QUANTUM THEORY OF A MASSLESS RELATIVISTIC SURFACE

AND A TWO-DIMENSIONAL BOUND STATE PROBLEM

by

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ABSTRACT

PART ONE

A massless relativistic surface is defined in a Lorentz invariant way by letting its action be proportional to the volume swept out in Minkowski space. The system is described in light cone coordinates and by going to a Hamiltonian formalism one sees that the dynamics depend only on the transverse coordinates X and Y. The Hamiltonian H is invariant under the group of area preserving reparametrizations whose Lie algebra can be shown to correspond in some sense to the Large N-limit of <u>SU(N)</u>. Using this one arrives at a SU(N) invariant, large N-two-matrix model with a quartic interaction $[X,Y]^2$.

PART TWO

The problem of N particles with nearest neighbors δ-function interactions is defined by regularizing the 2 body problem and deriving an eigenvalue integral equation that is equivalent to the Schrödinger equation (for bound states). The 3 body problem is discussed extensively and it is argued to be free of irregularities, in contrast with the known results in 3 dimensions. The crucial role of the dimension is displayed in looking at the limit of a short-range potential.

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PART ONE

QUANTUM THEORY OF A MASSLESS RELATIVISTIC SURFACE

INTRODUCTION

As a natural generalization of the massless string theory,* but also of interest in its own right, as an example in which geometry, classical relativity and quantum mechanics are deeply connected, one can define the dynamics of a massless closed M dimensional surface in a Lorentz- and coordinate invariant way by letting its action be proportional to the M+1 dimensional volume swept out in the D dimensional (generalized) Minkowski space $\mathcal{M}$. A particular observer with coordinate system $x^\mu=(t,x^\ell)$ would describe the shape he sees by $x^\ell(t,\lambda^1\ldots\lambda^M)$, where $\vec{\lambda}$ is a parametrization of the surface and the time like parameter λ^0 of the M+1--dimensional manifold was chosen to be t. Related to the arbitrariness of the choice of parametrization, not all of the x^μ and their conjugate momenta p^μ are independent.

It turns out to be extremely convenient to describe the system in terms of light cone coordinates $\tau(\equiv\frac{1}{2}(t+x^{D-1}))$ $\zeta(\equiv t-x^{D-1})$ and $\vec{x}(\equiv(x^1\ldots x^{D-2}))$, because the Hamiltonian turns out to be independent of ζ and** one can take $\vec{x}$ and the conjugate momentum $\vec{p}$ as the independent dynamical variables. In the classical theory ζ is determined via constraint equations, which are consistent provided $\{\vec{x},\vec{p}\}_{r,s}\equiv\frac{\partial\vec{x}}{\partial\lambda^r}\frac{\partial(\vec{p}/\omega(\lambda))}{\partial\lambda^s}-\frac{\partial(\vec{p}/\omega(\lambda))}{\partial\lambda^r}\frac{\partial\vec{x}}{\partial\lambda^s}=0$ where $\omega(\lambda)$ is a chosen density. These constraints fortunately do not cause a problem as their poisson bracket (commutator in the quantum theory) with the Hamiltonian is 0. (In the quantum theory they are interpreted as constraints acting on the wave functions ψ.)

*Goddard, Goldstone, Rebbi, Thorn, NP B56 (1973) "Quantum dynamics of a massless relativistic string".

**by picking a particular gauge, called orthonormal gauge.

$\{\vec{x},\vec{p}\}_{rs}$ are the generators of volume preserving (time independent) $\vec{\lambda}$-reparametrizations, which form a symmetry group that remains in orthonormal gauge.

After the general theory is described, everything else will be for the case M=2, D=4, with the parameter space (λ^1,λ^2) taken to have the topology of a 2-sphere. (Two examples of solutions to the equations of motion are given to become a little bit more familiar with the geometry of the problem and the parametrizations). The Hamiltonian which becomes

$$H = \int \sin\theta \, d\theta \, d\varphi \left\{ p_x^2 + p_y^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\varphi} - \frac{\partial y}{\partial\theta}\frac{\partial x}{\partial\varphi}\right)^2 \right\}$$

is invariant under the group G of areapreserving reparametrizations of S^2 (and $x+iy \to e^{i\alpha}(x+iy)$). The Lie algebra $\underline{G}$ consists of all smooth functions* of θ and φ, a basis of which one can take to be the usual spherical harmonics (leaving out Y_{00}).

In Part B it will be proved that the structure constants of $\underline{G}$ in the $Y_{\ell m}$-basis are in fact equal to the $N\to\infty$ limit of the structure constants of <u>SU(N)</u>, in a particular, properly chosen basis. This proof, which from a mathematical point of view turns out to be much more natural than the construction first seems to be, makes use of the fact that the $Y_{\ell m}$ are the harmonic polynomials (restricted to the unit sphere S^2) which one writes as $\sum_{i_\alpha=1}^{3} \overset{\circ}{a}{}^{(m)}_{i_1\ldots i_\ell} x_{i_1}\cdots x_{i_\ell}$. A basis of the fundamental representation of <u>SU(N)</u> can then be defined as $\overset{\circ}{T}_{\ell m} = \sum \overset{\circ}{a}{}^{(m)}_{i_1\ldots i_\ell} S_{i_1}\ldots S_{i_\ell}$ where S_i is a N-dimensional representation of <u>SO(3)</u>. A compact <u>formula for the</u> structure constants of <u>SU(N)</u> in this basis and

identifying any two differing just by a constant others differing from $\mathring{T}_{\ell m}$ by N and ℓ dependent normalization factors so to make the structure constants have a finite non-zero totally antisymmetric $N\to\infty$ limit, can be derived. The SU(N)-invariant Hamiltonian H_N one gets by replacing $x(\theta, \phi)$ by a hermitian N×N matrix x, { , } by $\frac{1}{i}[,]$, $\int d\Omega$ by Tr, is a good approximation to H for large N-in the sense that the degrees of freedom corresponding to $Y_{\ell m}$ with $\ell \leqslant N-1$ are represented correctly up to $O(\frac{1}{N})$, while the higher "frequencies" ($\ell \geqslant N$) have been cut off.

Note that both H and H_N are hamiltonians for a gauge theory in 2+1 dimensions with spatial derivatives = 0:

$$H_{(N)} = \sum_a \left((p_a^x)^2 + (p_a^y)^2 + \left(\sum_{b,c} f_{abc}^{(N)} x_b y_c \right)^2 \right)$$
$$= \mathrm{Tr} \left(E_x^2 + E_y^2 + B^2 \right)$$

where $x_b \leftrightarrow A_b^x$, and $B=[A^x,A^y]$. The conditions $f^{(N)}_{abc}\, \vec{x}_b \cdot \vec{p}_c = 0$ which are needed as a consistency condition for $H_{(N)}$ to be well defined translates into $[\vec{A},\vec{E}]=0$ which is exactly Gauss's law (when the spatial derivatives are 0). Bjorken** has looked at the analogue of this for SU(N=3) in 3 dimensions ($H=\mathrm{Tr}(\vec{E}^2+\vec{B}^2)$, with the vectors now having 3 components) and seems to have shown that the lowest

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*Please note the misleading notation: this transition has nothing to do with the transition from a classical theory with poisson bracket $\{\}_p$ to a quantum theory with $[x,p]=i\hbar$.

**"Elements of quantum chromodynamics", SLAC PUB 2372, Dec. '79.

lying set of energy levels is a rotational band corresponding to 3-dimensional rotations. We have so far been unable to confirm this result. The last chapter contains some work on or related to H_N.

One would hope to be able to find out much about the spectrum of H_N by using (or finding new) techniques for large N-matrix models.* The work on this during the past months, however, has provided puzzles rather than insight.

Though the original classical action is manifestly Lorentz invariant, we are quantizing in a particular Lorentz frame and will have to demonstrate the Lorentz-invariance of our theory. A satisfactory method would be to construct the generators of Lorentz transformations, but we have been unable to do this. A weaker method, which would give only a necessary condition, is to show that the spectrum is consistent with Lorentz invariance, i.e., that the states fall into multiplets characterized by mass and spin. We have not carried our study of the dynamics far enough to see if this is true, although there is some indication that $H_N (N\to\infty)$ will have a high degeneracy of its energy levels.

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*See e.g., "Planar Diagrams" CMP 59 p.35-51 (1978), by Brezin et al., and the review article about the 1/N expansion by Sidney Coleman: SLAC PUB 2484, 198.

A. THE ACTION AND THE HAMILTONIAN FORMALISM

I. The action S and an example

A massless M-dimensional closed surface moving in D-dimensional Minkowski space can be defined by letting its action be proportional to the M+1 dimensional volume swept out in Minkowski space (which is invariant under both Lorentz transformations and general reparametrizations ($\lambda^\alpha \to \lambda^{\alpha'}$) of the surface:

$$S = -T_0 \int_{\lambda^0_{\text{initial}}}^{\lambda^0_{\text{final}}} d\lambda^0 d^M\lambda \sqrt{G} \qquad \text{(A1)}$$

where G is $(-)^M$. the determinant of the metric $G_{\alpha\beta} \equiv \frac{\partial x^\mu}{\partial \lambda^\alpha} \frac{\partial x_\mu}{\partial \lambda^\beta}$ induced on the M+1 dimensional manifold M by Minkowski space; $x^\mu = x^\mu(\lambda^\alpha)$ are the space time coordinates of $\mathcal{M}$: $\mu=0, 1\ldots D-1$; $\alpha=0\ldots M$, $a^\mu b_\mu = a^0 b^0 - \sum_{\ell=1}^{D-1} a^\ell b^\ell$ for two D-vectors; and T_0 is the surface energy density (tension) of dim $\frac{\text{Energy}}{(\text{length})^M}$ which will from now on be put = 1 (one can always put it in on dimensional grounds). Using $\delta\sqrt{G} = \frac{1}{2}\sqrt{G}\, G^{\alpha\beta}\, \delta G_{\alpha\beta}$, where $G^{\alpha\beta}$ is defined via $G^{\alpha\beta} G_{\beta\gamma} = \delta^\alpha_\gamma$, one derives the equation of motion by setting the variation δS of the action = 0:

$$\delta S = -\frac{1}{2}\int d^{M+1}\lambda \sqrt{G}\, G^{\alpha\beta}\, \delta(\partial_\alpha x^\mu \partial_\beta x_\mu)$$

$$= \int d^{M+1}\lambda \sqrt{G}\, \delta x_\mu \frac{1}{\sqrt{G}} \partial_\alpha (\sqrt{G}\, G^{\alpha\beta} \partial_\beta x^\mu)$$

gives

$$\frac{1}{\sqrt{G}} \partial_\alpha (\sqrt{G}\, G^{\alpha\beta} \partial_\beta x^\mu) = 0 \qquad \text{(A2)}$$

Choosing the timelike parameter λ^0 of the manifold to be t, one has

$$G_{\alpha\beta} = \begin{pmatrix} 1-\dot{\vec{X}}^2 & -\dot{\vec{X}}\partial_r\vec{X} \\ -\dot{\vec{X}}\partial_r\vec{X} & -g_{rs} \end{pmatrix} \quad \text{where } \dot{\vec{X}} \equiv \frac{\partial\vec{X}}{\partial t}, \; \partial_r\vec{X} \equiv \frac{\partial\vec{X}}{\partial\lambda^r} \quad (r=1,\dots M)$$

$$\vec{X} = (x^1, \dots x^{D-1}) \qquad \vec{X}^2 = \sum_{\ell=1}^{D-1} x^\ell x^\ell \tag{A3}$$

$$g_{rs} \equiv -\partial_r X^\mu \partial_s X_\mu = +\partial_r\vec{X}\partial_s\vec{X} \; (\text{as } \partial_r t = 0 \text{ for } \lambda^0 \hat{=} t)$$

It is convenient to partially fix the parametrization by requiring

$$\text{(i) } G_{00} = 1-\dot{\vec{X}}^2 = +g \qquad (g \equiv \det g_{rs} \equiv |g_{rs}|)$$
$$\text{(ii) } G_{0r} = G_{r0} = \dot{\vec{X}}\,\partial_r\vec{X} = 0 \tag{A4}$$

This choice is possible provided x^μ satisfies the equations of motion: (ii) says that given the parametrization λ^r of the surface at time $t=t_0$, one chooses the parametrization at a slightly later time t_0+dt to be such that the intersections of any normal with the two surfaces are at equal λ^r. Further one certainly can choose the parametrization such that $g \equiv \left|\frac{\partial\vec{X}}{\partial\lambda^r}\frac{\partial\vec{X}}{\partial\lambda^s}\right|$ is $1-\dot{\vec{x}}^2$ at a given time. But given (ii) (for all times) the $\mu=0$ part of Eq. (A2) says that $\partial_t\left(\sqrt{\frac{g}{1-\dot{\vec{X}}^2}}\right)=0$, so (i) is true for all t. Note that (A4) is still invariant under volume preserving time independent reparametrizations of the surface (as those are exactly the ones that leave g invariant).

It is not difficult to find a solution of the classical equations of motion for M=2, D=4 (the physical case). The Ansatz

$$X^\mu = \begin{pmatrix} t \\ S(t)\vec{n} \end{pmatrix}, \quad \vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{A5}$$

with θ and φ being the usual angles of spherical coordinates, and defining $\lambda^1 \equiv -\cos\theta \equiv \mu$, $\lambda^2 = \varphi$ gives

$$G_{\alpha\beta} = \begin{pmatrix} 1-\dot{S}^2 & 0 & 0 \\ 0 & -S^2/\sin^2\theta & 0 \\ 0 & 0 & -\sin^2\theta\, S^2 \end{pmatrix}, \quad \sqrt{G} = S^2\sqrt{1-\dot{S}^2}$$

as $\partial_\varphi \vec{m} \cdot \partial_\mu \vec{m} = 0$, $(\partial_\mu \vec{m})^2 = \frac{1}{\sin^2\theta}$, $(\partial_\varphi \vec{m})^2 = \sin^2\theta$

The $\mu=0$ part of (A2) $(1-\dot{S}^2)^{-1/2} S^{-2} \partial_t S^2 (1-\dot{S}^2)^{-1/2} \partial_t t$ leads to

$$S^4 = (\mathrm{const})(1-\dot{S}^2) \tag{A6}$$

while the spatial part, which, using (A6) becomes

$$\left\{ 2 + \frac{1}{\sin\theta} \partial_\theta \sin\theta\, \partial_\theta + \frac{1}{\sin^2\theta} \partial_\varphi^2 \right\} \vec{m} = \vec{0}$$

is trivially satisfied by definition of $\vec{n}$. The solution of Eq. (A6), which is equivalent to $t/S_0 = \int_{S/S_0}^{1} \frac{dx}{\sqrt{1-x^4}}$ ($S =$ maximal radius), is a periodic elliptic function which can easily be expressed in terms of the standard Weierstrass-P-function.

II. General formalism in light cone coordinates

We define light cone coordinates by:

$$\left\{ \begin{array}{l} \tau \equiv \frac{1}{2}(t+z) \\ \zeta \equiv t - z \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} t = \tau + \zeta/2 \\ z = \tau - \zeta/2 \end{array} \right\} \tag{A7}$$

From now on $\vec{x}$ will always stand for $(x^1, \ldots, x^{D-2})$ and no distinction will be made between x_i and x^i $(i=1,\ldots N \equiv D-2)$ $x^\mu = (t, \vec{x}, z)$. Choosing $\lambda^0 = \tau$:

$$G_{\alpha\beta} \equiv \begin{pmatrix} G_{00} & G_{0r} \\ G_{r0} & -g_{rs} \end{pmatrix} = \begin{pmatrix} 2\dot{\zeta} - \dot{\vec{x}}^2 & (\partial_r \zeta - \dot{\vec{x}}\partial_r \vec{x}) \\ \partial_r \zeta - \dot{\vec{x}}\partial_r \vec{x} & -\frac{\partial \vec{x}}{\partial \lambda^r}\frac{\partial \vec{x}}{\partial \lambda^s} \end{pmatrix}$$

(Note that $\dot{\vec{x}}^2$ and $\vec{x}$ are differently defined from the $\dot{\vec{x}}^2$ and $\vec{x}$ appearing on p. 10 .) Now $\dot{\vec{x}}$ is a D-2-vector and $\cdot$ indicates differentiation with respect to τ.

$$G \equiv (-)^M \det G_{\alpha\beta} = \det \begin{pmatrix} G_{00} & -G_{0r} \\ G_{r0} & g_{rs} \end{pmatrix} = G_{00}\, g + G_{0r} G_{0s}\, g^{rs} g$$
$$= g\,(G_{00} + G_{0r} G_{0s}\, g^{rs}) \quad , \quad (g^{rs} g_{st} = \delta^r_t) \, ,$$

having used the fact that for a completely general square matrix $A = \begin{pmatrix} a_0 & \vec{a}^{tr} \\ \vec{b} & B \end{pmatrix}$ with invertible B one has $|A| = |B| \{ a_0 - \vec{a}^{tr} B^{-1} \vec{b} \}$

Therefore

$$\left.\begin{aligned} &\mathcal{L} \equiv -\sqrt{G} = -\sqrt{g\Gamma} \\ &\text{where } \Gamma \equiv 2\dot{\zeta} - \dot{\vec{x}}^2 + g^{rs} u_r u_s \\ &\text{and } u_r \equiv \dot{\vec{x}} \cdot \partial_r \vec{x} - \partial_r \zeta \quad (u^r \equiv g^{rs} u_s) \end{aligned}\right\} \quad \text{(A8)}$$

If we define canonical momenta by

$$\begin{aligned} \vec{p} &= \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = \sqrt{\frac{g}{\Gamma}}\,(\dot{\vec{x}} - u^r \partial_r \vec{x}) \\ \Pi &= \frac{\partial \mathcal{L}}{\partial \dot{\zeta}} = -\sqrt{g/\Gamma} \end{aligned} \quad \text{(A9)}$$

we find that

$$\vec{p} \cdot \partial_r \vec{x} + \Pi\, \partial_r \zeta \equiv 0 \quad \text{(A10)}$$

This constraint is a direct consequence of the invariance of S under τ-dependent reparametrization,

$$\delta \vec{x} = f^r(\lambda, \tau)\, \partial_r \vec{x} \quad , \quad \delta \zeta = f^r(\lambda, \tau)\, \partial_r \zeta$$

To go to a Hamiltonian formalism*, we express $\vec{p}\cdot\dot{\vec{x}}+\Pi\dot{\zeta}-\mathcal{L}=\mathcal{H}$ as a function of $\vec{p},\Pi,\vec{x},\zeta$ (This expression is, of course, not unique because of the relation (A10).):

$$\mathcal{H}\equiv\dot{\vec{x}}\,\vec{p}+\dot{\zeta}\Pi+\sqrt{g}\sqrt{2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r}$$

$$=\frac{\sqrt{g}}{\sqrt{2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r}}\left\{\dot{\vec{x}}^2-V_r u^r-\dot{\zeta}+(2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r)\right\}$$

$$=\frac{\sqrt{g}}{\sqrt{2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r}}\left\{\dot{\zeta}-\partial_r\zeta\, u^r\right\}\qquad (V_r\equiv\dot{\vec{x}}\,\partial_r\vec{x})$$

while
$$\frac{p^2+g}{-2\Pi}=\frac{\sqrt{2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r}}{\sqrt{g}}\,\frac{g}{(2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r)}\cdot$$

$$\cdot\left\{\dot{\vec{x}}^2-2V_r u^r+(\partial_r\vec{x}\,u^r)^2+(2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r)\right\}$$

$$=\frac{\sqrt{g}}{\sqrt{2\dot{\zeta}-\dot{\vec{x}}^2+u_r u^r}}\left\{\dot{\zeta}-V_r u^r+u_r u^r\right\}=\mathcal{H}\ \text{(see above)}$$

(In the last step we used: $\frac{1}{2}(\partial_r\vec{x}\,u^r)^2=\frac{1}{2}\partial_r\vec{x}\,\partial_{r'}\vec{x}\,u^r u^{r'}$
$=\frac{1}{2}u_r u^r$)

therefore
$$\mathcal{H}=\frac{p^2+g}{-2\Pi}\tag{A11}$$

We can then obtain the equations of motion from the Hamiltonian $\mathcal{H}'=\mathcal{H}+u^r(\vec{p}\cdot\partial_r\vec{x}+\Pi\partial_r\zeta)$ treating $\vec{x},\zeta,\vec{p},\Pi$ and u^r as independent variables:

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*For a general discussion of "constrained Hamiltonian systems" one could refer to the long article (with same title) of Hanson, Regge and Teitelboim. Academia Nazionale Dei Lincei, 1976.(Contributi del Centro Linceo Interdisciplinare Di Scienze Matematiche e Loro Applicazioni,N.22)

$$\frac{\delta H'}{\delta u^r} = \vec{\rho}\cdot\partial_r\vec{x} + \pi\,\partial_r\zeta = 0 \quad \left(H' \equiv \int d^M\lambda\ \mathcal{H}'\right)$$

$$\dot{\zeta} = \frac{\delta H'}{\delta \pi} = \frac{\rho^2 + g}{2\pi^2} + u^r\partial_r\zeta$$

$$\dot{\vec{x}} = -\vec{\rho}/\pi + u^r\partial_r\vec{x}\ , \quad \dot{\pi} = \partial_r(\pi u^r) \qquad \text{(A11')}$$

$$\dot{\vec{\rho}} = -\partial_r \frac{1}{\pi} g g^{rs}\partial_s\vec{x} + \partial_r(u^r\vec{\rho})$$

$$\left(\text{as } + \int\frac{\delta g}{2\pi} = +\int\frac{g g^{rs}}{2\pi}\delta g_{rs} = -\int\partial_r\left(\frac{1}{\pi}g g^{rs}\partial_s\vec{x}\right)\delta\vec{x}\ \right)$$

Note that H' = $\int\mathcal{H}' d^M\lambda$ and H = $\int\mathcal{H}\, d^M\lambda$ is invariant under reparametrization provided that p and π transform as densities*. Also as a consequence of Hamilton's equations, u^r is equal to u^r as defined in (A8) (Just calculate $\dot{\vec{x}}\partial_r\vec{x} - \partial_r\zeta$ from (A11'))

To discuss classical solutions, we can always choose the time variation of the parametrization so that u^r=0. Since $\mathcal{H}$ is independent of ζ, in this gauge $\dot{\pi} = 0$. We are still free to make a time-independent reparametrization. Since π transforms as a density we can make it equal to a constant times a specified λ-dependence, $\pi = -\eta w(\lambda)$. We are then left with the Hamiltonian

$$H = \frac{1}{2\eta}\int\frac{d^M\lambda}{\omega(\lambda)}\left(\vec{\rho}^{\,2} + g\right) \qquad \text{(A12)}$$

to determine the motion of $\vec{x}$. We call this gauge orthonormal gauge (ONG). The constraint (A10) becomes $\vec{\rho}\cdot\partial_r\vec{x} = \eta\,\omega(\lambda)\,\partial_r\zeta$ which we can solve for ζ provided

$$\partial_r\frac{1}{\omega(\lambda)}\vec{\rho}\cdot\partial_s\vec{x} - \partial_s\frac{\vec{\rho}}{\omega(\lambda)}\partial_r\vec{x} = 0 \qquad \text{A13)}$$

* i.e. $\delta\pi = \partial_r(f^r\pi)$, $\delta\vec{\rho} = \partial_r(f^r\vec{\rho})$; (while u^r transforms like a contravariant vector: $\delta u^r = (\partial_s u^r)f^s - (\partial_s f^r)u^s + \dot{f}^r$)

These constraints are consistent with the equations of motion derived from (A12) because H is still invariant under reparametrizations which leave the measure $w(\lambda)d^M\lambda$ invariant.

The constants of the motion P^μ may be obtained by comparing

$$P_\mu x^\mu = P^0 x^0 - P^z z - \vec{P}\cdot\vec{x}$$
$$= P^+ \zeta + P^- \tau - \vec{P}\cdot\vec{x}$$

We see that since $\vec{P}$ generates transverse translations, $-P^+$ must generate translations in ζ and P^- must be our H which generates the motion in τ. Thus $\vec{P} = \int \vec{p}\, d^M\lambda$

$$P^+ = -\int \Pi\, d^M\lambda = \eta \int w(\lambda)\, d^M\lambda$$

$$\text{and } P^- = \frac{1}{2\eta}\int (p^2 + g)\,\frac{d^M\lambda}{w(\lambda)} \qquad \text{(A14)}$$

If for a given choice $w(\lambda)$ (with $\int w(\lambda) d^M\lambda \equiv W$) we choose a complete orthonormal set of functions $\phi_n(\lambda)$, $\int \phi_n \phi_m w(\lambda) d^M\lambda = \delta_{nm}$

$$\vec{p} = \sum \vec{p}_m \phi_m w(\lambda) \qquad \vec{x} = \sum \vec{x}_m \phi_m$$

x_m and p_m will be canonically conjugate variables. If we take $\phi_0 = \frac{1}{\sqrt{W}}$, g which depends only on $\partial_r \vec{x}$ will be independent of $\vec{x}_0$ and we find

$$\vec{P} = \vec{p}_0 \sqrt{W}, \qquad P^+ = \eta W$$

$$P^- = \frac{1}{2\eta}\left(\vec{p}_0^{\,2} + \sum_{m>0} \vec{p}_m^{\,2} + \int g\,\frac{d^M\lambda}{w(\lambda)}\right)$$

$$= \frac{1}{2P^+}\left(\vec{P}^2 + W\left\{\sum_{m>0}\vec{p}_m^{\,2} + \int g\frac{d^M\lambda}{w(\lambda)}\right\}\right)$$

This relation is of the correct relativistic form,

$$2P^+P^- - \vec{P}^2 \left(= P^{0\,2} - P^{z\,2} - \vec{P}^2\right) = m^2 \quad \text{with}$$

$$m^2 = W\left\{\sum_{n>0} \vec{\rho}_n^{\,2} + \int g \frac{d^n\lambda}{\omega(\lambda)}\right\} \equiv H_{internal}$$

depending only on the degrees of freedom $\vec{x}_n$, $\vec{p}_n$, $n>0$.

Of the 6 homogeneous Lorentz transformations, 4 have remained explicit. H_{int} is clearly invariant under rotations about the z-axis, $x+iy \to e^{i\alpha}(x+iy)$. Boosts along the z-axis are generated by simply changing η to ηe^u, so that $P^\pm \to P^\pm e^{\pm u}$.* J_x+K_y and J_y-K_x correspond to the transformations $\vec{P}\to\vec{P}+\vec{v}P^+$, $P^+\to P^+$, $P^-\to P^- + \vec{v}\cdot\vec{P} + \frac{v^2}{2}P^+$. The remaining two, J_x-K_y and J_y+K_x must involve the internal degrees of freedom $\vec{x}_n$, $\vec{p}_n$.

In order to quantize this theory, we use the Hamiltonian

$$H = -\int \frac{\vec{\rho}^{\,2} + g}{2\pi}\, d^n\lambda$$

with $\vec{x}(\lambda)$, $\vec{p}(\lambda)$, $\zeta(\lambda)$, $\pi(\lambda)$ as canonical variables, obeying e.g. $[x_i(\lambda), \rho_j(\lambda')] = i\hbar\delta_{ij}\delta^n(\lambda-\lambda')$ with the constraints on the eigenstates of H corresponding to (A10)

$$\left(\partial_\tau \vec{x}\cdot\vec{\rho} + \partial_\tau \zeta\, \pi\right)|\psi\rangle = 0 \qquad \text{(A15)}$$

These constraints are consistent with each other and with $H\psi = E\psi$ since they are the generators of the group of reparametrizations. Since H is independent of ζ we can find eigenstates which are also eigenstates of $\pi(\lambda)$,

$$\pi(\lambda)|\psi\rangle = -\eta\,\omega(\lambda)|\psi\rangle \qquad \text{(A16)}$$

(*) $\tau \to \tau' = \sqrt{\frac{1+v}{1-v}}\,\tau \equiv e^u\tau$, $\zeta \to \zeta' = \sqrt{\frac{1-v}{1+v}}\,\zeta = e^{-u}\zeta$
$\vec{x} \to \vec{x}$.

