Topological Modular Forms, the Witten Genus, and the Theorem of the Cube

MICHAEL J. HOPKINS

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA 02139, USA

1. Introduction

There is a rich mathematical structure attached to the cobordism invariants of manifolds. In the cases described by the index theorem, a generalized cohomology theory is used to express the global properties of locally defined analytic objects. Hirzebruch's theory of multiplicative sequences gives an expression for these invariants in terms of characteristic classes, and brings to focus their remarkable arithmetic properties. Quillen's theory of formal groups and complex oriented cohomology theories illuminates the generalized cohomology theories themselves.

Around eight years ago a new invariant, the *elliptic genus*, was introduced [17]. It is a cobordism invariant of oriented manifolds that takes its values in a certain ring of modular forms. Witten [23], [22] proposed an analytic interpretation of the elliptic genus using analysis on loop spaces. Landweber, Ravenel, and Stong [13] constructed a corresponding cohomology theory (elliptic cohomology), and it is believed that there is an "index" theorem relating analysis on loop space to elliptic cohomology. So far, a satisfying mathematical theory is lacking.

In the same papers [23], [22] Witten introduced a variant of the elliptic genus, now known as the Witten genus. The Witten genus takes its values in modular forms when applied to Spin manifolds with $\frac{p_1}{2} = 0$. The cohomological significance of this invariant has remained unclear.

The point of this note is to describe a generalization of theories of Hirzebruch and Quillen to the cobordism of Spin manifolds with $\frac{p_1}{2} = 0$. It turns out that in the presence of an elliptic curve there is a canonical cobordism invariant. This invariant coincides with the Witten genus in the case where the elliptic curve is the Tate curve, though it is most natural to consider all elliptic curves at once. This leads to a cohomological expression for the modular invariance of the Witten genus (of a family), and to a new generalized cohomology theory. The coefficient ring of this new cohomology theory is the ring of *topological modular forms*. It is related to the ring of modular forms over $\mathbb{Z}$, but is not torsion free. The torsion groups in this ring represent new invariants of Spin manifolds with $\frac{p_1}{2} = 0$, and it would be interesting to describe these invariants in terms of geometry and analysis.

Most of this paper represents joint work with Matthew Ando and Neil Strickland. The construction and computations with the new cohomology theory are

Proceedings of the International Congress
of Mathematicians, Zürich, Switzerland 1994

joint work with Mark Mahowald and Haynes Miller. Some of the results described here represent work in progress.

2. Genera and their characteristic series

Let R be a commutative ring. An R-valued *genus* is a ring homomorphism Φ from some type of cobordism ring to R. Thus a genus is a function Φ that assigns to each manifold M an element $\Phi(M) \in R$, and that satisfies

$$\begin{aligned} \Phi(M_1 \coprod M_2) &= \phi(M_1) + \phi(M_2) \\ \Phi(M_1 \times M_2) &= \phi(M_1)\phi(M_2) \\ \Phi(\partial M) &= 0. \end{aligned}$$

The cobordism rings usually considered are the ring MU_* of cobordism classes of stably almost complex manifolds, and the ring MSO_* of cobordism classes of oriented manifolds. The structure of these rings has been determined [20], [14], [16], [21], and there are isomorphisms

$$\begin{aligned} MU_* \otimes \mathbb{Q} &\approx \mathbb{Q}[\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^2, \dots] \\ MSO_* \otimes \mathbb{Q} &\approx \mathbb{Q}[\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^4, \dots]. \end{aligned}$$

When R is torsion free, a genus is determined by its values on the complex projective spaces. There are two natural generating functions that collect these values

$$\log_\Phi(z) = \sum \Phi(\mathbb{C}\mathrm{P}^n) \frac{z^{n+1}}{n+1} \qquad \text{(logarithm)}$$

$$K_\Phi(z) = \frac{z}{\exp_\Phi(z)}, \qquad \text{(characteristic series)}$$

where $\exp_\Phi(z) = \log_\Phi^{-1}(z)$. A genus Φ with values in a torsion free ring factors through MSO_* if and only if the characteristic series is even

