Appendix A

Hermite polynomials and Hermite functions

Real Hermite polynomials are defined to be

$$H_n(u) \equiv (-1)^n e^{u^2/2} \frac{d^n}{du^n} e^{-u^2/2}, \quad u \in \mathbb{R}, n \in \mathbb{N}_0, \tag{A.1}$$

which are coefficients in expansion of power series for $\exp\{tu - t^2/2\}$ as function of t:

$$\exp\{tu - t^2/2\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u), \quad t, u \in \mathbb{R}. \tag{A.2}$$

By this expansion formula we have:

Theorem A.1 *Hermite polynomials have the following expression:*

$$H_n(u) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k u^{n-2k}}{2^k k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{A.3}$$

Conversely,

$$u^n = n! \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(u)}{2^k k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{A.4}$$

$\{H_n,\ n \in \mathbb{N}\}$ *satisfy the following differential equations*

$$H_n'(u) = nH_{n-1}(u), \qquad n \geq 1, \tag{A.5}$$

$$H_n''(u) - uH_n'(u) + nH_n(u) = 0, \qquad n \geq 0 \tag{A.6}$$

and recursion formula:

$$H_0(u) \equiv 1, \qquad H_1(u) = u,$$

$$H_{n+1}(u) = uH_n(u) - nH_{n-1}(u), \qquad n \geq 1, \tag{A.7}$$

as well as multiplication formula:

$$H_m(u)H_n(u) = \sum_{k=0}^{m\wedge n} k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}(u). \tag{A.8}$$

Moreover, for any $\lambda \in I\!R$ it holds that

$$H_n(\lambda u) = n! \sum_{k=0}^{[n/2]} \frac{(\lambda^2 - 1)^k \lambda^{n-2k}}{2^k k!(n-2k)!} H_{n-2k}(u). \tag{A.9}$$

Proof. Replacing the power series of e^{tu} and $e^{-t^2/2}$ with respect to t into eq. (A.2) and comparing the coefficients of t^n on both sides, we obtain eqs. (A.3) and (A.4). Differentiating eq. (A.2) with respect to u and comparing the coefficients of power series we get (A.5) and (A.6). Again from eq. (A.2) we know

$$\begin{aligned}\sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} H_m(u) H_n(u) &= \exp\left\{(s+t)u - \frac{(s+t)^2}{2} + st\right\} \\ &= \sum_{j=0}^{\infty} \frac{(s+t)^j}{j!} H_j(u) \sum_{k=0}^{\infty} \frac{s^k t^k}{k!} \\ &= \sum_{j,k=0}^{\infty} \frac{H_j(u)}{j!k!} \sum_{l=0}^{j} \binom{j}{l} s^{l+k} t^{j-l+k}.\end{aligned}$$

Letting $l + k = m,\ j - l + k = n$ in the last expression, we have

$$\sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}.$$

The multiplication formula (A.8) is obtained by comparing the coefficients of $s^m t^n$. In particular, the recursion formula (A.7) is obtained by letting $m = 1$ in eq. (A.8). Finally, it follows from eq. (A.2) that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\lambda u) &= \exp\left\{\lambda t u - \frac{t^2}{2}\right\} \\ &= \exp\left\{\lambda t u - \frac{\lambda^2 t^2}{2} + \frac{(\lambda^2 - 1)t^2}{2}\right\} \\ &= \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} H_j(u) \sum_{k=0}^{\infty} \frac{(\lambda^2 - 1)^k t^{2k}}{2^k k!}.\end{aligned}$$

Letting $j + 2k = n$ in the last expression, we obtain

$$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/2]} \frac{(\lambda^2 - 1)^k \lambda^{n-2k}}{2^k k!(n-2k)!} H_{n-2k}(u),$$

by comparing the coefficients of t^n, we then have eq. (A.9). ∎

Considering the Gaussian measure on $I\!R$:

$$\gamma(du) \equiv (2\pi)^{-1/2} \exp\{-u^2/2\} du$$

and the Hilbert space $L^2(\mathbb{R},\gamma)$, we have

Theorem A.2 *Hermite polynomials constitute an orthogonal system in* $L^2(\mathbb{R},\gamma)$:

$$\int_{\mathbb{R}} H_m(u)H_n(u)\gamma(du) = n!\delta_{mn}, \qquad m,n \in \mathbb{N}_0. \tag{A.10}$$

Denote $i = \sqrt{-1}$. *Then*

$$H_n(u) = \int_{\mathbb{R}} (u \pm iv)^n \gamma(dv), \qquad n \in \mathbb{N}_0, \tag{A.11}$$

moreover,

$$H_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} u^k H_{n-k}(v), \qquad n \in \mathbb{N}_0. \tag{A.12}$$

