Introduction to Intersection Theory

Preliminary Version July 2007

by

Günther Trautmann

Contents

0. Introduction

These notes are intended to provide an introduction to Intersection Theory and the algebraic theory of Chern classes. They grew out of several lectures on the subject in Kaiserslautern within the programme Mathematics International. It is supposed that the reader is familiar with the basic language of schemes and sheaves as presented in Harteshorne's book [9] or in sections of EGA.

Concerning the general Intersection Theory, the intention is to explain fundamental notions, definitions, results and some of the main constructions in Fulton's Intersection Theory [7] without trying to achieve an alternative approach. Often the reader is refered to [7] for a proof, when a statement has been made clear and the proof doesn't contain major gaps.

Besides the fundamentals of Intersection Theory, emphasis is given to the theory of Chern classes of vector bundles, related degeneracy classes and relative and classical Schubert varieties.

Most of the notation follows that of [7]. A scheme will always mean an algebraic scheme over a fixed field k, that is, a scheme of finite type over $\mathrm{Spec}(k)$. In particular, such schemes are noetherian. A variety will mean a reduced and irreducible scheme, and a subvariety of a scheme will always mean a closed subscheme which is a variety.

For a closed subscheme A of a scheme X we use the following notation. If $A \overset{i}{\hookrightarrow} X$ is the underlying continuous embedding, we identify the sheaves

$$i^*(\mathcal{O}_X/\mathcal{I}_A) = \mathcal{O}_A \quad \text{and} \quad i_*\mathcal{O}_A = \mathcal{O}_X/\mathcal{I}_A$$

such that we have an exact sequence

$$0 \to \mathcal{I}_A \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Given two closed subschemes A, B of X, the subscheme $A \cap B$ is defined by $\mathcal{I}_A + \mathcal{I}_B$ and there are isomorphisms

$$\mathcal{O}_{A\cap B} = \mathcal{O}_X/\mathcal{I}_A + \mathcal{I}_B \cong \mathcal{O}_X/\mathcal{I}_A \otimes \mathcal{O}_X/\mathcal{I}_B = \mathcal{O}_A \otimes \mathcal{O}_B.$$

1. Rational functions

Let U be an open subset of a scheme X and let Y be its complement. U is called s–dense (or schematically dense), if for any other open set V of X the restriction map

$$\Gamma(V, \mathcal{O}_X) \to \Gamma(V \cap U, \mathcal{O}_X)$$

is injective. Since the kernel is $\Gamma_Y(V, \mathcal{O}_X) = \Gamma(V, \mathcal{H}^0_Y\mathcal{O}_X)$, the condition is equivalent to $\mathcal{H}^0_Y\mathcal{O}_X = 0$, where $\mathcal{H}^0_Y$ denotes the subsheaf of germs supported on Y.

1.1. Lemma: *If U is s–dense it is also dense. The converse holds if X is reduced.*

Proof. Let U be s–dense and $V \neq \emptyset$. If $U \cap V = \emptyset$, then $\Gamma(V, \mathcal{O}_X) \to 0$ is not injective. Therefore $U \cap V \neq \emptyset$ and U is dense. Let conversely U be dense. In order to show that $\mathcal{H}^0_Y \mathcal{O}_X = 0$ we may assume that X is affine. Assume that there is a non–zero element $f \in \Gamma_Y(X, \mathcal{O}_X)$ with $f|U = 0$. Then $U \subset Z(f)$ and by density $X \subset Z(f)$. This implies $\mathrm{rad}(f) = \mathrm{rad}(0) = 0$ and then $f = 0$, contradiction. $\square$

1.2. Lemma: *Let X be an affine scheme and $g \in A(X)$. Then $D(g)$ is s–dense if and only if g is a NZD (non–zero divisor).*

Proof. Let g be a NZD. It is enough to show that $\Gamma(X, \mathcal{O}_X) \to \Gamma(D(g), \mathcal{O}_X)$ is injective. Let f be a section of $\mathcal{O}_X$ with $f \mid D(g) = 0$. Then there is an integer m with $g^m f = 0$. Since g is a NZD, $f = 0$. Conversely, if $D(g)$ is s–dense and $f \cdot g = 0$ in $A(X)$, then $f|D(g) = 0$ and by s–density $f = 0$. $\square$

1.3. Lemma: *Let X be an affine scheme and $g \in A(X)$. Then $D(g)$ is dense if and only if*

$$I(g) = \{a \in A(X) \mid g^m a = 0 \ \ \textit{for some } m > 0\}$$

is contained in the radical of 0.

Proof. If $I(g) \subset \mathrm{rad}(0)$ and $D(g)$ is not dense, there is an element $f \in A(X)$ such that

$$D(fg) = D(f) \cap D(g) = \emptyset$$

but $D(f) \neq \emptyset$. Then $fg \in \mathrm{rad}(0)$ or $f^m g^m = 0$ for some $m \geq 1$. Then $f^m \in I(g) \subset \mathrm{rad}(0)$ and $f^{mn} = 0$ for some $n \geq 1$. $\square$

The proof of the following Lemma is left to the reader.

1.4. Lemma: *Let X be a scheme and let $U, V \subset X$ be nonempty open parts.*

(i) *If U and V are dense (s–dense), then so is $U \cap V$.*
(ii) *If $U \subset V$ is dense (s–dense) in V, and V is dense (s–dense) in X, then U is dense (s–dense) in X.*

1.5. Lemma: *Let X be an affine scheme. Then the system of $D(g)$'s with g a NZD is cofinal with the system of all s–dense subsets, i.e. any open s–dense U contains a $D(g)$ for some NZD g.*

Proof. Let U be s–dense, $Y = X \smallsetminus U = V(\mathfrak{a})$. We have to show that there is a NZD $g \in \mathfrak{a}$. Then $D(g) \subset U$. Let $\mathfrak{p}_\nu = Ann(a_\nu)$ be the associated primes of $A(X)$, such that the set $ZD(A(X))$ of zero divisors is $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. If $\mathfrak{a} \subset ZD(A(X))$, then $\mathfrak{a} \subset \mathfrak{p}_\nu$ for some ν. Then $\mathfrak{a} \cdot a_\nu = 0$ and $a_\nu \mid U = 0$. Then $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ would not be injective. $\square$

1.6. Example: Let $X \subset \mathbb{A}^2_k$ be defined by the relations $xy = 0, y^2 = 0$ of the coordinate functions. So X is the affine line with an embedded point. The open set $D(x)$ is dense because $I(x) = (y) \subset \mathrm{rad}(0)$. But $D(x)$ is not s–dense because x is a ZD.

1.7. Example: Let $X \subset \mathbb{A}_k^2$ be the double line defined by $y^2 = 0$. Since $Z(y^2) = Z(y) = X$, we have $D(g) = \emptyset$. Now any $D(f) \neq 0$ is given by a NZD f because (y) is the set of zero divisors. Therefore in X the dense and s–dense open subsets coincide.

1.8. Definition: Let X be a scheme. Two pairs (f_1, U_1) and (f_2, U_2) of regular functions $f_\nu \in \mathcal{O}_X(U_\nu)$ on s–dense open sets U_ν are called s–equivalent if there is an s–dense open $U \subset U_1 \cap U_2$ such that $f_1|U = f_2|U$. We let $R_s(X)$ be the set of s–equivalence classes,

$$R_s(X) = \{[f, U]_s \mid f \in \mathcal{O}_X(U),\ U\, s\text{–dense}\}.$$

It is easy to see that $R_s(X)$ is a ring under the obvious definition of addition and multiplication. Similarly we define the ring $R(X)$ of rational functions using the usual dense open subsets.

$$R(X) = \{[f, U] \mid f \in \mathcal{O}_X(U),\ U\,\text{dense}\}$$

There is a natural ring homomorphism

$$R_s(X) \to R(X)$$

by $[f, U]_s \mapsto [f, U]$. If X is reduced, this is an isomorphism. Moreover, if $U \subset X$ is any open subset, we have natural restriction homomorphisms

$$R_s(X) \to R_s(U) \quad \text{and} \quad R(X) \to R(U).$$

We thus obtain presheaves of rational functions whose associated sheaves will be denoted by $\mathcal{R}_s$ and $\mathcal{R}$. There is a homomorphism $\mathcal{R}_s \to \mathcal{R}$, which is an isomorphism if X is reduced.

1.9. Lemma: *1) If U is dense, $R(X) \xrightarrow{\approx} R(U)$ is an isomorphism.*

2) If U is s-dense, $R_s(X) \xrightarrow{\approx} R_s(U)$ is an isomorphism.

Proof. only for 2). The map $[f, V]_s \to [f|U \cap V, U \cap V]_s$ is well-defined and injective. For, if there is an s–dense subset $W \subset U \cap V$ with $f|W = 0$, then W is also s–dense in X and so $[f, V]_s = 0$. Given $[f, V]_s$ with $V \subset U$ s–dense in U, V is also s–dense in X and $[f, V]_s$ is already in $R_s(X)$. □

1.10. Remark: If X is irreducible, the presheaf R is a sheaf, $R = \mathcal{R}$, and thus $\mathcal{R}(X) \to \mathcal{R}(U)$ is an isomorphism for any nonempty open subset U. So $\mathcal{R}$ is a simple sheaf in the sense of [1], 8.3.3 in this case.

1.11. Lemma: *If X is affine, then $R_s(X) \cong Q(A(X))$, the total ring of fractions of the coordinate ring.*

Proof. Let $Q = Q(A(X))$. We have a natural homomorphism $Q \to R_s(X)$ well defined by

$$\frac{f}{g} \mapsto \left[\frac{f}{g}, D(g)\right]_s$$

because for a NZD g as denominator, $D(g)$ is s–dense. This homomorphism is injective: if $f/g|U = 0$ as a function with $U \subset D(g)$ s–dense, there is a NZD h with $D(h) \subset U$, see 1.5. Because $D(h) \subset D(g), h^n = ag$ for some n, a. We get $h^p f = 0$ for some p, and then $f = 0$. Surjectivity: given $[\varphi, U]_s \in R_s(X)$, there is some $D(g) \subset U$ with g a NZD. Then $\varphi|D(g) \in A(X)_g$ and $\varphi|D(g) = f/g^m$ for some m, g. Now

$$\frac{f}{g^m} \mapsto [\varphi|D(g), D(g)]_s = [\varphi, U]_s. \quad \square$$

1.12. Remark: In general $R_s(X) \to R(X)$ is neither injective nor surjective. As an example consider the line $X \subset \mathbb{A}^2_k$ with embedded point as in 1.6. Let x, y be the generators of the coordinate ring $A(X)$ with relations $xy = 0, y^2 = 0$. Then $D(x)$ is dense but not s–dense. The element $[1/x, D(x)] \in R(X)$ is not in the image: Assume it is equal to some $[f/g, D(g)]$ with g a NZD. Then there is a dense subset $D(h) \subset D(xg)$ with $1/x = f/g$ in $A(X)_h$. Then there is an integer m with

$$h^m(xf - g) = 0.$$

But $g - xf$ is a NZD because the set of ZD of $A(X)$ is just the prime ideal (x, y). If $g - xf \in (x, y)$, then also $g \in (x, y)$, contradiction. Now $h = 0$ contradicting $D(h) \neq \emptyset$.

Now consider $[y, X]_s \in R_s(X)$. This is not 0. Otherwise there is a NZD $g \in A(X)$ and $y|D(g) = 0$ or $g^m y = 0$ for some m, and then $y = 0$. But $[y, X] = 0$ in $R(X)$, because $[y, X] = [y, D(x)] = 0$ since $D(x)$ is dense and $y|D(x) = 0$.

1.13. Lemma: *Let X be an integral scheme with generic point ξ. Then $R(X)$ is a field and isomorphic to $\mathcal{O}_{X,\xi}$.*

Proof. For any open affine subset $U \neq \emptyset$ we have $R(X) \xrightarrow{\approx} R(U) \xleftarrow{\approx} Q(\mathcal{O}_X(U))$ and $Q(\mathcal{O}_X(U))$ is a field since $\mathcal{O}_X(U)$ is a domain. On the other hand U is dense in X if and only if $\xi \in U$. By the definition of $R(X)$ we have $R(X) \cong \mathcal{O}_{X,\xi}$. $\square$

1.14. Examples:

(1) $R(\mathbb{A}^n_k) \cong k(x_1, \ldots, x_n)$ the field of rational functions in the indeterminants $x_1, \ldots, x_n$.

(2) $R(\mathbb{P}_{n,k}) \cong R(\mathbb{A}^n_k) \cong k(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$ where $x_0, \ldots, x_n$ are the standard homogeneous coordinates. We also have

$$R(\mathbb{P}_{n,k}) \cong \left\{ \frac{f}{g} \mid f, g \text{ homogeneous of the same degree with } g \neq 0 \right\}.$$

For that use

$$\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)} \longleftrightarrow \frac{f(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x0})}{g(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x0})}.$$

(3) $R(\mathbb{P}_{m,k} \times \mathbb{P}_{n,k}) \cong \left\{ \frac{f}{g} \mid f, g \text{ bihomogeneous of the same bidegree, } g \neq 0 \right\}$,

with $f = f(x_0, \ldots, x_m, y_0, \ldots, y_n)$ and $g = g(x_0, \ldots, x_m, y_0, \ldots, y_n)$.

(4) Let X be a reduced algebraic scheme over k and let $X_1, \dots, X_n$ be the irreducible components of X. Then

$$R(X) \cong \bigoplus_{\nu} R(X_\nu)\,.$$

Proof. $X'_\nu := X_\nu \smallsetminus \bigcup\limits_{\mu\neq\nu} X_\mu$ is dense in X_ν and $X' := \cup X'_\nu = \coprod X'_\nu$ is dense in X. Therefore

$$R(X) \cong R(X') \cong \bigoplus_{\nu} R(X'_n u) \cong \mathop{\oplus}_{\nu} R(X_\nu). \quad \square$$

1.15. The local ring of a subvariety

Let X be an algebraic scheme over k. A subvariety, i.e. an integral closed subscheme Y of X, has a unique generic point such that $Y = \overline{\{\eta\}}$. Therefore, for any open set $U \subset X$ we have $U \cap Y \neq \emptyset$ if and only if $\eta \in U$. In this case $U \cap Y$ is also dense in Y. It follows that

$$\mathcal{O}_{X,\eta} \cong \mathcal{O}_{Y,X} = \{[U,f] \mid f \in \mathcal{O}_X(U) \text{ and } U \cap Y \neq \emptyset\}.$$

Here the equivalence classes are defined as in the case of $R(X)$ under the additional assumption $\eta \in U$ for each representative. Similarly the maximal ideal $\mathfrak{m}_\eta$ of $\mathcal{O}_{X,\eta}$ can be described as

$$\mathfrak{m}_\eta \cong \mathfrak{m}_{Y,X} = \{[U,f] \mid f \in \mathcal{I}_Y(U) \text{ and } U \cap Y \neq \emptyset\}.$$

Note that $\mathcal{O}_{Y,X}$ is a noetherian local ring.

Lemma: *$\mathcal{O}_{Y,X}/\mathfrak{m}_{Y,X} \cong R(Y)$ for any subvariety $Y \subset X$.*

Proof. $[U,f] \mapsto [U \cap Y, \bar{f}]$ with $\bar{f} = f \mod \mathcal{I}_Y$ defines a homomorphism $\mathcal{O}_{Y,X} \to R(Y)$. It is surjective. To show this, let $[W,\varphi] \in R(Y)$ and choose an affine open subset U in X with $\emptyset \neq U \cap Y \subset W$. Then $[W,\varphi] = [U \cap Y, \varphi]$. Because $\Gamma(U, \mathcal{O}_X) \to \Gamma(U \cap Y, \mathcal{O}_Y)$ is surjective there is an element $f \in \mathcal{O}_X(U)$ with $\bar{f} = \varphi$. Now $[U,f] \mapsto [W,\varphi]$. On the other hand $\mathfrak{m}_{Y,X}$ is obviously the kernel of the homomorphism. $\square$

1.16. Dimension: Recall that the dimension of an algebraic scheme X can be characterized as the maximal length n of chains

$$\emptyset = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} \dots \underset{\neq}{\subset} V_n \subset X$$

of closed integral subschemes. If X is integral,

$$\dim X = \operatorname{trdeg}(R(X)/k).$$

If Y is a subvariety, the codimension $\operatorname{codim}_X Y$ is the maximum of integers d such that there is a chain

$$Y = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} \dots \underset{\neq}{\subset} V_d \subset X$$

of closed integral subschemes.

1.17. Lemma: *Let Y be a closed integral subscheme of X. Then for any open subset U of X with $U \cap Y \neq \emptyset$,*

$$\operatorname{codim}_U Y \cap U = \operatorname{codim}_X Y\,.$$

Proof. Given a chain $Y \cap U = Z_0 \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} Z_d \subset U$ of integral subschemes, also $\bar{Z}_\nu \cap U = Z_\nu$. Therefore

$$\operatorname{codim}_U Y \cap U \leq \operatorname{codim}_X Y.$$

On the other hand, if $Y = V_0 \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} V_d \subset X$ is a chain in X, then also $V_\nu \cap U \neq V_{\nu+1} \cap U$ because $V_\nu \cap U$ is dense in V_ν. Moreover $V_\nu \cap U$ is also integral. This implies that the two codimensions are equal. □

1.18. Lemma: *Let Y be an integral subscheme of X. Then*

$$\dim \mathcal{O}_{Y,X} = \operatorname{codim}_X Y.$$

If also X is integral, then $\mathcal{O}_{Y,X}$ is integral.

Proof. Let U be open and affine in $X, U \cap Y \neq \emptyset$. Let $A = \mathcal{O}_X(U)$ be the affine coordinate ring of U and $\mathfrak{p} \subset A$ the prime ideal of $Y \cap U$. Then

$$\mathcal{O}_{Y,X} \cong \mathcal{O}_{Y \cap U, U} \cong \mathcal{O}_{X,\eta} \cong A_{\mathfrak{p}}.$$

The prime ideals $\mathfrak{p}' \subset \mathfrak{p}$ correspond to varieties $U \supset Z' \supset Y \cap U$. Therefore the Krull dimension of $\mathcal{O}_{Y,X}$ equals the codimension of $Y \cap U$ in U or of Y in X. If X is also integral, then any $U \neq \emptyset$ is dense in X and we obtain a homomorphism $\mathcal{O}_{Y,X} \to R(X)$ by $[U, f]_Y \mapsto [U, f]$. This is injective. For, if $[U, f] = 0$, then $f|V = 0$ for some $V \subset U$. But $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is injective and hence $f = 0$. Since $R(X)$ is a field, $\mathcal{O}_{Y,X}$ has no zero divisors. □

1.19. Corollary: *If both Y and X are integral and Y has codimension 1, then $\mathcal{O}_{Y,X}$ is a 1–dimensional integral domain. Moreover,*

$$Q(\mathcal{O}_{Y,X}) \cong R(X).$$

Proof. It remains to verify the last statement. As in the previous proof we may assume that X is affine and Y corresponds to a prime ideal $\mathfrak{p} \subset A = A(X)$. Now $\mathcal{O}_{Y,X} = A_{\mathfrak{p}}$ and $R(X) = Q(A)$. By the assumption $A_{\mathfrak{p}} \subset Q(A)$ and it follows that $Q(A_{\mathfrak{p}}) = Q(A)$. □

1.20. Proposition: *Let $Y \subset X$ and both be integral with* $\operatorname{codim}_X Y = 1$. *If* $Y \not\subset \operatorname{Sing}(X)$, *then $\mathcal{O}_{Y,X}$ is a regular ring and a discrete valuation ring.*

Proof. The generic point η of Y is not in $\operatorname{Sing}(X)$ and therefore $\mathcal{O}_{X,\eta}$ is a regular ring. If U is an open affine subset of X with $\eta \in U \subset X \smallsetminus \operatorname{Sing}(X)$, then $U \cap Y$ is given by one equation in the smooth variety U, which is the generator of $\mathfrak{m}_{Y,X} = \mathfrak{m}_{Y \cap U, U} = \mathfrak{m}_\eta$. □

1.21. Corollary: *Let $Y \subset X$ be as in proposition 1.20 with $Y \not\subset \mathrm{Sing}(X)$, and let $r \in R(X)$. If $\mathrm{ord}_Y(r) \geq 0$ (see 3.3 for definition), then $r \in \mathcal{O}_{Y,X}$.*

Proof. We have $r = f/g$ with $f, g \in \mathcal{O}_{Y,X}$ and $f = ut^m, g = vt^n$ where u, v are units in $\mathcal{O}_{Y,X}$ and $\mathfrak{m}_{Y,X} = (t)$. Then

$$0 \leq \mathrm{ord}_Y(r) = \mathrm{ord}_Y(f) - \mathrm{ord}_Y(g) = m - n.$$

Therefore $r = uv^{-1}t^{m-n} \in \mathcal{O}_{Y,X}$. □

2. Meromorphic functions and divisors

Let X be an algebraic scheme over k and for an open set $U \subset X$ let

$$S(U) \subset \mathcal{O}_X(U)$$

be the subset of those f for which $f_x \in \mathcal{O}_{X,x}$ is a NZD for any point. Then S defines a subsheaf of $\mathcal{O}_X$ which is multiplicatively closed. If U is an affine open set, then $S(U)$ is the subset of NZD. For, if $A = \mathcal{O}_X(U)$ and $f \in A$ is a NZD, then $f_\mathfrak{p}$ is a NZD for any prime ideal $\mathfrak{p}$: Let $f \cdot (g/s) = 0$ with $s \notin \mathfrak{p}$. Then $tfg = 0$ for some $t \notin \mathfrak{p}$ and then $tg = 0$ or $g/s = 0$. Now we define the sheaf $\mathcal{M} = \mathcal{M}_X$ of meromorphic functions as the sheaf associated to the presheaf

$$U \mapsto M(U) = S(U)^{-1}\mathcal{O}_X(U).$$

This is a sheaf of $\mathcal{O}_X$–algebras. A reference for meromorhpic functions is [3] §20.

