ON THE HOMOTOPY GROUPS OF THE UNION OF SPHERES

P. J. HILTON†.

1. *Introduction.*

Let S_i be a sphere of dimension r_i+1, $r_i \geqslant 1$, $i = 1, \ldots, k$, and let T be the union of the spheres $S_1, \ldots, S_k$, with a single common point. Then T serves as a universal example for homotopy constructions (see [**1**]). The object of this paper is to compute the group $\pi_n(T)$, $n > 1$, as a direct sum of homotopy groups of spheres of appropriate dimensions‡. Each summand is embedded in $\pi_n(T)$ by a certain *multiple Whitehead product*; the products which appear will be called *basic products* and will now be defined.

Let $T_0 = S_{u_1} \vee S_{u_2} \vee \ldots \vee S_{u_m}$, where $1 \leqslant u_1 < u_2 < \ldots < u_m \leqslant k$. Then the injection $\pi_n(T_0) \to \pi_n(T)$ embeds $\pi_n(T_0)$ univalently as a direct summand in $\pi_n(T)$. We will identify elements of $\pi_n(T_0)$ with their images in $\pi_n(T)$, and an element in the image of $\pi_n(T_0)$ will be said to *involve* the spheres $S_{u_1}, \ldots, S_{u_m}$. With these conventions, we define and order the basic products as follows.

The basic products§ of weight 1 are the elements $\iota_1, \ldots, \iota_k$, where $\iota_1 < \iota_2 < \ldots < \iota_k$, ι_i being the positive generator of $\pi_{r_i+1}(S_i)$, $i = 1, \ldots, k$. Now suppose the basic products of weight $< w$ defined and ordered. Then a basic product of weight $w > 1$ is a Whitehead product $[a, b]$, where a is a basic product of weight u, b is a basic product of weight v, $u+v = w$, $a < b$, and if b is defined as the Whitehead product, $[c, d]$, of the basic products c, d, then $c \leqslant a$. The basic products of weight w are then ordered arbitrarily among themselves and are greater than any product of lesser weight.

It will be seen that a basic product of weight w is a string of symbols $\iota_{v_1} \ldots \iota_{v_w}$, suitably bracketed, where $1 \leqslant v_j \leqslant k$, $j = 1, \ldots, w$. Suppose ι_i occurs w_i times in this string. Then we will say that the basic product *involves* the sphere S_i w_i times, and the height of the basic product is defined as $\sum_{i=1}^{k} r_i w_i$.

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‡ T is, of course, simply-connected, but it is convenient to regard the trivial case $n = 1$ as excluded from the discussion. Thus all homotopy groups discussed in this paper are of dimension $n > 1$.

§ It is convenient for this definition to think of ι_i as a Whitehead product of minimum weight.

The definition of the basic products imitates P. Hall's definition of basic commutators (see §3 of this paper), with a minor modification which is unimportant algebraically, but which ensures that $[\iota_1, \iota_2]$ and not $[\iota_2, \iota_1]$ is a basic product. This appears to be natural in many applications of Theorem A (see §6 of this paper).

Let the basic products be written $p_1, p_2, \ldots, p_s, \ldots$, and let the height of p_s be q_s. Then the main theorem may be expressed as

THEOREM A. *$\pi_n(T) \cong \sum_{i=1}^{\infty} \pi_n(S^{q_i+1})$, where the direct summand $\pi_n(S^{q_i+1})$ is embedded in $\pi_n(T)$ by composition with the basic product $p_i \varepsilon \pi_{q_i+1}(T)$.*

Note that, for each n, there are only finitely many non-zero terms on the right-hand side, since the sequence $\{q_i\}$ tends to infinity. Clearly there is considerable choice in defining the basic products. However, according to a theorem due to E. Witt (see [**11**]), the number of basic products of weight w, involving the spheres S_i w_i times, $i = 1, \ldots, k$, is

$$\frac{1}{w} \sum_{d \mid w_i} \frac{\mu(d)(w/d)!}{(w_1/d)! \ldots (w_k/d)!},$$

where $\mu(d)$ is the Möbius inversion function. Each such basic product gives rise to a term $\pi_n(S^{q+1})$ in the direct sum decomposition of $\pi_n(T)$, where $q = \sum_{i=1}^{k} r_i w_i$. Thus the number of occurrences of the term $\pi_n(S^{q+1})$ on the right-hand side in Theorem A does not depend on the particular choice of basic products and a different choice would lead at most to a rearrangement of the summands and a change of embedding isomorphism.

Consider the case $T = S_1 \vee S_2 \vee S_3$. The basic products of weight 1 are $\iota_1, \iota_2, \iota_3$; those of weight 2 are $[\iota_1, \iota_2], [\iota_1, \iota_3], [\iota_2, \iota_3]$; those of weight 3 are $\big[\iota_1, [\iota_1, \iota_2]\big], \big[\iota_1, [\iota_1, \iota_3]\big], \big[\iota_2, [\iota_1, \iota_2]\big], \big[\iota_2, [\iota_1, \iota_3]\big], \big[\iota_2, [\iota_2, \iota_3]\big], \big[\iota_3, [\iota_1, \iota_2]\big], \big[\iota_3, [\iota_1, \iota_3]\big], \big[\iota_3, [\iota_2, \iota_3]\big]$. It will be observed that $\big[\iota_1, [\iota_2, \iota_3]\big]$ is not a basic product. It can therefore be expanded as a linear combination of terms involving basic products, and it is almost immediate that $\big[\iota_1, [\iota_2, \iota_3]\big]$ is a linear combination of $\big[\iota_2, [\iota_1, \iota_3]\big]$ and $\big[\iota_3, [\iota_1, \iota_2]\big]$. In fact, we deduce†

THEOREM B. *Let $\alpha \varepsilon \pi_p(X)$, $\beta \varepsilon \pi_q(X)$, $\gamma \varepsilon \pi_r(X)$. Then the "Jacobi identity" holds:*

$$(-1)^{pq}\big[[\beta, \gamma], \alpha\big] + (-1)^{qr}\big[[\gamma, \alpha], \beta\big] + (-1)^{rp}\big[[\alpha, \beta], \gamma\big] = 0.$$

We also deduce the "relative Jacobi identity". Further applications of Theorem A are given in the final section.

The author wishes to acknowledge the decisive contributions made by J.-P. Serre and J. A. Green; the fundamental idea in the proof of Theorem A is due to Serre, and the necessary extension of the algebraic method is due essentially to Green. The author is also grateful to J. C. Moore for his kind assistance.

† This theorem has also been proved by Hurewicz, G. W. Whitehead, Nakaoka, Toda, Uehara and probably others.

2. *Two topological lemmas.*

Let X be a path-connected space, and let Ω be the space of loops on X. Let η be the natural isomorphism $\eta:\pi_{p+1}(X)\cong\pi_p(\Omega)$, let h be the Hurewicz homomorphism $h:\pi_p(\Omega)\to H_p(\Omega)$ and let

$$\rho=h\eta:\pi_{p+1}(X)\to H_p(\Omega).$$

Now the composition of loops in Ω gives it the structure of an H-space and induces a Pontryagin multiplication into the homology classes of Ω. We write this multiplication as $\xi\,.\,\xi'\in H_{p+q}(\Omega)$, $\xi\in H_p(\Omega)$, $\xi'\in H_q(\Omega)$, or sometimes just as $\xi\xi'$.

Let Ω be the space of loops on T and let $e_i=\rho\iota_i$, $\iota_i\in\pi_{r_i+1}(T)$, $e_i\in H_{r_i}(\Omega)$, $i=1,\ldots,k$. Then Bott and Samelson have proved (see [2])

LEMMA 2.1. *The Pontryagin homology ring of Ω, the space of loops on T, is a free associative ring, freely generated by the elements $e_1,\ldots,e_k$.*

Revert to the general case and let $\alpha\in\pi_{p+1}(X)$, $\beta\in\pi_{q+1}(X)$ so that $[\alpha,\beta]\in\pi_{p+q+1}(X)$. Then Samelson has proved (see [8])

LEMMA 2.2. $\rho[\alpha,\beta]=(-1)^p\left(\rho\alpha\,.\,\rho\beta-(-1)^{pq}\rho\beta\,.\,\rho\alpha\right)$.