These will not satisfy (A15). However, (A15) is equivalent to the condition that the wavefunctions $\psi[\vec{x}(\lambda), \pi(\lambda)]$ are invariant when $\vec{x}$, π are transformed by reparametrization. (π transforms as a density.) We can always construct such a wave-function from a ψ satisfying (A16) and invariant under those reparametrizations which leave $w(\lambda)$ invariant. Furthermore we need only consider a single specified form of $w(\lambda)$ since all others may be reached by reparametrization and rescaling of η. This invariance condition is exactly (A13) interpreted as a constraint on ψ. The classical discussion is now exactly paralleled by the quantum theory. We must find the eigenstates of H_{int} subject to (A13). These will also be eigenstates of J_z. Clearly a necessary condition for Lorentz invariance is that for a given eigenvalue of H_{int} the states can be arranged into SO(3) multiplets (i.e., that the number of states <u>increases</u> as $|J_z|$ <u>decreases</u>). It is possible to see that in a certain sense this is also a sufficient condition, i.e., if it is satisfied unitary operators realizing Lorentz invariance can be constructed level by level of H_{int}. However, they would not necessarily be related in any simple way to the canonical variables.

The further discussion will be restricted to the case M=2, D=4, $w(\lambda)d^{n}\lambda = \sin\theta d\theta d\varphi$. It is convenient to define $\tilde{p} = p/\sin\theta$ so that (with $\mu \equiv -\cos\theta$)

$$[x(\theta,\varphi), \tilde{p}(\theta,\varphi)] = i/\sin\theta\, \delta(\theta'-\theta)\delta(\varphi'-\varphi) = i\,\delta(\mu'-\mu)\,\delta(\varphi'-\varphi)$$

Then
$$\frac{H_{int}}{8\pi} = \frac{1}{2}\int \sin\theta\, d\theta\, d\varphi \left\{ \tilde{p}^2 + g/\sin^2\theta \right\} \qquad \text{(A17)}$$

and $$g/\sin^2\theta = \left(\frac{1}{\sin\theta}\left(\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\varphi} - \frac{\partial y}{\partial\theta}\frac{\partial x}{\partial\varphi}\right)\right)^2 \equiv \{x,y\}^2$$

where we define the Lie bracket of two functions A,B by

$$\{A,B\} = \frac{1}{\sin\theta}\frac{\partial(A,B)}{\partial(\theta,\varphi)}\left(= \frac{\partial(A,B)}{\partial(\mu,\varphi)}\right) \qquad \text{- - - -} \quad (A18)$$

Area preserving transformations are of the form

$$\delta x = \frac{\partial x}{\partial\theta} f^\theta + \frac{\partial x}{\partial\varphi} f^\varphi = \frac{\partial x}{\partial\mu} f^\mu + \frac{\partial x}{\partial\varphi} f^\varphi$$

where $\partial_\mu f^\mu + \partial_\varphi f^\varphi = 0$ so that

$$f^\mu = \partial_\varphi f\,, \quad f^\varphi = -\partial_\mu f = -\frac{1}{\sin\theta}\partial_\theta f \quad \text{and } \delta x = \{x,f\}.$$

The constraints (A13) take the form $\{x,\tilde{p}_x\} + \{y,\tilde{p}_y\} \,/\!/\, 0$ on the states. It is seen that the whole theory now depends on the single algebraic structure {A,B}. Part B will depend essentially on this fact.

III. <u>Another example and a comparison</u>

The Ansatz $\vec{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix} = R(\tau,\mu)\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \equiv R\cdot\vec{m}$ (A19)

leads, in orthonormal gauge (*) to $\vec{p} = \eta\,\dot{R}\,\vec{m}$,

$$g\,g^{rs} = \begin{pmatrix} R^2 & 0 \\ 0 & R'^2 \end{pmatrix} \quad \left(R' \equiv \left.\frac{\partial R}{\partial\mu}\right|_{\text{constant }\tau}, \quad \dot{R} \equiv \left.\frac{\partial R}{\partial\tau}\right|_{\text{constant }\mu}\right)$$

and the equation of motion reads (*)

*See pg. 22.

$$\eta^2 \ddot{R} = R(RR')' \tag{A20}$$

The constraint $\{\vec{x},\vec{p}\} \equiv \frac{\partial\vec{x}}{\partial\mu}\frac{\partial\vec{p}}{\partial\varphi} - \frac{\partial\vec{p}}{\partial\mu}\frac{\partial\vec{x}}{\partial\varphi} = 0$ is satisfied, as $\vec{m}\,\partial_\varphi\vec{m} = 0$ (so both terms = 0). Equivalently one can see directly that the equations for ζ are integrable, for $\vec{x}$ of the form (A19):

$$\begin{aligned} \dot{\zeta} &= \frac{p^2+g}{2\eta^2} = \frac{1}{2}\dot{R}^2 + \frac{1}{2\eta^2}R^2R'^2 \\ \zeta' &= \frac{1}{\eta}\vec{p}\cdot\vec{x}' = \dot{R}R' \\ \partial_\varphi\zeta &= \frac{1}{\eta}\vec{p}\cdot\partial_\varphi\vec{x} = 0 \end{aligned} \tag{A21}$$

The integrability conditions involving derivatives of ζ with respect to φ are trivially satisfied (as ζ is independent of φ), the one involving $\dot{\zeta}'$ gives exactly (A20).

One particular solution of (A20) with μ-cosθ, is $R(\tau,\mu) = R(\tau)\sin\theta$, leading to*

$$\ddot{R} = -R^3/\eta^2 \left(\Leftrightarrow R^4 + 2\eta^2\dot{R}^2 = D\eta^2;\ D = \text{const}\right) \tag{A22}$$

and (A21) becomes

$$\begin{aligned} \dot{\zeta} &\overset{(i)}{=} \frac{1}{2}\dot{R}^2\sin^2\theta + \frac{1}{2\eta^2}R^4\cos^2\theta \\ \partial_\theta\zeta &\overset{(ii)}{=} R\dot{R}\sin\theta\cos\theta = \frac{1}{2}R\dot{R}\sin(2\theta) \end{aligned} \tag{A23}$$

This will now be integrated explicitly from the second equation

$$\zeta = -\frac{R\dot{R}}{4}\cos(2\theta) + f(\theta) \Rightarrow \dot{\zeta} = \dot{f} - \frac{1}{4}\cos 2\theta\left(\dot{R}^2 - R^4/\eta^2\right)$$

(using (A22)) which has to equal (A23i)

$$\frac{1}{2}\dot{R}^2\sin^2\theta + {}^1\!/_{2\eta^2}\,R^4\cos^2\theta$$

*Please note that θ is not any geometrical angle, in particular not the angle of the spherical coordinates.

Therefore f has to equal $\frac{1}{4}\dot{R}^2 + \frac{1}{4\eta^2} R^4$, which--using again (A22)--is $\frac{1}{4}\frac{d}{d\tau}(R\dot{R})$, so that

$$\zeta = \frac{R\dot{R}}{2}\sin^2\theta + h(\tau) \quad ; \quad \dot{h} = \frac{1}{2\eta^2} R^4(\tau) \qquad \text{(A19)}$$

Both because (A22) is exactly the equation found earlier for S(t) (the radius of the breathing solution in a regular Lorentz frame) and because both (A5) and $\vec{x}=R(\tau)\sin\theta\vec{m}$ are most simple and symmetric solutions, one would think that they are in fact the same solution, just looked at in different frames and with different variables. This appears to be wrong, i.e., the above solution $R=R(\tau)\cdot\sin\theta$ is <u>not</u> the $R(\tau,\mu)$ in $\vec{x}=R(\tau,\mu)e^{i\varphi}$ that corresponds to the solution $x(t,\vartheta,\varphi) = (t, S(t)\cdot\vec{m})$ nor a simple Lorentz transform of it.

One can, in fact, calculate the parameter μ as a function of t and the geometric angle with the z-axis ϑ

So far

$$\zeta=\zeta(\tau,\mu),\ R=R(\tau,\mu) \ ; \ dR=\dot{R}d\tau+R'd\mu,\ d\zeta=\dot{\zeta}d\tau+\zeta'd\mu$$

so that $d\mu = d\zeta - \dot{\zeta}/\zeta'\, d\tau$ On the other hand one could extract $\mu=\mu(\tau,\zeta)$ from $\zeta(\tau,\mu)$, $R(\tau,\mu)$ and think of R as $R(\tau,\zeta)$, so that $dR=R_\tau d\tau+R_\zeta d\zeta$, $d\mu=\mu_\tau d\tau+\mu_\zeta d\zeta$ where

$$X_\tau \equiv \frac{\partial X}{\partial \tau}\Big|_{\text{constant }\zeta} \ , \ X_\zeta \equiv \frac{\partial X}{\partial \zeta}\Big|_{\text{constant }\tau}$$

By comparing the two expressions for $d\mu$ one finds

$$\mu_\tau = -\dot{\zeta}/\zeta' \ , \ \mu_\zeta = 1/\zeta'$$

Noting that

$$\dot{R} \equiv \frac{\partial R}{\partial \tau}\Big|_\mu = R_\tau + R_\zeta\frac{\partial \zeta}{\partial \tau}\Big|_\mu = R_\tau + R_\zeta\dot{\zeta}$$
$$\text{and } R' = \frac{\partial R}{\partial \mu}\Big|_\tau = R_\zeta\zeta'$$

and putting this into (A21) one gets

$$\dot\zeta = \frac{1}{2}\left(R_\tau + R_\zeta\dot\zeta\right)^2 + \frac{1}{2\eta^2}R^2R_\zeta^2\,\zeta'^2$$

$$\zeta' = \left(R_\tau + R_\zeta\dot\zeta\right)R_\zeta\,\zeta'$$

from which one deduces

$$\dot\zeta = \frac{1 - R_\tau R_\zeta}{R_\zeta^2}$$

$$\zeta' = \frac{\eta}{RR_\zeta}\sqrt{2\dot\zeta - (R_\tau + R_\zeta\dot\zeta)^2} = \frac{\eta}{RR_\zeta^2}\sqrt{1 - 2R_\tau R_\zeta}$$

Therefore

$$\mu_\tau = \frac{R(R_\tau R_\zeta - 1)}{\eta\sqrt{1-2R_\tau R_\zeta}}\,,\quad \mu_\zeta = \frac{RR_\zeta^2}{\eta\cdot\sqrt{1-2R_\tau R_\zeta}} \qquad \text{(A25)}$$

This expression is true whenever $\vec{x} = R(\tau,\mu)\binom{\cos\varphi}{\sin\varphi}$. Now one specifies: the solution (to A20) $R(\tau,\mu)$ that corresponds to the solution (A5) ($x^\mu = (t, S(t)\vec{m})$) obeys $R^2 + z^2 = S^2$, $R^2 + (\tau - 3/2)^2 = S^2(\tau + 3/2)$. From this it follows (e.g. $R\sqrt{1-2R_\tau R_\zeta} = S^3$) that

$$\mu_\zeta = \frac{1}{4\eta S^3}\left(S\partial_t S + z\right)^2,\quad \mu_\tau = \frac{1}{2\eta S^3}\left(S^2(\partial_t S)^2 + z^2 - 2S^2\right)$$

and therefore $\qquad (z \equiv \tau - 3/2)$

$$\mu_z = -\frac{S + z\dot S}{2\eta S^2}\,,\quad \mu_t = \frac{1}{2\eta S^3}\left\{-S^6 + z^2 + zS\dot S\right\}$$

from which one can determine μ as a function of t and z, or t and ϑ; one finds $2\eta\mu = -\cos 2\vartheta \pm \frac{\sqrt{1-S^4}}{2}\sin^2 2\vartheta + \text{const.}$

(± for collapsing sphere).
growing

Summary of formulae in orthonormal gauge

For convenience, the important equations (in particular (A11')) are written out explicitly for orthonormal gauge:

$$\Pi = -\eta\,\omega(\lambda) \qquad , \quad u_r = 0$$

constraint : $\vec{p}\,\partial_s \vec{x} = \eta\,\omega(\lambda)\,\partial_r \zeta \,,\quad \left\{ \vec{p}/_{\omega(\lambda)}, \vec{x} \right\} = 0 \,;$

$$\dot{\zeta} = \frac{p^2 + g}{2\eta^2 \omega(\lambda)} \qquad \vec{p} = \eta\,\omega(\lambda)\,\dot{\vec{x}}$$

$$\dot{\vec{p}} = \frac{1}{\eta}\,\partial_r\,\frac{g g^{rs}}{\omega(\lambda)}\,\partial_s \vec{x}$$

$$\Longrightarrow \ddot{\vec{x}} = -\dot{\vec{p}}/_{\Pi} = +\frac{1}{\eta^2}\frac{1}{\omega(\lambda)}\,\partial_r\,\frac{g g^{rs}}{\omega(\lambda)}\,\partial_s \vec{x}$$

For M=2 a convenient choice is (used in AIII):

$$\lambda^1 \equiv \mu \equiv -\cos\theta \in [-1,+1] \quad (\theta \in [0,\pi])$$
$$\lambda^2 = \varphi \quad , \quad \omega(\lambda) = 1$$

(if $\mu = +1$ (or -1), all points with different φ-values have to be identified);

if $\lambda^1 = \theta$, then choose $\omega(\lambda) = \sin\theta$.

B. The surface problem as the limit of a large N matrix problem

I. The group of area preserving reparametrizations of S^2 and the structure of its Lie algebra in connection with the surface Hamiltonian

The Hamiltonian found in Section A may be written as:

$$H[x,y,p_x,p_y] = \frac{1}{2}\int_{S^2} d\Omega \left(p_x^2 + p_y^2 + \{x,y\}^2\right)$$

$$\text{where } d\Omega \equiv d\mu\, d\varphi \equiv \sin\theta\, d\theta\, d\varphi \tag{B1}$$

$$\text{and } \{x,y\} \equiv \frac{\partial x}{\partial \mu}\frac{\partial y}{\partial \varphi} - \frac{\partial y}{\partial \mu}\frac{\partial x}{\partial \varphi} = \frac{1}{\sin\theta}\left(\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \varphi} - \frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \varphi}\right)$$

H is invariant under the group G of area preserving diffeomorphisms of S^2 (that are connected to the identity)--meaning that the functional dependence of H (on $\vec{x}$ and $\vec{p}$) will not change under a smooth reparametrization of the parameter space (a 2-sphere): $(\mu,\varphi)\to(\mu',\varphi')$, with unit Jacobian. This can be seen by looking at infinitesimal transformations $\mu'=\mu+\delta\mu$, $\varphi'=\varphi+\delta\varphi$, for which the condition

$$J \equiv \begin{vmatrix} \frac{\partial \mu'}{\partial \mu} & \frac{\partial \mu'}{\partial \varphi} \\ \frac{\partial \varphi'}{\partial \mu} & \frac{\partial \varphi'}{\partial \varphi} \end{vmatrix} = 1$$

is satisfied (to first order) if $\delta\mu=+\partial_\varphi f$, $\delta\varphi=-\partial_\mu f$ with f being any smooth infinitesimal function (defined by these equations up to a constant); it follows that for any function $z(\mu,\varphi)$ one has

$$\delta z \equiv z(\mu',\varphi') - z(\mu,\varphi) = \partial_\mu z\, \partial_\varphi f - \partial_\mu f\, \partial_\varphi z + O(f^2)$$
$$= \{z,f\} + O(f^2)$$

and (to first order in f):

$$\delta H = \frac{1}{2}\int d\Omega \left(\{ p_x^2 + p_y^2, f \} + 2\{x,y\}\delta\{x,y\} \right) = 0,$$

as $\int d\Omega \{g, f\} = 0$ for any $g(\mu,\varphi)$ (integrate by parts!).
Using the Jacobi identity for $\{\ ,\ \}$ one has

$$\delta\{x,y\} = \{\delta x, y\} + \{x, \delta y\} = \{\{x,f\},y\} + \{x,\{y,f\}\}$$
$$= \{\{x,y\},f\}$$

so that

$$\delta V = \frac{1}{2}\int d\Omega \{x,y\}\delta\{x,y\} = \frac{1}{2}\int d\Omega \{\{x,y\}^2, f\} = 0$$

The equations of motion derived from H are*

$$\ddot{x} = \{\{x,y\},y\}$$
$$\ddot{y} = -\{\{x,y\},x\} \qquad \text{(B2)*}$$

The Lie algebra $\underline{G}$ of G is the space of all smooth functions $f(\theta,\varphi)$ with f and g identified if they differ only by a constant. The Lie bracket on G is

$$\{f,g\} \equiv \frac{1}{\sin\theta}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\varphi} - \frac{\partial g}{\partial\theta}\frac{\partial f}{\partial\varphi} \right)$$
$$\equiv \frac{\partial f}{\partial\mu}\frac{\partial g}{\partial\varphi} - \frac{\partial g}{\partial\mu}\frac{\partial f}{\partial\varphi} \qquad \text{(B3)}$$

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*A whole class of solutions is: $x+iy=\omega e^{i(\omega - \varphi t)}\sin\theta$; these solutions are, however, not consistent with the light cone description, as constraint(A13) is not satisfied.

[Note that for more than 2 parameters $\lambda^1 \ldots \lambda^r \ldots \lambda^M$, Jacobian≡1 would have still given $\partial_r \delta\lambda^r = 0$, which is solved by $f^r \equiv \delta\lambda^r \equiv \partial_s \hat{F}^{rs}$ provided $\hat{F}^{rs}$ is an anti-symmetric tensor; the lie bracket on the space of all divergence-free vector fields $\vec{f}(\vec{\lambda})$ is

$$\{\vec{f}, \vec{g}\}^r = f^s \frac{\partial g^r}{\partial \lambda^s} - g^s \frac{\partial f^r}{\partial \lambda^s} = \partial_s \left(f^s g^r - g^s f^r \right) \tag{B4}$$

For M=2 there is only one independent antisymmetric tensor (ϵ^{rs}) so that $f^r = \partial_s \epsilon^{rs} f$, so that (B4) translates to the Lie bracket (B3) for functions $f \in \underline{G}$.]

As an orthonormal basis of $\underline{G}$ one can take the usual spherical harmonics $Y_{\ell m}(\theta, \varphi)^{\ell \geq 1}_{|m| \leq \ell}$ and define structure constants $g_{\ell_1 m_1 \ell_2 m_2 \ell_3 m_3}$ by the equation:

$$\{Y_{\ell_1 m_1}, Y_{\ell_2 m_2}\} \equiv -i\, g_{\ell_1 m_1 \ell_2 m_2 \ell_3 m_3} Y^*_{\ell_3 m_3} \tag{B5}$$

(Summation over repeated indices is understood, unless stated otherwise; (ℓm) will often be abbreviated by a or α or just by ( ).) For definiteness, the definition* of spherical harmonics is given:

$$Y_{\ell m}(\theta\varphi) \equiv (-)^{\delta_m} N_{\ell m} P_{\ell m}(\cos\theta)\, e^{im\varphi} \tag{B6}$$

*This and all other conventions concerning angular momentum coupling—coefficients are those of A. Messiah Quantum Mechanics books, referred to as MI and MII (see especially Appendix C of MII).

where

$$\gamma_m \equiv \begin{cases} m & \text{if } m \geq 0 \\ 0 & \text{if } m \leq 0 \end{cases} \qquad N_{\ell m} \equiv \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}}$$

$$P_{\ell m}(-\mu) \equiv (-)^{\ell+|m|} \frac{(1-\mu^2)^{|m|/2}}{2^\ell \ell!} \frac{d^{\ell+|m|}}{d\mu^{\ell+|m|}} (\mu^2-1)^\ell \equiv (1-\mu^2)^{|m|/2} P_{\ell-|m|}(-\mu \tag{B6}$$

is an associated Legendre function of $-\mu \equiv \cos\theta$.)

Upon first inspection of (B5), one sees that

$$g = +i \int d\Omega\, Y \{Y, Y\} \text{ is real } (\partial_\varphi \text{ gives } im)$$

and totally antisymmetric (integration by parts!)

$$Y^*_{\ell m} = (-)^m Y_{\ell -m} \text{ (and } g \text{ real)} \Rightarrow g_{()_1()_2()_3} = -g_{\ell_1 -m_1\, \ell_2 -m_2\, \ell_3 -m_3} \tag{B7}$$

$$Y_{\ell m}(\pi-\theta, \varphi+\pi) = (-)^\ell Y_{\ell m}(\theta\varphi) \Rightarrow g_{()_1()_2()_3} = 0 \text{ if } \sum_{i=1}^{3} \ell_i \text{ even}$$

$$g \propto \int e^{i\sum m_j} \ldots \Rightarrow g = 0 \text{ unless } \sum m_j = 0$$

For later comparison it is useful to evaluate g for two simple cases; with $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$, $Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$ and $Y^*_{\ell m} = (-)^m Y_{\ell -m}$ one finds:

$$g_{\ell m \ell' m' 10} = +m(-)^m \sqrt{\frac{3}{4\pi}}\, \delta_{\ell\ell'}\, \delta_{m -m'}$$

$$g_{\ell m \ell' m' 20} = +m(-)^m \sqrt{\frac{5}{16\pi}}\, \delta_{m' -m}\, 6 \int d\Omega \cos\theta\, Y_{\ell m} Y^*_{\ell' -m} \tag{B8}$$

$$= 3\sqrt{\frac{5}{4\pi}}\, m(-)^m \delta_{m' -m} \left\{ \delta_{\ell' \ell+1} \sqrt{\frac{(\ell+1+m)(\ell+1-m)}{(2\ell+1)(2\ell+3)}} + \delta_{\ell' \ell-1} \sqrt{\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}} \right\}$$

where in the last step the decomposition of $\cos\theta Y_{\ell m}$ into the linear combination $\sqrt{\cdots}\, Y_{\ell+1,m} + \sqrt{\cdots}\, Y_{\ell-1,m}$ has been used.

The group G itself has been studied in the mathematical literature, and although not relevant for the further discussion of the surface problem contained in this thesis, some properties will be listed.

- G is simple,[1] i.e., has no nontrivial invariant subgroup H $\left(gHg^{-1} = H \quad \forall g \in G \right)$

- The Homotopy classes of G are those of SO(3) [Stephen Smale[2] proved this for the group of all diffeomorphisms; it then follows from a theorem by Moser[3] that the same thing is true for G]

- any $g \in G$ has at least two fixed points [N.A. Nikishin[4] and C.P. Simon[5]]

- given $P_1 \ldots P_R \in S^2$, $Q_1 \ldots Q_R \in S^2 \to \exists g \in G$ with $Q_i = g(P_i)$ and furthermore let $C_1 \ldots C_R$ be an arbitrary collection of

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[1] I would like to thank Augustin Banyaga for telling me this and other things about G; as a reference see: A.B. "Sur la structure du groupe des diffeomorphismes qui préservent une forme symplectique". Comment. Math. Helv. 53, 174-227 (1978).

[2] "Diffeomorphisms of the 2-sphere", Proc. Am. Math. Soc. 10, 1959.

[3] AMS Transactions 120, 1965, p. 287.

[4] Funct. Anal. Preloz. 8, 84-85 (1974) (in Russian).

[5] Inventiones Math. 26, 187-200 (1974)

[6] See "Transformation Groups" by Kobayashi-Nomizu.

disjoint closed curves on S^2, then there is a 1-parameter group of area preserving transformations with these curves as orbits.

Expanding x, y, p_x and p_y (θ,φ) in spherical harmonics

$$x = \sum x_{\ell m} Y_{\ell m}(\theta,\varphi) \qquad x^*_{\ell m} = (-)^m x_{\ell -m}$$

$$(y, p_x, p_y \text{ analogously})$$

one gets

$$T = \frac{1}{2}\sum\left(|p^x_{\ell m}|^2 + |p^y_{\ell m}|^2\right) \equiv \frac{1}{2}\sum_{\ell,m} |\vec{p}_{\ell,m}|^2$$

and, writing $\{x,y\}$ once as $-ig_{\ell m \ell' m' \ell'' m''} x_{\ell m} y_{\ell' m'} Y^*_{\ell'' m''}(\theta,\varphi)$ the other time as $=\{x,y\}^* = +ig_{(\ )(\ )'(\ )''} x^*_{(\ )} y^*_{(\ )'} Y_{\ell'' m''}$,

$$V = \frac{1}{2} g_{\ell m ()_1 ()_2}\, g_{\ell m ()_3 ()_4}\, x_{()_1} y_{()_2} x^*_{()_3} y^*_{()_4} \ .$$

One can think of H=T+V as describing infinitely many particles (labelled by ℓ and m) moving in two dimensions ($x_{\ell m}$ and $y_{\ell m}$) and interacting through the, not very symmetric, quartic potential V. The unitary transformation

$$\begin{pmatrix} \tilde{x}_{\ell |m|} \\ \tilde{x}_{\ell -|m|} \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} (-)^{|m|} & 1 \\ +i(-)^{|m|} & -i \end{pmatrix} \begin{pmatrix} x_{\ell |m|} \\ x_{\ell -|m|} \end{pmatrix} \text{(the same for } y, p_x, p_y)$$

corresponding to a real basis

$$\tilde{Y}_{\ell |m|} = \sqrt{2}\cos|m|\varphi\, N_{\ell m} P_{\ell m}\ , \quad \tilde{Y}_{\ell -|m|} = \sqrt{2}\sin|m|\varphi\, N_{\ell m} P_{\ell m}$$

will make $\tilde{x}_{\ell \pm |m|}$ real. The structure constants

$$\tilde{g}_{\alpha\beta\gamma} \equiv \int d\Omega\, \tilde{Y}_\alpha \{\tilde{Y}_\beta, \tilde{Y}_\gamma\}$$

are still totally antisymmetric (as the $\tilde{Y}_{\ell m}$ are orthonormal), but obey fewer selection rules than the $g_{\alpha\beta}$

Some properties of the real $\tilde{Y}$-basis:

from $Y_{\ell m} = (-)^{\gamma_m} N_{\ell m} P_{\ell m} e^{im\varphi}$ (see (B6))

$$\tilde{Y}_{\ell |m|} \equiv \frac{1}{\sqrt{2}}\left((-)^{|m|} Y_{\ell |m|} + Y_{\ell -|m|}\right) = N_{\ell m} P_{\ell m} \sqrt{2} \cos |m| \varphi$$

$$\tilde{Y}_{\ell -|m|} \equiv \frac{1}{\sqrt{2}\, i}\left((-)^{|m|} Y_{\ell |m|} - Y_{\ell -|m|}\right) = N_{\ell m} P_{\ell m} \sqrt{2} \sin |m| \varphi$$

$$\Rightarrow \int \tilde{Y}_{\ell m} \tilde{Y}_{\ell' m'} = \delta_{\ell \ell'} \delta_{m m'}$$

So that $\tilde{g} = \int \tilde{Y} \{\tilde{Y}, \tilde{Y}\}$

is still totally antisymmetric.

$$\begin{pmatrix} \tilde{Y}_{\ell |m|} \\ \tilde{Y}_{\ell -|m|} \end{pmatrix} = U \begin{pmatrix} Y_{\ell |m|} \\ Y_{\ell -|m|} \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} (-)^{|m|} & 1 \\ -i(-)^{|m|} & +i \end{pmatrix} \qquad \text{(B9)}$$

is unitary:

$$U^{+} U = \frac{1}{2} \begin{pmatrix} (-)^{|m|} & +i(-)^{|m|} \\ 1 & -i \end{pmatrix} \begin{pmatrix} (-)^{|m|} & 1 \\ -i(-)^{|m|} & +i \end{pmatrix} = \mathbb{1}$$

for $X(\theta,\varphi) = \sum x_{\ell m} Y_{\ell m} \overset{!}{=} \sum \tilde{x}_{\ell m} \tilde{Y}_{\ell m}$.

