FI-MODULES AND REPRESENTATION STABILITY

TREVOR HYDE

These are lectures notes for a talk I gave in the Student Representation Theory seminar at the University of Michigan on March 24th, 2016.

1. Representations of Categories

There are a few ways to think about k-representations of a group G.

(1) A vector space V with a linear G-action.
(2) A vector space V with a group homomorphism $\rho : G \to \mathrm{GL}(V)$.
(3) A module V over the group algebra kG.
(4) A functor $V : G \to k$-Vec from G viewed as a groupoid with one object to the category of k vector spaces.

These are all equivalent and useful perspectives. The last one is my favorite because it suggests many generalizations. We change the target...

- We could replace k-Vec with Sets to consider functors from G to Sets. These are called *G-sets* or *group actions*.
- We could replace k-Vec with Top to get *G-spaces*, or topological spaces with a continuous action of G.
- I think of the category G as the generic model of an object with G-symmetry and a representation of G as a concrete object in some category with G-symmetry.
- Some people would say you're only doing representation theory when the target category is vector spaces or maybe modules over a ring; however, I do not agree.

Or we can change the source...

- **Question:** What's the most general thing we could replace G by and still have this make sense? **Answer:** A general category C.
- A *linear representation* of C is a functor from C to k-Vec.
- A C-set would be a functor from C to Sets, etc.

We will focus on linear representations of C and we just call these *representations*. You might also call this a C-module

Note: One may construct a *category algebra* kC generalizing the notion of a group algebra as well as that of an *incidence algebra* of a poset.

- The objects of kC are finite k-linear combinations of morphisms from C.
- If α and β or morphism in C, we define their product by

$$\alpha\beta = \begin{cases} \alpha \circ \beta & \text{if } \mathrm{source}(\alpha) = \mathrm{target}(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

- Extend multiplication k-bilinearly.

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Then every representation of C corresponds to a kC-module, although not conversely when C has infinitely many objects.

2. The category FI

We will look specifically at representations of the category FI.

- FI is the category of finite sets with injections.
- If we let $[n] = \{1, 2, 3, \ldots, n\}$, then FI is equivalent to the category having $[n]$ as objects for each $n \geq 0$ and the injections between them.

What does FI look like?

$$\begin{array}{ccccccccc} \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ [0] & \longrightarrow & [1] & \rightrightarrows & [2] & \Rrightarrow & [3] & & \cdots \end{array}$$

- All morphisms "flow" upwards.
- $\mathrm{End}([n]) \cong S_n$ since any injection from a finite set to itself is a bijection.
- There are $\frac{m!}{(m-n)!}$ maps from $[n]$ to $[m]$ which we see by taking any listing of $[m]$ and selecting the first n.

What do representations of FI look like? Say $V : \mathrm{FI} \to k\text{-Vec}$ is an FI-representation.

- For each n, we get a vector space $V[n] = V_n$ with a linear action of S_n. Hence V_n is a representation of S_n.
- For each inclusion $[n] \hookrightarrow [m]$ we get a linear map $V_n \to V_m$, and these maps are compatible with the action of S_n and S_m.
- Note: we have representations of infinitely many groups all patched together in a consistent way; not all the same group.
- A representation of FI is called an FI-*module*.

What are the morphisms between FI-modules?

- For representations of groups, these are called "intertwining operators."
- The functor perspective gives a clue: morphisms between functors are natural transformations. Intertwining operators are exactly natural transformations.
- Hence, a morphism $f : U \to V$ between FI-modules is a natural transformation between the functors.
- More concretely, for each n, we get an S_n-map $f_n : U_n \to V_n$.
- For each injection $i : [n] \hookrightarrow [m]$ the following square commutes:

$$\begin{array}{ccc} U_n & \xrightarrow{i} & U_m \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_m} \\ V_n & \xrightarrow{i} & V_m \end{array}$$

An FI-module is a lot of information bundled together into a single mathematical object. Building them artificially could be a lot of work, but the point is that they often arise *naturally*.

3. An example

Let HP^4 be the FI-module of homogeneous degree 4 polynomials.

- HP^4_n is the vector space of homogeneous degree 4 polynomials in n variables $x_1, x_2, \ldots, x_n$.
- The maps in FI act on the subscripts of the variables.

- x_1^4 is an "element" of HP^4. We may view it as living in any HP^4_n, though it "comes from" HP^4_1.
- x_3^4, on the other hand, lives in any HP^4_n with $n \geq 3$. However it still "comes from" HP^4_1 in a natural way.
- $x_1^2x_2^1x_3^1$ lives in HP^4_n with $n \geq 3$, but unlike x_3^4, it is a "native." This monomial is not a relabelling of anything from HP^4_1 or HP^4_2.