3. *An algebraical theorem.*

Let R be a ring generated by $e_1,\ldots,e_k$, let ϵ be an arbitrary mapping of $R\times R$ into the set $(1,-1)$, and let λ be an arbitrary mapping of $R\times R$ into the integers. Define for $a,b\in R$,

$$a\circ b=\lambda(a,b)\,ab-\epsilon(a,b)\,ba.$$

We call $a\circ b$ the *quasi-commutator* (*qc*) of a and b, and define basic *qc*'s (*bqc*) exactly as for basic products, starting with the ordered set $e_1,\ldots,e_k$ of *bqc*'s of weight 1, and using the $\circ$ operation instead of the Whitehead product operation. Let the *bqc*'s be $b_1,b_2,\ldots,b_s,\ldots$. We define a *bqc-monomial* as a word M of the form $b_{i_1}b_{i_2}\ldots b_{i_r}$. The *weight* of M is its degree as a polynomial in $e_1,\ldots,e_k$, and the *disorder* of M is the number of pairs (u,v), $1\leqslant u<v\leqslant r$, with $i_u>i_v$. Then M has zero disorder if and only if it has the form $b_1^{n_1}\,b_2^{n_2}\,\ldots,\ n_i\geqslant 0$.

THEOREM 3.1. *Any monomial in $e_1,\ldots,e_k$ of degree w can be written as a linear combination of bqc-monomials of weight w and zero disorder.*

THEOREM 3.2. *If $e_1,\ldots,e_k$ are free generators of the free associative ring R, then the bqc-monomials of zero disorder constitute a free additive basis for R.*

THEOREM 3.3. *The number of bqc's of weight w is*

$$Q(w,k)=\frac{1}{w}\sum_{d\mid w}\mu(d)\,k^{w/d},$$

where μ is the Möbius function.

These theorems have all been proved† in the case $\epsilon \equiv -1$, $\lambda \equiv -1$ (see [11]). Clearly Theorem 3.3 cannot depend on the particular choice of "commutator". Assume Theorem 3.1. Then the *bqc*-monomials of zero disorder span R. Since R is graduated by degree and since the number of monomials spanning the subring of R consisting of homogeneous elements of degree d, say, is finite and the same for commutators as for quasi-commutators, Theorem 3.2 follows from the known facts for commutators. The proof of Theorem 3.1 in its generalized sense presents no new difficulties, and it should be sufficient to sketch the proof, which is modelled on the ideas of P. Hall and Magnus.

We define the *degree* of $M = b_{i_1} b_{i_2} \dots b_{i_r}$ to be r. Now let b be the first *bqc* (in the sequence $b_1, b_2, \dots$) to occur in disorder in M, in the sense that $b = b_{i_v}$ for a pair (u, v), $1 \leqslant u < v \leqslant r$, with $i_u > i_v$. Suppose that b first occurs in disorder in M as b_{i_v}, so that, certainly, $i_{v-1} > i_v$. Now $b_{i_v} \circ b_{i_{v-1}} = \lambda b_{i_v} b_{i_{v-1}} - \epsilon b_{i_{v-1}} b_{i_v}$ (writing λ, ϵ for short). Thus

$$M = \epsilon\lambda b_{i_1} \dots b_{i_{v-2}} b_{i_v} b_{i_{v-1}} b_{i_{v+1}} \dots b_{i_r} - \epsilon b_{i_1} \dots b_{i_{v-2}} (b_{i_v} \circ b_{i_{v-1}})\, b_{i_{v+1}} \dots b_{i_r}$$
$$= \epsilon\lambda M' - \epsilon M'', \text{ say.}$$

We now show that, if M arose from a monomial in the $e_1, \dots, e_k$ by successive applications of this process, then $b_{i_v} \circ b_{i_{v-1}}$ is a *bqc*. Certainly $b_{i_v} < b_{i_{v-1}}$. Now *qc*'s of weight > 1 arise by this process (they were not present at the start), so that, if $b_{i_{v-1}} = \alpha \circ \beta$, α must at an earlier stage have been the first *bqc* to be in disorder. Obviously this process, applied, say, to b, does not put into disorder any $b' < b$, so that $\alpha \leqslant b_{i_v}$ and $b_{i_v} \circ b_{i_{v-1}}$ is a *bqc*. Thus M is expressed as $\epsilon\lambda M' - \epsilon M''$, where M', M'' are *bqc*-monomials, M' has less disorder than M, and M'' has smaller degree. M' still has degree r, and, of course, the weights of M', M'' are the same as that of M.

We have now established the basis for an induction. For if we suppose that all *bqc* monomials of degree $< r$ may be expressed as a linear combination of monomials of zero disorder, we have a process for steadily reducing the disorder of a monomial of degree r. In this way the proof of Theorem 3.1 is completed.

4. *Proof of Theorem A.*

Let Ω be the space of loops on T. By (2.1), $H(\Omega)$ is the free associative ring freely generated by $e_1, \dots, e_k$. Let the $\circ$ operation in $H(\Omega)$ be specified for homogeneous elements by defining $\lambda(a, b) = (-1)^p$, $\epsilon(a, b) = (-1)^{p(q+1)}$, where $a \varepsilon H_p(\Omega)$, $b \varepsilon H_q(\Omega)$. It is then clear from (2.2) that ρ maps the basic products $p_1, p_2, \dots, p_s, \dots$ onto a complete set of *bqc*'s $b_1, b_2, \dots, b_s, \dots$, where $\rho p_i = b_i$, $i = 1, 2, \dots$.

† In P. Hall's definition of basic commutators, the commutator $(ab - ba)$ would only be basic if $a > b$. Thus we get the same definition as Hall if we define $a \circ b = ba - ab$ and admit $a \circ b$ as basic only if $a < b$. See the footnote on basic products.

Let q_i be the height of p_i. Use the symbol p_i for a map $S^{q_i+1} \to T$ in the class p_i. Then p_i induces a map $f_i : \Omega_i \to \Omega$, where Ω_i is the space of loops on S^{q_i+1}, and hence a homomorphism $f_{i*} : H(\Omega_i) \to H(\Omega)$. Let η_i be the natural isomorphism $\eta_i : \pi_{q_i+1}(S^{q_i+1}) \cong \pi_{q_i}(\Omega_i)$, and let h_i be the Hurewicz isomorphism $h_i : \pi_{q_i}(\Omega_i) \cong H_{q_i}(\Omega_i)$. Then $H(\Omega_i)$ is a free ring freely generated by $b_i' = h_i \eta_i \iota$, where ι is the positive generator of $\pi_{q_i+1}(S^{q_i+1})$.

LEMMA 4.1. *f_{i*} is a ring-homomorphism and $f_{i*} b_i' = b_i$.*

That f_{i*} is a ring-homomorphism follows from the more general proposition that a map $X \to Y$ always induces a homomorphism of the Pontryagin ring of $\Omega(X)$ into that of $\Omega(Y)$. That $f_{i*} b_i' = b_i$ follows from the commutativity of the diagram

$$\begin{array}{ccc} \pi_{q_i+1}(S^{q_i+1}) & \overset{p_{i*}}{\to} & \pi_{q_i+1}(T) \\ \eta_i \downarrow & & \downarrow \eta \\ \pi_{q_i}(\Omega_i) & \overset{f_{i\dagger}}{\to} & \pi_{q_i}(\Omega) \\ h_i \downarrow & & \downarrow h \\ H_{q_i}(\Omega_i) & \overset{f_{i*}}{\to} & H_{q_i}(\Omega), \end{array}$$

where p_{i*}, $f_{i\dagger}$ are induced by p_i, f_i. It follows from (4.1) that

$$f_{i*} b_i'^n = b_i^n.$$

Consider the maps $f_i : \Omega_i \to \Omega$, $f_j : \Omega_j \to \Omega$, induced by p_i, p_j. If we represent composition of loops in Ω by $\omega . \omega'$, $\omega, \omega' \in \Omega$, then we may define $f_{ij} : \Omega_i \times \Omega_j \to \Omega$ by $f_{ij}(\omega_i, \omega_j) = f_i \omega_i . f_j \omega_j$, $\omega_i \in \Omega_i$, $\omega_j \in \Omega_j$. Now let $\gamma_i \in H(\Omega_i)$, $\gamma_j \in H(\Omega_j)$. Then $\gamma_i \otimes \gamma_j \in H(\Omega_i \times \Omega_j)$.

LEMMA 4.2. $(f_{ij})_* (\gamma_i \otimes \gamma_j) = (f_{i*} \gamma_i) . (f_{j*} \gamma_j)$.