$$K_\Phi(z) = K_\Phi(-z). \tag{2.1}$$

The characteristic series determines a stable exponential characteristic class with values in $H^*(-; R \otimes \mathbb{Q})$ as follows. By the splitting principle, such a class is determined by its value on the complex line bundle L over BS^1 associated to the identity character. Setting $z = c_1(L)$, the characteristic class is then defined by

$$K_\Phi(L) = K_\Phi(z) \in H^*(BS^1; R \otimes \mathbb{Q}).$$

The following formula of Hirzebruch [10] expresses $\Phi(M)$ in terms of characteristic (Pontryagin or Chern) classes:

$$\Phi(M) = \langle K_\Phi(TM), [M] \rangle.$$

Here are some examples.

(1) The genus whose characteristic series is $z/(1 - e^{-z})$ is the Todd genus. The log of the Todd genus is the power series

$$-\log(1 - x) = \sum \frac{x^n}{n},$$

so its value on $\mathbb{C}P^n$ is 1. It can be shown that the Todd genus of a stably almost complex manifold is an integer.

(2) The genus with characteristic series

$$K(z) = \frac{z/2}{\sinh(z/2)} = \frac{z}{e^{z/2} - e^{-z/2}}$$

is the $\hat{A}$ genus. It is an invariant of oriented manifolds. The $\hat{A}$-genus has the property that it assumes integer values on manifolds that admit a Spin structure.

(3) The genus with logarithm

$$\log_{\Phi}(z) = \int_0^z (1 - 2\delta t^2 + \varepsilon t^4)^{-\frac{1}{2}}\, dt$$

is the *elliptic genus* of Ochanine [17]. The associated characteristic series is even, so it is an invariant of oriented manifolds.

(4) The *Witten genus* [23], [22] is the genus with characteristic series

$$\frac{z/2}{\sinh(z/2)} \prod_{n \geq 1} \frac{(1-q^n)^2}{(1-q^n e^z)(1-q^n e^{-z})}.$$

This is an even function of z, and so defines a cobordism invariant of oriented manifolds. The Witten genus takes values in $\mathbb{Z}[[q]]$ when applied to manifolds that admit a Spin structure.

There is a dimension 4 characteristic class of Spin bundles, twice which is p_1. Let's denote this class $\frac{p_1}{2}$. If M is a Spin manifold of dimension n, and $\frac{p_1}{2}(TM) = 0$, then the Witten genus of M turns out to be the q-expansion of a modular form for the group $SL_2(\mathbb{Z})$. This means that after setting $q = e^{2\pi i \tau}$, the Witten genus of M can be written as $f(\tau)$, where f is a holomorphic function on the upper half plane $\operatorname{Re}\tau > 0$, and satisfies the functional equation

$$f(-1/\tau) = (-\tau)^n f(\tau). \tag{2.2}$$

3. Genus of a family

The underlying geometry of a genus begins to be revealed when its definition is extended to families. Let M_s be a family of manifolds parameterized by the points of a space S. The manifolds M_s are allowed to transform through cobordisms, but are required to be equidimensional of dimension, say, n. Such a family defines an element of the generalized cohomology group $\mathbf{M}^{-\mathbf{n}}(\mathbf{S})$, where $\mathbf{M}$ is the cohomology theory associated to the type of cobordism being considered.

For a genus Φ the quantities $\Phi(M_s)$ form some kind of structure parameterized by the space S. It is best to think of this structure as representing an element of a generalized cohomology group $E^{-n}(S)$. A *genus for families* of manifolds is then a multiplicative map

$$\mathbf{M} \to \mathbf{E}$$

of generalized cohomology theories.

The process of extending the definition of a genus to families is not at all canonical, and is intimately connected with the expression of the genus in terms of geometry and analysis.