2.1. Lemma: *For any $x \in X$ the stalk $\mathcal{M}_x$ is the total ring of fractions of $\mathcal{O}_{X,x}$.*

Proof. Let $Q_x = Q(\mathcal{O}_{X,x})$ denote the total ring of fractions. For any open neighbourhood U of x let

$$M(U) \to Q_x$$

be defined by $f/g \mapsto f_x/g_x$. It is easy to see that this is well-defined and that it induces a homomorphism

$$\mathcal{M}_x \to Q_x.$$

It is also immediately verified that this map is bijective. □

2.2. Lemma: *$\mathcal{O}_X \hookrightarrow \mathcal{M}_X$ and $\mathcal{M}_X$ is a flat $\mathcal{O}_X$–module.*

Proof. The canonical homomorphism $\mathcal{O}_X \to \mathcal{M}_X$ is the embedding $\mathcal{O}_{X,x} \hookrightarrow Q(\mathcal{O}_{X,x})$ for any stalk. It is also well-known that $Q(\mathcal{O}_{X,x})$ is a flat $\mathcal{O}_{X,x}$–module for any x. □

2.3. Lemma: *On any scheme X there is a natural isomorphism $\mathcal{M}_X \xrightarrow{\approx} \mathcal{R}_s$.*

Proof. Let U be any open subset and $g \in S(U)$. Then the sheaf $\mathcal{H}^0_{Z(g)}(\mathcal{O}_U) = 0$ because g is a NZD at any point and $\mathcal{H}^0_{Z(g)}(\mathcal{O}_U)$ is annihilated by the powers of g locally. Therefore, $U_g = U \smallsetminus Z(g)$ is s–dense. Now the map $f/g \mapsto [U_g, f/g]_s$

$$S(U)^{-1}\mathcal{O}_X(U) \to R_s(U)$$

is well-defined and induces a sheaf homomorphism

$$\mathcal{M}_X \to \mathcal{R}_s.$$

For any affine open set U the composed homomorphism

$$S(U)^{-1}\mathcal{O}_X(U) \to R_s(U) \cong Q\big(\mathcal{O}_X(U)\big)$$

is the identity because $S(U)$ is then the system of NZD's. This proves that $\mathcal{M}_X \to \mathcal{R}_s$ is an isomorphism. □

2.4. Proposition: *Let X be integral and $U \subset X$ an affine open set. Then we have the commutative diagram*

$$\begin{tikzcd}
M(X) \arrow[rrr] \arrow[ddd] \arrow[dr] & & & R(X) \arrow[r, "\approx"] \arrow[dl, "1"', "\approx"] \arrow[ddd, "4"', "\approx"] & \mathcal{O}_{X,\xi} \\
& M(U) \arrow[r, "\approx", "3"'] \arrow[d, "\approx"', "\mu"] & R(U) \arrow[d, "4"', "\approx"] & & \\
& \mathcal{M}(U) \arrow[r, "\approx", "2"'] & \mathcal{R}(U) & & \\
\mathcal{M}(X) \arrow[ur, "\approx", "r"'] \arrow[rrr, "\approx", "2"'] & & & \mathcal{R}(X) \arrow[ul, "\approx"', "r"] &
\end{tikzcd}$$

of natural homomorphisms with indicated isomorphisms.

Proof. 1 is an isomorphism because U is dense. The arrows 2 are isomorphisms because $\mathcal{M} \simeq \mathcal{R}_s \simeq \mathcal{R}$. 3 is an isomorphism because U is affine and 4 are isomorphisms because $\mathcal{R}$ is a constant sheaf. If follows that r on $\mathcal{R}$ is an isomorphism as well as μ. □

A sheaf $\mathcal{A}$ of abelian groups on X is called **simple** if the restriction $\mathcal{A}(X) \to \mathcal{A}(U)$ is an isomorphism for any nonempty open subset U of X. It is shown in [1], Ch 0, 3.6.2, that $\mathcal{A}$ is already simple if it is locally simple. Any simple sheaf is also flabby.

2.5. Corollary: *If X is integral, then*

(i) *$\mathcal{M}_X$ is a simple and hence a flabby sheaf.*
(ii) *any stalk $\mathcal{M}_{X,x}$ and any $\mathcal{M}_X(U)$ for a nonempty open subset U of X is a field.*
(iii) *the sheaf $\mathcal{M}^*_X$ of invertible meromorphic functions is a simple and hence a flabby sheaf.*

Proof. (i) an arbitrary nonempty open subset U contains a nonempty open affine subset V, such that the composition $\mathcal{M}_X(X) \to \mathcal{M}_X(U) \to \mathcal{M}_X(V)$ is an isomorphism. Also the second restriction is an isomorphism because U is integral as well. Hence, the first restriction is an isomorphism, too.
(ii) follows from (i) because any $R(U)$ is a field.
(iii) $\mathcal{M}^*_X(X) \to \mathcal{M}^*_X(U)$ is injective because this is true for $\mathcal{M}$. Let $f \in \mathcal{M}^*(U)$ and let $F \in \mathcal{M}_X(X)$ with $F|U = f$. We have $(F|U)g = 1$ with $g = 1/f$. Let $G|U = g$. Then $FG|U = 1$. Because U is s–dense, $FG = 1$ on X. This proves that $\mathcal{M}^*_X(X) \to \mathcal{M}^*_X(U)$ is bijective. □

2.6. Remark: *Let $\mathcal{F}$ be a coherent sheaf on X, and X integral. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X \cong \mathcal{M}^r_X$ for some $r \geq 0$ (with $\mathcal{M}^0 = 0$), and the kernel of the canonical homomorphism $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X$ is the subsheaf $\mathcal{T}$ of $\mathcal{F}$ of all torsion elements.*
The number r is called the **rank** of $\mathcal{F}$.

Proof. (i) We use the abbreviations $\mathcal{M} = \mathcal{M}_X$, $\mathcal{O} = \mathcal{O}_X$, $\mathcal{F}_U = \mathcal{F}|U$ etc., and $\mathcal{M}(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X$. We first show that $\mathcal{M}(\mathcal{F})$ is locally simple, hence also globally simple. For that, notice, that any point of X has an affine open neighbourhood U with a presentation $\mathcal{O}^p_U \xrightarrow{F} \mathcal{O}^q_U \to \mathcal{F}_U \to 0$. After tensoring one obtains the exact sequence

$$\mathcal{M}^p_U \xrightarrow{F} \mathcal{M}^q_U \to \mathcal{M}(\mathcal{F})_U \to 0.$$

Let $r := q - \mathrm{rk}_{\mathcal{M}(U)}(F)$. Because $\mathcal{M}(U)$ is a field for any open U, the cokernel of F is an $\mathcal{M}(U)$–vector space of dimension r, so that we have an exact sequence

$$\mathcal{M}^p(U) \xrightarrow{F} \mathcal{M}^q(U) \to \mathcal{M}^r(U) \to 0.$$

It follows that $\mathcal{M}(\mathcal{F})_U \cong \mathcal{M}^r_U$, and then that $\mathcal{M}(\mathcal{F})_U$ is simple. In order to show that $\mathcal{M}(\mathcal{F}) = \mathcal{M}^r$ globally, we choose any open subset U_0 on which the two sheaves are isomorphic and consider the diagram

$$\begin{tikzcd}
\mathcal{M}(\mathcal{F})(U) \arrow[r, "\varphi(U)"] \arrow[d, "\approx"] & \mathcal{M}^r(U) \arrow[d, "\approx"] \\
\mathcal{M}(\mathcal{F})(U \cap U_0) \arrow[r, "\approx"] & \mathcal{M}^r(U \cap U_0)
\end{tikzcd}$$

for any other open subset U. It follows that the collection of the $\varphi(U)$ defines a global isomorphism $\mathcal{M}(\mathcal{F}) \cong \mathcal{M}^r$.

(ii) By defintion of the canonical homomorphism $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X$ each stalk $\mathcal{T}_x$ of the kernel consists of the germs t_x which are annihilated by some NZD $g_x \in \mathcal{O}_x$. Hence $\mathcal{T}$ is the subsheaf of all torsion germs of $\mathcal{F}$. If $r = 0$, then $\mathcal{T} = \mathcal{F}$. □

2.7. Cartier divisors and line bundles

Let $\mathcal{O}_X^* \subset \mathcal{O}_X$ and $\mathcal{M}_X^* \subset \mathcal{M}_X$ be the subsheaves of units of $\mathcal{O}_X$ and $\mathcal{M}_X$. For any open subset U of X we have

$$\begin{aligned} \mathcal{O}_X^*(U) &= \{f \in \mathcal{O}_X(U) | \quad f_x \text{ is a unit in } \mathcal{O}_{X,x} \text{ for any } x \in U\} \\ &= \{f \in \mathcal{O}_X(U) | \quad f \text{ is a unit in } \mathcal{O}_X(U)\}. \end{aligned}$$

and similarly for $\mathcal{M}_X^*$. The sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$ with multiplicative structure is called the sheaf of (Cartier–)divisors. We have the exact sequences

$$1 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{M}_X^*/\mathcal{O}_X^* \to 1$$

and

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X^*) \arrow[r] & \Gamma(X, \mathcal{M}_X^*) \arrow[r, "\mathrm{div}"] & \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*) \arrow[r, "\delta"] \arrow[d, equal] & H^1(X, \mathcal{O}_X^*) \arrow[d, equal] \\ & & \mathrm{Div}(X) & \mathrm{Pic}(X) \end{tikzcd}$$

Note here that for a sheaf $\mathcal{A}$ of abelian groups there is a canonical isomorphism between the first Čech cohomology group and the standard first cohomology group of $\mathcal{A}$, such that we have homomorphisms

$$H^1(\mathcal{U}, \mathcal{A}) \to \check{H}^1(X, \mathcal{A}) \simeq H^1(X, \mathcal{A})$$

for open coverings $\mathcal{U}$ compatible with refinements, see [9], Ch III, Ex. 4.4.

Any divisor $D \in \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ can be obtained by a system (f_α) of meromorphic functions $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ with $f_\alpha \mapsto D|U_\alpha$ for an open covering. This is the property of any quotient sheaf. Now $g_{\alpha\beta} = f_\alpha/f_\beta \in \mathcal{O}_X^*(U_{\alpha\beta})$ and $(g_{\alpha\beta})$ is a cocycle of a line bundle or invertible sheaf on X. Now $\delta(D)$ is the image of $[(g_{\alpha\beta})]$ under the canonical map

$$H^1(\mathcal{U}, \mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*).$$

The invertible sheaf can be directly defined as the $\mathcal{O}_X$–submodule

$$\mathcal{O}_X(D) \subset \mathcal{M}_X$$

which on U_α is generated by $1/f_\alpha$. It follows easily that $\mathcal{O}_X(D)$ is independent of the choice of the system (f_α) and the covering (U_α). Then $[(g_{\alpha\beta})] \leftrightarrow [\mathcal{O}_X(D)]$ under the isomorphism

$$H^1(X; \mathcal{O}_X^*) \cong \mathrm{Pic}(X).$$

2.8. Proposition: *1) The image of δ consists of the isomorphism classes of those invertible sheaves which are $\mathcal{O}_X$–submodules of $\mathcal{M}_X$.*

2) If X is integral, δ is surjective.

Proof. 1) Each $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$ submodule of $\mathcal{M}$. If conversely $\mathcal{L} \subset \mathcal{M}_X$, choose a trivializing covering (U_α) and let $g_\alpha : \mathcal{O}_X|U_\alpha \underset{\approx}{\to} \mathcal{L}|U_\alpha \hookrightarrow \mathcal{M}_X|U_\alpha$. Then g_α is a NZD at each point, because its homomorphism is injective, and $g_\alpha \in \Gamma(U_\alpha, \mathcal{M}_X^*)$. Let $f_\alpha = 1/g_\alpha$.

We have $f_\alpha = g_{\alpha\beta} f_\beta$ on $U_{\alpha\beta}$, where $(g_{\alpha\beta})$ is the cocycle of $\mathcal{L}$. Therefore (f_α) defines a divisor D on X, and $\mathcal{L} \cong \mathcal{O}_X(D)$ by definition.

2) As a simple sheaf $\mathcal{M}_X^*$ is flabby and hence $H^1(X, \mathcal{M}_X^*) = 0$. This implies that δ is surjective. $\square$

2.9. Effective divisors : Let X be a scheme and $D \in \Gamma(X, \mathcal{M}^*/\mathcal{O}^*)$ a Cartier divisor. D is called effective if it has a representing system (f_α) with $f_\alpha \in \Gamma(U_\alpha, \mathcal{O} \cap \mathcal{M}^*)$. Then any such system consists of regular functions. This means that the effective divisors are the sections of the sheaf $\mathcal{D}^+$ which is the image of $\mathcal{O} \cap \mathcal{M}^*$ in $\mathcal{M}^*/\mathcal{O}^*$. We write $D \geq 0$ if D is effective. If $D \geq 0$, then $\mathcal{O}(D)$ has a section, namely $1 \in \mathcal{M}(X)$, because locally $1 = f_\alpha(1/f_\alpha)$. This means that $\mathcal{O}_X \hookrightarrow \mathcal{M}_X$ factorizes through $\mathcal{O}_X(D)$. The section is also described by $f_\alpha = g_{\alpha\beta} f_\beta$. We thus have a homomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ and dually an ideal sheaf $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$. Its zero locus Z has the equation $f_\alpha = 0$ on U_α. For any Cartier divisor D there is the

$$\mathrm{Supp}(D) = |D| = \{x \in X | f_{\alpha x} \notin \mathcal{O}_{X,x}^* \ \text{ if } x \in U_\alpha\}$$

where (f_α) is a representing system. It is clear that this condition is independent of the choice of the system. If D is effective, then $|D|$ coincides with the zero locus Z of the canonical section of $\mathcal{O}_X(D)$, because $x \in Z$ if and only if $f_{\alpha x} \in \mathfrak{m}_x$.

3. Cycles and Weil divisors

If Y is a codimension 1 subvariety of a variety X, then $\mathcal{O}_{Y,X}$ is a local ring of dimension 1. If $Y \not\subset \mathrm{Sing}(X)$, then by 1.20 this ring is regular and $\mathfrak{m}_{Y,X} = (t)$ is generated by an element t. Then any $a \in \mathcal{O}_{Y,X}$ can uniquely be written as $a = ut^m$ with a unit u and $m \geq 0$. This defines an order function $R(X)^* = Q(\mathcal{O}_{Y,X}) \to \mathbb{Z}$ with $\mathrm{ord}(a/b) = \mathrm{exponent}(a) - \mathrm{exponent}(b)$. This order is the vanishing order of r along Y. If $\mathcal{O}_{Y,X}$ is not regular, the exponent can be replaced by the length of $\mathcal{O}_{Y,X}/a\mathcal{O}_{Y,X}$ for any $a \in \mathcal{O}_{Y,X}$, using the

3.1. Lemma: *Let A be a 1–dimension noetherian local integral domain. Then for any non–zero $a \in A$ the ring A/aA has finite length.*

Note that any noetherian A–module M has a composition series $M \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} \dots \underset{\neq}{\supset} M_k = 0$ with $M_i/M_{i+1} \cong A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is a prime ideal. If all the prime ideals equal the maximal ideal, M is said to have finite length k. In this case there are only finitely many prime ideals with $M_\mathfrak{p} \neq 0$ which are all maximal, and the number k is independent of the composition series. This number is called the length of M and denoted length (M) or $l(M)$.

Proof. Let $A/aA \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} \dots \underset{\neq}{\supset} M_k$ a composition series with $A/\mathfrak{p}_i \cong M_i/M_{i+1}$. Then $\mathfrak{p}_i \neq 0$. Otherwise there is a surjective homomorphism $M_i \to A$ with some $x_i \to 1$. Since

$M_i \subset A/aA, ax_i = 0$ and then $a = 0$. Because A is 1–dimensional, $\mathfrak{p}_i = \mathfrak{m}$ for each of the prime ideals. This proves the lemma. □

3.2. Lemma: *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A–modules with length(M'), length(M'') finite. Then M has finite length and*

$$\textit{length}\ (M) = \textit{length}\ (M') + \textit{length}\ (M'').$$

Proof. Consider composition series

$$M/M' \underset{\neq}{\supset} M_1/M' \underset{\neq}{\supset} \dots \underset{\neq}{\supset} M_k/M' = 0 \quad \text{and} \quad M' \underset{\neq}{\supset} M_{k+1} \underset{\neq}{\supset} \dots \underset{\neq}{\supset} M_{k+l} = 0.$$

Then $M_1 \underset{\neq}{\supset} \dots \underset{\neq}{\supset} M_{k+l}$ is a composition series of M. □

Let now A be as in 3.1 and $a \in A$. We define $\mathrm{ord}(a) := \mathrm{length}\ (A/aA)$. By 3.2 we obtain

$$\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$$

for two elements of A because there is the exact sequence

$$0 \to aA/abA \to A/abA \to A/aA \to 0$$

and $A/bA \cong aA/abA$. It follows from this formula that the order function

$$Q(A)^* \xrightarrow{\mathrm{ord}} \mathbb{Z}$$

given by

$$\mathrm{ord}(\frac{a}{b}) = \mathrm{ord}(a) - \mathrm{ord}(b)$$

is a well-defined homomorphism.

3.3. Vanishing order of rational functions and divisors

Let Y be a codimension 1 subvariety of a variety X, both integral by our convention. Then $\mathcal{O}_{Y,X}$ is a 1–dimensional local integral domain and $Q(\mathcal{O}_{Y,X}) \cong R(X)$. Therefore, we are given an order function

$$R(X)^* \underset{\mathrm{ord}_Y}{\longrightarrow} \mathbb{Z}$$

defined by $\mathrm{ord}_Y(\frac{f}{g}) = \mathrm{length}(\mathcal{O}_{Y,X}/f\mathcal{O}_{Y,X}) - \mathrm{length}(\mathcal{O}_{Y,X}/g\mathcal{O}_{Y,X})$ where $f, g \in \mathcal{O}_{Y,X}$.

We can as well write

$$\mathrm{ord}_\eta = \mathrm{ord}_Y$$

where η is the generic point of Y. Then

$$\mathrm{ord}_\eta\left(\frac{f}{g}\right) = \mathrm{length}(\mathcal{O}_{X,\eta}/f\mathcal{O}_{X,\eta}) - \mathrm{length}(\mathcal{O}_{X,\eta}/g\mathcal{O}_{X,\eta})$$

where f and g are germs in $\mathcal{O}_{X,\eta}$. For any rational function $r \in R(X)^*$ we can now define

$$\mathrm{cyc}(r) = \sum \mathrm{ord}_Y(r)Y,$$

the (finite, see 3.4 below) sum being taken over all 1–codimensional subvarieties.