When $t^2 < 1$, *we have*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u)H_n(v) = \frac{1}{\sqrt{1-t^2}} \exp\left\{-\frac{t^2u^2 - 2tuv + t^2v^2}{2(1-t^2)}\right\}. \tag{A.13}$$

Proof. It follows from eq. (A.2) that

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} \int_{\mathbb{R}} H_m(u)H_n(u)\gamma(du) \\
&= \int_{\mathbb{R}} \exp\left\{(s+t)u - \frac{s^2+t^2}{2}\right\}\gamma(du) \\
&= \exp\left\{-\frac{s^2+t^2}{2} + \frac{(s+t)^2}{2}\right\} = e^{st} \\
&= \sum_{n=0}^{\infty} \frac{(st)^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $s^m t^n$ we obtain eq. (A.10). Using contour integration we have

$$\int_{\mathbb{R}} \exp\{t(u \pm iv)\}\gamma(dv) = \exp\left\{tu - \frac{t^2}{2}\right\}.$$

By expansion in power series of t (using eq. (A.2) for right-hand side) and comparing the coefficients of t^n we prove eq. (A.11). From eq. (A.11) we know

$$\begin{aligned}
H_n(u+v) &= \int_{\mathbb{R}} (u+v+iy)^n \gamma(dy) \\
&= \sum_{k=0}^{n} \binom{n}{k} u^k \int_{\mathbb{R}} (v+iy)^{n-k} \gamma(dy),
\end{aligned}$$

which implies eq. (A.12). Again by eq. (A.11) we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u)H_n(v)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{t(u+ix)(v+iy)\}\gamma(dx)\gamma(dy).$$

A direct computation of the integral yields eq. (A.13). ∎

It follows from eq. (A.4) and multiplication formula (A.8) that Hermite polynomials constitute a linear base of polynomial ring. In view of eq. (A.10) and density of polynomials in $L^2(\mathbb{R}, \gamma)$, we know that $\{(n!)^{-1/2}H_n\}$ is an orthonormal base of $L^2(\mathbb{R}, \gamma)$. Now consider the Hilbert space $L^2(\mathbb{R}) = L^2(\mathbb{R}, du)$, where du is Lebesgue measure. For $f \in L^2(\mathbb{R})$, define

$$Jf(u) \equiv \pi^{1/4} e^{u^2/4} f(u/\sqrt{2}). \tag{A.14}$$

Then

$$||Jf||^2_{L^2(\mathbb{R},\gamma)} = ||f||^2_{L^2(\mathbb{R})}.$$

Moreover,

$$J^{-1}f(u) = \pi^{-1/4} e^{-u^2/2} f(\sqrt{2}u). \tag{A.15}$$

Hence $J : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}, \gamma)$ is an isomorphism for Hilbert spaces. Let

$$h_n(u) \equiv (n!)^{-1/2} J^{-1} H_n(u)$$
$$= (n!)^{-1/2} \pi^{-1/4} e^{-u^2/2} H_n(\sqrt{2}u). \tag{A.16}$$

Then $\{h_n, n \in \mathbb{N}_0\}$ constitute an orthonormal base of $L^2(\mathbb{R})$. They are called *Hermite functions.* By definition and properties of Hermite polynomials we have

$$h_n'(u) + uh_n(u) = \sqrt{2n} h_{n-1}(u), \qquad n \geq 1. \tag{A.17}$$

In addition, the following estimates are very useful, for the proof see Hille-Phillips[1] or G.Szegö[1].

Theorem A.3 *For any fixed* $u \in \mathbb{R}$, *we have*

$$h_n(u) = O(n^{-1/4}), \tag{A.18}$$

$$\int_0^u h_n(v)dv = O(n^{-3/4}). \tag{A.19}$$

Moreover,

$$||h_n||_{L^\infty} \equiv \sup_{u \in \mathbb{R}} |h_n(u)| = O(n^{-1/12}), \tag{A.20}$$

$$||h_n||_{L^1} \equiv \int_{\mathbb{R}} |h_n(u)|du = O(n^{1/4}). \tag{A.21}$$

Since $H_n(u) = (n!)^{1/2}\pi^{1/4}e^{u^2/4}h_n(u/\sqrt{2})$, it follows from (A.20) that

$$| H_n(u) | \leq c(n!)^{1/2}e^{u^2/4}. \tag{A.22}$$

More precisely, we may take $c = 1.2$ in the above inequality and (A.22) is then called Cramér's estimate (cf. Erdélyi[1], p.208).