Several kinds of cobordisms will be used in this paper. They are displayed below. The diagram on the left is a diagram of classifying spaces. The map labeled $\frac{p_1}{2}$ is the universal characteristic class of the same name, and the spaces $BU\langle 6\rangle$ and $BO\langle 8\rangle$ are the homotopy fibers of the map $\frac{p_1}{2}$ and its restriction to BSU, respectively. The diagram on the right is the corresponding diagram of cobordism theories. For example, a $BO\langle 8\rangle$-manifold is a manifold equipped with a lift to $BO\langle 8\rangle$ of the map classifying its stable tangent bundle, and $MO\langle 8\rangle$ is the cohomology theory associated to the cobordism of $BO\langle 8\rangle$-manifolds.

$$\begin{array}{ccccc} BU\langle 6\rangle & \longrightarrow & BO\langle 8\rangle & & \\ \downarrow & & \downarrow & & \\ BSU & \longrightarrow & B\mathrm{Spin} & \xrightarrow{\frac{p_1}{2}} & K(\mathbb{Z},4) \\ \downarrow & & \downarrow & & \\ BU & \longrightarrow & BSO & & \end{array} \qquad \begin{array}{ccc} MU\langle 6\rangle & \longrightarrow & MO\langle 8\rangle \\ \downarrow & & \downarrow \\ MSU & \longrightarrow & M\mathrm{Spin} \\ \downarrow & & \downarrow \\ MU & \longrightarrow & MSO. \end{array}$$

The "families" versions of the genera of Section 2 are as follows.

(1) The natural domain for the Todd genus is MU, the theory of complex cobordism. The target of the Todd genus can be taken to be ordinary cohomology with coefficients in the rational numbers. This, however, obscures the fact that the Todd genus of each individual manifold is an integer. If the Todd genus is thought of as a formula for the dimension (Euler characteristic) of certain cohomology sheaves, then the natural target appears as K-theory [4], [1].

(2) The $\hat{A}$ genus is most interesting when applied to Spin manifolds, making the natural domain the cohomology theory MSpin. Atiyah and Singer [2] showed that the $\hat{A}$-genus is the index of the Dirac operator, and portrayed the natural target of the "families $\hat{A}$-genus" as the cohomology theory KO (bundles of vector spaces over $\mathbb{R}$). This refinement represents more than an accounting of the integrality properties of the genus. The groups

$$KO^0\left(S^{8k+1}\right) \approx KO^0\left(S^{8k+2}\right) \approx \mathbb{Z}/2$$

correspond to torsion invariants of families of Spin-manifolds. These invariants can be described in terms of analysis but can not be calculated in terms of Pontryagin classes [3].

(3) In the case of the elliptic genus, it can be shown that the functor

$$\mathrm{Ell}^*(-) = MSO^*(-) \underset{MSO_*}{\otimes} \mathbb{Z}[\tfrac{1}{6}, \delta, \varepsilon, \Delta, \Delta^{-1}]/(2^6\varepsilon(\delta^2-\varepsilon)^2-\Delta)$$

defines a generalized cohomology theory [13], [12], [9] on the category of finite cell complexes. This represents a natural extension of the elliptic genus to families, but, at present, there is no known geometric interpretation of Ell (see, however, the exposé of Segal [19]).

(4) The natural domain for the Witten genus is the cohomology theory $MO\langle 8\rangle$. There is a map

$$MO\langle 8\rangle \to KO[[q]]$$

representing the Witten genus. It accounts for the integrality properties, and has some associated torsion invariants. On the other hand it factors through $M\mathrm{Spin}$, and so cannot possibly express the transformation properties with respect to the modular group. This is related to the fact that behavior with respect to the transformation $\tau \mapsto 1/\tau$ is very difficult to understand from the point of view of power series in q.

4. Cubical Structures

A deeper understanding of the Witten genus of a family requires investigating the genera attached to the cobordism theories $MU\langle 6\rangle$ and $MO\langle 8\rangle$. The result of Hirzebruch, that a genus can be calculated by integrating a stable exponential characteristic class over the manifold, remains valid for these theories. However, a stable characteristic class is not determined by its value on L. In fact it does not even have a value on L, as the structure group of L does not lift to $BO\langle 8\rangle$. On the other hand, the (virtual) bundle

$$V_3 = (L_1 - 1) \otimes (L_2 - 1) \otimes (L_3 - 1), \tag{4.1}$$

over $(BS^1)^3$, admits a canonical lift of its structure group to $BU\langle 6\rangle$. Furthermore, there is a "splitting principle" that allows one to formally express any $BU\langle 6\rangle$ bundle as a sum of trivial line bundles and bundles of this kind. The cohomology of $(BS^1)^3$ is a polynomial algebra in three variables, so one expects the characteristic series of an $MU\langle 6\rangle$ genus to be a function of three variables. This is indeed the case, and the series that arise satisfy a certain functional equation. There is a geometric interpretation of this functional equation that is particularly suited to the study of elliptic spectra. It is known as a *cubical structure*, and was introduced by Breen [5] in order to codify the the rich structure attached to line bundles on abelian varieties coming from the theorem of the cube.

