda] = [1]$,

$$
\rho_T^X(\lambda) = \sharp(P)^{-\frac{1}{2}\sharp(T_1)} [1]
$$

and similarly for $\rho_T^X(\mu|_S) = \sharp(P)^{-\frac{1}{2}\sharp(S_1)} [1]$.

We now can track λ, or, more exactly, the basis element labelled by λ, around the commutative square in the two possible ways.

- Down-then-right:
$$Z_{\mathsf{P}}(M)\rho_T^X(\lambda) = \sharp(P)^{-\frac{1}{2}\sharp(T_1)}Z_{\mathsf{P}}(M)[1].$$

- Right-then-down:

$$\begin{aligned}
\rho_S^Y Z_{\mathsf{P}}(M,T,S)(\lambda) &= \sharp(P)^{-\frac{1}{2}\sharp(S_1)}\frac{\sharp(P)^{\frac{1}{2}\sharp(T_1)}\sharp(P)^{\frac{1}{2}\sharp(S_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda)[1] \\
&= \frac{\sharp(P)^{\frac{1}{2}\sharp(T_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda)[1].
\end{aligned}$$

We therefore have that
$$Z_{\mathsf{P}}(M)[1] = \frac{\sharp(P)^{\sharp(T_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda)[1],$$

so we have to count the P-colourings of $\mathcal{T}$, which have value λ on T. We note that any assignment, $\mu : \mathcal{T}_1 \to P$, extends *uniquely* to a P-colouring, and conversely. We thus have $\sharp(\Lambda_{\mathsf{P}}(\mathcal{T}) = \sharp(P)^{\sharp(\mathcal{T}_1)}$, and so
$$\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda) = \frac{\sharp(P)^{\sharp(T_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}.$$

We thus have

Lemma 38
$$Z_{\mathsf{P}}(M) = 1.$$

■

and

Proposition 52 *For the crossed module, $\mathsf{P} = (P, P, =)$, $Z_{\mathsf{P}} \cong Z_1$, the trivial TQFT with constant value, $\mathbb{C}$.*   ■

7.3.5   Example 4: C, an inclusion crossed module

The next most obvious case to look at is that of inclusion crossed modules, i.e., when $\mathsf{C} = (C, P, \partial)$ has ∂ a monomorphism. We write $N = \partial C$, so N is just a normal subgroup of P, set $G = P/N$, and assume that ∂ is an actual inclusion, so $\mathsf{C} = (N, P, inc)$.

We can think of this as giving a morphism of crossed modules, $\mathsf{p} : \mathsf{C} \to \mathsf{G}$:

$$\begin{array}{ccc}
N & \xrightarrow{p_1} & 1 \\
\downarrow & & \downarrow \\
P & \xrightarrow{p_0} & G
\end{array}$$

that is, from C to what we will denote by G, or $K(G,0)$, the 'crossed module', $(1, G, inc)$. We saw (page 243) that $I_{\mathsf{C}}(M)$ and $I_G(M)$ are the same. We would expect there to be a 'very close' relationship between Z_{C} and Z_G as well, and, of course, the previous example showed this in the case where G is trivial.

Given any C-colouring of a triangulation, T, we clearly get a G-colouring by composing with p. The commutativity cocycle condition is easy to check. We thus get a function

$$\mathsf{p}_* : \Lambda_\mathsf{C}(T) \to \Lambda_G(T).$$

Is this a bijection? This is unlikely, except in very exceptional cases.

Is it a surjection? Can we reverse the process and build a C-colouring from a G-colouring?

We pick a transversal, $t : G \to P$, so $p_0 t(g) = g$ for all $g \in G$. We can assume that it is 'normalised', i.e., that $t(1_G) = 1_P$, but, of course, it need not be a homomorphism. Given any G-colouring, λ of T, we use t to try to build a C-colouring of T. If $\langle a, b \rangle \in T_1$, we try $\lambda' \langle a, b \rangle = t\lambda \langle a, b \rangle$ and, if $\langle a, b, c \rangle \in T_2$, we set

$$\lambda' \langle a, b, c \rangle = t\lambda \langle a, c \rangle .(t\lambda \langle b, c \rangle)^{-1}.(t\lambda \langle a, b \rangle)^{-1} \in N,$$

which automatically satisfies the boundary condition.

Lemma 39 *The assignment, λ', satisfies the cocycle condition.*

Proof: The proof is by direct calculation, so that is **left to you**. For a 3-simplex, $\langle a, b, c, d \rangle$, both composites in the cocycle 'square' come to $t\lambda \langle a, b \rangle .t\lambda \langle b, c \rangle .t\lambda \langle c, d \rangle .t\lambda \langle a, d \rangle^{-1}$. ∎

We thus have that λ' is a C-colouring and, as $\mathsf{p}_*(\lambda') = \lambda$, we have:

Proposition 53 *The function,*
$$\mathsf{p}_* : \Lambda_\mathsf{C}(T) \to \Lambda_G(T),$$
is a surjection. ∎

It is easy to see that each λ, in fact, gives rise to $\sharp(N)^{\sharp(T_1)}$ different C-colourings, since if we pick any function from T_1 to N, say, $\{n(a, b) \in N \mid \langle a, b \rangle \in T_1\}$, then we get a new C-colouring, $\{t\lambda(a, b)n(a, b) \mid \langle a, b \rangle \in T_1\}$. This effectively changes the chosen transversal, t, to a new one, so the result is a C-colouring. Conversely, if λ_1 and λ_2 are two C-colourings with $p\lambda_1 = p\lambda_2$, then $p(\lambda_1 \lambda_2^{-1})$ is trivial and setting $n(a, b) = \lambda_1(a, b).\lambda_2(a, b)^{-1}$ gives a function from T_1 to N, as before.

The idea of our attack will be to adapt techniques from our previous example almost, but not quite, as if attacking the $n(a, b)$s that link a given $\lambda \in \Lambda_\mathsf{C}(T)$ to one of the form $tp_*(\lambda)$. To see why this might work, we note that p induces an epimorphism

$$Z_\mathsf{C}(X, T) \to Z_G(X, T),$$

and also one,

$$Z_\mathsf{C}(X) \to Z_G(X),$$

compatibly with the projections to the colimits. (To see that the second of these is an epimorphism, it suffices to see how it is defined. Given an element of its domain, you pick a linear combination of colourings mapping to it, than map that across to a combination of G-colourings and finally back down to $Z_G(X)$. That will be well defined, as we will show.) As these maps p_* and t_*, are defined by mapping basis elements, it is simple to find a basis for $Ker\, \mathsf{p}_*$, namely all the $\lambda - tp(\lambda)$ that are not zero.

We have some observations which seem to be useful:

(i) if, for each λ, $\lambda \simeq_{thin} tp(\lambda)$, then all elements of $Ker\, p_*$ will vanish in the quotienting process. Put more simply and precisely: $Ker\, p_* \subseteq Ker\, \rho_T^X$.

(ii) If we prove that $Ker\, p_* \subseteq Ker\, \rho_T^X$, then the map, $Z_C(X) \to Z_G(X)$, will be well defined and epimorphic.

We therefore need to look closely at the situation for (i). We have already seen this in the case of trivial G, as that is just $\lambda \simeq_{thin} 1$ in our previous example. The obvious approach is, as suggested above, to try to adapt and extend the idea behind that proof to this more general context.

Suppose $\lambda \in \Lambda_C(X)$, so $\lambda : G(T) \to K(C)$. For each generating arrow $\langle a, b \rangle$ in $G(T)_0$, we get $\lambda(a, b) \in P$ and need a 1-simplex joining it to $tp\lambda(a, b)$.

We will be considering a 2-simplex, $\langle a, b, c \rangle$, of T shortly, so as $\langle a, b \rangle$ is that simplex's d_2-face, we will denote $p\lambda(a, b)$ by g_2, and later on use $g_1 = p\lambda(a, c)$ and $g_0 = p\lambda(b, c)$, as notation for the images, down in G, of the other faces.

There is a 1-simplex,
$$(t(g_2)\lambda(a, b)^{-1}, \lambda(a, b)) : \lambda(a, b) \to t(g_2),$$
and this looks good. This suggests that, for any $x \in P$, we use
$$(tp(x)x^{-1}, x) : x \to tp(x),$$
and this is almost what we want, however the homotopy is not just happening within $K(C)$, but also should involve the structure of $G(T)$, so, for instance, on the element $\lambda(a, c)\lambda(b, c)^{-1} \in P$ that we will use shortly, we will need to deform it along $(t(g_1)t(g_0)^{-1}\lambda(b, c)\lambda(a, c)^{-1}, \lambda(a, c)\lambda(b, c)^{-1})$ and not along the edge, $(t(g_1.g_0^{-1})\lambda(b, c)\lambda(a, c)^{-1}, \lambda(a, c)\lambda(b, c)^{-1})$. These will coincide if t is a splitting, but not necessarily otherwise. The first of these ends up where we want, not the second. We will return to this point shortly.

The 2-simplex, $\langle a, b, c \rangle$, of T gives rise to $\langle a, b, c \rangle \in G(T)_1$, (we will omit the overline that we have sometimes used here, as that notation gets burdensome in the formulae and diagrams below), and is mapped by λ to a 1-simplex of $K(C)$, that is, to an element of $N \rtimes P$. This, thus, has form $x, \lambda(a, b))$, and, as ∂ is a monomorphism, we can work out that $x = \lambda(a, c,)\lambda(b, c)^{-1}\lambda(a, b)^{-1} \in N$.

We have an embryonic 'local homotopy' as in the previous example:

$$
\begin{array}{ccc}
t(g_2) & \xrightarrow{\quad ? \quad} & t(g_1)t(g_0)^{-1} \\
{\scriptstyle (t(g_2)\lambda(a,b)^{-1}, \lambda(a,b))}\Big\uparrow & \nearrow{\scriptstyle ?} & \Big\uparrow{\scriptstyle (y, \lambda(a,c)\lambda(b,c)^{-1})} \\
\lambda(a, b) & \xrightarrow[{\scriptstyle (x, \lambda(a,b))}]{} & \lambda(a, c)\lambda(b, c)^{-1}
\end{array}
$$

where $y = t(g_1)t(g_0)^{-1}\lambda(b, c)\lambda(a, c)^{-1}$. As we hope to build the homotopy 'thinly', we use the unique thin filler for the $(2, 1)$-horn made up of the bottom and right-hand edges. (This is the composition 2-simplex in the (internal) nerve of the groupoid part of $\mathcal{X}(C)$, as before, and so we have, but will not need, explicit formulae.) This gives a thin filler and a resulting diagonal arrow given by the product (within $N \rtimes P$)

$$(x, \lambda(a, b)).(1, \lambda(b, c)\lambda(a, c)^{-1}).(y, \lambda(a, c)\lambda(b, c)^{-1}),$$

and **we can check** that this gives $(yx, \lambda(a.b))$. calculating yx then gives $(t(g_1)t(g_0)^{-1}\lambda(a, b)^{-1}$ as we might have guessed and hoped.

Across the top of the square, the natural candidate for the 1-simplex is $t(g_1)t(g_0)^{-1}t(g_2)^{-1}$, which is, incidentally, the factor set of the extension, $N \to P \to G$, evaluated on (g_2, g_0).

We now need to find a 2-simplex fitting into the top left of the square. We again have a (2,1)-horn, this time made up of the left edge and the top. We fill this thinly (to see if that will work!) and then **evaluate its d_1-face**. This gives the same value as our previous label on the diagonal, so we have our thin homotopy on this edge.

This does not handle all the cases we will need, but is a good start. We will see other cases later.

This looks, good, but we should glance at the cocycle condition for λ, that is, the relationship between the faces of the image of a tetrahedron, $\langle a, b, c, d \rangle$, considered as a generator in $G(T)_2$. The resulting 2-simplex in $K(\mathsf{C})$ is $\lambda(a, b, c, d)$, which looks like :

$$
\begin{array}{ccc}
 & \lambda(a,c)\lambda(b,c)^{-1} & \\
{\scriptstyle (x,\lambda(a,b))}\nearrow & & \searrow{\scriptstyle (y,\lambda(a,c)\lambda(b,c)^{-1})} \\
\lambda(a,b) & \xrightarrow[{\scriptstyle (z,\lambda(a,b))}]{} & \lambda(a,d)\lambda(b,d)^{-1}
\end{array}
$$

where $x = \lambda(a,c)\lambda(b,c)^{-1}\lambda(a,b)^{-1}$, $y = \lambda(a,d)\lambda(b,d)^{-1}\lambda(b,c)\lambda(a,c)^{-1}$ and $z = \lambda(a,d)\lambda(b,d)^{-1}\lambda(a,b)$.

In some ways, each side is exactly what we need to get from the source to the target, but you would be justified in wanting more indication of how these are obtained. This will also indicate why, just now, we used $(t(g_1)t(g_0)^{-1}\lambda(b,c)\lambda(a,c)^{-1}, \lambda(a,c)\lambda(b,c)^{-1})$. The point is that we need homomorphisms wherever possible. Here λ is a *morphism* of $\mathcal{S}$-groupoids. Earlier we need the homotopy to consist of morphisms. To see the effect this has on the calculation look at the edge from $\lambda(a,c)\lambda(b,c)^{-1}$ to $\lambda(a,d)\lambda(b,d)^{-1}$. We have a 1-simplex

$$(\lambda(a,d)\lambda(c,d)^{-1}\lambda(a,c)^{-1}, \lambda(a,c)) : \lambda(a,c) \to \lambda(a,d)\lambda(c,d)^{-1},$$

and another

$$(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c)) : \lambda(b,c) \to \lambda(b,d)\lambda(c,d)^{-1}.$$

Taking the group theoretic inverse of the second (not the groupoid one, and remember we have both in the group groupoid $\mathcal{X}(\mathsf{C})$), we get

$$(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c))^{-1} : \lambda(b,c)^{-1} \to \lambda(c,d)\lambda(b,d)^{-1},$$

and hence multiplying the two expressions term by term:

$$(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c)).(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c))^{-1}$$

from $\lambda(a,c)\lambda(b,c)^{-1}$ to $\lambda(a,d)\lambda(c,d)^{-1}.\lambda(c,d)\lambda(b,d)^{-1} = \lambda(a,d).\lambda(b,d)^{-1}$. Using the multiplication in $K(\mathsf{C})_1$, which is that of the semi-direct product $N \rtimes P$, we can easily **check** that this is $(\lambda(a,d)\lambda(b,d)^{-1}\lambda(b,c)\lambda(a,c)^{-1}, \lambda(a,c)\lambda(b,c)^{-1})$ as on the above figure.

Within the internal category structure of $\mathcal{X}(\mathsf{C})$, this diagram commutes. In terms of thin fillers, we can take the $(2,1)$-horn, form the thin 'composition' filler as we have done several time before, and then take its d_1-face, and, yes, this *is* $(z, \lambda(a,b))$ as **you can easily check**. In fact, as $K(\mathsf{C})$ is a T-complex, it just has thin elements in dimension 2, so this conclusion was 'obvious' for other reasons.

There is thus nothing to stop the construction of a thin homotopy between λ and $tp(\lambda)$, first locally and then extended up the skeletons. (There are some things you may want to check, so as

to convince yourself that there are no problems. For instance, if we take the above 2-simplex as base and build a thin homotopy over it, how do we know it has $tp(\lambda(a, b, c, d))$ at the top? It is a good idea to think about this sort of question, and from several different angles, as the answers use various features of the T-complex structure of $K(\mathsf{C})$ in a beautiful way.)

To sum up, we have proved:

Lemma 40 *For any $\lambda \in \Lambda_\mathsf{C}$, $\lambda \simeq_{thin} tp(\lambda)$.*   ∎

and thus have

Corollary 14

$$Ker\, \mathsf{p}_* \subseteq Ker\, \rho_T^X$$

∎

It now only needs a simple bit of diagram chasing to prove:

Proposition 54 *If $\mathsf{C} = (N, P, inc)$ is an inclusion crossed module and $G = P/N$, then the projection, p, induces a natural isomorphism*

$$p_* : Z_\mathsf{C}(X) \to Z_G(X)$$

∎

We thus have what we suspected, at least on objects. What about on the cobordisms?

We can use the general formulae that we recalled in the previous example, and the experience gained there will be useful, but we need to 'keep alert' as well!

We have the linear transformation,

$$Z_\mathsf{C}^!(M, T, S) : Z_\mathsf{C}(X, T) \to Z_\mathsf{C}(Y, S),$$

and also the geometrically significant bases given by the colourings. We therefore have a matrix representing $Z_\mathsf{C}^!(M, T, S)$ and will calculate the matrix entries. We thus pick $\lambda_T \in \Lambda_\mathsf{C}(X, T)$ and $\lambda_S \in \Lambda_\mathsf{C}(Y, S)$ and work out the corresponding entry. Referring back to the previous example, we get

$$Z_\mathsf{C}^!(M, T, S)_{\lambda_T, \lambda_S} = \frac{\sharp(G)^{\sharp(T_0)}\sharp(G)^{\sharp(S_0)}}{\sharp(G)^{\sharp(\mathcal{T}_0)}} \cdot \frac{\sharp(N)^{\sharp(T_1)}\sharp(N)^{\sharp(S_1)}}{\sharp(N)^{\sharp(\mathcal{T}_1)}} \cdot \sharp(\Lambda_\mathsf{C}(\mathcal{T})_{\lambda_T, \lambda_S}).$$

(Yes, we have used that $G = P/N$, so $\sharp(G) = \sharp(P)/\sharp(N)$, which simplifies things a lot!) We have set

$$\Lambda_\mathsf{C}(\mathcal{T})_{\lambda_T, \lambda_S} = \{\mu \in \Lambda_\mathsf{C}(\mathcal{T}) \mid \mu|_T = \lambda_T, \mu|_S = \lambda_S\}.$$

Our hope is to compare the result with that for Z_G, initially in the triangulated version, then, passing to the quotient, to compare $Z_\mathsf{C}(M)$ and $Z_G(M)$.

We have for $p(\lambda_T)$ and $p(\lambda_S)$,

$$Z_G(M, T, S)_{p(\lambda_T), p(\lambda_S)} = \frac{\sharp(G)^{\sharp(T_0)}\sharp(G)^{\sharp(S_0)}}{\sharp(G)^{\sharp(\mathcal{T}_0)}} \cdot \sharp(\Lambda_G(\mathcal{T})_{p(\lambda_T), p(\lambda_S)}).$$

This looks very hopeful as the first term is the same. Of course, life is not quite as simple as it would seem, as the quotient maps are not without a certain amount of complication and the above

refers to $Z_{\mathsf{C}}^!$ and $Z_G^!$, not to the final versions Z_{C} and Z_G. This being said, it still seems worth calculating $\sharp(\Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S})$ as explicitly as possible, and, for that there is a surjection,

$$\mathsf{p}_* : \Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S} \to \Lambda_G(\mathcal{T})_{p(\lambda_T),p(\lambda_S)},$$

induced by composition with $\mathsf{p} : \mathsf{C} \to \mathsf{G}$. At the start of the discussion of this example, we pointed out that, if two C-colourings have the same image after composition with p, then they differed by an element of $N^{\mathcal{T}_1}$. If the two colourings had the same values on the ends, then the corresponding element of $N^{\mathcal{T}_1}$ will be constant on T_1 and S_1, having value 1. The surjection, p_*, thus has all its fibres having the same size, namely

$$\frac{\sharp(N)^{\sharp(\mathcal{T}_1)}}{\sharp(N)^{\sharp(T_1)}\sharp(N)^{\sharp(S_1)}},$$

which is good! This means

$$\sharp(\Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S}) = \frac{\sharp(N)^{\sharp(\mathcal{T}_1)}}{\sharp(N)^{\sharp(T_1)}\sharp(N)^{\sharp(S_1)}} . \sharp(\Lambda_G(\mathcal{T})_{p(\lambda_T),p(\lambda_S)}).$$

This almost does the trick. It 'almost' proves that $Z_{\mathsf{C}}(M)$ and $Z_G(M)$ are 'the same', that is, after identification of $Z_{\mathsf{C}}(X)$ with $Z_G(X)$ and of $Z_{\mathsf{C}}(Y)$ with $Z_G(Y)$. 'Almost', but not quite... . The actual map from $Z_{\mathsf{C}}(X)$ to $Z_{\mathsf{C}}(Y)$ is defined using $Z_{\mathsf{C}}(M,T,S)$ and the quotients ρ_T^X and ρ_S^Y. The above uses $Z_{\mathsf{C}}^!(M,T,S)$ and r_T^X, r_S^Y, however if you **check** the scaling factors involved it becomes clear that they in fact cancel out. In other words, we have:

Proposition 55 *If* $\mathsf{C} = (N,P,inc)$ *and* $G = P/N$, *then* $Z_C \cong Z_G$. ∎

(You **should check** the last point in detail as it needs a certain amount of care.)

<div align="center">(To be continued)</div>

7.4   How can one construct TQFTs (continued)?

From these examples we can see what to expect and how to proceed with a general construction.

7.4.1   TQFTs from a finite simplicial group

It is natural to try to extend the methods of the above sections to a more general setting in which C is replaced by a finite simplicial group or 'finite crossed n-cube' or similar. How general would this be? Would it be useful?

Definition: A simplicial group, G, is said to be *finite* if each G_m is a finite group *and* there is some n such that NG_k is trivial for all $k > n$.

Clearly, for such a simplicial group, its homotopy groups are all trivial above some level. Also clearly, any finite simplicial group models an n-type for some n, since everything is generated by its group theoretic n-skeleton, by Conduché's decomposition result, Proposition 33. Any finite simplicial group will have NG of finite length and consisting of finite groups, so all the homotopy groups of G will be finite. Ellis, [74], proved a converse:

Theorem 22 *(Ellis, [74]) Suppose that $\pi_k(K)$ is trivial for all $k \geq c+1$, and that each of the homotopy groups, $\pi_k(K)$, is finite for $k \leq c$, then the homotopy type of K is faithfully represented by a simplicial group whose Moore complex is of length at most $c-1$ and whose group of n-simplices is finite for each $n \geq 0$.* ∎

We thus have that these finite simplicial groups are quite abundant!

The discussion in the previous section was given in such a way that the majority of the results and proofs made little or no use of the fact that $K(\mathsf{C})$ was other than just a finite simplicial group. Well, that is not quite true, as the compensating and scaling factors would presumably have needed more terms in general - just look at the extra terms involving $\sharp(C)$ in the case of C-colourings rather than the simpler G-colourings for a finite group G. That, however, does suggest what to do for a generalisation to G being a finite simplicial group. We would need more terms involving $\sharp(NG_n)$ for k more than just 0 or 1. The full details can be found in Porter, [138], but are not hard to derive, so generally will be **left to you**.

We thus assume that G is a finite simplicial group and construct a $Z_G : (d+1) - Cob_{PL} \to Vect$, that is, a TQFT of dimension $(d+1)$. The set-up is as before, with $(X, \mathbf{T})$, a $d-$manifold with ordered triangulation.

Definition: A *G-colouring* of $\mathbf{T}$ with values in a (finite) simplicial group, G, is a morphism,

$$\lambda : G(T) \to G,$$

of simplicial groupoids, or, equivalently, a simplicial map

$$\lambda : T \to \overline{W}G.$$

We write $\Lambda_G(\mathbf{T})$ for the set of such G-colourings and $Z_G(X, \mathbf{T})$ for the vector space with basis labelled by $\Lambda_G(\mathbf{T})$, as before. We go through the same process as previously:

(i) If $\mathbf{T}'$ is a subdivision of $\mathbf{T}$, composition with the map $r_{\mathbf{T}}^{\mathbf{T}'}$, coming from the strong deformation retraction data, induces a function,

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T}),$$

as above, and hence extends to a linear map from $Z_G(X, \mathbf{T}')$ to $Z_G(X, \mathbf{T})$.

(ii) If $(M, \mathcal{T})$ is a triangulated cobordism from $(X, \mathbf{T})$ to $(Y, \mathcal{S})$, then define a linear map, as before, by: for $\lambda \in \Lambda_G(\mathbf{T})$,

$$Z_G^!(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_G(\mathcal{T}) \\ \mu|\mathbf{T} = \lambda}} \mu|\mathbf{S}.$$

We set $g_i = \sharp NG_i$, the size of the ith Moore complex term.

Let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by edge stellar subdivision of an interior edge, e.

Lemma 41 *For any colouring μ of $\mathcal{T}$ fixed to be λ on $\mathcal{T}$ and λ' on $\mathbf{S}$, there are exactly $g_0 g_1^{s_2(e)} g_2^{s_3(e)} \ldots g_d^{s_d(e)}$ colourings of $\mathcal{T}'$ restricting to μ, where $s_k(e)$ is the number of k-simplices of $\mathcal{T}$ incident to e.* ∎

The proof is just a question of counting possible fillers.

(iii) Let $Z_G^!(M, \mathbf{T}, \mathbf{S}) = \prod_k g_k^{-\chi^{int}(\mathcal{T})} Z_G^!(M, \mathcal{T})$.

Proposition 56 *The linear map,*

$$Z_G^!(M, \mathbf{T}, \mathbf{S}) : Z_G(X, \mathbf{T}) \to Z_G(Y, \mathbf{S}),$$

is independent of the triangulation, $\mathcal{T}$, extending $\mathbf{T}$ and $\mathbf{S}$ to the cobordism. ∎

The proof follows the same lines as earlier results with some obvious replacements for lemmas valid in those cases for which generalisations are needed (as that above).

(iv) Now assume that we have $(Z, \mathbf{R})$ as another triangulated manifold and cobordisms M and N, as earlier. With the previous notation, we have:

Lemma 42

$$Z_G^!(N, \mathbf{S}, \mathbf{R}).Z_G^!(M, \mathbf{T}, \mathbf{S}) = \prod g_k^{\chi_k(S)} Z_G^!(M +_Y N, \mathbf{T}, \mathbf{R}).$$

∎

This gives that the linear maps

$$Z_G(M, \mathbf{T}, \mathbf{S}) = \prod g_k^{-\frac{1}{2}\chi_k^{\partial}(\mathcal{T})} Z_G(M, \mathbf{T}, \mathbf{S})$$

satisfy

Corollary 15

$$Z_G(N, \mathbf{S}, \mathbf{R}).Z_G(M, \mathbf{T}, \mathbf{S}) = Z_G(M +_Y N, \mathbf{T}, \mathbf{R}).$$

∎

(v) Following our now customary route, we look at rescaling the restriction maps. If $\mathbf{T}'$ is a subdivision of $\mathbf{T}$, both being triangulations of X, let

$$\mathrm{res}_{\mathbf{T}', \mathbf{T}} = \prod g_k^{\frac{1}{2}(\chi_k(T') - \chi_k(T))} res_{\mathbf{T}', \mathbf{T}}.$$

These adjusted maps are compatible with the cobordisms.

(vi) Finally let

$$Z_G(X) = colim_{\mathbf{T}} Z_G(X, \mathbf{T}),$$

using the adjusted restriction maps, and as expected, we get:

Theorem 23 *(Porter, [138]) For any dimension d, the construction above applied to $(d + 1) - Cob_{PL}$, gives a $(d + 1)$-dimensional TQFT.* ∎

The proof has the same form as for the low dimensional cases, using the above adjustments.

Remark: In our earlier discussions, we left the question of compatibility with the 'identities', i.e., the cobordisms $X \times [0, 1]$, for you to investigate. In case you need a hint, here is an idea to follow up on. We said cobordisms were considered 'up to homeomorphism', but if $M : X \to Y$ is a cobordism, then

$$(X \times I) +_X M \cong M$$

and, if $\mathbf{T}$ is a triangulation of X, the usual product triangulation of $X \times I$ glues to $(M, \mathcal{T}) : (X, \mathbf{T}) \to (Y, \mathbf{S})$ to give a triangulation of $(X \times I) +_X M \cong M$, so gives a new triangulation of M, but, from (iv), the induced map, $Z_G(M, \mathbf{T}, \mathbf{S})$, is independent of the triangulation. It is now easy to check that $Z_G(X \times I, \mathbf{T}, \mathbf{T})$ must be the identity. Alternatively, look at its construction in detail and do some 'sums'!

(To be continued)

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Chapter 8

Relative TQFTs: some motivation and some distractions

Before we introduce Turaev's Homotopy Quantum Field Theories, we will look at the motivation for wanting such things, and will examine two 'Case Studies' with that aim

8.1 Beyond TQFTs

One disadvantage of standard TQFTs is that the basic categories of form $d - Cob$ consist of orientable manifolds and cobordisms *without any extra structure beyond being 'Top', 'PL' or 'Diff'*. In many geometric situations, there is often a lot more structure around, for instance, if the basic situation is that of smooth manifolds and cobordisms, then each object, X, naturally has a tangent bundle, TX, and we mentioned, when we looked at bundles earlier (page **??**), this will have as basic structure group, $G\ell(d - 1, \mathbb{R})$, assuming, of course, that we are working with *real $d - 1$-dimensional* manifolds. As the manifolds are orientable, their tangent bundles will be orientable, i.e., the transition functions can be assumed to lie in $G\ell^+(d - 1, \mathbb{R})$, the subgroup of $G\ell(d - 1, \mathbb{R})$ consisting of the invertible matrices of positive determinant. If, now, we ask for extra geometric structure such as a Riemannian metric on our manifolds, then the group in which the transition functions live can be chosen to be the orthogonal group, $O(d - 1)$. If our manifolds are foliated in some way then the structure groups will correspond to some block decomposition of $G\ell^+(d-1, \mathbb{R})$, and so on. In general, this leads to the theory of *G-structures*.

8.1.1 Manifolds 'with extra structure'

Recall, from page **??**, that a vector bundle, V, of rank n on a space, B, is locally isomorphic to $\mathbb{R}_U^n := \mathbb{R}^n \times U$ for some open set U of B. The group of automorphisms of $\mathbb{R}_B^n$ is, of course, the trivial bundle of groups, $G\ell(n, \mathbb{R})_B := G\ell(n, \mathbb{R}) \times B$. The left $G\ell(n, \mathbb{R})_B$-torsor on B associated to V is $Isom(V, \mathbb{R}_B^n)$ and this is just the *frame bundle*, P_V, of V. Of particular note in our context of a $d - 1$-manifold is the *tangent frame bundle*. This is the associated $G\ell(d - 1, \mathbb{R})$-torsor of TX for X, one of our manifolds. Now assume given a potential structure group G, so we will assume it comes with a homomorphism $G \to G\ell(d-1, \mathbb{R})$, which may or may not be an inclusion / monomorphism.

> **Definition:** A *weak G-structure* on X is a principal sub-G-bundle of the tangent frame bundle.

Note reducibility of the structure group to G is usually coupled with an integrability condition when considering geometrically significant structures. The result is then a *G-structure.* This integrability condition is sometimes called the *solder form* of the G-structure. (We will not go into this in any more detail as this is only intended to motivate the constructions we will be considering. If you need this, check the literature, for instance, Ehresmann's original work, [71], or look at Kobayashi and Nomizu, [106].) We will be sloppy in our terminology and refer to weak G-structures as being just 'G-structures'.