Exercise: Let $C \subset \mathbb{A}_k^2$ be an integral curve and $F = Z(f), G = Z(g)$ two other curves which don't contain C as a component. Then $r = f/g$ has the order

$$\operatorname{ord}_p(r) = \mu(p, C, F) - \mu(p, C, G)$$

at any point, where μ denotes the intersection multiplicity, see 7.5, [7].

Let now $D \in \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ be a Cartier divisor and let $(f_\alpha), f_\alpha \in \mathcal{M}^*(U_\alpha)$ be a representing system of meromorphic functions. Then

$$\operatorname{ord}_Y(D) := \operatorname{ord}_\eta(f_{\alpha\eta}) = \operatorname{ord}_{Y \cap U_\alpha}(f_\alpha)$$

for $\eta \in U_\alpha$ is independent of α because different choices differ only by unit factors in $\mathcal{O}_X^*$ which have order 0. By definition

$$\operatorname{ord}_Y(f) = \operatorname{ord}_Y(\operatorname{div}(f))$$

where $f \in \mathcal{M}^*(X) = R(X)^*$ and $\operatorname{div}(f)$ is its image in $\Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$.

Lemma: Let X be an integral scheme. Then any non–zero divisor $D \in \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ has $|D| = \operatorname{Supp}(D) \neq X$.

Proof. Let (f_α) be a representing system and let $f_\alpha = a_\alpha/b_\alpha$ with $a_\alpha, b_\alpha \in \mathcal{O}_X(U_\alpha), U_\alpha$ affine, both NZD's. Then $|D| \cap U_\alpha \subset Z(a_\alpha) \cup Z(b_\alpha)$ because $x \notin Z(a_\alpha) \cup Z(b_\alpha)$ would imply that $a_{\alpha x}$ and $b_{\alpha x}$ are units in $\mathcal{O}_{X,x}$ and hence $f_{\alpha x} \in \mathcal{O}_{X,x}^*$ But $Z(a_\alpha) \cup Z(b_\alpha) \neq U_\alpha$, otherwise $\operatorname{rad}(a_\alpha b_\alpha) = (0)$ and $a_\alpha b_\alpha$ would be zero divisors. □

3.4. Associated cycles

We are now able to assign to any Cartier divisor on an integral scheme X a Weil–divisor. For that we denote by

$$Z_{n-1}(X)$$

the free Abelian group generated by the codimension 1 subvarieties of X, where $n = \dim X$. If $D \in \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$, then $\operatorname{ord}_Y(D) = 0$ for any $Y \not\subset \operatorname{Supp}(D)$, because then the generic point η of Y is not contained in $\operatorname{Supp}(D)$, i.e. $f_{\alpha\eta} \in \mathcal{O}_{X,\eta}^*$ and has order 0. Because $\operatorname{Supp}(D) \neq X$, there are only finitely many codimension 1 subvarieties $Y \subset \operatorname{Supp}(D)$, namely the components of $\operatorname{Supp}(D)$, for which $\operatorname{ord}_Y(D) \neq 0$. We put

$$\operatorname{cyc}(D) := \sum \operatorname{ord}_Y(D) Y \in Z_{n-1}(X).$$

We thus get a homomorphism

$$\Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*) = \operatorname{Div}(X) \xrightarrow{\operatorname{cyc}} Z_{n-1}(X).$$

For a rational function $r \in R(X)^* \cong \Gamma(X, \mathcal{M}_X^*)$ we take the composition and write

$$\operatorname{cyc}(r) = \operatorname{cyc}(\operatorname{div}(r)).$$

If D is an effective divisor, then $\operatorname{ord}_Y(D) \geq 0$ for any Y because the representing functions f_α are regular in this case. In this case the Weil–divisor $\operatorname{cyc}(D)$ is also called effective.

4. Chow groups $A_k(X)$

In the following X will always denote an algebraic scheme over a base field k and $Z_k(X)$ the group of k–cycles, the freely generated $\mathbb{Z}$–module over the set of integral subschemes (subvarieties) of dimension $k \geq 0$. If necessary to distinguish the subvariety $V \subset X$ from its basis element in $Z_k(X)$, we also write $[V]$. If $W \subset X$ is a $(k+1)$–dimensional subvariety and $r \in R(W)^*$ a rational function, we are given the cycle

$$\mathrm{cyc}(r) = \sum \mathrm{ord}_V(r) V \in Z_k(W) \subset Z_k(X)$$

A cycle $\alpha \in Z_k(X)$ is called rationally equivalent to 0, written as $\alpha \sim 0$, if $\alpha = 0$ or if there are finitely many $(k+1)$–dimensional subvarieties $W_1, \dots, W_n \subset X$ together with rational functions $r_\nu \in R(W_\nu)^*$ such that

$$\alpha = \sum_\nu \mathrm{cyc}(r_\nu).$$

These cycles form a subgroup $B_k(X) \subset Z_k(X)$. Note that $\mathrm{cyc}(r^{-1}) = -\mathrm{cyc}(r)$. We put

$$A_k(X) = Z_k(X)/B_k(X).$$

Since X and X_{red} have the same subvarieties, we have

$$A_k(X) = A_k(X_{\mathrm{red}})$$

for any k.

4.1. Example: Let $\mathbb{A}^n = \mathbb{A}^n_k$. Given an integral hypersurface $Y \subset \mathbb{A}^n$, we have $Y = Z(f)$ for a regular function (polynomial) f such that $Y = \mathrm{cyc}(f)$. Therefore $A_{n-1}(\mathbb{A}^n) = 0$. Similarly we have $A_0(\mathbb{A}^n) = 0$ by using a line as W through a given point. Later we will be able to show that $A_k(\mathbb{A}^n) = 0$ for all $k < n$. Clearly $A_k(\mathbb{A}^n) = 0$ for $k > n$. But $A_n(\mathbb{A}^n) \cong \mathbb{Z}$. Note that $[\mathbb{A}^n] \in Z_n(\mathbb{A}^n)$ must be a basis element, and is the only one. Therefore, $\mathbb{Z} \to Z_n(\mathbb{A}^n) = A_n(\mathbb{A}^n)$ is an isomorphism, $m \mapsto m[\mathbb{A}^n]$. We have $B_n(\mathbb{A}^n) = 0$ by definition. $[\mathbb{A}^n]$ is also called the fundamental class.

4.2. Example: If X is n–dimensional and $X_1, \dots, X_r$ are its irreducible n–dimensional components, then $\mathbb{Z}^r \cong A_n(X)$. This follows as in the case of $\mathbb{A}^n$ with $[X_1], \dots, [X_r]$ as a basis of $Z_n(X) = A_n(X)$.

4.3. Example: Let $\mathbb{P}_n = \mathbb{P}_{n,k}$. Since $\mathbb{P}_n$ is irreducible, $A_n(\mathbb{P}_n) \cong \mathbb{Z}$. However, also $A_{n-1}(\mathbb{P}_n) \cong \mathbb{Z}$. To verify this, let Y be an integral hypersurfaces, $Y = Z(g)$, of degree $d = \deg(g)$. Then $x_0^{-d} g$ is a rational function on $W = \mathbb{P}_n$ and

$$\mathrm{cyc}(x_0^{-d} g) = Y - dH_0$$

where H_0 is the hyperplane $x_0 = 0$. Therefore, $Y \sim dH_0$. It follows that $d \mapsto d[H_0]$ is a surjection $\mathbb{Z} \to A_{n-1}(\mathbb{P}_n)$. It is also injective: If $d[H_0] \sim 0$, there exist $r_1, \dots, r_m \in R(\mathbb{P}_n)^*$ such that

$$d[H_0] = \sum_\mu \mathrm{cyc}(r_\mu) = \mathrm{cyc}(r_1 \cdot \ldots \cdot r_m).$$

It follows that $r = r_1 \cdot \ldots \cdot r_m$ is a homogeneous polynomial of degree 0, and $d = 0$:

Let $r = f_1^{\mu_1} \cdots f_s^{\mu_s} / g_1^{\nu_1} \ldots g_t^{\nu_t}$ with irreducible forms f_σ, g_τ without common factor. Then

$$\operatorname{cyc}(r) = \sum \mu_\sigma Z(f_\sigma) - \sum \nu_\tau Z(g_\tau) = dH_0.$$

It follows that all $\nu_\tau = 0$ and all except one $\mu_\sigma = 0$. But

$$\sum \mu_\sigma \deg(f_\sigma) = \sum \nu_\tau \deg(g_\tau) = 0.$$

Then also the last $\mu_\sigma = 0$ and so $dH_0 = 0$ in $Z_{n-1}(\mathbb{P}_n)$. This implies $d = 0$.

With a similar argument using lines through two points we can show that $A_0(\mathbb{P}_n) \cong \mathbb{Z}$. It will be shown later that $A_k(\mathbb{P}_n) \cong \mathbb{Z}$ with generator $[\bar{E}], E$ any k–dimensional plane.

4.4. Example: Let $X \subset \mathbb{A}_k^3$ be the affine cone with equation $xy = z^2$, where x, y, z denote the residue classes of the coordinate functions as elements of $A(X) = k[x, y, z] = k[x, y, z]/(xy - z^2)$. Let

$$A = Z(x) = Z(x, z) \text{ and } B = Z(y) = Z(y, z)$$

The prime ideal of A respectively B is

$$\mathfrak{p} = (x, z) \text{ and } \mathfrak{q} = (y, z).$$

We also have $Z(z) = A \cup B$.

Claim 1: $\mathfrak{p}$ and $\mathfrak{q}$ are not principal.

Claim 2: $\operatorname{cyc}(x) = 2A, \operatorname{cyc}(y) = 2B, \operatorname{cyc}(z) = A + B$. Note that $A(X)$ ist not a UFD and that as rational functions we have $x/z = z/y$. Then $\operatorname{cyc}(x/z) = \operatorname{cyc}(z/y)$ mirrors $2A - (A + B) = (A + B) - 2B$.

Proof of Claim 1: Assume that $(x, z) = (f)$ for some $f \in A(X)$. Then $x = af, z = bf$ for some $a, b \in A(X)$, and therefore $bx - az = 0$. Now it is easy to prove that the relations of x and $-z$ are generated by the pairs (z, x) and (y, z). This implies $b = \alpha z + \beta y$ and $a = \alpha x + \beta z$. It follows that $x, z \in \mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of the origin of X. Similarly $y \in \mathfrak{m}^2$. Hence $\mathfrak{m} = \mathfrak{m}^2$ which is impossible.

Proof of Claim 2: Since $\mathfrak{p}$ is the generic point of A, we have

$$\mathcal{O}_{A,X} = \mathcal{O}_{X,\mathfrak{p}} \cong k[x, y, z]_{(x,z)}$$

and

$$\mathfrak{m}_{A,X} \cong (x, yz)k[x, y, z]_{(x,z)} = zk[x, y, z]_{(x,z)},$$

the last equality following from $xy = z^2$ with y a unit in the localized ring. We obtain the exact sequence

$$0 \to (x, z)/(x) \to \mathcal{O}_{A,X}/x\mathcal{O}_{A,X} \to \mathcal{O}_{A,X}/\mathfrak{m}_{A,X} \to 0$$

with an isomorphism $\mathcal{O}_{A,X}/\mathfrak{m}_{A,X} = \mathcal{O}_{A,X}/z\mathcal{O}_{A,X} \cong (x, z)/(x)$ because of $xy = z^2$. This proves that $\operatorname{ord}_A(x) = 2$. But $\operatorname{Supp}\, \operatorname{div}(x) = A$ and therefore $\operatorname{ord}_B(x) = 0$ for any other integral hypersurface. This proves $\operatorname{cyc}(x) = 2A$. By the same argument we obtain

$\mathrm{cyc}(y) = 2B$ and $\mathrm{cyc}(z) = A + B$. If $\bar{A}, \bar{B}$ denote the residue classes of A, B in $A_1(X)$, we have shown that

$$2\bar{A} = 0\ ,\ 2\bar{B} = 0\ ,\ \bar{A} + \bar{B} = 0.$$

We will show later that $\bar{A}$ generates $A_1(X)$ and that $\bar{A} \neq 0$. Then $\mathbb{Z}/2\mathbb{Z} \cong A_1(X)$. That $A_0(X) = 0$ and $A_2(X) \cong \mathbb{Z}$ can be shown as for $\mathbb{A}^n$.

4.5. Proposition: *Let Y be a closed subscheme of X. Then there is an exact sequence*

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(X \smallsetminus Y) \to 0$$

for any $k \geq 0$. The homomorphism i_* is induced by the inclusion $Y \overset{i}{\hookrightarrow} X$ and j^* is induced by restricting a subvariety V to $V \smallsetminus Y$, such that

$$j^*(\sum n_i V_i) = \sum_{V_i \not\subset Y} n_i (V_i \smallsetminus Y).$$

Proof. 1) Both maps are well-defined. For i_* this follows directly from the definition of $B_k(Y)$ and $B_k(X)$.

2) If $V \subset W$ are two subvarieties of X of dimension k and $k+1$ and if $\overline{\{\eta\}} = V \not\subset Y$, then $R(W) \cong R(W \smallsetminus Y) \cong Q(\mathcal{O}_{W,\eta})$, because $\eta \in W \smallsetminus Y$ and this set is open and dense in W. It follows from the definition of the order that then for any $r \in R(W)^*$ we have

$$\mathrm{ord}_V(r) = \mathrm{ord}_\eta(r) = \mathrm{ord}_{V \smallsetminus Y}(r|W \smallsetminus Y).$$

It follows that for any $W \subset X$ of dimension $k+1$ and any $r \in R(W)^*$

$$j^*\mathrm{cyc}(r) = j^* \sum \mathrm{ord}_V(r) V = \sum_{V \not\subset Y} \mathrm{ord}_V(r) V = \mathrm{cyc}(r|W \smallsetminus Y)$$

This proves $j^*(B_k(X) \subset B_k(X \smallsetminus Y)$.

3) If $V \subset X \smallsetminus Y$ is an integral subscheme of dimension k, then also its closure $\bar{V}$ in X is integral. It follows that j^* is surjective. To proves exactness, let $\alpha \in Z_k(X)$ and $j^*\alpha \sim 0$. Then

$$j^*\alpha = \sum_\nu \mathrm{cyc}(r_\nu)$$

with $r_\nu \in R(W_\nu)^*$ and $W_\nu \subset X \smallsetminus Y$ integral of dimension $k+1$. Let $\bar{W}_\nu$ be the closure in X. We have $R(\bar{W}_\nu) \cong R(W_\nu)$ and there are rational functions $\bar{r}_\nu \in R(\bar{W}_\nu)^*$ extending r_ν. As shown in 2), $j^*\mathrm{cyc}(\bar{r}_\nu) = \mathrm{cyc}(r_\nu)$.

Now $\beta = \alpha - \sum \mathrm{cyc}(\bar{r}_\nu)$ is a chain representing the class $\bar{\alpha}$ with $j^*\beta = 0$. This means that all the components of β are contained in Y and therefore $\bar{\alpha} = i_*\bar{\beta}$. $\square$

4.6. Example: Let $Y \subset \mathbb{P}_{n,k}$ be any reduced hypersurface of degree d. We then have the exact sequence

$$\begin{array}{ccccccc} A_{n-1}(Y) & \xrightarrow{i_*} & A_{n-1}(\mathbb{P}_n) & \longrightarrow & A_{n-1}(\mathbb{P}_n \smallsetminus Y) & \longrightarrow & 0 \\ \| \wr & & \| \wr & & & & \\ \mathbb{Z}^r & \xrightarrow{\alpha} & \mathbb{Z}\,. & & & & \end{array}$$

Let $Y_1, \dots, Y_r$ be the irreducible components of Y of degrees $d_1, \dots, d_r$. Since $i_*Y_\rho \sim d_\rho H$ for some hyperplane H, the map α is just $(n_1, \dots, n_r) \mapsto \sum n_\rho d_\rho$. It follows that

$$A_{n-1}(\mathbb{P}_n \smallsetminus Y) = \mathbb{Z}/(d_1, \dots, d_r)$$

where $(d_1, \dots, d_r)$ is the GCD of the degrees.

4.7. Example: Affine cone $X \subset \mathbb{A}^3_k$, continued. Let $A, B \subset X$ be the lines defined by $x = 0$ resp. $y = 0$. Then $X \smallsetminus B = \operatorname{Spec} k[x, y, z]_y$. Using the relation $xy = z^2$ we obtain isomorphisms

$$k[x, y, z]_< \cong k[y, y^{-1}, z] \cong k[y, z]_y.$$

Therefore $X \smallsetminus B \cong \mathbb{A}^2 \smallsetminus \mathbb{A}^1$. From 4.5 we have the exact sequence

$$A_1(\mathbb{A}^2) \to A_1(\mathbb{A}^2 \smallsetminus \mathbb{A}^1) \to 0$$

and therefore $A_1(X \smallsetminus B) = 0$. Again by 4.5 we have a surjection $\mathbb{Z} \cong A_1(B) \to A_1(X)$ given by $1 \leftrightarrow \bar{B} \to i_*\bar{B} = \bar{B}$ This proves that $\bar{B}$ or $\bar{A}$ generate $A_1(X)$. In order to show that $\bar{A} \neq 0$, assume that there is a rational function $r \in R(X)^*$ with $A = \operatorname{cyc}(r)$. Let $\mathfrak{p}$ be the prime ideal (x, z) of A in $A(X)$. Now $\operatorname{ord}_A(r) = 1$ and $\operatorname{ord}_Y(r) = 0$ for any integral curve $Y \subset X$ different from A.

Claim: $r \in \mathfrak{p} \subset A(X)$.

Proof: All local rings $\mathcal{O}_{Y,X} \cong \mathcal{O}_{X,\eta}$ are regular of dimension 1, hence discrete valuation rings. Let $(t) = \mathfrak{m}_\eta$. Then $r_\eta \in Q(\mathcal{O}_{X,\eta})$ can be written as $r_\eta = ut^m$ with a unit u in $\mathcal{O}_{X,\eta}$. Now $m = 1$ for $\eta = \mathfrak{p}$ and $m = 0$ for $\eta \neq \mathfrak{p}$. It follows that

$$r \in \cap A(X)_\mathfrak{q} \subset Q(A(X))$$

with the intersection taken over all prime ideals of height 1. But it is well known that this intersection equals $A(X)$. Since $A(X)_\mathfrak{p}/rA(X)_\mathfrak{p}$ has length 1, $r \in \mathfrak{p}$. □

Let now $g \in \mathfrak{p}$ be any element, $g \neq 0$. Then $\operatorname{ord}_A(g) \geq 1, \operatorname{ord}_Y(g) \geq 0$ for any other integral curve in X. Then the rational function g/r has $\operatorname{ord}_Y(g/r) \geq 0$ for any Y. By the same proof as for the claim we get $g/r = a \in A(X)$. Hence $g = ar$. Then $\mathfrak{p}$ would be a principal ideal $=(r)$, contradicting claim 1 of 4.4. This completes the proof of $A_1(X) \cong \mathbb{Z}/2\mathbb{Z}$.

4.8. Proposition: *Let X_1, X_2 be closed subschemes of X. Then for any $k \geq 0$ there is an exact sequence*

$$A_k(X_1 \cap X_2) \xrightarrow{a} A_k(X_1) \oplus A_k(X_2) \xrightarrow{b} A_k(X_1 \cup X_2) \to 0$$

Proof. 1) The mappings are induced by the natural mappings

$$Z_k(X_1 \cap X_2) \xrightarrow{a} Z_k(X_1) \oplus Z_k(X_2) \xrightarrow{b} Z_k(X_1 \cup X_2)$$

on the level of cycles, a as inclusion and b as difference of the inclusion. They are both well–define on the Chow groups.

2) If $\alpha \in Z_k(X_1 \cap X_2)$ then $a(\alpha) = \alpha \oplus \alpha$ and $b \circ a(\alpha) = \alpha - \alpha = 0$. So we have a complex.