The monomials form a basis for HP^4_n

HP^4_1	HP^4_2			HP^4_3				HP^4_4				
x_1^4	x_1^4	$x_1^3x_2^1$	$x_1^2x_2^2$	x_1^4	$x_1^3x_2^1$	$x_1^2x_2^2$	$x_1^2x_2^1x_3^1$	x_1^4	$x_1^3x_2^1$	$x_1^2x_2^2$	$x_1^2x_2^1x_3^1$	$x_1^1x_2^1x_3^1x_4^1$
1	2	2	1	3	6	3	3	4	12	6	12	1

- If we go to HP^4_n for any $n > 4$ we get "nothing new".
- If we were to compute the dimension of HP^4_n, we would do it by grouping the monomials based on the partition $\lambda \vdash 4$ as we have done in the tables.
- We can see the number of monomials in n variables with partition type λ are counted by a polynomial in n.
- For example, if $\lambda = [3, 1]$, then there are $\frac{n!}{(n-2)!} = n(n-1)$
- Another example, if $\lambda = [2, 1, 1]$, then there are $\frac{n!}{(n-3)!2!} = \frac{n(n-1)(n-2)}{2}$
- Since there are 5 partitions of 4, the total dimension of HP^4_n will be the sum of 5 polynomials in n, one counting monomials of each shape, for $n \geq 4$.

This natural grouping of basis elements we have observed may be compactly expressed as a structural decomposition of the FI-module HP^4.

- Given a partition λ, let HP^λ be the FI-module of homogeneous polynomials spanned by monomials of shape λ.
- This gives an FI-module because relabelling variables (injectively!) does not change the shape of monomials.
- If $\lambda \vdash 4$, then HP^λ is an FI-submodule of HP^4
- In fact, we have

$$\mathrm{HP}^4 \cong \mathrm{HP}^{[4]} \oplus \mathrm{HP}^{[3,1]} \oplus \mathrm{HP}^{[2,2]} \oplus \mathrm{HP}^{[2,1,1]} \oplus \mathrm{HP}^{[1,1,1,1]}$$

- Each HP^λ is cyclic, generated in $\mathrm{HP}^\lambda_{\ell(\lambda)}$ by a single monomial (the first monomial of shape λ).

This simple decomposition says a lot all at once.

- We get a simultaneous decomposition of HP^4_n as S_n-representations for all n. Hence, the characters decompose as well.
- All the S_n-representations HP^λ_n for $n \geq \ell(\lambda)$ are induced. Therefore, all of these representations are completely determined by the first non-zero one. For example, their characters may be computed from that of $\mathrm{HP}^\lambda_{\ell(\lambda)}$ using Frobenius reciprocity.
- Thus, while originally the FI-module HP^4 seemed like a large, infinite object, we see that it is completely determined by 5 representations of specific symmetric groups.
- All of this simple structure is a consequence of HP^4 being a *finitely generated* FI-module. We say an FI-module is finitely generated if, just as in the case examined, there are finitely many elements (in possibly different "degrees") which generate the entire module.

All the properties of HP^4 witnessed are examples of *representation stability*.

4. An Application to Number Theory

Let q be a prime power and $\mathbb{F}_q$ a finite field of size q. An interesting phenomenon in number theory is that counting problems over finite fields often have answers given by a polynomial in q, the size of the field, and which otherwise do not depend on the particular field or characterisitic.

One source of such counting problems are called *polynomial statistics*.

- Let $\mathrm{Conf}_n(\mathbb{F}_q)$ denote the collection of monic, degree n, squarefree polynomials defined over the finite field $\mathbb{F}_q$.
- A degree n, squarefree polynomial corresponds to a set of n distinct points in $\overline{\mathbb{F}}_q$. Being defined over $\mathbb{F}_q$ means the *set* is invariant under the action of Frobenius, or equivalently, that the coefficients of the polynomial are in $\mathbb{F}_q$.
- We call a function $F : \mathrm{Conf}_n(\mathbb{F}_q) \to \mathbb{Q}$ a polynomial statistic if $F(f)$ only depends on the factorization type of the polynomial f.
- The following are equivalent,

$$\{\text{Factorization type of } f \in \mathrm{Conf}_n(\mathbb{F}_q)\} \longleftrightarrow \{\text{Partitions } \lambda \vdash n\} \longleftrightarrow \{\text{Conjugacy classes } C_\lambda \subseteq S_n\}.$$

- CEF give a cool formula for computing the total value of a statistic F on $\mathrm{Conf}_n(\mathbb{F}_q)$.
- Let $\mathrm{PConf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j\}$ be the space of ordered configurations of n distinct points in $\mathbb{C}$.
- This gives an FI-space $\mathrm{PConf}(\mathbb{C})$. For each k we may compose $\mathrm{PConf}(\mathbb{C})$ with $H^k(-, \mathbb{Q})$ to get an FI-module.

$$\mathrm{FI} \xrightarrow{\mathrm{PConf}(\mathbb{C})} \mathrm{Man} \xrightarrow{H^k(-,\mathbb{Q})} \mathbb{Q}\text{-Vec}$$

- Let h_n^k be the character of $H^k(\mathrm{PConf}_n(\mathbb{C}), \mathbb{Q})$ as an S_n-representation. Then

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} F(f) = q^n \sum_{k=0}^{n} \langle F, h_n^k \rangle (-q)^{-k}$$

- This is the *Twisted Grothendieck-Lefschetz formula*. It shows that the answer to any such counting problem is given by a polynomial in q of degree at most n. Furthermore, the coefficients are integers representing the multiplicity of certain irreducible factors in $H^k(\mathrm{PConf}_n(\mathbb{C}), \mathbb{Q})$.
- Therefore, understanding the structure of the cohomology of $\mathrm{PConf}_n(\mathbb{C})$ as an S_n-representation is equivalent to understanding polynomial statistics over finite fields.
- For each k, the FI-module $H^k(\mathrm{PConf}(\mathbb{C}), \mathbb{Q})$ is known to be finitely generated, hence exhibit representation stability.
- The stability of these representations manifests as stability of polynomial statistics.