Let $u : I^p \to \Omega_i$ be a singular p-cube of Ω_i, and let $v : I^q \to \Omega_j$ be a singular q-cube of Ω_j. Then $(u, v) : I^p \times I^q \to \Omega_i \times \Omega_j$, given by $(u, v)(x, y) = (ux, vy)$, $x \in I^p$, $y \in I^q$, is a singular $(p+q)$-cube of $\Omega_i \times \Omega_j$, and we may restrict attention to such cubes of $\Omega_i \times \Omega_j$ in considering the homology of $\Omega_i \times \Omega_j$. Then

$$f_{ij}(u, v)(x, y) = f_i ux . f_j vy. \tag{4.3}$$

Thus, if we allow f_i, f_j, f_{ij} also to stand for the induced chain-mappings, (4.3) reads

$$f_{ij}(u \otimes v) = f_i u . f_j v,$$

the multiplication on the right being the multiplication of elements of the chain-group of Ω which induces the Pontryagin multiplication of homology classes. The lemma thus follows by passing to homology classes.

We have maps $f_i : \Omega_i \to \Omega$, $i = 1, 2, \ldots$. These induce maps

$${}_m f : \Omega_1 \times \ldots \times \Omega_m \to \Omega,$$

given by

$${}_m f(\omega_1, \ldots, \omega_m) = f_1 \omega_1 . \ldots . f_m \omega_m, \quad \omega_i \in \Omega_i,$$

where, for the sake of definiteness, we take the bracketing on the right-hand side to be the natural bracketing from the left. Let ω_i^0 be the null-loop in Ω_i. Then, if $l < m$, we may identify $\Omega_1 \times \ldots \times \Omega_l$ with the subspace $\Omega_1 \times \ldots \times \Omega_l \times \omega_{l+1}^0 \times \ldots \times \omega_m^0$ of $\Omega_1 \times \ldots \times \Omega_m$. Since each f_i maps ω_i^0 to the null-loop in Ω, it follows that

$$_mf \,|\, \Omega_1 \times \ldots \times \Omega_l \simeq {_lf} : \Omega_1 \times \ldots \times \Omega_l \to \Omega.$$

Let $\Omega^* = \prod_1^\infty \Omega_i$. Now $H(\Omega_i)$ is the free ring generated by b_i'. Since the dimension of b_i' tends to infinity with i, it follows that $H(\Omega^*)$ is additively generated by the (finite) tensor products $b_1'^{n_1} \otimes b_2'^{n_2} \otimes \ldots$ and the set of maps $_mf$ induce a well-defined homomorphism

$$\phi : H(\Omega^*) \to H(\Omega).$$

By (4.1) and an easy extension† of (4.2),

$$\phi(b_1'^{n_1} \otimes b_2'^{n_2} \otimes \ldots) = b_1^{n_1} \,.\, b_2'^{n_2} \,.\, \ldots \,. \tag{4.4}$$

Since the expressions $b_1^{n_1} b_2^{n_2} \ldots$ form a free additive basis of $H(\Omega)$, by Theorem 3.2, it follows that ϕ is in fact an additive isomorphism of $H(\Omega^*)$ onto $H(\Omega)$.

Let us suppose for simplicity of notation that the first t of the spheres in T have dimension 2. Then the set of maps $\{_mf\}$ induces an isomorphism of $\pi_1(\Omega^*)$ onto $\pi_1(\Omega)$, mapping each $\eta_i \iota_i$ onto $\eta\iota_i$, $i = 1, \ldots, t$. Let $\tilde{\Omega}$ be the universal cover of Ω and let $\tilde{\Omega}^*$ be the universal cover of Ω^*; then $\tilde{\Omega}^* = \tilde{\Omega}_1 \times \ldots \times \tilde{\Omega}_t \times \Omega_{t+1} \times \ldots$, where $\tilde{\Omega}_i$ is the universal cover of Ω_i, $i = 1, \ldots, t$. The maps $_mf$, $m = t, t+1, \ldots$, may be lifted uniquely to maps

$$_mg : \tilde{\Omega}_1 \times \tilde{\Omega}_1 \times \ldots \times \tilde{\Omega}_t \times \Omega_{t+1} \times \ldots \times \Omega_m \to \tilde{\Omega},$$

sending the class of null-loops on $\Omega_1 \times \ldots \times \Omega_m$ to the class of null-loops on Ω. Moreover, writing $_m\tilde{\Omega}^*$ for the space of arguments of $_mg$, and embedding $_l\tilde{\Omega}^*$ in $_m\tilde{\Omega}^*$ in the obvious way, $t \leqslant l \leqslant m$, we have

$$_mg \,|\, _l\tilde{\Omega}^* \simeq {_lg} : {_l\tilde{\Omega}^*} \to \tilde{\Omega}, \tag{4.5}$$

which induces a well-defined homomorphism

$$\tilde{\phi} : H(\tilde{\Omega}^*) \to H(\tilde{\Omega}).$$

LEMMA 4.6. *$\tilde{\phi}$ is an isomorphism onto $H(\tilde{\Omega})$.*

Let $\Omega^{*(0)} = \Omega^*$, $\Omega^{*(u)} = \tilde{\Omega}_1 \times \ldots \times \tilde{\Omega}_u \times \Omega_{u+1} \times \ldots$, $\Omega^{(0} = \Omega$, $\Omega^{(u)}$ the covering space of Ω with fundamental group $(\eta\iota_{u+1}, \ldots, \eta\iota_t)$, $u \leqslant t$. Then $\Omega^{*(t)} = \tilde{\Omega}^*$, $\Omega^{(t)} = \tilde{\Omega}$, $\Omega^{*(u)}$ is a covering space of $\Omega^{*(u-1)}$ with cover-

† Despite the appearance of (4.4), ϕ is not a ring-homomorphism. If we identify $b_1' \in H(\Omega_1)$ with $b_1' \otimes u_2 \in H(\Omega_1 \times \Omega_2)$, u_2 being the unit element of $H(\Omega_2)$, and similarly identify b_2' with $u_1 \otimes b_2'$, then $b_1' \,.\, b_2' = b_1' \otimes b_2'$, $b_2' \,.\, b_1' = \pm(b_1' \otimes b_2')$, but, of course, $b_1 b_2 \neq \pm b_2 b_1$ in $H(\Omega)$.

transformation group $(\eta_u \iota_u)$, and $\Omega^{(u)}$ is a covering space of $\Omega^{(u-1)}$ with cover-transformation group $(\eta\iota_u)$. Moreover, there exist sets of maps $\{_m f^{(u)}\}$,

$$_m f^{(u)}: \tilde{\Omega}_1 \times \ldots \times \tilde{\Omega}_u \times \Omega_{u+1} \times \ldots \times \Omega_t \times \ldots \times \Omega_m \to \Omega^{(u)},$$

such that $_m f^{(u)}$ covers $_m f^{(u-1)}$, $_m f^{(0)} = {_m f}$, $_m f^{(t)} = {_m g}$, and the set $\{_m f^{(u)}\}$ induces a homomorphism

$$\phi^{(u)}: H(\Omega^{*(u)}) \to H(\Omega^{(u)}),$$

and an isomorphism of $\pi_1(\Omega^{*(u)})$ onto $\pi_1(\Omega^{(u)})$ mapping $\eta\iota_s$ onto $\eta_s \iota_s$, $u+1 \leqslant s \leqslant t$. We assert that the appropriate cover-transformation groups operate trivially on the homology groups of the spaces $\Omega^{*(u)}$, $\Omega^{(u)}$. This follows from

LEMMA 4.7. *If X is an H-space and Y is a covering space of X, then Y is an H-space and the factor group $\pi_1(X)/\pi_1(Y)$ operates trivially on the homology groups of Y.*

The argument is almost exactly as on p. 478 of [9]. Writing π for the subgroup $\pi_1(Y)$ of $\pi_1(X)$ and E for the space of paths on X emanating from the distinguished point $e \varepsilon X$, we define an operation $f \vee g$ in E by

$$(f \vee g)t = ft \vee gt,$$

the operation on the right being that given by the H-structure of X.