The notion of G-structure is thus a general notion of geometric structure. It does not handle all geometric structures, but is a good start. (We could ask if it 'categorifies' nicely, and to some extent we will see some categorifications of it in the coming pages.) For the moment, we just need to accept that it is a step towards looking at manifolds with extra structure. That leaves the question of what to do about the cobordisms. This is a bit delicate as the structure on an $M : X \to Y$ between manifolds with G-structures will need to reflect the structure on its two ends. The general situation is too complex for us to handle here, but we can give an abstract version of this sort of set-up, which encompasses a fair number of the suggested cases, and, in fact, also includes a lot of more categorified geometric contexts, and yet is very easy to describe given the sort of discussion we have above.

Let us gather up things a bit. Almost all of this type of structure involves a group, G, and a reduction of the structural maps of some natural bundle on X (possibly analogous to, or linked to, TX), together with a principal G-bundle / G-torsor associated with that natural bundle.

We know that the G-torsor corresponds to a characteristic map from X to BG. We can extend this is in what should seem an obvious way. The torsor is 'really' given by transitions on an open cover and, given the link between open covers and triangulations, our characteristic map could be equally well specified by a simplicial map $T \to Ner\,G[1] = BG$. This assumes that G is a discrete group, but we will see how that restriction can be got around in a moment. (Here we are really using the simplicial approximation theorem and we are not going to give details - so **you should check them.** There *is* a slight 'fudge' here, but it is not that crucial as with more work we could get around it, but we would have to divert from our central themes.) We can extend this idea to one where G is a simplicial group and BG is $\overline{W}G$ and that is well known territory for us.

We thus might start with basic objects being $(d-1)$-manifolds with a structure map, $g : T \to \overline{W}(G)$, given locally on some open covering or triangulation of X. For the cobordisms, some of the motivating examples gives a slight problem. For instance, if G is $O_{d-1}(\mathbb{R})$, or rather a simplicial model for it such as its singular complex, then do we take cobordisms with the same G or with $O_d(\mathbb{R})$ as group, in which case what does the boundary condition of the cobordism look like? We take the 'cowards way out and keep the same G for all manifolds and cobordisms. The complications of the other situation look 'interesting', but without a good knowledge of the simpler case, they look 'too interesting' for the moment. In any case the cobordism should correspond intuitively to the way in which X *evolves* into Y, so perhaps having intermediate states which are G-structured manifolds is what is best.

Note: If here, as in previous sections, we are looking at simplicial groups, however we now make no restriction of finiteness on them as this would be not appropriate in this context.

8.1.2 Interpretation of simplicial groups from a geometric perspective

To concentrate attention on the interpretation of some of the simplicial groups that might be of interest, suppose G is a topological group of (linear) automorphisms of $\mathbb{R}^n$, so is essentially a subgroup of $G\ell_n(\mathbb{R})$. Typically, G might be asked to preserve some structure such as a specified quadratic form on $\mathbb{R}^n$. (We will look at this again in more detail in section **??**.)

   If we 'take apart' $Sing(G)$ in such a case, (cf. section 4.6), then each $\sigma \in Sing(G)_m$ is a continuous map,

$$\sigma : \Delta^m \to G,$$

but then σ is a continuously varying family of $n \times n$ invertible matrices over $\mathbb{R}$. (As a simple example, look at when $m = 1$, so that σ is just a path in G, corresponding to a t-indexed family, $\sigma(t)$ or σ_t, of invertible matrices for $0 \leq t \leq 1$, or equivalently, but perhaps more vividly, an $n \times n$ matrix of paths in $\mathbb{R}$ such that, for each t, $\sigma(t)$ is invertible. Such paths are often met in studies of the interaction of topological and algebraic properties of matrix groups, (cf. [57]). (If G is a Lie group, the singular simplex, σ, would usually be restricted to being smooth, or, at least, piecewise smooth, although the exact sense of smoothness at the endpoints 0 and 1 may vary in different contexts.)

   For each $\underline{t} \in \Delta^m$, we have an automorphism $\sigma_{\underline{t}}$ of $\mathbb{R}^n$. Each automorphism is assumed to be continuous. (Our assumption earlier was that they were linear, and continuity is certainly true in that context. 'Linearity' is mainly for expositional reasons.) As a result, we can think of G as being a subset of $Top(\mathbb{R}^n, \mathbb{R}^n)$, and, usually, as a subspace, depending on what topology is given to that space of continuous maps. We thus have that σ can be recast as a map

$$\sigma : \Delta^m \times \mathbb{R}^n \to \mathbb{R}^n,$$

(which is, of course, reminiscent of the definition of the $\mathcal{S}$-enrichment of Top; see page **??**.) The 'Δ^m-indexed family of automorphisms' viewpoint then enters here, as we can build a map

using $\tilde{\sigma}(\underline{t}, \underline{x}) = (\underline{t}, \sigma(\underline{t}, \underline{x}))$, so $\tilde{\sigma}$ will be a homeomorphism (over Δ^m). We have already visited this construction, briefly, on page 145 and a related idea on page 133. It would be a **good idea** to look back at those discussions now, although we will 'revise' that material below. We thus met this sort of construction first when discussing simplicial automorphisms in section 4.3. We supposed that Y was a simplicial set, and considered $\underline{\mathcal{S}}(Y, Y)$. In dimension m, this has simplicial maps

$$\sigma : \Delta[m] \times Y \to Y,$$

and, for composition of two such, σ and τ,

$$\tau \cdot \sigma := (\Delta[m] \times Y \overset{diag \times Y}{\longrightarrow} \Delta[m] \times \Delta[m] \times Y \overset{\Delta[m] \times \sigma}{\longrightarrow} \Delta[m] \times Y \overset{\tau}{\to} Y).$$

(This is $\tau \cdot \sigma$ or $\sigma \cdot \tau$ depending on your choice of composition convention.) The identity mapping from Y to Y, of course, lives in dimension zero, as the projection, $\Delta[0] \times Y \to Y$, but, of course,

has unique degenerate copies of itself in all dimensions and as maps from $\Delta[m] \times Y$ to Y, this degenerate copy of the identity is just the projection onto Y.

This composition makes life slightly awkward for deciding what it means for σ to be an automorphism, and, thus, to describe the simplicial group, $\mathsf{aut}(Y)$. We therefore rethink things a bit. We do not change the composition, that would be silly, but we do change our perspective on it by using the trick that we used above. (This is, in some ways, a *reprise* of earlier discussions, but given its importance, it does seem useful to review it here.) We will place ourselves in a more general context, as that will lead to greater simplicity and, hopefully, clarity.

We will assume that X, Y, and Z are simplicial sets and

$$f : \Delta[m] \times X \to Y,$$

and

$$g : \Delta[m] \times Y \to Z,$$

are m-simplices in $\underline{\mathcal{S}}(X, Y)$ and $\underline{\mathcal{S}}(Y, Z)$, respectively. We replace them by

$$\tilde{f} : \Delta[m] \times X \to \Delta[m] \times Y,$$

and

$$\tilde{g} : \Delta[m] \times Y \to \Delta[m] \times Z,$$

where $\tilde{f} = (p_1, f)$, etc., with p_1 being the first projection of the product, $p_1 : \Delta[m] \times X \to \Delta[m]$, (and we will not bother with any indicator of X, using p_1 indiscriminately for the first projection of the product regardless of the other factor). It is now easy to check

Lemma 43

$$\tilde{g} \circ \tilde{f} = (p_1, g \cdot f),$$

where $g \cdot f$ is the composite of g and f in the usual $\mathcal{S}$-category structure on $\mathcal{S}$. ∎

Remark: This is probably simplest to see using a (simplicial) set theoretic representation of $\tilde{f}$ as being given by a formula, $\tilde{f}_n(t, x) = (t, f_n(t, x))$ and so on, but it can also be seen using a diagrammatic argument and is valid in settings other than that of simplicial sets, in which elements are problematic. One such is that of simplicial maps in a (Grothendieck) topos.

Note that any simplicial map, $\tilde{f} : \Delta[m] \times X \to \Delta[m] \times Y$, over $\Delta[m]$, (so $p_1 \tilde{f} = p_1$), corresponds to a simplicial map, $f : \Delta[m] \times X \to Y$, in this way. One just sets $f = p_2(\tilde{f})$.

It is clear that this second description of the simplices in $\underline{\mathcal{S}}(X, Y)$, gives an easy solution to what it means for a $\sigma \in \underline{\mathcal{S}}(Y, Y)$ to be an automorphism.

Geometrically, this second perspective on the maps in $\underline{\mathcal{S}}(X, Y)$ is very bundle theoretic (as we saw when we first met it back in section 4.3). The object, $\Delta[m] \times X$, is an embryonic (trivial) simplicial bundle, a 'bundle patch', and if we have a 'base' simplicial set, B, we can form simplicial fibre bundles over B by gluing such patches together by automorphisms defined on faces, that is, on the overlaps between neighbouring patches. That is the whole point of the twisted Cartesian product construction and is thus at the heart of the $\overline{W}$ construction from this geometric point of view.

We thus have close links between geometric structure of a certain type and simplicial groups. In fact, the simplicial theory is the discrete analogue of the smooth theory of G-structures, (but without the integrality conditions), and as discrete analogues of physical theory are quite sought after and are difficult to do, simplicial techniques are one of several areas that seem well equipped to be exploited in developing such a 'discrete differential geometry'. Other ares include forms of n-category theory, and, as we have seen, that is very close to this one.

We can summarise the above as saying that one possible version of 'manifold with structure' would be a manifold, X, together with a structural map, $g : X \to B$, where B is a 'classifying space' for some sort of geometric structure on X. As we will be needing triangulations of X, we can be a bit more concrete and model this by a $g : T \to \overline{W}G$ for G, here, a simplicial group. This we can work with, and is so near to the G-colouring technology that we have been using that it is clearly worth exploring more thoroughly.

8.1.3 Case Study 1: Spin structures

To return to more specific examples, a metric on an n-dimensional vector bundle, $p : E \to X$, on X, is a bundle map $g : E \times_X E \to X \times \mathbb{R}$, such that the restriction of g to each fibre is a non-degenerate bilinear map, thus making each fibre, E_x, into an inner-product space. In this context, we will look in a bit more detail at the reduction of the structure group to the special linear group. We assume E is oriented, so already the transition functions of E could be given as having values in the subgroup $G\ell^+(n, \mathbb{R})$ of $G\ell(n, \mathbb{R})$. The oriented *orthonormal* frames in such an E form a principal $SO(n)$-bundle, $P_{SO}(E)$, since here the transition functions must preserve the orthonormality so must have determinant 1. An $SO(n)$-structure on X, an n-dimensional manifold, can thus be specified by an orientation together with a metric.

A related structure is a $Spin(n)$-structure. This has significant applications and interpretations in theoretical physics. The spin group, $Spin(n)$, is the double cover of the special orthogonal group, $SO(n)$. There is a short exact sequence:

$$1 \to C_2 \to Spin(n) \xrightarrow{\rho} SO(n) \to 1,$$

where C_2 is the cyclic group of order 2. (If $n > 2$, $Spin(n)$ is, in fact, simply connected and so coincides with the universal cover of $SO(n)$. The usual construction of the universal cover then gives $Spin(n)$ in terms of a quotient of the space of based paths in $SO(n)$. This idea is worth retaining for higher homotopy dimension analogues, later on.)

As before, E will be an oriented n-dimensional vector bundle on X, often, but not necessarily, a tangent bundle.

Definition: A *spin structure on E* is a lift of $P_{SO}(E)$ to a principal $Spin(n)$-bundle, $P_{Spin}(E)$.

This means that there is a ρ-equivariant bundle map and over X, that is, a map

$$\phi : P_{Spin}(E) \to P_{SO}(E)$$

of bundles over X such that, for $p \in P_{Spin}(E)$ and $\gamma \in Spin(n)$, $\phi(\gamma.p) = \rho(\gamma)\phi(p)$.

Definition (continued): If $E = TX$, the tangent bundle on X, then a spin structure on E is called a *spin structure on X* and X is said to be a *spin manifold*

We will not be using, nor proving, the following result, but it is an indication of some close links that we will not follow up on.

Proposition 57 *If T is a triangulation of a manifold X, a spin structure on X can be specified by a homotopy class of trivialisations of $TX|_{sk_1 T}$, that is, TX restricted to the 1-skeleton of the triangulation, that extends over the 2-skeleton.* ∎

As we said, we will not explore this point further here.

You should be thinking:

$$
\begin{array}{ccc}
 & & BSpin(n) \\
 & {}^{?}\nearrow & \downarrow {}^{B\rho} \\
X & \longrightarrow & BSO(n).
\end{array}
$$

You have a map classifying one type of structure and want to ask if that structure lifts to another 'finer' type of structure, and if so, to classify those 'finer' structures using some classification othe lifts.

You may have noticed the similarity between the ideas here and those discussed back in section **??** and, in particular, page **??**, where a somewhat similar problem was examined using the language of *bundle gerbes*. There is a connection, but also a slight, but significant, difference. In handling $Spin(n)$, we are using matrices over the real numbers and hence are considering *real* vector bundles, and the kernel of the extension is C_2, in the bundle gerbe case the kernel is $U(1)$, that is, the circle. The connection can be made stronger, but this needs the intermediate situation of $spin^c$-structures, for which look initially in Wikipedia (under spin-structure), and then at the work of Murray and his coauthors, see the bibliography, and in particular, [125]. Another link that is worth following up is that with Stiefel-Whitney classes, as the vanishing of these in low dimensions is related to the existence (or otherwise) of the lifts to $Spin(n)$, and hence to that of spin structures. You should note that, in both cases, that of the classical theory of characteristic classes and that using bundle-gerbe ideas, the exact sequence of the above extension plays a central role.

What *we* need to take from this discussion is that in this type of context, there will be a fibration of simplicial groups

$$p : H \to G$$

hopefully, as here, with finite fibre. (Remember that, for simplicial groups, fibrations are just the same as epimorphisms, so the fibre is just the kernel of p.) To end this initial discussion on 'motivation' we can sketch what a 'relative TQFT' might look like and how Yetter's construction might be adapted to give instances of this. We postpone a detailed look, until we discuss homotopy quantum field theories in later sections, but it is good to have the idea and motivation in front of us when introducing that new notion.

We could take a $p : H \to G$, as above, which would have finite fibre / kernel, K, (at least for the moment). The objects of study would be manifolds with structure maps $g_X : X \to BG$ and we would need triangulated simplicial versions of these, $g_T : T \to \overline{W}G$. Between these, we would have G-cobordisms, so, if $M : X \to Y$, with g_X and g_Y being the structure maps of X and Y, we would want $g_M : M \to BG$, restricting to them on the input and output ends.

A *p*-colouring of (T, g_T) would then be a simplicial map $\gamma_T : T \to \overline{W}H$, so that $\overline{W}p.\gamma_T = g_T$, in other words, a lift of the '*G*-structure' to an '*H*-structure'. The finite kernel assumption will mean that there are only finitely many such lifts, so we can form $\Lambda_p(T, g_T)$ and $Z_p(T, g_T)$. The route to check that *G*-cobordisms work then looks fairly clear (if perhaps slightly tortuous and long).

Will this give a TQFT? Well, that is the wrong question, as a TQFT is defined on $d-Cob$, and we have here some $d-Cob_G$, as the source. The better question is: what sort of structure does this construction give - if it 'works' at all and does not hit any 'snags'? We will return to this later.

8.1.4 Summary of Case Studies

We launched into the case studies to emphasise several point

- the usefulness of classifying spaces for encoding structure;

- lifts of structural maps correspond to classification of the 'finer' structure with respect to the 'coarser' one;

- simplicial techniques seem almost essential to encode these ideas, as they provide a link between geometric aspects and some, as yet ill perceived, ∞-categorical aspect, glimpsed via Kan complex and quasi-category structures.

The keys to these lifting and classification problems were in the (homotopy)fibres of maps between classifying spaces. It is exactly that aspect that our idea of a relative TQFT is meant to capture, but although, in the case of *Spin*-structures, the fibre was finite, in the Top/PL case, it is only 'homotopy finite', i.e., has finite total homotopy (recall the definition back in section 7.2.1, page 210). At present, our TQFT methods do not generalise to this second case, although where are certainly indication that they should do so.

Chapter 9

Homotopy Quantum Field Theories

In these 'Case studies', we have seen that one possible approach to handling 'manifolds with structure' might be using a 'classifying space', B, and, taking a 'manifold with B-structure' to be a map, or perhaps a homotopy class of maps, from the manifold to this space, B. We also suggested a situation that would potentially generate some TQFT-like structure from a fibration of simplicial groups having finite fibres / kernel, what we called a 'relative TQFT'.

In 1999-2000, Turaev, in two papers, [156, 157], (but that has only recently been formally published in full, see [158]), came up with exactly this sort of theory. He used the term *Homotopy Quantum Field Theory*. Everything is done 'over B' and the setup is such that in the case where B is a singleton space, the HQFTs are simply TQFTs. (Subsequently Turaev published several papers using that theory, or rather a slightly modified form of it.)

We thus want to study 'manifolds with extra structure' and that extra structure will be given by a 'characteristic map' from the manifold to the *target background* space, B. These 'B-manifolds' and 'B-cobordisms' are then studied using tools similar to those of Topological Quantum Field Theories. In those initial papers by Turaev, one axiom in the theory was unnecessarily strong and resulted in the elimination of any influence of the homotopy structure in B above its d-type, when the manifolds concerned were of dimension d. A modified version with change to one axiom (see below) was introduced by Rodrigues, [148]. This gave dependence of $(d+1)$-HQFTs over B on the $(d+1)$-type of B. This idea in a slightly different formulation was used by Brightwell and Turner, [27], and Bunke, Turner and Willerton, [45], to look at $(1+1)$-HQFTs with background space a simply connected space.

The initial results of [156] classified $(1+1)$-HQFTs with background spaces which were 1-types and the later results handled simply connected spaces, classification results there being in terms of the second homotopy group of B. It is therefore natural to try to classify such HQFTs for which the background space is a 2-type, a situation that would include both the previous cases. This was done in Porter-Turaev, [142]. We will summarise this once some of the earlier ideas have been explored in depth.

Remark: Turaev in [157] looked at 2+1 dimensional HQFTs and showed some neat structural classifications there as well.

We will start with a general development of HQFTs, explaining the main ideas of Turaev's work, before looking at $(1+1)$-HQFTs as these are the analogues of the TQFT situations that we studied earlier. The two cases of B being a $K(G,1)$ or a $K(A,2)$, will be looked at but as they will

be special cases of the general 2-type situation, the detailed cases can be handled via that route.

It is clear that (i) a good part of the basic theory works with little change if we did not restrict to (1+1)-HQFTs, allowing d-dimensional B-manifolds, and (ii) for $(d+1)$-HQFTs, we can assume B is, at least, the classifying space of a crossed complex, in the sense [36]. Some of the methods work in even greater generality namely when B is the classifying space of a $(d+1)$-truncated simplicial group, and thus was a general $(d+1)$-type. This leads to a concept of *simplicial formal map*, which provides an algebraic / combinatorial model for the characteristic map, $g : M \to B$, that specifies the basic background structure for the manifold, M. We will see that this idea is something that we have met before.

9.1   The category of B-manifolds and B-cobordisms

The basic objects on which an $(n+1)$-homotopy quantum field theory is built are compact, oriented n-manifolds together with maps to a 'background' or 'target' space, B. This space, B, will be path connected with a fixed base-point, $*$.

Definition: A *B-manifold* is a pair, (X, g), where X is a closed oriented n-manifold (with a choice of base-point, m_i, in each connected component X_i of X), and g is a continuous map $g : X \to B$, called the *characteristic map*, such that $g(m_i) = *$ for each base-point m_i.

A *B-isomorphism* between B-manifolds, $\varphi : (X, g) \to (Y, h)$, is an isomorphism, $\varphi : X \to Y$, of the manifolds, preserving the orientation, taking base-points into base-points and such that $h\varphi = g$.

Remark: The manifolds under consideration will often be differentiable and then 'isomorphism' is interpreted as 'diffeomorphism', but equally well we might position the theory in the category of PL-manifolds, or topological manifolds, with the obvious changes. In fact, for some of the time, we could develop constructions for simplicial complexes rather than manifolds, since, as for our earlier look at TQFTs, it is triangulations that provide the basis for the combinatorial descriptions of the structures that we will be using.

We will denote by $\mathbf{Man}(n, B)$, the category of n-dimensional B-manifolds and B-isomorphisms. We define a 'sum' operation on this category using disjoint union. The disjoint union of B-manifolds is defined by

$$(X, g) \amalg (Y, h) := (X \amalg Y, g \amalg h),$$

with the obvious characteristic map, $g \amalg h : X \amalg Y \to B$. With this 'sum' operation, $\mathbf{Man}(n, B)$ becomes a symmetric monoidal category with the unit being given by the empty B-manifold, $\emptyset$, with the empty characteristic map. Of course, this is an n-manifold by default.

It is important to note that $(X, g) \amalg \emptyset$ is not the same as (X, g), but merely isomorphic to it via the obvious B-isomorphism

$$l_{(X,g)} : (X, g) \amalg \emptyset \to (X, g).$$

Of course, there is a similar B-isomorphism, $r_{(X,g)} : \emptyset \amalg (X, g) \to (X, g)$. Likewise $(X, g) \amalg (Y, h)$ is a categorical coproduct, so is only determined up to natural (and universal) isomorphism. There are, of course, similar problems in most naturally arising monoidal structures such as the monoidal

category, $(Vect, \otimes)$, of finite dimensional vector spaces with tensor product as the monoidal structure.

For convenience, we recall that a $((n+1)$-dimensional) *cobordism*, $W : X_0 \to X_1$, is a compact oriented $(n+1)$-manifold, W, whose boundary is the disjoint union of pointed closed oriented n-manifolds, X_0 and X_1, such that the orientation of X_1 (resp. X_0) is induced by that on W (resp., is opposite to the one induced from that on W). (The manifold, W, is not considered as being pointed.) It may be convenient to write $\partial W = -X_0 \amalg X_1$ and also $\partial_- W = X_0$ and $\partial_+ W = X_1$.

Definition: A *B-cobordism*, (W, F), from (X_0, g) to (X_1, h), is a cobordism, $W : X_0 \to X_1$, endowed with a homotopy class of maps, $F : W \to B$, relative to the boundary, such that $F|_{X_0} = g$ and $F|_{X_1} = h$. (Generally, i.e., unless some confusion would ensue otherwise, we will not make a notational distinction between the homotopy class, F, and any of its representatives.) Finally, a *B-isomorphism of B-cobordisms*, $\psi : (W, F) \to (W', F')$, is an isomorphism, $\psi : W \to W'$, such that

$$\psi(\partial_+ W) = \partial_+ W',$$

$$\psi(\partial_- W) = \partial_- W',$$

and $F'\psi = F$, in the obvious sense of homotopy classes relative to the boundary.

We can glue B-cobordisms along their boundaries, or, more generally, along a B-isomorphism between their boundaries, in the usual way, see Turaev's [156], or Rodrigues, [148]. For each B-manifold, (X, g), there is a B-cobordism, $(I \times X, 1_g) : (X, g) \to (X, g)$, with $1_g(t, x) = g(x)$ and where, as usual, I denotes the unit interval. This cobordism will be called the *identity B-cobordism* on (X, g) and will be denoted $1_{(X,g)}$.

As for disjoint union of B-manifolds, we can define a disjoint union of B-cobordisms, in the obvious way.

Remark: The detailed structure of B-cobordisms and the resulting category, **HCobord**(n, B), is given in the Appendix to [148], at least in the important case of differentiable B-manifolds. This category is a monoidal category with strict duals.

9.2 The definitions of HQFTs

We will give two forms:

9.2.1 Categorical form

Definition (categorical form): A $(n+1)$-*dimensional homotopy quantum field theory with background, B,* is a symmetric monoidal functor, τ, from **HCobord**(n, B) to the monoidal category, $Vect_{\Bbbk}^{\otimes}$, of finite dimensional vector spaces over the field $\Bbbk$.

We may abbreviate the terminology in various ways, for instance, such a τ may be called a $(n+1)$-*dimensional HQFT with background, B* or a $(n+1)$-*dimensional B-HQFT* or even a $(n+1)$ *B-HQFT* The exact meaning of the abbreviation should usually be clear from the context and so

it is hoped will cause no problems. We may, abusively, also drop specification of the dimension or of the background from the terminology.

It is useful also to give here a more 'elementary' structural definition of a homotopy quantum field theory.

9.2.2   Structural form

Definition (structural form):

A $(n+1)$-*dimensional homotopy quantum field theory*, τ, *with background* B assigns

- to any n-dimensional B-manifold, (X, g), a vector space, $\tau(X, g)$;

- to any B-isomorphism, $\varphi : (X, g) \to (Y, h)$, of n-dimensional B-manifolds, a $\Bbbk$-linear isomorphism, $\tau(\varphi) : \tau(X, g) \to \tau(Y, h)$,

and

- to any B-cobordism, $(W, F) : (X_0, g_0) \to (X_1, g_1)$, a $\Bbbk$-linear transformation, $\tau(W) : \tau(X_0, g_0) \to \tau(X_1, g_1)$.

These assignments are to satisfy the following axioms:

(1) τ is functorial in $\mathbf{Man}(n, B)$, i.e., for two B-isomorphisms, $\psi : (X, g) \to (Y, h)$ and $\varphi : (Y, h) \to (P, j)$, we have
$$\tau(\varphi\psi) = \tau(\varphi)\tau(\psi),$$
and if $1_{(X,g)}$ is the identity B-isomorphism on (X, g), then $\tau(1_{(X,g)}) = 1_{\tau(X,g)}$.

(2) There are natural isomorphisms,
$$c_{(X,g),(Y,h)} : \tau((X, g) \amalg (Y, h)) \cong \tau(X, g) \otimes \tau(Y, h),$$
and an isomorphism, $u : \tau(\emptyset) \cong \Bbbk$, that satisfy the usual axioms for a symmetric monoidal functor.

(3) For B-cobordisms, $(W, F) : (X, g) \to (Y, h)$ and $(V, G) : (Y', h') \to (P, j)$ glued along a B-isomorphism, $\psi : (Y, h) \to Y', h')$, we have
$$\tau((W, F) \amalg_\psi (V, G)) = \tau(V, G)\tau(\psi)\tau(W, F).$$

(4) For the identity B-cobordism, $1_{(X,g)} = (I \times X, 1_g)$, we have
$$\tau(1_{(X,g)}) = 1_{\tau(X,g)}.$$

(5) For B-cobordisms, $(W, F) : (X, g) \to (Y, h)$, $(V, G) : (X', g') \to (Y', h')$ and $(P, J) : \emptyset \to \emptyset$, the following diagrams are commutative:

$$
\begin{array}{ccc}
\tau((X, g) \amalg (X', g')) & \xrightarrow{\ c\ } & \tau(X, g) \otimes \tau(X', g') \\
{\scriptstyle \tau((W,F)\amalg(V,G))}\downarrow & & \downarrow{\scriptstyle \tau(W,F)\otimes\tau(V,G)} \\
\tau((Y, h) \amalg (Y', h')) & \xrightarrow{\ c\ } & \tau(Y, h) \otimes \tau(Y', h')
\end{array}
\qquad
\begin{array}{ccc}
\tau\emptyset & \xrightarrow{\ u\ } & \Bbbk \ . \\
{\scriptstyle \tau(P,J)}\downarrow & \nearrow{\scriptstyle u} & \\
\tau\emptyset & &
\end{array}
$$

Remark: These axioms are slightly different from those given in the original paper, [156]. The really significant difference is in axiom 4 which is weaker than as originally formulated, where any B-cobordism structure on $I \times X$ was considered as trivial. The effect of this change is important for us in as much as it is now the case that the HQFT is determined by the $(n+1)$-type of B, cf. Rodrigues, [148]. Because of this, it is feasible to attempt a full classification of all $(1+1)$-HQFTs as there are simple algebraic models for 2-types, namely crossed modules. We will return to this later on.

9.2.3 Morphisms of HQFTs

To be able to discuss classification of HQFTs, it is first necessary to discuss some notion of map between different such theories.

Definition: Let τ and ρ be two $(n+1)$-HQFTs with background B, then a map, $\theta : \tau \to \rho$, is a family of maps, $\theta(X, g) : \tau(X, g) \to \rho(X, g)$, indexed by the B-manifolds, (X, g), such that for every B-isomorphism, $\psi : (X, g) \to (Y, h)$, and every B-cobordism, $(W, F) : (X, g) \to (Y, h)$, the maps $\theta(X, g)$ and $\theta(Y, h)$ satisfy the obvious naturality and conditions for compatibility with the structure maps, r, l, etc.

Using this, we can define a category, **HQFT**(n, B), with obvious objects and maps. Change of background space induces a functor between the corresponding categories and, extending a result of Turaev (for the initial form of HQFT), Rodrigues proved in [148] that the equivalence class of **HQFT**(n, B) depended only on the homotopy $(n+1)$-type of B. (We will examine this in more detail later.) One form of the classification problem is thus to start with an algebraic model of the $(n+1)$-type of B and to find an algebraic description of the category, **HQFT**(n, B). For instance, if B is a $K(G, 1)$, then Turaev showed that there is a bijective correspondence between the isomorphism classes of $(1+1)$-dimensional HQFTs with background $K(G, 1)$ and isomorphism classes of crossed G-algebras (see [156] and below). Brightwell and Turner, [27], for B a $K(G, 2)$ with, of course, G Abelian, showed that $(1+1)$-dimensional HQFTs, with such a background, form a category equivalent to that of G-Frobenius algebras, i.e., Frobenius algebras with a specified G-action.

Before we pass to consideration of examples, we note several consequences of the definition of a homotopy quantum field theory. One of the most important is that if τ is a $(n+1)$-HQFT and (X, g) and (X, h) are two B-manifolds with the same underlying manifold, X, and the two characteristic maps, g and h are freely homotopic, then a choice of homotopy, $F : g \simeq h$, gives a B-cobordism, $(I \times X, F)$, which induces an isomorphism between $\tau(X, g)$ and $\tau(X, h)$. (This is an easy exercise, but is also a consequence of Rodrigues, [148], Proposition 1.2.). Because of this, one can expect that some of the essential features of $\tau(X, g)$ can be gleaned from the homotopy class of g.

9.3 Examples of HQFTs

We will start with summarising some of the examples and constructions given by Turaev in his original paper and his monograph, [158].

9.3.1   Primitive cohomological HQFTs

For this, B is an Eilenberg-MacLane space, $K(G,1)$, so it has fundamental group isomorphic to a group G, and all other homotopy groups trivial. The description of this HQFT will need a few facts, which we have not yet met. We will give a description based on Turaev's paper, [156], but will also translate this to one involving triangulations, more akin to our treatment of TQFTs.

We will, as suggested before, work over a fixed field, $\Bbbk$, and, as usual in such situations, $\Bbbk^*$ will denote the group of invertible elements in $\Bbbk$. (We will usually be most interested in the case $\Bbbk = \mathbb{C}$, and in that case $\Bbbk^*$ can usually be replaced by $U(1)$, the circle group, thought of as the group of unit modulus complex numbers.)