3) b is surjective on the level of cycles and then surjective onto $A_k(X_1 \cup X_2)$: If $Y \subset X_1 \cup X_2$ is integral, then $Y \subset X_1$ or $Y \subset X_2$. Therefore, given a cycle α, it can be written as a difference $\alpha = \alpha_1 - \alpha_2$ with $\alpha_\nu \in Z_k(X_\nu)$.

4) Let now $\alpha_\nu \in Z_k(X_\nu)$ and $\alpha_1 - \alpha_2 \sim 0$ in $Z_k(X_1 \cup X_2)$. Then

$$\alpha_1 - \alpha_2 = \sum_\nu \operatorname{cyc}(r_\nu)$$

with $r_\nu \in R(W_\nu)^*, W_1, \ldots, W_n \subset X_1 \cup X_2$ integral of dimension $k+1$. We may assume that

$$W_1, \ldots, W_p \subset X_1 \ \text{ and } W_{p+1}, \ldots, W_n \not\subset X_1.$$

Then

$$\alpha := \alpha_1 - \sum_{\nu \leq p} \operatorname{cyc}(r_\nu) = \alpha_2 + \sum_{p < \nu} \operatorname{cyc}(r_\nu)$$

has all its components in $X_1 \cap X_2$: If Y is a component of the left hand side,then $Y \subset X_1$, by the choice of p. If $Y \not\subset X_2$, it cannot occur in the right hand side, because $W_\nu \subset X_2$ for $p < \nu$ and α_2 has its components in X_2. Now $[\alpha]$ is mapped to $([\alpha]_1, [\alpha]_2) = ([\alpha_1]_1, [\alpha_2]_2)$. □

5. Affine Bundles

For the affine space $\mathbb{A}^n = \mathbb{A}^n_k$ we distinguish the following groups of automorphisms

$$GL_n(k) \underset{\neq}{\subset} \operatorname{Aff}(\mathbb{A}^n) \underset{\neq}{\subset} \operatorname{Aut}(\mathbb{A}^n)$$

which are all different. For simplicity we assume that k is algebraically closed and that $\mathbb{A}^n$ and all its subschemes are determined by $\mathbb{A}^n(k) = k^n$ and its subscheme of closed points. The group $\operatorname{Aff}(\mathbb{A}^n)$ consists of all transformations

$$v \mapsto gv + \xi$$

with $g \in \mathrm{GL}_n(k)$ and $\xi \in k^n$. But $\operatorname{Aut}(\mathbb{A}^n)$ contains transformations which are not affine. For example $(x, y) \mapsto (y + f(x), x)$ with any polynomial f is an automorphism of k^n and defines an automorphism of $\mathbb{A}^n$.

5.1. Affine bundles*:* A morphism $E \xrightarrow{p} X$ of schemes over k is called a **general affine bundle** of rank n if each point of X admits an open neighbourhood U together with an isomorphism $E_U = p^{-1}(U) \to U \times_k \mathbb{A}^n$ which is compatible with the projections. It is locally trivial, but the coordinate transformations need not be affine in the fibres.

If in addition the local isomorphisms $E_U \to U \times_k \mathbb{A}^n$ can be chosen to be affine on the fibres or if the coordinate transformations are of the type

$$(x, v) \mapsto \big(x, g_{ij}(x)v + \xi_{ij}(x)\big),$$

using only the k–valued points, $E \xrightarrow{p} X$ is called an **affine bundle**. The cocycle condition then splits into the two conditions

$$g_{ij}g_{jk} = g_{ik} \quad \text{ and } \quad g_{ij}\xi_{jk} + \xi_{ij} = \xi_{ik},$$

where $U_{ij} \xrightarrow{g_{ij}} \mathrm{GL}_n(k)$ and $U_{ij} \xrightarrow{\xi_{ij}} k^n$. They are equivalent to the condition

$$\begin{pmatrix} g_{ij} & \xi_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_{jk} & \xi_{jk} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{ik} & \xi_{ik} \\ 0 & 1 \end{pmatrix}. \tag{B}$$

This means that together with $E \xrightarrow{p} X$ we are given two locally free sheaves $\mathcal{E}$ and $\mathcal{F}$, the first defined by the cocycle (g_{ij}) and the second defined by the cocycle (B), together with an extension sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O} \to 0.$$

We let $P(\mathcal{E})$ and $P(\mathcal{F})$ denote the associated projective bundles of lines with fibres $\mathbb{P}\mathcal{E}(x)$ and $\mathbb{P}\mathcal{F}(x)$ respectively, where $\mathcal{E}(x) = \mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$.

5.2. Lemma*:* $E \cong P(\mathcal{F}) \smallsetminus P(\mathcal{E})$.

Proof. We consider only the geometric points. Let (U_i) with (g_{ij}) and (ξ_{ij}) be the trivializing covering for E. We have the natural embedding $k^n \cong \mathbb{P}_n(k) \smallsetminus \mathbb{P}_{n-1}(k)$ given by $v \leftrightarrow \langle v, 1\rangle$, and the isomorphisms $U_i \times k^n \xrightarrow{\varphi_i} U_i \times (\mathbb{P}_n(k) \smallsetminus \mathbb{P}_{n-1}(k))$. Let $\alpha_{ij}(x,v) = (x, g_{ij}(x)v + \xi_{ij}(x))$, and let

$$a_{ij} = \begin{pmatrix} g_{ij} & \xi_{ij} \\ 0 & 1 \end{pmatrix} \quad \text{modulo } k^*$$

be the cocycle of $P(\mathcal{F}) \smallsetminus P(\mathcal{E})$. Then

$$\varphi_i \circ \alpha_{ij} = a_{ij} \circ \varphi_j$$

and therefore the system (φ_i) defines an isomorphism $E \cong P(\mathcal{F}) \smallsetminus P(\mathcal{E})$. $\square$

The system (ξ_{ij}) of translations of the cocycle of E can be interpreted as a cocycle in $Z^1(\mathcal{U}, \mathcal{E})$. Namely, if $\mathcal{E}|U_i \xrightarrow{\sigma_i} \mathcal{O}^n|U_i$ is the trivialization of $\mathcal{E}|U_i$ with $\sigma_i \circ \sigma_j^{-1} = g_{ij}$, and if $\zeta_{ij} = \sigma_i^{-1}\xi_{ij}$ over U_{ij}, we have

$$\zeta_{ij} + \zeta_{jk} = \zeta_{ik}.$$

Then (ζ_{ij}) defines a class in $H^1(\mathcal{U}, \mathcal{E}) \cong H^1(X, \mathcal{E})$ which corresponds to the extension class $[\mathcal{F}] \in \mathrm{Ext}^1(X, \mathcal{O}, \mathcal{E})$ under the canonical isomorphism between the two groups. The class of (ζ_{ij}) is zero if and only if there is a chain (ζ_i) with $\zeta_{ij} = \zeta_j - \zeta_i$ and if and only if the extension sequence splits. Indeed, if $\xi_i = \sigma_i\zeta_i$ in this case, we have

$$\xi_{ij} = g_{ij}\xi_j - \xi_i \tag{S}$$

and therefore

$$\begin{pmatrix} 1_n & -\zeta_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_{ij} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & \zeta_j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{ij} & \zeta_{ij} \\ 0 & 1 \end{pmatrix}.$$

which means that $\mathcal{F} \cong \mathcal{E} \oplus \mathcal{O}$. On the other hand, condition (S) means that the affine bundle with cocycle α_{ij} has a section s with local components ξ_i. But his in turn means that as an affine bundle E is isomorphic to the affine bundle associated to the bundle space $E - 0$ 0f $\mathcal{E}$: If $\gamma_{ij}(x,v) = (x, g_{ij}(x)v)$ and $\alpha_i(x,v) = (x, v + \xi_i, (x))$ we have

$$\alpha_{ij} = \alpha_i^{-1}\gamma_{ij}\alpha_j.$$

Altogether we have the

5.3. Lemma: *For an affine bundle $E \xrightarrow{p} X$ with associated extension sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O} \to 0$ the following conditions are equivalent.*

(1) *$\mathcal{F} \cong \mathcal{E} \oplus \mathcal{O}$ as an extension*
(2) *$E \cong P(\mathcal{F}) \setminus P(\mathcal{E})$ has a section*
(3) *E is a vector bundle*

Let now $E \xrightarrow{p} X$ be a general affine bundle of rank n. Then the fibres of p are isomorphic to the affine space $\mathbb{A}_k^n$ and for any k–dimensional subvariety Y of X we obtain a $(k+n)$–dimensional subvariety $p^{-1}(Y)$ of E. We thus obtain a homomorphism

$$Z_k(X) \xrightarrow{p^*} Z_{k+n}(E).$$

If $W \subset X$ is a $(k+1)$–dimensional subvariety and $r \in R(W)^*$, then $p^*\text{cyc}(r) = \text{cyc}(r \circ p)$ as can be easily verified. This implies that p^* is well-defined as a homomorphism

$$A_k(X) \xrightarrow{p^*} A_{k+n}(E).$$

If $E \xrightarrow{p} X$ is an affine bundle with $E \cong P(\mathcal{F}) \setminus P(\mathcal{E})$ we get a diagram

$$\begin{tikzcd} A_{k+n}(P(\mathcal{E})) \arrow[r] & A_{k+n}(P(\mathcal{F})) \arrow[r, "j^*"] & A_{k+n}(E) \arrow[r] & 0 \\ & A_k(X) \arrow[u, "p^*"] \arrow[ur, "p^*"'] & & \end{tikzcd}$$

5.4. Theorem: *For a general affine bundle $E \xrightarrow{p} X$ of rank n the homomorphism $A_k(X) \xrightarrow{p^*} A_{k+n}(E)$ is surjective for any $k \geq 0$.*

Proof. **Step 1**: We first check the simplest but essential case where X is affine and integral and $E = X \times_k \mathbb{A}_k^1$ and where $k = n-1, n = \dim X$.

We are going to show that for any integral subscheme $Y \subset E$ of dimension n there is a cycle $\xi \in Z_{n-1}(X)$ and a rational function $r \in R(E)^*$ such that

$$Y = p^*\xi + \text{cyc}(r).$$

Then $A_{n-1}(X) \to A_n(E)$ is surjective. To find ξ and r we distinguish the cases $\overline{p(Y)} = X$ or $\overline{p(Y)}$ a subvariety of dimension $n-1$. Because p is locally trivial with fibre $\mathbb{A}^1$, we have $n - 1 \leq \dim \overline{p(Y)} \leq n$.

If $n-1 = \dim \overline{p(Y)}$, we have $Y = p^{-1}\overline{p(Y)}$ because both are n–dimensional and irreducible. In this case there is nothing to prove. So we assume that $\overline{p(Y)} = X$ and Y dominates X. Let

$$\mathfrak{p} \subset A(E) = A(X)[t] \subset R(X)[t]$$

be the prime ideal of Y. Because $R(X)$ is a field, the ideal $\mathfrak{p}R(X)[t]$ is generated by an element $r \in R(X)[t]$. There is an element $0 \neq b \in A(X)$ such that

$$r = \frac{f}{b}$$

with $f \in \mathfrak{p}$. Then also $f = br$ generates $\mathfrak{p}R(X)[t]$ and we may assume that $r \in \mathfrak{p}$. Next we observe that

$$\mathfrak{p} \cap A(X) = 0.$$

Otherwise there would be an element $0 \neq f$ in $\mathfrak{p} \cap A(X)$ and then $p^*Z(f) \supset Z(\mathfrak{p}) = Y$ with $Z(f) \neq X$. We are going to show now that r vanishes along Y in order 1. This is equivalent to

$$\mathfrak{p}A(E)_{\mathfrak{p}} = rA(E)_{\mathfrak{p}},$$

because the quotient modulo $\mathfrak{p}A(E)_{\mathfrak{p}}$ has length 1. For that let $f/g \in \mathfrak{p}$ with $f \in \mathfrak{p}$ and $g \notin \mathfrak{p}$. Because r generates $\mathfrak{p}R(X)[t]$, $f = rh/b,\ 0 \neq b \in R(X),\ h \in A(E)$. Then $f/g \in rA(E)_{\mathfrak{p}}$. Now we have

$$\operatorname{cyc}(r) = Y + \sum n_i Y_i$$

with $Y_i \neq Y$. We show that no Y_i can dominate X. Assume that $\overline{p(Y_i)} = X$. Let as before $f \in \mathfrak{p}$ with $bf = hr, b \neq 0$. Since r is a regular function and $n_i \neq 0$, r vanishes along Y_i. Because b doesn't vanish identically on X, f vanishes along Y_i (use generic points). But this implies that $Y_i \subset Z(\mathfrak{p}) = Y$, contradicting $Y_i \neq Y$ as both are of the same dimension. Now $\dim \overline{p(Y_i)} = n-1$ and $Y_i = p^{-1}\overline{p(Y_i)}$. This finally proves

$$Y = -\sum n_i p^* \overline{p(Y_i)} + \operatorname{cyc}(r)$$

which completes step 1.

Step 2: If X is affine and integral and $k \leq n-1$ arbitrary, then $A_k(X) \to A_{k+1}(X \underset{k}{\times} \mathbb{A}^1_k)$ is still surjective. To see this, let $Y \subset E$ be integral of dimension $k+1$. If $\dim \overline{p(Y)} = k$ then $Y = p^{-1}\overline{p(Y)}$ and there is nothing to prove. If, however, $\dim \overline{p(Y)} = k+1$, we consider

$$\begin{array}{ccc} A_{k+1}(E_{\overline{p(Y)}}) & \longrightarrow & A_{k+1}(E) \\ \uparrow & & \uparrow \\ A_k\big(\overline{p(Y)}\big) & \longrightarrow & A_k(X) \end{array}$$

By step 1 there exists a k–chain $\eta \in Z_k(\overline{p(Y)}) \subset Z_k(X)$ with $[Y] = p^*(\eta)$.

Step 3: The theorem is true for integral affine X and $E = X \underset{k}{\times} \mathbb{A}^n_k$

Proof: By induction n. We have

$$E = X \underset{k}{\times} \mathbb{A}^{n-1}_k \underset{k}{\times} \mathbb{A}^1_k$$

and hence that p^* as the composition

$$A_k(X) \twoheadrightarrow A_k(X \underset{k}{\times} \mathbb{A}^{n-1}_k) \twoheadrightarrow \mathbb{A}_k(X \underset{k}{\times} \mathbb{A}^n_k)$$

of two surjective maps is surjective.

Step 4: The theorem is true for any affine X and $X \underset{k}{\times} \mathbb{A}^n_k$.

Proof: If $X = X_1 \cup X'$ is a decomposition with X_1 irreducible we use the diagram

$$\begin{tikzcd} A_{k+1}(E_1) \bigoplus A_{k+n}(E') \arrow[r, two heads] & A_{k+n}(E) \\ A_k(X_1) \bigoplus A_k(X') \arrow[r, two heads] \arrow[u, two heads] & A_k(X) \arrow[u] \end{tikzcd}$$

where E_1 resp. E' are the restrictions of the bundle E to the components and do induction on the number of components.

Step 5: The theorem is true in general. We do induction on the dimension of X. We may assume that the theorem is true for $\dim X < m$. If $\dim X = m$, we may assume that X is irreducible by step 4. Then choose an affine open set $U \subset X$ such that $E_U \cong U \underset{k}{\times} \mathbb{A}^n_k$. Let $Z = X \smallsetminus U$. Then $\dim Z < m$. The exact diagram

$$\begin{tikzcd} A_{k+n}(E_Z) \arrow[r] & A_{k+n}(E) \arrow[r] & A_{k+n}(E_U) \arrow[r] & 0 \\ A_k(Z) \arrow[r] \arrow[u, "p_Z^*"'] & A_k(X) \arrow[r] \arrow[u, "p^*"'] & A_k(U) \arrow[r] \arrow[u, "p_U^*"'] & 0 \end{tikzcd}$$

gives the result for X by the surjectivity of p_Z^* and p_U^* □

5.5. Remark: We shall see later that for a rank n vector bundle $E \xrightarrow{p} X$ all the maps $A_k(X) \xrightarrow{p^*} A_{k+n}(E)$ are isomorphisms. In particular

$$A_k(X) \to A_{k+n}(X \underset{k}{\times} \mathbb{A}^n_k)$$

are isomorphisms. All this follows from the existence of a section $X \xrightarrow{i} X \underset{k}{\times} \mathbb{A}^n$ which gives rise to a diagram

$$\begin{tikzcd} A_k(X) \arrow[dr, equal] & & A_{k+n}(X \underset{k}{\times} \mathbb{A}^n_k) \arrow[ll, "i^*"'] \\ & A_k(X) \arrow[ur, two heads] & \end{tikzcd}$$

5.6. Remark: If $E \xrightarrow{p} X$ is a locally trivial fibration with typical fibre an open set $U \subset \mathbb{A}^n_k$, then also $A_k(X) \to A_{k+n}(E)$ is surjective for any k. This can be shown with a similar proof.

6. Examples

In this section theorem 5.4 will be applied to get information about the Chow groups of affine and projective spaces, of Grassmannians and more generally of cellular varieties. All schemes will be defined over k.

6.1. Proposition: *Let U be a nonempty open set of $\mathbb{A}^n$. Then $A_k(U) = 0$ for $k < n$ and $A_n(U) \cong \mathbb{Z}$.*

Proof. By induction n. For $n = 1$ this is known. For $n \geq 2$ there is a projection $\mathbb{A}^n \to \mathbb{A}^{n-1}$ with fibre $\mathbb{A}^1$. It follows that $A_{k-1}(\mathbb{A}^{n-1}) \to A_k(\mathbb{A}^n)$ is surjective for $1 \leq k \leq n$. If $k < n$, the groups are zero. If $k = n$, both groups are isomorphic to $\mathbb{Z}$. It had already been shown that $A_0(\mathbb{A}^n) = 0$. If U is an open set of $\mathbb{A}^n$, we have a surjection $A_k(\mathbb{A}^n) \to A_k(U)$ for any k. $\square$

6.2. Proposition: *$A_k(\mathbb{P}_n) \cong \mathbb{Z}$ for any $0 \leq k \leq n$ and this group is generated by the class $[H_k]$ of any k–plane $H_k \subset \mathbb{P}_n$.*

Proof. The result had been shown for $k = n - 1$ and n. We proceed by induction on n with $k < n$. Let $H \subset \mathbb{P}_n$ be any hyperplane. We have the exact sequence

$$\begin{tikzcd}
A_k(H) \arrow[r] & A_k(\mathbb{P}_n) \arrow[r] & A_k(\mathbb{P}_n \setminus H) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z} \arrow[u, "\approx"] \arrow[ur] & & 0 &
\end{tikzcd}$$

with $H \cong \mathbb{P}_{n-1}$ and $\mathbb{P}_n \setminus H \cong \mathbb{A}^n$. Then $A_k(\mathbb{P}_n)$ for $k < n$ is generated by the class of any k–plane H_k. It remains to show that $\mathbb{Z} \to A_k(\mathbb{P}_n)$ is injective. Let $d\mathbb{Z}$ be the kernel. There are $(k+1)$–dimensional subvarieties V_μ and rational functions $r_\mu \in R(V_\mu)^*$ such that

$$dH_k = \sum_\mu \mathrm{cyc}(r_\mu).$$

Let $Z = V_1 \cup \ldots \cup V_m$. There is a linear subspace L of dimension $n - k - 2$ such that $L \cap Z = \emptyset$. (If $k = n = 1$ there is nothing to prove). If $d \neq 0$, the formula implies that $H_k \subset Z$. Now $Z \subset \mathbb{P}_n \setminus L$ and there is the central projection

$$\pi : Z \to \mathbb{P}_n \setminus L \to \mathbb{P}_{k+1}$$

as composition. The morphism $Z \xrightarrow{\pi} \mathbb{P}_{k+1}$ is proper with finite fibres. Because $H_k \cap L = \emptyset$ we find that $\pi(H_k) = H'_k$ is a k–plane in $\mathbb{P}_{k+1}$, with $H_k \underset{\approx}{\to} H'_k$.