Now Y is obtained from E by identifying $f, f' \varepsilon E$ when and only when the loop h, given by

$$h(t) = f(2t), \quad 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$= f'(2-2t), \quad \tfrac{1}{2} \leqslant t \leqslant 1,$$

represents an element of π. Writing this equivalence $f \underset{\pi}{\simeq} f'$, we have to show that, if $f \underset{\pi}{\simeq} f'$ and $g \underset{\pi}{\simeq} g'$ then $f \vee g \underset{\pi}{\simeq} f' \vee g'$. Now the loop k, given by

$$k(t) = g(2t), \quad 0 \leqslant t < \tfrac{1}{2},$$
$$= g'(2-2t), \quad \tfrac{1}{2} \leqslant t \leqslant 1,$$

represents an element of π, and the loop $h \vee k$ represents the product of the elements represented by h, k; but

$$(h \vee k)(t) = (f \vee g)(2t), \quad 0 \leqslant t \leqslant \tfrac{1}{2}$$
$$= (f' \vee g')(2-2t), \quad \tfrac{1}{2} \leqslant t \leqslant 1,$$

so that Y carries a multiplication induced by that in E. Let us write $e \varepsilon E$ for the null-loop on $e \varepsilon X$ and let $\rho_t : X, e \to X, e$ be a homotopy such that $\rho_0 = 1$, $\rho_1 x = x \vee e$. Then $\rho_t' : E, e \to E, e$, given by $(\rho_t' f)(u) = \rho_t f(u)$, is such that $\rho_0' = 1$, $\rho_1' f = f \vee e$. Moreover, it is easy to veryify that ρ_t' induces a homotopy $Y \to Y$ deforming $\{f\}$ to $\{f\} \vee \{e\}$, rel $\{e\}$, where $\{f\}$,

$\{e\}$ stand for the equivalence classes (points of Y) containing f, e. Similarly $\{f\}$ may be deformed to $\{e\} \vee \{f\}$, rel $\{e\}$, and Y is an H-space.

That $\pi_1(X)/\pi_1(Y)$ operates trivially on the homology groups of Y also follows almost exactly as on p. 478 of [9]. It is only necessary to observe that Serre's second step remains valid in our more general situation because $\pi_1(Y)$ is normal in $\pi_1(X)$, the fundamental group of an H-space being abelian. This completes the proof of Lemma 4.7.

Since ϕ is an isomorphism of $H(\Omega^*)$ onto $H(\Omega)$, Lemma 4.6 now follows from t applications of

LEMMA 4.8. *Let $f: X_1 \to X_2$ be a map inducing isomorphisms*

$$\phi: H(X_1) \cong H(X_2), \quad \pi_1(X_1) \cong \pi_1(X_2).$$

Let π be a normal subgroup of $\pi_1(X_1)$ such that $\pi^1 = \pi_1(X_1)/\pi$ is cyclic infinite and let $\pi^2 = \pi_1(X_2)/\phi\pi$. Let Y_1, Y_2 be covering spaces of X_1, X_2 with cover-transformation groups π^1, π^2, which act trivially on the homology groups of Y_1, Y_2, and let $g: Y_1 \to Y_2$ be the unique map lifting f and sending the class of null-loops to the class of null-loops. Then g induces isomorphisms

$$\psi: H(Y_1) \cong H(Y_2), \quad \pi_1(Y_1) \cong \pi_1(Y_2).$$

It is trivial that g induces the isomorphism $\pi_1(Y_1) \cong \pi_1(Y_2)$. Let σ_i generate π^i and let k_i be the projection $Y_i \to X_i$, $i = 1, 2$. Serre shows on p. 503 of [9], that the sequence

$$0 \to C(Y_i) \xrightarrow{1-\sigma_i} C(Y_i) \xrightarrow{k_i} C(X_i) \to 0 \tag{4.9}$$

is exact, where C stands for chain-group. Moreover, we clearly have commutativity in the diagram

$$\begin{array}{ccccccccc} 0 \to & C(Y_1) & \xrightarrow{1-\sigma_1} & C(Y_1) & \xrightarrow{k_1} & C(X_1) & \to 0 \\ & g\downarrow & & g\downarrow & & \downarrow f & \\ 0 \to & C(Y_2) & \xrightarrow{1-\sigma_2} & C(Y_2) & \xrightarrow{k_2} & C(X_2) & \to 0 \end{array}$$

It follows from (4.9) and the fact that π^i acts trivially on the homology groups of Y_i that there is an exact sequence

$$0 \to H_n(Y_i) \xrightarrow{k_{i*}} H_n(X_i) \xrightarrow{\partial_i} H_{n-1}(Y_i) \to 0, \tag{4.10}$$

and from the commutativity of the previous diagram that the diagram

$$\begin{array}{ccccccccc} 0 \to & H_n(Y_1) & \xrightarrow{k_{1*}} & H_n(X_1) & \xrightarrow{\partial_1} & H_{n-1}(Y_1) & \to 0 \\ & \psi\downarrow & & \phi\downarrow & & \downarrow \psi & \\ 0 \to & H_n(Y_2) & \xrightarrow{k_{2*}} & H_n(X_2) & \xrightarrow{\partial_2} & H_{n-1}(Y_2) & \to 0 \end{array}$$

is commutative. Thus $\phi k_{1*} = k_{2*}\psi$, $\psi\partial_1 = \partial_2\phi$. Since ϕ, k_{1*} are (1-1), the first equation shows that ψ is (1-1), and since ∂_2, ϕ are onto, the second equation shows that ψ is onto. Thus Lemma 4.8 is established and with it Lemma 4.6.

It should be noted that our method of proof of (4.6) enables us to calculate the homology groups of $H(\tilde{\Omega})$ and, of course, of $H(\tilde{\Omega}^*)$; for it follows from (4.10) that the homology groups of Y_i are free abelian if and only if those of X_i are free abelian, in which case

$$H_n(X_i) \cong H_n(Y_i) + H_{n-1}(Y_i).$$

Now the homology groups of Ω are free abelian, so that the homology groups of all the $\Omega^{(u)}$ are free abelian, and, moreover, finitely generated. Thus if $p_n^{(u)}$ is the n-th Betti number of $\Omega^{(u)}$, $p_n^{(0)} = p_n$, $p_n^{(t)} = \tilde{p}_n$, we have

$$p_n^{(u-1)} = p_n^{(u)} + p_{n-1}^{(u)},$$

whence

$$p_n = \tilde{p}_n + t\tilde{p}_{n-1} + \ldots + \binom{t}{r}\tilde{p}_{n-r} + \ldots + \tilde{p}_{n-t},$$

and, inverting this formula,

$$\tilde{p}_n = p_n - tp_{n-1} + \ldots + (-1)^r \frac{t(t+1)\ldots(t+r-1)}{r!} p_{n-r} + \ldots + (-1)^n \frac{t(t+1)\ldots(t+n-1)}{n!} p_0.$$

After this digression we return to the proof of Theorem A.

By an obvious extension of the Whitehead theorem (see [**10**]), we deduce from (4.6) that the set of maps $_mg$ induces isomorphisms of the homotopy groups of $\tilde{\Omega}^*$ onto those of $\tilde{\Omega}$, so that the set of maps $_mf$ induces isomorphisms of the homotopy groups of Ω^* onto those of Ω. But $\pi_{n-1}(\Omega^*) = \sum_{i=1}^{\infty} \pi_{n-1}(\Omega_i)$. Thus $\pi_{n-1}(\Omega) \cong \sum_{i=1}^{\infty} \pi_{n-1}(\Omega_i)$; moreover $\pi_{n-1}(\Omega_i)$ is embedded in $\pi_{n-1}(\Omega)$ by composition with the homotopy class of f_i, so that, in the induced isomorphism

$$\pi_n(T) \cong \sum_{i=1}^{\infty} \pi_n(S^{q_i+1}),$$

$\pi_n(S^{q_i+1})$ is embedded in $\pi_n(T)$ by composition with p_i. This completes the proof of Theorem A.

Theorem 3.3 does not play a prominent role in the statement of Theorem A. $Q(w, k)$ is the number of basic products of weight w, but, in general, different basic products of the same weight belong to homotopy groups of T of different dimensions. However, in the important special case when $r_1 = r_2 = \ldots = r_k = r$, we have

COROLLARY 4.10. *Let T be the union of k $(r+1)$-spheres with a single common point, $r \geqslant 1$. Then*

$$\pi_n(T) \cong \sum_{w=1}^{\infty} \left(\textit{sum of } Q(w, k) \textit{ copies of } \pi_n(S^{wr+1})\right),$$

where $Q(w, k) = \dfrac{1}{w} \sum_{d|w} \mu(d) k^{w/d}$.