The original cohomological approach to the cohomology of groups was via the 'spatial' cohomology of a $K(G,1)$, so $H^n(G,A)$ was thought of as $H^n(K(G,1),A)$ and, although now almost purely algebraic approaches to group cohomology are often given, the link with that 'spatial' origin is still strong, (see, for instance, K. Brown's book, [29], that we have mentioned before). Here, if we are considering d-dimensional oriented manifolds for our HQFT, then we will specify a cohomology class, $\theta \in H^{d+1}(G,\Bbbk^*) \cong H^{d+1}(B,\Bbbk^*)$. Such a θ will enable us to define of a $(d+1)$-HQFT, with target, B, having each $\tau(X,g)$ of dimension 1.

The next ingredient that we need is the notion of a *fundamental class* of a d-manifold. This is well known, standard material and uou may have already met it, but, just in case, we will 'recall' it in brief. For the detailed background theory, we refer to standard books on algebraic topology.

If X is a d-dimensional connected orientable manifold without boundary, then its dth homology group, $H_d(X) = H_d(X,\mathbb{Z})$ (with integer coefficients), is an infinite cyclic group. There are, of course, two choices of generator, the second being the inverse of the first, corresponding to a choice of orientation, or of the reverse orientation. In fact, the *definition* of *orientation* is exactly that, a choice of generator for $H_d(X)$. If we look into this a bit more geometrically, we can intuitively think of X as being triangulated by some simplicial complex, T, and, form $C_d(T)$, the Abelian group of formal sums (over $\mathbb{Z}$) of the d-dimensional oriented simplices. (Think of a surface, which is orientable, and 'add up' all the simplices.) As there are no $(d+1)$-simplices around, $C_{d+1}(T)$ is the trivial group, and in $C_d(T)$, the sum of all the d-simplices has trivial boundary, as each boundary bit of a simplex will be matched, exactly, by another, of an adjacent simplex, which will have the opposite orientation and hence the opposite sign. (To get a feel for this, if you have not met it before, go to your surface picture and try it out!) Here, we need that X itself has no boundary, so there is nothing 'left over' from that sum.

Definition: A *fundamental class*, $[X]$, of a d-dimensional connected orientable closed manifold, X, is a generator of $H_d(X)$. If a choice of such a class is made, it specifies an *orientation* of X, which is then referred to as an *oriented*, rather than just an orientable, manifold and $[X]$ is then *the* fundamental class of X.

Remarks: (i) We will not need it, but there is a notion of fundamental class for a non-orientable manifold, made by working over the 'integers mod 2', i.e., $\mathbb{Z}_2$. There the difference between +1 and -1 has been eliminated, so the oriented simplices of a triangulation always fit together correctly.

(ii) If X is not connected, but is orientable, a *fundamental class* for X is a choice of fundamental class for each component of X, and hence an element of $H_d(X)$, which is a free Abelian group of rank the number of components of X. This is important for us as cobordisms, of course, usually

have disconnected boundaries.

We will also need fundamental classes for $(d+1)$-cobordisms between d-dimensional manifolds, and, of course, if $W : X_0 \to X_1$ is such a thing, "$\partial W = X_1 - X_0$", that is, the boundary of W has an inward part, X_0, and an outward part, X_1, both of which may be disconnected, so to link the fundamental classes of X_0 and X_1, we need to have a fundamental class for the probably *non-closed* $(d+1)$-dimensional manifold, W. For this, we need the $(d+1)$-dimensional relative homology group, $H_{d+1}(W, \partial W)$, so we will briefly handle this next.

Suppose we have a space, X, and a subspace, A. Considering either both to be simplicial complexes, or using singular simplices, we get, as usual, a chain complex, $C(X)$, of X, and a corresponding one consisting of the chains within A, $C(A)$. We have a short exact sequence of chain complexes,

$$0 \to C(A) \to C(X) \to C(X)/C(A) \to 0.$$

The nth relative homology group, $H_n(X, A)$, of the pair, (X, A), is the nth homology of $C(X)/C(A)$, i.e., the quotient $Ker \, \partial_n / Im \, \partial_{n+1}$.

If we now go to our $(d+1)$-cobordism, $W : X_0 \to X_1$, (so $\partial W = (-X_0) \coprod X_1$, and think of $H_{d+1}(W, \partial W)$, our earlier 'handwave' suggests that, if we take the sum of the oriented $(d+1)$-simplices, the bits of the boundary that will not cancel out with other parts of the expression will be those that lie in ∂W, *but* in $C(X)/C(\partial W)$, we have 'killed' those pieces off. This makes it feasible that $H_{d+1}(W, \partial W)$ will be infinite cyclic as well - of course, we will need W to be a *connected* $(d+1)$-manifold for this to work, but the extension to the non-connected case is as before. We write $[W]$ for its (chosen) generator.

The relative homology forms part of a long exact sequence giving in the critical dimensions

$$0 \to H_{d+1}(W, \partial W) \xrightarrow{\delta} H_d(\partial W) \to H_d(W) \to \dots,$$

and the linking map, δ, joining the different dimensions, sending $[W]$ to $[X_1] - [X_0]$. More exactly, the conditions on the orientations that we imposed on W relative to X_0 and X_1, were (recalled from pages 200 and 267) that 'the orientation of X_1 (resp. X_0) is induced by that on W (resp., is opposite to the one induced from that on W.' This translates, more precisely, to:

$$\delta[W] = [X_1] - [X_0].$$

Note that here $H_d(\partial W)$ will consist of a direct sum of infinite cyclic groups, one for each component of ∂W, and $[X_1]$ and $[X_0]$ will be the sums of the chosen fundamental / orientation classes.

Remark: One additional point to note is that there is a choice that can be made here, one that was hinted at above. If we do not want to assume that we have triangulations of all manifolds and cobordisms, we can use singular complexes throughout. This avoids some of the work later that is used to eliminate the dependence on the triangulations. On the other hand, it comes at a slight price as there will be non-trivial chains in dimensions greater than the dimension of the manifolds or cobordism. In fact, in many places, this enables the singular complex based theory to be 'sleaker' than the simplicial complex / triangulation based one. The two approaches yield the same result as they both give HQFTs based around the cohomology theory of the manifolds and cobordisms and, of course, singular and simplicial based cohomology theories give isomorphic cohomology groups. The singular complex has a side which could be exploited here and will be

later on. For a space, X, $Sing(X)$ is a Kan complex and behaves like an ∞-groupoid. It is the fundamental (weak) ∞-groupoid of X. The 'quasi-algebraic' structure allows analogues of many algebraic constructions to be given; see the nLab, [134].

The input into a primitive cohomological HQFT is a space B. (This is often given by a group, G, and then B is a corresponding Eilenberg-MacLane space, $B = K(G,1)$.) This will act as the 'target' space, and we assume given a $(d+1)$-dimensional cohomology class, $\theta \in H^{d+1}(B, \Bbbk^*)$. To each B-manifold, (X,g), of dimension d, we will assign a 1-dimensional $\Bbbk$-vector space, which we will often write as $A_{(X,g)}$, and sometimes as $\tau^\theta(X,g)$, generated by a vector $\langle a \rangle$, corresponding to a singular cycle, $a \in C_d(X)$, which represents the fundamental class $[X]$. We will loosely say that a is a *fundamental d-cycle*. Different choices of a within the fundamental class, $[X]$, will give related basis elements, thus involving the homotopy type of the space, B, and the cocycle, θ. Suppose $c \in C_{d+1}(X)$ is such that $\partial c = a - b$ for $b \in C_d(X)$, therefore, b is another choice of 'fundamental d-cocycle' for X. We require

$$\langle a \rangle = g^*(\theta)(c)\langle b \rangle.$$

This formula needs some 'deconstruction'. We have $\theta \in H^{d+1}(B, \Bbbk^*)$, (which, as was mentioned above, is the same as $H^{d+1}(G, \Bbbk^*)$), but we have $g : X \to B$, so this gives $g^*(\theta) \in H^{d+1}(X, \Bbbk^*)$, and this is represented by some homomorphism, which we also call $g^*(\theta)$, from $C_{d+1}(X)$ to $\Bbbk^*$. This gives $g^*(\theta)(c) \in \Bbbk^*$, thus a non-zero element of the field (or, if you need $\Bbbk$ to be a commutative ring and are replacing 'vector space' by finite rank free module, then it will be a unit of $\Bbbk$). It is worth noting that this 'scalar' does not depend on the choice of c, merely on the properties that c has, as, if c' was another such element of $C_{d+1}(X)$ satisfying $\partial c' = a - b$, then, as the homology of X is trivial in dimension $d+1$, c and c' are homologous, i.e., there is some $e \in C_{d+2}(X)$ with $\partial e = c - c'$. **You are left to check** that this implies $g^*(\theta)(c) = g^*(\theta)(c')$. (Alternatively there is a proof (one equation) in [156] or in Chapter 1, section 2.1. of [158].)

Because of the above, we can think of $A_{(X,g)}$ as having as basis element, the fundamental class of X, 'twisted' by the characteristic map $g : X \to B$ and 'weighted' by the element θ. That is just 'words', as the 'twisting' is subtle and to understand it, we do need to see its interaction with the other structure.

Now, let $f : (X,g) \to (Y,h)$ be a B-homomorphism, then we can obtain an isomorphism, f_*, from $A_{(X,g)}$ to $A_{(Y,h)}$, by mapping the basis element $\langle a \rangle$ to $\langle f_*(a) \rangle \in A_{(Y,h)}$. Here $f_* : H_d(X) \to H_d(Y)$, of course, and **you should check** that the resulting $\langle f_*(a) \rangle$ is independent of the choice of representing $a \in C_d(X)$.

If (X,g) is a disjoint union of the B-manifolds, (X_1, g_1) and (X_2, g_2), then $a \in C_d(X)$ can be written as the sum of the images of fundamental cycles of X_1 and X_2 under the induced maps, $i_1 : H_d(X_1) \to H_d(X)$, etc., thus we can assume $a = i_{1,*}a_1 + i_{2,*}a_2$. Clearly, as all of the vector spaces, $A_{(X,g)}$, $A_{(X_1,g_1)}$ and $A_{(X_2,g_2)}$, are of dimension 1, they are just copies of the 'vector space', $\Bbbk^1$, and the tensor product, $A_{(X_1,g_1)} \otimes A_{(X_2,g_2)}$, will be isomorphic to $A_{(X,g)}$, but what is important is the description and specification of that isomorphism. It does need to be checked that matching $\langle a_1 \rangle \otimes \langle a_2 \rangle$ with $\langle a \rangle$, is the 'right' isomorphism, compatible with the 'twisting', etc. (This will again be **left to you**. The **only problem** is to work out exactly what has to be checked, so, in some sense, what 'right' means! Clearly we need the isomorphism to be 'well defined', so, if $\langle a_1 \rangle = g^*(\theta)(c_1)\langle b_1 \rangle$, etc., we need it to match $\langle b_1 \rangle \otimes \langle b_2 \rangle$ with $\langle b \rangle$. This needs some fairly routine calculations that are better done **by the reader**.)

Formally we take $\tau^\theta(X, g)$ to be this $A_{(X,g)}$, and we will often omit θ from the notation.

The next point is to see how to define the hoped for HQFT, τ^θ, on B-cobordisms. Here again the 'twisting' comes in to play again. Let (W, F) be a B-cobordism from (X_0, g_0) to (X_1, g_1). Pick a fundamental $(d+1)$-cycle, $b \in C_{d+1}(W, \partial W)$, so that $\delta[b] = [a_1] - [a_0]$, the difference of fundamental cocycles for X_0 and X_1, where we use square brackets, here, to denote homology classes. The B-cobordism defines a map $\tau(W, F) : \tau(X_0, g_0) \to \tau(X_1, g_1)$ by mapping the basis element, $\langle a_0 \rangle$ to $(F^*(\theta)(b))^{-1}\langle a_1 \rangle$. This looks neat and loosely corresponds to manipulations that we have seen in our earlier discussions on how to define a TQFT starting with a finite group, etc., but again we *do* need to take it apart, as there is a lot happening in a short space. We first need to back-track to $\delta[b] = [a_1] - [a_0]$, and to recall how δ is constructed.

The construction of the connecting map in the homology long exact sequence is well known, but is worth recalling. We look at the general case in a homological algebra situation first. It is a particular case of the Puppe type sequence argument.

Suppose

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is a short exact sequence of chain complexes of modules over some commutative ring, then we want the connecting homomorphism, δ, from $H_{n+1}(C)$ to $H_n(A)$. We start with some $(n + 1)$-cycle, $c \in C_{n+1}$ (so $\partial c = 0$). As B_{n+1} maps down to C_{n+1}, we can find a $b \in B_{n+1}$ with $\beta b = c$. (In our case of chains on X, we could split the exact sequence,

$$0 \to A_{n+1} \xrightarrow{\alpha_{n+1}} B_{n+1} \xrightarrow{\beta_{n+1}} C_{n+1} \to 0,$$

as we have just vector spaces, but this would obscure the use of choices and a certain indeterminacy within the construction.) We know β is a chain, so $\beta \partial b = \partial \beta b = \partial c = 0$ and $\partial b \in Ker\,\beta_n = Im\,\alpha_n$. We can thus find a unique $a \in A_n$ with $\alpha a = \partial b$. We, tentatively, define $\delta[c] = [a]$. Checking that this gives a well defined homomorphism is then **left to you**. (Even if you have met this within homological algebra, **do check** you recall how that verification goes.) Interpreting this, now, in our situation, it is clear that the connecting homomorphism sends a fundamental cycle of W to the difference, $[X_1] - [X_0]$, in $C_d(\partial W)$.

We next bring in F, the characteristic map of the cobordism. It induces

$$F^* : H^{d+1}(G, \Bbbk^*) \to H^{d+1}(W, \Bbbk^*),$$

so $F^*(\theta)$ is a cohomology class in this latter group. We will also repeat our earlier (ab)use of notation and write the same thing for a (choice of) $(d + 1)$-cocycle representing that class, so $F^*(\theta) : C_{d+1}(W) \to \Bbbk^*$. This means that we can form $F^*(\theta)(b)$, an invertible element of $\Bbbk$. We divide $\langle a_1 \rangle$ by this element and that will define

$$\tau^\theta(W, F) : \tau^\theta(X_0, g_0) \to \tau^\theta(X_1, g_1),$$

as in our formula

$$\tau^\theta(W, F)\langle a_0 \rangle = (F^*(\theta)(b))^{-1}\langle a_1 \rangle.$$

Of course, it really needs to be checked that this is independent of the choices of representative made. (This is examined in Turaev's paper, [156] and his book, [158], but it is a good idea to **look at it yourself** first.)

Remark: We will look in more detail a bit later on, at the relationship between this construction and the 'labelled triangulation' approach to TQFTs that we sketched out earlier, however it is worth looking at this briefly now. We had the hint that a fundamental class, whether of a closed d-manifold or a $(d+1)$-cobordism (hence in a relative homology group) was, more-or-less, the sum of the top dimensional oriented simplices. We thus can think of this $(F^*(\theta)(b))^{-1}$ as dividing by a "θ-weighted" sum of $\Bbbk^*$-valued cochains, the sum being over all the $(d+1)$-simplices of W. This is analogous to the weighting factors we used in constructing our TQFTs earlier.

It is worth noting that Turaev, in his introduction of this construction states; "This construction is inspired by the work of Freed and Quinn, [82], on TQFTs associated with finite groups." This work of Freed and Quinn was also the inspiration for Yetter's work on TQFTs from homotopy 2-types.

Much of the checking that τ^θ gives a HQFT can be **left to the reader**, either to work out themselves or to check up in the sources referred to above. We will, however, glance at the construction's interaction with the composition of B-cobordisms as this is a little more tricky than some of the other parts. It uses the Mayer-Vietoris sequence, which we now recall, but refer to standard texts such as Spanier, [150], or Hatcher, [92], for detailed proofs.

We suppose given a space, X, and two subspaces, A and B, whose interiors cover X, then there is a long exact sequence

$$\ldots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots$$

The idea of the proof is to form

$$0 \to C(A \cap B) \xrightarrow{\varphi} C(A) \oplus C(B) \xrightarrow{\psi} C(A + B) \to 0,$$

where $C(A + B)$ is shorthand for the subcomplex of $C(X)$ consisting of chains that are sums of chains wholly in A and chains wholly in B. (If, as here, we use singular chains then the terms of an element of $C_n(A + B)$ are in A or in B (or in both, of course). This is clearly the image of the obvious map from $C_n(A) \oplus C_n(B)$ to $C_n(X)$.) The chain maps, φ and ψ, are given by

$$\varphi(x) = (x, -x)$$

for x, a chain in $C(A \cap B)$, and

$$\psi(x, y) = x + y.$$

The exactness of the Mayer-Vietoris sequence then follows from the induced long exact sequence for the homology of these chain complexes, together with a proof that the inclusion of $C(A + B)$ into $C(X)$ induces an isomorphism on homology.

We next go to our B-cobordisms:

$$(W_0, F_0) : (X_0, g_0) \to (X_1, g_1),$$

$$(W_1, F_1) : (X_1, g_1) \to (X_2, g_2),$$

and their composite

$$(W, F) = (W_0 \amalg_{X_0} W_1, F_0 \amalg F_1) : (X_0, g_0) \to (X_2, g_2).$$

(Turaev, [156], does consider a slightly more complex situation, namely that the input, (X_1, g_1), of the second cobordism is replaced by another B-manifold, (X_1', g_1'), with a specified B-isomorphism, $f : (X_1, g_1) \to (X_1', g_1')$, which is then used in the 'gluing together' of the two B-cobordisms. The simpler case that we will consider can be shown to be equivalent to this, but, in any case, will suffice for our exposition here. The argument we will give adapts easily to cover his case.)

In our situation, what are A and B? Clearly X should be $W = W_0 \amalg_{X_0} W_1$, and 'obviously' we should take $A = W_0$, and $B = W_1$, but that will not work as their interiors do not cover W. Here we use a slightly technical detail that was not explicitly mentioned earlier. In the cobordisms, the boundary does need to have a cylindrical neighbourhood nicely embedded in the larger manifold. For instance, this is needed to ensure a reasonable smooth structure on composites if we are in the smooth case, or to ensure a nice triangulation if we are in the PL case, etc. It is not difficult to obtain, but is usually assumed as it aids the definition and construction of the gluing / composition. Here we need to take a small open neighbourhood, N_0 of X_1 in W_0, and another one, N_1 of the copy of X_1 in W_1. These must be retractable into X_1 in each case. We then take $A = N_1 \cup_{X_1} W_0$ and $B = N_0 \cup_{X_1} W_1$, so A retracts to W_0 and B retracts to W_1. Now we can use the Mayer-Vietoris sequence, together with these retractions to get what we need, namely a composition of the fundamental classes of W_0 and W_1 and the right sort of behaviour on the boundary. The resulting long exact sequence is then:

$$\ldots \to H_n(X_1) \to H_n(W_0) \oplus H_n(W_1) \to H_n(W) \to H_{n-1}(X_1) \to \ldots$$

Now let b_0 be a fundamental $(d+1)$-cycle on W_0, so that

$$\delta[b_0] = [a_1] - [a_0],$$

where a_i is, as before, a fundamental cycle on X_i, $i = 0, 1$. Similarly, suppose $b_1 \in C_{p+1}(X_1)$ is a fundamental $(d+1)$-cycle on W_1, with

$$\delta[b_1] = [a_2] - [a_1].$$

In the Mayer-Vietoris sequence, $\psi([b_0], [b_1]) = [b_0 + b_1] \in H_{d+1}(W, \partial W)$, and $\delta[b_0 + b_1]$ is what we would expect, namely $[a_2] - [a_0]$. (A bit of easy element chasing around various interlocking exact sequences will **check this** for you.) Moreover, that $b = b_0 + b_1$ is a fundamental $(d+1)$-cycle for W, easily drops out of the same sequence in the top dimension.

The composite,

$$\tau(W_1, F_1) \sharp \tau(W_0, F_0) : \tau(X_0, g_0) \to \tau(X_2, g_2),$$

sends $\langle a_0 \rangle$ to $(F_1^*(\theta)(b_1)^{-1} F_0^*(\theta)(b_0)^{-1}) \langle a_2 \rangle$.

By definition, $\tau(W, F) : \tau(X_0, g_0) \to \tau(X_2, g_2)$ sends $\langle a_0 \rangle$ to $(F^*(\theta)(b))^{-1} \langle a_2 \rangle$. The operation in $\Bbbk^*$ is multiplication, so another diagram chase shows that

$$F_1^*(\theta)(b_1) F_0^*(\theta)(b_0) = F^*(\theta)(b),$$

and hence that

$$\tau(W_1, F_1) \sharp \tau(W_0, F_0) = \tau(W, F).$$

For verification of the other conditions necessary for τ^θ to be an HQFT, we refer to the original source, [156], the monograph, [158], or **to your own resourcefulness!**

9.3.2   Operations on HQFTs

These primitive cohomological HQFTs provide a very basic set of examples, but these can be used as building blocks for more complex ones, provided we have some operations for combining arbitrary HQFTs and we turn to these next.

Duals: We suppose τ is a HQFT with background, B. Its dual, $\overline{\tau}$, is given by:

- for a B-manifold, (X, g), $\overline{\tau}(X, g) = Hom_{\Bbbk}(\tau(X, g), \Bbbk) = \tau(X, g)^*$, the dual space of $\tau(X, g)$;

- for any B-isomorphism, $\varphi : (X, g) \to (Y, h)$, $\overline{\tau}(\varphi)$ is the transpose of $\tau(\varphi)$;

- for $(W, F) : (X_0, g_0) \to (X_1, g_1)$, a B-cobordism, we first take the opposite B-cobordism $(-W, X_1, X_0, -F) : (X_1, g_1) \to (X_0, g_0)$, and then take $\overline{\tau}(W, F)$ to be the transpose of τ of this.

Verification of the axioms is straightforward.

Tensor product: If τ and τ' are two $(n + 1)$ B-HQFTs, then $\tau \otimes \tau'$, defined in the obvious way (so, for instance, $(\tau \otimes \tau')(X, g) = \tau(X, g) \otimes \tau'(X, g)$), defines a $(n + 1)$ B-HQFT.

Direct sum: A similar construction gives $\tau \oplus \tau'$, by using the direct sum of the vector spaces, etc.

Rescaling: HQFTs can be *rescaled* using numerical invariants of B-cobordisms. Suppose we have a $\mathbb{Z}$-va;ued assignment, ρ, sending a B-cobordism, $\underline{W}$, (a shorthand for (W, X_0, X_1, F)), to an integer $\rho(\underline{W})$. We say it is an *invariant* if it respects B-isomorphism and homotopies of the characteristic map, F, relative to the boundary. The invariant ρ is said to be *additive* if

(i) it sends disjoint union to addition:

$$\rho(\underline{W} \text{ II II} \underline{W'}) = \rho(\underline{W}) + \rho(\underline{W'}),$$

and

(ii) sends 'gluing' to addition, as well, so if $\underline{W} = (W, X_0, X_1, F)$ and $\underline{W'} = (W', X_1, X_2, F')$, then on forming $\underline{W} +_{X_1} \underline{W'}$, the B-cobordism obtained by gluing the two given B-cobordisms along X_1, we have

$$\rho(\underline{W} +_{X_1} \underline{W'}) = \rho(\underline{W}) + \rho(\underline{W'}).$$

An example of such an additive invariant is the *relative Euler characteristic*, $\chi(W, X_0) = \chi(\underline{W}) = \chi(W) - \chi(X_0)$. (It is clear that this is additive under disjoint union and to see that it is also additive under gluing, think of it as the alternating sum of the numbers of vertices, edges, faces, etc., of a triangulation of W, but in which the contributions from simplices in X_0 are not counted. In $W +_{X_1} W'$, the contribution of X_1, which is in $\chi(W) + \chi(W')$ twice, is correctly counted. (This is, of course, a similar point to one we saw in Lemma 32 on page 217, when handling Yetter's construction.)

Now suppose that ρ is such an invariant, and $a \in \Bbbk^*$ is an invertible element of $\Bbbk$. If τ is a $(n + 1)$-dimensional B-HQFT, then we can form the a^ρ-scaled HQFT, $a^\rho \tau$, which has the same

vector spaces as its 'values' on B-manifolds as does τ itself, but if $(W, F) : (X_0, g_0) \to (X_1, g_1)$ is a B-cobordism, then

$$a^\rho \tau(W, F) = a^{\rho(\underline{W})} \tau(W, F),$$

that is, the images of $\tau(W, F)$ are *scaled* by a factor $a^{\rho(\underline{W})}$.

9.3.3 Geometric transfer:

Suppose that we have a finite sheeted covering space, $p : E \to B$, of B. We collapse the fibre over the base-point, $* \in B$ to a point getting a new space, E'. Suppose τ is a E'-HQFT. Let $q : E \to E'$ be the projection and $(X, g : X \to B)$ a B-manifold. Looking at all lifts $\overline{g} : X \to E$, we have $(M, q\overline{g})$ is a E'-manifold, so we can set

$$A_{(M,g)} = \bigoplus \{ \tau(M, q\overline{g}) \mid \overline{g} : X \to E, p\overline{g} = g \}.$$

It is then not hard to extend this assignment to give a B-HQFT structure.

9.4 Change of background

The operation of transfer, given above, uses a 'transfer' of background. There was an HQFT with background E', the result of collapsing the fibre of the covering, $p : E \to B$, over the base-point of B. The method used the induced map from E' to B to build an HQFT on B. We will see this process and related ones several times in the following. it is clearly related to the relative TQFT construction sketched earlier. To examine it in detail, we will need to study the general problem of change of background under a pointed continuous map, and, in fact, that will give us very valuable information about B-HQFTs and their dependence on B.

9.4.1 The induced functor on HCobord(n, B)

Suppose $f : B \to B'$ is a base-point preserving continuous map between two 'background' spaces, by which we mean that they satisfy the conditions given earlier, (page 266).

Let $(X, g : X \to B)$ be an n-dimensional B-manifold, then it is clear that (X, fg) is an n-dimensional B'-manifold. This process clearly respects B-isomorphisms, sending then the B'-isomorphisms.

If $(W, F : W \to B)$ is a B-cobordism, then, similarly, (W, fF) is a B'-cobordism, and, as the process of gluing takes place without any use of the characteristic maps, this 'induction' or 'post-composition' process will respect composition. (This deserves a bit more detail, but can be left at that on a first read. The point is composition of morphisms in $HCobord(n, B)$ is *derived* from gluing of B-cobordisms, but the B-cobordisms themselves are not 'officially' the morphisms as those are B-isomorphism classes of B-cobordisms (relative to their boundaries). **Writing down all that in detail** and then post-composing with f, gives the proof of the claim just made.) We thus have that 'post-composition' with f gives a functor, f_* from $HCobord(n, B)$ to $HCobord(n, B')$.

Proposition 58 *The functor, f_*, is monoidal.*

Proof: The tensor product in $HCobord(n, B)$ is given by disjoint union / coproduct of manifold and uses the universal property of coproduct, so it is easy to check that

$$f_*((X, g) \amalg (X', g')) \cong f_*(X, g) \amalg f_*(X', g'),$$

and that these isomorphisms are compatible with composition.   ■

In fact, the functor, f_*, can be considered to be strictly monoidal provided the disjoint union is suitably rigidly specified.

As the categorical form of the definition of an $(n + 1)$-dimensional homotopy quantum field theory with background, B, is very simply a symmetric monoidal functor, τ, from $HCobord(n, B)$ to $Vect_{\mathbb{k}}^{\otimes}$, and a morphism between two such is a monoidal transformation, we obtain the following by applying just a bit of simple monoidal category theory.

Proposition 59 *A continuous map, $f : B \to B'$, of backgrounds induces a functor,*

$$f^{\sharp} : HQFT(n, B') \to HQFT(n, B).$$

Proof: This is simply defined by $f^{\sharp}(\tau) = \tau f_*$.   ■

We next consider what happens if $H : f_0 \simeq f_1 : B \to B'$, so we have two homotopic maps between B and B'.

If (X, g) is a B-manifold, then $(X \times I, Hg) : (X, f_0 g) \to (X, f_1 g)$ is a B'-cobordism. In the following lemma, we look at a slightly more general result:

Lemma 44 *Let (X, g) and (X, h) be two B-manifolds such that there is a (free) homotopy between g and h, then there is an isomorphism $(X, g) \overset{\cong}{\to} (X, h)$ in $HCobord(n, B)$.*

Proof: Let $F : X \times I \to B$ be a homotopy between g and h, and $(X \times I, F)$ the corresponding cobordism. We also have the reverse homotopy, which we denote by $\overline{F} : h \simeq g$, so $\overline{F}(x, t) = f(x, 1 - t)$ for $t \in I$, and $(X, \overline{F})$ is also a B-cobordism. We know that the composite homotopies $F \circ \overline{F}$ and $\overline{F} \circ F$ are homotopic to the constant 'identity' homotopy on the respective maps, h or g. The corresponding glued B-cobordisms are therefore B-isomorphic to the identities. Hence $(X \times I, F)$ is an isomorphism in $HCobord(n, B)$. (We have skimmed over some of the details here. They can safely be **left to you** and in the smooth case are given by Rodrigues, ([148], Proposition 1.2.).   ■

Using this lemma, we have that in our previous setting, if (X, g) is a B-manifold, then the homotopy yields a natural isomorphism between $(X, f_0 g)$ and $(X, f_1 g)$. It is easy, then, to see that the following holds:

Corollary 16 *If $H : f_0 \simeq f_1 : B \to B'$, then H induces (i) a monoidal natural isomorphism $f_{0*} \overset{\cong}{\to} f_{1*}$, and (ii) a natural isomorphism $f_0^{\sharp} \overset{\cong}{\to} f_{1*}^{\sharp}$*   ■

Corollary 17 *If $f : B \to B'$ is a homotopy equivalence, then the induced functor, f_*, is an equivalence of categories, as also is $f^{\sharp}$.*   ■

This basically says that the properties of a HQFT only depend on the homotopy type of its background. We will shortly see how it depends on the dimension, n, of the manifolds involved. Before that we will use the above to explore the relationship between HQFTs and TQFTs.

9.4.2 HQFTs and TQFTs

Any manifold, X, has a trivial B-manifold structure, since we can always take the constant characteristic map, $g : X \to B$, $g(x) = *$, the base point of B. The same goes for unadorned cobordisms between manifolds. We thus, after **doing a bit of checking**, have a monoidal functor from $(n + 1)-Cob$ to $HCobord(n, B)$, and this functor is an inclusion on the objects. It is not full, in general, the obvious case being a closed $(n + 1)$-manifold thought of as a cobordism from $\emptyset$ to $\emptyset$. This could have other than a trivial characteristic map to B. Of course, if B is contractible, then this could not happen and $HCobord(n, B)$ will be equivalent to $(n + 1)-Cob$.