$\begin{pmatrix} \tilde{x}_{\ell |m|} \\ \tilde{x}_{\ell -|m|} \end{pmatrix}$

has to transform with the complex conjugate of U:

$$\begin{pmatrix} \tilde{x}_{\ell |m|} \\ \tilde{x}_{\ell -|m|} \end{pmatrix} = U^* \begin{pmatrix} x_{\ell |m|} \\ x_{\ell -|m|} \end{pmatrix} ;$$

in shorthand notation:

$$\vec{\tilde{x}} = U^* \vec{x} \quad , \quad \vec{\tilde{Y}} = U \vec{Y} \quad ,$$

so that

$$\tilde{x}_{\ell m} \tilde{Y}_{\ell m} + \tilde{x}_{\ell -m} \tilde{Y}_{\ell -m} = \vec{\tilde{x}}^{tr} \vec{\tilde{Y}} = \vec{x}^{tr} U^{+} U \vec{Y} = \vec{x}^{tr} \vec{Y} ;$$

written out :

$$\tilde{x}_{\ell |m|} = \frac{(-)^m}{\sqrt{2}} \left(x_{\ell |m|} + x^*_{\ell |m|} \right) = \frac{1}{\sqrt{2}} \left((-)^{|m|} x_{\ell |m|} + x_{\ell -|m|} \right)$$

$$\tilde{x}_{\ell -|m|} = \frac{-(-)^{|m|}}{\sqrt{2} \cdot i} \left(x_{\ell |m|} - x^*_{\ell +|m|} \right) = \frac{1}{\sqrt{2} i} \left(-(-)^{|m|} x_{\ell |m|} + x_{\ell -|m|} \right) .$$

One has

$$\left\{ \tilde{x}^i_{\ell m} , \tilde{p}^j_{\ell' m'} \right\}_{Poisson} = \delta_{\ell \ell'} \delta_{m m'} \delta_{ij}$$

where

$$\left(\tilde{x}^1_{\ell m} , \tilde{x}^2_{\ell m} \right) = \left(\tilde{x}_{\ell m} , \tilde{y}_{\ell m} \right) \equiv \vec{\tilde{x}}_{\ell m} \quad ,$$

and $\{\ \}_p$ is now the poisson bracket for functions of the canonical variables $x^i_{\ell m}$ and $p^i_{\ell m}$. The invariance of H under G is now expressed as

$$\left\{ H, \tilde{g}_{\alpha\beta\gamma} \tilde{\vec{X}}_\beta \cdot \tilde{\vec{P}}_\gamma \right\}_p \equiv \left\{ H, K_\alpha \right\}_p = 0$$

(which one can verify explicitly) The constants of the motion

$$K_\alpha \equiv \int d\Omega \, \tilde{Y}_\alpha \left\{ \vec{x}, \vec{P} \right\} \tag{B10}$$

are the generators of area preserving transformations, and, for the light cone coordinate description to be consistent, one has to have $K_{\ell m}=0 \ \forall_{\ell m}$. Note that, of course,

$$\left\{ K_\alpha, K_\beta \right\} = \tilde{g}_{\alpha\beta\gamma} K_\gamma$$

(see also below). As mentioned in part A, one can proceed to the quantum theory via the correspondence $\{,\}_p (-)-i[,]$, i.e.,

$$\left[\tilde{X}^i_\alpha , \tilde{P}^j_\beta \right] = i \, \delta_{ij} \, \delta_{\alpha\beta} \quad (\hbar = 1)$$

for the hermitian operators $\tilde{x}^i_\alpha$ and $\tilde{p}^j_\beta$. (From now on drop ~ for the quantum mechanical operators.)

One finds that

$$[K_a, K_b] = i\,\tilde{g}_{abc} K_c$$

as it must (they are a basis of the representation of G as operators on Hilbert space), since

$$[K_a, K_b] = [\tilde{g}_{a\alpha\beta}\vec{x}_\alpha \vec{p}_\beta, \tilde{g}_{b\gamma\delta}\vec{x}_\gamma \vec{p}_\delta]$$

$$= \tilde{g}_{a\alpha\beta}\tilde{g}_{b\gamma\delta}\{\vec{x}_\alpha[\vec{p}_\beta, \vec{x}_\gamma\vec{p}_\delta] + [\vec{x}_\alpha, \vec{x}_\gamma\vec{p}_\delta]\vec{p}_\beta\}$$

$$= \tilde{g}_{a\alpha\beta}\tilde{g}_{b\beta\gamma}\{-i\vec{x}_\alpha\vec{p}_\delta\delta_{\beta\gamma} + i\vec{x}_\gamma\vec{p}_\beta\,\delta_{\alpha\delta}\}$$

$$= -i\,\tilde{g}_{a\alpha\beta}\tilde{g}_{b\beta\delta}\vec{x}_\alpha\vec{p}_\delta + i\,\tilde{g}_{a\alpha\beta}\tilde{g}_{b\gamma\alpha}\vec{x}_\gamma\cdot\vec{p}_\beta$$

$$= +i\,(-\tilde{g}_{a\alpha\gamma}\tilde{g}_{b\gamma\delta} + \tilde{g}_{a\gamma\delta}\tilde{g}_{b\alpha\gamma})\vec{x}_\alpha\cdot\vec{p}_\delta$$

$$= -i\,(g^{\gamma}_{ba}\,g^{\delta}_{\gamma\alpha})\vec{x}_\alpha\cdot\vec{p}_\delta \quad (\text{Jacobi identity!})$$

$$= +i\,g_{abc}\,g_{c\alpha\delta}\,\vec{x}_\alpha\cdot\vec{p}_\delta$$

$$= +i\,g_{abc}K_c$$

Also one can check that $[K_\alpha,H]=0$. The consistency condition (A13) requires physical states $|\psi\rangle$ to be singlets under the symmetry group, i.e., $K_\alpha|\psi\rangle = 0 \quad \alpha=(\ell m)$. The change of a wave-functional $\psi[x]$ under an infinitesimal are preserving reparametrization characterized by a function $f(\mu,\phi)$ is:

$$
\begin{aligned}
\delta_f \psi &= \int d\Omega \; \delta x_i \; \frac{\partial \psi}{\partial x_i} \\
&\hat{=} \int d\Omega \; \{x_i, f\} \; \frac{\delta \psi}{\delta x_i} \\
&= +i \int d\Omega \; \{x_i, f\} \, p_i \, \psi \\
&= -i \int d\Omega \; f(\mu,\varphi) \, \{x_i, p_i\} \, \psi \\
&= -i \int d\Omega \; (f_\alpha \hat{Y}_\alpha)(K_\beta \hat{Y}_\beta) \, \psi \\
&= -i \; f_\alpha K_\alpha \psi \qquad .
\end{aligned}
$$

II. Explicit construction and proof that a basis of the fundamental representation of SU(N) can be chosen such that for the structure constants: $\lim_{N\to\infty} \hat{f}^{(N)}_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$.

The aim of this section is to establish a correspondence between the Lie algebra G* of area preserving transformations and the Lie algebra SU(N) for $N\to\infty$. This correspondence allows one to transform the problem of finding the spectrum of the surface Hamiltonian H to that of finding the spectrum of a large N-matrix Hamiltonian

$$H_N \equiv \frac{1}{2} \operatorname{Tr} \left\{ P_x^2 + P_y^2 - \frac{1}{N} [x,Y]^2 \right\}$$

(x, y, p_x and p_y traceless hermitian N×N=matrices). Going from H to H_N is a sort of renormalization as one is cutting off the degrees of freedom corresponding to $Y_{\ell m}$ with $\ell \geqslant N$ ("High frequencies") while representing the low frequencies ($\ell \leqslant N-1$) correctly up to $O(1/N)$.

(BII) is subdivided into 5 sections as follows.

1. By a correspondence to the solid spherical harmonics $r^\ell Y_{\ell m}$ (written as harmonic polynomials) one defines N^2-1 linearly independent real, traceless N×N matrices $\overset{0}{T}_{\ell m}$ ($\ell=1...N-1$, $|m|<\ell$). They are a basis of the N-dimensional representation of SU(N). Also they are, for given ℓ, tensor operators of degree ℓ; so is any $T_{\ell m}$ differing from $\overset{0}{T}_{\ell m}$ by N and ℓ dependent (but m-independent!) factor.

*The underlining always denotes the Lie algebra of the corresponding group.

2. Using the Wigner Eckart theorem, the structure constants of $\underline{SU(N)}$, defined by the relation $[T_{\ell m}, T_{\ell' m'}] = f^{(N)}_{\substack{\ell\ell'\ell'' \\ mm'm''}} T^{+}_{\ell'' m''}$, can be calculated in terms of the reduced matrix elements $R_N(\ell)$. The answer also involves Wigner 3j- and 6j-symbols.

3. Instead of actually calculating the structure constants $g_{\alpha\beta\gamma}$ of $\underline{G}$ (α is a short-hand notation for (ℓm)), a proof is given that

$$\hat{T}_{\ell m} \equiv \frac{\overset{\circ}{T}_{\ell m}}{\left(\frac{N^2-1}{4}\right)^{\ell-1}}$$

must lead to structure constants $\hat{f}_{\alpha\beta\gamma}$ that in the $N\to\infty$ limit are equal to the $g_{\alpha\beta\gamma}$ $\forall_{\alpha\beta\gamma}$. <u>This proof is the central part of (BII)</u>.

4. Knowing this one can deduce the corresponding choice $\hat{R}_N(\ell)$, when calculating the $N\to\infty$ limit of the structure constants derived in (2). This limit then is the formula for $g_{\alpha\beta\gamma}$. In (5) the correct choice $\hat{R}_N(\ell)$ is derived without using 3.

1. Definition of $\overset{\circ}{T}_{\ell m}$:

Let S_i be an N-dimensional representation of the Lie algebra $\underline{SO(3)}$, the spin $S=(N-1)/2$ representation. Conventionally one chooses a basis S_1, S_2, S_3 with

$$\langle Sm' | S_3 | Sm \rangle = m\, \delta_{m'm}$$

$$\langle m' | S_1 \pm i S_2 | m \rangle = \sqrt{S(S+1) - m(m\pm 1)}\; \delta_{m', m\pm 1} \qquad \text{(B11)}$$

S_3 and $S_\pm \equiv S_1 \pm iS_2$ are real. One then defines N×N matrices $\overset{0}{T}_{\ell m}$ as polynomials of degree ℓ in the S_i which correspond in some sense to the $Y_{\ell m}(\theta,\varphi)$. One does this by remembering that $r^\ell Y_{\ell m}$ are homogeneous, in fact harmonic, polynomials of degree ℓ in the variables $x_1(\equiv r\cos\varphi \sin\theta)$, $x_2(\equiv r\sin\theta\sin\varphi)$ and $x_3(\equiv r\cos\theta)$

$$\mathcal{Y}_{\ell m} \equiv r^\ell Y_{\ell m}(\theta,\varphi)$$

$$\equiv \sum_{\substack{\Sigma = \ell \\ (j_\rho = 1,2,3)}} \overset{0}{a}{}^{(m)}_{j_1 j_2 j_3} x_1^{j_1} x_2^{j_2} x_3^{j_3} = \sum_{\substack{i_\alpha=1 \\ (\alpha=1,2,\ldots\ell)}}^{3} \overset{0}{a}{}^{(m)}_{i_1\ldots i_\ell} x_{i_1}\cdots x_{i_\ell} \tag{B12}$$

The $\overset{0}{a}{}^{(m)}_{i_1\ldots i_\ell}$ defined this way are traceless between any two indices ($\leftrightarrow \nabla^2 \mathcal{Y}_{\ell m} = 0$) and totally symmetric. For given ℓ there are $2\ell+1$ independent ones. Then define:

$$\overset{0}{T}_{\ell m} \equiv \sum_{i_\alpha=1}^{3} \overset{0}{a}{}^{(m)}_{i_1\cdots i_\ell} S_{i_1} \cdots S_{i_\ell} \tag{B13}$$

The first few ones are:

$$\overset{0}{T}_{10} = \sqrt{\frac{3}{4\pi}}\, S_z \,,\quad \overset{0}{T}_{11} = -\sqrt{\frac{3}{8\pi}}\,(S_x + iS_y) \,,\quad \overset{0}{T}_{1-1} = \sqrt{\frac{3}{8\pi}}\,(S_x - iS_y)$$

$$\overset{\circ}{T}_{2\pm1} = \mp\sqrt{\frac{15}{32\pi}}\left(S_xS_z+S_zS_x \pm i\left(S_yS_z+S_zS_y\right)\right)$$

$$\overset{\circ}{T}_{2\pm2} = \sqrt{\frac{15}{32\pi}}\left(S_x^2-S_y^2 \pm i\left(S_xS_y+S_yS_x\right)\right)$$

$$\overset{\circ}{T}_{20} = \sqrt{\frac{5}{16\pi}}\left(2S_z^2-S_x^2-S_y^2\right)$$

All $\overset{\circ}{T}_{\ell m}$ are by definition real and traceless, but not hermitian:

$$\left(\overset{\circ}{T}_{\ell m}\right)^{\dagger} = \left(\overset{\circ}{T}_{\ell m}\right)^{tr} = (-)^m \overset{\circ}{T}_{\ell -m}$$

$$\left(\overset{\circ}{a}{}^{(m)}_{i_1\cdots i_\ell}\right)^{*} = (-)^m a^{(-m)}_{i_1\cdots i_\ell}$$

For fixed ℓ, the $\overset{\circ}{T}_{\ell m}$ form a set of tensor operators of rank ℓ, i.e., for a rotation R:

$$U(R)\overset{\circ}{T}_{\ell m}U(R)^{-1} = \sum_{m'=-\ell}^{+\ell} \overset{\circ}{T}_{\ell m'} R^{\ell}_{m'm}(R) \qquad \text{(B14)}$$

where $R^{\ell}_{m'm}$ are the rotation matrices for angular momentum ℓ (see for instance Messiah II, p. 1070) and U(R) is a N-dimensional representation of the rotation R). [If R$\in$SO(3), N would have to be odd, and later one would take $\lim\limits_{N\to\infty} f^{(N)}$;
(N odd!)

but one might as

well take $R \in SU(2)$ which does not alter anything as the two Lie algebras $\underline{SU(2)}$ and $\underline{SO(3)}$ are the same.] Changing the normalization of the $\overset{0}{T}_{\ell m}$'s in an m-independent way will not alter the transformation properties. Therefore any $T_{\ell m}=U(\ell,N)\overset{0}{T}_{\ell m}$ will obey the Wigner Eckart theorem* :

$$\langle s m_1 | T^{(N)}_{\ell m} | s m_2 \rangle = (-)^{s-m_1} \begin{pmatrix} s & \ell & s \\ -m_1 & m & m_2 \end{pmatrix} R_N(\ell) \tag{B15}$$

where ( ) denotes the 3j-symbol* and $R_N(\ell)$ the reduced matrix element (real for real $U(\ell,N)$). $R_N(\ell) \equiv \overset{0}{R}_N(\ell)\cdot U(N,\ell)$ has been left general, as different normalizations will be useful in different situations.

One may now define structure constants $f^{(N)}_{\substack{\ell\ell'\ell'' \\ mm'm''}}$ by:

$$[T_{\ell m}, T_{\ell' m'}] = f^{(N)}_{\substack{\ell\ell'\ell'' \\ mm'm''}} T^{\dagger}_{\ell'' m''} \tag{B16}$$

By using (B15) and standard formulae concerning coupling of angular momenta one can proceed to calculate $\mathrm{Tr}(T_\alpha T^{\dagger}_\beta)$ and $f^{(N)}$. This is done in the next section.

---

*See e.g., MII. p. 1056.

2. Calculation of $Tr(TT^+)$, $Tr(TTT)$, and choice of $R_N(\ell)$

From (B15) one has

$$\mathrm{Tr}\, T_{\ell m} T^{\dagger}_{\ell' m'} = \sum_{m_1 m_2} \langle m_1 | T_{\ell m} | m_2 \rangle \langle m_2 | T^{\dagger}_{\ell' m'} | m_1 \rangle$$

$$= R_N(\ell) R_N(\ell') \sum (-)^{2S-m_1-m_2+m'} \begin{pmatrix} S & \ell & S \\ -m_1 & m & m_2 \end{pmatrix} \begin{pmatrix} S & \ell' & S \\ -m_2 & -m' & m_1 \end{pmatrix}$$

As the second 3j-symbol is 0 unless $m_1=m_2+m'$, $2S-m_1-m_2+m'$ has to be even and $(-)^{2S-m_1-m_2+m'}$ therefore $=+1$. Further

$$\sum (\ )(\ ) = \sum_{m_1 m_2} \begin{pmatrix} S & \ell & S \\ m_1 & m & m_2 \end{pmatrix} \begin{pmatrix} S & \ell' & S \\ -m_2 & -m' & -m_1 \end{pmatrix}$$

$$= \sum \begin{pmatrix} S & S & \ell \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} S & \ell' & S \\ m_2 & m' & m_1 \end{pmatrix} (-)^{\ell+\ell'}$$

$$= (-)^{\ell+\ell'} \sum \begin{pmatrix} S & S & \ell \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} S & S & \ell' \\ m_1 & m_2 & m' \end{pmatrix}$$

$$= (-)^{\ell+\ell'} \delta_{\ell\ell'} \delta_{mm'} \frac{1}{2\ell+1} ,$$

where in the first step m_1 was changed to $-m_1$; in the second step 2nd and 3rd column of the first 3j-symbol were interchanged (giving factor $(-)^{2S+\ell}$) and in the second 3j-symbol the sign of the lower row was changed (i.e., $-m_\alpha \to +m_\alpha$, giving a factor $(-)^{2S+\ell'}$); in the third step invariance of ( ) under cyclic permutations of the 3 columns was used; the last step is true because of Eq. (C15a), p. 1057, MII. Therefore

$$\mathrm{Tr}\left(T_{\ell m} T^{\dagger}_{\ell' m'}\right) = \delta_{\ell'\ell}\,\delta_{m'm}\,\frac{R_N^{\,2}(\ell)}{(2\ell+1)} \tag{B17}$$

i.e., the $T_{\ell m}$'s are orthogonal (with the choice $R_N^{(\ell)} \equiv \sqrt{2\ell+1}$ they would be orthonormal.) Note that $T_{\ell m} \equiv 0$ for $\ell \geqslant N=2S+1$, as $\begin{pmatrix} S & \ell & S \\ \cdot & \cdot & \cdot \end{pmatrix} = 0$ then.

But this means that one has constructed this way exactly N^2-1 ($3+5+...+N-1$) independent traceless real $N\times N$ matrices. They, therefore, furnish a basis of the fundamental (i.e., N-dimensional) representation of the Lie algebra <u>SU(N)</u>, and the $f^{(N)}_{\alpha\beta\gamma}$ defined via (B16) are the structure constants of <u>SU(N)</u> in this basis. They will now be calculated:

$$\mathrm{Tr}\left(T_{\ell_1 m_1} T_{\ell_2 m_2} T_{\ell_3 m_3}\right) =$$

$$= \sum_{m\,m'\,m''} (-)^{3S-m-m'-m''} \prod_{i=1}^{3} R_N(\ell_i) \begin{pmatrix} S & \ell_1 & S \\ -m & m_1 & m' \end{pmatrix} \begin{pmatrix} S & \ell_2 & S \\ -m' & m_2 & m'' \end{pmatrix} \begin{pmatrix} S & \ell_3 & S \\ -m'' & m_3 & m \end{pmatrix}$$

Now change summation variables to $M_2 \equiv -m$ $M_3 \equiv -m'$, $M \equiv -m''$, in all three 3-j symbols interchange 2nd and 3rd row, picking up a factor of $(-)^{6S+\Sigma\ell_i} = (-)^{2S+\Sigma\ell_i}$ altogether;

use formula (C33), p. 1064 in MII, with the identification $(\ell_i\, m_i) \leftrightarrow (j_i\, m_i)$ and $J_1 = J_2 = J_3 \equiv S$, $M_i \leftrightarrow M_i$ to get:

$$\mathrm{Tr}\left(T_{\ell_1 m_1} T_{\ell_2 m_2} T_{\ell_3 m_3}\right)$$

$$= \prod R_N(\ell_i)\,(-)^{2S+\Sigma\ell_i} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ S & S & S \end{Bmatrix}$$

where {} denotes the Wigner 6j-symbol.

$$\mathrm{Tr}\left(T_{\ell_2 m_2} T_{\ell_1 m_1} T_{\ell_3 m_3}\right)$$

$$=\left(\prod R_N(\ell_i)\right)(-)^{2S} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ S & S & S \end{Bmatrix}$$

as {} is invariant under interchange of two columns, while () has to be multiplied by $(-)^{\ell_1+\ell_2+\ell_3}$. Therefore,

$$f^{(N)}_{\substack{\ell_1 \ell_2 \ell_3 \\ m_1 m_2 m_3}} = \begin{cases} \dfrac{R_N(\ell_1) R_N(\ell_2)}{R_N(\ell_3)} (2\ell_3+1)\, 2 \cdot (-)^N \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ S & S & S \end{Bmatrix} & (\text{if } \Sigma \ell_i \text{ odd}) \\ 0 & (\text{if } \Sigma \ell_i \text{ even}) \end{cases} \tag{B18}$$

Note that $f \equiv 0$ if one $\ell_i > N$. Also $f=0$ unless $\Sigma m_i = 0$ and the ℓ_i satisfy the triangle inequalities. (B18 was obtained from (B16):

$$[T_{\alpha_1}, T_{\alpha_2}] = f_{\alpha_1 \alpha_2 \alpha_3} T^{+}_{\alpha_3}$$

$$\Rightarrow \mathrm{Tr}\, T_{\alpha_3} [T_{\alpha_1}, T_{\alpha_2}] = f_{\alpha_1 \alpha_2 \alpha_3} R_N^2(\ell_3) (2\ell_3+1)^{-1} \Big)$$

(B18) is a formula for the structure constants of <u>SU(N)</u> in the basis $T_{\ell m} = \overset{0}{T}_{\ell m} \cdot U(N,\ell)$.

Particular choices for $R_N(\ell) = R_N^{O}(\ell) \cdot U(N,\ell)$ are:

i) $R=R_0$

ii) R = $\overline{R}_N(\ell) \equiv \sqrt{2\ell+1}$

This choice will make $f^{(N)}$ totally antisymmetric for all N. ($\hat{=}T_{\ell m}$-basis orthonormal) (any R differing from $\bar{R}$ by an ℓ-dependent factor will not have this property).

iii) R = $\frac{(-)^N\sqrt{2\ell+1}}{2}$ leads to

$$f^{(N)}_{\substack{\ell_1\ell_2\ell_3\\m_1m_2m_3}} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ S & S & S \end{Bmatrix}$$

iv) R = $\sqrt{2\ell+1}\cdot N^{3/2} = \overline{R}_N(\ell)\cdot N^{3/2}$ is (B19)

totally antisymmetric ($\forall_N$), but in addition, the corresponding $f^{(N)}$ will have a finite, non-zero limit as $N\to\infty$, as

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ S & S & S \end{Bmatrix} \propto N^{-3/2}$$

for $S\to\infty$ (see below). Therefore, if there is <u>any</u> choice of R_N (at all) for which

$$\lim_{N\to\infty} f^{(N)}_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$$

one can use the above choice $R = N^{3/2}\sqrt{2\ell+1}$ to calculate $g_{\alpha\beta\gamma}$ (up to a constant independent of $\alpha\beta$ and γ-- which turns out to be $(16\pi)^{-1/2}$). The "correct" choice is

$$\hat{R}_N(\ell) = \sqrt{\frac{(N+\ell)!}{(N-\ell-1)!}}\ \sqrt{\frac{N^2-1}{16\pi}}^{\,-\ell+1}\ \sqrt{2\ell+1} \qquad \text{(B20)}$$

and is based on the proof given in the next section. (As $N\to\infty$, $\hat{R}_N$ differs from (B19) only by $1/\sqrt{4\pi}$

3. There is a basis $\hat{T}^{(N)}_{\ell m}$ of SU(N) with $\lim_{N\to\infty} \hat{f}^{(N)}_{\alpha\beta\gamma}=g_{\alpha\beta\gamma}$ (constructive proof)

First look at the process that determines $g_{\alpha\beta\gamma}$:

$$y_{\ell m} \equiv r^\ell Y_{\ell m} \equiv \sum \mathring{a}^{(m)}_{i_1\cdots i_\ell} x_{i_1}\cdots x_{i_\ell}$$

are harmonic polynomials, $\mathring{a}$ traceless (and symmetric of course).

$$\{f(\theta\varphi), g(\theta\varphi)\} \equiv \frac{1}{\sin\theta}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\varphi} - \frac{\partial g}{\partial\theta}\frac{\partial f}{\partial\varphi}\right)$$

is in fact what one gets when one restricts the space of all polynomial functions $f(x_1x_2x_3)$ with the Lie bracket defined as

$$\{f, g\} \equiv \epsilon_{ijk} x_i \partial_j f \, \partial_k g \qquad \text{(B21)}$$

to functions on the unit sphere $(x_1^2+x_2^2+x_3^2=\vec{x}^2=1)$

That (B21) really defines a Lie bracket (i.e., { } satisfies the Jacobi identity; {f,f}=0 is trivial) is shown in III; there in fact for ϵ_{ijk} being replaced by any c^i_{jk} antisymmetric in j and k and satisfying the Jacobi identity. Therefore:

$$\{y_{\ell m}, y_{\ell' m'}\} = \sum \mathring{a}^{(m)}_{i_1\cdots i_\ell}\, \mathring{a}^{(m')}_{j_1\cdots j_{\ell'}} \{x_{i_1}\cdots x_{i_\ell}, x_{j_1}\cdots x_{j_{\ell'}}\}$$

$$= \sum \mathring{a}^{(m)}_{i_1\cdots i_\ell}\, \mathring{a}^{(m')}_{j_1\cdots j_{\ell'}} \sum_{\alpha=1}^{\ell}\sum_{\beta=1}^{\ell'} x_{i_1}\cdots x_{i_{\alpha-1}} x_{j_1}\cdots x_{j_{\beta-1}} \{x_{i_\alpha}, x_{j_\beta}\} x_{j_{\beta+1}}\cdots x_{j_{\ell'}} x_{i_{\alpha+1}}\cdots x_{i_\ell} \tag{B22}$$

using $\{fg, h\} = f\{g, h\} + \{f, h\} g$.