Let G be an abelian group, and $\mathfrak{L}$ a line bundle over G. The group G might be a discrete group, an algebraic group, a topological group, or a group of some other kind. The line bundle $\mathfrak{L}$ consists of a collection of lines $\mathfrak{L}_x$ for $x \in G$, and should be thought of as varying discretely, algebraically, continuously, or in some other manner, depending on the kind of group.

Given G and $\mathfrak{L}$, let $\Theta(\mathfrak{L})$ be the line bundle over G^3 whose fiber at (x, y, z) is

$$\Theta(\mathfrak{L})_{(x,y,z)} = \frac{\mathfrak{L}_{x+y+z}\mathfrak{L}_x\mathfrak{L}_y\mathfrak{L}_z}{\mathfrak{L}_{x+y}\mathfrak{L}_{x+z}\mathfrak{L}_{y+z}\mathfrak{L}_e},$$

where $e \in G$ is the identity element. In this expression, multiplication and division are meant to indicate tensor product of lines and their duals.

The functor Θ is a kind of "second difference" operator. If the terms "line bundle" and "tensor product" are replaced with "function" and "addition," then Θ becomes the operator whose kernel consists of quadratic functions.

A *cubical structure* on $\mathfrak{L}$ is a section s of $\Theta(\mathfrak{L})$ satisfying

$$\begin{array}{lrcl} \text{(rigid)} & s(e,e,e) & = & 1 \\ \text{(symmetry)} & s(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}) & = & s(x_1,x_2,x_3) \\ \text{(cocycle)} & s(w+x,y,z)s(w,x,z) & = & s(w,x+y,z)s(x,y,z). \end{array}$$

The two sides of these equations are sections of different bundles, so a comment is in order. In each case a canonical identification can be made. For example, the section $s(e,e,e)$ is an element of $\Theta(\mathfrak{L})_{(e,e,e)}$, which is the tensor product of four copies of $\mathfrak{L}_e$ with its dual. Contracting the lines with their duals gives an identification of this with the trivial line, and it is via this identification that the equation labeled "rigid" takes place. There are similar canonical identifications that need to be made for the other equations.

The set of cubical structures on $\mathfrak{L}$ will be denoted $C^3(G;\mathfrak{L})$.

If the line bundle $\mathfrak{L}$ comes equipped with a symmetry isomorphism

$$t : \mathfrak{L}_x \approx \mathfrak{L}_{-x}$$

then the fiber of $\Theta(\mathfrak{L})$ over the point $(x,y,-x-y)$ admits a canonical trivialization. A Σ-*structure* on $\mathfrak{L}$ is a cubical structure s with the property that

$$s(x,y,-x-y)=1. \tag{4.2}$$

The set of Σ-structures on $\mathfrak{L}$ will be denoted $C^3_0(G;\mathfrak{L},t)$.

5. Formal groups and complex orientable spectra

The group that arises in homotopy theory is the formal group attached to a complex orientable cohomology theory [18]. Recall that a cohomology theory E is *complex orientable* if there is a class $x \in E^*BS^1$ whose restriction to $E^*S^2 \approx E^{*-2}(\text{pt})$ is a unit. A choice of such an x gives rise to a very rich structure, and in particular, to a theory of E-valued Chern classes for complex vector bundles.