In general, we have a unique pointed change of background from the one point pointed space, $*$, to B and, similarly, from B to $*$, with the composite in one sense giving the identity on $*$. We thus get, on identifying $HCobord(n, *)$ and $(n + 1)-Cob$, monoidal functors

$$(n + 1)-Cob \to HCobord(n, *) \to (n + 1)-Cob,$$

with composite the identity functor. (The left hand one is the inclusion we gave earlier.)

Similarly, at the level of HQFTs, we have

$$(n + 1)-TQFT \to HQFT(n, B) \to (n + 1)-TQFT,$$

so every TQFT extends, trivially, to an HQFT. This is useful, as it gives us information about models for HQFTs. We have, in certain cases, characterisations, even classifications, in terms of algebraic models, of the TQFTs and the above relationship strengthens the intuition that introducing B into the picture may not perturb the basic constructions those models too much.

The discussion suggests a means of attack on B-HQFTs, so as to analyse them and interpret what they tell us, both about B and about the structured B-manifolds. If we can decompose the homotopy type of B in a sensible way, we might be able to build up a picture of how a B-HQFT depended on related, hopefully simpler ones. We saw this to some extent with transfer, where we could use E'-HQFTs to construct certain B-HQFTs. The next ingredient in the discussion will thus be one such decomposition namely that involving n-types,

9.4.3 Change along an $(n + 1)$-equivalence

Our next aim is to see what an $(n+1)$-dimensional HQFT records about B by seeing how changing the background along a $(n + 1)$-equivalence influences things.

Suppose that $f : B \to B'$ is an $(n + 1)$-equivalence, and look at the induced functor,

$$f_* : HCobord(n, B) \to HCobord(n, B').$$

If f is an $(n + 1)$-equivalence, then we know that for any n-manifold, X,

$$[X, f] : [X, B] \to [X, B']$$

is a bijection (cf. section 3.1.1, page 75).

Lemma 45 *For any B'-manifold structure, $g' : X \to B'$ on X, there is a B-manifold structure, (X, g), so that $f_*(X, g) \cong (X, g')$.*

Proof: This is clear, since there is a $g : X \to B$ such that $[X, f][g] = [g']$, whilst by Lemmas 44, this means that $(X, g') \cong (X, fg)$ in $HCobord(n, B')$. ∎

We thus have that f_* is essentially surjective on objects. this suggests that we check if f_* is full and faithful.

Suppose W is a cobordism between X_0 and X_1 and we look at the set, $[W, B]_{rel\, \partial W}$, of homotopy classes (relative to the boundary) of maps from W to B. The induced map, given by composition with f, does give

$$[W, f]_{rel\, \partial W} : [W, B]_{rel\, \partial W} \to [W, B']_{rel\, \partial W},$$

and we can adapt the arguments of section 3.1.1, page 75, to show that $[W, f]_{rel\, \partial W}$ is also a bijection.

As a morphism in $HCobord(n, B)$ is a B-isomorphism class of B-cobordisms, (W, F), where F 'is' a homotopy class $rel\, \partial W$ of maps $F : W \to B$, it is now a simple matter to check that f_* is full and faithful, and we have proved:

Theorem 24 *(Rodrigues, [148]) If f is an $(n+1)$-equivalence, then the induced functor, f_*, is an equivalence of categories.* ∎

The consequences of this result for HQFTs is clear.

Corollary 18 *If f is an $(n + 1)$-equivalence, then the induced functor, $f^\sharp$, is an equivalence of categories.* ∎

We can, thus, restrict attention to background spaces which are m-types for $m \leq n + 1$, as any n-dimensional HQFT is isomorphic to one with a $(n + 1)$-type as background; you simply replace B by $P_{n+1}B$, a Postnikov $(n + 1)$-section of B.

As an instance of this, for any simply connected space, B, there is equivalence of categories:

$$HCobord(1, B) \stackrel{\cong}{\Rightarrow} HCobord(1, K(A, 2)),$$

where $A \cong \pi_2(B)$.

This raises the possibility of studying $HCobord(1, K(A, 2))$ directly obtaining algebraic and categorical classifications of it. Rodrigues, in [148], uses this to give a description of $HCobord(1, K(A, 2))$, up to equivalence, as the free symmetric monoidal category with strict duals on an 'A-Frobenius object'. (An A-Frobenius object is a Frobenius object (cf. page 208), a, together with and action $A \to End(a)$, satisfying some compatibility conditions. We will consider them in more detail a bit later on.) This, in turn, implies that $HCobord(1, K(A, 2))$ is equivalent to the category of A-Frobenius algebras (over $\Bbbk$). This result was first discovered by Brightwell and Turner, [27], and extends the classification of TQFTs that we mentioned briefly on page 207.

This is just one of the potential instances of the theorem, but its link with a categorical characterisation of $HCobord(1, B)$, in this case, suggests a host of generalisations and potential applications / interpretations of HQFTs. (For this beyond what we will discuss in these notes, see the discussion the entries on the nLab, [134], on the 'Cobordism Hypothesis', and the preprint of Lurie, [113].)

If we ;look at the simple case of $HCobord(1, B)$, but with no additional constraint on B, then we know that, up to equivalence, this is the same as $HCobord(1, P_2B)$, so we may assume $\pi_i B = 0$ for $i > 2$. In other words, we can suppose that B is a 2-type, hence is the classifying space of some crossed module, C. If we know *algebraic* information abnout C, can we glean categorical

information, and perhaps, eventually, geometric information about $HCobord(1, B\mathsf{C})$. Of course, that question is just the start, as we can ask similar questions about $HCobord(n, B\mathsf{C})$, where C is a model for a homotopy $(n + 1)$-type, something like a crossed n-cube, or an $(n+1)$-hypergroupoid. How does $HCobord(n, B\mathsf{C})$ reflect properties of C, for instance, if C is a $(n+1)$-truncated k-crossed complex? In each case, there are also two excellent questions to ask. Firstly, are there 'interesting' examples (for various values of 'interesting')? and secondly, what does it all mean?

9.5 Simplicial approaches to HQFTs.

Before we ask for answers in more detail, we will look at some of the simplicial aspects of all this. We can adapt ideas from our treatment of (relative)simplicially generated TQFTs to give a common generalisation of the Brightwell-Turner classification theorem and the complementary classification of $HCobord(1, K(G, 1))$ in terms of G-graded crossed algebras, which we have not yet seen in any detail, so, in the next few sections, we will look at some of the ways in which the intuitions gained from the Yetter approach to TQFTs can be applied to HQFTs, initially just in low dimensions. Later we will discuss the general cases. (The treatment given is adapted from [140, 142]. We may repeat some of the points from our treatment of the Yetter construction for convenience.)

9.5.1 Background

As we have seen, in the construction of models for topological quantum field theories, one can use a (finite) group G, and a triangulation of the manifolds, Σ, etc., involved, and one assigns labels from G to each (oriented) edge of each (oriented) triangle, for example in the diagram below.
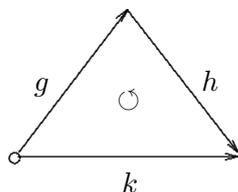
with the boundary/cocycle condition that $kh^{-1}g^{-1} = 1$, so $k = gh$.

(Here the orientation is given as anticlockwise, which may seem unnatural given the ordering, but this is necessary as we are using the 'path order convention' on composition of labels on edges. The other convention also leads to some inelegance at times.)

The geometric intuition behind this is that 'integrating' the labels around the triangle yields the identity. This intuition corresponds to problems where a G-bundle on Σ is specified by charts and the elements g, h, k, etc. are transition automorphisms of the fibre. The methods then use manipulations of the pictures as the triangulation is changed by subdivision, etc.

Another closely related view of this is to consider continuous functions, $f : \Sigma \to BG$, to the classifying space of G. If we triangulate Σ, we can assume that f is a cellular map using a suitable cellular model of BG and at the cost of replacing f by a homotopic map and perhaps subdividing the triangulation. From this perspective the previous model is a combinatorial description of such a continuous 'characteristic' map, f. The edges of the triangulation pick up group elements since the end points of each edge get mapped to the base point of BG, and $\pi_1 BG \cong G$, whilst the faces

give a realisation of the cocycle condition. Likewise we can use a labelled decomposition of the objects as CW-complexes, cf. [112, 156].

Let B be a CW-complex model for a 2-type (so $\pi_k B$ is trivial for $k > 2$). Assume it is reduced, so has a single vertex, then, denoting by B_1, the 1-skeleton of B, the crossed module, $(\pi_2(B, B_1), \pi_1(B_1), \partial)$, will represent the 2-type of B . For any B-manifold, the characteristic map, $g : \Sigma \to B$, or for a B-cobordism, $F : M \to B$, can be replaced, up to homotopy, by a cellular map, so, in general, we can think of a combinatorial model for the B-manifolds and B-cobordisms, in terms of combining labelled triangles with $g, h, k \in \pi_1(B_1)$ and $c \in \pi_2(B, B_1)$, and where the
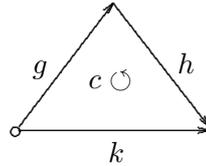
cocycle condition is replaced by a boundary condition

$$\partial c = kh^{-1}g^{-1}.$$

Usually $\pi_1(B_1)$ will be free and it will be useful to replace this particular crossed module by a general one.

The approach that we will explore is via 'formal' HQFTs. These can be seen as being analogous to combinatorial or lattice approaches to TQFTs, and thus, via some duality, to state-sum approaches. They rely initially on triangulations, but subsequently on cell decompositions of the manifolds and cobordisms, which, of course, then have to be shown not to influence the theories unduly.

9.5.2   Formal C maps, circuits and cobordisms

From now on, we fix a crossed module, $C = (C, P, \partial)$, as given. Our formal C-maps will initially be introduced via C-labelled / coloured triangles, as above, but will then be replaced by a cellular version as soon as the basic results are established confirming some basic intuitions. The labelled triangles, tetrahedra, etc., will all need a base point as a 'start vertex'. The need for this can be seen in an elementary way as follows: if we have the situation below, we get the boundary condition
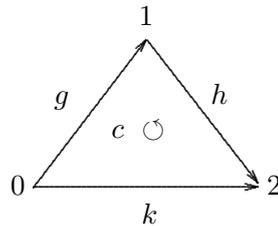
$\partial c = kh^{-1}g^{-1}$, which was read off starting at vertex 0: first k, back along h giving h^{-1}, then the same for g giving g^{-1}. The element c is assigned to this 2-simplex with this ordering / orientation, but if we tried to read off the boundary starting at vertex 1, we would get $g^{-1}kh^{-1}$, which is not ∂c, but is $\partial(^{g^{-1}}c)$. We thus have that the P-action on C is precisely encoding the change of starting vertex.

Remark: Our simplices will have a marked vertex to enable the boundary condition, and later on a cocycle condition, to be read off unambiguously. We could equally well work with a pair of marked vertices corresponding to 'start' and 'finish' or 'source' and 'target'. For triangles this would give, for instance, the above with start at 0 and finish at 2, and would give a boundary condition read off as $k = \partial c \cdot gh$. This can lead to a 2-categorical formulation of formal C-maps, which is connected with the way in which a crossed module C is equivalent to a strict 2-group.

Formal C maps are a combinatorial and algebraic model of the characteristic maps, mirroring in many ways the use of colourings in the Yetter approach to generation of TQFTs, and as our main initial use of formal C-maps will be in low dimensions, we will first describe them for closed 1-manifolds, then for surfaces, etc.

Let C_n denote an oriented n-circuit, that is, a triangulated oriented circle with n-edges and a choice of start-vertex. A *formal* C-*map* on C_n is a sequence of elements of P, $\mathbf{g} = (g_1, \ldots, g_n)$, thought of as labelling the edges in turn. We will also call this a *formal* C-*circuit*. Two formal C-circuits will be isomorphic if there is a simplicial isomorphism between the underlying circuits preserving the orientation and labelling.

If S is a closed 1-manifold, it will be a k-fold disjoint union of circles and an oriented triangulation of S gives a family of n-circuits for varying n. A *formal* C-*map* on S will be a family of formal C-maps on the various C_ns. (This includes the empty family as an instance where S is the empty 1-manifold.) As with the construction of TQFTs, it will be technically useful to have chosen an ordering of the vertices in any 1-manifold or, later, cobordism/triangulated surface. This ordering may be a total order, in which case it can be used to replace the orientation, but a partial order in which the vertices of each simplex and equally the base points of components, are totally ordered, will suffice. With such an order on the vertices of a 1-manifold, we have that a formal C-map on it is able to be written as an *ordered* family of formal C-circuits, that is, a list of lists of elements of P. Of course, the end result depends on that order and care must be taken with this, just as care needs to be taken with the order of the constituent spaces in a vector product decomposition - and for the same reasons.

Given two formal C-maps $\mathbf{g}$ on S_1, $\mathbf{h}$ on S_2, we can take their disjoint union to obtain a C-map $\mathbf{g} \sqcup \mathbf{h}$ on $S_1 \sqcup S_2$. We note that $\mathbf{g} \sqcup \mathbf{h}$ and $\mathbf{h} \sqcup \mathbf{g}$ are not identical, merely 'isomorphic', via an action of the symmetric group of suitable order, but, of course, this can be handled in the usual ways, depending to some extent on taste, for instance via the standard technical machinery of symmetric monoidal categories.

Now let M be an oriented (triangulated) cobordism between two such 1-manifolds S_0 and S_1, and suppose given formal C-maps, $\mathbf{g}_0$, $\mathbf{g}_1$, on S_0 and S_1 respectively. A *formal* C-*map*, $\mathbf{F}$, on M consists of a family of elements $\{c_t\}$ of C, indexed by the triangles t of M, a family, $\{p_e\}$ of elements of P indexed by the edges of M and for each t, a choice of base vertex, $b(t)$, such that the boundary condition below is satisfied:
in any triangle t,
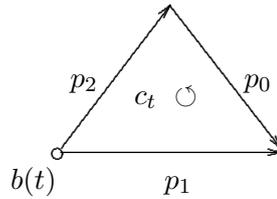
we have

$$\partial c_t = p_1 p_0^{-1} p_2^{-1}.$$

We call such a formal C-map on M a *formal C-cobordism* from $(S_0, \mathbf{g}_0)$ to $(S_1, \mathbf{g}_1)$ if it restricts to these formal C-maps on the boundary 1-manifolds. We will denote it $(M, \mathbf{F})$.

To be able to handle manipulation of formal C-cobordisms 'up to equivalence', so as to be able to absorb choices of triangulation, base vertices, etc. and eventually to pass to regular cellular decompositions, we need to consider triangulations of 3-dimensional simplicial complexes and formal C-maps on these. We, in fact, can use a common generalisation to all simplicial complexes.

9.5.3   Simplicial formal maps and cobordisms

We make things a bit more abstract (and 'formal'!)

Definition: Let K be a simplicial complex. A *(simplicial) formal C-map*, λ, on K consists of families of elements
(i) $\{c_t\}$ of C, indexed by the set, K_2, of 2-simplices of K,
(ii) $\{p_e\}$ of P, indexed by the set of 1-simplices, K_1, of K,
and a partial order on the vertices of K, so that each simplex is totally ordered. The assignments of c_t and p_e, etc. are to satisfy
(a) the boundary condition

$$\partial c_t = p_1 p_0^{-1} p_2^{-1},$$

where the vertices of t, labelled v_0, v_1, v_2 in order, determine the numbering of the opposite edges, e.g., e_0 is between v_1 and v_2, and p_{e_i} is abbreviated to p_i;
and
(b) the cocycle condition:
in a tetrahedron yielding two composite faces
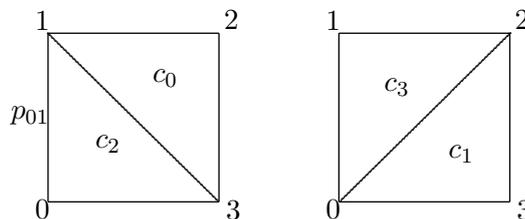
we have

$$c_2{}^{p_{01}} c_0 = c_1 c_3.$$

Explanation of the cocycle condition: The left hand and right hand sides of the cocycle condition have the same boundary, namely the boundary of the square, so $c_2{}^{p_{01}}c_0(c_1c_3)^{-1}$ is a cycle. There is no reason for this to be non trivial, so we should expect it to be 1. A crossed module has elements in dimensions 1 and 2, but nothing in dimension 3, therefore, just as the case where $B = BG$ for G, a group led to a cocycle condition in dimension 2, so when labelling with elements of a crossed module, we should expect the cocycle condition to be a 'tetrahedral equation', hence in dimension 3.

When 'integrating' a labelling over a surface corresponding to three faces of a tetrahedron, the composite label is on the remaining face, so given a formal C-map on the tetrahedron, and a specification of p_{01}, any one of $c_0, \ldots, c_3$ is determined by the others. (For example, if all but c_0 are given, then

$$^{p_{01}}c_0 = c_2^{-1}c_1c_3,$$

and then acting throughout with p_{01}^{-1} yields c_0.)

A third related view is that coming from the homotopy addition lemma, [37], which loosely says that any one face of an n-simplex is a (suitably defined) composite of the others.

Remark: For the moment, we will restrict attention to 1+1 HQFTs and to formal C-maps on 1-manifolds, surfaces and 3-manifolds. If a higher dimensional theory was being considered based on B manifolds of dimension d, the cocycle condition would naturally occur in dimension $d + 2$. In that case, the natural coefficients would be in one of the higher dimensional analogues of a crossed module such as crossed complexes, truncated hypercrossed complexes (or, equivalently, simplicial groups). We will look at this later.

9.5.4 Equivalence of formal C-maps

Suppose X is a polyhedron with a given family of base points, $\mathbf{m} = \{m_i\}$, and K_0, K_1 are two triangulations of X, i.e., K_0 and K_1 are simplicial complexes with geometric realisations homeomorphic to X (by specified homeomorphisms) with the given base points among the vertices of the triangulation.

Definition: Given two formal C-maps, (K_0, λ_0), (K_1, λ_1), then we say they are *equivalent* if there is a triangulation, T, of $X \times I$ extending K_0 and K_1 on $X \times \{0\}$ and $X \times \{1\}$ respectively, and a formal C-map, Λ, on T extending the given ones on the two ends and respecting the base points, in the sense that T contains a subdivided $\{m_i\} \times I$ for each basepoint m_i and Λ assigns the identity element 1_P of P to each 1-simplex of $\{m_i\} \times I$.

We will use the term 'ordered simplicial complex' for a simplicial complex, K, together with a partial order on its set of vertices such that the vertices in any simplex of K form a totally ordered set. If we give the unit interval, I, the obvious structure of an ordered simplicial complex with $0 < 1$, then the cylinder $|K| \times I$ has a canonical triangulation as an ordered simplicial complex and we will write $K \times I$ for this.

If we are given two formal C-maps defined on the same ordered K, (K, λ_0), and (K, λ_1), we say they are *simplicially homotopic* if there is a formal C-map defined on the ordered simplicial complex $K \times I$ extending them both.

The following is fairly easy to prove.

Lemma 46 *Equivalence is an equivalence relation.*   ∎

Equivalence combines the intuition of the geometry of triangulating a (topological) homotopy, where the triangulations of the two ends may differ, with some idea of a combinatorially defined simplicial homotopy of formal maps. We have:

Lemma 47 *If (K, λ_0), and (K, λ_1) are two formal maps, which are simplicially homotopic as formal C-maps, then they are equivalent.*   ∎

There are several possible proofs of the following result. The one in [142] is amongst the longer ones as it illustrates the processes of combination of labellings of simplices given by a formal C-map by explicitly constructing the required extension. (As it is quite long we will **leave it to you** to check up on, after hopefully **attempting to give a sketch proof yourself**.)

Proposition 60 *Given a simplicial complex, K, with geometric realisation $X = |K|$, and a subdivision K' of K.*
(a) Suppose λ is a formal C-map on K, then there is a formal C-map, λ' on K' equivalent to λ.
(b) Suppose λ' is a formal C-map on K', then there is a formal C-map, λ on K equivalent to λ'. ∎

Remarks: (i) To help with the understanding of what needs to be done, we can see it as a series of nested inductions 'up the skeleton' of various parts of the structure. To handle higher dimensions, we continue that process only handling $\sigma \in K_n$ when all its faces have been done, then using inverse induction and a join formulation of the triangulation, which is easy to see for the case $n = 2$.

(ii) There is a simplicial set formulation of the above in terms of the Kan complex condition on the simplicial nerve of C. This is useful for the extensions of this theory to higher dimensions and we will develop them shortly.

(iii) Remember that the idea of a formal C-map is to represent, combinatorially, the characteristic map of a B-manifold or B-cobordism, and from this perspective, equivalent formal maps will correspond to homotopic characteristic maps.

Proposition 61 *A change of partial order on the vertices of K or a change in choice of start vertices for simplices, generates an equivalent formal C-map.*

Proof: More formally, let K_0 be K with the given order and K_1 the same simplicial complex with a new ordering. Construct a triangulation T of $|K| \times I$ having K_0 and K_1 on the two ends. (Inductively, we can suppose just one pair of elements has been transposed in the order.) It is now easy to extend any given λ_0 on K_0 over T and then to restrict to get an equivalent λ_1 on K_1.   ∎

Note if $\langle v_0, v_1 \rangle$ is an ordered edge of K_0 and, with the reordering, $\langle v_1, v_0 \rangle$ is the corresponding one in K_1, then if λ_0 assigns p to $\langle v_0, v_1 \rangle$, λ_1 assigns p^{-1} to $\langle v_1, v_0 \rangle$ as is clear for the simplest assignment scheme:

$$
\begin{array}{ccc}
(v_0, 1) & \xleftarrow{\;p^{-1}\;} & (v_1, 1) \\
\Big\uparrow{\scriptstyle 1_P} & \nwarrow{\scriptstyle p^{-1}} & \Big\uparrow{\scriptstyle 1_P} \\
(v_0, 0) & \xrightarrow[\;p\;]{} & (v_1, 0)
\end{array}
$$

(The triangulation T assumes here that vertices of $|K| \times \{1\}$ are always listed after those of $|K| \times \{0\}$.) A similar, but more complex, observation is valid for higher dimensional simplices. Once the use of the boundary and cocycle conditions is understood, the choice of local ordering within the triangulation easily determines the simplest choice of extension. That extension can be perturbed or deformed by changing the choice of fillers for the 2-simplices in the faces of the prisms however.

9.5.5 Cellular formal C-maps

We can use the cocycle condition to combine formal C-data given locally on simplices into cellular blocks, up to equivalence. Combining simplices provides a simplification process which allows us to replace triangulated manifolds by manifolds with a given regular cellular decomposition. These are much easier to handle and given what we have discussed before the definitions and some of the proofs just 'fall out'. We still will need base points in each 1-manifold and start vertices in each cell.

Assume given a regular CW-complex, X, having, for each cell, a specified 'start 0-cell' among which are a set of distinguished base points. Assume further that each cell has a specified orientation.

Definition: A *cellular formal* C-*map* λ on X consists of families of elements
(i) $\{c_f\}$ of C indexed by the 2-cells, f, of X, and
(ii) $\{p_e\}$ of P indexed by the 1-cells, e, of X such that
   a) the boundary condition
$$\partial c_f = \text{the ordered product of the edge labels of } f$$
is satisfied;
and
   b) the cocycle condition is satisfied for each 3-cell.
   (In words b) gives, for each 3-cell σ, that the product of the labels on the boundary cells of σ is trivial.)

For a connected 1-manifold, S, decomposed as a CW-complex, (thus a subdivided circle), there is no difference from the simplicial description we had before. We have notions of formal C-circuit given by a sequence of elements of P and, more generally, if S is not connected, we have a list of such formal C-circuits.

A *cellular formal* C-*cobordism* between cellular formal C-maps is the obvious thing. It is a cellular cobordism between the underlying 1-manifolds endowed with a formal C-map that agrees with the two given C-maps on the two ends of the cobordism. Here the important ingredient is the cocycle condition and before going further we will say something more about both this and the boundary condition.

The algebraic-combinatorial description of the cellular version formal C-map is less explicitly given above than for the simplicial version as a full description would require the introduction of some additional detail, but this is not essential for the *intuitive* development of the ideas. We will, however, briefly sketch this extra theory. (This is probably not needed on a first reading.)

Recall the following ideas from earlier in the notes:

- *Crossed complex*: (cf. Section 2.1) The main example for us is the crossed complex of X, a CW-complex as above. This has $C_n = \pi_n(X_n, X_{n-1}, \mathbf{x})$, X_n, being as usual, the n-skeleton

of X and with ∂ the usual boundary map. Here we really need the many-object /groupoid version working with the multiple base points $\mathbf{x}$, but we will omit the detailed changes to the basic idea. We write $\pi(\mathbf{X})$ for this crossed complex.

- *Free crossed module*: (cf. Section 1.2.2 for the single vertex case.) The case of a 2-dimensional CW-complex, X, is of some importance for our theory as the B-cobordisms will be surfaces and hence 2-dimensional regular CW-complexes once a decomposition is given. Any such 2-dimensional CW-complex yields a *free* crossed module,

$$\pi_2(X_2, X_1, X_0) \to \pi_1(X_1, X_0)$$

with $\pi_1(X_1, X_0)$, the fundamental groupoid of the 1-skeleton X_1 of X based at the set of vertices X_0 of X. Each 2-cell of X gives a generating element in $\pi_2(X_2, X_1, X_0)$ and the assignment of the data for a cellular formal C-map satisfying the boundary condition, is equivalent to specifying a morphism, λ, of crossed modules,

$$
\begin{array}{ccc}
\pi_2(X_2, X_1, X_0) & \xrightarrow{\ \partial\ } & \pi_1(X_1, X_0) \ . \\
{\scriptstyle \lambda_2}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\
C & \xrightarrow[\ \partial\ ]{} & P
\end{array}
$$

The boundary condition just states $\lambda_1 \partial = \partial \lambda_2$.

- *Free crossed complex*: (See Brown-Higgins-Sivera, [37] for a detailed discussion of the concept.) The idea of free crossed complex is an extension of the above and $\pi(\mathbf{X})$ is free on the cells of X. (In particular, $C_3 = \pi_3(X_3, X_2, \mathbf{x})$ is a collection of free $\pi_1(X)$-modules over the various basepoints. The generating set is the set of 3-cells of X.)

A formal C-map, λ, is equivalent to a morphism of crossed complexes,

$$\lambda : \pi(\mathbf{X}) \to \mathsf{C},$$

or, expanding this, to

$$
\begin{array}{ccccc}
\longrightarrow \pi_3(X_3, X_2, X_0) & \xrightarrow{\ \partial\ } & \pi_2(X_2, X_1, X_0) & \xrightarrow{\ \partial\ } & \pi_1(X_1, X_0) \ . \\
{\scriptstyle \lambda_3}\downarrow & & {\scriptstyle \lambda_2}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\
\longrightarrow 1 & \xrightarrow[\ \partial\ ]{} & C & \xrightarrow[\ \partial\ ]{} & P
\end{array}
$$

Each 3-cell gives an element of $\pi_3(X_3, X_2, X_0)$. More exactly, if σ is a 3-cell of X, then it can be specified by a characteristic map $\varphi_\sigma : (B^3, S^2, \mathbf{s}) \to (X_3, X_2, X_0)$ and thus we get an induced crossed complex morphism, which in the crucial dimensions gives

$$
\begin{array}{ccccc}
\longrightarrow \pi_3(B^3, S^2, \mathbf{s}) & \xrightarrow{\ \partial\ } & \pi_2(S^2, \varphi_\sigma^{-1}(X_1), \mathbf{s}) & \xrightarrow{\ \partial\ } & \pi_1(\varphi_\sigma^{-1}(X_1), \mathbf{s}) \ . \\
{\scriptstyle \varphi_{\sigma,3}}\downarrow & & {\scriptstyle \varphi_{\sigma,2}}\downarrow & & \downarrow{\scriptstyle \varphi_{\sigma,1}} \\
\longrightarrow \pi_3(X_3, X_2, X_0) & \xrightarrow{\ \partial\ } & \pi_2(X_2, X_1, X_0) & \xrightarrow{\ \partial\ } & \pi_1(X_1, X_0) \\
{\scriptstyle \lambda_3}\downarrow & & {\scriptstyle \lambda_2}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\
\longrightarrow 1 & \xrightarrow[\ \partial\ ]{} & C & \xrightarrow[\ \partial\ ]{} & P
\end{array}
$$

We have $\pi_3(B^3, S^2, \mathbf{s})$ is generated by the class of the 3-cell, $\langle e^3 \rangle$ and $\varphi_{\sigma,3}(\langle e^3 \rangle) = \langle \sigma \rangle$. The cocycle condition is then explicitly given by $\lambda_2 \partial \langle \sigma \rangle = 1$.

The explicit *combinatorial* form of the cocycle condition for σ will depend on the decomposition of the boundary, S^2, given by $\varphi_\sigma^{-1}(X_1)$. (This type of argument was first introduced in the original paper by J. H. C. Whitehead, [164]. It can also be found in the forthcoming book by Brown and Sivera, [37], work by Brown and Higgins, [34, 36] and by Baues, [16, 17], where, however, crossed complexes are called *crossed chain complexes*.) Our use of this cocycle condition does not require such a detailed description so we will not attempt to give one here.

The next ingredient is to cellularise 'equivalence'. We can do this for arbitrary formal C-maps specialising to 1- or 2-dimensions (cobordisms) afterwards. We use a regular cellular decomposition of the space $X \times I$, with possibly different regular CW-complex decompositions on the two ends, but with the base points 'fixed' so that $\mathbf{x} \times I$ is a subcomplex of $X \times I$.

Definition: Given cellular formal C-maps, λ_i, on $X_i = X \times \{i\}$, for $i = 0, 1$, they will be *equivalent* if there is a cellular formal C-map, Λ, on a cellular decomposition of $X \times I$ extending λ_0 and λ_1 and assigning 1_p to each edge in $\mathbf{x} \times I$.

Again equivalence is an equivalence relation. It allows the combination and collection processes examined in the previous subsection to be made precise. In other words:

- *if we triangulate each cell of a CW-complex, X, in such a way that the result gives a triangulation, K, of the space, then a formal C-map, λ, on K determines a cellular formal C-map on X;*

- *equivalent simplicial formal C-maps on (possibly different) such triangulations yield equivalent formal C-maps on X;*

- *given any cellular formal C-map, μ, on X and a triangulation, K, of X subdividing the cells of X, there is a simplicial formal C-map on K that combines to give μ.*

Remarks: (i) Full proofs of these would use cellular and simplicial decompositions of $X \times I$, but would also need the introduction of far more of the theory of crossed modules, crossed complexes and their classifying spaces than we have available in these notes. Because of that, the proofs are omitted here in order to make this introduction to formal C-maps easier to approach.

(ii) Any simplicial formal C-map on K is, of course, a cellular one for the obvious regular CW-structure on $|K|$.

The notion of equivalent cellular formal C-cobordisms can now be formulated. Given the obvious set-up with $\mathbf{F}$ and $\mathbf{G}$, two such cobordisms between $\mathbf{g_1}$ and $\mathbf{g_2}$, they will be equivalent if they are equivalent as formal C-maps by an equivalence that is constant on the two 'ends'.