The order is, of course, completely irrelevant, as everything commutes, and with $\{x_{i_\alpha}, x_{j_\beta}\} = \epsilon_{i_\alpha j_\beta k}\, x_k$ one gets (B21) as one must. There were/are two reasons for having written down (B22). The first is that it makes slightly more apparent the following decomposition of $\{y_{\ell m}, y_{\ell' m'}\}$ — which is a homogeneous but no longer harmonic polynomial $P_{\ell+\ell'-1}$ into a sum of harmonic polynomials of degree $\ell+\ell'-1$ and lower:

$$P_{\ell+\ell'-1} = \sum_{\substack{m'', \ell''=1 \\ (\ell+\ell'-1-\ell''\ \mathrm{even})}}^{\ell+\ell'-1} d_{\ell\ell'\ell''}^{\,m m' m''} (x^2)^{\frac{\ell+\ell'-1-\ell''}{2}} \cdot y^*_{\ell'' m''} \tag{B23}$$

(corresponding to making $\mathring{a}^{(m)}_{i_1\cdots i_\ell}\, \mathring{a}^{(m')}_{j_1\cdots j_{\ell'}} \epsilon_{i_\alpha j_\beta k}$ traceless and totally symmetric). By restriction to the unit sphere, one sees that $d_{\ell\ell'\ell''} = -i g_{\ell\ell'\ell''}$ (see B5). The second reason is that (B22) stresses the connection between $Y_{\ell m}$ and $\mathring{T}_{\ell m}$ as the expression for $[\mathring{T}_{\ell m}, \mathring{T}_{\ell' m'}]$ will be exactly like (B22) just with

$$x_i \to S_i \text{ and } \{x_{i\alpha}\, x_{j\beta}\} \to [S_{i\alpha},\, S_{j\beta}]$$
$$(= i\,\epsilon_{i\alpha j\beta k}\, S_k).$$

$[\overset{\circ}{T}_{\ell m},\, \overset{\circ}{T}_{\ell' m'}] \equiv Q_{\ell+\ell'-1}$ is a homogeneous polynomial in the S_i of degree $\ell+\ell'-1$. The decomposition of $Q_{\ell+\ell'-1}$ into

$$\sum_{\substack{m''\ \ell''=1 \\ (\ell+\ell'-1-\ell''\ \text{even})}}^{\ell+\ell'-1} \overset{\circ}{f}{}^{(N)}_{\substack{\ell\ell'\ell'' \\ mm'm''}}(\chi_N)\, \overset{\circ}{T}{}^{\dagger}_{\ell'' m''} \quad \left(\text{with } \chi_N \equiv S(S+1) = \frac{N^2-1}{4}\right) \tag{B23'}$$

however is more complicated, as the S_i are non commuting objects so that the process of making $\overset{\circ}{a}{}^{(m)}\overset{\circ}{a}{}^{(m')}\,\epsilon_{i\alpha j\beta k}$ traceless and symmetric, which involves moving the S_i around,

$$\text{using } S_i S_j = S_j S_i + i\epsilon_{ijk} S_k \tag{B24}$$

will give lower order polynomials. Therefore

$$\overset{\circ}{f}{}^{(N)}_{\substack{\ell\ell'\ell'' \\ mm'm''}}(\chi_N) = \sum_{\substack{\alpha=0 \\ (\alpha\ \text{even})}}^{\ell+\ell'-1-\ell''} \overset{\circ}{f}{}^{[\alpha]}_{\substack{\ell\ell'\ell'' \\ mm'm''}} \cdot \chi_N^{\alpha/2} \tag{B25}$$

with highest order term

$$\overset{\circ}{f}{}^{[\ell+\ell'-1-\ell'']}_{\substack{\ell\ell'\ell'' \\ mm'm''}} \sqrt{\chi_N}^{\,\ell+\ell'-1-\ell''} \equiv f_{\substack{\ell\ell'\ell'' \\ mm'm''}} \sqrt{\chi_N}^{\,\ell+\ell'-1-\ell''}$$

will contain lower powers of N. $\chi_N \mathbb{1} = S_1^2 + S_2^2 + S_3^2 = \frac{N^2-1}{4}\mathbb{1}$ of course arises from the trace contributions) But what is important is that all terms in (B23') of degree $\ell+\ell'-1$ (X_N has degree 2, $T_{\ell''}$ degree ℓ''), i.e., all terms

$$f_{\substack{\ell\ell'\ell''\\ mm'm''}} \chi_N^{\frac{\ell+\ell'-1-\ell''}{2}} T^{\dagger}_{\ell''m''} \quad (\text{no summation})$$

arose from always picking up the first term in (B24), i.e., treating the S_i as commuting objects, in effect. Therefore:

$$f_{\substack{\ell\ell'\ell''\\ mm'm''}} = i\, d_{\substack{\ell\ell'\ell''\\ mm'm''}} \tag{B26}$$

(the i, as [,] gives an extra i compared with {}). This means that for $\hat{T}_{\ell m} \equiv \overset{\circ}{T}_{\ell m} \Big/ \chi_N^{\frac{\ell-1}{2}}$ leading to

$$[\hat{T}_{\ell m}, \hat{T}_{\ell' m'}] = \sum_{\ell'' m''} \hat{f}^{(N)}_{\substack{\ell\ell'\ell''\\ mm'm''}} \hat{T}^{\dagger}_{\ell'' m''}$$

one has

$$\hat{f}^{(N)}_{\substack{\ell\ell'\ell''\\ mm'm''}} = \frac{\chi_N^{\frac{\ell+\ell'-\ell''-1}{2}} f_{\substack{\ell\ell'\ell''\\ mm'm''}}}{\chi_N^{\frac{\ell+\ell'-\ell''-1}{2}}} + O\left(\frac{1}{\chi_N}\right) = i\, d_{\substack{\ell\ell'\ell''\\ mm'm''}} + O\left(\frac{1}{\chi_N}\right)$$

$$i d = i(-ig) = g \qquad \lim_{N\to\infty} \hat{f}^{(N)}_{\substack{\ell\ell'\ell''\\ mm'm''}} = g_{\substack{\ell\ell'\ell''\\ mm'm''}} \tag{B28}$$

Thus one has the desired result.

(B28)

4. Calculation of $g_{\alpha\beta\gamma}$

Because of (B28) one can now use (B18) (for properly chosen $R_N(\ell)$) to calculate $g_{\alpha\beta\gamma}$. First one has to find out the behavior of $\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ s & s & s \end{Bmatrix}$ as $N\to\infty$.

Racah's formula [see e.g. MII, p. 1065,
for $1 \le \ell_1 \le \ell_2 \le \ell_3$, $\ell_1+\ell_2 \le 2s = N-1$
is

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ s & s & s \end{Bmatrix} = \ell_1!\,\ell_2!\,\ell_3!\,\sqrt{\frac{(\ell_1+\ell_2-\ell_3)!(\ell_1+\ell_3-\ell_2)!(\ell_2+\ell_3-\ell_1)!}{(\ell_1+\ell_2+\ell_3+1)!}}\,(-)^{\ell_3-1}$$

$$\cdot\sqrt{\frac{(2s-\ell_1)!}{(2s+1+\ell_1)!}}\,\sqrt{\frac{(2s-\ell_2)!}{(2s+1+\ell_2)!}}\,\sqrt{\frac{(2s-\ell_3)!}{(2s+1+\ell_3)!}}\,(-)^{2s+1}$$

$$\cdot\sum_{x=0}^{\ell_1+\ell_2-\ell_3} \frac{(2s+\ell_3+x+1)!}{(2s+x-\ell_1-\ell_2)!}\,\frac{(-)^x}{x!(\ell_1+\ell_2-\ell_3-x)!(\ell_1-x)!(\ell_2-x)!(\ell_3+x-\ell_1)!(\ell_3+x-} \tag{B29}$$

$$\equiv J(\ell_i)\,H_N(\ell_i)(-)^N \sum_{x=0}^{\ell_1+\ell_2-\ell_3} G_N(x,\ell_i)\,\frac{(-)^x}{F(x,\ell_i)} \tag{B29'}$$

[i.e., J includes all N and x-independent factors; H_N consists of the remaining x-independent factors (apart from $(-)^N$) G_N depends on both x and N, F is independent of N] as $N \to \infty$:

$$H_N = \prod_{i=1}^{3} \sqrt{\frac{(N-(\ell_i+1))!}{(N+\ell_i)!}} = \prod_{i=1}^{3} \left((N+\ell_i)(N+\ell_i-1)\cdots(N-\ell_i)\right)^{-1/2} \to N^{-3/2} N^{-\ell_1-\ell_2-\ell_3},$$

the leading term in G_N is $N^{\ell_1+\ell_2+\ell_3+1}$. However, any independent term in G_N will give 0, as $F(x;\ell_i)$ is invariant under $x \to (\ell_1+\ell_2-\ell_3)-x$, the # of terms in $\sum$ is even (as $\ell_1+\ell_2-\ell_3$ is odd), and therefore $\sum_x \frac{(-)^x}{F(x)} = 0$.

The leading contributing term in

$$G_N = \frac{(N+x+\ell_3)!}{(N+x-(\ell_1+\ell_2+1))!}$$ is therefore

$$N^{\ell_1+\ell_2+\ell_3}(\ell_1+\ell_2+\ell_3+1)\cdot x$$

The leading term in $\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ S & S & S \end{Bmatrix}$ as $N\to\infty$, is therefore:

$$J \cdot N^{-3/2}(-)^N(\ell_1+\ell_2+\ell_3+1)\sum_{x=0}^{\ell_1+\ell_2-\ell_3} \frac{x(-)^x}{F(x)} \qquad \text{(B29")}$$

and

$$f^{(N)}_{\substack{\ell_1\ell_2\ell_3 \\ m_1m_2m_3}} = \frac{R_N(\ell_1)R_N(\ell_2)}{N^{3/2}\, R_N(\ell_3)}(2\ell_3+1)\cdot 2\cdot\left(\Sigma\ell_i+1\right)\cdot J\cdot\sum_{x=0}^{\ell_1+\ell_2-\ell_3}\frac{x(-)^x}{F(x;\ell_i)}\cdot\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\left(1+O\left(\tfrac{1}{N}\right)\right)$$

where $J \equiv \ell_1!\,\ell_2!\,\ell_3!\,(-)^{\ell_3-1}\sqrt{\frac{(\ell_1+\ell_2-\ell_3)!\,(\ell_1+\ell_3-\ell_2)!\,(\ell_2+\ell_3-\ell_1)!}{(\ell_1+\ell_2+\ell_3+1)!}}$

(B30)

and $F(x) \equiv x!\,(\ell_1+\ell_2-\ell_3-x)!\,(\ell_1-x)!\,(\ell_2-x)!\,(x+\ell_3-\ell_1)!\,(x+\ell_3-\ell_2)!$;

$1 \le \ell_1 \le \ell_2 \le \ell_3$, $\ell_1+\ell_2 \le N-1$

(these two conditions are slightly artificial as they are only necessary to write down $\left\{\begin{matrix} \ell_1 & \ell_2 & \ell_3 \\ s & s & s \end{matrix}\right\}$ as explicitly as in (B29)).

Because of (B18), and because $\hat{R}_N$ has to behave like $N^{3/2}\sqrt{2\ell+1}$ const. for large N (so to make (B30) finite and totally antisymmetric as $N\to\infty$), $g_{\alpha\beta\gamma}$ can be calculated as:

$$\lim_{N\to\infty} f^{(N)}_{\alpha\beta\gamma}\left[R_N(\ell) = N^{3/2}\sqrt{2\ell+1}\cdot\text{const.}\right]$$

where the constant can be determined by comparing g and $\lim_{N\to\infty} f^{(N)}$ (calculated via (B30)) in just one simple case, e.g., $\begin{bmatrix} 1 & 1 & 1 \\ 0 & +1 & -1 \end{bmatrix}$. As not more work is involved one calculates $f^{(N)}$ for the case $\begin{bmatrix} 1 & \ell & \ell \\ 0 & m & -m \end{bmatrix}$: with

$$\begin{pmatrix} 1 & \ell & \ell \\ 0 & m & -m \end{pmatrix} \overset{*}{=} (-)^{\ell-m}\frac{m}{\sqrt{(2\ell+1)\ell(\ell+1)}}$$

(B30) gives, for $R=N^{3/2}\sqrt{2\ell+1}\cdot(\text{const.})$:

$$f_{\substack{1\ell\ell \\ 0\,m\,-m}} = (\text{const.})(2\ell+1)\sqrt{3}(2\ell+2)(\ell!)^2\cdot\sqrt{\frac{(2\ell-1)!}{(2\ell+2)!}\frac{1}{\ell!(\ell-1)!}}\cdot\frac{(-)^m m}{\sqrt{\ell(\ell+1)(2\ell+1)}}$$

$$= 2\sqrt{3}\,m(-)^m\cdot\text{const.}$$

---

*See e.g. MII, p. 1060.

which agrees with

$$g_{\substack{1\ell\ell \\ 0m-m}} = m(-)^m \sqrt{\frac{3}{4\pi}} \quad (\text{from B8})$$

provided const = $1/\sqrt{16\pi}$, and provides already a first check, as ℓ and m are general in $\begin{bmatrix} 1 & \ell & \ell \\ 0 & m & -m \end{bmatrix}$ So

$$R_N(\ell) = \frac{1}{\sqrt{16\pi}} N^{3/2} \sqrt{2\ell+1} \tag{B31}$$

can be used to calculate ~~$g_{\alpha\beta\gamma}$~~

$$g_{\substack{\ell_1\ell_2\ell_3 \\ m_1m_2m_3}} = \lim_{N\to\infty} f^{(N)}_{\substack{\ell_1\ell_2\ell_3 \\ m_1m_2m_3}}\left[R_N(\ell) = \frac{1}{\sqrt{16\pi}} N^{3/2}\sqrt{2\ell+1}\right] \equiv f_{\substack{\ell_1\ell_2\ell_3 \\ m_1m_2m_3}}$$

$$= \frac{1}{\sqrt{4\pi}}(\ell_1+\ell_2+\ell_3+1)\prod_i\sqrt{2\ell_i+1}\cdot J\cdot\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3\end{pmatrix}\sum_{x=0}^{\ell_1+\ell_2-\ell_3}\frac{x(-)^x}{F(x; \ell_i)} \tag{B32}$$

$(1 \le \ell_1 \le \ell_2 \le \ell_3,\ \sum \ell_i \text{ odd},\ J \text{ and } F \text{ as in B30})$

Since $g_{\alpha\beta\gamma}$ is totally antisymmetric, it can be calculated via (B32) for all $(\alpha\beta\gamma)$. We check (B32) for $\begin{bmatrix} 2 & \ell & \ell+1 \\ 0 & m & -m \end{bmatrix}$: Using

$$\begin{pmatrix} 2 & \ell & \ell+1 \\ 0 & m & -m \end{pmatrix} \overset{*}{=} (-)^{\ell-m}\, 2m \sqrt{\frac{6(\ell+m+1)(\ell-m+1)}{(2\ell+4)(2\ell+3)(2\ell+2)(2\ell+1)2\ell}}$$

one finds that $f_{\substack{2\,\ell\,\ell+1 \\ 0\,m\,-m}}$ calculated via (B32), agrees with

$$g_{\substack{2\,\ell\,\ell+1 \\ 0\,m\,-m}} = 3\sqrt{\frac{5}{4\pi}}\, m\, (-)^m \sqrt{\frac{(\ell+m+1)(\ell+1-m)}{(2\ell+1)(2\ell+3)}} \qquad \text{from (B8')}$$

Finally it is useful to calculate f^N for $\ell_3 = \ell_1+\ell_2-1$: with the notation as in (B29') one has for this case

$$\sum_{x=0}^{\ell_1+\ell_2-\ell_3} G_N(x;\ell_i)\frac{(-)^x}{F(x;\ell_i)} = \sum_{x=0}^{1} \frac{(N+x+\ell_3)!}{(N+x-(\ell_1+\ell_2+1))!}\,\frac{(-)^x}{F(x;\ell_i)}$$

$$= \frac{1}{F(0)}\left(\frac{(N+\ell_3)!}{N-(\ell_1+\ell_2+1)!} - \frac{(N+\ell_3+1)!}{(N+1-(\ell_1+\ell_2+1))!} \right)$$

$$= \frac{1}{F}\,\frac{(N+\ell_3)!}{(N-\ell_1-\ell_2)!}\,(N-\ell_1-\ell_2-N-\ell_3-1)$$

$$= \frac{-2}{F}\,\frac{(N+\ell_1+\ell_2-1)!}{(N-\ell_1-\ell_2)!}\,(\ell_1+\ell_2)$$

Therefore, using (B18) and (B29):

$$f^{(N)}_{\substack{\ell_1 \ell_2 \ell_3 = \ell_1+\ell_2-1 \\ m_1 m_2 m_3}} = \frac{R_N(\ell_1) R_N(\ell_2)(2\ell_1+2\ell_2-1)}{R_N(\ell_1+\ell_2-1)} 2 \begin{pmatrix} \ell_1 & \ell_2 & \ell_1+\ell_2-1 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$\cdot \left(-\frac{2(\ell_1+\ell_2)}{F(0)}\right) \cdot J(\ell_3 = \ell_1+\ell_2-1) \quad \text{(B33)}$$

$$\cdot \sqrt{\frac{(N-\ell_1-1)!}{(N+\ell_1)!}} \sqrt{\frac{(N-\ell_2-1)!}{(N+\ell_2)!}} \sqrt{\frac{(N-\ell_1-\ell_2)!}{(N+\ell_1+\ell_2-1)!}} \cdot \frac{(N+\ell_1+\ell_2-1)!}{N-\ell_1-\ell_2}$$

The proof in (BII3) showed in particular that R corresponding to

$$\hat{T} = \frac{\overset{\circ}{T}}{\sqrt{\frac{N^2-1}{4}}^{\,\ell-1}}$$

leads to $\hat{f}^{(N)}$ that is independent of N for $\ell_3 = \ell_1 + \ell_2 - 1$

A factor

$$\sqrt{\frac{(N+\ell)!}{(N-\ell-1)!}}$$

must therefore be contained in $\hat{R}_N$. This factor $\approx N^{\ell+1/2}$ as $N \to \infty$. To have $\hat{R}_N \approx N^{3/2}$ (as $N \to \infty$), which is needed so that $\hat{f}^{(N)}$ has a finite non-zero limit, one must include another factor that is $\approx N^{1-\ell}$ (as $N \to \infty$). Because of

$$\hat{T} = \overset{\circ}{T} \Big/ \sqrt{\frac{N^2-1}{4}}^{\,\ell-1}$$

one is led to choose this factor to be $\sqrt{N^2-1}^{\,1-\ell}$.

Putting all together

$$\hat{R}_N = \frac{\sqrt{2\ell+1}}{\sqrt{16\pi}} \sqrt{\frac{(N+\ell)!}{(N-\ell-1)!}} \sqrt{N^2-1}^{\,1-\ell} \tag{B20}$$

5. Direct calculation of $\hat{R}_N$

Rather than by making heavy use of the proof BII3 and deducing $\hat{R}_N$ by the above arguments, $\hat{R}_N$ can be derived directly from the correspondence to the $Y_{\ell m}$ and the properties of that $\hat{R}$ will then provide a check on the proof, rather than relying on it!

$$\overset{\circ}{T}_{\ell\ell} = \frac{(-)^\ell}{\ell!\,2^\ell} \sqrt{\frac{(2\ell+1)!}{4\pi}} \left(S_x + iS_y\right)^\ell \qquad \text{(from (B12))}$$

$$\Rightarrow$$

$$\frac{\overset{\circ}{R}{}^{2}_N(\ell)}{2\ell+1} = \mathrm{Tr}\left(\overset{\circ}{T}_{\ell\ell}\,\overset{\circ}{T}{}^{\dagger}_{\ell\ell}\right) = \frac{(2\ell+1)!}{(4\pi)(\ell!)^2\,2^{2\ell}} \mathrm{Tr}\left(S_+^{\ell} S_-^{\ell}\right) \tag{B17}$$

$$\mathrm{Tr}(S_+^{\ell} S_-^{\ell})$$

$$= \sum_m \langle m|S_+|m-1\rangle\langle m-1|S_+|m-2\rangle \cdots \langle m+1-\ell|S_+|m-\ell\rangle \cdot \langle m-\ell|S_-|m+1-\ell\rangle \cdots \langle m-1|S_-|m\rangle$$

$$= \sum_{m=\ell-s}^{+s} \Big(s(s+1)-m(m+1)\Big)(\ldots)\cdots\Big(s(s+1)-(m-\ell+1)(m-\ell)\Big)$$

(B11)

$m=\ell-s+\alpha$

$N=2s+1$

$$= \sum_{\alpha=0}^{N-\ell-1} \Big(s(s+1)-\underbrace{(\ell-s+\alpha)(\ell-s+\alpha-1)}_{=\,(s-(\ell+\alpha))(s+1-(\ell+\alpha))}\Big)\cdots\Big(s(s+1)-\underbrace{(-s+\alpha+1)(-s+\alpha)}_{=\,(s-(\alpha+1))(s+1-(\alpha+1))}\Big)$$

$$= \sum_{\alpha=0}^{N-\ell-1} \Big(N(\ell+\alpha)-(\ell+\alpha)^2\Big)\cdot\ldots\cdot\Big(N(\alpha+1)-(\alpha+1)\Big)$$

$$= \sum_{\alpha=0}^{N-\ell-1} (\alpha+1)(\alpha+2)\cdots(\alpha+\ell)\cdot\big(N-(\alpha+1)\big)\cdots\big(N-(\alpha+\ell)\big)$$

$$= \sum_{\alpha=0}^{N-\ell-1} \prod_{\beta=1}^{\ell} (\alpha+\beta)\big(N-(\alpha+\beta)\big)$$

$$= (\ell!)^2 \sum_{\alpha=0}^{N-\ell-1} \binom{\alpha+\ell}{\ell}\binom{N-1-\alpha}{\ell} = (\ell!)^2 \binom{N+\ell}{2\ell+1} \qquad \text{(B34)}$$

$$= \frac{(\ell!)^2}{(2\ell+1)!}\,\frac{(N+\ell)!}{(N-\ell-1)!}$$

(B34) can be proved in the following way: Since one has

$$(1-x)^{-m-1} = \sum_r \binom{m+r}{r} x^r$$

one has $(1-x)^{-m-1}(1-x)^{-n-1} = \sum_{r,s} \binom{m+r}{r} x^{r+s} \binom{n+s}{s}$

but also

$$= (1-x)^{-m-n-2} = \sum_t \binom{m+n+1+t}{t} \mathbf{x}^t$$

so that $\Rightarrow \sum_{r+s=t} \binom{m+r}{r}\binom{n+s}{s} = \binom{m+n+1+t}{t}$

$$= \binom{m+n+1+t}{m+n+1}$$

Since

$$\binom{\alpha+\ell}{\ell}\binom{N-1-\alpha}{\ell} \equiv \binom{\alpha+\ell}{\alpha}\binom{N-1-\alpha}{N-1-\ell-\alpha}$$

one obtains (B34) by identifying

$$\alpha \leftrightarrow r,\ \ell \leftrightarrow m,\ N-1-\ell-\alpha \leftrightarrow s$$

$$n \leftrightarrow \ell,\ r+s=t \leftrightarrow N-1-\ell,\ m+n+1 \leftrightarrow 2\ell+1$$

Using (B30) one has

$$\mathring{R}_N(\ell) = \sqrt{\frac{(N+\ell)!}{(N-\ell-1)!}}\ \frac{1}{\sqrt{4\pi}}\ \frac{\sqrt{2\ell+1}}{2^\ell}$$

and

$$\hat{R}_N = \frac{\mathring{R}_N}{\sqrt{\chi_N}^{\,\ell-1}} =$$

$$= \frac{\sqrt{2\ell+1}}{\sqrt{16\pi}} \sqrt{\frac{(N+\ell)!}{(N-\ell-1)!}} \sqrt{N^2-1}^{\,1-\ell}$$

(which agrees with (B20))

As was done in the previous section, one can explicitly see, that for this choice, the structure constants of the stretched position ($\ell_3=\ell_1+\ell_2-1$) are independent of N, a fact whose significance appears in the next section.

III. AN UNDERLYING MATHEMATICAL REASON FOR THE ABOVE CONSTRUCTION

Having found explicitly the correspondence between the Lie algebra of area preserving transformations and an N-dimensional representation of SU(N) ($N\to\infty$) by constructing a basis (the $T_{\ell m}$) as polynomials in the S_i (a basis of the N-dimensional representation of SO(3)) one might wonder whether there is not an underlying mathematical reason for this construction to work. This would provide some additional understanding and also possibly lead to generalizations. In particular, most statements would be independent of a particular representation.

It turns out that it is the space of the $Y_{\ell m}$'s with $\{,\}$ and the role of the abstract Lie algebra SO(3) which have a natural generalization, while SU(N) arises as the space in which N-dimensional unitary representation of SO(3) lie. (In this sense SO(3) *is* special, as for a general Lie algebra there will not be exactly one irreducible inequivalent representation for each N. Also there will be in general more than one Casimir operator that when going to an N-dimensional representation will carry the N-dependence.)

Let G be a Lie algebra over the complex numbers, whose adjoint representation is completely reducible, and G be the adjoint group.* Let $x_1 \ldots x_n$ be a basis of G. The enveloping

*Note: G will *not* correspond to the group of area preserving reparametrizations of S^2 (which was called G in BI), but rather to SO(3).

algebra U(G), which is defined in rather abstract terms *, can be taken** to be the tensor algebra $\tau(G)$ (i.e., the space of all polynomials $a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$) with two elements identified if they are equal using the commutation relations $[x_i, x_j] = c_{ij}^k x_k$; the set of all

$$X_1^{j_1} X_2^{j_2} \cdots X_m^{j_m} \quad (j_\gamma \geq 0 \,,\; \Sigma j_\gamma > 0)$$

is therefore a basis of $U(\underline{G})$, and $u \in U$ will be written as

$$\sum a^{(u)}_{j_1 \cdots j_m} x_1^{j_1} \cdots x_m^{j_m} \quad .$$

Define U_ℓ as the space of all $u \in U$ with degree

$$(\equiv \Sigma j_\gamma) \leq \ell \quad . \qquad U_\ell U_k \subseteq U_{\ell+k}$$

and the U_ℓ are called a filtration of U. There is a natural Poisson bracket defined on U: $[U,U'] = uu' - u'u$; then $[U_k, U_\ell] \subseteq U_{k+\ell-1}$.

The symmetric algebra S(G) is defined as the space of all polynomials in n commuting objects $x_1 \ldots x_n$. This space also has

$$\left\{ x_1^{j_1} \cdots x_m^{j_m} \;\middle|\; j_\gamma \geq 0 \,,\; \Sigma j_\gamma > 0 \right\}$$

*See e.g. "Lie Algebras" by Jacobson (Interscience, 1962).

**Poincaré-Birkhoff-Witt theorem, see *.

as a basis, but $xx'-x'x=0$ in $S(\underline{G})$. $S(\underline{G})$ can (and will from now on) be regarded as the space of pol nomial functions f on the dual space $\underline{G}' = \mathbb{R}^n$, (then $s \in S$ is a polynomial in n real variables,with complex coefficients). Let $S_k \subset S$ be the set of all homogeneous polynomials of degree=k. One can define a Poisson bracket {} on S, with $\{S_k, S_\ell\} \in S_{k+\ell-1}$, by defining the following surjective homomorphism $T_k : U_k \to S_k$ (which has U_{k-1} as kernel):

$$U_k = \sum_{\Sigma i_s \leq k} a^u_{j_1 \cdots j_m} x_1^{j_1} \cdots x_m^{j_m} \to \sum_{\Sigma i_s = k} a^u_{j_1 \cdots j_m} x_1^{j_1} \cdots x_m^{j_m} \in S_k$$

and letting

$$\{S_{k_1}, S_{k_2}\} \equiv_{\text{def.}} T_{k_1+k_2-1}([u_{k_1}, u_{k_2}])$$

where the u_{k_i} are some elements of U_{k_i} with $T_{k_i}(u_{k_i}) = S_{k_i}$. {} is well defined, as u_{k_i} is ambiguous only in the terms of degree k_i, so that $[u_{k_1}, u_{k_2}]$ is some $u_{k_1+k_2-1}$, with an ambiguity only in the terms of degree k_1+k_2-1, which makes $T_{k_1+k_2-1}(u_{k_1+k_2-1}) \in S_{k_1+k_2-1}$ unambiguous (uniquely defined). The so defined Poisson bracket $\{f,g\}$ of two polynomial functions $f,g \in S(G)$ is in fact equal to

$$C^i_{jk}\, x_i\, \partial_j f\, \partial_k g \tag{B35}$$

where c^i_{jk} are the (not necessarily totally antisymmetric) structure constants * of $\underline{G}$. One can verify explicitly that (C9) defines a Poisson bracket, i.e.,

$$\{f, f\} = C^i_{jk} X_i \partial_j f \partial_k f = 0 \quad (\text{as } C^i_{jk} = -C^i_{kj})$$

and

$$\sum_{\nu} \{\{f_{\nu_1}, f_{\nu_2}\}, f_{\nu_3}\}$$

(ν cyclic permutation of $(1,2,3)$)

$$= C^i_{jk} X_i \partial_j \left(C^r_{st} X_r \partial_s f_{\nu_2} \partial_t f_{\nu_3} \right) \partial_k f_{\nu_1}$$

$$= \sum_{\nu} C^i_{jk} C^r_{st} X_i \delta_{rj} \partial_s f_{\nu_2} \partial_t f_{\nu_3} \partial_k f_{\nu_1}$$

$$+ \sum_{\nu} C^i_{jk} C^r_{st} X_i X_r (\partial^2_{js} f_{\nu_2}) \partial_t f_{\nu_3} f_k f_{\nu_1}$$

$$+ \sum_{\nu} C^i_{jk} C^r_{st} X_i X_r \partial_s f_{\nu_2} (\partial^2_{jt} f_{\nu_3}) \partial_k f_{\nu_1}$$

$$\text{the first term} = X_i (C^j_{st} C^i_{jk} + C^j_{ks} C^i_{jt} + C^j_{tk} C^i_{js}) \partial_s f_{\nu_2} \partial_t f_{\nu_3} \partial_k f_{\nu_1}$$

$$= 0 \quad (\text{by Jacobi identity of } C^{\alpha}_{\beta\gamma})$$

One then sees that the second and third term cancel, using only $c^i_{jk} = -c^i_{kj}$.