Suppose that E is a multiplicative, complex orientable cohomology theory with the additional properties that

$$E_*(\text{pt}) \text{ is commutative} \tag{5.1}$$

$$E_2(\text{pt}) \text{ contains a unit.} \tag{5.2}$$

With these assumptions, the ring $E^0(BS^1)^n$ is isomorphic to a formal power series ring in n variables over $E^0(\text{pt})$. The multiplication map

$$BS^1 \times BS^1 \to BS^1$$

gives the formal spectrum $G = \operatorname{spf} E^0BS^1$ the structure of a formal group. In terms of "physical" groups, it provides the abelian group structure on the functor $G = \operatorname{Hom}(E^0BS^1, -)$, from the category of augmented $E^0(\text{pt})$-algebras with nilpotent augmentation ideal, to the category of abelian groups.

The formal group G is the one of interest. The ring of functions on G is isomorphic to a formal power series ring in one variable over $E^0(\text{pt})$. Let $\mathfrak{L}$ be the line bundle $\mathcal{O}(-e)$, whose local sections are functions that vanish at the unit. The module of global sections of $\mathfrak{L}$ is the reduced cohomology group $\widetilde{E}^0BS^1$. This line bundle comes with an obvious symmetry isomorphism $t : \mathfrak{L}_x \approx \mathfrak{L}_{-x}$.

6. A homology calculation

Now let R be a commutative ring, and suppose E is as above. A map

$$E_0 BU\langle 6\rangle \to R$$

can be composed with the map classifying V_3 (4.1), to yield an $E^0(\mathrm{pt})$-module map from $E_0\left(BS^1\right)^3$ to R. This can be thought of as an R-valued function f on G^3. It satisfies the following equations (in which the symbol "+" refers to addition in the group G):

$$\begin{aligned} f(e,e,e) &= 1 \\ f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) &= f(x_1,x_2,x_3) \\ f(w+x,y,z)f(w,x,z) &= f(w,x+y,z)f(x,y,z). \end{aligned}$$

The first two of these equations are obvious. The last arises from the tensor product of (L_4-1) with the equation

$$(L_1L_2-1)(L_3-1)+(L_1-1)(L_2-1) \\ = (L_1-1)(L_2L_3-1)+(L_2-1)(L_3-1).$$

Stated another way, the function f defines a cubical structure on the trivial line bundle $\mathcal{O}_G$.

THEOREM 6.1. *The map described above gives rise to a natural isomorphism*

$$\operatorname{spec} E_0 BU\langle 6\rangle \approx C^3(G;\mathcal{O}_G)$$

of functors on the category of multiplicative complex orientable cohomology theories E satisfying (5.1) *and* (5.2).

For multiplicative cohomology theories E and F, let $\mathrm{Mult}(E,F)$ be the set of multiplicative transformations from E to F. In terms of the representing spectra, this is the set of homotopy classes of homotopy multiplicative maps.

The following theorem is proved by applying the Thom isomorphism to Theorem 6.1.

THEOREM 6.2. *The map described above gives rise to a natural isomorphism*

$$\mathrm{Mult}(MU\langle 6\rangle, E) \approx C^3(G;\mathfrak{L})$$

of functors on the category of multiplicative complex orientable cohomology theories E satisfying (5.1) *and* (5.2), *with associated formal group G. If $\frac{1}{2}\in E^0(pt)$, or if E is $K(n)$-local for some Morava K-theory $K(n)$, with $n\le 2$, then this descends to a natural isomorphism*

$$\mathrm{Mult}(MO\langle 8\rangle, E) \approx C^3_0(G;\mathfrak{L},t).$$

There are even more general criteria guaranteeing the validity of the second assertion, but they involve a lengthy discussion.

Theorem 6.2 is analogous to the result that a genus with values in a torsion free ring is determined by its characteristic series. The role of the characteristic series is played by a cubical structure. The analogue of the symmetry condition (2.1) is condition (4.2).

7. Elliptic spectra

Theorem 6.2 is most interesting when the formal group G is extended to an elliptic curve. For the purposes of this paper, an *elliptic curve* is a generalized elliptic curve in the sense of [8, Définition 1.12], all of whose geometric fibers are irreducible.

DEFINITION 7.1. An *elliptic spectrum* consists of

(1) a complex orientable spectrum E satisfying (5.1) and (5.2), with associated formal group G;

(2) an elliptic curve $\mathbf{E}$ over $E^0(\mathrm{pt})$;

(3) an isomorphism $t : G \to \mathbf{E}^{\mathrm{f}}$ from G to the formal completion of $\mathbf{E}$.