5. *The Jacobi identity*

As mentioned in the introduction, if $T = S_1^{p+1} \vee S_2^{q+1} \vee S_3^{r+1}$, then $\left[\iota_1, [\iota_2, \iota_3]\right]$ is not a basic product. This means that, in the direct sum decomposition,

$$\pi_{p+q+r+1}(T) \cong \sum_{i=1}^{\infty} \pi_{p+q+r+1}(S^{q_i+1}),$$

we get a non-trivial representation of $\left[\iota_1, [\iota_2, \iota_3]\right]$,

$$\left[\iota_1, [\iota_2, \iota_3]\right] = \sum_{i=1}^{\infty} p_i \circ \alpha_i, \quad \alpha_i \in \pi_{p+q+r+1}(S^{q_i+1}).$$

Now suppose that $p_{j_1}, \ldots, p_{j_\lambda}, \ldots$ are the basic products not involving S_3. Shrinking S_3 to a point, we get

$$0 = \sum_{\lambda=1}^{\infty} p_{j_\lambda} \circ \alpha_{j_\lambda},$$

whence $\alpha_{j_\lambda} = 0$, $\lambda = 1, 2, \ldots$. Similarly, we see that $\alpha_i = 0$ for any basic product p_i which does not involve all of S_1, S_2, S_3. For a basic product p_i involving S_1, S_2, and S_3, we have $q_i+1 > p+q+r+1$, except for $\left[\iota_2, [\iota_1, \iota_3]\right]$ and $\left[\iota_3, [\iota_1, \iota_2]\right]$. Thus

$$\left[\iota_1, [\iota_2, \iota_3]\right] = a\left[\iota_2, [\iota_1, \iota_3]\right] + b\left[\iota_3, [\iota_1, \iota_2]\right], \tag{5.1}$$

where a, b are integers. Moreover, only one such relation can exist, since there is no non-trivial relation between the basic products. Thus, if ρ is the homomorphism $\rho : \pi_{p+q+r+1}(T) \to H_{p+q+r}(\Omega)$, then $\rho\left[\iota_1, [\iota_2, \iota_3]\right]$ is uniquely expressible as a linear combination of $\rho\left[\iota_2, [\iota_1, \iota_3]\right]$ and $\rho\left[\iota_3, [\iota_1, \iota_2]\right]$, and the relation

$$\rho\left[\iota_1, [\iota_2, \iota_3]\right] = a\rho\left[\iota_2, [\iota_1, \iota_3]\right] + b\rho\left[\iota_3, [\iota_1, \iota_2]\right]$$

subsists in $H_{p+q+r}(\Omega)$ if and only if (5.1) subsists in $\pi_{p+q+r+1}(T)$. The problem is therefore to find a linear relation between $\rho\left[\iota_1, [\iota_2, \iota_3]\right]$, $\rho\left[\iota_2, [\iota_1, \iota_3]\right]$ and $\rho\left[\iota_3, [\iota_1, \iota_2]\right]$ in which the coefficient of $\rho\left[\iota_1, [\iota_2, \iota_3]\right]$ is ± 1.

Let us change notation slightly (in view of the result we wish to prove). Let $T = S_1^p \vee S_2^q \vee S_3^r$, and let us look for a linear relation between $\rho\left[[\iota_2, \iota_3], \iota_1\right]$, $\rho\left[[\iota_3, \iota_1], \iota_2\right]$ and $\rho\left[[\iota_1, \iota_2], \iota_3\right]$ in which the coefficient of $\rho\left[[\iota_2, \iota_3], \iota_1\right]$ is ± 1; it is then clear that the same linear relation will hold† between $\left[[\iota_2, \iota_3], \iota_1\right]$, $\left[[\iota_3, \iota_1], \iota_2\right]$ and $\left[[\iota_1, \iota_2], \iota_3\right]$.

† The ensuing calculation is implicit in the last remark of [8].

Now, by (2.2),

$$\rho\Big[[\iota_2, \iota_3], \iota_1\Big] = (-1)^{q+r}\Big(\rho[\iota_2, \iota_3].e_1 - (-1)^{(q+r)(p-1)} e_1 . \rho[\iota_2, \iota_3]\Big)$$
$$= (-1)^{q+r}\Big\{(-1)^{q-1}\Big(e_2 e_3 - (-1)^{(q-1)(r-1)} e_3 e_2\Big) e_1$$
$$- (-1)^{(q+r)(p-1)} (-1)^{q-1} e_1 \Big(e_2 e_3 - (-1)^{(q-1)(r-1)} e_3 e_2\Big)\Big\}$$
$$= (-1)^{r+1} e_2 e_3 e_1 + (-1)^{qr+q+1} e_3 e_2 e_1$$
$$+ (-1)^{pq+rp+q} e_1 e_2 e_3 + (-1)^{pq+rp+qr+r} e_1 e_3 e_2.$$

Similarly,

$$\rho\Big[[\iota_3, \iota_1], \iota_2\Big] = (-1)^{p+1} e_3 e_1 e_2 + (-1)^{rp+r+1} e_1 e_3 e_2$$
$$+ (-1)^{qr+pq+r} e_2 e_3 e_1 + (-1)^{qr+pq+rp+p} e_2 e_1 e_3,$$

and

$$\rho\Big[[\iota_1, \iota_2], \iota_3\Big] = (-1)^{q+1} e_1 e_2 e_3 + (-1)^{pq+p+1} e_2 e_1 e_3$$
$$+ (-1)^{rp+qr+p} e_3 e_1 e_2 + (-1)^{rp+qr+pq+q} e_3 e_2 e_1.$$

We observe that

$$(-1)^{pq} \rho\Big[[\iota_2, \iota_3], \iota_1\Big] + (-1)^{qr} \rho\Big[[\iota_3, \iota_1], \iota_2\Big] + (-1)^{rp} \rho\Big[[\iota_1, \iota_2], \iota_3\Big] = 0,$$

whence

$$(-1)^{pq}\Big[[\iota_2, \iota_3], \iota_1\Big] + (-1)^{qr}\Big[[\iota_3, \iota_1], \iota_2\Big] + (-1)^{rp}\Big[[\iota_1, \iota_2], \iota_3\Big] = 0.$$

Theorem B now follows by mapping $S_1^p \vee S_2^q \vee S_3^r$ into X by a map which agrees on S_1^p with a representative of $\alpha \in \pi_p(X)$, on S_2^q with a representative of $\beta \in \pi_q(X)$, and on S_3^r with a representative of $\gamma \in \pi_r(X)$.

We note that any further relation between Whitehead products would imply a relation between quasi-commutators; by the same argument as that used by Magnus in the case of ordinary ring commutators (see [**7**]), we have

COROLLARY 5.2. *All identical relations between Whitehead products follow from the skew-commutative law* $[\alpha, \beta] = (-1)^{pq}[\beta, \alpha]$, $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, *and the Jacobi identity by application of the laws of addition and the distributivity of the Whitehead product.*

We now prove

THEOREM 5.3. *Let* $[\xi, \eta]$, $[\eta, \xi] \in \pi_{s+t}(X, A)$ *be generalized Whitehead products of elements* $\xi \in \pi_{s+1}(X, A)$, $\eta \in \pi_t(A)$ *in the sense of* [**1**], $s, t > 1$, *so that the symbol* [ ] *stands for the Whitehead product in whatever sense is relevant. Then, if* $\alpha \in \pi_{p+1}(X, A)$, $\beta \in \pi_q(A)$, $\gamma \in \pi_r(A)$, *we have*

$$(-1)^{(p+1)q}\Big[[\beta, \gamma], \alpha\Big] + (-1)^{qr}\Big[[\gamma, \alpha], \beta\Big] + (-1)^{r(p+1)}\Big[[\alpha, \beta], \gamma\Big] = 0.$$

To prove this, we propose, for convenience, to modify the Blakers-Massey definition of the generalized Whitehead product in a manner due to M. G. Barratt. Consider the homotopy sequence

$$\ldots \to \pi_n(E_1^{s+1} \vee S_2^t,\ S_1^s \vee S_2^t) \xrightarrow{d} \pi_{n-1}(S_1^s \vee S_2^t) \xrightarrow{i} \pi_{n-1}(E_1^{s+1} \vee S_2^t) \to \ldots,$$

where E_1^{s+1} is a cell bounded by S_1^s. Since i is onto in all dimensions, d is (1-1), and it is clear that $d\pi_{s+t}(E_1^{s+1} \vee S_2^t,\ S_1^s \vee S_2^t)$ contains $[\iota_1, \iota_2] \in \pi_{s+t-1}(S_1^s \vee S_2^t)$. Let $\theta = d^{-1}[\iota_1, \iota_2]$, and let (ξ, η) be the class of a map $E_1^{s+1} \vee S_2^t,\ S_1^s \vee S_2^t \to X, A$, which agrees on E^{s+1} with a representative of ξ and on S_2^t with a representative of η. We then define

$$[\xi, \eta]' = (\xi, \eta) \circ \theta, \quad [\eta, \xi]' = (-1)^{st}[\xi, \eta]'.$$