---

*in the basis $x_1 \ldots x_n$

Following B. Kostant*, one can characterize the structure of U and S and the relation between them in the following way:

1. $S=J \otimes H$ (every element of S can be written as

$$\sum j_\alpha h_\alpha \text{ with } j_\alpha \in J, h_\alpha \in H \text{)}$$

where J is defined as the space of all polynomials invariant under the group action [which is induced by the adjoint action of G on $\underline{G}$; for matrices:

$$x \in \underline{G} \to g^{-1} x g \in \underline{G} \text{]}$$

and $H \equiv$ the set of all G-harmonic polynomials, i.e., all $f \in S$ such that $\partial f=0$ for every homogeneous differential operator ∂ with constant coefficients, that commutes with the group action. In our case:

the only such ∂ is ∇^2 (and functions of ∇^2),

$H \equiv$ space of harmonic polynomials in the usual sense ($\nabla^2 h=0$),

any nonconstant $f(x_1 x_2 x_3)$ can be written as

$\sum_{\ell=1}^{\infty} j_\ell(r)\left(\sum_m a_{\ell m} r^\ell Y_{\ell m}(\theta\varphi)\right)$, which is the usual separation of variables.

2. Let O_x denote the G-orbit in $\underline{G}$ of $x \in \underline{G}$, and let $S(O_x)$ be the ring of all functions on O_x defined by restricting S to O_x; let $\underline{r}$ be the rank of $\underline{G}$; then $\dim O_x \leq n-\underline{r}$ and

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*"Lie group representations on polynomial rings", Am. J.M. 85, 1963, p. 327-404. I would like to thank Prof. Kostant, Alex Uribe and Robin Ticciati very much for several discussions and much patience. This Section (BIII) would not exist without their ideas and help.

for every $x \in \underline{G}$ such that $\dim O_x = n - \underline{r}$, H and $S(O_x)$ are isomorphic as G-modules [a G-module is a vector space V together with a map $G \to GL(V)$, $g_1(g_2 V) = (g_1 g_2)V$]. For $G = SO(3)$, $\dim O_x = 3 - 1 = 2 \ \forall_{x \in \underline{G}}$.

3. $U = Z \otimes E$ where $Z \equiv$ Center of U (i.e., all Z with $[z, u] = 0$ $\forall u \in U$) and $E \equiv$ space spanned by all powers x^k, for all nilpotent elements $x \in \underline{G}$ ($x \in \underline{G}$ is called nilpotent if $(\mathrm{ad}x)^M = 0$ for some M, where adx is the adjoint representation of x, which is a nxn matrix) [for G=SO(3):

$$Z \equiv \text{all polynomials in } X \equiv X_1^2 + X_2^2 + X_3^2$$

$$\mathrm{ad}\, X_i \equiv \tilde{S}_i \quad \text{with } (\tilde{S}_i)_{jk} = -i\,\epsilon_{ijk} \,,$$

$$A = a_i \tilde{S}_i = -i \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}$$

satisfies

$$A^3 + A\,(\underbrace{a_1^2 + a_2^2 + a_3^2}_{\equiv \vec{a}^2}) = 0$$

as

$$\det(A - \lambda \mathbb{1}) = -(\lambda^3 + \lambda \vec{a}^2)$$

has to vanish for $\lambda = A$. Therefore $A^3 = 0$ for $\vec{a}^2 = 0$ and one has:

$$X^k = \left(\sum a_i x_i\right)^k$$
$$= \sum (a_{i_1} a_{i_2} \cdots a_{i_k}) x_{i_1} x_{i_2} \cdots x_{i_k}$$
$$\equiv \sum \mathcal{O}_{i_1 \cdots i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \; .$$

x nilpotent $\Leftrightarrow \vec{a}^2=0 \Leftrightarrow \mathcal{O}$ totally traceless. As $\mathcal{O}$ is by definition a symmetric tensor, one has: $E(\underline{SO(3)}) \subset U(\underline{SO(3)})$ is the space of all u with $\mathcal{O}^u$ traceless (and symmetric).

$E_k \equiv U_k \cap E$ is $(2k+1)$ dimensional.]

4. $J \otimes H$ and $Z \otimes E$ are isomorphic as G-modules.

5. Now look at the Poisson structures of S and U; using 1 and 3 one finds

$$[e_{km_k}, e_{\ell m_\ell}] = u_{k+\ell-1}$$
$$= \sum_{i=1}^{k+\ell-1} \sum_{m_i=1}^{\dim(E_i)} d^{\,i m_i}_{k m_k \ell m_\ell}(x_\alpha)\, e_{i m_i}$$

where the

$$e_{j m_j} \quad (1 \le m_j \le \dim(E_j))$$

denote a basis of E_j and $d^{\,i m_i}_{k m_k \ell m_\ell}(x_\alpha) \in Z$ is a polynomial in the independent Casimir operators x_α

$$\left\{h_{km_k}, h_{\ell m_\ell}\right\} = S_{k+\ell-1} = \sum_{i=1}^{k+\ell-1} \sum_{m_i} \tilde{d}_{\substack{k\ell \\ m_k m_\ell}}^{im_i} h_{im_i}$$

where $h_{jm_j} \in H_j \equiv S_j \cap H$

is a basis of H_j which one chooses to be the one given by the isomorphism between E and H and $\tilde{d}^{im_i}_{km_k\ell m_\ell} \in J$

is just a set of complex numbers when restricting S(G) to $S(O_x)$. [The Y 's are a basis of $S(O_x)$ for G=SO(3) and $|\vec{x}| = 1$.] Because of the way $\{\ \}$ was defined via Υ_j in terms of [,] one has $\tilde{d}^{i=\ell+k-1,m_i}_{km_k\ell m_\ell} = d^{i=k+\ell-1,m_i}_{km_k\ell m_\ell}$ [if one has chosen $e_{jm_j} \leftrightarrow h_{jm_j}$ according to the isomorphism between E and H]. $d^{k+\ell-1}_{k\ell}$ is, of course, independent of χ_α anyway (just counting powers); it is therefore also the same for all representations of U. Let the mapping Π_N from U into the set of all complex NxN matrices be such a N-dimensional representation of U.

$$\Pi_N(\chi_\alpha) \quad \text{and} \quad d^{im_i}_{km_k\ell m_\ell}(\Pi_N(\chi_\alpha))$$

then just become a set of numbers and $\Pi_N(E)$ is a Lie algebra with structure constants d, that depend on N via $\chi^\alpha_N \equiv \Pi_N(\chi_\alpha)$.

The earlier proof that $\lim_{N\to\infty} \hat{f}^{(N)} = g$ relied on the fact that for SO(3) there is only one independent Casimir operator χ ($\equiv S_1^2+S_2^2+S_3^2$), (exactly) one irreducible representation for each N, and $\Pi_N(\chi) = \frac{N^2-1}{4} \to \infty$ as $N \to \infty$.

C. THE NATURE OF THE SPECTRUM OF H_N

I. Some general remarks

In Section B it was shown that the structure constants $g_{\alpha\beta\gamma}$ appearing in

$$H = \frac{1}{2}\int d\Omega \left(p_x^2 + p_y^2 + \{x,y\}^2 \right)$$
$$= \frac{1}{2}\sum_{\alpha=(\ell m)} \left(\vec{p}_\alpha \cdot \vec{p}_\alpha^{\,*} + g_{\alpha\beta\gamma}\, g_{\alpha\delta\epsilon}\, x_\beta y_\gamma x_\delta^* y_\epsilon^* \right)$$

are equal to the $N\to\infty$ limit of the SU(N) structure constants $f^{(N)}_{\alpha\beta\gamma}$. The Hamiltonian

$$\frac{1}{2}\sum_{\alpha=1}^{N^2-1} \left(\vec{p}_\alpha \cdot \vec{p}_\alpha^{\,*} + \hat{f}^{(N)}_{\alpha\beta\gamma}\, \hat{f}^{(N)}_{\alpha\delta\epsilon}\, x_\beta y_\gamma x_\delta^* y_\epsilon^* \right)$$

involving only a finite number of degrees of freedom is, therefore, a good approximation to H as $N\to\infty$. It is invariant under the finite group SU(N). Defining traceless hermitian $N\times N$ matrices $X = x_\alpha \hat{T}_\alpha$, $Y = \ldots$, the above Hamiltonian becomes

$$\hat{C}\cdot\left\{ \frac{1}{2}\,\mathrm{Tr}\left(P_x^2 + P_y^2 - [X,Y]^2 \right) \right\}$$

$$\text{where } \mathrm{Tr}\left(\hat{T}_\alpha \hat{T}_{\alpha'}^{\dagger} \right) \equiv \hat{C}^{-1}\delta_{\alpha\alpha'} = \left(\frac{N^3}{16\pi} + O(N^2) \right)$$

see B17 and B20

One is, of course, always free to change the relative strength of potential to kinetic energy by rescaling X and Y. (See B17 and B20).

One could have gone directly from the surface Hamiltonian H to the above matrix hamiltonian noticing that H depends only on the algebraic structure $\{,\}$ which is preserved when replacing $\{x(\theta,\varphi), y(\theta,\varphi)\}$ by $\frac{1}{i}[X,Y]$ (and $\int d\Omega \to \hat{C}\cdot \mathrm{Tr}$).

Note that, as already mentioned in the introduction, this transition has nothing to do with the transition from the classical surface Hamiltonian to the quantum theory, although the 1/i formally comes from the extra i in $[S_i,S_j]=i\epsilon_{ijk}S_k$ compared to $\{x_i,x_j\}_P = \epsilon_{ijk}x_k$ (compare page 44/5)

In order to obtain a sensible $N\to\infty$ limit one rescales X and Y by $N^{1/6}$, absorbs the overall factor $\hat{C}N^{1/3}$ in the surface tension T_0 and defines the SU(N) invariant Hamiltonian

$$H_N \equiv \frac{1}{2}\mathrm{Tr}\left(P_X^2 + P_Y^2 - \frac{1}{N}[X,Y]^2\right) \quad \text{(C1)}$$

From what is known about large N-matrix models in general,* H_N will have a groundstate with energy of $0(N^2)$ (which one subtracts) and the level spacing of the excited states will be of 0(1).

From now on the matrices X, Y ... are most conveniently expanded in hermitian orthonormal generators T_a (i.e. $X = x_a T_a$, ..., $\mathrm{Tr}(T_a T_b) = \delta_{ab}$) with real coefficients. With $[T_a,T_b] = if_{abc}T_c$ one then has, e.g., for the potential

$$V = \frac{1}{2N} f_{abc} f_{ade}\, x_b y_c x_d y_e \, ; \quad \text{(C2)}$$

for SU(N=2) this is $V = \frac{1}{4}(\vec{x}\times\vec{y})^2$

---

*Following the work of Brezin, Itzykson, Parisi, Zuber, Communications in mathematical physics 59, p. 35-51 (1978).

The generators of SU(N) symmetry transformations are

$$K_a \equiv \frac{1}{i} \mathrm{Tr}\left\{ T_a \left([X, P_X] + [Y, P_Y] \right) \right\}$$

and one is interested in $K_a=0$ (classically), $K_a |\psi\rangle = 0$ (for the quantum theory). [These constraints, unfortunately, exclude the class of solutions $X+iY=e^{i\omega t}\,\omega(S_x+iS_y)\cdot\sqrt{N}$ which solve the classical equations of motion derived from (C1):

$$\ddot{X} = \frac{1}{N}[Y[X,Y]] \;,\; \ddot{Y} = \frac{1}{N}[X,[Y,X]] \qquad (C3)$$

The S_i (i=1,2,3) denote $3N\times N$ matrices satisfying $[S_i, S_j] = i\epsilon_{ijk} S_k$

$$K_a = -2\omega^3 N\, \mathrm{Tr}(T_a S_z) \neq 0 \;.$$

One can further see that, at least for SU(N=2) that these solutions are unstable against small perturbations. Note that one can rewrite (C3) in the slightly more compact form:

$$\ddot{Q} = \frac{1}{2N}[Q,[Q,Q^{+}]] \quad \text{where} \quad Q \equiv X + iY \;]$$

Although $V \geq 0$ (as $A=[x,y]=-A^{+}$, $V=+\frac{g^2}{4N}\mathrm{Tr}(A^{+}A) \geq 0$) one might wonder whether the potential C2 confines* or not, as V=0 for a rather large subspace of configuration space (for fixed X, all matrices Y that commute with X). The simplest case, SU(N=2)

$$V_2 = \frac{1}{4}(\vec{X} \times \vec{Y})^2 ,$$

which is 0 for $x \parallel y$ (the classical partition function diverges as a result) . The simplest quartic potential of type (C2) one could possibly think of is $V=x^2y^2$ (in fact, one is lead to something very similar for O(2)xO(3) singlet states of V_2) which will be looked at in the next section. As the answer there is that V confines, one is led to believe that the spectrum of H_N is discrete.

---

* i.e. has a purely discrete spectrum

II. The x^2y^2-problem and the "B.O." approximation

We consider the spectrum of $H = p_x^2 + p_y^2 + x^2y^2$ (C4)

Although there is a short mathematical proof* that the spectrum of H is discrete**

[$H > H' \equiv \frac{1}{2}(p_x^2+p_y^2+|x|+|y|)$; spectrum of H' discrete $\Longrightarrow$ spectrum of H discrete] it might be worth looking at the problem in the following way. As the question of binding should not have much to do with the shape of the potential in a finite region, assume $V=\infty$ for $x \leq \Lambda$, $\Lambda \gg 1$ and try to solve the problem

$$\left(-(\partial_x^2+\partial_y^2)+x^2y^2\right)\psi(x,y) = E\psi \text{ f. } x \geq \Lambda, \qquad \text{(C5)}$$
$$\psi = 0 \text{ f. } x = \Lambda$$

Changing variables to $\xi > 0$ and η by writing $x = \Lambda + \xi$, $y = \eta/\sqrt{\Lambda}$ one gets

$$H\tilde{\psi}(\xi,\eta) = E\tilde{\psi}; \quad \tilde{\psi} = 0 \text{ on } \xi = 0$$

$$H = \Lambda\left\{-\frac{1}{\Lambda}\partial_\xi^2 + \left(-\partial_\eta^2 + \tilde{V}(\xi,\eta)\right)\right\}$$

$$\tilde{V}(\xi,\eta) = (1+\xi/\Lambda)^2\eta^2 \equiv \omega^2(\xi)\cdot\eta^2$$

Now one first solves the η-dependent part (as $\Lambda \gg 1$):

$$\left(-\partial_\eta^2 + \omega^2(\xi)\eta^2\right)\psi(\eta) = E\psi$$

gives

$$E = E_m(\xi) = 2(m-\tfrac{1}{2})\,\omega(\xi); \quad m = 1,2,\ldots$$

In the same sense as Born and Oppenheimer treated the electron energy (calculated as a function of the nuclei distance) as a potential for the two nuclei, $E_m(\xi)$ will now be treated as a potential for the ξ-coordinate, i.e., for given m solve for the eigenvalues and eigenstates of

*pointed out by Barry Simon in private communication with R.Jackiw.

**as $Z = \int e^{-x^2y^2}dx\,dy \sim \int^\infty \frac{dx}{x}$ still diverges (logarithmically) one could say that the discreteness is due to the uncertainty principle.

$$H(m) \equiv (2m-1)\Lambda + (2m-1)^{2/3}\left(-\partial_u^2 + u\right)$$

$$u \equiv (2m-1)^{1/3}\}$$

Calling the eigenvalues of $(-\partial_u^2 + u)$, as before, E_i,

$$H = -(\partial_x^2 + \partial_y^2) + x^2 y^2, \text{ with } \psi(x \leq \Lambda) = 0$$

will therefore (within the Born Oppenheimer approximation) have the eigenvalues

$$E_i^m = (2m-1)\Lambda + (2m-1)^{2/3} E_i$$

One can show quite generally that the Born-Oppenheimer approximation gives a lower bound for the true ground state energy [so that $E_{B.O.} \leq$ true $E_o \leq E^o_{var}$]. [Proof: Consider a general Hamiltonian $H = H(p,q;p',q') = p^2 + H(q;q',p')$ where p' and q' are abbreviating all degrees of freedom different from q and p. Define $H_{B.O.}$ to be the Hamiltonian obtained from H by replacing $H(q;q',p')$ for fixed q by its eigenvalues $E_m(q)$, i.e., $H_{B.O.} = p^2 + E_m(q)$. Using $(\psi_{(q)}, \psi_{(q)})$ as an abbreviation for integrating $\psi(q,q')$ only over q'-coordinates, one has $E_o(q) \leq (\psi(q), H(q)\psi(q))/(\psi(q),\psi(q))$ by the variational principle and, therefore, for all ψ:

$$\langle\psi|H|\psi\rangle = \int dq\,(\psi(q), H\,\psi(q))$$

$$= \int dq \left\{ \left(\frac{\partial\psi}{\partial q}, \frac{\partial\psi}{\partial q}\right) + \frac{(\psi(q), H(q)\psi(q))}{(\psi(q),\psi(q))}(\psi(q),\psi(q)) \right\}$$

$$\geq \int dq\,(\psi(q), (p^2 + E_o(q))\psi(q)) = \langle\langle\psi|H_{B.O.}|\psi\rangle\rangle_{q.e.d.}$$]

For our case one can do an

Explicit calculation and comparison of $E_{B.O.}$ and E_{var}:

a) taking $e^{-1/2\omega(x^2+y^2)}$ as trial wave function and minimizing with respect to ω gives $E_{var} = 2\left(\sqrt[3]{1/16} + \sqrt[3]{1/128}\right) \approx 1.2$

b) $E_{B.O.} \equiv$ lowest eigenvalue of $(-\partial_x^2 + |x|)$. One therefore has to find the smallest E for which

$$\left\{ f''(z) - z f(z) = 0,\ z \equiv (|x| - E) \in [-E, +\infty),\ f(+\infty) = 0,\ f'(-E) = 0 \right\}$$

has a solution. For $z>0$ one takes $f(z) = i^{4/3} \sqrt{z}\, H^{(1)}_{1/3}\left(i \frac{2}{3} z^{3/2}\right) \in \mathbb{R}$

and by analytical continuation ($H^{(1)}$ and J defined as in Jahnke Emde.)

$$\left.\frac{df}{dz}\right|_{z=-E} = \frac{2}{3 \sin\frac{2\pi}{3}}\, E \left\{ J_{-2/3}\left(\tfrac{2}{3} E^{3/2}\right) - J_{+2/3}\left(\tfrac{2}{3} E^{3/2}\right) \right\} = 0$$

$$\Longrightarrow E_{B.O.} \approx 1 \quad ;$$

$$\text{So} \quad 1 \lessapprox E_0^{true} \lessapprox 1.2$$

III. Calculating $\tilde{Z}(g^2) = \int dX\, dY\, e^{-\frac{1}{2}\mathrm{Tr}\left(X^2+Y^2-\frac{g^2}{2N}[X,Y]^2\right)}$

Although it does not provide any information about the spectrum of H_N, the integral $\tilde{Z}(g^2) \equiv \int dX\, dY\, e^{-H_g}$ will be calculated below, where

$$H_g \equiv \frac{1}{2}\mathrm{Tr}\left(X^2+Y^2-g^2/_{2N}[X,Y]^2\right)$$

X and Y hermitian $N\times N$ Matricies

and $dX \equiv \prod_{i=1}^{N} dX_{ii} \prod_{i<j} (d\,\mathrm{Re}\, X_{ij})(d\,\mathrm{Im}\, X_{ij})$

This integral is interesting in its own right as, at least to the best of my knowledge, integrals of this type (i.e a two-matrix-model with coupled quartic interaction) have not been calculated so far in the literature, while the one-matrix-model with quartic self-interaction, and the multi-matrix-problem with quartic self- but only quadratic nearest neighbour interactions have been solved*.

In the case at hand, one first integrates over all but N of the original $2N^2$ (real) variables explicitely (arriving at (C7)). The resultant integral is $\int \prod_{i=1}^{N} d\lambda_i\, e^{-W[\{\lambda_i\}]}$ where W is of $O(N^2)$. Therefore, as $N \to \infty$, the integral will be $\approx e^{-W[\{\bar{\lambda}_i\}]}$ where $\{\bar{\lambda}_i\}$ minimizes W. By defining the density $u(\lambda) \equiv \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)$ the problem of minimizing W becomes that of solving a singular integral equation for $u(\lambda)$ (see C 10) One can do so, but instead of calculating (e.g.) the first moment of u (i.e. $\int \lambda^2 u(\lambda)\, d\lambda$) as a function of g, we are only able to explicitely calculate it as a function of a parameter b, where b is given as a function of g via an implicit equation involving complete elliptic integrals (see C 18iii). The formula for $\langle g^2 [X,Y]^2 \rangle$, (the expectation value of the potential) is given in terms of $\int \lambda^2 u(\lambda)\, d\lambda$ (see C 11).

* [Brezin et al] and Mehta et al, J.Phys.A.Math.Gen. 14 (1981) p.579-586.

$$\tilde{Z} \equiv \int dX\, dY\, e^{-H_g}$$
$$= \int dX\, dY\, e^{-\mathrm{Tr}\left\{\frac{1}{2}X^2 + \frac{1}{2}Y^2 - g^2/_{4N}[X,Y]^2\right\}}$$
$$= C\int \prod dx_i\, dY\, e^{-\frac{1}{2}\Sigma x_i^2}\, e^{-\frac{1}{2}\sum_{i,j}|y_{ij}|^2\left(1 + g^2/_{2N}(x_i - x_j)^2\right)}$$

where x_i are the eigenvalues of X and then, using X diagonal and $y^+ = y$: $\mathrm{Tr}[x,y] = 2\mathrm{Tr}(xyxy - x^2y^2)$

$$= 2\left(\sum_{i,j} x_i x_j |y_{ij}|^2 - x_i^2 |y_{ij}|^2\right) = -\sum_{ij}(x_i - x_j)^2 |y_{ij}|^2$$

and writing the exponent as

$$-\frac{1}{2}\Sigma x_i^2 - \frac{1}{2}\sum_1^N y_{ii}^2 - \sum_{i<j}\left((\mathrm{Re}(y_{ij}))^2 + (\mathrm{Im}\, y_{ij})^2\right)\left(1 + g^2/_{2N}(x_i - x_j)^2\right),$$

the integral $\int dY$ is simply a product of gaussian integrals so that (with $\lambda_i \equiv x_i/\sqrt{2N}$)

$$\tilde{Z} = C' \int_{-\infty}^{+\infty} \prod_{i=1}^N d\lambda_i\, e^{-N\sum_{i=1}^N \lambda_i^2} \prod_{i<j} \frac{(\lambda_i - \lambda_j)^2}{1 + g^2(\lambda_i - \lambda_j)^2}.$$

One can also calculate the integral in a more symmetrical way by introducing an auxiliary matrix ϕ--to get rid of the qua-r-tic interaction, then integrating over x and the λ_i appearing in the above formula are then the eigenvalues of

$$\phi/\sqrt{2}\,N$$

(C6)

$$Z \equiv \frac{1}{\mathcal{N}(g=0)} \int dX\, dY\, e^{-\frac{1}{2}\left\{ \mathrm{tr}(X^2+Y^2) - \frac{g^2}{2N} \mathrm{tr}[X,Y]^2 \right\}}$$

$$\overset{(Q=X+iY)}{=} \frac{1}{\mathcal{N}'(0)} \int dQ\, e^{-\frac{1}{2}\mathrm{tr}\, Q^\dagger Q - \frac{1}{2} g^2/_{8N}\, \mathrm{tr}[Q^\dagger, Q]^2}$$

$$\overset{(\phi^\dagger=\phi)}{=} \frac{1}{\mathcal{N}''(0)} \int dQ\, d\phi\, e^{-\frac{1}{2}\mathrm{tr}\, Q^\dagger Q - \frac{1}{2} g^2/_{8N}\, \mathrm{tr}[Q^\dagger, Q]^2 - \frac{1}{2N}\mathrm{tr}\left(\phi - \frac{ig}{\sqrt{2}2}[Q^\dagger, Q]\right)^2}$$

$$\overset{(\phi = U\Lambda U^\dagger)}{\underset{(q_{rs}=q'_{rs}+iq''_{rs})}{=}} \frac{1}{\mathcal{N}'''(0)} \int \prod_{r,s} dq'_{rs}\, dq''_{rs} \prod_{t=1}^{N} d\lambda_t \prod_{r<s} (\lambda_r-\lambda_s)^2 \cdot$$

$$\cdot\, e^{-\frac{1}{2}\sum |q_{rs}|^2 + \frac{ig}{\sqrt{2}2N}\sum_{r,s} |q_{rs}|^2 (\lambda_r-\lambda_s) - \frac{1}{2N}\sum_1^N \lambda_r^2}$$

$$= \frac{1}{\mathcal{N}^{(IV)}(0)} \int_{-\infty}^{+\infty} \prod d\lambda_r \prod_{r<s} (\lambda_r-\lambda_s)^2 \prod_{r<s}\left(1+ \frac{g^2}{2N^2}(\lambda_r-\lambda_s)^2\right)^{-1} e^{-\frac{1}{2N}\sum_1^N \lambda_r^2}$$

$$\overset{(\lambda_r \to \sqrt{2}N\lambda_r)}{=} \frac{1}{\mathcal{N}^{(V)}(0)} \int_{-\infty}^{+\infty} \prod d\lambda_r \prod_{r<s} \frac{(\lambda_r-\lambda_s)^2}{1+g^2(\lambda_r-\lambda_s)^2}\; e^{-N\sum_1^N \lambda_r^2}$$

$$= \frac{1}{\mathcal{N}^{(V)}(0)} \int \prod d\lambda_r\, e^{-N^2\left\{\frac{1}{N}\sum \lambda_r^2 + \frac{1}{2N^2}\sum_{r\neq s} \ln\left(1+g^2(\lambda_r-\lambda_s)^2\right) - \frac{1}{N^2}\sum_{r\neq s} \ln|\lambda^r-\lambda^s|\right\}}$$

$$\equiv \frac{1}{\mathcal{N}^{(V)}(0)} \int d\Lambda\, e^{-W(\Lambda)} = \frac{\int d\Lambda\, e^{-W(g,\Lambda)}}{\int d\Lambda\, e^{-W(0,\Lambda)}} \qquad \text{(C7)}$$

with

$$W \equiv -\sum_{r<s}{}' \ln \frac{(\lambda_r-\lambda_s)^2}{1+g^2(\lambda_r-\lambda_s)^2} + N\sum_{r=1}^{N} \lambda_r^2 \qquad \text{(C8)}$$

W is of $O(N^2)$, so that Z can be computed, in the large N-limit, by minimizing W with respect to the λ_i:

$$0 = \frac{\partial W}{\partial \lambda^t} = 2N\lambda_t - 2\sum_{s\neq t} \frac{1}{\lambda_t - \lambda_s} + 2\sum_{s\neq t} \frac{g^2(\lambda_t-\lambda_s)}{1+g^2(\lambda_t-\lambda_s)^2} \quad \text{(C9)}$$