The third condition requires explanation. The elliptic curve $\mathbf{E}$ gives rise to an abelian group-valued functor on $E^0(\mathrm{pt})$-algebras, by associating to an algebra R the abelian group of R-valued points of $\mathbf{E}$. Restricting this functor to the category of augmented $E^0(\mathrm{pt})$-algebras with nilpotent augmentation ideal gives a formal group $\mathbf{E}^{\mathrm{f}}$. This is the *formal completion* of $\mathbf{E}$. The isomorphism $G \to \mathbf{E}^{\mathrm{f}}$ is then an isomorphism of formal groups.

The collection of elliptic spectra forms a category, in which a map consists of a multiplicative map of cohomology theories, and a map of elliptic curves that is compatible with the associated map of formal groups.

THEOREM 7.2. *Attached to each elliptic spectrum E is a multiplicative map*

$$\sigma_E : MU\langle 6\rangle \to E.$$

This map is modular *in the sense that if $f : E \to F$ is a map of elliptic spectra, then $\sigma_F = f \circ \sigma_E$. If $\frac{1}{2} \in E$, or if E has the property that $E^*(pt)$ is torsion free and concentrated in even degrees, then $MU\langle 6\rangle$ can be replaced with $MO\langle 8\rangle$.*

In the case where $E = K[[q]]$, and $\mathbf{E}$ is the Tate curve, the map

$$\pi_* \sigma_E : MO\langle 8\rangle_* \to \mathbb{Z}[[q]]$$

can be shown to be the Witten genus. The *modular* invariance of the genus σ_E is an expression of the modular invariance of the "families" Witten genus. In the next section it will be explained how this reduces to "modular invariance" in the classical sense, when the parameter space S consists of only one point.

The main tool used to deduce Theorem 7.2 from Theorem 6.2 is the theorem of the cube.

THEOREM 7.3 Theorem of the cube. *If $\mathfrak{L}$ is a line bundle over an abelian variety, then $\Theta(\mathfrak{L})$ is trivial.*

Topologically this result follows from the facts that line bundles are classified by $H^2(-;\mathbb{Z})$, and H^2 is a quadratic functor. The theorem of the cube is the analogue of this assertion for algebraic line bundles.

It follows from the theorem of the cube that any line bundle over an abelian variety has a canonical cubical structure. Indeed, the only sections of $\Theta(\mathfrak{L})$ are constants, and any potential cubical structure must assume the value 1 at the unit. The "rigid," "symmetry," and "cocycle" conditions become identities between constant functions that assume a prescribed value at the unit.

Proof of Theorem 7.2, given Theorem 6.2. The unique section s of $\Theta(\mathfrak{L})$ satisfying $s(e,e,e)=1$, and extending to a section of $\Theta(\mathcal{O}_{\mathbf{E}}(-e))$, is automatically an element of $C_0^3(G;\mathfrak{L},t)$. Take σ_E to be the multiplicative map associated to s by Theorem 6.2. □

8. Modularity

The point of this section is to relate the "modular" invariance of the maps σ_E to modular forms. This leads naturally to two new cohomology theories.

Let $\mathcal{M}_{\mathrm{Ell}}$ be the category whose objects are elliptic curves

$$\mathbf{E} \xrightarrow{\mathbf{p}} \mathbf{S}$$

with identity section e, and in which morphisms are cartesian squares

$$\begin{array}{ccc} \mathbf{E}' & \longrightarrow & \mathbf{E} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

This is the *elliptic moduli stack* (see [15], [8], [7]), as is denoted $\mathcal{M}_1$ in [8].