These definitions are related to those of Blakers-Massey by

$$[\xi, \eta]' = -[\xi, \eta], \quad [\eta, \xi]' = (-1)^{t-1}[\eta, \xi]. \tag{5.4}$$

Moreover, using the symbol $[\,]'$ temporarily to denote also the ordinary Whitehead product, we have immediately

$$d[\xi, \eta]' = [d\xi, \eta]', \quad d[\eta, \xi]' = [\eta, d\xi]'. \tag{5.5}$$

Now consider the homotopy sequence

$$\ldots \to \pi_n(E_2^{p+1} \vee S_2^q \vee S_3^r,\ S_1^p \vee S_2^q \vee S_3^r) \xrightarrow{d} \pi_{n-1}(S_1^p \vee S_2^q \vee S_3^r) \xrightarrow{i}$$
$$\to \pi_{n-1}(E_1^{p+1} \vee S_2^q \vee S_3^r) \to \ldots.$$

Again, i is onto in all dimensions, so that d is (1-1), and the image of d obviously contains $\big[[\iota_2, \iota_3]', \iota_1\big]'$, $\big[[\iota_3, \iota_1]', \iota_2\big]'$ and $\big[[\iota_1, \iota_2]', \iota_3\big]'$. Let κ_1 be the positive generator of $\pi_{p+1}(E_1^{p+1}, S_1^p)$. Then it is clear from (5.5) that

$$d\Big\{(-1)^{pq}\big[[\iota_2, \iota_3]', \kappa_1\big]' + (-1)^{qr}\big[[\iota_3, \kappa_1]', \iota_2\big]' + (-1)^{rp}\big[[\kappa_1, \iota_2]', \iota_3\big]'\Big\}$$
$$= (-1)^{pq}\big[[\iota_2, \iota_3]', \iota_1\big]' + (-1)^{qr}\big[[\iota_3, \iota_1]', \iota_2\big]' + (-1)^{rp}\big[[\iota_1, \iota_2]', \iota_3\big]' = 0.$$

It now follows from the univalence of d, by mapping $E_1^{p+1} \vee S_2^q \vee S_3^r$ into X in the obvious way, that

$$(-1)^{pq}\big[[\beta, \gamma]', \alpha\big]' + (-1)^{qr}\big[[\gamma, \alpha]', \beta\big]' + (-1)^{rp}\big[[\alpha, \beta]', \gamma\big]' = 0.$$

The theorem as stated results from this, using (5.4).

6. *Applications*

Let $\gamma \in \pi_n(S^r)$ and let Φ be the homomorphism $\Phi : \pi_n(S^r) \to \pi_n(S_1^r \vee S_2^r)$ obtained by pinching an equatorial S^{r-1} to a point. Let H'_{i-1}, $i = 1, 2, \ldots$, project $\pi_n(S_1^r \vee S_2^r)$ onto its $(i+2)$-nd direct summand in the decomposition given by Theorem A and let $H_{i-1} = H'_{i-1}\Phi$. Thus we have

$$H_0 : \pi_n(S^r) \to \pi_n(S^{2r-1}),$$
$$H_1,\ H_2 : \pi_n(S^r) \to \pi_n(S^{3r-2}),\ \ldots$$

and

$$\Phi\gamma = (\iota_1+\iota_2)\circ\gamma = \iota_1\circ\gamma+\iota_2\circ\gamma+[\iota_1, \iota_2]\circ H_0\gamma$$
$$+\big[\iota_1, [\iota_1, \iota_2]\big]\circ H_1\gamma+\big[\iota_2, [\iota_1, \iota_2]\big]\circ H_2\gamma+\dots \quad (6.1)$$

The homomorphism H_0 is a generalization of the Hopf invariant, and may easily be seen to generalize that defined in [3] if $n \leqslant 4r-4$. In fact, we have

THEOREM 6.2. *$H^* = EH_0$, where $H^*: \pi_n(S^r)\to\pi_{n+1}(S^{2r})$ is the homomorphism defined in* [3], *and E is the suspension.*

This follows immediately from

LEMMA 6.3. *Let $T = S_1^r \vee S_2^s$, let p_i be the i-th basic product, of height q_i, and let $\chi: \pi_{n+1}(S_1^r\times S_2^s,\ S_1^r\vee S_2^s)\to\pi_{n+1}(S^{r+s})$ be induced by shrinking $S_1^r\vee S_2^s$ to a point. Then $\chi d^{-1}(p_i\circ\alpha) = 0$, $\alpha\in\pi_n(S^{q_i+1})$, $i>3$, $\chi d^{-1}(p_3\circ\alpha) = E\alpha$, $\alpha\in\pi_n(S^{r+s-1})$.*

Of course $d: \pi_{n+1}(S_1^r\times S_2^s,\ S_1^r\vee S_2^s)\to\pi_n(S_1^r\vee S_2^s)$ is univalent, and

$$d\pi_{n+1}(S_1^r\times S_2^s,\ S_1^r\vee S_2^s)\cong\sum_{i=3}^{\infty}\pi_n(S^{q_i+1}).$$

The proof is now just as in the proof of Lemma 3 of [4]. In fact, we draw the analogous conclusion, generalizing that of Lemma 3 of [4], namely

THEOREM 6.4. *$\chi\pi_{n+1}(S_1^r\times S_2^s,\ S_1^r\vee S_2^s) = E\pi_n(S^{r+s-1})$.*

The projections H'_{i-1} $(i\geqslant 1)$ are not canonical since they depend essentially on the choice of basic products. However, the basic products p_3, p_4, p_5 would be affected at most by changes of sign and the interchange of p_4, p_5 if the spheres S^r, S^s were reordered and are, to that extent, canonical. We revert later to this question. Meanwhile we prove

LEMMA 6.5. *Let $\alpha, \beta\in\pi_r(X)$, $[\alpha, \beta] = 0$. Then, if $\gamma\in\pi_n(S^r)$,*

$$(\alpha+\beta)\circ\gamma = \alpha\circ\gamma+\beta\circ\gamma.$$

For it follows immediately from (6.1) that†

$$(\alpha+\beta)\circ\gamma = \alpha\circ\gamma+\beta\circ\gamma+[\alpha, \beta]\circ H_0\gamma+\big[\alpha, [\alpha, \beta]\big]\circ H_1\gamma$$
$$+\big[\beta, [\alpha, \beta]\big]\circ H_2\gamma+\dots,$$

and all the later terms disappear since $[\alpha, \beta] = 0$.

THEOREM 6.6. *Let η be the homomorphism induced by the map $S^r\vee S^r\to S^r$ in the class (ι, ι), and let $d': \pi_{n+1}(S^r\times S^r,\ S^r\vee S^r)\to\pi_n(S^r)$ be given*

† As pointed out by Serre, this lemma admits an elementary proof.

by $d' = \eta d$. *Then all elements of* $d' \pi_{n+1}(S^r \times S^r,\ S^r \vee S^r)$ *are expressible as*

$$[\iota,\ \iota] \circ \sigma, \quad \sigma \in \pi_n(S^{2r-1}), \quad r \text{ odd},$$

$$[\iota,\ \iota] \circ \sigma + \big[\iota,\ [\iota,\ \iota]\big] \circ \tau, \quad \sigma \in \pi_n(S^{2r-1}), \quad \tau \in \pi_n(S^{3r-2}), \quad r \text{ even}.$$

In particular, all Whitehead products in $\pi_n(S^r)$ *are so expressible. Moreover,*

$$2([\iota,\ \iota] \circ \sigma) = 0, \quad r \text{ odd},$$

$$3\Big(\big[\iota,\ [\iota,\ \iota]\big] \circ \tau\Big) = 0, \quad r \text{ even}.†$$

We recall from [5] that $\big[\iota,\ [\iota,\ \iota]\big] = 0$, r odd, and that $3\big[\iota,\ [\iota,\ \iota]\big] = 0$ and $\big[\iota,\ [\iota,\ \iota]\big] \in E\pi_{3r-3}(S^{r-1})$, r even. It follows easily that all q-tuple Whitehead products involving only the element ι vanish in S^r, where $q \geqslant 3$ if r is odd, and $q \geqslant 4$ if r is even‡. The first half of the theorem now follows from the fact that, if $\alpha \in d' \pi_{n+1}(S^r \times S^r,\ S^r \vee S^r)$, then

$$\alpha = [\iota,\ \iota] \circ H_0{}' \alpha' + \big[\iota,\ [\iota,\ \iota]\big] \circ H_1{}' \alpha' + \big[\iota,\ [\iota,\ \iota]\big] \circ H_2{}' \alpha' + \dots,$$

where $\alpha = \eta\alpha'$, $\alpha' \in d\pi_{n+1}(S^r \times S^r,\ S^r \vee S^r)$.