Introducing the eigenvalue density $u(\lambda) \equiv \frac{1}{N}\sum_{r=1}^{N} \delta(\lambda-\lambda_r)$ the above equation can be written as

$$\lambda = \fint_{-a}^{+a} \frac{u(\mu)}{\lambda-\mu}\,d\mu - \int_{-a}^{+a} \frac{u(\mu)\,(\lambda-\mu)}{1/g^2 + (\lambda-\mu)^2}\,d\mu \quad \text{(C10)}$$

which is a singular integral equation for $u(\lambda)$, which has to be solved subject to the constraint $\int_{-a}^{+a} u(\lambda)\,d\lambda = 1$. Before outlining how to solve equation (C10) note how one can, e.g., determine $\langle V\rangle$ once $u(\lambda)$ is known:

$$\langle V\rangle = \left\langle -\frac{g^2}{4N}\,\mathrm{tr}[X,Y]^2 \right\rangle = -g^2\frac{\partial Z}{\partial g^2} = \left\langle g^2 \frac{\partial W}{\partial g^2}\right\rangle$$

$$= + \sum_{r<s} g^2 \frac{(\lambda_r-\lambda_s)^2}{1+g^2(\lambda_r-\lambda_s)^2} \quad (\lambda_r \text{ satisfying C9})$$

$$= N^2\left\{\frac{1}{2} - \int_{-a}^{+a} u(\lambda)\lambda^2\,d\lambda - \cancel{\tfrac{1}{2N}} \xrightarrow{} \text{discard}\right\} \cdots \text{(C11)}$$

The last step could be made because $\frac{\partial W}{\partial \lambda_t} = 0 \implies$

$$0 = \sum \lambda_t \frac{\partial W}{\partial \lambda_t}$$

$$= 2N\sum \lambda_t^2 - (N^2 - N) + 2\sum_{t<s} \frac{g^2(\lambda_t-\lambda_s)^2}{1+g^2(\lambda_t-\lambda_s)^2}$$

Solution of (C10), <u>first for g=0</u>: Defining $F(z) \equiv \int_{-a}^{+a} \frac{u(\lambda)\,d\lambda}{z-\lambda}$ which is real for real $z \notin [-a,+a]$, behaves like $1/Z$ for $|z|\to\infty$, is analytic in the complex Z-plane except for a cut along $[-a,+a]$, and—approaching the cut from above / below—:

$$\lim_{\epsilon\to 0} \mathrm{Re}\, F(\lambda \pm i\epsilon)$$

$$= \frac{1}{2}\lim \int_{-a}^{+a} d\mu\, u(\mu)\left(\frac{1}{\lambda-\mu+i\epsilon} + \frac{1}{\lambda-\mu-i\epsilon}\right) \equiv \fint_{-a}^{+a} \frac{u(\mu)\,d\mu}{\lambda-\mu} \overset{\text{(C10)}}{=} \lambda$$

while $\lim_{\epsilon\to 0} \operatorname{Im} F(\lambda \pm i\epsilon) = \frac{1}{2i} \lim \int d\mu\, u(\mu) \left(\frac{1}{\lambda-\mu \pm i\epsilon} - \frac{1}{\lambda - \mu \mp i\epsilon} \right)$

$$= \frac{1}{2i} \int d\mu\, u(\mu) \left(\mp 2i\pi\, \delta(\lambda-\mu) \right) = \mp \pi u(\lambda)$$

The (unique) function having these properties is

$$F(z) = z - \sqrt{z^2 - 2}$$

as is easy to see and even simpler to check:

$$F(z) = \frac{2}{a^2} \left(z - \sqrt{z^2 - a^2} \right)$$

satisfies the first 3 criteria, while $\lim_{\epsilon \to 0} \operatorname{Re}(\lambda \pm i\epsilon) = \frac{2\lambda}{a^2} \stackrel{!}{=} \lambda$ gives $a = \sqrt{2}$. Then one calculates $u(\lambda)$ as

$$u(\lambda) = \frac{-1}{\pi} \lim_{\epsilon \to 0} \operatorname{Im} F(\lambda + i\epsilon) = \frac{-1}{2\pi i} \lim \left(-2\sqrt{(\lambda + i\epsilon)^2 - 2} \right)$$

$$= + \frac{\sqrt{2-\lambda^2}}{\pi} \qquad \text{(C12)}$$

As a check one can calculate

$$\int_{-a}^{+a} u(\lambda)\, d\lambda = \frac{1}{\pi} \int_{-\sqrt{2}}^{+\sqrt{2}} \sqrt{2-\lambda^2}\, d\lambda = \frac{4}{\pi} \int_0^{\pi/2} \cos^2\theta\, d\theta = 1$$

as it must be. Also, according to the general formula (C11) for $\langle V \rangle$, $\int \lambda^2 u(\lambda)\, d\lambda$ has to be +1/2 for g=0, so that $\langle V \rangle = 0$; indeed:

$$\frac{1}{\pi} \int_{-\sqrt{2}}^{+\sqrt{2}} \sqrt{2-\lambda^2}\, \lambda^2\, d\lambda = \frac{(\sqrt{2})^2 2 \cdot 2}{\pi} \int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta = \frac{8}{\pi} \frac{\pi}{2} \frac{1}{8} = \frac{1}{2} \; .$$

For g≠0, define $G(z) \equiv -i \left(F(z + i/2g) - F(z - i/2g) \right)$

$$= -\frac{1}{g} \int_{-a}^{+a} \frac{u(t)\, dt}{(z-t)^2 + 1/4g^2} \quad , \text{ which}$$

assuming $u(\lambda) = u(-\lambda)$ behaves like $\frac{1}{gz^2} - \frac{\left(3\int \lambda^2 u(\lambda) d\lambda - \frac{1}{4g^2}\right)}{g z^4} + O\left(\frac{1}{z^6}\right)$ and has, because of (C10), the property:

$$\operatorname{Im} G(\lambda \pm i/2g) = \pm \lambda \text{ for } \lambda \in [-a, +a]$$

(irrespective of approach from above or below). Defining $G' = -gz^2 + G$ this translates to:

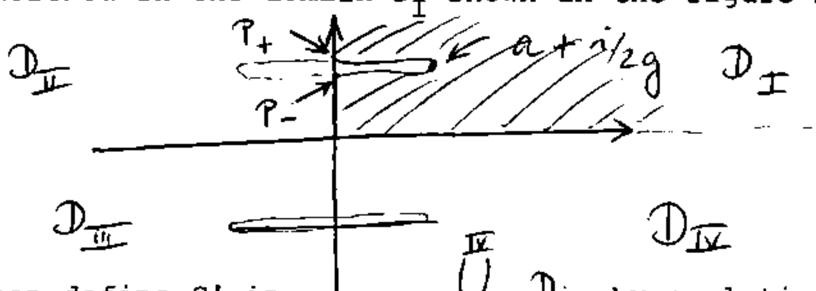
$\mathrm{Im}\, G' = 0$ for $z = \lambda \pm i/2g$, $\lambda \in [-a, +a]$ (from above or below)

also: $G'(z) \overset{(ii)}{=} G'(-z)$, $G'^{*}(z^{*}) \overset{iii}{=} G'(z)$ (as is real) (C13)

and at ∞ : $G'(z) \approx -g z^2 + \gamma/z^2 + \delta/z^4 + \cdots$

(where γ necessarily = $-1/g$ and $\int \lambda^2 u(\lambda)\, d\lambda = \frac{1}{3}\left(\frac{1}{4g^2} - g\delta\right)$ so that the knowledge of δ will yield $\langle V \rangle$ via (C10').)

In order to find such a function G', analytic everywhere except at the two cuts $[-a,+a] \pm i/2g$, think of G' as being first only defined in the domain D_I shown in the figure below:



$(p^{\pm} = i^{\pm}/2g)$

and then define G' in $\bigcup_{i=I}^{IV} D_i$ by analytic continuation which, using (C13ii) and (iii) gives

for $z \in D_{IV}$: $G'(z) \equiv G'^{*}(z^{*})$

for $z \in D_{III}$: $G'(z) \equiv G'(-z)$

for $z \in D_{II}$: $G'(z) \equiv G'^{*}(-z^{*})$.

This shows that, in fact, $\mathrm{Im}(G')$ vanishes on the entire boundary of D_I. Therefore G'(z) can in fact be taken to be, up to real constants, the conformal transformation ($z \to \zeta$) mapping D_I onto the upper half plane. This transformation $(\zeta(z))$, mapping $P_+ \to -1$, $a + i/2g \to -c$, $P_- \to -b < -c < 0$, real z into real ζ, is given implicitly by the equation(s)*

$$z = A \cdot \int_0^{\zeta} \frac{(t+c)\,dt}{\sqrt{t(t+1)(t+b)}}$$

*See, e.g., Fuchs and Shabat "Functions of a complex variable", Vol. 1, Problem 9 in Ch. 8, but note the mistakes in the last two lines before Problem 10.

$$\frac{1}{2g} \overset{(i)}{=} A\int_0^1 \frac{(c-\rho)\,d\rho}{\sqrt{(b-\rho)(1-\rho)\rho}}\,,\quad a \overset{(ii)}{=} A\int_1^c \frac{(c-\rho)\,d\rho}{\sqrt{(b-\rho)(\rho-1)\rho}}$$

$$a \overset{(iii)}{=} \int_c^b \frac{(\rho-c)\,d\rho}{\sqrt{(b-\rho)(\rho-1)\rho}} \qquad \text{(C14)}$$

Although it would be nice (and Simplify to know $\zeta(z)$, which is an elliptic function, in closed form, e.g., expressed in terms of the Weierstraß function P(z) of the same periods (and, possibly, $\wp'(z)$), one can calculate δ, the coefficient of $1/z$ in ζ (as $z\to\infty$) also directly and therefore give a formula for $\langle v\rangle$ (which, however, will be very complicated and not much less implicit than (C14), as the g-dependence of c and b can only be given implicitly). From C14:

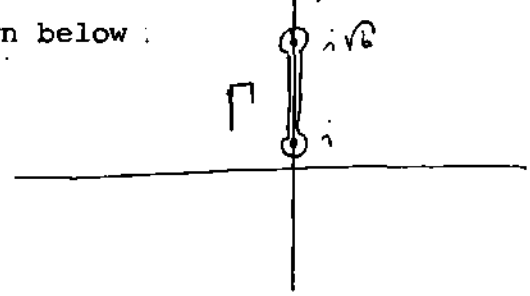
$$z/A = \int_0^\zeta \left(\frac{t+c}{\sqrt{t(t+1)(t+b)}} - \frac{1}{\sqrt{t}}\right)dt + 2\sqrt{\zeta}$$

$$= 2\sqrt{\zeta} + \int_0^\infty \left(\frac{t+c}{\sqrt{t(t+1)(t+b)}} - \frac{1}{\sqrt{t}}\right)dt - \int_\zeta^\infty \left(\frac{t+c}{\sqrt{\cdots}} - \frac{1}{\sqrt{t}}\right) \qquad \text{(C15)}$$

The third term will be expanded, the second term $(\equiv \mathcal{Z})$ can be shown to be $=0$, using (C14) (i)-(iii):

$$\mathcal{Z} \underset{(x\equiv\sqrt{t})}{\overset{\uparrow}{=}} 2\int_0^\infty dx\left(\frac{x^2+c}{\sqrt{(x^2+1)(x^2+b)}} - 1\right) \equiv \int_{-\infty}^{+\infty} dx\,\mathcal{J}$$

As the integrand $\mathcal{J}$ behaves like $1/x^2$ at ∞, one can close the contour (at ∞) without altering $\mathcal{Z}$, and, as $\mathcal{J}$ is analytic in the upper half-plane except for a cut between i and $i\sqrt{b}$, alter the contour to the closed path Γ shown below:



$\Gamma \oint dx=0$, and therefore, with $\rho \equiv -x^2$

$$\mathcal{L} = 2 \int_1^b \frac{c-\rho}{\sqrt{\rho(\rho-1)(b-\rho)}}\, d\rho$$

which is 0 because of (C14ii) + (iii) (added together). The constant term in the expansion of z/2A for large ζ is therefore 0, and

$$z/2A = \sqrt{\zeta} + D/\sqrt{\zeta} + E/\zeta^{3/2} + F/\zeta^{5/2} + \cdots$$

(higher order terms will not be needed to calculate δ). From (C15) the coefficients D,E and F can be calculated ;

$$D = \left(\frac{b+1}{2} - c\right) \geq 0 \quad , \quad E = -\left(\frac{b^2+1}{8} - \frac{c(b+1)}{6} + \frac{b}{12}\right) \leq 0 \qquad \text{(C16)}$$

$$F = +\frac{1}{5}\left\{\frac{5}{16}(b^3+1) - \frac{bc}{4} - \frac{3}{8}b^2(c-\tfrac{1}{2}) - \tfrac{3}{8}(c-b/2)\right)$$

from which

$$z^2/4A^2 = \zeta' + \alpha/\zeta' + \beta/\zeta'^2 + \cdots$$

$$\left(\alpha \equiv D^2 + 2E < 0, \beta \equiv 2F + 6DE + 2D^3, \zeta' \equiv \zeta + 2D\right) \qquad \text{(C16')}$$

and therefore

$$\zeta' = \frac{z^2}{4A^2} - \frac{4\alpha A^2}{z^2} - \frac{16(\alpha^2+\beta)A^4}{z^4} + \cdots \qquad \text{(C16'')}$$

so that

$$G'(\zeta(z)) \equiv -4gA^2(\zeta(z) + 2D)$$

$$= -gz^2 + \frac{16A^4\alpha g}{z^2} + \frac{64gA^6(\alpha^2+\beta)}{z^4} + \cdots \qquad \text{(C17)}$$

will have the required behavior at ∞. Also it must be that

$$\gamma \equiv 16A^4\alpha \cdot g = -\frac{1}{g} \qquad \text{(C17')}$$

and, extracting δ as the coefficient of $1/z^4$ in (C17), one has, using (C11):

$$\lim_{N\to\infty}\left\langle \frac{V}{N^2}\right\rangle = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{4g^2} - 64g^2A^6(\alpha^2+\beta)\right) \tag{C17''}$$

The problem with the above formula(e) is that they are rather useless unless one can determine b, c and A (a is not needed) as functions of g via (C14i)-(ii)--which seems to be very difficult. What one can do without much work, however, is to derive equations for C, A and g as functions of b: (C14ii)-(iii) gives

$$C = C(b) = b\,\frac{\int_0^{\pi/2}\sqrt{1-k^2\sin^2\alpha}\,d\alpha}{\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}} \equiv b\cdot\frac{E(k)}{K(k)} \qquad (k'^2 \equiv 1-k^2 \equiv 1/b)$$

(i) gives:

$$A = A(b) = \left(4g\sqrt{b}\left\{\frac{E(k)}{K(k)}K(k') + E(k') - K(k')\right\}\right)^{-1} \tag{C18}$$

$$\equiv \left(4g\,f(b)\right)^{-1}$$

and from (C17'):

$$g^2 = \left(16 f^4(b)\right)^{-1}\cdot\left(b^2\left(\frac{2}{3}\frac{E(k)}{K(k)} - \frac{E^2(k)}{K^2(k)}\right) + b\left(\frac{2}{3}\frac{E(k)}{K(k)} - \frac{1}{3}\right)\right)$$

One can look at the limits $g\to 0$ ($g\to\infty$), corresponding to $b>c\to\infty$ ($c<b\to 1$), using the expansions of the complete elliptic integrals $E(x)$ and $K(x)$ for $x\to 0$ and $x\to 1$

$$K(x) = \begin{cases} \frac{\pi}{2}\left(1+\frac{x^2}{4}+\frac{9}{64}x^4+\dots\right) & \text{f. } x\to 0 \\ \ln\frac{4}{x'} + \frac{x'^2}{4}\left(\ln\frac{4}{x'}-1\right)+\dots & \text{f. } x'^2 = 1-x^2\to 0 \end{cases}$$

$$E(x) = \begin{cases} \frac{\pi}{2}\left(1-\frac{x^2}{4}-\frac{3}{64}x^4-\dots\right) & \text{f. } x\to 0 \\ 1+\frac{x'^2}{2}\left(\ln\frac{4}{x'}-\frac{1}{2}\right)+\dots & \text{f. } x\equiv\sqrt{1-x'^2}\to 1 \end{cases}$$

SUMMARY

The Lorentz-invariant action and the transition to a Hamiltonian formalism are given for a closed M-dimensional surface moving in D-dimensional Minkowski space. The definition of the system, the use of light cone coordinates and much more is in close analogy to the theory of a massless relativistic string, although the important role which the group of volume preserving reparametrizations plays is new. For the case M=2, D=4 this group and in particular its Lie algebra are studied, and the latter can be shown to correspond in some sense to the limit of SU(N) (as $N \rightarrow \infty$). This fact is used to transform the surface Hamiltonian into a large N two-matrix hamiltonian with the quartic interaction $[X,Y]^2$, a problem formulated in a much more familiar language. However, we have been so far unable to find out much about the spectrum of this Hamiltonian, apart from being almost certainly purely discrete, and some hints that its levels are highly degenerate which is needed for the theory to be Lorentz invariant. We hope that the states of each energy level of H_N could be arranged into multiplets of total spin S. As the "energy" is really the square of the restmass, the states would then be characterized by spin and mass, as they should in a relativistic theory.

PART TWO

A TWO DIMENSIONAL BOUND STATE PROBLEM

INTRODUCTION

Attempts to relate field theories of the strong interactions, in particular QCD, to string models of hadrons lead one* to study the nonrelativistic system of N distinguishable particles of equal mass (labelled 1 to N) moving in two dimensions with an attractive δ-function potential between particles r and r+1:

$$H_N^{\mathrm{ur}} \equiv \frac{1}{2}\sum_{r=1}^{N} \vec{\Pi}_r^{\,2} - (2\pi\lambda)\sum_r \delta^{(2)}(\vec{\xi}_r - \vec{\xi}_{r+1}),$$

where the second sum runs from either 1 to N-1 ("open case") or 1 to N ("closed case", (N+1)$\equiv$(1)). Solving the two-body problem one encounters divergences which are regularized by introducing a cut-off Λ to the divergent integral(s) and choosing the coupling constant λ in a cut-off dependent way so to make the two-body binding energy Δ_2 of the bound state independent of Λ:

$$\Lambda^2 e^{-2/\lambda(\Lambda)} = \Delta_2$$

(which one then sets =1). The question then is what happens to the N(>2)-body problem, with λ given by the above equation? While in 3 dimensions the spectrum of the 3-body problem will not be

*See e.g. C.B. Thorn, Phys. Rev. D19 (1979) 639 [Thorn] J.L. Gervais, A. Neveu, N.P. B163 (1980) 189.

bounded from below (when regularizing the 2-body problem in an analogous way) , the answer for D=2 seems to be that the open (closed) 3-body system has only one (two) bound state(s) at energy -2.5* (-16 and -1.5), and is free of any irregularities. One can conclude this by deriving an eigenvalue-integral equation that is equivalent to the Schrödinger equation for bound states (but no longer contains λ nor Λ)

How delicate a border case D=2 is (note that for D<2 no regularization is necessary at all) can be illustrated by looking at the δ-function as a limit of a short-range potential

$$V = \frac{S}{a^2} f(r/a), \quad f(\rho) = 0 \text{ f. } \rho \geq 1, \quad a \to 0$$

One finds out how the choice of S=S(a), that will give one bound state at finite energy (-1, say) depends crucially on the dimension:

$$S = O(a^{2-D}) \text{ f. } D < 2 \quad (\cong a \text{ f. } D = 1)$$
$$S \cong \frac{2}{|\ln a|} \quad \text{f. } D = 2$$
$$S \cong 2\epsilon \quad \text{f. } D = 2+\epsilon \ (\epsilon \ll 1)$$
$$S \cong \pi^2/4 \quad \text{f. } D = 3$$

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*See also Bruch/Tjon, Phys. Rev. A19, No. 2 (79) p. 425-432, and I.V. Simenog (1980) "Regularisation of the zero range interaction limit in a one- and two-dimensional many-particle problem."

D=2, looked at it this way, is more like D<2 as $\lim_{a\to 0} S(a)=0$

f. $D \leq 2$, while $\lim_{a \to 0} S(a) = \text{const} \neq 0$ f. $D > 2$

For $D \geq 2$ both kinetic and potential energy diverge (logarithmically for D=2 but with the kinetic energy contained in the classically allowed region $r < a$ finite; as a negative power for $D > 2$) and the total energy (-1) arises from a delicate cancellation between them.

For the general N-body problem one can, using the consistency relation for λ, again derive an integralequation that does not contain λ nor Λ and is equivalent to the Schrödinger equation for bound states.

In an earlier work* the following results were derived for the open case (they will only be stated here in the introduction)

The N-body system binds

$$\Delta_{M+N} \geq \Delta_M + \Delta_N + 1$$

and in a random phase approximation is found to have phonon like excitations that come arbitrarily close to the ground state energy as $N \to \infty$:

$$E_\ell(N) = E_0(N) + \sqrt{3}\,\frac{\pi \ell}{N} \quad (\ell = 1, 2, \ldots)$$

---

*J.H. Master Thesis, MIT, 1980.

When this result is used in the hadron models, one obtains[1] a relation between the slope of the Regge trajectories and the QCD perturbation theory scale parameter Λ. $E_\ell(N)$ will be the same for any short-range potential, while for an arbitrary interaction

$$\sum_{r=1}^{N-1} V(\vec{\xi}_r - \vec{\xi}_{r+1}) \, , \qquad \sqrt{3}$$

has to be replaced by $(-g_{xx}(0))^{-1/2}$ where $g_{xx}(w)$ is a response function for the corresponding two-body problem. A (diagrammatic) random phase approximation is used to obtain

$$E_0(N) \approx -1.4\, N + (2.06) - \frac{\pi\sqrt{3}}{12}\frac{1}{N} + O\left(\frac{1}{N^2}\right)$$

as an approximation to the ground state energy, which should be compared with a second order perturbation theory result: $E_0(N) \approx -(1.3)N + 1.6$.

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[1] See [Thorn], which also contains some of the above mentioned results.

A. The two-body problem (exact solution)

In two dimensions tne Hamiltoni n is

$$H_2^{\text{Ur}} = \frac{1}{2}\left(\vec{\Pi}_1^2 + \vec{\Pi}_2^2\right) - (2\pi\lambda)\,\delta^{(2)}(\vec{\xi}_1 - \vec{\xi}_2)$$

As the potential depends only on the relative coordinate, the problem separates in the center of mass system:

$$H_2^{\text{Ur}} = \frac{1}{4}\vec{P}^2 + \left(\vec{\rho}^2 - (2\pi\lambda)\,\delta^{(2)}(\vec{x})\right)$$

where $\vec{X} \equiv \vec{\xi}_1 - \vec{\xi}_2$, $\vec{\rho} = \frac{1}{2}(\vec{\Pi}_1 - \vec{\Pi}_2)$

and $\vec{P}$ is the total momentum $\vec{\Pi}_1 + \vec{\Pi}_2$. The problem is therefore reduced to finding the spectrum of

$$h = \vec{\rho}^2 - (2\pi\lambda)\,\delta^{(2)}(\vec{X}) .$$

The equation for a bound state is $h|B\rangle = -\Delta|B\rangle$. Multiply by $\langle\vec{p}|$ to get

$$\vec{\rho}^2\langle\vec{\rho}|B\rangle - (2\pi\lambda)\langle\vec{\rho}|\,\delta^{(2)}(\vec{x})\,|B\rangle = -\Delta\langle\vec{\rho}|B\rangle,$$

insert a complete set of states, use $\langle\vec{\rho}\,|\,\delta^{(2)}(\vec{x})\,|\,\vec{\rho}\,'\rangle = 1$ and rearrange terms to get

$$(\vec{\rho}^2 + \Delta)\langle\vec{\rho}|B\rangle = (2\pi\lambda)\int\frac{d^2\vec{\rho}\,'}{(2\pi)^2}\langle\vec{\rho}\,'|B\rangle = \text{const}.$$

Therefore there is only one bound state $|B\rangle$ of the two-body system (with binding energy $\Delta \equiv \Delta_2$)

$$\tilde{\psi}_B(\vec{p}) = \langle \vec{p} | B \rangle = \frac{(\text{const.})}{(\vec{p}^{\,2} + \Delta_2)} = \frac{\sqrt{4\pi \Delta_2}}{p^2 + \Delta_2}$$

$$\left(\text{demanding } \langle B | B \rangle = 1 \right)$$

Putting $\langle p | B \rangle$ back into the original equation gives the consistency relation for λ:

$$(2\pi\lambda) \int \frac{d^2\vec{p}\,'}{(2\pi)^2} \frac{1}{p^2 + \Delta_2} = 1$$

The integral $\left(= \frac{1}{4\pi} \int_0^\infty \frac{dE}{E + \Delta_2} \right)$ diverges; introducing a cutoff Λ^2 it becomes equal to $\frac{1}{4\pi} \ln \left(\Lambda^2 / \Delta_2 \right)$ and therefore

$$\Delta_2 = \Lambda^2 e^{-2/\lambda}$$

In order to have Δ_2 finite, λ has to go to 0 as $\Lambda \to \infty$. The parameter of this model problem is therefore not λ, but the two-body binding energy Δ_2.* From now on all energies will be measured in units of Δ_2, i.e., $\Delta_2 = 1$.

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*In a slightly more mathematical treatment Δ_2 would appear as the one real parameter of the class of self adjoint extensions of $h_0 = p^2$. For a mathematically precise treatment of point interactions in general see: Albeverio, Fenstadt (cont.)

and the self consistency relation for λ is

$$(2\pi\lambda)\int_0^\infty \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+1} = 1 \qquad (A1)$$

Because $v(x)=\delta^2(x)$ is a (special case of a) separable potential, the scattering problem $h|\gamma\rangle = \epsilon_\gamma|\gamma\rangle$ can be solved exactly by using the Lippman Schwinger equation. One finds

$$\langle \vec{p}\,|\gamma^\pm\rangle = 2\pi^2\,\delta^{(2)}(\vec{p}-\vec{p}_\gamma) - \frac{4\pi}{(p_\gamma^2-p^2\pm i\epsilon)(\ln p_\gamma^2 \mp i\pi)}$$

---

and Hoegh Krohn, "Singular Perturbation and Nonstandard Analysis", Transac. AMS, Vol. 252, August 1979, and for 3 dimensions, the good review article by G. Flamand, "Mathematical Theory of Nonrelativistic 2- and 3-particle Systems with Point Interactions" in Cargese lectures in Theor. Phys., Gordon and Breach, N.Y., 1967, Lurcat, ed.. Also I would like to thank Prof. T.T. Wu for interesting discussions.

from which

$$\langle B|\vec{p}|\gamma\rangle = \sqrt{4\pi}\,\frac{\vec{p}_\gamma}{p_\gamma^2+1} \qquad (A2)$$

$\langle \delta|\vec{p}|\gamma\rangle$ will not be used, $\langle B|\vec{p}|B\rangle$ is =0.

$$1 = |B\rangle\langle B| + \int \frac{d^2 p_\gamma}{(2\pi)^2} |\gamma^\pm\rangle\langle\gamma^\pm|$$

The normalisations of position and momentum eigenstates and the definition of Fourier transformation are listed below:

$$\langle \vec{x}|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}}$$

$$\langle \vec{x}|\vec{x}'\rangle = \delta^{(2)}(\vec{x}-\vec{x}') \quad , \langle \vec{p}|\vec{p}'\rangle = (2\pi)^2\,\delta^{(2)}(\vec{p}-\vec{p}')$$

$$\int d^2x\, |\vec{x}\rangle\langle\vec{x}| = 1 = \int \frac{d^2p}{(2\pi)^2} |\vec{p}\rangle\langle\vec{p}|$$

$$\tilde{f}(\vec{p}) = \langle \vec{p}|f\rangle = \int d^2x\, e^{-i\vec{p}\cdot\vec{x}} f(\vec{x})$$

$$f(\vec{x}) = \langle \vec{x}|f\rangle = \int \frac{d^2p}{(2\pi)^2} e^{+i\vec{p}\cdot\vec{x}} \tilde{f}(\vec{p})$$

<u>The δ-function as the limit of a short-range potential</u>

Instead of looking at a "δ-function" with cutoff Λ in the limit $\Lambda \to \infty$, one can look at a short-range radially symmetric potential ($V(r)=0$ for $r \equiv |\vec{x}| > a$, $a \ll 1$) in the limit $a \to 0$. On dimensional grounds

$$V = \frac{S}{a^2} f(r/a) \equiv \bar{V}/a^2$$

with S and f dimensionless, and f normalized to $\int_0^\infty f(\rho)\,d\rho = -1$; i.e., f determines the shape of $\bar{V}$, S its strength. By defining a rescaled variable $\rho \equiv r/a$ one writes the two-body hamiltonian

$$h_2 = -\nabla^2 + V = -\nabla^2 + \frac{S}{a^2} f(r/a) \text{ as}$$

$$h_2 = \frac{1}{a^2}\left(-\nabla_\rho^2 + S f(\rho)\right) \equiv \frac{\bar{h}_2}{a^2} \qquad \text{(A3)}$$

For h_2 to have exactly one bound state at a given finite energy ($-\delta$ say) as $a \to 0$, S has to be chosen appropriately as a function of a (and δ) so that $\bar{V}$ just binds ($\bar{h}_2$ with bound state at energy $-\delta a^2 \to 0$, as $a \to 0$).