For an elliptic curve $\mathbf{E}/\mathbf{S}$ let $\omega_{\mathbf{E}/\mathbf{S}} = e^*\Omega^1_{\mathbf{E}/\mathbf{S}}$ be the line bundle over S consisting of invariant 1-forms along the fibers. For each $k\in\mathbb{Z}$, let ω^k be the functor on $\mathcal{M}_{\mathrm{Ell}}$ whose value on $\mathbf{E}/\mathbf{S}$ is the abelian group of global sections of ${\omega_{\mathbf{E}/\mathbf{S}}}^k$. The collection of functors ω^k forms a functor ω^* on $\mathcal{M}_{\mathrm{Ell}}$ with values in graded rings. The *ring of modular forms over* $\mathbb{Z}$ is the graded ring

$$R_* = \varprojlim{}_{\mathcal{M}_{\mathrm{Ell}}} \omega^*.$$

This ring has been determined [6, Prop. 6.1], and there is an isomorphism

$$R_* \approx \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

The grading is such that the class c_n is homogeneous of degree n.

The ring R maps to the classical ring of modular forms by restricting to the inverse limit over the full subcategory of $\mathcal{M}_{\mathrm{Ell}}$ whose only object is the usual family of elliptic curves over the upper half plane. The automorphism group of this object is the group $SL_2(\mathbb{Z})$. This map sends c_4 to $2^4\cdot 3^2\cdot 5\cdot E_2$ and c_6 to $2^5\cdot 3^3\cdot 5\cdot 7\cdot E_3$, where E_2 and E_3 are the Eisenstein series of weights 4 and 6 respectively. The element Δ maps to the discriminant.

Attached to each elliptic spectrum E is the elliptic curve $\mathbf{E}$ over $S=\operatorname{spec}\pi_0 E$. The isomorphism $G\approx \mathbf{E}^{\mathrm{f}}$ determines an isomorphism

$$E^0(S^{2k}) = E^{-2k}(\mathrm{pt}) \approx \omega^k(\mathbf{E}/\mathbf{S}).$$

It turns out that there are enough elliptic spectra that there are isomorphisms

$$\varprojlim_{\mathrm{E\ elliptic}} E^{-2k}(\mathrm{pt}) \approx R_k,$$

$$\varprojlim_{\mathrm{E\ elliptic}} E^{2k+1}(\mathrm{pt}) \approx 0.$$

Moreover, Theorem 7.2 shows that the orientations σ_E give rise to a map

$$MO\langle 8\rangle_* \to \varprojlim_{\text{E elliptic}} E_*(\text{pt}).$$

This proves that the Witten genus takes its values in R_*. One can, however, hope for a more refined statement. This is the subject of the next section.

9. Topological modular forms and eo_2

The category of elliptic spectra is closely related to the elliptic moduli stack. There is one important difference. Whereas there is no "good" colimit of the objects in $\mathcal{M}_{\text{Ell}}$, the homotopy inverse limit in spectra, of the category of elliptic spectra, can be formed. The resulting spectrum is no longer elliptic, but it still represents an interesting cohomology theory.

In practice it is necessary to "rigidify" the category of elliptic spectra by working with a certain subcategory of A_∞ etale elliptic spectra. The A_∞ condition has to do with higher homotopy associativity of E, and the etale condition is that the map $\operatorname{spec} \pi_0 E \to \mathcal{M}_{\text{Ell}}$ which classifies **E** is etale and open. The other conditions defining this subcategory arise from obstruction theory and will remain unspecified. Though the notation is slightly misleading, the homotopy inverse limits that follow are taken over this subcategory.

Define eo_2 to be the connected cover of

$$\underset{E,\ A_\infty \text{ etale elliptic}}{\underleftarrow{\text{holim}}} E$$

and let EO_2 be the spectrum

$$\underset{\substack{E,\ A_\infty \text{ etale elliptic} \\ \text{E smooth}}}{\underleftarrow{\text{holim}}} E.$$

These spectra are topological models for the moduli space of elliptic curves. There is a spectral sequence

$$\varprojlim{}^s_{\mathcal{M}_{\text{Ell}}} \omega^k \Rightarrow \pi_{2k-s} eo_2, \tag{9.1}$$

so it makes sense to call the ring $eo_{2*}(\text{pt})$ the ring of *topological modular forms.*

The spectrum EO_2 is closely related to a spectrum constructed by the author and Miller, and the spectrum eo_2 is closely related to one constructed by the author and Mahowald [11].