By 7.4 and 7.1 we have $\pi_*[H_k] = [H'_k]$ and $d[H'_k] = 0$. But from $\mathbb{Z} \xrightarrow{\approx} A_k(\mathbb{P}_{k+1})$ we conclude that $d = 0$. $\square$

6.3. Question: Let $S \subset \mathbb{P}_n$ be a hypersurface with components $S_1, \ldots, S_r$ of degrees $d_1, \ldots, d_r$. We had shown in 4.6 that $A_{n-1}(\mathbb{P}_n \setminus S)$ is isomorphic to $\mathbb{Z}/(d_1, \ldots, d_r)$. What can be said about $A_k(\mathbb{P}_n \setminus S)$ for $k < n - 1$? As an example let $S \subset \mathbb{P}_3$ be a quadric surface, $S \cong \mathbb{P}_1 \times \mathbb{P}_1$. We shall see later that $A_1(S) \cong \mathbb{Z} \times \mathbb{Z}$ with generators the classes of a line in each system of lines in S. Then the homomorphism $A_1(S) \to A_1(\mathbb{P}_3)$ is given as $\mathbb{Z}^2 \to \mathbb{Z}$ by $(a, b) \mapsto a + b$. Therefore, the cokernel $A_1(\mathbb{P}_3 \setminus S) = 0$.

6.4. Projective cones: Let $\mathbb{P}_{n+1} \smallsetminus \{pt\} \xrightarrow{\pi} \mathbb{P}_n$ be the central projection from a point, which is a line bundle. If $Y \subset \mathbb{P}_n$ is a subvariety, let X be the closure of $\pi^{-1}(Y)$. Then $X \smallsetminus \{pt\} \to Y$ is also a line bundle and we have isomorphisms

$$A_k(Y) \underset{\approx}{\longrightarrow} A_{k+1}(X \smallsetminus pt) \cong A_{k+1}(X).$$

Let in particular $X \subset \mathbb{P}_3$ be the subvariety by $x_2^2 - x_0x_1 = 0$. It is a cone over the smooth conic $\{x_2^2 - x_0x_1 = 0\} \cap \{x_3 = 0\} = C$. Because $C \cong \mathbb{P}_1$ we have isomorphisms

$$\mathbb{Z} \cong A_0(C) \cong A_1(X \smallsetminus \{p\}) \cong A_1(X)$$

where $p = \langle 0, 0, 0, 1\rangle$. We have $C \subset X$ and the exact sequence

$$\begin{array}{ccccccc} A_1(C) & \longrightarrow & A_1(X) & \longrightarrow & A_1(X \smallsetminus C) & \longrightarrow & 0 \\ \| \wr & & \| \wr & & & & \\ \mathbb{Z} & \xrightarrow{h} & \mathbb{Z}\,. & & & & \end{array}$$

Let $L = \overline{\pi^{-1}(p)}$ be one of the lines of X. Then $[L]$ is the generator of $A_1(X)$. The zero scheme of x_0 in X is the union of two lines $L_1, L_2 \subset X$. Now the cycle

$$0 \sim \operatorname{cyc}(x_0/x_3) = L_1 + L_2 - C$$

and we get $[C] = [L_1] + [L_2] = 2[L]$.

Therefore $h(a) = 2a$ and $A_1(X \smallsetminus C) \cong \mathbb{Z}/2\mathbb{Z}$.

6.5. Cellular varieties: As we have already realized, it is often easier to determine generators of the Chow groups $A_k(X)$ but more difficult to determine the relations. Generators can also easily be found for so-called cellular varieties. These are varieties X with a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

by closed reduced subschemes such that

$$X_\nu \smallsetminus X_{\nu-1} = \coprod_\mu U_{\nu\mu}$$

with $U_{\nu\mu} \cong \mathbb{A}^{n_{\nu\mu}}$ or more generally $U_{\nu\mu}$ open in some affine space. Let $Z_{\nu\mu} = \bar{U}_{\nu\mu}$ the closure. Then the classes $[Z_{\nu\mu}]$ generate the group

$$A_*(X) = \bigoplus_{k \geq 0} A_k(X).$$

The proof follows by induction from the graded exact sequence

$$A_*(X_{\nu-1}) \to A_*(X_\nu) \to A_*(X_\nu \smallsetminus X_{\nu-1}) \to 0.$$

Let us consider the special case with X_ν of pure dimension ν. Then we have

$$\begin{array}{ccccccc} A_n(X_{n-1}) & \longrightarrow & A_n(X) & \underset{\approx}{\longrightarrow} & A_n(X \smallsetminus X_{n-1}) & \longrightarrow & 0. \\ \| & & & & & & \\ 0\,. & & & & & & \end{array}$$

Now $A_n(X \smallsetminus X_{n-1})$ is generated by the open fundamental cycles $U_{n\mu}$ and then $A_n(X)$ is generated by the closures $Z_{n\mu}$. In this case we even have $A_n(X) = \mathbb{Z}^{p_n}$ where p_n is the number of the $U_{n\mu}$. Next we have the exact sequence

$$\begin{array}{ccccccc} A_k(X_{n-1}) & \longrightarrow & A_k(X) & \longrightarrow & A_k(X \smallsetminus X_{n-1}) & \longrightarrow & 0 \\ & & & & \| & & \\ & & & & 0\,. & & \end{array}$$

for $k < n$. By induction we may assume that $A_k(X_{n-1})$ is generated by the classes $[Z_{k\mu}]$, which then also generate $A_k(X)$. Note that $\mathbb{P}_n$ is a cellular variety of this type.

6.6. The Grassmannian $\mathbf{G}_{2,4}$: Let $G = G(2, V) \subset \mathbb{P}\Lambda^2 V$ be the Grassmannian of 2–dimensional subspaces of a 4–dimensional k–vector space V. Let $e_0, \ldots, e_3$ be a basis of V with induced basis $e_i \wedge e_j$ of $\Lambda^2 V$. Let $p_{01}, \ldots, p_{23}$ be the dual basis for $\Lambda^2 V^*$, also called Plücker coordinates. Then $G \subset \mathbb{P}\Lambda^2 V$ is given by the non–degenerate quadratic equation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0 \tag{$*$}$$

($\xi \in \Lambda^2 V$ is decomposable if and only if $\xi \wedge \xi = 0$ in $\Lambda^4 V$).

Let $Q \subset G$ be the hyperplane section given by $p_{01} = 0$. Then Q is the set of all lines in $\mathbb{P}V = \mathbb{P}_3$ meeting the line $\mathbb{P}\langle e_2, e_3\rangle \leftrightarrow \langle e_2 \wedge e_3\rangle$. Now $G \smallsetminus Q \cong \mathbb{A}^4$ is an affine chart of G with local coordinates $p_{02}/p_{01}, p_{03}/p_{01}, p_{12}/p_{01}, p_{13}/p_{01}$ (p_{23} is determined by $(*)$).

Next we consider α–planes and β–planes (classical names). Let $P_\alpha \subset G$ be the set of all lines through $\langle e_3\rangle$. It is determined by the equations $p_{01} = p_{02} = p_{12} = 0$ and hence $P_\alpha \cong \mathbb{P}_2$. Dually we have the set $P_\beta \subset G$ of all lines contained in the plane $H = \mathbb{P}\langle e_1, e_2, e_3\rangle$ spanned by $\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle$. It has the equations $p_{01} = p_{02} = p_{03} = 0$ and so $P_\beta \cong \mathbb{P}_2$. Now $P_\alpha \cup P_\beta$ is determined by the equations $p_{01} = p_{02} = 0$. It follows that $P_\alpha \cup P_\beta \subset Q$ and

$$Q \smallsetminus P_\alpha \cup P_\beta \cong \mathbb{A}^3$$

with local coordinates $p_{12}/p_{02}, p_{03}/p_{02}, p_{23}/p_{02}$. Finally $P_\alpha \cap P_\beta = L_{\alpha\beta}$ is the set of lines in the plane H through $\langle e_3\rangle$. It is isomorphic to $\mathbb{P}_1$ by intersecting each line $l \in L_{\alpha\beta}$ with the line $\mathbb{P}\langle e_1, e_2\rangle \subset H$. We have the open sets

$$U_\alpha = P_\alpha \smallsetminus L_{\alpha\beta} \cong \mathbb{A}^2 \;\text{ and }\; U_\beta = P_\beta \smallsetminus L_{\alpha\beta} \cong \mathbb{A}^2$$

and we have

$$P_\alpha \cup P_\beta \smallsetminus L_{\alpha\beta} = U_\alpha \dot{\cup} U_\beta.$$

Finally, there is the point $p = \langle e_2 \wedge e_3\rangle \in L_{\alpha\beta}$ and $L_{\alpha\beta} \smallsetminus \{p\} \cong \mathbb{A}^1$. Altogether we have the filtration

$$G \supset Q \supset P_\alpha \cup P_\beta \supset L_{\alpha\beta} \supset \{p\}$$

with

$$G \smallsetminus Q \cong \mathbb{A}^4, \quad Q \smallsetminus P_\alpha \cup P_\beta \cong \mathbb{A}^3, \quad P_\alpha \cup P_\beta \smallsetminus L_{\alpha\beta} \cong \mathbb{A}^2 \dot{\cup} \mathbb{A}^2, \quad L_{\alpha\beta} \smallsetminus \{p\} \cong \mathbb{A}^1.$$

By the procedure above we find:

$$\begin{array}{lll} [G] & \text{generates} & A_4(G) \\ [Q] & \text{generates} & A_3(G) \\ [P_\alpha], [P_\beta] & \text{generates} & A_2(G) \\ [L_{\alpha\beta}] & \text{generates} & A_1(G) \\ [p] & \text{generates} & A_0(G)\,. \end{array}$$

The subvarieties $Q, P_\alpha, P_\beta, L_{\alpha\beta}$ are the classical Schubert cycles in this case. One can even prove that

$$A_4(G) \cong \mathbb{Z},\ A_3(G) \cong \mathbb{Z},\ A_2(G) \cong \mathbb{Z}^2,\ A_1(G) \cong \mathbb{Z},\ A_0(G) \cong \mathbb{Z}$$

with the above generators.

6.7. Künneth map: Let X and Y be two algebraic schemes over k. If $V \subset X$ and $W \subset Y$ are subvarieties of dimension i and j respectively, then $V \times W$ is one of $X \times Y$ of dimension $i + j$. Then

$$([V], [W]) \mapsto [V \times W]$$

defines a homomorphism

$$Z_i(X) \otimes Z_j(Y) \xrightarrow{\times} Z_{i+j}(X \times Y)$$

also called Künneth homomorphism.

If $\alpha \sim 0$ in $Z_i(X)$ and $\beta \sim 0$ in $Z_j(Y)$, it follows from 7.16 below that then also $\alpha \times \beta \sim 0$. Thus we are given homomorphisms

$$S_k(X, Y) = \bigoplus_{i+j=k} A_i(X) \otimes A_j(Y) \to A_k(X \times Y)$$

for any k. It is an easy exercise to show that this homomorphism is surjective if X is cellular.

7. Push forward and pull–back

It is not clear how to define push forward of cycles for general morphisms. Proper morphisms allow this in an easy way. We refer to Hartshorne's book II, §4 for proper morphisms. A morphism $X \xrightarrow{f} Y$ of schemes is called proper it it is separated, of finite type and universally closed. The following rules are useful:

(a) closed immersions are proper
(b) projective morphisms are proper
(c) properness is stable under base extension
(d) products $f \times g$ of proper morphisms f and g are proper
(e) compositions of proper morphism are proper
(f) If $f \circ g$ is defined and proper and if g is separated, then f is proper
(g) properness is a local property with respect to the base space.

An algebraic scheme/k is called complete if $X \to \operatorname{Spec}(k)$ is proper.

If $X \xrightarrow{f} Y$ is a proper morphism and $V \subset X$ an integral subscheme with $W = f(V)$, then we have

$$\begin{tikzcd}
R(W) \arrow[r, hook, "f^*"] & R(V) \\
\mathcal{O}_{W,Y/\mathfrak{m}_{W,Y}} \arrow[u, "\approx"'] \arrow[r, hook] & \mathcal{O}_{V,X/\mathfrak{m}_{V,X}} \arrow[u, "\approx"'] \\
\mathcal{O}_{W,Y} \arrow[u] \arrow[r, hook, "f^*"] & \mathcal{O}_{V,X} \arrow[u]
\end{tikzcd}$$

and $R(V)$ is a field extension of $R(W)$. If V and W have equal dimension, there is an open dense subset $W' \subset W$ over which $f_V = f|V$ has finite fibres. Since also f_V is proper, f_V is finite, see [11], prop. 6.25. The open set W' can be chosen to be affine. Then also $V' = f_V^{-1}(W')$ is affine and $A(W') \hookrightarrow A(V')$ is a finite integral extension and therefore $R(W) = R(W') \hookrightarrow R(V') = R(V)$ is a finite field extension. The degree of this extension is used to define the multiplicity of W in $Z_k(Y)$ when $k = \dim V = \dim W$.

7.1. Let $X \xrightarrow{f} Y$ be a proper morphism and $k \geq 0$. The homomorphism

$$Z_k(X) \xrightarrow{f_*} Z_k(Y)$$

is defined by

$$f_*V = \begin{cases} 0 & \text{if } \dim f(V) < k \\ \deg(V/f(V)) \cdot f(V) & \text{if } \dim f(V) = k \end{cases}$$

where $\deg(V/f(V)) = \deg(R(V) : R(f(V))$ and V is a subvariety of X of dimension k.

7.2. Let $Q \subset \mathbb{P}_n$ be a nonsingular quadric with equation $x_0^2 + \ldots + x_n^2 = 0$ and let $p = \langle 1, 0, \ldots, 0 \rangle$. Then $p \notin Q$ and the composition $f : Q \subset \mathbb{P}_n \smallsetminus \{p\} \xrightarrow{\pi} \mathbb{P}_{n-1}$ of the inclusion and the central projection is a $2:1$ proper morphism which is surjective. For the field extension

$$R(\mathbb{P}_{n-1}) \hookrightarrow R(Q)$$

we have the minimal equation

$$\left(\frac{x_n}{x_0}\right)^2 = -\frac{x_0^2 + \cdots + x_{n-1}^2}{x_0^2}$$

such that $R(Q)$ is an algebraic extension of degree 2.

For $k = n - 1$ there is the diagram

$$\begin{tikzcd}
 & A_{n-1}(\mathbb{P}_n) \arrow[d, "\approx"'] \\
A_{n-1}(Q) \arrow[ur, "i_*"] \arrow[r] & A_{n-1}(\mathbb{P}_n \smallsetminus \{p\}) .
\end{tikzcd}$$

In this diagram each of the groups is isomorphic to $\mathbb{Z}$ and both i_* and f_* correspond to multiplication with 2. Moreover, if $V \subset Q$ is a linear subspace of dimension k, then also $f(V) = \pi(V)$ is a linear subspace of the same dimension and we would have $f_*V = f(V)$.

7.3. Lemma: *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be proper morphisms. Then* $(g \circ f)_* = g_* \circ f_*$ *on the level of cycles.*

Proof. Let $U \subset X$ a subvariety of dimension k, $V = f(U), W = g(V)$. If all of the three subvarieties have the same dimension, then

$$\deg(U|W) = \deg(U|V). \deg(V|W)$$

because

$$R(W) \subset R(V) \subset R(U)$$

are both finite field extensions. If the dimensions are not equal, one of f_*U or g_*V would be zero and then $(g \circ f)_*U = 0$. In either case we have $(g \circ f)_*U = g_*f_*U$. □

7.4. Theorem: *Let* $X \xrightarrow{f} Y$ *be a proper morphism and* $\alpha \in Z_k(X)$*. If* $\alpha \sim 0$*, then also* $f_*\alpha \sim 0$*.*

Proof. If we replace X by the $(k+1)$–dimensional subvariety on which the rational function r of a component $\mathrm{cyc}(r)$ of α is defined, the following has to be shown. Let $X \xrightarrow{f} Y$ be a proper morphism of (integral) varieties and $r \in R(X)^*$. Then $f_*\mathrm{cyc}(r)$ is 0 or equal to $\mathrm{cyc}(s)$ for some $s \in R(Y)^*$. In fact, we prove

(i) if $\dim X = \dim Y$, then $f_*\mathrm{cyc}(r) = \mathrm{cyc}(N(r))$ where $N(r)$ is the determinant of the multiplication map $R(X) \xrightarrow{\cdot r} R(X)$ as an $R(Y)$–linear isomorphism

(ii) if $\dim X > \dim Y$, then $f_*\mathrm{cyc}(r) = 0$.

a) Since $f_*\mathrm{cyc}(r) = \sum \mathrm{ord}_V(r) \deg(V/f(V))f(V)$ with the sum taken over all codimension 1 subvarieties V of X with $\dim f(V) = \dim V$, (i) will follow if for any codimension 1 subvariety $W \subset Y$

$$\sum_{f(V)=W} \mathrm{ord}_V(r) \deg(V/W) = \mathrm{ord}_W(N(r)). \tag{1}$$

For now fixed W we may assume that there are components $V_1, \dots, V_k$ of $f^{-1}(W)$ which dominate W and have the same dimension. Otherwise $f^{-1}(W)$ would equal X and f could not be surjective. Then the generic point ω of W has the finite fibre $\{\xi_1, \dots, \xi_k\}$ where ξ_i is the generic point of V_i ($W = f(V_i) = f(\overline{\{\xi_i\}}) = \overline{\{f(\xi_i)\}}$ and hence $f(\xi_i) = \omega$. If $f(\xi) = \omega$, then $f(\{\bar{\xi}\}) = \overline{\{f(\xi)\}} = \overline{\{f(\xi)\}} = W$ because f is closed and $\overline{\{\xi\}}$ is one of the V_i). Therefore there is an affine open neighbourhood Y' of ω in Y over which f is finite. Since (1) is unchanged when we replace Y by Y', we may assume that both X and Y are affine, f is finite and

$$f^{-1}(W) = V_1 \cup \dots \cup V_k.$$

b) Now ω is the prime ideal $\mathfrak{p} \subset A(Y)$ of W and ξ_i is the prime ideal $\mathfrak{q}_i \subset A(X)$ of V_i with $\mathfrak{q}_i \cap A(Y) = \mathfrak{p}$, and the $\mathfrak{q}_i$ are all prime ideals with this property. Let $A(X)_\mathfrak{p}$ be the ring $A(X)(A(Y) \smallsetminus \mathfrak{p})^{-1}$. The natural map

$$A(Y)_\mathfrak{p} \hookrightarrow A(X)_\mathfrak{p} \cong A(X) \otimes_{A(Y)} A(Y)_\mathfrak{p}$$

is also injective and a finite integral extension. Then we have the pull–back diagram

$$\begin{tikzcd}
\mathrm{Spec} A(X)_{\mathfrak{p}} \arrow[r, tail] \arrow[d, "f"] & X \arrow[d, "f"] \\
\mathrm{Spec} A(Y)_{\mathfrak{p}} \arrow[r, tail] & Y
\end{tikzcd}$$

with injective horizontal morphisms and finite vertical morphisms. The ideals $\mathfrak{m}_i = \mathfrak{q}_i A(X)_{\mathfrak{p}}$ satisfy

$$\mathfrak{m}_i \cap A(Y)_{\mathfrak{p}} = \mathfrak{p} A(Y)_{\mathfrak{p}}$$

and are maximal by the "going up" theorem, see e.g. [11], 6.8, p. 102. By this theorem we also conclude that the ideals

$$(0), \mathfrak{m}_1, \ldots, \mathfrak{m}_k$$

are the only prime ideals of $A(X)_{\mathfrak{p}}$ because (0) and $\mathfrak{p}A(Y)_{\mathfrak{p}}$ are the only prime ideals of $A(Y)_{\mathfrak{p}}$.

c) We put $B = A(X)_{\mathfrak{p}}$, $B_i = B_{\mathfrak{m}_i}$, $A = A(Y)_{\mathfrak{p}}$ and $\mathfrak{q} = \mathfrak{p}A(Y)_{\mathfrak{p}}$. With this notation we have

(c.1) $A(X)_{\mathfrak{q}_i} \cong B_i \quad$ and $R(X) \cong A(X)_{(0)} \cong B_{(0)} \cong Q(B)$

(c.2) $R(Y) \cong Q(A), \quad R(V_i) \cong B_i/\mathfrak{m}_i B_i, \quad R(W) \cong A/\mathfrak{q}$

(c.3) $A(X) \otimes_{A(Y)} R(Y) \cong R(X)$

The isomorphisms of (c.1) are induced by $A(X) \to B$ and the definition of the localizations. (c.2) follows from $A = A(Y)_{\mathfrak{p}} \cong \mathcal{O}_{W,Y}$ and $R(V_i) = \mathcal{O}_{X,\xi_i}/\mathfrak{m}_{\xi_i} \cong A(X)_{\mathfrak{q}_i}/\mathfrak{q}_i A(X)_{\mathfrak{q}_i} = B_i/\mathfrak{m}_i B_i$, and $R(W) \cong A(Y)_{\mathfrak{p}}/\mathfrak{p}A(Y)_{\mathfrak{p}} = A/\mathfrak{q}$. (c.3) is induced by $A(X) \otimes_{A(Y)} A(Y) \cong A(X)$ and the fact that $A(Y) \subset A(X)$ is an integral extension, because any nonzero $G \in A(X)$ satisfies an equation

$$\frac{1}{G} + \alpha_0 + \alpha_1 G + \cdots + \alpha_m G^m = 0$$

with $\alpha_\mu \in Q(A(Y)) \cong R(Y)$.

d) Formula (1) follows if for any $b \in B$

$$\sum_i l_{B_i}(B_i/bB_i)[B_i/\mathfrak{m}_i B_i : A/\mathfrak{q}] = \mathrm{ord}_A(\det(b)) \tag{2}$$

where for $\det(b) = \alpha/\beta \in Q(A)$ we have $\mathrm{ord}_A(\alpha/\beta) = l_A(A/\alpha A) - l_A(A/\beta A)$.