As the dimensionality of the problem turns out to be an interesting point, one defines the problem in $2+\epsilon$ dimensions $(-1 \leq \epsilon \leq +\epsilon)$ by writing down the Schrödinger equation for radially symmetric bound state wavefunctions $\psi(\rho)$ in $2+\epsilon$ dimensions:

$$\frac{1}{a^2}\left(\psi'' + \frac{1+\epsilon}{\rho}\psi' - s\, f(\rho)\psi(\rho)\right) = +\delta\,\psi(\rho) \qquad \text{(A4)}$$

from now on $f(\rho)$ will be taken to be simply $-\Theta(1-\rho)$. (A4) is, of course, solved by solving for the regions $\rho < 1$ and $\rho > 1$ (from now on referred to just as $<$ and $>$) and then matching function and logarithmic derivatives at $\rho = 1$ (giving a condition on δ)

For $r < a$ (A4) becomes

$$\psi'' + \frac{1+\epsilon}{\rho}\psi' + (s - \delta a^2)\psi(\rho) = 0 \qquad \text{(A51)}$$

with solution*

$$\rho^{-\epsilon/2} J_{\epsilon/2}(\sqrt{t}\,\rho)$$

assuming

$$t \equiv s - \delta a^2 > 0$$

for $r > a$ (A4) becomes

$$\psi''(r) + \frac{1+\epsilon}{r}\psi'(r) - \delta\,\psi(r) = 0 \qquad \text{(A52)}$$

with solution*

$$r^{-\epsilon/2} K_{\epsilon/2}(r\sqrt{\delta})$$

---

*Conventions used, in particular for Bessel functions J and K, are those of "Table of integrals series and products", Gradsteyn and Ryzhik.

matching

$$\frac{\psi'}{\psi} \text{ at } r=a \; : \; \frac{\sqrt{t}}{a} \frac{J_{1+\epsilon/2}(\sqrt{t})}{J_{\epsilon/2}(\sqrt{t})} = \frac{\sqrt{\delta}\, K'_{1+\epsilon/2}(a\sqrt{\delta})}{K_{\epsilon/2}(a\sqrt{\delta})} \qquad \text{(A6)}$$

Requiring that (A6) has only $\delta=1$ as a solution independent of $a(\to 0)$, which is equivalent to h having exactly one bound state with energy $-a^2$, one finds:

for

$$\begin{aligned} D \le 2 &: \quad \delta = O(a^{2-D}) \quad (\cong a \text{ f. } D=1) \\ D=2 &: \quad \delta \cong \frac{2}{|\ln a|} \qquad\qquad \text{(A7)} \\ D = 2+\epsilon \; (0<\epsilon\ll 1) &: \quad \delta \cong 2\epsilon \\ D=3 &: \quad \delta \cong \pi^2/4 + 2a \cong \pi^2/4 \end{aligned}$$

(where $x \cong y$ for two functions of a means that $x=y\,(1+h(a))$ with $\lim_{a\to 0} h(a) = 0$) ---- (A8)

(A7) shows that for $D \le 2$, $\lim_{a\to 0} \delta(a) = 0$ while $\lim_{a\to 0} \delta(a) > 0$

for D>2; this is of interest as it suggests that--despite the fact that

$$\int \frac{d^D \rho}{\rho^2+\Delta} \text{ diverges for } D \geq 2$$

while for D<2 everything is finite--the binding in 2 dimensions is more like D<2 rather than D>2, and, therefore, a more regular phenomenon than for D>2, in particular for D=3 where the spectrum of the corresponding 3-body problem is not bounded from below, both "Thomas"- and Efimov-effect are known to occur.*

It is interesting to calculate the expectation values of the potential, the kinetic energy contained in the inside region r<a and the outside region r>a, and where the wave-function is concentrated. With $\cong$ defined by (A8): for D=2 take $(\mathcal{S} \cong \frac{2}{|\ln a|})$:

$$\psi(r) \cong \frac{1}{\sqrt{2\pi}} \begin{cases} K_0(r) & \text{outside} \\ |\ln a| \, J_0(\sqrt{\mathcal{S}} \, \frac{r}{a}) & \text{inside} \end{cases} \tag{A9}$$

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*See e.g., L.H. Thomas "The Interaction between a neutron and a proton and the structure of H^3", Phys. Rev. 47, 1935. Minlos and Faddeev [M F] "Comment on the problem of three particles with point interactions", Soviet Physics JETP Vol. 14, No. 6, 1962. S. Albevenio, Hoegh-Krohn and Tsai Tsun Wu "A class of exactly solvable three-body quantum mechanical problems and the universal low energy behavior", Phys. Lett. 83A, No. 3, 1981, and [Flamand].

then $\int_{>} |\psi|^2 r\,dr\,d\varphi \cong \pi/4$, $\int_{<} |\psi|^2 r\,dr\,d\varphi \cong \frac{1}{2} a^2 \ln^2 a$

$$\langle V \rangle = \int V |\psi|^2 r\,dr\,d\varphi \cong \frac{-2}{a^2 |\ln a|} \int_{<} |\psi|^2 r\,dr\,d\varphi = -|\ln a| + \text{const.}$$

$$(\vec{\nabla}\psi)^2 = \frac{1}{2\pi} \begin{cases} (-K_1(r))^2 & \text{inside} \\ \ln^2 a \left(-\sqrt{s}/a \, J_1(\sqrt{s}\,\frac{r}{a})\right)^2 & \text{outside} \end{cases}$$

and so

$$T_< \equiv \int_{<} (\vec{\nabla}\psi)^2 dr\,r\,d\varphi \cong \frac{s \ln^2 a}{a^2} \int_0^a \left(\frac{1}{2}\sqrt{s}\,\frac{r}{a}\right)^2 r\,dr$$

$$= \frac{s^2 \ln^2 a}{4a^4} \frac{1}{4} a^4 = \frac{1}{4} \qquad (A9)$$

$$T_> \equiv \int_{>} (\vec{\nabla}\psi)^2 r\,dr\,d\varphi \cong \int_a^\infty K_1^2(r)\, r\,dr \cong \int_a \frac{dr}{r}$$

$$\cong + |\ln a| + \text{const.}$$

$$\left(\begin{array}{l} \text{using } J_0'(x) = -J_1(x) \approx -\frac{x}{2} \, , \; J_0 \to 1 \\ K_0'(x) = -K_1(x) \approx \frac{1}{x} \, , \\ K_0 \to |\ln x| + \ln 2 - (\text{Eulers constant } \gamma), \text{ as } x \to 0 \end{array} \right) \qquad (A10)$$

One sees that both $T_>$ and $\langle V\rangle$ diverge logarithmically as $a \to 0$ and the finite binding energy (-1) arises as a delicate cancellation between them.

On the other hand for D=3 one has

$$\psi \cong \frac{1}{\sqrt{2\pi}} \begin{Bmatrix} e^{-r}/r & \text{outside} \\ 1/r \,\sin\left(\frac{r}{a}\,\pi/2\right) & \text{inside} \end{Bmatrix} \qquad (A11)$$

Thus(the approximation lies in taking $\pi/2$ instead of $\sqrt{s}$ in the expression for $\psi_<$; $\langle\psi_>\rangle$ and $\langle T_>\rangle$ are exact however):

$$\int_> |\psi|^2 r^2 dr \underbrace{\sin\theta\, d\theta\, d\varphi}_{\equiv d\Omega} = 1 \quad , \quad \int_< |\psi|^2 r^2 dr\, d\Omega \cong a$$

$$\int V |\psi|^2 r^2 dr\, d\Omega = \frac{-\pi^2}{4a^2} \int_< |\psi|^2 r^2 dr\, d\Omega = -\pi^2/4a$$

Since
$$\psi'(r) \cong \frac{1}{\sqrt{2\pi}} \begin{cases} -e^{-r}/r\,(1+1/r) & \text{outside} \\ \frac{\pi}{2a}\frac{1}{r}\left(\cos u - \frac{\sin u}{u}\right) & \text{inside} \end{cases}$$
$$(u \equiv r/a\;\pi/2)$$

one finds

$$\int_> (\vec\nabla\psi)^2 r^2 dr\, d\Omega = \frac{2}{a} - 3 + O(a)$$

$$\int_< (\vec\nabla\psi)^2 r^2 dr\, d\Omega \cong 2\frac{\pi^2}{4a^2}\cdot a\cdot\frac{2}{\pi}\int_0^{\pi/2}\left(\cos^2 u - \frac{2\sin u\cos u}{u} + \frac{\sin^2 u}{u^2}\right)du$$

$$= \frac{\pi^2}{4a} + \frac{\pi}{a}\left[-\frac{\sin^2 u}{u}\right]_0^{\pi/2} = \frac{\pi^2}{4a} - \frac{2}{a} \quad .$$

One sees that in 3 dimensions not only V and $T_>$ but also $T_<$ diverge,all like $1/a$,and one can check that,again,the divergent terms cancel in the expression for the total energy

$$T_< + T_> + \langle V\rangle$$

B. The 3-body problem

As in the two-body case, one can separate the center of mass motion also in the open 3-body problem by going to relative coordinates

$$\vec{X}_1 \equiv \vec{\xi}_1 - \vec{\xi}_2 \quad \text{and} \quad \vec{X}_2 \equiv \vec{\xi}_2 - \vec{\xi}_3$$

The Hamiltonian becomes

$$H_3 = \vec{P}_1^{\,2} + \vec{P}_2^{\,2} - \vec{P}_1 \cdot \vec{P}_2 - (2\pi\lambda)\left(\delta^{(2)}(\vec{X}_1) + \delta^{(2)}(\vec{X}_2)\right)$$

Multiplying the equation for a bound state

$$H_3 |\psi\rangle = -\Delta |\psi\rangle \quad \text{by} \quad \langle \vec{P}_1 \vec{P}_2 |$$

gives

$$\left(\vec{P}_1^{\,2} + \vec{P}_2^{\,2} - \vec{P}_1 \cdot \vec{P}_2 + \Delta\right) \tilde{\psi}(\vec{P}_1, \vec{P}_2)$$

$$= (2\pi\lambda) \int \frac{d^2 p'}{(2\pi)^2} \left(\tilde{\psi}(\vec{P}_1{}', \vec{P}_2) + \tilde{\psi}(\vec{P}_1, \vec{P}') \right)$$

$$\equiv g_2(\vec{P}_2) + g_1(\vec{P}_1)$$

Because H_3 is invariant under interchange of 1 and 2, one can use

$$\tilde{\psi}(\vec{p},\vec{q}) = \pm\tilde{\psi}(\vec{q},\vec{p})$$

i.e., $g_1=\pm g_2\equiv g$ so that

$$\tilde{\psi}(\vec{p},\vec{q}) = \frac{g(\vec{p})\pm g(\vec{q})}{p^2+q^2-\vec{p}\,\vec{q}+\Delta} ,$$

and from above

$$g(\vec{\mathbf{p}}_1) \equiv 2\pi\lambda\int\frac{d^2p_2}{(2\pi)^2}\tilde{\psi}(\vec{p}_1,\vec{p}_2) = 2\pi\lambda\int\frac{d^2p_2}{(2\pi)^2}\frac{g(\vec{p}_1)\pm g(\vec{p}_2)}{p_1^2+p_2^2-\vec{p}_1\cdot\vec{p}_2+\Delta}$$

$$= (2\pi\lambda)\,g(\vec{p}_1)\int\frac{d^2p_2}{(2\pi)^2(p_1^2+p_2^2-\vec{p}_1\cdot\vec{p}_2+\Delta)}$$

$$\pm(2\pi\lambda)\int\frac{d^2p_2}{(2\pi)^2}\frac{g(\vec{p}_2)}{p_1^2+p_2^2-\vec{p}_1\cdot\vec{p}_2+\Delta}$$

Dividing by $2\pi\lambda$, using the consistency relation (A1) for λ, and subtracting the first term on the right hand side gives:

$$g(\vec{p}_1)\left\{\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+1}-\int\frac{d^2p_2}{(2\pi)^2}\frac{1}{p_1^2+p_2^2-\vec{p}_1\cdot\vec{p}_2+\Delta}\right\}$$

$$= \pm\int\frac{d^2p_2}{(2\pi)^2}\frac{g(\vec{p}_2)}{p_1^2+p_2^2-\vec{p}_1\cdot\vec{p}_2+\Delta}$$

Changing variables from $\vec{p}_2$ to $\vec{p} \equiv \vec{p}_2 - (1/2)\vec{p}_1$ on the left hand side and then from p^2 to E, the curly bracket becomes

$$\lim_{\Lambda\to\infty}\left\{ \frac{1}{4\pi}\int_0^{\Lambda^2}\frac{dE}{E+1} - \frac{1}{4\pi}\int_0^{\Lambda^2}\frac{dE}{E+(\frac{3}{4}p_1^2+\Delta)} \right\}$$

$$= \frac{1}{4\pi}\ln\left(\frac{3}{4}p_1^2+\Delta\right) = \frac{1}{4\pi}\left(\ln\Delta + \ln\left(1+\frac{3}{4}p_1^2/\Delta\right)\right)$$

Defining rescaled variables $\vec{p} \equiv \vec{p}_1/\sqrt{\Delta}$, $\vec{q} \equiv \vec{p}_2/\sqrt{\Delta}$ and $f(\vec{p}) \equiv g(\vec{p}\cdot\sqrt{\Delta})$, the resulting equation is:

$$-\ln\Delta\, f(\vec{p}) = \ln\left(1+\frac{3}{4}p^2\right)f(\vec{p}) \mp \frac{1}{\pi}\int\frac{f(\vec{q})\,d^2q}{p^2+q^2-\vec{p}\vec{q}+1} \qquad (B1)$$

$$\equiv (Hf)(\vec{p})$$

which can be rewritten as

$$f(\vec{p}) = \pm\frac{1}{\pi}\int\frac{f(\vec{q})\,d^2q}{\ln\left(\Delta(1+\frac{3}{4}p^2)\right)(p^2+q^2-\vec{p}\vec{q}+1)}$$

$$\equiv \int K(\vec{p},\vec{q})f(\vec{q})\frac{d^2q}{(2\pi)^2} \equiv (Kf)(\vec{p}) \qquad \text{(B1')}$$

These equations are equivalent to the Schrödinger equation for bound states

$$H_3|\psi\rangle = -\Delta|\psi\rangle \tag{B2}$$

in the sense that

if $f(\vec{p})$ satisfies (B1') then
$$\left\{\begin{array}{l} |\psi\rangle \text{ with} \\ \tilde{\psi}(\vec{p},\vec{q}) \equiv \dfrac{f(\vec{p}/\sqrt{\Delta}) \pm f(\vec{q}/\sqrt{\Delta})}{p^2+q^2-\vec{p}\cdot\vec{q}+\Delta} \\ \text{Satisfies (B2)} \end{array}\right\} \tag{B3}$$

Although λ and Λ and δ-functions do not appear in the equation(s) ($B1^{(\prime)}$), which on a naive level might suggest that with the two-body system also the 3-body (and hopefully N-body) problem has been successfully regularized, one really still has to show that ($B1^{(\prime)}$) is free of irregularities,- preferably that there is only a finite number of bound states, i.e., that: "the values of Δ for which ($B1^{(\prime)}$) can be solved, form a finite discrete set" } (B4)

Neither the question per se nor the task of actually proving (B4) are of academic nature, as the following discussion — which is an uncompleted attempt to rigorously answer (B4) positively for D=2 and the fact that (B4) is in fact wrong for D=3 (although the corresponding equation is also free of the naive divergencies) show.

For D=3 the equation corresponding to (B1) is

$$\left(\sqrt{1+\frac{3}{4}p^2}-\frac{1}{\sqrt{\Delta}}\right)f(\vec{p}) = +\frac{1}{2\pi^2}\int\frac{d^3q\, f(\vec{q})}{p^2+q^2+\vec{p}\cdot\vec{q}+1} \qquad \text{(B5)}$$

which at least for S-waves[1] ($f=f(|\vec{p}|)$) has been studied extensively in the literature.[2] Even after a continuum of solutions is removed by orthogonality conditions,[3] (B5) still admits solutions for an infinite set of values for Δ, that extends to $+\infty$, so that there is no ground state.[4] These results sharpen the difficulty pointed out as early as 1935 by L.H. Thomas[5], who--in the formulation of the problem as the limit of particles interacting by short-range potentials--constructed a complicated trial wavefunction (whose derivatives are not everywhere continuous e.g.) for

$$h_3 = -\frac{1}{a^2}\left(-\nabla_1^2-\vec{\nabla}_1\cdot\vec{\nabla}_2-\nabla_2^2+S f(\rho_1)+S f(\rho_2)\right)$$

which has infinite Binding energy as a 0. (The attempt to find the analogous trial wavefunction for D=2 leads to one containing Bessel functions and complete elliptic integrals; however, E_{var} turns out to go to $+\infty$ (rather than $-\infty$) as $a\to 0$)

[1]Then equations (S1) p. 259 in [F], with $\alpha \leftrightarrow 1$, $\lambda^2 \leftrightarrow \Delta$, an extra 1/2 in front of the integral (open case!) and $\chi_{\beta=c}(k) \leftrightarrow f(\vec{p}/\sqrt{\Delta})$. [F] contains a long discussion of (B5).

[2]First derived by Skornyakov and Ter-Martirosjan, JETP 4, 648 (1957)

[3]Danilov, J.E.T.P. 13, 349 (1961).

[4]Minlos and Faddeev, J.E.T.P. Vol. 14, No. 6, 1962.

[5]L.H. Thomas, Phys. Rev. 47, 903 (1935).

This article is often quoted but never cursed at for its misprints at crucial places.*

After this brief discussion of the 3-body problem in 3-dimensions, (B1') will be discussed (trying to prove (B4)): It is not too difficult to prove that <u>Eq. (B1') has no solution $f \in L^2$</u>

$$\left(f \in L^2 \Leftrightarrow \|f\|_{(L^2)} \equiv \left(\int |f|^2 \frac{d^2\rho}{(2\pi)^2} \right)^{1/2} < \infty \right)$$

if $\Delta > e^{4/3}$. One does this by noting that,

with $k^2(x) \equiv \int K^2(x,y)\,dy$, $\int K(x,y) f(y)\,dy \leq k(x)\,\|f\|$
(because of Schwartz's inequality)

Therefore:

$$f = Kf \Rightarrow \|f\| = \|Kf\| \leq |K|\ \|f\|,$$

where $|K| \equiv \|k\|$ $\left(\text{and } \begin{matrix} x \leftrightarrow \vec{\rho} \\ dx \leftrightarrow \frac{d^2\rho}{(2\pi)^2} \end{matrix} \right)$

---

*In particular, Eq. (28) should read:

$$I_{(ii)} = \ominus \int 4\pi K_0^2(\mu s) \left\{ \frac{\pi^2}{4s^2 a} + \frac{\pi^2}{4s^2}\lambda + \frac{2\frac{\pi}{2}\left(\frac{2\pi}{3^{3/2}-1}\right)}{s^3} + O\left(\frac{a}{s^4}\right) dV_2 \right.$$

and eq. (27): $J_{ii} = \ominus \int \cdots$

$$A_s \quad \|f\| \leq |K| \, \|f\| \qquad (B6)$$

f=Kf cannot have a solution $f \neq 0$ ($\in L^2$) if $|K| < 1$. As K is clearly a monotonically decreasing function of Δ for the kernel of (B1'), one in fact needs only to show that $|K|$ is finite (then for some big enough $\Delta = \tilde{\Delta}$, $|K| < 1$, and there cannot be a bound state with binding energy $\Delta > \tilde{\Delta}$) However, accidentally $|K|$ can be computed exactly (as a function of Δ) for

$$K(\vec{p},\vec{q}) = \frac{\pm 1}{\pi} \frac{1}{\ln\left(\Delta\left(1+\frac{3}{4}p^2\right)\right)\left(p^2+q^2-\vec{p}\cdot\vec{q}+1\right)}$$

$$|K|^2 = \frac{1}{\pi^2}(2\pi)\left(\frac{1}{2}\right)^2 \int_0^\infty \frac{dx\,dy}{\ln^2\Delta\left(1+\frac{3}{4}x\right)} \int_0^{2\pi} \frac{d\varphi}{\left(x+y+1-\sqrt{xy}\cos\varphi\right)^2}$$

$$= \int \frac{dx}{\ln^2\Delta\left(1+\frac{3}{4}x\right)} \int \frac{(x+y+1)\,dy}{\left((x+y+1)^2-xy\right)^{3/2}}$$

using $\displaystyle\int \frac{dy}{(y^2+by+c)^{3/2}} = \frac{2(2y+b)}{(4c-b^2)\sqrt{y^2+by+c}}$

and

$$\int \frac{y\,dy}{(y^2+by+c)^{3/2}} = -2\,\frac{2c+by}{(4c-b^2)\sqrt{y^2+by+c}}$$

(with $b=(x+2)$, $c=(x+1)^2$, $(4c-b^2)=4x(1+3x/4)\geqslant 0$) one gets

$$\left.\begin{aligned} |K|^2 &= \int_0^\infty \frac{dx}{\ln^2\Delta\,(1+\frac{3}{4}x)}\,\frac{1}{(1+\frac{3x}{4})} = \frac{4}{3}\,\frac{1}{\ln\Delta} \\ \text{so}\quad |K| &< 1 \quad \text{for } \Delta > e^{4/3} \approx 3.79 \end{aligned}\right\} \quad \text{(B7)}$$

Unfortunately one has to allow for a larger class of functions than L^2--because

$$\|\psi\|^2 \equiv \int |\tilde{\psi}(\vec{\rho},\vec{q})|^2 \frac{d^2\rho\, d^2q}{(2\pi)^4}$$

$$\underset{\substack{\uparrow \\ \text{(using B3)}}}{=} \frac{1}{2\pi}\int \frac{|f(\vec{\rho})|^2}{1+\frac{3}{4}\rho^2}\frac{d^2\rho}{(2\pi)^2} + 2\,\mathrm{Re}\int \frac{f^*(\vec{\rho})f(\vec{q})}{(\rho^2+q^2-\vec{\rho}\cdot\vec{q}+1)}\frac{d^2\rho\, d^2q}{(2\pi)^4} \quad \text{(B8)}$$

is finite for a larger class of functions L. L includes, e.g., L^{1+p^2}, defined as the space of functions f with

$$\| f \|_{1+\rho^2} \equiv \left(\int \frac{|f|^2}{1+\rho^2} \frac{d^2\rho}{(2\pi)^2} \right)^{1/2} < \infty$$

For this space one would write (B1') as

$$\begin{aligned} f(\vec{\rho}) &= \int \frac{d^2 q}{(2\pi)^2 (1+q^2)} \tilde{K}(\vec{\rho},\vec{q}) f(\vec{q}) \\ &= \pm \frac{1}{\pi} \int \frac{(1+q^2) f(\vec{q})}{\left(\ln \Delta (1+\frac{3}{4}\rho^2)\right)\left(\rho^2+q^2-\vec{\rho}\,\vec{q}+1\right)} \frac{d^2 q}{(2\pi)^2(1+q^2)} \\ &\equiv (\tilde{K} f)(\vec{\rho}) \end{aligned}$$

and

$$|\tilde{K}|^2_{(1+\rho^2)} \equiv \int |\tilde{K}|^2 \frac{d^2\rho \, d^2 q}{(2\pi)^4 (\rho^2+1)(q^2+1)}$$

no longer converges, so that the proof based on (B6) ceases to hold. (However, the fact that $|\tilde{K}|_{1+\rho^2}$ is infinite, does not necessarily mean that (B4) is wrong.)

<u>Looking at (B1$^{(\prime)}$) for rotationally symmetric functions</u> $f(\vec{\rho}) \equiv h(\rho^2)$ simplifies the formulae a little bit, but does not help much :

$$h(x) = \pm \int_0^\infty \frac{h(y)\,dy}{\ln \Delta (1+\frac{3}{4}x) \sqrt{(x+y+1)^2 - xy}} \equiv (K_0 h)(x) \qquad \text{(B9)}$$

The bound $\Delta < e^{4/3}$ (B7) for L^2-functions is not much improved: instead of getting

$$|K|^2 = \frac{4}{3} \int_0^\infty \frac{dt}{(\ln\Delta + t)^2} = \frac{4}{3}\,\frac{1}{\ln\Delta}$$

$$\left(\text{compare B7},\ 1+\tfrac{3}{4}x = e^t \right)$$

one gets

$$|K_0|^2 = \frac{4}{3} \int_0^\infty \frac{dt}{(\ln\Delta + t)^2} \cdot \left\{ \frac{\cos^{-1}\left(\frac{2+e^{-t}}{4-e^{-t}}\right)}{\sqrt{4/3\,(1-e^{-t})}} \right\} \qquad \text{(B10)}$$

With $\dfrac{2+e^{-t}}{4-e^{-t}} \equiv \cos\theta$ the curly bracket becomes

$$\left\{ \frac{\theta}{2\tan\theta/2} \right\}$$

which, instead of being =1 (in the calculation for $|K|^2$), varies slightly, but not much: Its minimal value in the range of integration is $\frac{\pi}{2\sqrt{3}} \approx 0.907$.

Rewriting (B9) as

$$-\ln\Delta\, h(x) = \ln\left(1+\tfrac{3}{4}x\right) h(x) - \int_0^\infty \frac{h(y)\,dy}{\sqrt{(x+y+1)^2 - xy}} \qquad \text{(B9')}$$

(now restricting oneself also to symmetric wavefunctions ; for every antisymmetric $|\psi\rangle$ there is always a symmetric $|\psi\rangle$ with lower energy) one could naively apply the variational principle by thinking of the right hand side as a Hamiltonian $\tilde{H}$ acting on h: $(\tilde{H}h)(x)$ with eigenvalue $-\ln\Delta$. It is not difficult to find normalized trial wavefunctions $h \in L^{1+x}$ with arbitrarily large binding energy: take

$$h(x) = \sqrt{2\epsilon}\,(1+x)^{-\epsilon} \qquad (B11)$$

then $h \in L^{1+x}$, and in fact

$$\|h\|_{1+x} \equiv \left(\int_0^\infty \frac{h^2(x)\,dx}{1+x}\right)^{1/2} = \left(2\epsilon\int_0^\infty \frac{dx}{(1+x)^{1+2\epsilon}}\right)^{1/2} = 1$$

independent of ϵ. then

$$\langle\tilde{H}\rangle_h = 2\epsilon\int_0^\infty \frac{\ln(1+\frac{3}{4}x)\,dx}{(1+x)^{1+2\epsilon}} - 2\epsilon\int_0^\infty \frac{(1+x)^{-\epsilon}dx}{(1+x)}\,\frac{dy\,(1+y)^{-\epsilon}}{\sqrt{(1+x+y)^2-xy}}$$

$$< 2\epsilon\int_0^\infty \frac{\ln(1+x)\,dx}{(1+x)^{1+2\epsilon}} - 2\epsilon\int_0^\infty \frac{dx\,dy}{(1+x)^{1+\epsilon}(1+x+y)^{\epsilon}\sqrt{(1+x+y)^2}}$$

$$= 2\epsilon\int_0^\infty t\,e^{-2\epsilon t}\,dt - 2\epsilon\int_0^\infty \frac{dx}{(1+x)^{1+\epsilon}}\int_{(x+1)}^\infty \frac{dy}{y^{1+\epsilon}}$$

$$= \frac{1}{2\epsilon} - 2\int_0^\infty \frac{dx}{(x+1)^{1+2\epsilon}} = \frac{1}{2\epsilon} - \frac{1}{\epsilon} = -\frac{1}{2\epsilon} \longrightarrow -\infty \quad (\text{as } \epsilon \to 0)$$

However, $\tilde{H}$ acting on L^{1+x} is not a self-adjoint operator, so that the "variational principle" (i.e., the statement that the true ground state energy $E_0 < \langle \tilde{H} \rangle_h \ \forall_{h \in L^{1+x}}$ does not hold.