9.5.6 2-dimensional formal C-maps.

Now that we have cellular descriptions, it is easy to describe a set of 'building blocks' for all cellular formal C-maps on orientable surfaces and thus all cobordisms between 1-dimensional formal C-maps.

Again we want to emphasise the fact that these models provide *formal combinatorial models* for the characteristic maps with target a 2-type.

We will shortly introduce the formal version of 1+1 HQFTs with a 'background' crossed module, C, which is a model for a 2-type, B, represented by that crossed module. As the basic manifolds are 1-dimensional, they are just disjoint unions of pointed oriented circles, and so a formal C-map on a 1-manifold, as we saw earlier (page 283), is specified by a list of lists of elements in P, one list for each connected component. Cellularly we can assume that the lists have just one element in them, obtained from the simplicial case by multiplying the elements in the list together in order. The corresponding cellular cobordisms are then compact oriented surfaces, W, with pointed oriented boundary endowed with a formal C-map Λ as above. Since such surfaces can be built up from three basic models, the disc, annulus and disc with two holes (pair of trousers), we need only examine what formal C-maps look like on these basic example spaces and how they compose and combine, as any formal $1 + 1$ 'C-HQFT' will be determined completely by its behaviour on the formal maps on these basic surfaces.

Formal C-Discs: The only formal C-maps that makes sense on the disc must have an element $c \in C$ assigned to the interior 2-cell with the boundary ∂c assigned to the single 1-cell, i.e.,
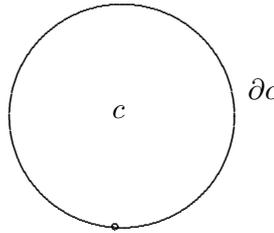
$$Disc(c) : \emptyset \to \partial\mathbf{c}$$

Later we will see that these give the crucial difference between the formal C-theory and the standard form of [156, 158].

Formal C-Annuli: Let Cyl denote the cylinder/annulus, $S^1 \times [0, 1]$. We fix an orientation of Cyl once and for all, and set $Cyl^0 = S^1 \times (0) \subset \partial Cyl$ and $Cyl^1 = S^1 \times (1) \subset \partial Cyl$. We provide Cyl^0 and Cyl^1 with base points $z^0 = (s, 0)$, $z^1 = (s, 1)$, respectively, where $s \in S^1$. As in [156, 158], let $\varepsilon, \mu = \pm$, and denote by $Cyl_{\varepsilon,\mu}$ the triple $(Cyl, Cyl^0_\varepsilon, Cyl^1_\mu)$. This is an annulus with oriented pointed boundary,

$$\partial Cyl_{\varepsilon,\mu} = (\varepsilon Cyl^0_\varepsilon) \cup (\mu Cyl^1_\mu),$$

where, by $-X$, we mean X with opposite orientation. A formal C-map, Λ, on Cyl can be drawn diagrammatically as:
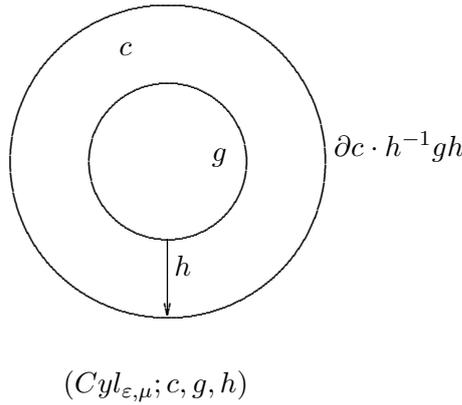
$$(Cyl_{\varepsilon,\mu}; c, g, h)$$

with initial vertex, s, for the 2-cell at the start of h. This diagram will represent the cobordism that we will denote $(Cyl_{\varepsilon,\mu}; c, g, h)$. Similar notation may be used in other contexts without further comment. (We omit the orientations on the boundary circles, so as to avoid the need to repeat more or less the same diagram several times.)

The loop, $\Lambda|_{Cyl^1_\mu}$, represents $(\partial c \cdot h^{-1}gh)$ or its inverse depending on the sign of μ. There are two special cases that generate all the others: (i) $c = 1$, which corresponds to the case already handled in [156], and (ii) $h = 1$, where the base point s does not move during the cobordism. The general case, illustrated in the figure, is the composite of particular instances of the two cases.
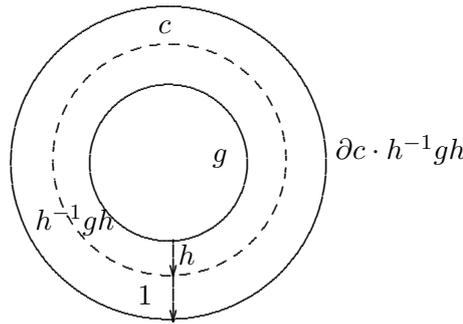
Combination of cobordisms is more or less obvious, so we will not give details.

Formal C-Disc with 2 holes: Let D be an oriented 2-disc with two holes. We will denote the boundary components of D for convenience by Y, Z, and T and provide them with base points y, z and t respectively. For any choice of signs ε, μ, $\nu = \pm$, we denote by $D_{\varepsilon,\mu,\nu}$ the tuple $(D, Y_\varepsilon, Z_\mu, T_\nu)$. This is a 2-disc with two holes with oriented pointed boundary. By definition,

$$\partial D_{\varepsilon,\mu,\nu} = (\varepsilon Y_\varepsilon) \cup (\mu Z_\mu) \cup (\nu T_\nu).$$

Finally we fix two proper embedded arcs, yt and zt, in D leading from y and z to t. A formal C-map λ on $D_{\varepsilon,\mu,\nu}$ will, in general, assign elements of P to each boundary component and to each arc. As for the annulus we may assume that the formal map assigns 1_P to both yt and zt, as the general case can be generated by this one together with cylinders. In addition, the single 2-cell will be assigned an element c of C.

$$\partial c \cdot g_1 \cdot g_2$$

c

g_1 g_2

y z

t

$$(D_{\varepsilon,\mu,\nu}; c, g_1, g_2)$$
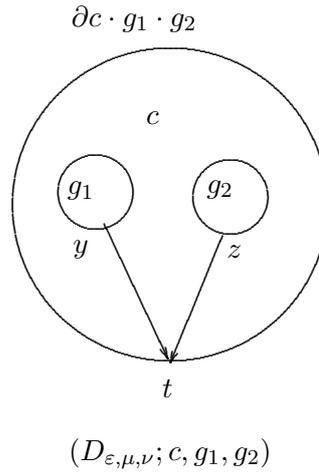
This situation leads to an interesting relation. If we have a formal C-map on $D_{\varepsilon,\mu,\nu}$ in which, for simplicity, we assume that the 2-cell is assigned the element 1_C and then add suitable cylinders, labelled with c_1 and c_2 respectively, to the boundary components, Y and Z, then the resulting cobordism can be rearranged to give a labelling with the 2-cell coloured $c_1 \cdot {}^{g_1} c_2$ as shown in the following diagram:

$$\partial c_1 \cdot g_1 \cdot \partial c_2 \cdot g_2 \qquad\qquad \partial c_1 \cdot g_1 \cdot \partial c_2 \cdot g_2$$

c_1 c_2

g_1 g_2

$\sim$

$c_1 \cdot {}^{g_1} c_2$

g_1 g_2
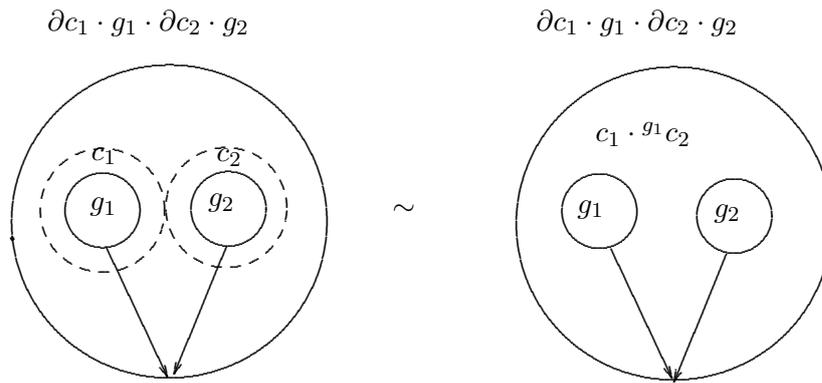
Figure 9.5.6 : From the case $c = 1$ to the general one.

The importance of this element $c_1 \cdot {}^{g_1} c_2$ is that it is the C-part of the product of the two cylinder labels in the semidirect product, $C \rtimes P$, more exactly, the elements (c_1, g_1) and $(c_2, g_2) \in C \rtimes P$ correspond to the two added cylinders and within that semi-direct product $(c_1, g_1) \cdot (c_2, g_2) = (c_1 \cdot {}^{g_1} c_2, g_1 g_2)$.

9.6   Formal HQFTs

To start with we will restrict attention to modelling 1+1 HQFTs and so, here, will give a definition of a formal HQFT only for that case. First we introduce some notation.

If we have formal C-cobordisms,

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \to \mathbf{g}_2,$$

then we will denote the composite C-cobordism by $\mathbf{F} \#_{\mathbf{g}_1} \mathbf{G}$.

For $\mathbf{g}$ as before, the trivial identity C-cobordism on $\mathbf{g}$ will be denoted $1_{\mathbf{g}}$.

9.6.1 The definition

Fix, as before, a crossed module, $\mathsf{C} = (C, P, \partial)$, and also fix a ground field, $\Bbbk$.

Definition: A *formal HQFT* with background C assigns

- to each formal C-circuit, $\mathbf{g} = (g_1, \ldots, g_n)$, a $\Bbbk$-vector space, $\tau(\mathbf{g})$, and by extension, to each formal C-map on a 1-manifold S, given by a list, $\mathbf{g} = \{\mathbf{g}_i \,|\, i = 1, 2, \ldots, m\}$ of formal C-circuits, a vector space $\tau(\mathbf{g})$ and an identification,

$$\tau(\mathbf{g}) = \bigotimes_{i=1,\ldots,m} \tau(\mathbf{g}_i),$$

  giving $\tau(\mathbf{g})$ as a tensor product;

- to any formal C-cobordism, $(M, \mathbf{F})$ between $(S_0, \mathbf{g}_0)$ and $(S_1, \mathbf{g}_1)$, a $\Bbbk$-linear transformation

$$\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_1).$$

These assignments are to satisfy the following axioms:

(i) Disjoint union of formal C-maps corresponds to tensor product of the corresponding vector spaces via specified isomorphisms:

$$\tau(\mathbf{g} \sqcup \mathbf{h}) \overset{\cong}{\to} \tau(\mathbf{g}) \otimes \tau(\mathbf{h}),$$

$$\tau(\emptyset) \overset{\cong}{\to} \Bbbk$$

for the ground field $\Bbbk$, so that a) the diagram of specified isomorphisms

$$
\begin{array}{ccc}
\tau(\mathbf{g}) & \overset{\cong}{\longrightarrow} & \tau(\mathbf{g} \sqcup \emptyset) \\
\cong \downarrow & & \downarrow \cong \\
\tau(\mathbf{g}) \otimes \Bbbk & \underset{\cong}{\longleftarrow} & \tau(\mathbf{g}) \otimes \tau(\emptyset)
\end{array}
$$

for $\mathbf{g} \to \emptyset \sqcup \mathbf{g}$, commutes and similarly for $\mathbf{g} \to \mathbf{g} \sqcup \emptyset$, and b) the assignments are compatible with the associativity isomorphisms for $\sqcup$ and $\otimes$, (so that τ satisfies the usual axioms for a symmetric monoidal functor).

(ii) For formal C-cobordisms

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \to \mathbf{g}_2$$

with composite $\mathbf{F} \#_{\mathbf{g}_1} \mathbf{G}$, we have

$$\tau(\mathbf{F} \#_{\mathbf{g}_1} \mathbf{G}) = \tau(\mathbf{G})\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_2).$$

(iii) For the identity formal C-cobordism on $\mathbf{g}$,

$$\tau(1_{\mathbf{g}}) = 1_{\tau(\mathbf{g})}.$$

(iv) Interaction of cobordisms and disjoint union is transformed correctly by τ, i.e., given formal C-cobordisms

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{h}_0 \to \mathbf{h}_1$$

the following diagram

$$
\begin{array}{ccc}
\tau(\mathbf{g}_0 \sqcup \mathbf{h}_0) & \xrightarrow{\ \cong\ } & \tau(\mathbf{g}_0) \otimes \tau(\mathbf{h}_0) \\
{\scriptstyle \tau(\mathbf{F} \sqcup \mathbf{G})} \downarrow & & \downarrow {\scriptstyle \tau(\mathbf{F}) \otimes \tau(\mathbf{G})} \\
\tau(\mathbf{g}_1 \sqcup \mathbf{h}_1) & \xrightarrow[\ \cong\ ]{} & \tau(\mathbf{g}_1) \otimes \tau(\mathbf{h}_1)
\end{array}
$$

commutes, compatibly with the associativity structure.

9.6.2   Basic Structure

Formal C-maps can be specified by composing / combining the basic building blocks outlined in section 9.5.6. As a formal 1+1 HQFT transforms the formal C-maps to vector space structure, to specify a formal HQFT, we need only give it on the connected 1-manifolds, thus on formal C-circuits, and on the building blocks mentioned before. We assume τ is a formal HQFT with C as before.

On a formal C-circuit, $\mathbf{g} = (g_1, \ldots, g_n)$, we can assume $n = 1$, since the obvious formal C-cobordism between $\mathbf{g}$ and $\{(g_1 \ldots g_n)\}$, based on the cylinder yields, an isomorphism

$$\tau(\mathbf{g}) \xrightarrow{\ \cong\ } \tau(g_1 \ldots g_n),$$

and, consequently, a decomposition of $\tau(g_1 \ldots g_n)$ as a tensor product. For any element $g \in P$, we thus have the formal C-circuit $\{(g)\}$ and a vector space $\tau(g)$. In fact, for later use it will be convenient to change notation to L_g (or, for the more general case, $L_{\mathbf{g}}$).

For a general $\mathbf{g} = (g_1, \ldots, g_n)$, we now have

$$L_{\mathbf{g}} = \bigotimes_{i=1}^{n} L_{g_i}.$$

The special case when $\mathbf{g}$ is empty gives $L_{\emptyset} = \Bbbk$ and the isomorphisms in section 9.13.2 and here, above, are compatible with these assignments.

The basic formal C-cobordisms give us various structural maps:

- the formal C-disc with $c \in C$ gives

$$\tau(Disc(c)) : \tau(\emptyset) \to \tau(\partial c),$$

that is, a linear map, which we will write as

$$\ell_c : \Bbbk \to L_{\partial c}.$$

(We write $\tilde{c} := \ell_c(1) \in L_{\partial c}$.)

- the formal C-annuli of the two basic types yield
  (a) $(Cyl_{\varepsilon,\mu}; 1, g, h) : \{(g)\} \to \{(h^{-1}gh)\}$, and hence a linear isomorphism

$$L_g \to L_{h^{-1}gh}$$

  (cf. [156]), or a related one, depending on the sign of μ;
  or
  (b) $(Cyl_{\varepsilon,\mu}; c, g, 1) : \{(g)\} \to \{(\partial c \cdot g)\}$ and a linear isomorphism,

$$L_g \to L_{\partial c \cdot g},$$

  again with variants for other signs.

- the formal C-disc with 2 holes,

$$(D_{\varepsilon,\mu,\nu}; c, g_1, g_2) : \{(g_1), (g_2)\} \to \{(\partial c \cdot g_1 \cdot g_2)\},$$

  giving a bilinear map,

$$L_{g_1} \otimes L_{g_2} \to L_{\partial c \cdot g_1 \cdot g_2}.$$

  Again, the key case is $c = 1$, and, consequently,

$$L_{g_1} \otimes L_{g_2} \to L_{g_1 g_2}.$$

  The general case can be obtained from that and a suitable formal C-annulus,

$$L_{g_1} \otimes L_{g_2} \to L_{g_1 g_2} \to L_{\partial c \cdot g_1 g_2}.$$

  This can be done, as here, by adding the annulus after the 'pair of pants' or adding it on the first component. The two formal C-cobordisms are equivalent.

  We can, of course, reverse the orientation to get

$$L_{g_1 g_2} \to L_{g_1} \otimes L_{g_2},$$

  a 'comultiplication'. It is fairly standard that this comultiplication is 'redundant', as it can be recovered from the annuli and a suitable 'positive pair of pants', see, for instance, the brief argument given in section 5.1 of [156].

The treatment here is designed to mirror Turaev's discussion in [156], which handles the case when B is a $K(G, 1)$. That way we can see that the passage from that case, here, with 'background' a 1-type, $B = K(P, 1)$ to the model for a general 2-type, $B = BC$, merely requires the addition of extra linear isomorphisms $,L_g \to L_{\partial c \cdot g}$ for c in the top group of the crossed module, C. This structure is therefore very similar to that of a π-algebra (cf. [156]) for $\pi = P/\partial C$.

9.7 Crossed C-algebras: first steps

In [156], Turaev classified (1+1)-HQFTs with background a $K(G, 1)$ in terms of *crossed group-algebras*. These were generalisations of classical group algebras with many of the same features, but 'twisted' by an action. In [27], M. Brightwell and P. Turner examined the analogous case when the background is a $K(A, 2)$ for A an Abelian group, and classified them in terms of A-Frobenius algebras, that is, Frobenius algebras with a A-action. In this section we will look in some detail at both these types of algebra before introducing the more general type, 'crossed C-algebras', which combines features of both and which will classify formal HQFTs as above.

9.7.1   Crossed G-algebras

Here G will be a group corresponding to $\pi_1(B)$ if B is a 1-type.

Definition: A *graded G-algebra* or *G-algebra* over a field $\Bbbk$ is an associative algebra, L, over $\Bbbk$ with a decomposition,

$$L = \bigoplus_{g \in G} L_g,$$

as a direct sum of projective $\Bbbk$-modules of *finite* type such that
(i) $L_g L_h \subseteq L_{gh}$ for any $g, h \in G$ (so, if ℓ_1 is graded g, and ℓ_2 is graded h, then $\ell_1 \ell_2$ is graded gh), and
(ii) L has a unit $1 = 1_L \in L_1$ for 1, the identity element of G.

Example: (i) The group algebra, $\Bbbk[G]$, has an obvious G-algebra structure in which each summand of the decomposition is free of dimension 1.
(ii) For any associatve $\Bbbk$-algebra, A, the algebra, $A[G] = A \otimes_{\Bbbk} \Bbbk[G]$ is also G-algebra. Multiplication in $A[G]$ is given by $(ag)(bh) = (ab)(gh)$ for $a, b \in A$, $g, h \in G$, in the obvious notation.

Definition: A *Frobenius G-algebra* is a G-algebra, L, together with a symmetric $\Bbbk$-bilinear form,

$$\rho : L \otimes L \to \Bbbk$$

such that
(i) $\rho(L_g \otimes L_h) = 0$ if $h \neq g^{-1}$;
(ii) the restriction of ρ to $L_g \otimes L_{g^{-1}}$ is non-degenerate for each $g \in G$;
and
(iii) $\rho(ab, c) = \rho(a, bc)$ for any $a, b, c \in L$.

We note that (ii) implies that $L_{g^{-1}} \cong L_g^*$, the dual of L_g.

Example continued: The group algebra, $L = \Bbbk[G]$, is a Frobenius G-algebra with $\rho(g, h) = 1$ if $gh = 1$, and 0 otherwise, and then extending linearly. (Here we write g both for the element of G labelling the summand L_g and the basis element generating that summand.)

Finally the notion of crossed G-algebra combines the above with an action of G on L, explicitly:

Definition: A *crossed G-algebra* over $\Bbbk$ is a Frobenius G-algebra over $\Bbbk$ together with a group homomorphism,

$$\varphi : G \to Aut(L)$$

satisfying:
(i) if $g \in G$ and we write $\varphi_g = \varphi(g)$ for the corresponding automorphism of L, then φ_g preserves ρ, (i.e. $\rho(\varphi_g a, \varphi_g b) = \rho(a, b)$) and

$$\varphi_g(L_h) \subseteq L_{ghg^{-1}}$$

for all $h \in G$;
(ii) $\varphi_g|_{L_g} = id$ for all $g \in G$;

(iii) for any $g, h \in G$, $a \in L_g$, $b \in L_h$, $\varphi_h(a)b = ba$;

(iv) for any $g, h \in G$ and $c \in L_{ghg^{-1}h^{-1}}$,

$$Tr(c\varphi_h : L_g \to L_g) = Tr(\varphi_{g^{-1}}c : L_h \to L_h),$$

where Tr denotes the $\Bbbk$-valued trace of the endomorphism. (The homomorphism $c\varphi_h$ sends $a \in L_g$ to $c\varphi_h(a) \in L_g$, whilst $(\varphi_{g^{-1}}c)(b) = \varphi_{g^{-1}}(cb)$ for $c \in L_h$.)

Example: concluded The group algebra, $L = \Bbbk[G]$, is a crossed G-algebra with $\varphi_g(h) = hgh^{-1}$, and extended linearly.

9.7.2 Morphisms of crossed G-algebras

We clearly need to have a notion of morphism of crossed G-algebras. We start with a fixed group, G.

Definition: Suppose L and L' are two crossed G-algebras. A $\Bbbk$-algebra morphism, $\theta : L \to L'$, is a *morphism of crossed G-algebras* if it is compatible with the extra structure. Explicitly:

$$\begin{aligned}
\theta(L_g) &\subseteq L'_g \\
\rho'(\theta a, \theta b) &= \rho(a, b), \\
\varphi'_g(\theta a) &= \theta(\varphi_g(a)),
\end{aligned}$$

for all $a, b \in L$, $g \in G$, where, when necessary, primes indicate the structure in L'.

We will also need a version of this relative to a 'change of groups'. Suppose that $f : G \to H$ is a homomorphism of groups, that L is a crossed G-algebra, and L', a crossed H-algebra and let $\theta : L \to L'$ be a $\Bbbk$-algebra homomorphism.

Definition: We say that θ is *compatible* with f, or is a *morphism of crossed algebras over f*, if

$$\begin{aligned}
\theta(L_g) &\subseteq L'_{f(g)} \\
\rho'(\theta a, \theta b) &= \rho(a, b), \\
\varphi'_{f(g)}(\theta a) &= \theta(\varphi_g(a)),
\end{aligned}$$

for all $a, b \in L$, $g \in G$, where primes indicate the structure in L'.

We will use this definition shortly when looking at pullbacks and related construction. For the moment, we just record that there will be category, $Crossed.G-Alg$, of crossed G-algebras and the corresponding morphisms and a larger category consisting of all crossed algebras (over any group), denoted $Crossed.Alg$, and morphisms over group homomorphisms. As might be expected, there is a functor from $Crossed.Alg$ to the category of groups, and we would also expect this to form a fibered category in such a way that the fibre over a group G would, of course, be $Crossed.G-Alg$. We will investigate this slightly later.

Returning to the category $Crossed.G-Alg$ itself, this category is a groupoid as all morphisms are isomorphisms. (If **you looked at the proof** of the corresponding fact for TQFTs then the proof is more or less obvious, as it is 'the same'.)

9.8 Constructions on crossed G-algebras

Before looking at the G-Frobenius algebras that model the case of $1 + 1$-dimensional HQFTs where the background is a $K(G, 2)$, (and, therefore, where G must be Abelian), we should look at analogues of some of the constructions on HQFTs that we saw earlier, but this time on the crossed algebras that we have just introduced.

9.8.1 Cohomological crossed G-algebras

(The sources for this material are Turaev's book, [158], and his earlier paper, [156].)

The cohomological crossed algebras mentioned in the title of this section are the analogues of the cohomological HQFTs that we met in section 9.3.1.

We let $\theta = \{\theta_{g,h} \in \Bbbk^*\}$ be a normalised 2-cocycle for G, representing a cohomology class $[\theta] \in H^2(G, \Bbbk^*)$. Recall that this means that $\theta : G \times G \to \Bbbk^*$, and we are writing $\theta_{g,h}$ for $\theta(g, h)$. That θ is a 2-cocycle means

$$\theta_{g,h}\theta_{gh,k} = \theta_{g,hk}\theta_{h,k}$$

for all triples, g, h, k, of elements of G and that θ is normalised means that $\theta_{1,1} = 1$.

Define a G-graded algebra, $L = L^\theta$, as follows:

for $g \in G$, L_g is a free $\Bbbk$-module of rank 1, with a generating vector denoted ℓ_g, so $L_g = (k)\ell_g$.

This resembles the group algebra, $\Bbbk[G]$, in which the multiplication would be given on generators by $\ell_g \ell_h = \ell_{gh}$ and extended linearly. Here we twist this using the 2-cocycle. We take in L^θ, the product to be defined on basis elements by

$$\ell_g \ell_h = \theta_{g,h} \ell_{gh}$$

and then, of course, extend linearly.

Associativity of the multiplication is exactly the cocycle condition. It is easy to check also that ℓ_1 is the 1 of L^θ.

Lemma 48 *Cohomologous 2-cocycles determine isomorphic graded G-algebras.*   ∎

The proof is a routine manipulation.

This lemma says that the isomorphism class of L^θ only depends on $[\theta] \in H^2(G, \Bbbk^*)$.

Proposition 62 *The graded G-algebra, L^θ, is a Frobenius G-algebra.*

Proof: We define $\rho : L \otimes L \to \Bbbk$ by $\rho(\ell_g, \ell_h) = 0$ unless $g = h^{-1}$, and $\rho(\ell_g, \ell_{g^{-1}}) = \theta_{g,g^{-1}}$. The verification that this satisfies (ii) and (iii) of the definition of a Frobenius G-algebra is **left as an exercise**. (It can be found in [158].)   ∎

Finally we want to show that L^θ is a crossed G-algebra, so we need to give or find a $\varphi : G \to Aut(L^\theta)$ satisfying the axioms. We first note that the multiplication

$$L_h \otimes L_g \to L_{hg}$$

is clearly an isomorphism of $\Bbbk$-modules and, of course, equally well $L_{hgh^{-1}} \otimes L_h \to L_{hg}$ is one. This means that $\ell_h \ell_g$ must also be $\varphi_h(\ell_g)\ell_h$ for some *unique* $\varphi_h(\ell_g) \in L_{hgh^{-1}}$, and we use this to define φ_h, extending linearly as usual. This φ_h is an automorphism of L^θ.

Proposition 63 $L^\theta = (L, \eta, \rho)$ *is a crossed G-algebra.* ■

The proof is quite long, as it has to verify a fair number of conditions, but it is not that difficult. It is therefore omitted. It can be found in Turaev, [158], p. 26 - 27.

Again the isomorphism class of L^θ only depends on the cohomology class of θ in $H^2(G, \Bbbk^*)$. We also note that $L^{\theta_1 + \theta_2} \cong L^{\theta_1} \otimes L^{\theta_2}$ and that if $\theta = 0$ then the resulting crossed algebra is the group algebra, $\Bbbk[G]$.

9.8.2   Pulling back a crossed G-algebra

A morphism, as above in section 9.7.2, over a group morphism, $f : G \to H$, can be replaced by a morphism of crossed G-algebras, $L \to f^*(L')$, where $f^*(L')$ is obtained by pulling back L' along f.

If $f : G \to H$ is a group homomorphism, given a crossed H-algebra, L, we can obtain a crossed G-algebra, $f^*(L)$, by pulling back using f. The structure of $f_0^*(L)$ is given by:

- $(f^*(L))_g$ is $L_{f(g)}$, by which we mean that $(f^*(L))_g$ is a copy of $L_{f(g)}$ with grade g, and we note that, if $x \in L_{f(g)}$, it can be useful to write it $x_{f(g)}$ with x_g denoting the corresponding element of $(f^*(L))_g$;

- if x and y have matching grades, g and g^{-1}, respectively, so $x \in (f^*(L))_g$, then $\rho(x, y)$ is the same as in $L_{f(g)}$, but if x and y have non-matching grades, then $\rho(x, y) = 0$;

- $\varphi_{g'}(x_p) := \varphi_{f(g')}(x_{f(g)})$.

We have:

Proposition 64 *The algebra, $f^*(L)$, has a crossed G-algebra structure given by the above.* ■

The construction of $f^*(L)$, then, makes it clear that, if $f : G \to H$, L is a crossed G-algebra, and L', a crossed H-algebra, we have:

Proposition 65 *There is a bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed G-algebra morphisms from L to $f^*(L')$.* ■

(Both the proofs are fairly routine, so **just check up that they make sense and try to sketch some details**.) As with most such operations, this pullback construction gives a functor from the category of crossed H-algebras to that of crossed G-algebras.

Example: If $f : G \to H$ is the inclusion of a subgroup, then the operation of pulling back along f corresponds to restricting the crossed H-algebra to G.

9.8.3   Pushing forward crossed G-algebras

We have shown that, given $f : G \to H$ and a $\theta : L \to L'$ over f, we can pull back L' over G to get a map from L to $f^*(L')$ that encodes almost the same information as L'. (The exception to this is if f is not an epimorphism, as then outside the image of f, we do not retain information on the 'fibres' of L'.) The obvious question to ask is whether there is some 'adjoint' push-forward construction with θ corresponding to some morphism from $f_*(L)$ to L' over H.

CAUTIONARY NOTE: Quite often in this sort of construction, there may have to be finiteness conditions imposed. This is usually due to the finite type conditions on the summands of the type of graded algebras being considered. Other conditions may also be needed on f. We are not guaranteeing that this adjoint exists for 'any old' f!

To start with we will look at the overall situation, with f and θ, as set out above, but we note that, if there is an adjoint f_* construction, we should expect θ to factorise via $f_*(L)$ and to have an image in L'.

Given this context, first set $N = Ker f$, then we have that, for $n \in N$, and $a \in L_g$,

$$\varphi_n(a) - a \in Ker\,\theta,$$

as $\theta(\varphi_n(a)) = \varphi'_{f(n)}(\theta(a)) = \varphi'_1(\theta(a)) = \theta(a)$. We therefore form the ideal, K, generated by elements of this form, $\varphi_n(a) - a$, and note that it will be in the kernel of *any* θ, however this is not a G-graded ideal, but that, in fact, is exactly what we need. We have that L/K is an associative algebra, and we will have to give it a H-graded algebra structure. We note that, in order to get an H-grading and an action, we have to kill off the action of N on L and, of course, this is exactly what quotienting by K does.

There has to be a universal morphism (over f) from L to the conjectured $f_*(L)$, and this must be compatible with the grading. This more or less forces one to look at the following:

For each $h \in H$, let
$$\mathsf{L}_h = \oplus\{L_g \mid g \in G, f(g) = h\},$$

and
$$\mathsf{L} = \oplus_{h \in H}\mathsf{L}_h.$$

This makes sense from the grading point of view, but we should note that it only works if N is finite, or more exactly, if the interaction of L with N only involves finitely many non-zero summands of L, as otherwise the vector space, or $\Bbbk$-module, L_h, may not be of finite type. Of course, for a given h, there may be no g satisfying $f(g) = h$, in which case that L_h will be trivial.