Proof of (2): By (c.1) $l_{B_i}(B_i/bB_i) = \mathrm{ord}_{V_i}(b)$ with b considered as an element of $R(X)$, and by (c.2) $[B_i/\mathfrak{m}_i B_i : A/\mathfrak{q}] = [R(V_i) : R(W)]$, while $\mathrm{ord}_A(\det(b)) = \mathrm{ord}_W(N(b))$. If $r \in R(X)^*$ is general, it is the quotient b/a with $b, a \in B$. Since the order functions and determinant are homomorphisms, (2) implies (1).

e) Finally (2) follows from fundamental properties of the length. Because $(0), \mathfrak{m}_1, \ldots, \mathfrak{m}_k$ are the only prime ideals of B and since $B_i/bB_i = B_{\mathfrak{m}_i}/bB_{\mathfrak{m}_i}$ have finite length over A and

$B_{(0)} = bB_{(0)}$ (note that the multiplication operator b is injective and hence an injective operator of the finite dimensional $R(Y)$–vector space $R(X)$), B/bB has finite length and

$$l_A(B/bB) = \sum_i l_A(B_i/bB_i).$$

By [8], appendix A.2.3,

$$l_A(B_i/bB_i) = l_{B_i}(B_i/bB_i)[B_i/\mathfrak{m}_i B_i : A/\mathfrak{q}]$$

and hence finite. On the other hand, [8], appendix A.3, guarantees that

$$l_A(B/bB) = \mathrm{ord}_A(\det(b)).$$

This proves (2) and finally (i).

f) In case $n = \dim Y < \dim X$ we may assume that $\dim X = n+1$. Otherwise $\dim f(V) < \dim V$ for any n–dimensional subvariety $V \subset X$. Now

$$f_*\mathrm{cyc}(r) = \sum_{f(V)=Y} \mathrm{ord}_V(r) \deg(V/Y)Y$$

because for any component V of $\mathrm{cyc}(r)$ with $f(V) \neq Y$ we have $\dim f(V) < \dim V$. Let $V_1, \ldots, V_s$ be the components of $S = \mathrm{Supp}\ \mathrm{div}(r)$ which are mapped onto Y. We have to show that

$$\sum_i \mathrm{ord}_{V_i}(r) \deg(V_i/Y) = 0.$$

Let now ξ_i resp. η be the generic points of V_i resp. Y, and let X_η be the (1–dimensional) fibre of η. Then

$$\mathrm{ord}_{V_i}(r) = \mathrm{ord}_{\xi_i}(r \mid X_\eta)$$

and

$$\deg(V_i/Y) = [R(V_i)/R(Y)] = [(\mathcal{O}_{X_\eta,\xi_i}/\mathfrak{m}_{\xi_i})/\mathcal{O}_{Y,\eta}]\,.$$

Therefore we may assume that X is a complete curve over $\mathrm{Spec}(K)$ with $K = R(Y)$.

g) We consider first the case where $X = \mathbb{P}_{1,K}$ and $R(X) \cong K(t)$ with $t = x_1/x_0$. Now we may assume that $r \in K[t]$ is an irreducible polynomial because the order function is a homomorphism. Let $P \in \mathbb{P}_{1K}$ be the prime ideal $(r) \subset K[t]$. Then $\mathrm{ord}_P(r) = 1$ and the only other point Q with $\mathrm{ord}_Q(r) \neq 0$ is $Q =< 0, 1 >$. In the affine neighbourhood of Q the local coordinate function is $s = 1/t$ and we have

$$\mathrm{ord}_Q(r) = -d$$

with $d = \deg(r)$. On the other hand, the field of P is $R(P) = K[t]/(r)$ while the field of Q is $R(Q) \cong K$. Therefore,

$$\mathrm{cyc}(r) = P - dQ$$

and then

$$f_*\mathrm{cyc}(r) = dY - dY = 0.$$

h) If X is a general complete curve over $\mathrm{Spec}(K)$, we consider the normalization $\widetilde{X} \xrightarrow{g} X$ for which we have $R(X) \cong R(\widetilde{X})$ and $\mathrm{cyc}(r) = g_*\mathrm{cyc}(\widetilde{r})$, where $\widetilde{r}$ is the rational function corresponding to r. There is now a finite morphism $\widetilde{X} \xrightarrow{h} \mathbb{P}_{1K}$ over Spec(K) with $f \circ g = p \circ h$, where p is the structural morphism of $\mathbb{P}_{1K}$. Now

$$f_*\mathrm{cyc}(r) = f_*g_*\mathrm{cyc}(\widetilde{r}) = p_*h_*\mathrm{cyc}(\widetilde{r}).$$

By (i) $h_*\mathrm{cyc}(\widetilde{r}) = \mathrm{cyc}(N(\widetilde{r}))$ and by g) $p_*\mathrm{cyc}(N(\widetilde{r})) = 0$. □

Theorem 7.4 says that a proper morphism $X \xrightarrow{f} Y$ defines a homomorphism

$$A_k(X) \xrightarrow{f_*} A_k(Y)$$

for any k and that $f \mapsto f_*$ is a functor on the category of proper maps. The theorem also provides a new proof of **Bezout's theorem** for plane projective curves:

7.5. Example: If X is complete, $X \to \mathrm{Spec}(k)$ is a proper map, then $A_0(X) \to A_0(pt) \cong \mathbb{Z}$ is nothing but the degree map. Let now $F \subset \mathbb{P}_2$ (over k) be an integral curve and L a line, which is not a component of F. Then the intersection multiplicity

$$\mu(p, F, L) = l(\mathcal{O}_{L,p}/f_p\mathcal{O}_{L,p})$$

is defined at any closed point $p \in L \cap F$, where f is the equation of F. If L' is any other line with equation z, then $r = f/z^n$ is a rational function on $\mathbb{P}_2$, where $n = \deg(f)$. Then

$$\mathrm{cyc}(r) = \sum_{p \in L} \mu(p, F, L)p - np_0 \in Z_0(\mathbb{P}_2),$$

where p_0 is the intersection point $L \cap L'$. By theorem 7.4

$$0 = \sum_{p \in L} \mu(p, F, L) - n.$$

Let now m be any integer and $G = Z(g)$ a curve of degree m. Then

$$\mu(p, F, G) = l(\mathcal{O}_{F,p}/g_p\mathcal{O}_{F,p})$$

where g_p is the germ of the local function of g at p. Similarly

$$\mu(p, F, L_m) = l(\mathcal{O}_{F,p}/u_p^m\mathcal{O}_{F,p}) = m \cdot l(\mathcal{O}_{F,p}(u_p\mathcal{O}_{F,p}) = m \cdot \mu(p, F, L),$$

where u denotes the equation of L and L_m is the multiple line $u^m = 0$. Now $s = g/u^m$ is a rational function and

$$\mathrm{cyc}(s) = \sum_{p \in F} \mu(p, F, G)p - \sum_{p \in F} \mu(p, F, L_m)p.$$

It follows that

$$\sum_{p \in F} \mu(p, F, G) = m \cdot \sum_{p \in F} \mu(p, F, L) = m \cdot n\,.$$

7.6. Flat morphisms: A morphism $X \xrightarrow{f} Y$ of schemes is called flat if the local ring $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$–module for any $x \in X$. It is shown in commutative algebra that this is equivalent to

$$\mathrm{Tor}_1(\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}, \mathcal{O}_{X,x}) = 0$$

for any x. Then

$$\mathrm{Tor}_1(M, \mathcal{O}_{X,x}) = 0$$

for any finitely generated $\mathcal{O}_{Y,f(x)}$–module. In terms of exact sequences this can be expressed as follows. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent $\mathcal{O}_Y$–modules on an open set $U \subset Y$, then the lifted sequence $0 \to f^*\mathcal{F}' \to f^*\mathcal{F} \to f^*\mathcal{F}'' \to 0$ is exact over $f^{-1}(U)$. Here $f^*\mathcal{F}$ denotes the sheaf $f^\bullet\mathcal{F} \otimes_{f^\bullet\mathcal{O}_Y} \mathcal{O}_X$ where $f^\bullet\mathcal{F}$ is the topological pull–back. It is enough to test this for resolutions for the ideal sheaves $\mathfrak{m}(y)$ of points $y \in Y$,

$$0 \to \mathcal{R} \to \mathcal{O}_Y^p \to \mathfrak{m}(y) \to 0.$$

7.7. Example: Let $X \subset \mathbb{A}_k^2$ be the subvariety defined by $xy = 0$, and let $X \xrightarrow{f} \mathbb{A}_k^1$ be the first projection. Then f is not flat along the fibre of 0. Here we have the resolution $0 \to k[t]_{(t)} \xrightarrow{t} k[t]_{(t)} \to k \to 0$ of $\mathcal{O}_{\mathbb{A}^1,0}/\mathfrak{m}_0 \cong k$ and the lifted homomorphism at any closed point $(0, b) \in f^{-1}(0)$ is the localization of the complex

$$0 \to k[x,y]/(xy) \xrightarrow{x} k[x,y]/(x,y) \to$$

which is not injective.

7.8. Example: Instead let $X \subset \mathbb{A}_k^2$ now be given by $y^2 - x^2 = 0$. Here the lifted sequence is the localization of the complex

$$0 \to k[x,y]/(y^2 - x^2) \xrightarrow{x} k[x,y]/(y^2 - x^2) \to$$

which is exact. The same can be said for any other point of $\mathbb{A}^1$ or of X.

7.9. Example: Let V be a finite dimensional vector space and let $X \subset \mathbb{P}V \times \mathbb{P}S^dV^*$ be defined by pairs $(\langle v\rangle, \langle f\rangle)$ with $f(v) = 0$. Then the induced projection $X \to \mathbb{P}S^dV^*$ is flat. This is also called the universal hypersurface. If $z_0, \ldots, z_n$ is a basis of V^*, i.e. homogeneous coordinates of $\mathbb{P}V$ and if $t_{\nu_0\ldots\nu_n}$ with $\nu_0 + \cdots + \nu_n = d$ are the homogeneous coordinates of $\mathbb{P}S^dV^*$ (dual to the basis $z_0^{\nu_0} \ldots z_n^{\nu_n}$ of S^dV^*), then X is the hypersurface defined by the $(1, d)$–homogeneous equation

$$f = \sum t_{\nu_0\ldots\nu_n} z_0^{\nu_0} \ldots z_n^{\nu_n} = 0.$$

This is a section of the line bundle

$$\mathcal{O}_{\mathbb{P}S^dV^*}(1) \boxtimes \mathcal{O}_{\mathbb{P}V}(d) = \mathcal{O}_{\mathbb{P}S^dV^* \times \mathbb{P}V}(1, d)$$

and we have the resolution

$$0 \to \mathcal{O}_{\mathbb{P}S^dV^* \times \mathbb{P}V}(-1, -d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}S^dV^* \times \mathbb{P}V} \to \mathcal{O}_X \to 0.$$

For a fixed point

$$a = \left\langle \sum a_{\nu_0\ldots\nu_n} z_0^{\nu_0} \cdots z_n^{\nu_n} \right\rangle = \langle f_a\rangle \in \mathbb{P}S^dV^* = Y$$

the structure sheaf of the fibre X_a ist obtained as the tensor product

$$\mathcal{O}_X \otimes_{f^\bullet\mathcal{O}_Y} f^\bullet\mathcal{O}_Y / f^\bullet\mathcal{M}(a)$$

where $\mathcal{M}(a)$ is the ideal shaef of a. Tensoring the above sequence with $f^\bullet\mathcal{O}_Y/f^\bullet\mathcal{M}(a)$ we get the sequence

$$0 \to \mathcal{O}_{\mathbb{P}V}(-d) \xrightarrow{f_a} \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_{X_a} \to 0,$$

which is exact. This proves flatness at any point of X.

7.10. Example: Let $X \subset \mathbb{A}^3_k$ be the hypersurface $ty - x^2 = 0$ and let $X \to \mathbb{A}^1_k$ be defined by the projection to the t–axis. We have the exact sequence

$$0 \to \mathcal{O}_{\mathbb{A}^3} \xrightarrow{ty-x^2} \mathcal{O}_{\mathbb{A}^3} \to \mathcal{O}_X \to 0$$

and for any fixed t_0 the exact sequence

$$0 \to \mathcal{O}_{\mathbb{A}^2} \xrightarrow{t_0y-x^2} \mathcal{O}_{\mathbb{A}^2} \to \mathcal{O}_{X_{t_0}} \to 0$$

because $t_0y - x^2 \neq 0$. This shows that $X \to \mathbb{A}^1_k$ is flat. Here X_t is a parabola for any $t \neq 0$ and a double line for $t = 0$, see [9] II, Example 3.3.1.

7.11. Proposition: *(see* [9] *III, 9.5, 9.6)*

If $X \xrightarrow{f} Y$ is a flat morphism of finite type between noetherian schemes, then

$$\dim_x X_y = \dim_x X - \dim_y Y$$

for any $x \in X$ with $y = f(x)$. In particular, if X and Y are pure dimensional, then all fibres are pure dimensional.

7.12. Flat morphisms of fixed relative dimension:

In [7] only flat morphisms of fixed relative dimension (or fixed fibre dimension) are considered for pulling back cycles.

This means that for any subvariety $V \subset Y$ and any irreducible component V' of $f^{-1}(V)$,

$$\dim V' = \dim V + n$$

where n is fixed. By the above, this is fulfilled if f is flat between integral algebraic schemes over some field. Then f is of finite type. The following are flat morphisms of fixed relative dimension.

- open immersions
- projections of fibre bundles onto a pure–dimensional base scheme
- dominant morphisms from an integral scheme to a non–singular curve.

7.13. Fundamental cycle of a scheme

Let X be an algebraic scheme over k and let $X_1, \ldots, X_r$ be the irreducible components of X_{red}. Each X_ρ has a generic point ξ_ρ which is not contained in any other component $X_\sigma, \sigma \neq \rho$. Then the local rings

$$\mathcal{O}_{X_\rho,X} = \mathcal{O}_{X,\xi_\rho} = \mathcal{O}_{Y_\rho,\xi_\rho}$$

where $Y_\rho = X \smallsetminus \underset{\sigma\neq\rho}{\cup} X_\sigma$, have finite length, because

$$\dim \mathcal{O}_{Y_\rho,\xi_\rho} = \mathrm{codim}_{Y_\rho} X_\rho = 0.$$

Let μ_ρ be the length of $\mathcal{O}_{X_\rho,X}$. This can be interpreted as the multiplicity of X_ρ in X.

If all the X_ρ have the same dimension n we obtain the cycle

$$[X] = \sum \mu_\rho [X_\rho] \in A_n(X).$$

If there are different dimensions, we consider the direct sum

$$A_*(X) = \bigoplus_{k\geq 0} A_k(X)$$

and obtain a fundamental class $[X] \in A_*(X)$.

7.14. Example $X \subset \mathbb{A}_k^2$ with equation $xy^2 = 0$. $X_1 = Z(y), X_2 = Z(x)$ and $[X] = 2[X_1] + [X_2]$. If $A(X)$ is the coordinate ring, we have the exact sequence

$$0 \to yA(X)_{(y)} \to A(X)_{(y)} \to A(X)_{(y)}/yA(X)_{(y)} \to 0$$

with $yA(X)_{(y)} \cong A(X)_{(y)}/yA(X)_{(y)}$ because $y^2 = 0$ in $A(X)_{(y)}$.

Hence length $A(X)_{(y)} = 2$.

7.15. pull–back by flat morphisms

Let $X \xrightarrow{f} Y$ be a flat morphism of relative dimension n. Given a subvariety $V \subset Y$ of dimension k. Then $f^{-1}(V)$ has pure dimension $k+n$, but need not be reduced. Then the cycle

$$f^*V = [f^{-1}V] = \sum a_\rho V_\rho$$

is defined as the fundamental cycle of $f^{-1}V$ where $V_1, \ldots, V_r$ are the irreducible components of $(f^{-1}V)_{\mathrm{red}}$ with $a_\rho =$ length $\mathcal{O}_{V_\rho,f^{-1}V}$. We thus obtain a homomorphism

$$Z_k(Y) \xrightarrow{f^*} Z_{k+n}(X).$$

7.16. Theorem: *Let $X \xrightarrow{f} Y$ be flat of relative dimension n and $\alpha \in Z_k(Y)$. If $\alpha \sim 0$, then also $f^*\alpha \sim 0$ in $Z_{k+n}(X)$.*

For a proof see [8], section 1.7. The theorem says that f defines a homomorphism

$$A_k(Y) \xrightarrow{f^*} A_{k+n}(X).$$

It follows from the definition that for two flat morphism $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ of relative dimensions m and $n, (gf)^* = f^*g^*$.

7.17. Projection formula: Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a Cartesian diagram with g flat of relative dimension n and f proper. Then also f' is proper and g' is flat of the same relative dimension n and for any cycle $\alpha \in Z_k(X)$

$$\boxed{f'_*g'^*\alpha = g^*f_*\alpha}$$

For the easy proof see [8], prop. 1.7.

8. Intersection with Cartier divisors

As before X will denote an algebraic scheme over k and D a Cartier divisor on X. We are going to define an intersection class $D \cap V = D.V$ in $A_{k-1}(X)$ for each k–dimensional subvariety V and by this an intersection homomorphism

$$A_k(X) \xrightarrow{D.} A_{k-1}(X).$$

The image of this will be contained in $A_{k-1}(|D|)$. To begin with $D.V$, let $V \overset{j}{\hookrightarrow} X$ be the inclusion. Then $j^*\mathcal{O}_X(D)$ is a line bundle (invertible sheaf) on V. Since V is integral, there is a Cartier divisor C on V with

$$j^*\mathcal{O}_X(D) \cong \mathcal{O}_V(C),$$

see 2.8. This divisor is only determined modulo principal divisors. If $\mathcal{O}_V(C) \cong \mathcal{O}_V(C')$, there is a rational function r on V such that

$$C' = C + \mathrm{div}(r) \text{ and then } \mathrm{cyc}(C') = \mathrm{cyc}(C) + \mathrm{cyc}(r).$$

Therefore

$$D.V = D.[V] := [\mathrm{cyc}(C)] \in A_{k-1}(V)$$

is uniquely determined.

8.1. Lemma: *$D.V \in A_{k-1}(V \cap |D|)$.*

Proof. If $V \subset |D|$, there is nothing to prove. If $V \not\subset |D|$, we can define a Cartier divisor j^*D as follows. Let (f_α) represent D, each f_α being a rational function in $\mathcal{M}^*(U_\alpha)$.