One final argument will be given, strongly suggesting that the 3-body spectrum is bounded from below: leaving the cutoff parameter Λ in the integral equation, instead of taking $\Lambda \to \infty$ — once λ has disappeared and the appearing expressions are finite as $\Lambda \to \infty$ — one has, for S-waves:

$$g(x) = \int_0^{\Lambda^2} \frac{g(y)\, dy}{F(x,\Lambda)\sqrt{(x+y+\Delta)^2 - xy}} \equiv (K_\Lambda g)(x) \qquad \text{(B12)}$$

where

$$\begin{aligned} F(x,\Lambda) &\equiv F(\rho^2, \Lambda) \\ &= \frac{1}{\pi} \int_{q^2<\Lambda^2} d^2q \left\{ \frac{1}{q^2+1} - \frac{1}{\rho^2 + q^2 - \vec{\rho}\,\vec{q} + \Delta} \right\} \\ &= \int_0^{\Lambda^2} dx \ldots = \ln\left(\tfrac{3}{4}x + \Delta\right) + \ln\left(1 + \frac{1}{\Lambda^2}\right) - \\ &\quad - \ln\left(\frac{1}{2}\sqrt{1 + \frac{x+2\Delta}{\Lambda^2} + \frac{(x+\Delta)^2}{\Lambda^4}} + \frac{1}{2} + \frac{x+2\Delta}{\Lambda^2} \right) \end{aligned}$$

and g(x) is assumed to be Lebesgue-integrable on $[0, \Lambda^2]$.

With

$$\| g \|_\Lambda^2 \equiv \int_0^{\Lambda^2} dx\, |g|^2$$

and

$$|K_\Lambda|^2 \equiv \iint_0^{\Lambda^2} |K_\Lambda(x,y)|^2 dx\, dy$$

one has, as before (compare Eq. (B6)):

$$g = K_\Lambda g \Rightarrow \| g \|_\Lambda \leq |K_\Lambda| \, \| g \|_\Lambda .$$

As $\Lambda \to \infty$, $F(x,\Lambda)$ is dominated by $\ln\left(\frac{3}{4}x + \Delta\right)$ (for all x!)*, so that as $\Lambda \to \infty$

$$|K_\Lambda|^2 < 4/_{3 \ln \Delta}$$

(see (B7) and (B10)), which is independent of Λ for large Λ, so that (B12) cannot have a solution $g \neq 0$ for any large Λ, if $\Delta \geq e^{4/3}$. — From now on (B4) will be assumed to be true with $\Delta_{max} \leq e^{4/3}$.

Strengthened by the above argument, one performs a variational calculation for H (defined in B1), with

$$f(\vec{p}) = \frac{\sqrt{4\pi a}}{p^2 + a} \qquad \left(\| f \|_{L^2} = 1 \right)$$

as trial wavefunctions (a as parameter). One finds:

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* i.e. $F(x,\Lambda) = \left(\ln\left(\frac{3}{4}x+\Delta\right)\right)\left(1 + G(x,\Lambda)\right)$, where $\lim_{\Lambda\to\infty} G(x,\Lambda) = 0$ even if one allows $x (< \Lambda^2)$ to be a diverging function of Λ.

$$\left.\begin{aligned} T &\equiv \int \frac{d^2p}{(2\pi)^2}\, \frac{\ln\left(1+\frac{3}{4}p^2\right) 4\pi a}{(p^2+a)^2} = \frac{\ln b}{b-1} \qquad (b \equiv 4/3a) \\ W &\equiv -\frac{1}{4\pi^3} \int \frac{d^2p\, d^2q\, (4\pi a)}{(p^2+a)(q^2+a)(p^2+q^2-\vec{p}\,\vec{q}+1)} \\ &= 4(\beta-1) \int_0^1 \frac{dx}{x^2+2\beta-3}\, \ln\left(\frac{(\beta-1)(x+3)}{(x+1)(\beta-x)}\right) \\ &\qquad (\beta \equiv 1/1-a) \end{aligned}\right\} \quad \text{(B13)}$$

In order to arrive at the above form of W, Feynman's trick of combining denominators was used first. The results of a numerical calculation* for different values of a, which are listed below, gave $a \approx 3/4$ to be the value which leads to a maximal lower bound, on Δ_3, giving $\approx$ 2.4.

a	W	T	$W-T$	$\Delta_3 \geq$	$2W-T$	Δ_3'
1/2	1.443	0.588	0.855	2.350	2.298	
3/4	1.611	0.740	0.871	2.389	2.482	
1	1.726	0.863	0.863	2.370	2.589	
4/3	1.836	1			2.672	
5/3	1.918	1.116			2.720	
2	1.981	1.216			2.746	
5/2	2.054	1.347			2.762	
11/4	2.084	1.405			2.763	15.848
3	2.111	1.460			2.762	
4	2.193	1.648			2.738	

(2W-T has been listed, as it turns out to be the lower bound for ln Δ_3')

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*I would like to thank Slobodan Tepić for having done this computation

Finally it will be shown that the closed 3-body problem (i.e., all 3 particles mutually interacting) is exactly the same as the open case, apart from a factor of 2 in front of the integral in the integral equation(s) (B1$^{(1)}$):

$$H_3^{ur} = \frac{1}{2}\left(\vec{\Pi}_1^2 + \vec{\Pi}_2^2 + \vec{\Pi}_3^2\right) - (2\pi\lambda)\left(\delta^{(2)}(\vec{\xi}_1 - \vec{\xi}_2) + \delta^2(\vec{\xi}_2 - \vec{\xi}_3) + \delta^2(\vec{\xi}_3 - \vec{\xi}_1)\right)$$

Multiplying $H_3^{ur}|\psi\rangle = -\Delta'|\psi\rangle$ by $\langle \vec{\Pi}_1 \vec{\Pi}_2 \vec{\Pi}_3 |$

gives

$$\left(\frac{1}{2}\left(\Pi_1^2 + \Pi_2^2 + \Pi_3^2\right) + \Delta'\right)\tilde{\psi}(\vec{\Pi}_1 \vec{\Pi}_2 \vec{\Pi}_3)$$
$$= g_3(\vec{\Pi}_3) + g_2(\vec{\Pi}_2) + g_1(\vec{\Pi}_1) \quad \text{(B14)}$$

where

$$g_r \equiv (2\pi\lambda)\langle \vec{\Pi}_1 \vec{\Pi}_2 \vec{\Pi}_3 | \delta^2(\vec{\xi}_{r+1} - \vec{\xi}_{r+2})|\psi\rangle$$

can be shown to be a function of $\vec{\Pi}_r$ only $\left(\text{for} \sum_{r=1}^{3} \vec{\Pi}_r = \vec{0}\right)$:

$$g_1 = (2\pi\lambda) \int d^2\xi_1 d^2\xi_2 d^2\xi_3 \, e^{-i(\vec{\Pi}_1\vec{\xi}_1 + \vec{\Pi}_2\vec{\xi}_2 + \vec{\Pi}_3\vec{\xi}_3)} \cdot \delta^2(\vec{\xi}_2 - \vec{\xi}_3)\,\psi(\vec{\xi}_1\vec{\xi}_2\vec{\xi}_3),$$

with

$$\vec{\Pi}_2\vec{\xi}_2 + \vec{\Pi}_3\vec{\xi}_3 = (\vec{\Pi}_2 + \vec{\Pi}_3)\left(\frac{\vec{\xi}_2 + \vec{\xi}_3}{2}\right) + \left(\frac{\vec{\Pi}_2 - \vec{\Pi}_3}{2}\right)(\vec{\xi}_2 - \vec{\xi}_3)$$

$$= (\vec{\Pi}_2 + \vec{\Pi}_3)\left(\frac{\vec{\xi}_2 + \vec{\xi}_3}{2}\right),$$

and

$$\delta^2(\vec{\xi}_2 - \vec{\xi}_3) = \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}(\vec{\xi}_2 - \vec{\xi}_3)}$$

one gets:

$$g_1 = (2\pi\lambda)\int \frac{d^2 q}{(2\pi)^2}\, \tilde{\varphi}\left(\vec{\pi}_1,\ \frac{\vec{\pi}_2+\vec{\pi}_3}{2}+\vec{q},\ \frac{\vec{\pi}_2+\vec{\pi}_3}{2}-\vec{q}\right)$$

$$= (2\pi\lambda)\int \frac{d^2 q}{(2\pi)^2}\, \tilde{\varphi}\left(\vec{\pi}_1,\ \vec{q}-\vec{\pi}_1/2,\ -\vec{q}-\vec{\pi}_1/2\right)$$

$$= g_1(\vec{\pi}_1)$$

g_2 and g_3 are given by the same expression with the arguments of $\tilde{\varphi}$ being cyclicly permuted. Restricting oneself to totally symmetric solutions $|\psi\rangle$, $g_1=g_2=g_3\equiv g$ therefore, and--using (B14)-one thus has:

$$g(\vec{\pi}_1) = (2\pi\lambda)\int d^2 q/(2\pi)^2\, \frac{g(-\vec{q}-\vec{\pi}_1/2)+g(\vec{q}-\vec{\pi}_1/2)+g(\vec{\pi}_1)}{\vec{q}^{\,2}+\frac{3}{4}\vec{\pi}_1^{\,2}+\Delta'}$$

$$g(\vec{\pi}_1) = 2\pi\lambda\, g(\vec{\pi}_1) \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + (\frac{3}{4}\pi_1^2 + \Delta')}$$

$$+ (2\pi\lambda)\cdot 2 \cdot \int \frac{d^2q}{(2\pi)^2} \frac{g(\vec{q} - \vec{\pi}_1/2)}{q^2 + (\frac{3}{4}\pi_1^2 + \Delta')}$$

and

$$\tilde{\varphi} = \frac{g(\vec{\pi}_1) + g(\vec{\pi}_2) + g(-\vec{\pi}_1 - \vec{\pi}_2)}{\pi_1^2 + \pi_2^2 + \vec{\pi}_1\cdot\vec{\pi}_2 + \Delta'} .$$

Changing $\vec{q}$ to $-\vec{q}$, assuming g to be an even function* and with the identification

$$\Delta \leftrightarrow \Delta', \quad \vec{\pi}_1 \leftrightarrow \vec{p}_1, \quad \vec{q} \leftrightarrow \vec{p}_2 - \vec{\pi}_1/2 = \vec{p}_2 - \tfrac{1}{2}\vec{p}_1$$

this is, apart from a factor of 2 in front of the second term,

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$$^{*}\; g(-\vec{\pi}_1) = 2\pi\lambda \int \frac{d^2q}{(2\pi)^2} \tilde{\varphi}(-\vec{\pi}_1, \vec{q} + \vec{\pi}_1/2, -\vec{q} + \vec{\pi}_1/2)$$

$$= 2\pi\lambda \int \frac{d^2q'}{(2\pi)^2} \tilde{\varphi}(-\vec{\pi}_1, -(\vec{q}\,' - \vec{\pi}_1/2), -(-\vec{q}\,' - \vec{\pi}_1/2))$$

$$= g(+\vec{\pi}_1)$$

assuming $|\psi\rangle$ to have positive parity, i.e., $\tilde{\varphi}(---) = +\tilde{\varphi}(+++)$; Note that H_3 is invariant under $\vec{\xi}_i \to -\vec{\xi}_i \;\forall i$

the same as the equation considered in the open case and the lower bounds on the ground state binding energy for the closed system (Δ_3'), corresponding to trial wave functions of the form $\sqrt{4\pi a}/\rho^2+a$ are now given as e^{2W-T} instead of e^{W-T}. $a \approx 11/4$ led to a maximal bound on Δ_3': $\Delta_3' > 15.8$. That the binding energy of the closed three-body system comes out so large might be explained by noting that $\Delta_2' = \infty$ in a sense, because the coupling strength had been adjusted to make Δ_2 come out finite.

Because of the additional factor of 2 multiplying the kernel of the integral equation, one has

$$|K|^2 = 4 \cdot \left(\frac{4}{3} \frac{1}{\ln \Delta'} \right)$$

for the L^2-case, so that one knows that for $\Delta' > e^{16/3}$ there is no square integrable solution of

$$f(\vec{\rho}) = \pm \frac{2}{\pi} \int \frac{f(\vec{q})\, d^2q}{\ln\left(\Delta\left(1+\frac{3}{4}\rho^2\right)\right)\left(\rho^2+q^2-\vec{\rho}\,\vec{q}+1\right)} \quad \text{(B15)}$$

Bruch and Tjon* have in fact calculated numerically the eigenvalues of (B15) as $\Delta_3'=16.1$ (± 0.2) (so that the above variational calculation gave in fact an astonishingly good bound) and a second eigenvalue at $\Delta'=1.25$ (± 0.05). For the open case it follows from their numerical calculation that $\Delta \approx 2.5$, which is in very good agreement with the above variational calculation.

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*Phys. Rev. 19, No. 2; Only after having done the work presented in this thesis did I find this article. I believe the numerical calculation, although in the theoretical treatment they take the calculation corresponding to (B7) for the closed case (and for S-waves) as proof of (B4) without worrying about functions not in L^2 (which does not convince me). Maybe they assume even in the numerical calculation, that the eigenfunctions are square integrable.

C. The N-body problem

Changing variables to relative coordinates $\vec{X}_r \equiv \vec{\xi}_r - \vec{\xi}_{r+1}$ in

$$H_N^{ur} \equiv \frac{1}{2}\sum_{r=1}^{N} \vec{\pi}_r^{\,2} - 2\pi\lambda \sum_{r=1}^{N-1} \delta^{(2)}(\vec{\xi}_r - \vec{\xi}_{r+1}) \qquad \text{(C1)}$$

and setting the total momentum $\vec{P} = \sum_1^N \vec{\pi}_r$ (conjugate to $\vec{X} \equiv \frac{1}{N}\sum_1^N \vec{\xi}_r$) equal to $\vec{0}$ gives

$$H_N = \sum_1^{N-1}\left(\vec{p}_r^{\,2} - (2\pi\lambda)\,\delta^{(2)}(\vec{x}_r)\right) - \sum_{r=1}^{N-2} \vec{p}_r \cdot \vec{p}_{r+1} \qquad \text{(C2)}$$

For the closed case (i.e., particles 1 and N also interacting) one can show that

$$\begin{aligned} H_N' &\equiv \sum_1^N \left(\vec{p}_r^{\,2} - (2\pi\lambda)\,\delta^{(2)}(\vec{x}_r)\right) - \sum_1^N \vec{p}_r \cdot \vec{p}_{r+1} \\ &= \sum_1^N \frac{1}{2}(\vec{p}_r - \vec{p}_{r+1})^2 - (2\pi\lambda)\sum_1^N \delta^{(2)}(\vec{x}_r) \\ &\equiv T(\vec{p}_1 \cdots \vec{p}_N) + V \qquad (\vec{p}_{N+1} \equiv \vec{p}_1) \end{aligned}$$

is equivalent to "H_N^{ur} (closed case) with $\sum \vec{\pi}_r = \vec{0}$", provided that one restricts oneself to states with $\sum_1^N \vec{x}_r = \vec{0}$

(Note: $[H_N^{ur}, \sum_1^N \vec{\pi}_r] = 0$, $[H_N', \sum_1^N \vec{x}_r] = 0$)

As was done for the 3-body system, one can eliminate λ and derive an integral equation from the Schrödinger equation for bound states: $H_N^{(1)} |\psi\rangle = -\Delta |\psi\rangle$. Multiply by a momentum Eigen-bra $\langle p_1 \cdots p_N|$ to get

$$\left(T(p_1 \cdots p_N) + \Delta\right)\tilde{\psi}(p_1 \cdots p_N) = 2\pi\lambda \sum_r \int \frac{d^2 q_r}{(2\pi)^2} \tilde{\psi}(p_1 \cdots q_r \cdots p_N) \tag{C3}$$

where $T(p_1 \cdots p_N) \equiv \sum_1^N \frac{1}{2}(\vec{p}_r - \vec{p}_{r+1})^2$ and the vector notation $\rightharpoonup$ will from now on be dropped. Defining the right hand side of (C3) to be $\sum_r g_r(p_1 \cdots \not{p}_r \cdots p_N)$ where $\not{p}_r$ indicates that this variable does not occur, one has

$$g_r \equiv (2\pi\lambda) \int \frac{d^2 q_r}{(2\pi)^2} \tilde{\psi}(p_1 \cdots q_r \cdots p_N)$$

$$= 2\pi\lambda \sum_s \int \frac{d^2 q_r}{(2\pi)^2} \frac{g_s(p_1 \cdots q_r \not{p}_s \cdots p_N)}{T(p_1 \cdots q_r \cdots p_N) + \Delta} \tag{C4}$$

$$g_r = (2\pi\lambda) \int d^2x_1 \cdots \cancel{d^2x_r} \cdots d^2x_N \, e^{-i\sum_{s\neq r} \vec{p}_s \cdot \vec{x}_s} \psi(x_1 \cdots x_r = 0 \cdots x_N)$$

is $(2\pi\lambda)$ times the Fourier transform of the wave function in position space, with the r-th coordinate x_r fixed at the origin. Separating out the diagonal term in the above equation for g_r one has:

$$g_r(p_1 \cdots \not{p}_r \cdots p_N)\left(\frac{1}{2\pi\lambda} - \int \frac{d^2 q_r}{(2\pi)^2} \frac{1}{T(p_1 \cdots q_r \cdots p_N) + \Delta} \right)$$

$$= \sum_{s \neq r} \int \frac{d^2 q_r}{(2\pi)^2} \frac{g_s(p_1 \cdots q_r \not{p}_s \cdots p_N)}{T(\dots) + \Delta}$$

which, using the consistency relation for λ (Eq. A1), leads to an integral equation for the g_r, not containing λ :

$$\frac{1}{2\pi\lambda} - \int \frac{d^2 q}{(2\pi)^2} \frac{1}{T+\Delta}$$

$$= \frac{1}{4\pi} \int \frac{dE}{E+1} - \int \frac{d^2 q_r}{(2\pi)^2} \frac{1}{T_1^{r-2} + \frac{1}{2}(p_{r-1} - q_r) + \frac{1}{2}(q_r - p_{r+1}) + T_{r+1}^N + \Delta}$$

where

$$T_i^j \equiv \begin{cases} \sum_{s=i}^{j} \frac{1}{2}(\vec{p}_s - \vec{p}_{s+1})^2 = T(p_i, \dots, p_{j+1}) & \text{for } N \geq j > i \geq 1 \\ 0 & \text{otherwise} \end{cases} ;$$

in the second term change integration variable from $\vec{q}_r$ to $\vec{q} \equiv \vec{q}_r - 1/2(\vec{p}_{r-1} + \vec{p}_{r+1})$ (note that the integral is only logarythmically diverging) and then to $E \equiv \vec{q}^{\,2}$, so that the denominator becomes

$$T_1^{r-2} + T_{r+1}^N + \frac{1}{4}(p_{r+1} - p_{r-1})^2 + E$$

and, by combining the two integrals, one gets

$$\frac{1}{4\pi}\ln\left(T_1^{r-2}+T_{r+1}^{N}+\frac{1}{4}(p_{r+1}-p_{r-1})^2+\Delta\right)$$

so that

$$g_r\cdot\ln\left(\Delta+T_1^{r-2}+T_{r+1}^{N}+\frac{1}{4}(p_{r+1}-p_{r-1})^2\right.$$

$$=\frac{1}{\pi}\sum_{s\neq r}\int d^2q_r\,\frac{g_s(p_1\cdots q_r\cancel{p_s}\cdots p_N)}{T(p_1\cdots q_r\cdots p_N)+\Delta}$$

Scaling all momenta by $\sqrt{\Delta}$ and with $f_r(\ldots p_s\ldots)\equiv g_r(\ldots p_s\sqrt{\Delta}\ldots)$ one finally arrives at

$$-\ln\Delta\cdot f_r(p_1\cdots\cancel{p_r}\cdots p_N)$$

$$=\ln\left(1+T_1^{r-2}+\frac{1}{4}(p_{r+1}-p_{r-1})^2+T_{r+1}^{N}\right)\cdot f_r$$

$$-\frac{1}{\pi}\sum_{s\neq r}\int d^2q_r\,\frac{f_s(p_1\cdots q_r\cancel{p_s}\cdots p_N)}{1+T(p_1\cdots q_r\cdots p_N)} \qquad \text{(C5)}$$

$$\equiv H_{rs}f_s\equiv(H\vec{f})_r(p_1\cdots\cancel{p_r}\cdots p_N)\quad;\ r=1,2,\ldots N.$$

Also, by definition of g_r:

$$\int d^2q_r\,f_s(p_1\cdots q_r\cancel{p_s}\cdots p_N)=\int d^2q_s\,f_r(p_1\cdots\cancel{p_r}q_s\cdots p_N) \qquad \text{(C6)}$$

Although the above derivation is written out for the closed case, all corresponding equations for the open case can be obtained by simply setting $p_N \equiv 0$ everywhere (p_N non-existing).

For the closed case one can simplify (C5) considerably by making use of the fact that H_N' is invariant under cyclic permutations ($r \to r+1$) and also reflections ($N \leftrightarrow 1$, $N-1 \leftrightarrow 2, \ldots$). Restricting oneself to states that are singlets under these transformation i.e.,

$$\tilde{\psi}(p_1 \cdots p_N) = \tilde{\psi}(p_N p_1 \cdots p_{N-1}) = \tilde{\psi}(p_N p_{N-1} \cdots p_1)$$

one has

$$\begin{aligned} g_r(p_1 \cdots \cancel{p_r} \cdots p_N) &= 2\pi\lambda \int \tilde{\psi}(p_1 \cdots q_r \cdots p_N) \frac{d^2 q_r}{(2\pi)^2} \\ &= 2\pi\lambda \int \tilde{\psi}(p_N p_1 \cdots q_r \cdots p_N) \frac{d^2 q_r}{(2\pi)^2} \\ &\overset{(i)}{=} g_{r+1}(p_N p_1 \cdots \cancel{p_r} \cdots p_{N-1}) \end{aligned} \qquad \text{(C7)}$$

(analogously)

$$\begin{aligned} &= g_{r-1}(p_2 \cdots \cancel{p_r} \cdots p_N p_1) \\ &\overset{(iii)}{=} g_{N+1-r}(p_N \cdots \cancel{p_r} \cdots p_1) \end{aligned}$$

Using (C7) (in fact only (i)), (C5) becomes

$$-\ln\Delta\cdot f(\vec{p}_2\vec{p}_3\cdots\vec{p}_N)$$
$$=\ln\Big(1+\sum_{r=2}^{N-1}\tfrac{1}{2}(p_{r+1}-p_r)^2+\tfrac{1}{4}(p_N-p_2)^2\Big)\cdot f \qquad \text{(C8)}$$
$$-\frac{1}{\pi}\int d^2q_1\,\frac{f(p_3p_4\cdots p_N q_1)+f(p_4p_5\cdots p_N q_1 p_2)+\dots+f(q_1p_2\cdots p_{N-1})}{1+\frac{1}{2}(p_N-q_1)^2+\frac{1}{2}(q_1-p_2)^2+\sum_{r=2}^{N-1}\frac{1}{2}(p_{r+1}-p_r)^2}$$

($f\equiv f_1$, all other f_r are obtained from f via C7). (C8) is a single Schrödinger-like equation for a function f of N-1 variables $\vec{p}_r$. It is important, however, to remember that (C8) (and also (C5), for the closed case) is subject to the constraint $\sum_1^N \vec{x}_r=\vec{0}$ which translates to

$$f(p_2\cdots p_N)=f(p_2+k,\dots,p_N+k) \qquad \text{(C9)}$$

(in general $f_r(p_s\dots)=f_r(\dots p_s+k\dots)$ $\forall r$). Also one must not forget the condition (C6), which e.g., for N=4 says that $\int d^2p\,f(p,q,q')$ is invariant under all permutations of the arguments of f. For the case N=3 can (C9) be used to further reduce the number of variables explicitly: for N=3:

$$-\ln\Delta\cdot f(p_2p_3)=\ln\Big(1+\tfrac{3}{4}(p_3-p_2)^2\Big)f$$
$$-\frac{1}{\pi}\int d^2q_1\,\frac{f(p_3,q_1)+f(q_1,p_2)}{1+\frac{1}{2}(p_3-p_2)^2+\frac{1}{2}(p_3-q_1)^2+\frac{1}{2}(q_1-p_2)^2},$$

(C9) $\Rightarrow$ $f(p_2,p_3) = f(p_3-p_2)$; by shifting the integration variable in the first term to q_1-p_3, in the second to q_1-p_2 (and using $f(x)=f(-x)$, from parity invariance of H_N'*) one sees that both terms are, in fact, equal to

$$-\frac{1}{\pi}\int d^2q_1 \frac{f(\vec{q}_1)}{1+(\vec{p}_3-\vec{p}_2)^2+\vec{q}_1^{\,2}-\vec{q}_1\cdot(\vec{p}_3-\vec{p}_2)}$$

which agrees with Eq. (B15) ($\vec{p}\equiv\vec{p}_3-\vec{p}_2$).

N=3 is a special case : As for a function of two variables, reflection invariance is equivalent to invariance under cyclic permutations, (C8) is the correct equation also for the open case (which has only reflection symmetry) putting $\vec{p}_{N=3}=0$ (which up to Eq. (C5) was the simple and correct procedure of getting the corresponding equation for the open case). (C8) then is

$$-\ln\Delta\, f(\vec{p}_2) = \ln\left(1+\frac{3}{4}p_2^2\right)f$$
$$-\frac{1}{\pi}\int d^2q_1 \frac{f(q_1)}{p_2^2+q_1^2-\vec{p}_2\cdot\vec{q}_1+1}$$

which is exactly B1. The important new feature of (C8) for $N>3$ is that it cannot be brought into the form $f= Kf$ with K nonsingular. As the f in the different terms in the integral of (C8) contains all the variables $p_2 \ldots p_N$ and the integration variable q_1 (in cyclic permutations), K necessarily involves many δ-functions.

*Or use (C7iii) for N=3, r=2 $g_2(p)=g_2(-p) \Rightarrow g_1(p)=g_1(-p)$.

SUMMARY

Two particles attracting each other by a δ-function will have infinite binding energy in 2 (or more) dimensions, unless one chooses the coupling constant to be infinitesimal and regularizes the δ-function by introducing a cutoff to the divergent integrals. Equivalently, one can define the δ-function as a limit of a short-range potential. It turns out that then 2 dimensions are more similar to lower dimensions ($D < 2$), where there is no regularization needed in the first place.

For the N-body problem, one can derive an integral equation for the Schrodinger equation for bound states, which is free of any naive divergencies. However, one has to make sure that this equation cannot be solved for arbitrarily large binding energy.

For the 3-body case this is argued not to happen (in contrast to the analogous equation in 3 dimensions, where there are eigenfunctions explicitly known for any large binding energy). The major problem is that one has to allow for a rather large class of functions f in the integral equation, as the physical wavefunction will be square integrable even if f falls off much slower at ∞ (in momentum space).

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