Let, now,
$$\mathsf{K}_h = \mathsf{L}_h \cap \mathsf{K}.$$

The underlying H-graded vector space of $f_*(L)$ will be
$$f_*(L) = \oplus_{h \in H}\mathsf{L}_h/\mathsf{K}_h.$$

This is an associative algebra, as it is exactly L/K, but we have to check that this grading will be compatible with that multiplication.

Suppose $a + \mathsf{K} \in f_*(L)_{h_1}$, and $b + \mathsf{K} \in f_*(L)_{h_2}$, then $a \in L_{g_1}$ and $b \in L_{g_2}$ for some $g_1, g_2 \in G$ with $f(g_i) = h_i$, for $i = 1, 2$, but then $ab + \mathsf{K} \in f_*(L)_{h_1 h_2}$ as required.

We next define the bilinear form giving the inner product. Clearly, with the same notation,

$$\rho(a + \mathsf{K}, b + \mathsf{K}) := 0 \quad \text{if } h_1 \neq h_2^{-1}. \tag{9.1}$$

If $h_1 = h_2^{-1}$, then we can assume that $g_1 = g_2^{-1}$, and, after changing the element b representing $b + \mathsf{K}$ if necessary, that $b \in L_{g_2}$. Finally we set

$$\rho(a + \mathsf{K}, b + \mathsf{K}) := \rho(a, b).$$

This is easily seen to be independent of the choices of a and b, since, once we have a suitable pair (a, b) with $a \in L_{g_1}$ and $b \in L_{g_1^{-1}}$, any other will be related by isometries induced by composites of φs. Clearly ρ, thus defined, is a symmetric bilinear form and, on restricting to $f_*(L)_{h_1} \otimes f_*(L)_{h_2}$, it is essentially the original inner product restricted to $L_{g_1} \otimes L_{g_2}$, so is non-degenerate and satisfies

$$\rho(ab + \mathsf{K}, c + \mathsf{K}) = \rho(a + \mathsf{K}, bc + \mathsf{K}).$$

The next structure to check is the crossed H-algebra action

$$\varphi : H \to Aut(f_*(L)).$$

The obvious formula to try is

$$\varphi_h(a + \mathsf{K}) := \varphi_g(a) + \mathsf{K}$$

where $f(g) = h$. It is easy to reduce the proof that this is well defined to checking independence of the choice of g, but, if g' is another element of $f^{-1}(h)$, then $g' = ng$ for some $n \in N$ and $\varphi_{g'}(a) = \varphi_n \varphi_g(a) \equiv_\mathsf{K} \varphi_g(a)$, so the action is well defined. Of course, this definition will give us immediately that the $\varphi_h(a)b = ba$ axiom holds and that $\varphi_h|_{f_*(L)_h} = id$, etc.

The trace axiom follows somewhat similarly via a fairly routine calculation.

Proposition 66 *With the above structure, $f_*(L)$ is a crossed H-algebra.* ■

Now return to the morphism, θ over f. It is clear that as θ killed K, then θ induces a unique morphism $\bar\theta : f_*(L) \to L'$ over H. We thus have

Proposition 67 *If f has finite kernel, there is a bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed H-algebra morphisms from $f_*(L)$ to L'.* ■

Of course, f_* is a functor from $Crossed.G - Alg$ to $Crossed.H - Alg$.

As a corollary of the proposition and the earlier result on f^*, we have:

Corollary 19 *If f has finite kernel, then f_* is left adjoint to f^*.* ■

Example: A neat example of this construction occurs when in addition to being finite, N is central. Taking $L = \Bbbk[G]$, then we have seen that L is a crossed G-algebra with L_g generated by some single element also labelled g. With this for $n \in N$, φ_n is the identity automorphism, and L becomes a crossed H-algebra without any further bother. (The construction in this case is given in Chapter II of [158].)

Another instance occurs when f is a monomorphism as, of course, its kernel is then finite! In that context, we can sometimes say more.

9.8.4 Algebraic transfer

For a change of groups, $f : G \to H$, we thus have a pullback functor $f_* : Crossed.H - Alg \to Crossed.G - Alg$, and, if f has a finite kernel, this functor has a 'pushforward', that is, a right adjoint. In geometric situation, induced pullback functors such as f_* sometimes have left adjoints as well. Conditions on f for such a left adjoint to exist might, at a guess, involve a 'cofiniteness' condition and the 'cokernel' of f, although cokernels are not that simple to use if the groups involved

are non-Abelian. This suggested situation is more-or-less the case, as f_* has a left adjoint when f is a monomorphism, i.e., essentially an inclusion of a subgroup, which is 'cofinite' in as much as it is of finite index. This sort of situation, of course, corresponds to transfer at the geometric level. (Recall the context, from section 9.3.3, where the inclusion of G into H corresponds to a finite sheeted covering space of $K(H, 1)$.)

We set up the algebraic analogue of this as follows:

Let H be a group and $G < H$, a subgroup of finite index, $n = [H : G]$.

9.8.5 G-Frobenius algebras

The prime sources for this are the two papers, [27, 148]. In what follows in this section, G will denote an Abelian group. We have already defined a Frobenius object in a symmetric monoidal category, $\mathcal{A}$, (see page 208).

Definition: A G-*Frobenius object* in $\mathcal{A}$ is a Frobenius object, A, together with a homomorphism,

$$G \to End(A).$$

Definition: When $\mathcal{A} = (Vect, \otimes)$, or $(ModR, \otimes)$, the corresponding concept is that of a G-*Frobenius algebra.*

Examination of the action shows that, if we write $g \cdot a$ for the action of g on an element $a \in A$,

$$a(g \cdot b) = g \cdot (ab) = (g \cdot a)b$$

and

$$\rho(a, g \cdot b) = \rho(g \cdot a, b).$$

As A is a unital algebra

$$g \cdot v = g \cdot 1v = (g \cdot 1)v,$$

so the action actually comes from a morphism of monoids

$$G \to A$$

$$g \to g \cdot 1,$$

and $g \cdot 1$ is in the center of A.

Recall that this is the variant of a Frobenuis algebra adapted for the classification and characterisation of 2-dimensional HQFTs with background a simply connected space.

9.8.6   Crossed C-algebras

We now turn to the general case with $\mathsf{C} = (C, P, \partial)$, as earlier. The additional structure is thus that given by the annuli or cylinders $(Cyl_{\epsilon,\mu}; c, g_1, g_2)$. We saw earlier that this collection of operations could be reduced further to the case $g_2 = 1$ and $c \neq 1$, and, in fact, the only ones that we actually

need are with $g_1 = 1$ as well, as the general case is a composite of this with the unit on the left and the 'pair of pants' multiplication. (The general case gives an isomorphism

$$\theta_{(c,g)} : L_g \to L_{\partial c \cdot g}$$

and we can build this up by

$$L_g \to \Bbbk \otimes L_g \to L_1 \otimes L_g \overset{\theta_{(c,1)} \otimes L_g}{\to} L_{\partial c} \otimes L_g \overset{\mu}{\to} L_{\partial c \cdot g},$$

where the third morphism is that given by that special case $g = 1$. We say that $\theta_{(c,g)}$ is obtained by 'translation' from $\theta_{(c,1)}$.)

The extra structure can be thought of a collection of isomorphisms,

$$\Theta_{\mathsf{C}} = \{\theta_{(c,1)} : L_1 \to L_{\partial c}\}.$$

It is worth noting that if $C = \{1\}$, the resulting structure reduces to that of a crossed P-algebra and, if $P = 1$ and C is just an Abelian group, then the $\theta_{(c,1)} : L_1 \to L_1$ are just automorphisms of L_1, which is itself just a Frobenius algebra.

This structure of extra specified automorphisms does not immediately tell us how to retrieve the structure given by the C-discs. Those gave linear maps,

$$\ell_c : \Bbbk \to L_{\partial c}.$$

We can, however, recover them from $\ell_1 : \Bbbk \to L_1$, which was part of the crossed P-algebra structure, together with $\theta_{(c,1)} : L_1 \to L_{\partial c}$, but, conversely, given the ℓ_c, we can recover the $\theta_{(c,g)}$:

Proposition 68
The composite

$$L_g \overset{\cong}{\to} \Bbbk \otimes L_g \overset{\ell_c \otimes L_g}{\to} L_{\partial c} \otimes L_g \overset{\mu}{\to} L_{\partial c \cdot g}$$

is equal to $\theta_{(c,g)}$.

Proof
We can realise this composite by a C-cobordism
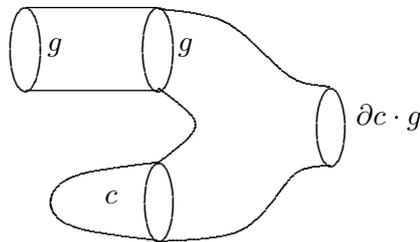
but this is equivalent to the C-annulus that gives us $\theta_{(c,g)}$. ∎

As before we will write $\tilde{c} = \ell_c(1) \in L_{\partial c}$.

Corollary 20
For any $c \in C$, $g \in P$ and for $x \in L_g$,

$$\theta_{(c,g)}(x) = \tilde{c} \cdot x,$$

where $\cdot$ denotes the product in the algebra structure of $L = \bigoplus_{h \in P} L_h$. ∎

Abstracting this extra structure, we get:

Definition: Let $\mathsf{C} = (C, P, \partial)$ be a crossed module. A *crossed C-algebra* consists of a crossed P-algebra, $L = \bigoplus_{g \in P} L_g$, together with elements $\tilde{c} \in L_{\partial c}$, for $c \in C$, such that

(a) $\tilde{1} = 1 \in L_1$;

(b) for $c, c' \in C$, $\widetilde{(c'c)} = \tilde{c}' \cdot \tilde{c}$;

(c) for any $h \in P$, $\varphi_h(\tilde{c}) = \widetilde{{}^h c}$.

We note for future use that the first two conditions make 'tilderisation' into a group homomorphism $(\tilde{\ }) : C \to U(L)$, the group of units of the algebra, L.

The two special cases with (i) $C = 1$ and, (ii) for Abelian C, with $P = 1$, correspond, of course, to crossed P-algebras and C-Frobenius algebras respectively. An interesting special case of the general form is when C is a P-module and ∂ sends every element in C to the identity of P. In this case, we have an object that could be described as a C-crossed P-algebra! It consists of a crossed P-algebra together with a C-action by multiplication by central elements. This results in a very weak mixing of the two structures. The important thing to note is that the general form is more highly structured as the twisting in the crossed modules, in general, can result in non-central elements amongst the $\tilde{c}$s.

9.9 A classification theorem for formal C-HQFTs

9.9.1 Main Theorem

Theorem 25 *There is a canonical bijection between isomorphism classes of formal 2-dimensional HQFTs based on a crossed module, C, and isomorphism classes of crossed C-algebras.*

More explicitly:

Theorem 26 *a) For any formal 2-dimensional HQFT, τ, based on C, the crossed P-algebra, $L = \bigoplus_{g \in P} L_g$, having $L_g = \tau(g)$, is a crossed C-algebra, where for $c \in C$, $\tilde{c} = \ell_c(1)$ (notation as above). b) Given any crossed C-algebra, $L = \bigoplus_{g \in P} L_g$, there is a formal 2-dimensional HQFT, τ, based on C yielding L as its crossed C-algebra, up to isomorphism.*

9.9.2 Proof of Main Theorem

We start by identifying the geometric behaviour of the isomorphisms, $\theta_{(c,g)}$. We know that, from the special case of $C = 1$, L is a crossed P-algebra, so we need to look at the extra structure:

- **Vertical composition of C-cobordisms.**

  The basic condition is that the composite,

  $$L_g \xrightarrow{\theta_{(c,g)}} L_{\partial c \cdot g} \xrightarrow{\theta_{(c',\partial c \cdot g)}} L_{\partial(c'c) \cdot g},$$

  is $\theta_{(c'c,g)}$:

  $$\boxed{\theta_{(c'c,g)} = \theta_{(c',\partial c \cdot g)} \circ \theta_{(c,g)} : L_g \to L_{\partial(c'c) \cdot g}}$$

  since τ must be compatible with the 'vertical composition' of C-cobordisms. Evaluating this on an element gives

  $$\widetilde{(c'c)} = \tilde{c}' \cdot \tilde{c},$$

  where $\tilde{c}' \cdot \tilde{c} = \mu(\tilde{c}', \tilde{c})$. Similarly $\tilde{1} = 1$.

- **'Horizontal' composition of C-cobordisms.** Using the interchange law/Peiffer rule and the 'pair of pants' to give the multiplication, we have

  $$(1, g_1) \#_0 (c, g_2) = ({}^{g_1}c, g_1 g_2) = ({}^{g_1}c, g_1) \#_0 (1, g_2).$$

  (Here the useful notation $\#_0$ corresponds to the horizontal composition in the associated strict 2-group of C.) We thus have two composite C-cobordisms giving the same result, and hence,

  $$L_{g_1} \otimes L_{g_2} \xrightarrow{L_{g_1} \otimes \theta_{(c,g_2)}} L_{g_1} \otimes L_{\partial c \cdot g_2} \xrightarrow{\mu} L_{g_1 \cdot \partial c \cdot g_2} = L_{g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{g_1 g_2} \xrightarrow{\theta_{({}^{g_1}c, g_2)}} L_{g_1 \cdot \partial c \cdot g_2}.$$

  In general, for $d \in C$, the second type of composite will be

  $$L_{g_1} \otimes L_{g_2} \xrightarrow{\theta_{(d,g_1)} \otimes L_{g_2}} L_{\partial d \cdot g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{\partial d \cdot g_1 \cdot g_2} = L_{g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{g_1 g_2} \xrightarrow{\theta_{(d, g_1 g_2)}} L_{\partial d \cdot g_1 \cdot g_2},$$

  and we need this for $d = {}^{g_1}c$, for which the corresponding composite cobordisms are equal. Geometrically these rules correspond to a pair of pants with g_1, g_2 on the trouser cuffs and the 2-cell colored c. We can push c onto either leg, but in so doing may have to conjugate by g_1.

  Summarising, for given $c \in C$, $g_1, g_2 \in G$,

  $$\boxed{\mu(id_{L_{g_1}} \otimes \theta_{(c,g_2)}) = \theta_{({}^{g_1}c, g_1 g_2)} \circ \mu = \mu(\theta_{({}^{g_1}c, g_1)} \otimes id_{L_{g2}}).}$$

  As we have reduced $\#_0$ to 'whiskering' and the vertical composition, $\#_1$, and have already checked the interpretation of $\#_1$, we might expect this pair of equations to follow from our earlier calculations, however we have invoked here the interchange law and that was not used earlier. The above equations reduce to

  $$x \cdot \tilde{c} = \widetilde{{}^g c} \cdot x,$$

but this is implied by axiom c) of a crossed C-algebra, since we have

$$\widetilde{{}^g c} \cdot x = \varphi_h(\tilde{c})x = x \cdot \tilde{c},$$

using the third axiom (page 297) of the crossed P-algebra structure on L. Thus the combination of these two rules corresponds, in part, to the Interchange Law. Conversely this rule in either form is clearly implied by the axioms for a formal HQFT.)

To complete the proof, we would have to check that the inner product structure of L and action of P via φ are compatible with the new structure. The compatibility of the isomorphisms $\theta_{(c,g)}$ defined via the $\tilde{c}$ will follow both from the geometry of the HQFT and from the axioms of crossed C-algebras. These parts of the proof are similar to the parts we have already given, so are **left to you** either to try to prove yourselves, or to look up in [142].

The formal details of the reconstruction of τ from L follow the same pattern as for the case $C = 1$ and, for the most part, are exactly the same as the only extra feature is the 'tilde' operation. The details are left to you to adapt from Turaev's book, [158]. ∎

Remark: It is sometimes useful to have the extra rules of the $\tilde{c}$s written in the intermediate language of the family of isomorphisms,

$$\theta_{(c,g)} : L_g \to L_{\partial c \cdot g}.$$

The first two conditions are easily so interpreted and the last corresponds to the compositions given earlier and also to the equality of

$$L_g \xrightarrow{\theta_{(c,g)}} L_{\partial c \cdot g} \xrightarrow{\varphi_h} L_{h \cdot \partial c \cdot g \cdot h^{-1}},$$

and

$$L_g \xrightarrow{\varphi_h} L_{hgh^{-1}} \xrightarrow{\theta_{({}^h c, {}^h g)}} L_{h \partial c \cdot g \cdot h^{-1}},$$

and thus to

$$\boxed{\varphi_h \circ \theta_{(c,g)} = \theta_{({}^h c, {}^h g)} \circ \varphi_h.}$$

Collectively these boxed equations in their various forms give compatibility conditions for the various structures.

Algebraic interpretation: There is a very neat algebraic interpretation of these conditions. Let L be an associative algebra and $U(L)$ be its group of units. There is a homomorphism of groups $\delta = \delta_L : U(L) \to Aut(L)$ given by $\delta(u)(x) = u \cdot x \cdot u^{-1}$.

Lemma 49 *With the obvious action of $Aut(L)$ on the group of units, $(U(L), Aut(L), \delta)$ is a crossed module.* ∎

The proof is simple, although quite instructive, and will be **left to the reader**. We will denote this crossed module by $\mathfrak{Aut}(L)$. If L has extra structure such as being a Frobenius algebra or being graded, the result generalises to have the automorphisms respecting that structure.

Proposition 69 *Suppose that L is a crossed C-algebra. The diagram*

$$
\begin{array}{ccc}
C & \xrightarrow{\;(\tilde{\;})\;} & U(L) \\
{\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\delta} \\
P & \xrightarrow[\;\varphi\;]{} & Aut(L)
\end{array}
$$

is a morphism of crossed modules from C to $\mathfrak{Aut}(L)$.

Proof: First, we check commutativity of the square above. Let $c \in C$, going around clockwise gives $\delta(\tilde{c})$ and on an element $x \in L$, this gives $\tilde{c} \cdot x \cdot \tilde{c}^{-1}$. We compare this with the other composite, again acting on $x \in L$. If we multiply $\varphi_{\partial c}(x)$ by $\tilde{c}$, then we get $\varphi_{\partial c}(x)\tilde{c} = \tilde{c} \cdot x$, but therefore $\varphi_{\partial c}(x) = \tilde{c} \cdot x \cdot \tilde{c}^{-1}$ as well.

The other thing to check is that the maps are compatible with the actions of the bottom groups on the top ones, but this is exactly what the third condition on the 'tilde' gives. ∎

9.10 Constructions on formal HQFTs and crossed C-algebras.

As formal HQFTs correspond to crossed C-algebras by our main result above, the category of crossed C-algebras needs to be understood better if we are to understand the relationships between formal HQFTs. We clearly also need some examples of crossed C-algebras.

First we note that the usual constructions of direct sum and tensor product of graded algebras extend to crossed C-algebras in the obvious way.

9.10.1 Examples of crossed C-algebras

As usual we will fix a crossed module $\mathsf{C} = (C, P, \partial)$. We assume, for convenience, that $Ker\partial$ is a finite group, although this may not always be strictly necessary.

The group algebra $\Bbbk(C)$ as a crossed C-algebra: We take $L = \Bbbk(C)$ and will denote the generator corresponding to $c \in C$ by e_c rather than merely using the symbol c itself, as we will need a fair amount of precision when specifying various types of related elements in different settings. Define $L_p = \Bbbk\langle\{e_c : \partial c = p\}\rangle$, so, if $p \in P - \partial C$, this is the zero dimensional $\Bbbk$-vector space, otherwise it has dimension the order of $Ker\partial$ (whence our requirement that this be finite).

Lemma 50 *With this grading structure, L is a crossed P-algebra.*

Proof:

- L is P-graded: this follows since $e_c \cdot e_{c'} = e_{cc'}$, ∂ is a group homomorphism and $e_1 \in L_1$.

- There is an inner product:

$$\rho : L \otimes L \to \Bbbk$$

$$\rho(e_c \otimes e_{c'}) = \begin{cases} 0 & \text{if } c^{-1} \neq c' \\ 1 & \text{otherwise} \end{cases}$$

and this is clearly non-degenerate. Moreover

$$\rho(e_{c_1}e_{c_2} \otimes e_{c_3}) = \rho(e_{c_1 c_2} \otimes e_{c_3}) = 0$$

unless $c_3 = c_2^{-1} c_1^{-1}$ when it is 1, whilst

$$\rho(e_{c_1} \otimes e_{c_2 c_3}) = 0$$

unless $c_1^{-1} = c_2 c_3$, etc., so the inner product satisfies the third condition for a Frobenius P-algebra.

- Finally there is a group homomorphism

$$\varphi : P \to Aut(L),$$

  given by $\varphi_g(e_c) = e_{{}^g c}$, which permutes the basis, compatibly with the multiplication and innerproduct structures.

As $\partial({}^g c) = g \cdot \partial c \cdot g^{-1}$, φ clearly satisfies $\varphi_g(L_h) \subseteq L_{ghg^{-1}}$, and the Peiffer identity implies ${}^{\partial c} c = c$, so $\varphi_g | L_g$ is the identity. The Peiffer identity in general gives

$$^{\partial c} c' = c c' c^{-1},$$

so $e_c e_{c'} = e_{\partial c\, c'} e_c$, i.e., $\varphi_h(a) b = ba$ if $a \in L_g, b \in L_h$.   ∎

As we want this to be a crossed C-algebra, the remaining structure that we have to specify is the 'tildefication'

$$\tilde{\ } : C \to \Bbbk C.$$

The obvious mapping gives $\tilde{c} = e_c$, and, of course,

$$\delta(\tilde{c})(e_{c'}) = e_c e_{c'} e_{c^{-1}} = e_{cc'c^{-1}} = \varphi_{\partial c}(e_{c'}),$$

as above. We thus have

Proposition 70 *With the above structure, $\Bbbk(C)$ is a crossed C-algebra.*   ∎

By its construction $\Bbbk(C)$ records little of the structure of P itself, only the way the P-action permutes the elements of C, but, of course, it records C faithfully. The next example give another extreme.

The group algebra, $\Bbbk(P)$, as a crossed C-algebra: We first note the following result from [156]:

Lemma 51 *$\Bbbk(P)$ has the structure of a crossed P-algebra with $(\Bbbk(P))_p = \Bbbk e_p$, the subspace generated by the basis element labelled by $p \in P$.*   ∎

The one thing to note is that the axiom

$$\varphi_h(a) b = ba$$

for any $g, h \in P$, $a \in L_g, b \in L_h$ implies that

$$\varphi_h(e_g) = e_h e_g e_{h^{-1}} = e_{hgh^{-1}},$$

since e_h is a unit of $\Bbbk(P)$ with inverse $e_{h^{-1}}$.

Proposition 71
For $c \in C$, defining $\tilde{c} = e_{\partial c}$, gives $\Bbbk(P)$ the additional structure of a crossed C-*algebra.*

Proof: The grading is as expected and $\delta(\tilde{c}) = \varphi_{\partial c}$, by construction.   ■

Of course, $\Bbbk(P)$ does not encode anything about the kernel of $\partial : C \to P$. In fact, it basically remains a crossed P-algebra as the extra crossed C-structure is derived from that underlying algebra.

We will give further examples of crossed C-algebras shortly.

9.10.2 Morphisms of crossed algebras, for a crossed module background

We clearly need to have a notion of morphism of crossed C-algebras. We start with a fixed crossed module $\mathsf{C} = (C, P, \partial)$.

(This will need editing later to just add in the extra parts.)

Definition: Suppose L and L' are two crossed C-algebras. A $\Bbbk$-algebra morphism $\theta : L \to L'$ is a *morphism of crossed* C-*algebras* if it is compatible with the extra structure. Explicitly:

$$
\begin{aligned}
\theta(L_p) &\subseteq L'_p \\
\rho'(\theta a, \theta b) &= \rho(a, b), \\
\varphi'_h(\theta a) &= \theta(\varphi_h(a)), \\
\theta(\tilde{c}) &= \tilde{c}
\end{aligned}
$$

for all $a, b \in L$, $h \in P$, $c \in C$, where, when necessary, primes indicate the structure in L'.

We know that a given crossed module represents a homotopy 2-type, but that different crossed modules can give equivalent 2-types, so it will also be necessary to compare crossed algebras over different crossed modules. We need this not just to move within a 2-type, but for various constructions linking different 2-types. We therefore put forward the following definition. First some preliminary notation:
Suppose $f : \mathsf{C} \to \mathsf{D}$ is a morphism of crossed modules. The morphism f gives a commutative square of group homomorphisms

$$
\begin{array}{ccc}
C & \xrightarrow{f_1} & D \\
\partial \downarrow & & \downarrow \partial' \\
P & \xrightarrow{f_0} & Q.
\end{array}
$$

We want to define a morphism of crossed algebras over f, i.e., an algebra morphism, $\theta : L \to L'$, where L is a crossed C-algebra and L', a crossed D-algebra.

Definition: Suppose L and L' are two crossed algebras over C and D, respectively. A $\Bbbk$-algebra morphism $\theta : L \to L'$ is a *morphism of crossed algebras over f* if it is compatible with the extra

structure. Explicitly:

$$\theta(L_p) \subseteq L'_{f_0(p)} \tag{9.2}$$

$$\rho'(\theta a, \theta b) = \rho(a, b), \tag{9.3}$$

$$\varphi'_{f_0(h)}(\theta a) = \theta(\varphi_h(a)), \tag{9.4}$$

$$\theta(\tilde{c}) = \widetilde{f_1(c)} \tag{9.5}$$

for all $a, b \in L$, $h \in P$, $c \in C$, where, when necessary, primes indicate the structure in L'.

9.10.3 Pulling back a crossed C-algebra

A morphism, as above, over f can be replaced by a morphism of crossed C-algebras, $L \to f_0^*(L')$, where $f_0^*(L')$ is obtained by pulling back L' along f. We will consider this construction independently of any particular θ.

If $P \to Q$ is a group homomorphism, we know, from [156] that given a crossed Q-algebra, L, we obtain a crossed P-algebra $f_0^*(L)$, by pulling back using f_0. The structure of $f_0^*(L)$ is given by:

- $(f_0^*(L))_p$ is $L_{f_0(p)}$, by which we mean that $(f_0^*(L))_p$ is a copy of $L_{f_0(p)}$ with grade p and we note that if $x \in L_{f_0(p)}$, it can be useful to write it $x_{f_0(p)}$ with x_p denoting the corresponding element of $(f_0^*(L))_p$;

- if x and y have the same grade, say $x, y \in (f_0^*(L))_p$, then $\rho(x, y)$ is the same as in $L_{f_0(p)}$, but if x and y have different grades then $\rho(x, y) = 0$;

- $\varphi_h(x_p) := \varphi_{f_0(h)}(x_{f_0(p)})$.

If, in addition, we consider the crossed C-structure assuming that L' is a crossed D-algebra, then defining $\tilde{c} := \widetilde{f_1(c)}_{\partial c}$ gives us:

Proposition 72 *The crossed P-algebra $f_0^*(L)$ has a crossed C-algebra structure given by the above.*
∎

The construction of $f_0^*(L)$, then, makes it clear that

Proposition 73 *There is a bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed C-algebra morphisms from L to $f_0^*(L')$.*   ∎

Of course, as with most such operations, this pullback construction gives a functor from the category of crossed D-algebras to that of crossed C-algebras (up to isomorphism in the usual way).

9.10.4 Applications of pulling back

Consider our crossed module $C = (C, P, \partial)$ and let $G = P/\partial C$. We can realise this as a morphism of crossed modules:

$$
\begin{array}{ccc}
C & \longrightarrow & 1 \\
\partial \downarrow & & \downarrow \\
P & \xrightarrow{\ q\ } & G.
\end{array}
$$

If ∂ was an inclusion then this would be a weak equivalence of crossed modules as then both the kernel and cokernels of the crossed modules would be mapped isomorphically by the induced maps. In that case, thinking back to our original motivations for introducing formal C-maps, we would really be in a situation corresponding to a HQFT with background a $\Bbbk(G,1)$ and by [156], we know such theories are classified by crossed G-algebras. Thus it is of interest to see what the pullback algebra of a crossed G-algebra along this morphism will be. We will look at the obvious example of $\Bbbk(G)$, the group algebra of G with its usual crossed G-algebra structure (cf., [156]). We will assume that the crossed module, C, is finite.

Writing $N = \partial C$, for convenience, we have an extension

$$N \longrightarrow P \xrightarrow{\;q\;} G \; .$$

Pick a section s for q and define the corresponding cocycle $f(g,h) = s(g)s(h)s(gh)^{-1}$, so $f : G \times G \to N$ is naturally normalised, $f(1,h) = f(g,1) = 1$ and satisfies the cocycle condition:

$$f(g,h)f(gh,k) = {}^{s(g)}f(h,k)f(g,hk). \tag{9.6}$$

Take $L = \Bbbk(G)$, the group algebra of G considered with its crossed G-algebra structure and form the crossed P-algebra, $q^*(L)$. We will give a cohomological proof of the following to illustrate some of the links between cohomology and constructions on crossed algebras.

Proposition 74
The two crossed C-algebras $\Bbbk(P)$ and $q^*(\Bbbk(G))$ are isomorphic.

Proof
We first note that
$$q^*(L)_p = L_{q(p)} = \Bbbk e_{q(p)}.$$
We will write $g = q(p)$, so $p \in P$ has the form $p = ns(g)$. (We will need to keep check of which $e_{q(p)}$ is which and will later introduce notation which will handle this.)

Recall the description of the product in P in terms of the cocycle and the section:

$$
\begin{aligned}
n_1 s(g_1) \cdot n_2 s(g_2) &= n_1 {}^{s(g_1)} n_2 s(g_1) s(g_2) & (9.7)\\
&= (n_1 {}^{s(g_1)} n_2 f(g_1, g_2)) s(g_1 g_2). & (9.8)
\end{aligned}
$$

Each unit e_g of $\Bbbk(G)$ gives $\#(N)$ copies in $q^*(L)$. Write $(e_g)_n$ for the copy of e_g in $q^*(L)_{ns(g)}$ and examine the multiplication in $q^*(L)$ in this notation:

$$(e_{g_1})_{n_1} \cdot (e_{g_2})_{n_2} = (e_{g_1 g_2})_{(n_1 {}^{s(g_1)} n_2 f(g_1, g_2))}.$$

(That this gives an associative multiplication corresponds to the cocycle condition (9.6).)

We next have to ask : what is φ_p? Of course as $p = ns(g)$, we can restrict to examining φ_n and $\varphi_{s(g)}$.

- φ_n links the two copies $q^*(L)_p$ and $q^*(L)_{npn^{-1}}$ of $L_{q(p)}$ via what is essentially the identity map between the two copies;

- $\varphi_{s(g)}$ restricts to $\varphi_{s(g)} : q^*(L)_p \to q^*(L)_{s(g)ps(g)^{-1}}$, but on identifying these two subspaces as $L_{q(p)}$ and $L_{gq(p)g^{-1}}$, this is just φ_g.