If $V \cap U_\alpha \neq \emptyset$, then $V \cap U_\alpha \smallsetminus |D| \neq \emptyset$ because $V \smallsetminus |D| \neq \emptyset$ and V is irreducible. Then the residue class $\bar{f}_\alpha = f_\alpha | V \cap U_\alpha \smallsetminus |D|$ is defined and

$$\bar{f}_\alpha \in \mathcal{O}_V^*(V \cap U_\alpha \smallsetminus |D|) \subset \mathcal{M}_V^*(V \cap U_\alpha).$$

The system $(\bar{f}_\alpha), V \cap U_\alpha \neq \emptyset$, defines a Cartier divisor j^*D on V. It has cocycle $(\bar{g}_{\alpha\beta})$ where $g_{\alpha\beta}$ is the cocycle of D. Therefore

$$j^*\mathcal{O}_X(D) \cong \mathcal{O}_V(j^*D).$$

Now $\operatorname{cyc}(j^*D) \in Z_{k-1}(V \cap |D|)$ because each $\bar{f}_\alpha$ is in $\mathcal{O}_V^*(V \cap U_\alpha \smallsetminus |D|)$. This proves the Lemma. $\square$

8.2. Intersection with D

Given a cycle $\alpha = \sum n_i V_i$ in $Z_k(X)$, we can define

$$D.\alpha = \sum n_i D.V_i \in A_{k-1}(|D| \cap |\alpha|) \subset A_{k-1}(|D|) \subset A_{k-1}(X)$$

where $|\alpha|$ is the union of the V_i. This defines a homomorphism

$$Z_k(X) \xrightarrow{D.} A_{k-1}(|D|) \subset A_{k-1}(X).$$

We are going to show that this is defined on $A_k(X)$, i.e. if $\alpha \sim 0$, then $D.\alpha \sim 0$, see 8.6.1

This intersection pairing $(D, \alpha) \mapsto D.\alpha$ satisfies the rules

(a) $D.(\alpha + \alpha') = D.\alpha + D.\alpha'$
(b) $(D + D').\alpha = D.\alpha + D'.\alpha$
(c) $\operatorname{div}(r).\alpha = 0$ for rational functions $r \in \mathcal{M}^*(X)$.

which follows directly from the definition.

If X is a smooth surface and $V \subset X$ an irreducible curve, then $\operatorname{Div}(X) = \Gamma(X, \mathcal{M}^*/\mathcal{O}^*) \cong Z_1(X)$ and we obtain the pairing $Z_1(X) \times Z_1(X) \to A_0(X) \underset{\deg}{\longrightarrow} \mathbb{Z}$ written as $(\alpha, \beta) \to \deg(D_\alpha.\beta)$ where D_α is the Cartier divisor defined by the Weil divisor α.

8.3. Chern classes of a line bundle

For an invertible sheaf $\mathcal{L}$ on X and a k–dimensional subvariety $V \subset X$ there is also a Cartier divisor C on V with $j^*\mathcal{L} \cong \mathcal{O}_V(C)$ and a unique class $c_1(\mathcal{L}) \cap V \in A_{k-1}(V) \subset A_{k-1}(X)$. As before we obtain a homomorphism

$$Z_k(X) \to A_{k-1}(X) \text{ denoted } \alpha \mapsto c_1(\mathcal{L}) \cap \alpha.$$

This operator is also called the first Chern class of $\mathcal{L}$. If X is itself integral of dimension n, the intersection with the fundamental cycle X in $Z_n(X)$ gives the class

$$c_1(\mathcal{L}) := c_1(\mathcal{L}) \cap X \in A_{n-1}(X)$$

which is nothing but the class $[\operatorname{cyc}(C)]$ where $\mathcal{L} \cong \mathcal{O}_X(C)$. Note that this is only defined modulo rational equivalence. If $X = \mathbb{P}_n$ (over k), we have isomorphisms

$$\operatorname{Pic}(\mathbb{P}_n) \underset{\approx}{\xrightarrow{c_1}} A_{n-1}(\mathbb{P}_n) \underset{\approx}{\longrightarrow} \mathbb{Z}$$

and the isomorphism class $[\mathcal{L}]$ is determined by an integer.

8.4. Projection formula: Let $X' \xrightarrow{f} X$ be a proper morphism, let $D \in \operatorname{Div}(X)$ a Cartier divisor, and $\alpha \in Z_k(X')$. Then the induced morphism

$$f^{-1}(|D|) \cap |\alpha| \xrightarrow{g} |D| \cap f(|\alpha|)$$

on the closed subscheme is also proper. If f^*D can be defined as g^*D, e.g. in case X' is integral and $f^{-1}(|D|) \subsetneq X'$, then

$$\text{(d)} \qquad g_*((f^*D).\alpha) = D.f_*(\alpha) \text{ in } A_{k-1}(X).$$

For a proof see [7], 2.3. If f^*D cannot be defined as a divisor, it is defined as a **pseudo–divisor**. This is the reason why pseudo–divisors had been introduced in [7], 2.2. However, (d) is true in general in the form

$$\boxed{g_*(c_1(f^*\mathcal{O}_X(D)) \cap \alpha) = D.f_*(\alpha)}$$

8.5. Flat pull–back formula: Let $X' \xrightarrow{f} X$ be flat of relative dimension $n, D \in \mathrm{Div}(X), \alpha \in Z_k(X)$. Then the induced morphism

$$f^{-1}(|D| \cap |\alpha|) \xrightarrow{g} |D| \cap |\alpha|$$

is also flat of relative dimension n and

$$\text{(e)} \qquad (f^*D).(f^*\alpha) = g^*(D.\alpha) \text{ in } A_{k+n-1}(X')$$

if f^*D is defined. In general the formula reads

$$\boxed{c_1(f^*\mathcal{O}_X(D)) \cap f^*\alpha = g^*(D.\alpha)}$$

8.6. Theorem: *Let X be an n–dimensional integral scheme and let D, D' be divisors on X. Then*

$$D.\mathrm{cyc}(D') = D'.\mathrm{cyc}(D).$$

For a proof see [7], 2.4.

8.6.1. Corollary: *Let D be a divisor on an algebraic scheme/k and $\alpha \in Z_k(X)$. If $\alpha \sim 0$, then $D.\alpha = 0$.*

Proof. Let $V \subset X$ be a $(k+1)$-dimensional subvariety, $r \in R(V)^*$ and $\alpha = \mathrm{cyc}(r)$. We have to show that $D.\alpha = 0$. Now on V we have $\mathrm{cyc}(r) = \mathrm{cyc}(\mathrm{div}(r))$ and for any Cartier divisor C on V: $C.\mathrm{cyc}(r) = C.\mathrm{cyc}(\mathrm{div}(r)) = \mathrm{div}(r).\mathrm{cyc}(C) = 0$.
If $j^*\mathcal{O}_X(D) = \mathcal{O}_V(C)$, then

$$D.\alpha = D.\mathrm{cyc}(r) = C.\mathrm{cyc}(r) = 0. \quad \square$$

8.6.2. Corollary: *For two Cartier divisors D and D' on X and any $\alpha \in Z_k(X)$,*

$$D.(D'.\alpha) = D'.(D.\alpha) \text{ in } A_{k-2}(|D| \cap |D'| \cap |\alpha|).$$

Proof. We may assume $\alpha = V$ for a k–dimensional subvariety $V \xhookrightarrow{j} X$. Let $j^*\mathcal{O}_X(D) \cong \mathcal{O}_V(C)$ and $j^*\mathcal{O}_X(D') = \mathcal{O}_V(C')$. Then $D.\alpha = [\mathrm{cyc}(C)]$ and $D'.\alpha = [\mathrm{cyc}(C')]$ and

$$\begin{aligned} D.(D'.\alpha) = D.\mathrm{cyc}(C') &= C.\mathrm{cyc}(C') \\ &= C'.\mathrm{cyc}(C) = D'.\mathrm{cyc}(C) = D'.(D.\alpha) \end{aligned}$$

in $A_{k-2}(V \cap |D| \cap |D'|)$. $\square$

8.7. Intersection with polynomials of divisors: By the preceding corollaries we are now able to define intersections $D.[\alpha] = D.\alpha$ for classes $[\alpha] \in A_k(X)$ and iterated intersections

$$(D_1 \cdot \ldots \cdot D_n).[\alpha] = D_1.(D_2 \cdot \ldots \cdot D_n).[\alpha]$$

by induction. This product is multilinear and commutative in the D's. This identity holds in

$$A_{k-n}(|\alpha| \cap |D_1| \cap \ldots \cap |D_n|) \ \text{ if } \alpha \in Z_k(X).$$

More generally, if $P(T_1, \ldots, T_n) \in \mathbb{Z}[T_1, \ldots, T_n]$ is a homogeneous polynomial of degree d,

$$P(T_1, \ldots, T_n) = \sum a_{\nu_1 \ldots \nu_n} T_1^{\nu_1} \cdot \ldots \cdot T_n^{\nu_n},$$

we obtain a class

$$P(D_1, \ldots, D_n).\alpha = \sum a_{\nu_1 \ldots \nu_n} (D_1^{\nu_1} \cdot \ldots \cdot D_n^{\nu_n}).\alpha \in A_{k-d}(X)$$

for any k–cycle α and any subscheme Y containing $(|D_1| \cup \ldots \cup |D_n|) \cap |\alpha|$.

Examples: see [8], 2.4.4 to 2.4.9.

8.8. Intersection formulas with line bundles:

Let $\mathcal{L}$ be an invertible sheaf on an algebraic scheme X over k. By 8.3 and 8.6.1 there is the intersection operator

$$c_1(\mathcal{L})\cap : A_k(X) \to A_{k-1}(X)$$

for any k defined by $c_1(\mathcal{L}) \cap V = [\mathrm{cyc}(C)]$ if $\mathcal{L}|V \cong \mathcal{O}_V(C)$. It is clear that the formulas for the intersection with divisors transcribe into

(a) $c_1(\mathcal{L}) \cap c_1(\mathcal{L}') \cap \alpha = c_1(\mathcal{L}') \cap c_1(\mathcal{L}) \cap \alpha$

(b) (projection formula) If $X' \xrightarrow{f} X$ is a proper morphism, $\mathcal{L}$ is a line bundle on X and α a k–cycle on X', then

$$f_*(c_1(f^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap f_*\alpha \ \text{ in } A_{k-1}(X).$$

(c) (flat pullback) If $X' \xrightarrow{f} X$ is a flat morphism of relative dimension n, and $\mathcal{L}$ and α are given on X, then

$$c_1(f^*\mathcal{L}) \cap f^*\alpha = f^*(c_1(\mathcal{L}) \cap \alpha) \ \text{ in } A_{k+n-1}(X').$$

(d)

$$\begin{aligned} c_1(\mathcal{L} \otimes \mathcal{L}') \cap \alpha &= c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{L}') \cap \alpha \\ c_1(\mathcal{L}^*) \cap \alpha &= -c_1(\mathcal{L}) \cap \alpha \end{aligned}$$

If $P(T_1, \ldots, T_n) \in \mathbb{Z}[T_1, \ldots, T_n]$ is a homogeneous polynomial of degree d, then there is the intersection operator

$$P(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_n))\cap : A_k(X) \to A_{k-d}(X).$$

Examples: see [8], 2.5.2 to 2.5.6.

On $\mathbb{P}_n$ we have $c_1(\mathcal{O}_{\mathbb{P}_n}(H)) \cap [H_k] = [H_{k-1}]$ for projective linear subspaces H, H_k, H_{k-1} of dimensions $n-1, k, k-1$. Since $[H_k]$ is the free generator of $A_k(\mathbb{P}_n) \cong \mathbb{Z}$,

$$c_1(\mathcal{O}_{\mathbb{P}_n}(H))\cap : A_k(\mathbb{P}_n) \underset{\approx}{\longrightarrow} A_{k-1}(\mathbb{P}_n)$$

is an isomorphism.

9. The Gysin homomorphism

Given an effective divisor $D \in \mathrm{Div}^+(X)$ we can consider D also as a scheme structure on $|D| = D_{\mathrm{red}}$ and define $A_k(X) \xrightarrow{i^*} A_{k-1}(D)$ as above by $\alpha \mapsto D.\alpha$ with the Cartier divisor D. This is the Gysin homomorphism. We are going to describe its rules.

9.1. Normal bundle: Let X be any scheme and let $\mathcal{D}^+$ be the image of $\mathcal{O} \cap \mathcal{M}^*$ in $\mathcal{M}^*/\mathcal{O}^*$. It is called the sheaf of effective divisors, see 2.9. $\mathrm{Div}^+(X) = \Gamma(X, \mathcal{D}^+)$ is the group of effective divisors. If $D \in \mathrm{Div}^+(X)$, then $\mathrm{cyc}(D)$ has only positive coefficients, see 3.3. We thus have a homomorphism $\mathrm{Div}^+(X) \to Z^+_{n-1}(X)$ if X is a variety of dimension n. When D if effective, the line bundle $\mathcal{O}_X(D)$ has a regular section $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ vanishing exactly on $|D|$. By abuse of notation we denote the zero scheme of this section also by D. It has the ideal sheaf $\mathcal{O}_X(-D)$ with exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

The cokernel of the dual sequence is called the normal bundle $\mathcal{N} = \mathcal{N}_{D/X}$ of D in X with exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{N}_{D/X} \to 0.$$

Denoting $\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}_X(D)$ for any sheaf, we get, by tensoring this sequence with $\mathcal{O}_D$:

$$\mathcal{T}\mathrm{or}_1^{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_D) \cong \mathcal{O}_D \quad \text{and} \quad \mathcal{O}_D(D) \cong \mathcal{N}.$$

9.2. Zero section of a line bundle: Let $\mathcal{L}$ be an invertible sheaf on an algebraic scheme X over k and let $L \xrightarrow{p} X$ be its bundle space. Then X has an embedding $X \overset{i}{\hookrightarrow} L$ as the zero section. As such, X is an effective divisor: if (U_α) is a trivializing covering of $\mathcal{L}$ or L such that $L_{U_\alpha} \cong U_\alpha \times_k \mathbb{A}^1$, let t_α be the pull bak of the coordinate function of $\mathbb{A}^1$, which is the equation of $X \cap L_{U_\alpha}$. On $L_{U_\alpha} \cap L_{U_\beta}$ we have

$$t_\alpha = (g_{\alpha\beta} \circ p) t_\beta$$

where $(g_{\alpha\beta})$ is the cocycle of $\mathcal{L}$, and therefore $\mathcal{O}_L(X)$ has the cocycle $(g_{\alpha\beta} \circ p)$.

This means that

$$\mathcal{O}_L(X) \cong p^*\mathcal{L}.$$

Moreover,

$$\mathcal{N}_{X/L} = i^*\mathcal{O}_L(X) \cong i^*p^*\mathcal{L} \cong \mathcal{L}.$$

9.3. Gysin homomorphism

Let $D \in \mathrm{Div}^+(X)$ on an algebraic scheme over k and let $D \overset{i}{\hookrightarrow} X$ be the inclusion as a subscheme. Then $\alpha \mapsto D.\alpha \in A_{k-1}(|D|) = A_{k-1}(D_{\mathrm{red}}) = A_{k-1}(D)$ defines a homomorphism

$$A_k(X) \xrightarrow{i^*} A_{k-1}(D),$$

called the Gysin homomorphism. For this intersection operator we have the following rules

(a) $i_* i^*(\alpha) = c_1(\mathcal{O}_X(D)) \cap \alpha \qquad \alpha \in A_k(X)$
(b) $i^* i_*(\alpha) = c_1(\mathcal{N}_{D/X}) \cap \alpha \qquad \alpha \in A_k(D)$
(c) If X is purely n–dimensional, $i^*[X] = [D]$
(d) If $\mathcal{L}$ is a line bundle on X,

$$i^*(c_1(\mathcal{L}) \cap \alpha) = c_1(i^*\mathcal{L}) \cap i^*\alpha \quad \text{in} \quad A_{k-2}(D).$$

Proof. (a) follows from the definition. If $\alpha = [V]$ is the class of a subvariety of dimension k,

$$i^*(\alpha) = D.[V] = [\mathrm{cyc}(C)] = c_1(\mathcal{O}_X(D)) \cap [V] \text{ in } A_{k-1}(V)$$

where $j^*\mathcal{O}_X(D) \cong \mathcal{O}_V(C)$. Then $i_* i^*(\alpha)$ is the same class in $A_{k-1}(X)$. To prove (b), let $V \overset{\varepsilon}{\hookrightarrow} D$ with $j = i \circ \varepsilon$. Then $i_*[V] = [V]$ in $A_k(X)$ and

$$i^* i_*[V] = D.[V] = [\mathrm{cyc}(C)] \in A_{k-1}(V) \subset A_{k-1}(D)$$

with

$$\mathcal{O}_V(C) \cong j^*\mathcal{O}_X(D) \cong \varepsilon^* i^* \mathcal{O}_X(D) \cong \varepsilon^* \mathcal{N}_{D/X}.$$

(c) Let X_ν be the irreducible components of X, all of dimension $n = \dim X$. Then $[X] = \sum\limits_\nu m_\nu [X_\nu]$ with multiplicities m_ν, see 7.13. Then

$$i^*[X] = \sum m_\nu D.[X_\nu].$$

Let $C_\nu \subset X_\nu$ be defined by $j_\nu^* \mathcal{O}_X(D) \cong \mathcal{O}_{X_\nu}(C_\nu)$. Then C_ν can be chosen as the component of D in X_ν and it has the same multiplicity m_ν with respect to D, see [8], 1.7.2. Therefore,

$$i^*[X] = \sum m_\nu [\mathrm{cyc}(C_\nu)] = [D].$$

(d) follows from $c_1(\mathcal{O}_X(D)) \cap (c_1(\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap (c_1(\mathcal{O}_X(D)) \cap \alpha)$ and the observation that

$$c_1(\mathcal{L}) \cap \beta = c_1(i^*\mathcal{L}) \cap \beta \quad \text{for } \beta \in A_{k-1}(D). \quad \square$$

9.4. Chow groups of line bundles:

Let $L \xrightarrow{p} X$ be the bundle space of an invertible sheaf $\mathcal{L}$ on X and let $X \xhookrightarrow{i} L$ be the zero section. If V is a k–dimensional subvariety, we have the pull–back diagram

$$\begin{tikzcd} L_V \arrow[r, hook, "j_L"] \arrow[d, "p_V"] & L \arrow[d, "p"] \\ V \arrow[r, hook, "j"] & X. \end{tikzcd}$$

Claim: $i^*p^*[V] = [V]$

Proof: $p^*[V] = [p^{-1}V] = [L_V]$ and $i^*[L_V]$ is defined as $[\mathrm{cyc}(C)]$ where $j_L^*\mathcal{O}_L(X) \cong \mathcal{O}_{L_V}(C)$. But $\mathcal{O}_{L_V}(C) \cong j_L^*\mathcal{O}_L(X) \cong j_L^*p^*\mathcal{L} \cong p_V^*j^*\mathcal{L} \cong \mathcal{O}_{L_V}(V)$ and this proves that $[\mathrm{cyc}(C)] = [V]$. As a conclusion we get

9.4.1. Proposition: *Let $L \xrightarrow{p} X$ be a line bundle on an algebraic scheme X over k. Then the flat pull–back homomorphism $A_k(X) \xrightarrow{p^*} A_{k+1}(L)$ is an isomorphism for any k.*

Proof. By 5.4 p^* is surjective. Because $i^*p^* = id$, it is also injective. □

9.4.2. Corollary: With the same notation

$$c_1(\mathcal{L}) \cap \alpha = i^*i_*\alpha \qquad \text{for any } \alpha \in A_{k+1}(X).$$

Proof. There is the exact diagram

$$\begin{tikzcd} A_{k+1}(X) \arrow[r, "i_*"] \arrow[dr, "c_1(\mathcal{L})\cap"'] & A_{k+1}(L) \arrow[r] \arrow[d, shift left, "i^*"] & A_{k+1}(X \smallsetminus L) \arrow[r] & 0 \\ & A_k(X)\,. \arrow[u, shift left, "p^*"] & & \end{tikzcd}$$

By 9.3, (b), we have $i^*i_*\alpha = c_1(\mathcal{N}_{X/L}) \cap \alpha$ and by 9.2 $\mathcal{N}_{X/L} \cong \mathcal{L}$. □

10. Chern classes of vector bundles

In this section $E \to X$ denotes an algebraic vector bundle of rank $e+1$ over an algebraic scheme over k and $P(E) \xrightarrow{p} X$ the associated projective bundle whose fibre at a closed point is the projective space $\mathbb{P}(E_x)$ of 1–dimensional subspaces of E_x, which is isomorphic to $\mathbb{P}_e(k)$. We let $\mathcal{E}$ denote the locally free sheaf corresponding to E. There is a tautological line subbundle $\mathcal{O}_E(-1) \subset p^*\mathcal{E}$ whose restriction to $\mathbb{P}(E_x)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(E_x)}(-1) \subset E_x \otimes \mathcal{O}_{\mathbb{P}(E_x)}$. The cokernel of $\mathcal{O}_E(-1)$ is the locally free sheaf $\mathcal{T}_{P(E)/X} \otimes \mathcal{O}_E(-1)$ of relative tangent vectors in twist -1. The dual sequence is the relative Euler sequence

$$0 \to \Omega^1_{P(E)/X}(1) \to p^*\mathcal{E}^\vee \to \mathcal{O}_E(1) \to 0.$$

Note that $\mathcal{O}_E(1)$ depends on $\mathcal{E}$ and not only on the scheme $P(E)$. If L is a line bundle on X, then $P(E \otimes L) = P(E)$ but $\mathcal{O}_{E\otimes L}(1) \cong \mathcal{O}_E(1) \otimes p^*\mathcal{L}^\vee$. If X is a variety, then also $P(E)$ is integral and there is a divisor $H \subset P(E)$ such that $\mathcal{O}_E(1) \cong \mathcal{O}_{P(E)}(H)$. Then

H induces the hyperplane divisor $H_x \subset P(E_x)$ for each x on the fibre. The divisor H is effective because locally $P(E)_{U_\alpha} \cong U_\alpha \times \mathbb{P}_e$ and $\mathcal{O}_E(1) \mid P(E)_{U_\alpha}$ is the pull–back of $\mathcal{O}_{\mathbb{P}_e}(1)$. Therefore there are locally regular equations defining H.