We can potentially use the twisted Grothendieck-Lefschetz formula in both directions. For example, it is well-known that there are $q^n - q^{n-1}$ monic, degree n, squarefree polynomials in $\mathbb{F}_q[x]$. This corresponds to the polynomial statistic 1, which is constant equal to 1 on $\mathrm{Conf}_n(\mathbb{F}_q)$.

- As a class function, this is the trivial character.
- Hence $\langle 1, h_n^k \rangle$ is the multiplicity of the trivial representation in $H^k(\mathrm{PConf}_n(\mathbb{C}), \mathbb{Q})$.
- Comparing both sides of the TGL, we see that

$$\langle 1, h_n^0 \rangle = 1$$
$$\langle 1, h_n^1 \rangle = 1$$
$$\langle 1, h_n^k \rangle = 0, \text{ for } k \geq 2.$$

Next we have an example in the other direction. Let QR and QI be the polynomial statistics "quadratic reducible factors" and "quadratic irreducible factors", let $QE = QR - QI$ be "quadratic excess". Then we can compute the total value of QE on $\mathrm{Conf}_n(\mathbb{F}_q)$ for some small values of n:

n	$\sum_{f\in \mathrm{Conf}_n(\mathbb{F}_q)} QE(f)$
5	$q^4 - 4q^3 + 5q^2 - 2q$
6	$q^5 - 4q^4 + 7q^3 - 7q^2 + 3q$
7	$q^6 - 4q^5 + 7q^4 - 8q^3 + 8q^2 - 4q$
8	$q^7 - 4q^6 + 7q^5 - 8q^4 + 9q^3 - 10q^2 + 4q$

Each coefficient corresponds to a different level of cohomology. That these values appear to be eventually constant is a consequence of the stability of $H^k(\mathrm{PConf}_n(\mathbb{C}), \mathbb{Q})$. Note that the point at which we stabilize depends on k.

5. Representation Stability

The notion of representation stability was introduced by Church and Farb in their 2010 paper *Representation theory and homological stability*.

- The notion of homological stability had been observed and studied for some time. This is a phenomenon where one has a sequence of topological spaces X_n with maps $X_n \to X_{n+1}$, if the induced maps on kth homology are eventually isomorphisms for each k, then we say the sequence of spaces is homologically stable.
- Of course, not every sequence of spaces with such maps will be homologically stable. However, in some cases, our spaces have more structure, like an action of a symmetric group. This action makes the homology into a group representation. One can then ask how these representations decompose to obtain more refined information.
- What Church and Farb observed is that in many cases, the structure of these decompositions "stabilizes" in the sense that the irreducibles and their multiplicities eventually fall into a family which does not change. (Elaborate with an example if time.)
- They describe these families by their corresponding partitions.
- Another observation about these representations is that their dimensions are eventually given by a single polynomial in n.
- These properties were taken as the definition of representation stability.

Soon after, Church, Ellenberg, and Farb introduced the concept of an FI-module to simplify the original notion of a family of representations of different symmetric groups with compatible maps.

- They demonstrated that any finitely generated FI-module will exhibit all the properties originally used to characterize representation stability.
- For example, the stability of the partitions associated to irreducible components can be seen as a consequence of Pieri's rule for computing the irreducible decomposition of an induced representation.
- Church, Ellenberg, and Farb proved a beautiful result, later improved with Ellenberg's student Nagpal: If R is a Noetherian ring, and V is a finitely generated FI-module over R, then every FI-submodule W of V is finitely generated.
- Finitely generated FI-modules are closed under many other natural properties. This provides a tool for proving a sequence of compatible S_n-representations is stable: prove the

corresponding FI-module is finitely generated by embedding it into another FI-module which we know to be finitely generated.

- Using this tool, they were able to prove many of the conjectures on representation stable sequences.
- From a different perspective, Snowden proved a result about Δ-modules from which one many also deduce their original Noetherian result.

The first definition of representation stability appears in the paper of Church and Farb [4]. The theory of FI-modules is introduced and developed in Church, Ellenberg, Farb [2]. Their results are extended to representations in positive characteristic with Nagpal in [3]. Another approach to these results on stability is through Δ-modules, beginning in the paper of Snowden [6]. The twisted Grothendieck-Lefschetz formula and applications to number theory appear in a follow-up paper of Church, Ellenberg, and Farb [1]. A recent survey on representation stability by Khomenko and Kesari [5] appeared on the *arXiv*, giving a good overview of the rapidly developing state of the theory.

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DEPT. OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043,
E-mail address: tghyde@umich.edu