In fact we can be more explicit if we look at the basic units and, as these do form a basis, behaviour on them determines the automorphisms:

$$\varphi_n((e_{g_1})_{n_1})(e_1)_n = (e_1)_n(e_{g_1})_{n_1},$$

so

$$
\begin{align}
\varphi_n((e_{g_1})_{n_1}) &= (e_{g_1})_{nn_1}(e_1)_n^{-1} \tag{9.9} \\
&= (e_{g_1})_{nn_1}(e_1)_{n^{-1}} \tag{9.10} \\
&= (e_{g_1})_{nn_1 s(g_1) n^{-1}} \tag{9.11}
\end{align}
$$

that is, conjugation by $(e_1)_n$.

This leads naturally on to noting that $\tilde{c} = (e_1)_{\partial c}$, so we have explicitly given the crossed C-algebra structure on $q^*(L)$. Sending e_p to $(e_{q(p)})_n$ (using the same notation as before) establishes the isomorphism of the statement without difficulty. ∎

Remark: In this identification of $q^*(\Bbbk(G))$ as $\Bbbk(P)$, it is worth noting that

$$q^*(L)_1 = L_1 = \Bbbk e_1 \cong \Bbbk,$$

as a vector space, but also that $q^*(L)_n \cong \Bbbk$ for each $n \in N$. The notation $(e_g)_n$ used and the behaviour of these basis elements suggests that $q^*(\Bbbk(P))$ behaves like some sort of twisted tensor product with basis $e_n \otimes e_g$, with that element corresponding to $(e_g)_n$, and with multiplication

$$(e_{n_1} \otimes e_{g_1})(e_{n_2} \otimes e_{g_2}) = (e_{n_1 s(g_1) n_2 f(g_1, g_2)} \otimes e_{g_1 g_2}).$$

We have not yet investigated how general this construction may be.

9.10.5　Pushing forward

We have shown that, given $f : \mathsf{C} \to \mathsf{D}$ and a $\theta : L \to L'$ over f, we can pull L' back over C to get a map from L to $f^*(L')$ that encodes the same information as L' (provided f is an epimorphism and all crossed modules are finite). The obvious question to ask is whether there is an 'adjoint' push-forward construction with θ corresponding to some morphism from $f_*(L)$ to L' over D. This is what we turn to next keeping the same assumptions of finiteness, etc.

Given such a context, setting, as before, $N = Ker f_0$, $B = Ker f_1$, we have

$$\boxed{\varphi_n(a) - a \in Ker\theta,} \tag{9.12}$$

as $\theta(\varphi_n(a)) = \varphi'_{f_0(n)}(\theta(a)) = \varphi'_1(\theta(a)) = \theta(a)$. Similarly, since

$$\theta(\tilde{c}) = \widetilde{f_1(c)},$$

if $b \in B = Ker f_1$,

$$\theta(\tilde{b}) = \tilde{1},$$

so

$$\boxed{\tilde{b} - 1 \in Ker\theta.} \tag{9.13}$$

We therefore form the ideal K generated by elements of these forms, (9.12) and (9.13). Note this is not a P-graded ideal, but that, in fact, is exactly what is needed. We have that L/K is an associative algebra and we give it a Q-graded algebra structure as follows.

For each $q \in Q$, let

$$\mathsf{L}_q = \oplus_p \{ L_p \mid f_0(p) = q \},$$

and

$$\mathsf{K}_q = \mathsf{L}_q \cap \mathsf{K}.$$

The underlying Q-graded vector space of $f_*(L)$ will be

$$f_*(L) = \oplus_{q \in Q} \mathsf{L}_q / \mathsf{K}_q.$$

This is an associative algebra as it is exactly L/K, but we have to check that this grading is compatible with that multiplication.

Suppose $a + \mathsf{K} \in f_*(L)_{q_1}$, and $b + \mathsf{K} \in f_*(L)_{q_2}$, then $a \in L_{p_1}$ and $b \in L_{p_2}$ for some $p_1, p_2 \in P$ with $f_0(p_i) = q_i$, for $i = 1, 2$, but then $ab + \mathsf{K} \in f_*(L)_{q_1 q_2}$ as required.

We next define the bilinear form giving the inner product. Clearly, with the same notation,

$$\rho(a + \mathsf{K}, b + \mathsf{K}) := 0 \quad \text{if } q_1 \neq q_2^{-1}. \tag{9.14}$$

If $q_1 = q_2^{-1}$, then we can assume that $p_1 = p_2^{-1}$, and, after changing the element b representing $b + \mathsf{K}$ if necessary, that $b \in L_{p_2}$. Finally we set

$$\rho(a + \mathsf{K}, b + \mathsf{K}) := \rho(a, b). \tag{9.15}$$

This is easily seen to be independent of the choices of a and b, since, once we have a suitable pair (a, b) with $a \in L_{p_1}$ and $b \in L_{p_1^{-1}}$, any other will be related by isometries induced by composites of φs and $\tilde{b}$s. Clearly ρ thus defined is a symmetric bilinear form and restricting to $f_*(L)_{q_1} \otimes f_*(L)_{q_2}$, it is essentially the original inner product restricted to $L_{p_1} \otimes L_{p_2}$, so is non-degenerate and satisfies

$$\rho(ab + \mathsf{K}, c + \mathsf{K}) = \rho(a + \mathsf{K}, bc + \mathsf{K}).$$

The next structure to check is the crossed Q-algebra action

$$\varphi : Q \to Aut(f_*(L)).$$

The obvious formula to try is

$$\varphi_q(a + \mathsf{K}) := \varphi_p(a) + \mathsf{K} \tag{9.16}$$

where $f_0(p) = q$. It is easy to reduce the proof that this is well defined to checking independence of the choice of p, but if p' is another element of $f_0^{-1}(q)$, then $p' = np$ for some $n \in N$ and $\varphi_{p'}(a) = \varphi_n \varphi_p(a) \equiv_\mathsf{K} \varphi_p(a)$, so (9.16) is well defined. Of course, this definition will give us immediately that the $\varphi_q(a)b = ba$ axiom holds and that $\varphi_q|_{f_*(L)_q} = id$, etc.

The trace axiom follows from this definition by arguments similar to that used in the corresponding result for crossed G-algebras in [156] §10.3; the requirement, there, that the kernel be central is avoided since $\varphi_n(a) - a$ is defined to be in K.

Proposition 75
With the above structure, $f_*(L)$ is a crossed D-*algebra.*

Proof: The above argument shows it is a crossed Q-algebra, so we only have to define the tilde. The obvious definition is

$$\tilde{d} := \tilde{c} + \mathsf{K}, \tag{9.17}$$

where $f_1(c) = d$. This works. It is well defined as each $\tilde{b} - 1$ is in K, and the equation

$$\varphi_q(\tilde{d}) = \widetilde{{}^q d}$$

follows from the corresponding one in L. ■

Proposition 76 *There is a natural bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed D-algebra morphisms from $f_*(L)$ to L'.* ■

The proof is obvious given our construction of $f_*(L)$. We note that this, with its companion result on pulling back, give a pairs of adjoint functors determined by $f : \mathsf{C} \to \mathsf{D}$ between the categories of crossed C-algebras and crossed D-algebras.

9.11 Back to formal maps

So far, we have only considered formal maps where the background 'coefficient system' is a crossed module / 2-group, C. From what we saw for the Yetter construction, in section 7.4, where a similar approach was being used, the use of simplicial techniques allows an extension of these methods to higher dimensions.

9.11.1 Simplicial formal maps

The notion described in section 9.5.3, is a 'bare-hands' version of a more general one which can be given in terms of simplicial theory and which applies to a general simplicial group, G, as coefficients. (The corresponding cellular theory works best, for the moment, with a crossed complex, C, as base / coefficients.) This approach is appropriate for modelling characteristic maps in higher dimension HQFTs and for developing general theory.

The full version of the simplicial theory will be developed slightly later on after we have looked at the case of C being a reduced crossed complex.

Let K be a simplicial complex and $\mathsf{C} = (C_i, \partial_i)$, a reduced crossed complex. We recall from section 4.2.3, that $Ner(\mathsf{C})$ denotes the nerve of C.

Definition: A *simplicial formal C-map* on K is a pair consisting of an ordering, $\leq$, on the vertices of K, so that each simplex is totally ordered, and a simplicial map

$$\lambda : K \to Ner(\mathsf{C}).$$

Remarks: (i) As usual, the ordering, $\leq$, on K_0 endows K with the structure of a simplicial set, which we will usually also write as K. The simplices of the *simplicial set K* are the ordered sets, $\langle v_0 \leq \ldots \leq v_k \rangle$, where, after deletion of any repetitions, the resulting set, $\{v_0, \ldots, v_k\}$, is a simplex of the simplicial complex K, cf., for example, [56] p.111.

(ii) The term 'formal map' is suggested as recalling two images. The first is that of a formally defined 'mapping' from the realisation of K to the classifying space, $B\mathsf{C}$, of C, and we will explore this in more detail later. The second is that of a map on a surface, being an embedded graph with complement a disjoint union of discs, as in the idea of *coloring* a map on a surface with elements of a group or other structure.

(iii) Ner is right adjoint to π, the functor from simplicial sets to crossed complexes, (or if you prefer to go all the way to simplicial groups, to the Dwyer-Kan 'loop groupoid' functor, $G : \mathcal{S} \to$ $\$\mathcal{S} - Grpds$,(recall from section 4.1), so we could specify λ by a map

$$\overline{\lambda} : \pi(K) \to \mathsf{C},$$

or, alternatively, by a morphism of $\mathcal{S}$-groupoids

$$\overline{\lambda} : GK \to \mathsf{C},$$

where we are thinking of the crossed complex, C, as the corresponding simplicial group.

Suppose that $\lambda : K \to Ner(\mathsf{C})$ is a formal map with an order, $\leq$, given on the vertices of K. What happens if the order of the vertices is changed? As with both the low dimensional case that we looked at earlier and the contructions when discussing TQFTs, this dependence on the order vaishes later on.

Clearly any formal map, $\lambda : K \to Ner(\mathsf{C})$, induces a continuous mapping on the realisations, i.e., $|\lambda| : |K| \to B\mathsf{C}$, and, in fact, the homotopy class of this is independent of the ordering on the vertices.

9.11.2   Formal maps on a manifold.

As we really want to look at $B\mathsf{C}$-manifolds, we can as a first step easily extend the idea of a formal map to one on a manifold *relative to a triangulation, T*.

Let X be a d-manifold, or, more generally, a d-dimensional polyhedron, and $\mathbf{T} = (T, \phi : |T| \to X)$ be an ordered triangulation of X, so ϕ is a homeomorphism between the realisation of the simplicial complex T and X.

Definition: A *(simplicial) formal C-map on X relative to $\mathbf{T}$* is a formal map, $\lambda : T \to Ner(\mathsf{C})$, and hence, notationally, may be specified by (T, ϕ, λ) or, more briefly, $(\mathbf{T}, \lambda)$.

Remember that the data for a triangulation includes explicit mention of the homeomorphism, as the simplicial complex T by itself is not enough to specify $\mathbf{T}$. Alternatively, we can usefully view the simplicial complex as arising as the Čech nerve of an open cover of X and the extra data needed is an ordering on the open sets making up the open cover.

Although our manifolds are oriented, the orientation only needs careful attention occasionally as many of the constructions do not use it explicitly, being special cases of more general ones.

In addition to formal maps, we will need formal cobordisms between them. This can be done in more generality than we will give here, where we restrict to manifolds, as the notion of cobordism between manifolds is well known and well understood. (A suitable setting for the extension to

complexes can be given using, for instance, the domain categories of Quinn, [145].) Following the recipe that we have now seen several times, we will need to consider triangulated manifolds, X_i, $i = 1, 2$, with triangulations, T_i, and a cobordism, M, between them, triangulated by $\mathcal{T}$, compatibly with the incoming and outgoing boundary triangulations. We adapt our considerations from earlier to this more general situation.

If $\lambda_1 : T_1 \to Ner(\mathsf{C})$ and $\lambda_2 : T_2 \to Ner(\mathsf{C})$ are two formal maps on the manifolds X_1 and X_2, then a *formal* C-*cobordism*, $\boldsymbol{\Lambda} : \lambda_1 \to \lambda_2$, consists of a triangulated cobordism, $(M, \mathcal{T})$, between them, and a formal C-map, $\boldsymbol{\Lambda} : \mathcal{T} \to Ner(\mathsf{C})$, defined compatibly with the λ_i on the incoming and outgoing boundaries. (Again we will not give this condition explicitly.) We will usually be concerned with such formal cobordisms up to equivalence relative to the boundaries, in a sense that generalises our earlier versions and which will be made precise shortly. (Some of that earlier discussion will be reviewed for convenience.)

The generalisation beyond crossed complexes: In the definition of simplicial formal map, we have taken the background to be $Ner(\mathsf{C})$ for C a reduced crossed complex. We could equally well have taken, as background here, any $BG = |\overline{W}G|$ for a general simplicial group G, and this is an important generalisation to make. Certain aspect of our discussion later on become more difficult, and so we will do this generalisation 'in stages'.

9.11.3   Cellular formal maps

If we are considering a regular CW-decomposition of a space X, then there is an obvious generalisation of simplicial formal maps, extending the notion of cellular formal maps introduced earlier.

Definition: A *(cellular) formal* C-*map* on a regular CW-complex X is a crossed complex morphism

$$\lambda : \pi(\mathbf{X}) \to \mathsf{C},$$

where $\mathbf{X}$ denotes the space, X, with the skeletal filtration on it.

If the CW-structure came from a triangulation of X, then this coincides with the previous definition. We may sometimes omit the qualifying 'simplicial' or 'cellular', in the term 'simplicial (or cellular) formal map'. 'Formal map' can therefore refer to either situation without real ambiguity. One reason for working with crossed complexes rather than simplicial groups is that the transition from a CW-structure or any similar cell or handle decomposition of a space to the algebraic model (crossed complex) does not need the intervention of a supplementary triangulation, followed by elimination of the spurious effects that imposing that the triangulation brings. We can usually go directly to the algebraic model of the geometry. Of course, in the process we *do* lose some generality.

9.11.4   Equivalence of simplicial formal maps

The idea behind the definition of a formal map is that it provides a good approximation to a characteristic map of a B-manifold, but is specified in an algebraic/combinatorial form. It, of course, needs the triangulation or regular CW-decomposition on the underlying space, but clearly we will need to be able to subdivide or combine simplices or cells to get new decompositions and

new 'equivalent' formal maps. More precisely, suppose $\lambda : K \to Ner(\mathsf{C})$ is a formal map on an ordered triangulation of the space, X, and K' is another ordered triangulation of X, we need to have a notion of equivalent formal C-maps and a technique for constructing such maps, so that (i) we can construct a new formal C-map λ' on K' and that (ii) λ and λ' are 'equivalent'.

In section 9.5.4, and thus for the case of 1-dimensional spaces, and 'surfaces as cobordisms', but with C being a crossed module, we achieved this by using the 3-dimensional cocycle condition, however in general that is not available to us so we need to use an alternative method. We will follow the treatment of section 9.5.4 wherever possible and will usually quote results if the proof goes across to the general case. This will allow us to indicate more clearly where any differences occur.

Although we have not yet defined formal HQFTs in this generality, we will put ourselves in the context that will be needed later by supposing that X is a polyhedron with a given family of base points $\mathbf{m} = \{m_i\}$. This will correspond either to having at least one basepoint in each connected component of the object, or in each boundary component, if X is a cobordism between two objects.) Let K_0, K_1 be two triangulations of X, i.e., K_0 and K_1 are simplicial complexes with geometric realisations homeomorphic to X (by specified homeomorphisms) with the given base points among the vertices of the triangulation.

Once again we present this for reduced crossed complexes, but there is a clear analogue for the more general case of a simplicial group.

Definition: Given two formal, C-maps (K_0, λ_0), (K_1, λ_1), then we say they are *equivalent* if there is a triangulation, T, of $X \times I$ extending K_0 and K_1 on $X \times \{0\}$ and $X \times \{1\}$ respectively, and a formal C-map, Λ, on T extending the given ones on the two ends and respecting the base points, in the sense that T contains a subdivided $\{m_i\} \times I$ for each basepoint m_i and Λ assigns the identity element 1_P of P to each 1-simplex of $\{m_i\} \times I$.

Equivalence combines the intuition of the geometry of triangulating a (topological) homotopy, where the triangulations of the two ends may differ, with some idea of a combinatorially defined simplicial homotopy of formal maps. There are fairly obvious cellular variants of the above, which we will use later without making a formal definition.

We collect below some of the results on equivalence that generalise those of section 9.5.4

Lemma 52 *(i) Equivalence is an equivalence relation.*
(ii) If (K, λ_0) and (K, λ_1) are two formal maps, which are simplicially homotopic as formal C-maps, then they are equivalent.
(iii) A change of order on the vertices of K generates an equivalent formal C-map. ∎

The proofs can be constructed fairly easily by looking back at those of the earlier discussion, or can be found in [140].

This leads to:

Proposition 77 *Given a simplicial complex, K, with geometric realisation $X = |K|$, and a subdivision K' of K.*
(a) Suppose λ is a formal C-map on K, then there is a formal C-map, λ' on K' equivalent to λ.
(b) Suppose λ' is a formal C-map on K', then there is a formal C-map, λ on K equivalent to λ'. ∎

Again the proof follows the line of the earlier ones in section **??** with a bit of our earlier chapter, especially section 7.4 (which still need a bit more detail to be filled completed). The idea is that we triangulate a copy of the cylinder $|K| \times I$, so that we have the triangulation K on $|K| \times \{0\}$ and K' on $|K| \times \{1\}$. This can be done so that the simplices in K that are unaffected by the subdivision yield prisms with the standard simplicial set structure. In particular we have that the base points m_i give 1-simplices $m_i \times I$ in the triangulated cylinder. Now assume given λ defined on K and thinking of K as $K \times \{0\}$, we seek to *extend* λ to a formal C-map, say Λ, on the triangulated cylinder. If we can do that we will be able to restrict Λ to the copy of K' on $|K| \times \{1\}$ to get a formal C-map, λ', and, by definition, this will be equivalent to λ proving (a). Reversing the roles of the two ends a similar argument will prove (b). A proof is included in [140].

Remarks. (i) As is usual with Kan complexes, we can think of filling simplices or extending maps as generalised or weak compositions. Thus using the Kan property of $Ner(\mathsf{C})$, we can compose values of a formal map on adjacent simplices. As we have unique canonical 'thin' fillers for all horns in $Ner(\mathsf{C})$, these compositions could in principle be written down exactly.

(ii) There is an alternative proof of the extension part of the above result. It is very neat but less constructive so does not suggest that the composition process is algebraic as does the one used above: consider the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ \lambda\ } & Ner(\mathsf{C}) \\
\Big\downarrow & \ \ \overset{\Lambda}{\nearrow} & \Big\downarrow \\
L & \xrightarrow{\hspace{1cm}} & 1
\end{array}
$$

where L denotes the triangulated cylinder. The inclusion of the end K into L is a trivial cofibration, and $Ner(\mathsf{C}) \to 1$ is a Kan fibration, so the dashed diagonal exists as required.

Given any cellular formal C-map, we can triangulate the cell complex and find a simplicial formal C-map that is cellularly equivalent to it. Conversely given a simplicial formal map, λ, on a triangulation of a regular CW-complex, then we can 'integrate' λ over each cell, inductively up the skeleton, to get a cellular formal map equivalent to it. The process in each case is to decompose the cylinder on the complex compatibly with the CW decomposition on one end and the triangulation on the other.

(From here on editing in progress.)

9.12 Formal maps as models for $B\mathsf{C}$-manifolds

It is reasonable to expect the combinatorial mechanism of formal maps to accurately to reflect the notion of a map from a polyhedral space, or manifold, to $B = B\mathsf{C}$, where, as before, C is, at least, a reduced crossed complex if not the classifying space of a simplicial group.

9.12.1 From 'formal' to 'actual'

Given any formal C-map,

$$\lambda : K \to Ner(\mathsf{C}),$$

we can take its geometric realisation to get a map

$$|\lambda| : |K| \to |Ner(\mathsf{C})| = B\mathsf{C}.$$

We thus have a $B\mathsf{C}$-space and if, for instance, K was an ordered triangulation of a manifold M, we could compose with the homeomorphism ϕ, say, between $|K|$ and M to get a $B\mathsf{C}$-manifold or cobordism. It is clear that other choices of ϕ correspond to the action of the automorphism group of M on the set of maps from M to $B\mathsf{C}$ and so are already accounted for in the theory.

If $\lambda : K \to Ner(\mathsf{C})$ and $\lambda' : K' \to Ner(\mathsf{C})$ are equivalent formal maps, then the equivalence (i.e., the formal map on the cylinder) gives a reversible $B\mathsf{C}$-cobordism between the two resulting $B\mathsf{C}$-manifolds. Again this is accounted for within the HQFT.

Going from 'formal' maps to 'actual' maps thus causes no problems. One just uses geometric realisation. To go in the other direction, one expects to use simplicial and cellular approximation theory.

9.12.2 Simplicial and CW-approximations and the passage to Crossed Complexes

Suppose K is an n-dimensional simplicial complex, then simplicial / CW-approximation theory implies that the space of maps from $|K|$ to $B\mathsf{C}$ is weakly homotopy equivalent to $|\mathcal{S}(K, Ner(\mathsf{C}))|$. Thus any characteristic map $g : |K| \to B\mathsf{C}$ is in the same connected component of this mapping space as a realisation, $|\lambda|$, of a formal C-map. Moreover any two ways of connecting g to such a $|\lambda|$ will be mirrored by a pair of paths in $|\mathcal{S}(K, Ner(\mathsf{C}))|$.

The simplicial set $\mathcal{S}(K, Ner(\mathsf{C}))$ is itself equivalent to $Ner(\mathsf{CRS}(\pi K, \mathsf{C}))$, where $\mathsf{CRS}(\mathsf{C}, \mathsf{D})$ denotes the Brown-Higgins crossed complex of morphisms from a crossed complex C to another one D. This means that $|\mathsf{S}(K, Ner(\mathsf{C}))|$ is weakly equivalent to the classifying space of $\mathsf{CRS}(\pi K, \mathsf{C})$. (These results are special cases of results of Brown and Higgins in the papers, [34, 36].)

Now suppose that we are considering a $(d + 1)$-HQFT, τ, then all the objects, manifolds and cobordisms have dimension less than or equal to $d + 1$. We know that τ really only depends on the $d + 1$-type of $B\mathsf{C}$ and this can clearly be seen in the algebra in the following way.

The images of all formal C-maps will be trivial in dimensions greater than $d + 1$, since if K has dimension n, the crossed complex, $\pi(\mathbf{K})$ will be trivial in dimensions greater than n. (Recall that $\pi(\mathbf{K})_p = \pi_p(K_p, K_{p-1}, K_0)$ and is a free $\pi_1(K_1 K_0)$-module on the p-cells of K.) We may, thus, replace C by its $(d+1)^{th}$ 'coskeleton' or 'truncation', $tr_{d+1}\mathsf{C}$. To do this we replace each C_n by the trivial group above dimension $d + 1$ and replace C_{d+1} by $C_{d+1}/\partial C_{d+2}$. Any formal C-map on K corresponds uniquely to a formal $tr_{d+1}\mathsf{C}$-map and conversely.

The setting is now clear when it comes to equivalence of formal C-cobordisms. If we have two equivalent formal C-cobordisms between two formal C-maps, then the equivalence corresponds to a $(d + 2)$-dimensional simplicial complex in the form of a cylinder, (so the highest dimensional simplices must be labelled by the identity elements of C_{d+2} and hence correspond to a cocycle condition in this dimension). As a result, the two induced maps under geometric realisation will be homotopic and the resulting induced maps under the HQFT will be equal.

Remarks: (i) Note that $\pi(\mathbf{K})$ can be given as a colimit, over the category of simplices of K, of the various $\pi(k)$, that is the crossed complex of a k-dimensional simplex. For each k, and each k-simplex $\sigma \in K_k$, a formal map λ yields a map from $\pi(k)$ to C and thus specifies an element in C_k. These different elements are related by face formulae to the corresponding elements in C_{k-1}. We thus have that a formal map encodes a generalisation of the notion of a π-system as introduced by the Turaev in [156].

(ii) This sort of analysis can also be given at the purely simplicial level leading to a homotopy of simplicial maps from K to $Ner(\mathsf{C})$. The advantage of the crossed complex approach is that we can replace πK, defined simplicially, by $\pi|K|$ defined via any regular CW-decomposition of $|K|$, which will be completely independent of the choice of order on the vertices of the underlying simplicial complex and may be much smaller and nearer to the 'geometry'. The simplicial approach, however, also has its advantages, in particular because of the similarity with lattice based models in TQFTs and the explicit combinatorial / geometric gadgetry available.

9.13   Formal HQFTs with background a crossed complex

The notion of a simplicial formal C-map and the corresponding formal C-cobordisms allow us to extend the definition of formal HQFT that we introduced above to all dimensions and a general crossed complex, C.

9.13.1   Formal structures of formal C-maps

Before we can give the definition of a formal HQFT, we need to describe some of the constructions we will use.

Supposing that we are working with d-dimensional manifolds, we will need to consider these together with the corresponding cobordisms. First we note that if K is the empty simplicial complex, for instance, triangulating the empty d-dimensional manifold, then there is a unique formal C-map defined on K. Next if $\lambda_i : K_i \to Ner(\mathsf{C})$ for $i = 1, 2$ are two formal C-maps, then they naturally give a formal C-map, $\lambda_1 \sqcup \lambda_2 : K_1 \sqcup K_2 \to Ner(\mathsf{C})$, given by the universal property of the coproduct, and unique up to isomorphism given a choice of that coproduct in the usual way. We say this is the *sum* of the two formal maps.

We will say that a formal C-map, $\lambda : K \to Ner(\mathsf{C})$, is *connected* if the underlying domain, K, is a connected simplicial complex. Given a general formal C-map $\lambda : K \to Ner(\mathsf{C})$, and an ordered decomposition of K as a disjoint union of its connected components, then, naturally, we get a decomposition of λ as a sum of connected formal maps.

If $\mathbf{\Lambda} : \lambda_0 \to \lambda_1$ and $\mathbf{\Gamma} : \lambda_1 \to \lambda_2$ are two formal C-cobordisms (with suitable triangulating simplicial complexes subsummed in the notation), then we can construct a composite formal C-cobordisms in the obvious way, which we will denote by $\mathbf{\Lambda}\#_{\lambda_1}\mathbf{\Gamma}$. (If extra structure (e.g., differential manifold structures) is being considered on the manifolds, it will be necessary to use cobordisms with a collar neighbourhood of the boundaries to ensure composition works at the deeper level. Ways of handling this are well known for TQFTs and cause no real problem.)

9.13.2   The definition

Fix, as before, a crossed complex, C, and also fix a ground field, $\mathbb{K}$.

A *(simplicial) formal HQFT* with background C assigns

- to each connected (simplicial) formal C-map, λ, a $\mathbb{K}$-vector space $\tau(\lambda)$, and by extension, to each formal C-map on a d-manifold X, given by a list $\lambda = \{\lambda_i \,|\, i \in I\}$ of formal connected C-maps, a tensor product

$$\tau(\lambda) = \bigotimes_{i \in I} \tau(\lambda_i);$$

- to any equivalence class of (simplicial) formal C-cobordisms, $(M, \boldsymbol{\Lambda})$ between (X_0, λ_0) and (X_1, λ_1), a K-linear transformation

$$\tau(\boldsymbol{\Lambda}) : \tau(\lambda_0) \to \tau(\lambda_1),$$

These assignments are to satisfy the following axioms:

(i) Disjoint union of formal C-maps corresponds to tensor product of the corresponding vector spaces via specified isomorphisms:

$$\tau(\lambda_0 \sqcup \lambda_1) \overset{\cong}{\to} \tau(\lambda_0) \otimes \tau(\lambda_1),$$

$$\tau(\emptyset) \overset{\cong}{\to} \mathbb{K}$$

for the ground field $\mathbb{K}$, so that a) the diagram of specified isomorphisms

$$
\begin{array}{ccc}
\tau(\lambda) & \overset{\cong}{\longrightarrow} & \tau(\lambda \sqcup \emptyset) \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \cong} \\
\tau(\lambda) \otimes \mathbb{K} & \underset{\cong}{\longleftarrow} & \tau(\lambda) \otimes \tau(\emptyset)
\end{array}
$$

for $\lambda \to \emptyset \sqcup \lambda$, commutes and similarly for $\lambda \to \lambda \sqcup \emptyset$, and b) the assignments are compatible with the associativity isomorphisms for $\sqcup$ and $\otimes$, so that τ satisfies the usual axioms for a symmetric monoidal functor.

(ii) For formal C-cobordisms

$$\boldsymbol{\Lambda} : \lambda_0 \to \lambda_1, \quad \boldsymbol{\Gamma} : \lambda_1 \to \lambda_2$$

with composite $\boldsymbol{\Lambda} \#_{\lambda_1} \boldsymbol{\Gamma}$, we have

$$\tau(\boldsymbol{\Lambda} \#_{\lambda_1} \boldsymbol{\Gamma}) = \tau(\boldsymbol{\Gamma})\tau(\boldsymbol{\Lambda}) : \tau(\lambda_0) \to \tau(\lambda_2).$$

(iii) For the identity formal C-cobordism on λ,

$$\tau(1_\lambda) = 1_{\tau(\lambda)}.$$

(iv) Interaction of cobordisms and disjoint union is transformed correctly by τ, i.e., given formal C-cobordisms

$$\boldsymbol{\Lambda} : \lambda_0 \to \lambda_1, \quad \boldsymbol{\Gamma} : \gamma_0 \to \gamma_1,$$

the following diagram

$$
\begin{array}{ccc}
\tau(\lambda_0 \sqcup \gamma_0) & \overset{\cong}{\longrightarrow} & \tau(\lambda_0) \otimes \tau(\gamma_0) \\
{\scriptstyle \tau(\boldsymbol{\Lambda} \sqcup \boldsymbol{\Gamma})} \downarrow & & \downarrow {\scriptstyle \tau(\boldsymbol{\Lambda}) \otimes \tau(\boldsymbol{\Gamma})} \\
\tau(\lambda_1 \sqcup \gamma_1) & \underset{\cong}{\longrightarrow} & \tau(\lambda_1) \otimes \tau(\gamma_1)
\end{array}
$$

commutes, compatibly with the associativity structure.

Remark. Replacing the 'simplicial' by 'cellular' etc. gives a wider definition of formal HQFT and, of course, this has an advantage of allowing smaller calculations for manifolds as there are fewer cells in a CW-decomposition than simplices in a triangulation, in general.