This spectral sequence (9.1) has been computed by the author and Mahowald. It terminates at a finite stage. One interesting feature is that the discriminant Δ is not a permanent cycle, whereas the forms 24Δ and Δ^{24} are. The form Δ^{24} is not a divisor of zero. There is an isomorphism

$$EO_{2*}(-) \approx (\Delta^{24})^{-1} eo_{2*}(-).$$

The cohomology theory EO_2 is periodic with period $24^2 = 576$.

The torsion in eo_{2*} is annihilated by 24. It has a very rich structure. The cohomology theory eo_2 can be used to account for nearly everything that is known about the stable homotopy groups of spheres in dimensions less than 60.

Regarding the Witten genus, the more refined statement for which one can hope is that the maps σ_E assemble to a multiplicative map

$$MO\langle 8\rangle \to eo_2.$$

This is consistent with many calculations, and is the subject of work in progress. It is the most natural target for the "families" Witten genus, and would define new torsion invariants of Spin-manifolds with $\frac{p_1}{2} = 0$. It would be very interesting to have an explanation of these invariants in terms of geometry and analysis.

References

[1] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281.

[2] M. F. Atiyah and I. M. Singer, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. **69** (1963), 422–433.

[3] ———, *The index of elliptic operators: V*, Ann. of Math. **93** (1971), 139–149.

[4] A. Borel and J.-P. Serre, *Le théorème de Riemann-Roch (d'après Grothendieck)*, Bull. Soc. Math. France **86** (1958), 97–136.

[5] L. Breen, Fonctions théta et théorème du cube, Lecture Notes in Math. **980**, Springer-Verlag, Berlin and New York, 1983.

[6] P. Deligne, *Courbes elliptiques: formulaire (d'après J. Tate)*, Modular Functions of One Variable III, Lecture Notes in Math. **476**, Springer-Verlag, Berlin and New York, 1975, pp. 53–73.

[7] P. Deligne and D. Mumford, *Irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.

[8] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular Functions of One Variable II, Lecture Notes in Math. **349**, Springer-Verlag, Berlin and New York, 1973, pp. 143–316.

[9] J. Franke, *On the construction of elliptic cohomology*, Max Planck Institut preprint, 1991.

[10] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer-Verlag, New York, 1956.

[11] M. J. Hopkins and M. Mahowald, *The spectrum eo_2*, to appear.

[12] P. S. Landweber, *Elliptic cohomology and modular forms*, Elliptic Curves and Modular Forms in Algebraic Topology (New York) (P. S. Landweber, ed.), Lecture Notes in Math. **1326**, Springer-Verlag, 1988, pp. 55–68.

[13] P. S. Landweber, D. C. Ravenel, and R. E. Stong, *Periodic cohomology theories defined by elliptic curves*, To appear.

[14] J. W. Milnor, *On the cobordism ring Ω^* and a complex analogue, Part I*, Amer. J. Math. **82** (1960), 505–521.

[15] D. Mumford, *Picard groups of moduli problems*, Arithmetic Algebraic Geometry (O. F. G. Schilling, ed.), Harper & Row, New York, 1965.

[16] S. P. Novikov, *Some problems in the topology of manifolds connected with the theory of Thom spaces*, Soviet Mathematics Dokl. **1** (1960), 717–720.

[17] S. Ochanine, *Sur les genres multiplicatifs definis par des integrals elliptiques*, Topology **26** (1987), 143–151.

[18] D. G. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

[19] G. Segal, *Elliptic cohomology*, Séminaire Bourbaki 1987/88, Astérisque, vol. 161-162, Societe Mathematique de France, Fév. 1988, pp. 187–201.

[20] R. Thom, *Quelques propriétés globales des variétés differentiables*, Comm. Math. Helv. **28** (1954), 17–86.

[21] C. T. C. Wall, *Determination of the cobordism ring*, Ann. Math. (2) **72** (1960), 292–311.

[22] E. Witten, *Elliptic genera and quantum field theory*, Comm. Math. Phys. **109** (1987), 525–536.

[23] ———, *The index of the Dirac operator in loop space*, Elliptic Curves and Modular Forms in Algebraic Topology (New York) (P. S. Landweber, ed.), Lecture Notes in Math. **1326**, Springer-Verlag, Berlin and New York, 1988, pp. 161–181.