The second half of the theorem now follows from Lemma 6.5.

THEOREM 6.7. *Let* $\alpha \in \pi_n(S^r)$, $\beta \in \pi_r(X)$, *and let* k *be an integer. Then, if* r *is odd,*

$$k\beta \circ \alpha = k(\beta \circ \alpha), \quad k = 4m \text{ or } 4m+1,$$

$$= k(\beta \circ \alpha) + [\beta,\ \beta] \circ H_0 \alpha, \quad k = 4m-2 \text{ or } 4m-1.$$

Since $k\beta = \beta \circ k\iota$, $k(\beta \circ \alpha) = \beta \circ k\alpha = \beta \circ k(\iota \circ \alpha)$, and $[\beta,\ \beta] = \beta \circ [\iota,\ \iota]$, it is obviously sufficient to prove the theorem if $\beta = \iota$. Assume k positive, and suppose we have proved that

$$k\iota \circ \alpha = k\alpha + \frac{k(k-1)}{2} [\iota,\ \iota] \circ H_0 \alpha. \tag{6.8}$$

Then $(k\iota + \iota) \circ \alpha = k\iota \circ \alpha + \alpha + [k\iota,\ \iota] \circ H_0 \alpha$, since all the "higher" products vanish. By (6.5), we deduce that

$$(k+1)\iota \circ \alpha = (k+1)\alpha + \frac{(k+1)k}{2} [\iota,\ \iota] \circ H_0 \alpha.$$

Thus (6.8) is proved if k is positive (it is, of course, trivial if $k = 0$ or 1).

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† Some of the results of this section and further results of the same type have been obtained by I. M. James and will be published in a forthcoming paper, "On the suspension triad".

‡ See Theorem 6.10 for a stronger result. By a "q-tuple Whitehead product", we mean an element obtained by $(q-1)$ applications of the Whitehead product operation. Thus a basic product of weight w is a w-tuple Whitehead product.

Again, $0=(k\iota-k\iota)\circ\alpha=k\iota\circ\alpha+(-k\iota)\circ\alpha-k^2[\iota,\iota]\circ H_0\alpha$, so that, k being positive,

$$(-k\iota)\circ\alpha=-k\alpha-\frac{k(k-1)}{2}[\iota,\iota]\circ H_0\alpha+k^2[\iota,\iota]\circ H_0\alpha$$

$$=-k\alpha+\frac{-k(-k-1)}{2}[\iota,\iota]\circ H_0\alpha.$$

This establishes (6.8) for all k, and the theorem follows from the fact that† $2([\iota,\iota]\circ H_0\alpha)=0$.

The analogous result, in case r is even, may be proved similarly. We will merely state it.

THEOREM 6.9. *Let $\alpha\in\pi_n(S^r)$, $\beta\in\pi_r(X)$, and let k be an integer. Then if r is even,*

$$k\beta\circ\alpha=k(\beta\circ\alpha)+\frac{k(k-1)}{2}[\beta,\beta]\circ H_0\alpha+\frac{(k+1)k(k-1)}{3}\big[\beta,[\beta,\beta]\big]\circ H_1\alpha.$$

Note that $3\Big(\big[\beta,[\beta,\beta]\big]\circ H_1\alpha\Big)=0$ and‡

$$\frac{(k+1)k(k-1)}{3}\equiv \;\;0 \bmod 3,\quad k=9m-1,\ 9m,\ 9m+1,$$

$$\equiv -1 \bmod 3,\quad k=9m+2,\ 9m+3,\ 9m+4,$$

$$\equiv \;\;1 \bmod 3,\quad k=9m-4,\ 9m-3,\ 9m-2.$$

Of course, we use in (6.7) and (6.9) the fact that, for any integer q, $q[\iota,\iota]\circ\alpha=q([\iota,\iota]\circ\alpha)$, which follows immediately from (6.5).

THEOREM 6.10. *Let $\alpha\in\pi_l(S^r)$, $\beta\in\pi_m(S^r)$, $\gamma\in\pi_n(S^r)$. Then*

$$\big[\alpha,[\beta,\gamma]\big]=0,\ r \textit{ odd},$$

$$3\big[\alpha,[\beta,\gamma]\big]=0,\ r \textit{ even},$$

and all q-tuple products in S^r vanish, $q>3$.

First, assume r odd. Then $[\beta,\gamma]=[\iota,\iota]\circ\sigma$, by (6.6), and

$$\big[\alpha,[\beta,\gamma]\big]=\big[\iota\circ\alpha,\ [\iota,\iota]\circ\sigma\big].$$

The result now follows from Theorem (2.1) of [**6**] and the fact that $\big[\iota,[\iota,\iota]\big]=0$. Next, assume r even. Then

$$[\beta,\gamma]=[\iota,\iota]\circ\sigma+\big[\iota,[\iota,\iota]\big]\circ\tau \quad\text{and}\quad 3[\beta,\gamma]=3[\iota,\iota]\circ\sigma.$$

† We know of no case in which $[\iota,\iota]\circ H_0\alpha\neq 0$, with r odd.

‡ The author and I. M. James have now proved (independently) that the term $\big[\beta,[\beta,\beta]\big]\circ H_1\alpha$ is zero.

Thus $3\big[\alpha, [\beta, \gamma]\big] = \big[\alpha, 3[\beta, \gamma]\big] = \big[\iota \circ \alpha, 3[\iota, \iota] \circ \sigma\big]$, and the result again follows from Theorem (2.1) of [6] and the fact that $\big[\iota, 3[\iota, \iota]\big] = 0$.

Now, obviously, the vanishing of all q-tuple products implies the vanishing of all q'-tuple products, $q' > q$. Thus the last assertion of the theorem is proved if r is odd and it remains to show that all quadruple products in S^r vanish if r is even. We first prove a lemma.

LEMMA 6.11. $\big[\alpha, [\beta, \gamma]\big] = \big[\iota, [\iota, \iota]\big] \circ \tau$, *for some* $\tau \in \pi_{l+m+n-2}(S^{3r-2})$.

Consider first $\big[\alpha_1, [\beta_1, \gamma_2]\big]$, $\alpha_1 \in \pi_l(S_1^r)$, $\beta_1 \in \pi_m(S_1^r)$, $\gamma_2 \in \pi_n(S_2^r)$. Then

$$\big[\alpha_1, [\beta_1, \gamma_2]\big] = [\iota_1, \iota_2] \circ \sigma + \big[\iota_1, [\iota_1, \iota_2]\big] \circ \tau' + \big[\iota_2, [\iota_1, \iota_2]\big] \circ \tau'' + \dots, \quad (6.12)$$

where $\sigma \in \pi_{l+m+n-2}(S^{2r-1})$, τ', $\tau'' \in \pi_{l+m+n-2}(S^{3r-2})$. Now apply χd^{-1} to (6.12). By an adaptation of the argument of Lemma 3 of [4], we have

$$0 = E\sigma.$$

By Corollary 2 on p. 282 of [10], the order of σ is a power of 2. Now apply η to (6.12). Then $\big[\alpha, [\beta, \gamma]\big] = [\iota, \iota] \circ \sigma + \big[\iota, [\iota, \iota]\big] \circ \tau$, where $\tau = \tau' + \tau''$, and, multiplying by 3, we have $0 = 3([\iota, \iota] \circ \sigma)$. Since the order of σ, and hence of $[\iota, \iota] \circ \sigma$, is a power of 2, it follows that $[\iota, \iota] \circ \sigma = 0$ and the lemma is proved.

Reverting to the theorem, we see that

$$\Big[\delta, \big[\alpha, [\beta, \gamma]\big]\Big] = \Big[\iota \circ \delta, \big[\iota, [\iota, \iota]\big] \circ \tau\Big] = 0,$$

since $\Big[\iota, \big[\iota, [\iota, \iota]\big]\Big] = 0$, and, using the Jacobi identity, all quadruple products vanish, and the theorem is proved.