9.13.3   The category of formal C-maps

One idea of a homotopy quantum field theory is that it is a representation of the monoidal category of B-cobordisms. This was made explicit by Rodrigues, [148], who proved that the category, **HCobord**(d, B), of d-dimensional B-manifolds and (homotopy) B-cobordisms is a symmetric monoidal category. (A similar observation had been made by Brightwell and Turner [27] on the low dimensional case of the homotopy surface category, linked to constructions of Segal, Tillmann and others.) With that interpretation, a $(d + 1)$-HQFT is a means of studying **HCobord**(d, B) via a representation, i.e., a monoidal functor from **HCobord**(d, B) to the category of vector space over some field or, more generally, to any well understood and nicely behaved symmetric monoidal category.

Given the motivation of these papers, it seems clear that there should be a symmetric monoidal category of (simplicial) formal C-maps so that a formal HQFT with C as base was a symmetric monoidal functor from it to $Vect$. This is more or less clear but needs a little care in the setting up.

We let **FHCobord**(d, C) have the following claimed categorical structure:

- its objects are oriented d-dimensional manifolds X, each together with a triangulation **T** and a formal C-map $\lambda : T \to Ner(\mathsf{C})$;

- its morphisms are equivalence classes of formal C-cobordisms between such formal C-maps;

- its composition is given by gluing of cobordisms in the obvious way;

- for a given $(X, \mathbf{T}, \lambda)$, the corresponding identity is the equivalence class of the cylinder cobordism on $X \times I$ with triangulation and C-coloring as considered earlier;

- the monoidal category structure is given by 'coproduct over $Ner(\mathsf{C})$', that is, given $(X_i, \mathbf{T}_i, \lambda_i)$ for $i = 1, 2$, we take the disjoint union of the manifolds $X_1 \sqcup X_2$ with the obvious induced triangulation giving a simplicial complex $T_1 \sqcup T_2$ and then use the universal property for coproduct / disjoint union to give the map to $Ner(\mathsf{C})$;

- the unit of the monoidal structure is the empty formal C-map.

Theorem 27 *The above definition makes* **FHCobord**(d, C) *into a symmetric monoidal category.*

Proof. Most of this is routine as similar arguments are well represented in the literature on TQFTs. One point of note is that the category structure, and in particular, the identities of that structure, is where it becomes necessary to work with equivalence classes of cobordisms, and not just with the formal C-cobordisms themselves. The sort of argument is well known. Attaching a cylinder to an incoming or outgoing boundary of a cobordism changes the cobordism, but does keep within the equivalence class.   ∎

The following is now an obvious reformulation / corollary of this result. In the case that C is a crossed complex with an abelian group A in dimension 2 and trivial groups everywhere else, the formal HQFTs on C are exactly the HQFTs with background $K(A, 2)$ considered by Brightwell and Turner in [27] and so this result extends the corresponding observation in their work.

Theorem 28 *A (simplicial) formal HQFT, τ, with background* C *corresponds to a representation*

$$\tau : \mathbf{FHCobord}(d, \mathsf{C}) \to Vect.$$

∎

No essential role is played by any simplicial hypothesis here and so one should expect similar result for theories based on cellular or handle decompositions on the one hand and ones in which C is replaced by a simplicial group on the other.

It is worth noting that equivalent d-dimensional formal C-maps on a manifold give isomorphic objects in **FHCobord**(d, C), so effectively are independent of the decomposition used.

Bibliography

[1] L. Abrams, *Two dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory and its Ramifications, 5, (1996), 569 –587. 208

[2] J. F. Adams, 1972, *Algebraic Topology - A Student's Guide*, number 4 in London Mathematical Society Lecture Note Series, Cambridge University Press. 173, 332

[3] E. Aldrovandi, 2005, *2-Gerbes bound by complexes of gr-stacks, and cohomology*, URL http://www.arXiv.org:math/0512453. 124

[4] J. W. Alexander, *The combinatorial theory of complexes*, Annals of Math, 31, (1930), 294 –322. 214, 220

[5] L. Anderson and N. Mnev, *Triangulations of Manifolds and Combinatorial Bundle Theory: an Announcement*, Journal of Mathematical Sciences, 113, (2000), 755 – 758. 220, 224

[6] M. André, 1967, *Méthode simpliciale en algèbre homologique et algèbre commutative*, number 32 in Lecture Notes in Maths, Springer-Verlag, Berlin. 54, 55, 56, 57, 58

[7] M. André, 1970, *Homology of simplicial objects*, in *Proc. Symp. on Categorical Algebra*, 15 – 36, American Math. Soc. 54, 55

[8] M. Artin, A. Grothendieck and J. L. Verdier, 1973, *Théorie des Topos et Cohomologie Etale des Schémas*, volume 269, 270,305, of *Springer Lecture Notes in Mathematics*, Springer, available from: http://www.math.mcgill.ca/~archibal/SGA/SGA.html. 54, 80

[9] M. Artin and B. Mazur, *On the van Kampen Theorem*, Topology, 5, (1966), 179 – 189. 116

[10] M. Artin and B. Mazur, 1969, *Étale homotopy*, number 100 in Lecture Notes in Maths., Springer-Verlag, Berlin. 80, 96

[11] N. Ashley, *Simplicial T-Complexes: a non abelian version of a theorem of Dold-Kan*, Dissertations Math., 165, (1989), 11 – 58. 127, 130, 132

[12] J. C. Baez and J. Dolan, 1998, *Categorification*, URL http://www.arXiv.org:math/9802029. 23

[13] M. Barr and J. Beck, 1969, *Homology and Standard Constructions*, in *Seminar on triples and categorical homology*, number 80 in Lecture Notes in Maths., Springer-Verlag, Berlin. 53, 54

[14] M. G. Barratt, V. Gugenheim and J. Moore, *On semisimplicial fibre-bundles*, American Journal of Mathematics, 81, (1959), 639–657. 157

[15] J. Barrett, *Holonomy and path structures in General Relativity and Yang-Mills theory*, Int. J. Theor. Phys., 30, (1991), 1171 – 1215. 241

[16] H. J. Baues, 1989, *Algebraic Homotopy*, volume 15 of *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press. 31, 289

[17] H. J. Baues, 1991, *Combinatorial homotopy and 4-dimensional complexes*, Walter de Gruyter. 289

[18] H. J. Baues, 1995, *Homotopy Types*, in I.M.James, ed., *Handbook of Algebraic Topology*, 1 – 72, Elsevier. 31, 74, 75

[19] T. Beke, *Higher Cech Theory*, K Theory, 32, (2004), 293 – 322, University of Michigan, preprint. 80, 85, 86, 191, 192

[20] G. Biedermann, *On the homotopy theory of n-types*, Arxiv preprint math/0604514 (to appear in Homology, Homotopy and Applications). 91

[21] V. Blanco, M. Bullejos and E. Faro, *Categorical non-abelian cohomology and the Schreier theory of groupoids*, Math. Z., 251, (2005), 41 – 59. 122

[22] F. Borceux and G. Janelidze, 2001, *Galois theories*, volume 72 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press. 156

[23] L. Breen, 1990, *Bitorseurs et cohomologie non abélienne*, in *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, 401 – 476, Birkhäuser Boston, Boston, MA. 124, 174, 186

[24] L. Breen, *Théorie de Schreier supérieure*, Ann. Sci. École Norm. Sup. (4), 25, (1992), 465 – 514.

[25] L. Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque, 225, (1994), 160, ISSN 0303-1179. 124

[26] L. Breen, 2009, *Notes on 1-and 2-gerbes*, in J. Baez and J. May, eds., *Towards Higher Categories*, volume 152 of *The IMA Volumes in Mathematics and its Applications*, 193–235, Springer. 174

[27] M. Brightwell and P. Turner, *Representations of the homotopy surface category of a simply connected space*, J. Knot Theory and its Ramifications, 9, (2000), 855 – 864. 265, 269, 280, 295, 302, 322

[28] K. Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Trans. Amer. Math. Soc, 186, (1973), 419 – 458. 20

[29] K. Brown, 1982, *Cohomology of groups*, number 87 in Graduate Texts in Mathematics, Springer-Verlag. 270

[30] R. Brown, *From groups to groupoids: a brief survey*, Bull. London Math. Soc., 19, (1987), 113 –134. 221

[31] R. Brown, 2009, *Groupoids in Mathematics*, http://www.bangor.ac.uk/ mas010/gpdsweb.html. 221

[32] R. Brown, K. A. Hardie, K. H. Kamps and T. Porter, *A Homotopy Double Groupoid of a Hausdorff Space*, Theory Appl. Categ, 10, (2002), 71–93. 241

[33] R. Brown and P. J. Higgins, 1982, *Crossed complexes and non-abelian extensions*, in *Int. Conf. Cat. Theory, Gummersbach*, volume 962 of *Lecture Notes in Maths*, Springer-Verlag. 33

[34] R. Brown and P. J. Higgins, *Tensor products and homotopies for ω-groupoids and crossed complexes,*, J. Pure Appl. Alg., 47, (1987), 1 – 33. 31, 175, 289, 319

[35] R. Brown and P. J. Higgins, *Crossed complexes and chain complexes with operators*, Math. Proc. Camb. Phil. Soc., 107, (1990), 33 – 57. 31, 34, 47

[36] R. Brown and P. J. Higgins, *The classifying space of a crossed complex*, Math. Proc. Cambridge Philos. Soc., 110, (1991), 95 – 120. 126, 132, 266, 289, 319

[37] R. Brown, P. J. Higgins and R. Sivera, 2010, *Nonabelian Algebraic Topology: Filtered spaces, crossed complexes, cubical homotopy groupoids*, volume 15 of *EMS Monographs in Mathematics*, European Mathematical Society. 14, 31, 32, 34, 105, 241, 285, 288, 289

[38] R. Brown and J. Huebschmann, 1982, *Identities among relations*, in R. Brown and T. L. Thickstun, eds., *Low Dimensional Topology*, London Math. Soc Lecture Notes, Cambridge University Press. 16, 18, 19, 46, 50, 51

[39] R. Brown and J.-L. Loday, *Homotopical excision, and Hurewicz theorems, for n-cubes of spaces*, Proc. London Math. Soc., (3)54, (1987), 176 – 192. 111, 119

[40] R. Brown and J.-L. Loday, *Van Kampen Theorems for diagrams of spaces*, Topology, 26, (1987), 311 – 337. 94, 111, 119

[41] R. Brown and T. Porter, *On the Schreier theory of non-abelian extensions: generalisations and computations*, Proc. Roy. Irish Acad. Sect. A, 96, (1996), 213 – 227. 123

[42] R. Brown and C. Spencer, *G-groupoids, crossed modules and the fundamental groupoid of a topological group*, Proc. Kon. Ned. Akad. v. Wet, 79, (1976), 296 – 302. 22

[43] M. Bullejos and A. Cegarra, *On the Geometry of 2-Categories and their Classifying Spaces*, K-Theory, 29, (2003), 211 – 229. 231

[44] M. Bullejos, A. M. Cegarra. and J. Duskin, *On cat^n-groups and homotopy types*, J. Pure Appl. Alg., 86, (1993), 135 – 154. 116

[45] U. Bunke, P. Turner and S. Willerton, *Gerbes and homotopy quantum field theories*, Algebr. Geom. Topol., 4, (2004), 407 – 437. 265

[46] S. Buoncristiano, 2003, *Fragments of geometric topology from the sixties*, http://www.emis.de/journals/GT//gtmcontents6.html. 145

[47] S. Buoncristiano and C. Rourke, 2003, *Fragments of geometric topology from the sixties*, volume 6 of *Geometry & Topology Monographs*, Mathematical Sciences Publishers, University of California, Berkeley. 145

[48] A. Caetano and R. Picken, *An axiomatic definition of holonomy*, Int. J. Math., 5, (1994), 835 – 848. 241

[49] P. Carrasco, 1987, *Complejos Hipercruzados, Cohomologia y Extensiones*, Ph.D. thesis, Universidad de Granada. 98, 233

[50] P. Carrasco and A. M. Cegarra, *Group-theoretic Algebraic Models for Homotopy Types*, J. Pure Appl. Alg., 75, (1991), 195 – 235. 98, 127, 233

[51] A. Cegarra and J. Remedios, *The relationship between the diagonal and the bar constructions on a bisimplicial set*, Topology and its Applications, 153, (2005), 21 – 51. 116

[52] D. Conduché, *Modules croisés généralisés de longueur 2*, J. Pure Appl. Alg., 34, (1984), 155 – 178. 97, 98, 126, 127

[53] D. Conduché, *Simplicial Crossed Modules and Mapping Cones*, Georgian Math. J., 10, (2003), 623 – 636. 103, 116

[54] R. H. Crowell, *The derived module of a homomorphism*, Advances in Math., 5, (1971), 210 – 238. 36, 40, 42

[55] R. H. Crowell and R. H. Fox, 1977, *Introduction to Knot Theory*, number 57 in Graduate Texts, Springer. 43

[56] E. Curtis, *Simplicial Homotopy Theory*, Advances in Math., 6, (1971), 107 – 209. 128, 144, 157, 165, 173, 314

[57] M. L. Curtis, 1984, *Matrix Groups*, Universitext, Springer-Verlag. 259

[58] R. Debremaeker, 1976, *Cohomologie met Waarden in een Gekruiste Groepenschoof op ein Situs*, Ph.D. thesis, Katholieke Universiteit te Leuven. 124

[59] R. Debremaeker, *Cohomologie à valeurs dans un faisceau de groupes croisés sur un site. I*, Acad. Roy. Belg. Bull. Cl. Sci. (5), 63, (1977), 758 – 764.

[60] R. Debremaeker, *Cohomologie à valeurs dans un faisceau de groupes croisés sur un site. II*, Acad. Roy. Belg. Bull. Cl. Sci. (5), 63, (1977), 765 – 772.

[61] R. Debremaeker, *Non abelian cohomology*, Bull. Soc. Math. Belg., 29, (1977), 57 – 72. 124

[62] P. Dedecker, *Les foncteurs Ext_Π, $H_\Pi{}^2$ et $H_\Pi{}^2$ non abéliens*, C. R. Acad. Sci. Paris, 258, (1964), 4891 – 4894. 123

[63] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Commun. Math. Phys., 129. 212

[64] J. Duskin, 1975, *Simplicial methods and the interpretation of "triple" cohomology*, number 163 in Mem. Amer. Math. Soc., 3, Amer. Math. Soc. 54, 55, 80, 82, 97, 192, 195

[65] J. Duskin, 1979, *Higher-dimensional torsors and the cohomology of topoi: the abelian theory*, in *Applications of Sheaves*, number 753 in Lecture Notes in Maths., Springer-Verlag. 85, 192

[66] J. Duskin, *A simplicial matrix approach to higher dimensional category theory I: nerves of bicategories*, Theory and Applications of Categories, 9, (2002), 198 – 308, URL http://www.tac.mta.ca/tac/volumes/9/n10/9-10abs.html. 80, 90, 91

[67] W. G. Dwyer and D. M. Kan, *Homotopy theory and simplicial groupoids*, Nederl. Akad. Wetensch. Indag. Math., 46, (1984), 379 – 385. 125

[68] P. Ehlers and T. Porter, *Varieties of simplicial groupoids I: Crossed complexes*, J. Pure Appl. Alg., 120, (1997), 221 – 233. 127

[69] P. Ehlers and T. Porter, *Erratum to "Varieties of simplicial groupoids I: Crossed complexes"*, J. Pure Appl. Alg., 134, (1999), 207 – 209. 127

[70] P. J. Ehlers, 1991, *Simplicial Groupoids as Models for Homotopy Type*, M.Sc. thesis, University College of North Wales, Bangor. 187

[71] C. Ehresmann, 1995, *Les connexions infinitésimales dans un espace fibré différentiable*, in *Séminaire Bourbaki, Vol. 1*, Exp. No. 24, 153–168, Soc. Math. France, Paris. 258

[72] S. Eilenberg and S. M. Lane, *Cohomology theory in abstract groups, II*, Annals of Math., 48, (1957), 326 – 341. 60

[73] G. J. Ellis, *Crossed Squares and Combinatorial Homotopy*, Math. Z., 214, (1993), 93 – 110. 118

[74] G. J. Ellis, *Spaces with finitely many non-trivial homotopy groups all of which are finite*, Topology, 36, (1997), 501 – 504. 253, 254

[75] G. J. Ellis and R.Steiner, *Higher dimensional crossed modules and the homotopy groups of $(n+1)$-ads*, J. Pure Appl. Alg., 46, (1987), 117–136. 112, 113, 115

[76] T. Everaert, M. Gran and T. Van der Linden, *Higher Hopf formulae for homology via Galois Theory*, Advances in Math., 217, (2008), 2231 – 2267. 111

[77] J. Faria Martins, *On the homotopy type and the fundamental crossed complex of the skeletal filtration of a CW-complex*, Homology, Homotopy Appl., 9, (2007), 295–329 (electronic), ISSN 1532-0073. 242

[78] J. Faria Martins, *The fundamental crossed module of the complement of a knotted surface*, Trans. Amer. math. Soc., 361, (2009), 4593–4360. 242

[79] J. Faria Martins and T. Porter, *On Yetter's Invariant and an Extension of the Dijkgraaf-Witten Invariant to Categorical Groups*, Theor. Appl. Categor., 18, (2007), 118–150. 229, 242, 243

[80] Y. Felix, S. Halperin and J.-C. Thomas, 2001, *Rational Homotopy Theory*, volume 205 of *Graduate Texts in Mathematics*, Springer. 178

[81] R. H. Fox, *Free differential Calculus, I: Derivation in the free group ring*, Ann. of Maths, 57, (1953), 547 – 560. 43

[82] D. S. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*, Commun. Math. Phys., 156, (1993), 435 – 472. 274

[83] P. Gabriel and M. Zisman, 1967, *Calculus of Fractions and Homotopy Theory*, number 35 in Ergebnisse der Math. und ihrer Grenzgebeite, Springer. 126, 185

[84] P. Glenn, *Realization of cohomology classes in arbitrary exact categories*, J. Pure. Appl. Alg., 25, (1982), 33 – 105. 85, 192

[85] P. G. Goerss and J. F. Jardine, 1999, *Simplicial Homotopy Theory*, number 174 in Progress in Mathematics, Birkhauser. 91, 157, 187, 190, 192

[86] M. Gran and V. Rossi, *Galois Theory and Double Central Extensions*, Homology, Homotopy and Applications, 6, (2004), 283 – 298. 111

[87] A. Grothendieck, 1983, *Pursuing Stacks*, manuscript, 600 + pages. 151, 174

[88] A. Grothendieck, 2003, *Revêtements étales et groupe fondamental (SGA1)*, Documents Mathématiques (Paris), 3, Société Mathématique de France, Paris, séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; (50 #7129)]. 155

[89] D. Guin-Walery and J.-L. Loday, 1981, *Obstructions à l'excision en K-théorie algèbrique*, in *Evanston Conference on Algebraic K-theory, 1980*, volume 854 of *Lecture Notes in Maths.*, 179 – 216, Springer. 94, 107

[90] K. A. Hardie, K. H. Kamps and R. W. Kieboom, *A Homotopy 2-Groupoid of a Hausdorff Space*, Applied Categorical Structures, 8, (2000), 209–234. 241

[91] K. A. Hardie, K. H. Kamps and R. W. Kieboom, *A Homotopy Bigroupoid of a Topological Space*, Applied Categorical Structures, 9, (2001), 311–327. 241

[92] A. Hatcher, 2002, *Algebraic topology*, Cambridge University Press. 76, 87, 88, 91, 274

[93] P. Hilton and U. Stammbach, 1971, *A course in Homological Algebra*, Graduate Texts in Mathematics, Springer Verlag. 68

[94] P. J. Hilton, 1953, *An Introduction to Homotopy Theory*, Cambridge University Press. 14

[95] M. Hovey, 1998, *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*, Amer. Math. Soc. 185

[96] J. Huebschmann, *Crossed n-fold extensions of groups and cohomology*, Comm. Math. Helv., 55, (1980), 302 – 313. 33, 59

[97] D. Husemoller, 1975, *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*, Springer-Verlag, New York-Heidelberg, second edition. 155, 157

[98] L. Illusie, 1971, *Complexe cotangent et déformations I*, volume 239 of *Lecture Notes in Maths*, Springer-Verlag. 95, 96, 193, 195

[99] L. Illusie, 1972, *Complexe cotangent et déformations II*, volume 283 of *Lecture Notes in Maths*, Springer-Verlag. 95, 193, 195

[100] G. Janelidze, *What is a double central extension ? (the question was asked by Ronald Brown)*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 32, (1991), 191 – 201. 111

[101] J. F. Jardine, 1989, *Universal Hasse-Witt classes.*, in *Algebraic K-theory and algebraic number theory*, volume 83 of *Contemporary Mathematics*, 83 – 100, American Mathematical Society, AMS. 193

[102] B. Jurčo, 2005, *Crossed Module Bundle Gerbes; Classification, String Group and Differential Geometry*. 93

[103] K. H. Kamps and T. Porter, 1997, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific Publishing Co. Inc., River Edge, NJ. 92, 125, 175, 179

[104] M. Karoubi, 1978, *K-theory*, volume 226 of *Grundlehren der math. Wiss. $N°$*, Springer-Verlag Berlin. 157

[105] F. Keune, 1972, *Homotopical Algebra and Algebraic K-theory*, Ph.D. thesis, Universiteit van Amsterdam. 56, 58

[106] S. Kobayashi and K. Nomizu, 1996, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, reprint of the 1969 original, A Wiley-Interscience Publication. 258

[107] J. Kock, 2003, *Frobenius Algebras and 2-D Topological Quantum Field Theories*, number 59 in London Mathematical Society Student Texts, Cambridge U.P., Cambridge. 199, 202, 207, 210

[108] N. H. Kuiper and R. K. Lashof, *Microbundles and Bundles I. Elementary Theory*, Invent. Math., 1, (1966), 1 – 17. 145

[109] N. H. Kuiper and R. K. Lashof, *Microbundles and Bundles II. Semisimplicial Theory*, Invent. Math., 1, (1966), 243 – 259. 145

[110] J.-L. Loday, *Spaces with finitely many homotopy groups*, J.Pure Appl. Alg., 24, (1982), 179 – 202. 28, 31, 107

[111] J.-L. Loday, 2000, *Homotopical Syzygies*, in *Une dégustation topologique: Homotopy theory in the Swiss Alps*, volume 265 of *Contemporary Mathematics*, 99 – 127, AMS. 18, 19, 47

[112] A. Lundell and S. Weingram, 1969, *The topology of CW complexes*, New York, Van Nostrand Reinhold Co. 282

[113] J. Lurie, 2009, *On the Classification of Topological Field Theories*, URL http://arxiv.org/abs/0905.0465. 239, 280

[114] S. Mac Lane, 1967, *Homology*, number 114 in Grundlehren, Springer. 20, 53, 54, 59, 68, 171

[115] S. Mac Lane, 1978, *Categories for the Working Mathematician*, number 5 in Graduate Texts, Springer. 54, 55, 81

[116] S. Mac Lane, *Historical Note*, J. Alg., 60, (1979), 319 – 320, appendix to D. F. Holt, An interpretation of the cohomology groups, $H^n(G, M)$, J. Alg. 60 (1979) 307-318. 20

[117] S. Mac Lane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A., 36, (1950), 41 – 48. 16, 63, 79

[118] M. Mackaay and R. Picken, *Holonomy and parallel transport for abelian gerbes,*, Adv. Math., 170, (2002), 287 – 339. 241

[119] J. P. May, 1967, *Simplicial objects in algebraic topology*, number 11 in Math. Studies, van Nostrand, Princeton. 135, 144, 157, 163, 165

[120] J. P. May, 1999, *A Concise Course in Algebraic Topology*, Chicago Lectures in Mathematics, University of Chicago Press. 176, 185

[121] J. S. Milne, 2003, *Gerbes and abelian motives*, URL http://www.arXiv.org:math/0301304. 124

[122] J. W. Milnor, 1956, *On the construction FK*, mimeographed note from Princeton University, in Adams, [2]. 173

[123] J. W. Milnor, 1971, *Introduction to Algebraic K-theory*, Annals of Math. Studies, Princeton University Press. 13

[124] J. C. Moore, 1956, *Seminar in Algebraic Homotopy*, Princeton. 56

[125] M. Murray and M. A. Singer, *Gerbes, Clifford Modules and the Index Theorem*, Annals of Global Analysis and Geometry, 26, (2004), 355–367. 262

[126] A. Mutlu and T. Porter, *Applications of Peiffer pairings in the Moore complex of a simplicial group*, Theory and Applications of Categories, 4, (1998), 148 – 173. 98

[127] A. Mutlu and T. Porter, *Freeness Conditions for 2-Crossed Modules and Complexes*, Theory and Applications of Categories, 4, (1998), 174 – 194.

[128] A. Mutlu and T. Porter, *Free crossed resolutions from simplicial resolutions with given CW-basis*, Cahiers Top. Géom. Diff. catégoriques, 50, (1999), 261 – 283. 56

[129] A. Mutlu and T. Porter, *Freeness Conditions for Crossed Squares and Squared Complexes*, K-Theory, 20, (2000), 345 – 368.

[130] A. Mutlu and T. Porter, *Iterated Peiffer pairings in the Moore complex of a simplicial group*, Applied Categorical Structures, 9, (2001), 111 – 130. 98

[131] G. Nan Tie, *A Dold-Kan theorem for crossed complexes*, J. Pure Appl. Alg., 56, (1989.), 177 – 194. 132, 192

[132] G. Nan Tie, *Iterated W and T-groupoids*, J. Pure Appl. Alg., 56, (1989), 195 – 209. 132, 192

[133] A. Neeman, 2001, *Triangulated Categories*, Princeton University Press Press. 185

[134] nLab team, *nLab*, URL http://ncatlab.org/nlab/show/HomePage. 3, 80, 81, 91, 92, 272, 280

[135] S. Paoli, *Weakly globular catn-groups and Tamsamani's model*, Advances in Mathematics, 222, (2009), 621–727. 117

[136] T. Porter, *n-types of simplicial groups and crossed n-cubes*, Topology, 32, (1993), 5 – 24. 115, 116, 117

[137] T. Porter, *Interpretations of Yetter's notion of G-coloring : simplicial fibre bundles and non-abelian cohomology*, J. Knot Theory and its Ramifications, 5, (1996), 687 – 720. 212, 242

[138] T. Porter, *TQFTs from Homotopy n-types*, J. London Math. Soc., 58, (1998), 723 – 732. 212, 241, 254, 255

[139] T. Porter, 2004, *S-categories, S-groupoids, Segal categories and quasicategories*, URL http://arxiv.org/abs/math/0401274. 125

[140] T. Porter, *Formal Homotopy Quantum Field Theories, II: Simplicial Formal maps*, Contemporary Math., 431, (2007), 375 – 403. 281, 317

[141] T. Porter, 2012, *Profinite Algebraic Homotopy*, Algebra and Applications, Springer-Verlag. 39, 49, 105, 111, 115, 117, 119, 130

[142] T. Porter and V. Turaev, *Formal Homotopy Quantum Field Theories, I: Formal Maps and Crossed C-algebras.*, Journal of Homotopy and Related Structures, 3, (2008), 113 –159. 265, 281, 286, 306

[143] D. G. Quillen, 1967, *Homotopical algebra*, number 53 in Lecture Notes in Maths., Springer-Verlag. 54

[144] D. G. Quillen, 1970, *On the (co-)homology of commutative rings*, in *Proc. Symp. on Categorical Algebra*, 65 – 87, American Math. Soc. 54

[145] F. Quinn, 1995, *Lectures on axiomatic topological quantum field theory*, in D. Freed and K. Uhlenbeck, eds., *Geometry and Quantum Field Theory*, volume 1 of *IAS/Park City Mathematics Series*, AMS/IAS,. 199, 202, 210, 211, 315

[146] K. Reidemeister, 1950, *Einführung die kombinatorische Topologie*, Braunschweig, Berlin (1932); reprint Chelsea, New York. 221

[147] D. Rodelo and T. van der Linden, 2008, *The Third Cohomology group classifies double central extensions*, Pré-Publicacões do Departamento de Matematica 08-55, Universidade de Coimbra. 111

[148] G. Rodrigues, *Homotopy Quantum Field Theories and the Homotopy Cobordism Category in Dimension 1 + 1*, J. Knot Theory and its Ramifications, 12, (2003), 287 – 317. 208, 265, 267, 269, 278, 280, 302, 322

[149] C. Schommer-Pries, *Central extensions of smooth 2-groups and a finite dimensional string 2-group*, Available as arXiv 0911.2483v3. 93

[150] E. H. Spanier, 1966, *Algebraic Topology*, McGraw Hill. 221, 225, 227, 228, 234, 274

[151] N. Steenrod, 1974, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, ninth edition. 155

[152] J. Stillwell, 1980, *Classical Topology and Combinatorial Group Theory*, number 72 in Graduate Texts in Mathematics, Springer Verlag. 224, 225

[153] R. Street, *The algebra of oriented simplexes*, J. Pure Appl. Algebra, 49, (1987), 283 – 335. 241

[154] Z. Tamsamani, *Sur des notions de n-catégorie et n-groupoïde non-strictes via des ensembles multi-simpliciaux*, K. Theory, 16, (1999), 51 – 99. 117

[155] A. P. Tonks, 1993, *Theory and applications of crossed complexes*, Ph.D. thesis, University of Wales, Bangor. 71

[156] V. Turaev, 1999, *Homotopy field theory in dimension 2 and group-algebras*, arXiv.org:math/9910010. 265, 267, 269, 270, 272, 273, 275, 282, 290, 291, 295, 298, 308, 310, 313, 319

[157] V. Turaev, 2000, *Homotopy field theory in dimension 3 and crossed group-categories*, arXiv.org:math/0005291. 265

[158] V. Turaev, 2010, *Homotopy Quantum Field Theory (with Appendices by Michael Muger and Alexis Virelizier)*, number 10 in Tracts in Mathematics, European Math. Society. 265, 269, 272, 273, 275, 290, 298, 299, 301, 306

[159] D. Verity, 2006, *Weak complicial sets I: basic homotopy theory*, available from : arXiv.org:math/0604414. 241

[160] D. R. Verity, 2005, *Complicial Sets*, URL http://arxiv.org/abs/math.CT/0410412. 241

[161] D. R. Verity, 2007, *Weak complicial sets. II. Nerves of complicial Gray-categories*, in *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, 441 – 467, Amer. Math. Soc., Providence, RI. 241

[162] D. R. Verity, 2011, *Weak complicial sets. III. Enriched and internal quasi-category theory*, (in preparation). 241

[163] C. A. Weibel, 1995, *An Introduction to Homological Algebra*, number 38 in Cambridge Studies in Advanced Mathematics, Cambridge University Press. 68

[164] J. H. C. Whitehead, *Combinatorial Homotopy II*, Bull. Amer. Math. Soc., 55, (1949), 453 – 496. 31, 289

[165] D. Yetter, *Topological Quantum Field Theories Associated to Finite Groups and Crossed G-Sets*, J. Knot Theory Ramifications, 1, (1992), 1 – 20. 212, 215, 220, 222, 223

[166] D. Yetter, *TQFT's from Homotopy 2-Types*, J. Knot Theory Ramifications, 2, (1993), 113 – 123. 212, 215, 219, 223, 228, 234, 236, 239, 245

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