Next, we take up again the question of the homomorphisms H_{i-1}. We recall from [4] that $2H^* \alpha = 0$ if $\alpha \in \pi_n(S^r)$, r odd. It follows† that $2H_0 \alpha = 0$ if r is odd and $n \leqslant 4r-4$. The difficulty of extending this result to general values of n is due to the fact that the homomorphisms H_{i-1} are not canonical. We demonstrate the difficulty if $n \leqslant 5r-5$. Then $(\iota_1 + \iota_2) \circ \alpha = \iota_1 \circ \alpha + \iota_2 \circ \alpha + p_3 \circ H_0 \alpha + \dots + p_8 \circ H_5 \alpha$, where

$$H_3, H_4, H_5 : \pi_n(S^r) \to \pi_n(S^{4r-3}),$$

and

$$p_6 = \Big[\iota_1, \big[\iota_1, [\iota_1, \iota_2]\big]\Big], \quad p_7 = \Big[\iota_2, \big[\iota_1, [\iota_1, \iota_2]\big]\Big], \quad p_8 = \Big[\iota_2, \big[\iota_2, [\iota_1, \iota_2]\big]\Big].$$

† I. M. James has communicated to me a proof of this for arbitrary n, based on a theorem in his paper, "On the suspension triad".

Let λ be the automorphism $\pi_n(S^r \vee S^r) \cong \pi_n(S^r \vee S^r)$ induced by the map which exchanges the two r-spheres. Then, applying λ, we get, if r is even,

$$(\iota_2+\iota_1)\circ\alpha = \iota_2\circ\alpha+\iota_1\circ\alpha+p_3\circ H_0\,\alpha+p_5\circ H_1\,\alpha+p_4\circ H_2\,\alpha+p_8\circ H_3\,\alpha$$
$$+p_7'\circ H_4\,\alpha+p_6\circ H_5\,\alpha,$$

where, by the Jacobi identity,

$$p_7' = \Big[\iota_1, \big[\iota_2, [\iota_1, \iota_2]\big]\Big] = -p_7+[p_3, p_3]$$
$$= -p_7+p_3\circ[\iota^*, \iota^*], \quad \text{where } \iota^* \text{ generates } \pi_{2r-1}(S^{2r-1}).$$

We deduce, if r is even, $n \leqslant 5r-5$, that

$$H_1 = H_2, \quad H_3 = H_5, \quad 2H_4 = 0, \quad [\iota^*, \iota^*]\circ H_4 = 0. \tag{6.12}$$

Now let r be odd. Again applying λ, we get

$$(\iota_2+\iota_1)\circ\alpha = \iota_2\circ\alpha+\iota_1\circ\alpha+(-p_3)\circ H_0\,\alpha-p_5\circ H_1\,\alpha-p_4\circ H_2\,\alpha$$
$$-p_8\circ H_3\,\alpha-p_7'\circ H_4\,\alpha-p_6\circ H_5\,\alpha,$$

and, by the Jacobi identity, $p_7' = p_7+p_3\circ[\iota^*, \iota^*]$. Also

$$(-p_3)\circ H_0\,\alpha = -(p_3\circ H_0\,\alpha)+[p_3, p_3]\circ H_0(H_0\,\alpha)$$
$$= -(p_3\circ H_0\,\alpha)+p_3\circ[\iota^*, \iota^*]\circ H_0(H_0\,\alpha).$$

We deduce, if r is odd, $n \leqslant 5r-5$, that

$$H_1 = -H_2, \quad H_3 = -H_5, \quad 2H_4 = 0, \quad 2H_0 = [\iota^*, \iota^*]\circ(H_0^2+H_4). \tag{6.13}$$

Note that $4H_0\,\alpha = 0$; on general grounds we may deduce that, if r is odd, $H_0\,\alpha$ is always of order a power of 2. We hope to revert to these questions in a later paper.

We close this section with a theorem which generalizes an original result due to Hopf. Let us say that H_{i-1} has weight w if the basic product p_{i+2} has weight w. Then, if ι stands for the positive generator of $\pi_q(S^q)$ for any q, we have

THEOREM 6.14. *$H_i(k\iota\circ\alpha) = k^w\iota\circ H_i\,\alpha$, where H_i has weight w. In particular*

$$H_i\big((-\iota)\circ\alpha\big) = H_i\,\alpha \text{ if } H_i \text{ has even weight,}$$
$$= -\iota\circ H_i\,\alpha \text{ if } H_i \text{ has odd weight,}$$

For

$$(\iota_1+\iota_2)\circ k\iota\circ\alpha = (k\iota_1+k\iota_2)\circ\alpha = k\iota_1\circ\alpha+k\iota_2\circ\alpha+[k\iota_1, k\iota_2]\circ H_0\,\alpha+\ldots,$$

the general term being $k^w p_{i+3}\circ H_i\,\alpha$, where H_i is of weight w. Since $k^w p_{i+3}\circ H_i\,\alpha = p_{i+3}\circ k^w\iota\circ H_i\,\alpha$, the result follows.

Notice that, if H_i has weight w, then H_i maps $\pi_n(S^r)$ into $\pi_n(S^{w(r-1)+1})$. Now $w(r-1)+1$ is odd if w is even or if w and r are odd. We then deduce, as a corollary, using 6.7,

COROLLARY 6.15. *$H_i(k\iota \circ \alpha) = k^w H_i \alpha$, where H_i has weight w, if*

(i) w is even,

or (ii) w is odd, r is odd, k is even,

or (iii) w is odd, r is odd, $k \equiv 1 \bmod 4$.

7. *Appendix to Section* 3

It seems to be of some interest to consider the general operation of quasi-commutation and certain special cases. The Jacobi identity for triple products proved in Section 5 arises from the existence of a Jacobi identity for the particular quasi-commutators chosen. In general, one would not expect a Jacobi identity to subsist. However, (3.1) tells us that it is always possible to express $a \circ (b \circ c)$ as a linear combination of monomials in the *bqcs* a, b, c, $a \circ b$, $a \circ c$, $b \circ c$, $a \circ (a \circ b)$, $a \circ (a \circ c)$, $b \circ (a \circ b)$, $b \circ (a \circ c)$, $b \circ (b \circ c)$, $c \circ (a \circ b)$, $c \circ (a \circ c)$, $c \circ (b \circ c)$, with zero disorder, and (3.2) tells us that this representation is unique if a, b, c are algebraically independent in R. One would then say that a Jacobi identity subsists if, in fact, $a \circ (b \circ c)$ is expressible as a linear combination of $b \circ (a \circ c)$ and $c \circ (a \circ b)$. The argument adopted in Section 5 may be adapted to show that, if a, b, c are independent, the monomials occurring in the expansion of $a \circ (b \circ c)$ must each contain all of the symbols a, b, c, so that we have proved

THEOREM 7.1. *In a ring R, with quasi-commutators $a \circ b$, we always have*

$$a \circ (b \circ c) = m_1(abc) + m_2 a(b \circ c) + m_3 b(a \circ c) + m_4 c(a \circ b) + m_5 b \circ (a \circ c) + m_6 c \circ (a \circ b),$$

where the m_i are integers, $1 \leqslant i \leqslant 6$, and the representation is unique if a, b, c, are algebraically independent.

If we carry out the process described in Section 3 for the operation $a \circ b = ab + ba$, we find

THEOREM 7.2. *If $a \circ b = ab + ba$, then*

$$a \circ (b \circ c) = 2a(b \circ c) + 2b(a \circ c) + 2c(a \circ b) - b \circ (a \circ c) - c \circ (a \circ b).$$

The case $\lambda = 0$, $\epsilon = -1$ is of combinatorial interest; for then $a \circ b = ba$ and (3.1) allows us to insert brackets into any monomial in $e_1, \ldots, e_k$ in such a way that the bracketed terms constitute a monomial in the *bqcs* of zero disorder. Thus, for example, $e_2^2 e_1^2 = e_2(e_2 e_1^2)$, $e_3 e_2 e_1 = e_3(e_2 e_1)$. The latter example shows all that is left of the "Jacobi identity" in this special case.

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328 Milton Road,
Cambridge.

THE ISOMORPHISM BETWEEN $LF(2, 3^2)$ AND $\mathfrak{A}_6$

W. L. EDGE*.

The isomorphism of the linear fractional group $LF(2, 3^2)$, of order 360, and the alternating group $\mathfrak{A}_6$ is on record ([1], 309; and [5], with *PSL* for *LF*, 8 and 9). If one seeks not merely for record but for proof one might take the conclusion to Chapter XII of [1], in which chapter the subgroups of $LF(2, q)$ for any Galois field $GF(q)$ are obtained and catalogued; representations of $LF(2, q)$ as permutation groups have degrees equal to the indices of these subgroups whenever $LF(2, q)$ is simple, as it is known to be if $q > 3$. For $q = 3^2$ there are subgroups of index 6. But while the fact of the isomorphism has been common knowledge for so long, and while it cannot be gainsaid that a proof has been available, it may be questioned whether an essential reason for the existence of these subgroups of index 6 has been perceived. $LF(2, q)$ may be handled as

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