10.1. Segre classes $s_i(E)$. Because p is a proper and flat morphism, for any class $\alpha \in A_k(X)$ the class

$$s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*\alpha)$$

is well-defined in $A_{k-i}(X)$. We thus have defined an operator

$$A_k(X) \xrightarrow{s_i(E)} A_{k-i}(X)$$

for any i and any k, called the i-th Segre class of E. When V is a k–dimensional subvariety of X, then $p^*[V] = [p^{-1}V]$ is a subvariety of dimension $k+e$. The Segre operator means cutting $p^{-1}V \quad (e+i)$ times with H to arrive at a $(k-i)$–dimensional cycle and projecting it down again to X.

10.2. Proposition: *With the above notation the Segre classes satisfy the following rules*

(a) $s_0(E) \cap \alpha = \alpha$ *and* $s_i(E) \cap \alpha = 0$ *for* $-e \leq i < 0$.
(b) $s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha)$ *for any two vector bundles* E *and* F *on* X.
(c) *projection formula: given a proper morphism* $X' \xrightarrow{f} X$ *and* $\alpha \in A_k(X')$, *then*

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*\alpha$$

(d) *pull–back formula: given a flat morphism of fixed relative dimension* $X' \xrightarrow{f} X$ *and a class* $\alpha \in A_k(X)$, *then*

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha)$$

(e) *If* $E = L$ *is a line bundle, then*

$$s_1(L) \cap \alpha = -c_1(L) \cap \alpha.$$

Proof. We are going to prove (c) first. Because $P(f^*E)$ is the pull–back of $P(E)$, we have the diagram

$$\begin{array}{ccc} P(f^*E) & \xrightarrow{f'} & P(E) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

with $f'^*\mathcal{O}_E(1) \cong \mathcal{O}_{f^*E}(1)$. Now we get the chain of equalities

$$\begin{aligned}
& f_*(s_i(f^*E) \cap \alpha) \\
= \;& f_* p'_*(c_1(\mathcal{O}_{f^*E}(1))^{e+i} \cap p'^*\alpha) && \text{by definition} \\
= \;& p_* f'_*(c_1(f'^*\mathcal{O}_E(1))^{e+i} \cap p'^*\alpha) && \text{because } p \circ f' = f \circ p', \text{ see 7.3} \\
= \;& p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap f'_* p'^*\alpha) && \text{by 8.8 for } f' \\
= \;& p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^* f_*\alpha) && \text{by 7.17} \\
= \;& s_i(E) \cap f_*\alpha && \text{by definition.}
\end{aligned}$$

The formula (d) can be proved by a similar chain using 8.8, (c). For the proof of (a) we may assume that X is integral and $\alpha = [X]$ is the fundamental class, because, if $\alpha = [V]$ with $V \overset{j}{\hookrightarrow} X$, we have

$$\begin{aligned}
s_0(E) \cap [V] &= s_0(E) \cap j_*[V] \\
&= j_*(s_0(j^*E) \cap [V]) \text{ by (c)} \\
&= j_*[V] \qquad \text{if true for } [V] \text{ and } j^*E \\
&= [V]\,.
\end{aligned}$$

When $U \overset{\varepsilon}{\hookrightarrow} X$ is an open affine subset and $E_U = \varepsilon^* E$, then

$$s_0(E_U) \cap [U] = \varepsilon^*(s_0(E) \cap [X])$$

by (d). If the left hand side equals $[U]$, then $s_0(E) \cap [X] = [X]$ because $\varepsilon^*[X] = [U]$ and $A_n(X) \xrightarrow{\varepsilon^*} A_n(U)$ is an isomorphism. ($A_n(X \smallsetminus U) = 0$ because $\dim(X \smallsetminus U) < \dim X = n$). Now we may assume that X is affine and integral and that E is a trivial bundle or $P(E) = X \times \mathbb{P}_e$. Then $\mathcal{O}_E(1) = q^*\mathcal{O}_{\mathbb{P}_e}(1)$ where q is the second projection, or

$$\mathcal{O}_E(1) \cong \mathcal{O}_{P(E)}(X \times H_{e-1})$$

where H_{e-1} is a hyperplane in $\mathbb{P}_e$. But now

$$c_1(\mathcal{O}_E(1)) \cap [X \times \mathbb{P}_e] = [X \times H_{e-1}].$$

Continuing $e-1$ times with a flag of planes, we arrive at

$$c_1(\mathcal{O}_E(1))^e \cap [X \times \mathbb{P}_e] = [X \times \{pt\}] = [X]\,.$$

This proves the first part of (a). If $-e \leq i < 0$ and V is a subvariety of dimension k, then a cycle ξ representing $c_1(\mathcal{O}_E(1))^{e+i} \cap [p^{-1}V]$ is of dimension $k+i$ and has its support over V. Then $p_*\xi = 0$ by definition, see 7.1. This finishes the proof of (a).

The formula (b) is a consequence of the commutativity relation 8.8, (a). In order to derive it, we consider the pull–back diagram

$$\begin{tikzcd}
Y \arrow[r, "p'"] \arrow[d, "q'"] & P(F) \arrow[d, "q"] \\
P(E) \arrow[r, "p"'] & X
\end{tikzcd}$$

defined by two vector bundles. In this diagram all maps are proper and flat of fixed relative dimension. Let $e+1$ and $f+1$ be the ranks of the bundles. Then

$$
\begin{aligned}
& s_i(E) \cap (s_j(F) \cap \alpha) \\
= \;& p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^* q_*(c_1(\mathcal{O}_F(1))^{f+j} \cap q^*\alpha)) && \text{by definition} \\
= \;& p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap q'_* p'^*(c_1(\mathcal{O}_F(1))^{f+j} \cap q^*\alpha)) && \text{by 7.17} \\
= \;& p_* q'_*(c_1(q'^*\mathcal{O}_E(1))^{e+i} \cap p'^*(c_1(\mathcal{O}_F(1))^{f+j} \cap q^*\alpha) && \text{by 8.4} \\
= \;& p_* q'_*(c_1(q'^*\mathcal{O}_E(1))^{e+i} \cap (c_1(p'^*\mathcal{O}_F(1))^{f+j} \cap p'^* q^*\alpha) && \text{by 8.5.}
\end{aligned}
$$

In the last expression we can interchange the two operators by 8.8, (a). Using then the same chain of equalities, we obtain the formula (b). Finally, if $E = L$ is a line bundle, we have $P(L) = X$ and $\mathcal{O}_L(-1) = \mathcal{L}$ or $\mathcal{O}_E(1) = \mathcal{L}^\vee$. Then

$$
s_1(L) \cap \alpha = c_1(\mathcal{O}_E(1)) \cap \alpha = c_1(\mathcal{L}^\vee) \cap \alpha = -c_1(\mathcal{L}) \cap \alpha. \quad \square
$$

10.3. Corollary: *Let $P(E) \xrightarrow{p} X$ be the projective bundle of a vector bundle of rank $e+1$ over X. Then*

$$
A_k(X) \xrightarrow{p^*} A_{k+e}(P(E))
$$

is a split monomorphism.

Proof. Let ρ be defined by $\rho(\beta) = p_*(c_1(\mathcal{O}_E(1))^e \cap \beta)$ for classes $\beta \in A_{k+e}(P(E))$. Then ρ is a homomorphism $A_{k+e}(P(E)) \to A_k(E)$. If $\beta = p^*\alpha$, then

$$
\rho(p^*\alpha) = p_*(c_1(\mathcal{O}_E(1))^e \cap p^*\alpha) = s_0(E) \cap \alpha = \alpha. \quad \square
$$

10.4. Exercise: Let E be a vector bundle of rank $e+1$ on X and let L be a line bundle on X. Then for any j

$$
s_j(E \otimes L) = \sum_{i=0}^{j} (-1)^{j-i} \binom{e+j}{e+i} s_i(E) c_1(L)^{j-i}.
$$

Here $s_i(E) \cap c_1(L)$ is written as $s_i(E).c_1(L)$ because the intersection operation is commutative.

Proof. We have $P(E \otimes L) = P(E)$ but

$$
\mathcal{O}_{E\otimes L}(-1) \cong \mathcal{O}_E(-1) \otimes p^*\mathcal{L}
$$

because the universal line subbundle of $E \otimes L$ is

$$
\mathcal{O}_{E\otimes L}(-1) \subset p^*\mathcal{E} \otimes p^*\mathcal{L}
$$

and therefore

$$
\mathcal{O}_{E\otimes L}(-1) \otimes p^*\mathcal{L}^\vee \subset p^*\mathcal{E}
$$

is isomorphic to $\mathcal{O}_E(-1)$. Now

$$
c_1\mathcal{O}_{E\otimes L}(1) = c_1\mathcal{O}_E(1) - c_1 p^*\mathcal{L}
$$

and we get

$$
s_j(E \otimes L) \cap \alpha = p_*((c_1(\mathcal{O}_E(1)) - c_1(p^*\mathcal{L}))^{e+j} \cap p^*\alpha).
$$

The formula follows now from the binomial formula for the difference of the c_1–operators.

□

10.5. Recursion formulas. Let R be a commutative ring and $R[\![t]\!]$ the ring of formal power series in one variable. Any series

$$1 + s_1 t + s_2 t^2 + \cdots$$

with first coefficient 1 is a unit in $R[\![t]\!]$. Let

$$(1 + s_1 t + s_2 t^2 + \cdots)^{-1} = 1 + c_1 t + c_2 t^2 + \cdots .$$

The coefficients c_ν can be computed by the recursion formulas

$$c_n + c_{n-1} s_1 + \cdots + s_n = 0 \qquad (SC)$$

The relation $(s_\nu) \leftrightarrow (c_\nu)$ will be referred as the correspondence between Segre and Chern coefficients.

10.6. Chern classes. Let again E be a vector bundle on the algebraic scheme X of rank $e+1$. Its Segre classes $s_i(E)$ are commuting operators on $A_*(X)$. We let $c_i(E)$ be the operators defined by the recursion formulas (SC). Then

$$\begin{array}{rcl} c_1(E) & = & -s_1(E) \\ c_2(E) & = & s_1(E)^2 - s_2(E) \\ c_3(E) & = & -s_1(E)^3 + 2s_1(E)s_2(E) - s_3(E) \\ & \vdots . & \end{array}$$

Thus each $c_i(E)$ is an intersection operator

$$A_k(X) \xrightarrow{c_i(E)\cap} A_{k-i}(X)$$

for all k. The $c_i(E)$ are called the Chern classes of E. The rules for the Segre classes turn into the following rules for Chern classes.

10.7. Proposition: *Let E and F be vector bundles on the algebraic scheme X. Then*

(a) $c_i(E) = 0$ *for* $i > \mathrm{rk}(E)$
(b) $c_i(E).c_j(F) = c_j(F).c_i(E)$
(c) *projection formula: given a proper morphism* $X' \xrightarrow{f} X$ *and* $\alpha \in A_k(X')$, *then*

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha$$

(d) *pull–back formula: given a flat morphism of fixed relative dimension* $X' \xrightarrow{f} X$ *and a class* $\alpha \in A_k(X)$, *then*

$$c_i(f^*(E)) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$

(e) *If E is a line bundle L with sheaf $\mathcal{L} = \mathcal{O}_X(D)$ and X is equi–dimensional, then*

$$c_1(L) \cap [X] = [D]$$

(f) *Whitney's sum formula: given an exact sequence* $0 \to E' \to E \to E'' \to 0$ *of vector bundles on* X*, then*

$$c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$$

for any k*.*

10.8. Remark: On a variety X of dimension n one can define $A^i(X) = A_{n-i}(X)$ using the codimension of the cycles as index. We get classes

$$c_i(E) = c_i(E) \cap [X] \in A^i(X)$$

by intersecting the fundamental class. It is shown in 13.6, that the operators $c_i(E)$ are already determined by the classes $c_i(E)$ if X is a smooth variety.

The rules (b), (c), (d), (e) follow directly from the corresponding formulas for the Segre classes. The rules (a) and (f) will be proved after the theorem of the splitting principle. For that we need the next two lemmata.

10.9. Exercise: *Let* $\mathcal{E}$ *be locally free of rank* $e+1$*, let* $P(\mathcal{E}) \xrightarrow{p} X$ *be the projective bundle and let* $\mathcal{Q}$ *be the tautological quotient bundle with exact sequence*

$$0 \to \mathcal{O}_E(-1) \to p^*\mathcal{E} \to \mathcal{Q} \to 0.$$

Then

$$c_k(\mathcal{Q}) = \sum_{i=0}^{k} c_1(\mathcal{O}_E(1))^i c_i(p^*\mathcal{E})$$

and for any class $\alpha \in A_*(X)$

$$p_*(c_k(\mathcal{Q}) \cap p^*\alpha) = \begin{cases} 0 & k < e \\ \alpha & k = e\,. \end{cases}$$

10.10. Lemma: *Let* $\mathbb{B}$ *be a finite set of vector bundles on* X*. There is a proper and flat morphism* $X' \xrightarrow{f} X$ *of fixed relative dimension such that for any* $E \in \mathbb{B}$ *the pull–back* f^*E *has a filtration*

$$f^*E = E_r \supset E_{r-1} \supset \ldots \supset E_0 = 0$$

by subbundles such that any quotient E_i/E_{i-1} *is a line bundle* L_i*, and such that* $A_*(X) \xrightarrow{f^*} A_*(X')$ *is injective.*

Proof. We proceed by induction on the sum of the ranks of the bundles of $\mathbb{B}$. Starting with one bundle $E \in \mathbb{B}$ we get

$$P(E) \xrightarrow{p} X$$

with a line subbundle $\mathcal{O}_E(-1) \subset p^*\mathcal{E}$ or $L_E \subset p^*E$. By 10.3 the mapping p^* is an injection $A_*(X) \hookrightarrow A_*(P(E))$. Now we can proceed with $p^*(E)/L_E$ by induction, to arrive at a complete flag of subbundles of a lifting of E on $Y \xrightarrow{g} X$. If F is a second bundle on X, we can proceed with g^*F and $P(g^*F) \to Y$. $\square$

10.11. Lemma: *Let $E = E_r \supset \ldots \supset E_0 = 0$ be a filtration with invertible quotients $\mathcal{L}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ and let $0 \neq s \in \Gamma(X, \mathcal{E})$ with zero scheme $Z = Z(s)$. Then for any cycle $\alpha \in Z_k(X)$ there is a cycle $\beta \in Z_{k-r}(Z)$ with*

$$\prod_1^r c_1(L_i) \cap [\alpha] = [\beta] \in A_{k-r}(X)$$

(i.e. the class $\prod c_1(L_i) \cap [\alpha]$ is represented by a cycle with support in Z). In particular, if $Z(s) = \emptyset$, then $\prod c_1(L_i) = 0$.

Proof. The exact sequence $0 \to \mathcal{E}_{r-1} \to \mathcal{E}_r \to \mathcal{L}_r \to 0$ induces the exact sequence

$$0 \to H^0(X, \mathcal{E}_{r-1}) \to H^0(X, \mathcal{E}_r) \to H^0(X, \mathcal{L}_r)$$

and we let $\bar{s}$ denote the image of s in $H^0(X, \mathcal{L}_r)$. If $\bar{s} \neq 0$, then $Y = Z(\bar{s})$ is an effective divisor with $\mathcal{L}_r = \mathcal{O}_X(Y)$. Let $Y \overset{j}{\hookrightarrow} X$ be the inclusion. Then

$$c_1(\mathcal{L}_r) \cap \alpha = j_*(Y.\alpha)$$

where $Y.\alpha \in A_{k-1}(Y)$ is induced by a cycle on Y. Then

$$\begin{aligned} \prod_1^r c_1(L_i) \cap \alpha &= \prod_1^{r-1} c_1(L_i) \cap c_1(L_r) \cap \alpha \\ &= \prod_1^{r-1} c_1(L_i) \cap j_*(Y.\alpha) \\ &= j_*(\prod_1^{r-1} c_1(j^* L_i) \cap (Y.\alpha)). \end{aligned}$$

Because $\bar{s}|Y = 0$ there is a section $t \in H^0(Y, j^*\mathcal{E}_{r-1})$ which is mapped to $s|Y \in H^0(Y, j^*\mathcal{E}_r)$. Because $Z(s) \subset Z(\bar{s}) = Y$, we get $Z(t) = Z \subset Y$. By induction

$$\prod_1^{r-1} c_1(j^*\mathcal{L}_i) \cap (Y.\alpha) = [\beta]$$

with $\beta \in Z_{(k-1)-(r-1)}(Z)$.

If, however, $\bar{s} = 0$, then $s \in H^0(X, \mathcal{E}_{r-1})$ and the zero scheme is the same. Now there is an index ρ such that $s \in H^0(X, \mathcal{E}_\rho)$ and $\bar{s} \neq 0$ in $H^0(X, \mathcal{L}_\rho)$ and $Z = Z(s)$. Let then

$$\gamma = \prod_{\rho+1}^r c_1(\mathcal{L}_i) \cap \alpha \in A_{k-r+\rho}(X).$$

By the first part there is a cycle $\beta \in Z_{k-r}(Z)$ with

$$\prod_1^r c_1(\mathcal{L}_i) \cap \alpha = \prod_1^\rho c_1(\mathcal{L}_i) \cap \gamma = [\beta]. \quad \square$$

10.12. Proposition: *Let E and $P(E) \xrightarrow{p} X$ be as before and suppose that E has a filtration $E = E_r \supset \ldots \supset E_0 = 0$ with invertible quotients $\mathcal{L}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. Then*

$$1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots = (1 + c_1(\mathcal{L}_1)t) \cdot \ldots \cdot (1 + c_1(\mathcal{L}_r)t).$$

In particular, $c_i(\mathcal{E}) = 0$ for $i > r$.

Proof. $\mathcal{O}_E(-1) \hookrightarrow p^*\mathcal{E}$ corresponds to a nowhere vanishing section of $p^*(\mathcal{E}) \otimes \mathcal{O}_E(1)$. By 10.11

$$\prod_1^r c_1(p^*(\mathcal{L}_i) \otimes \mathcal{O}_E(1)) = 0. \tag{1}$$

This equation will be transposed into the formula of the proposition. To do this, let f be any homogeneous polynomial in $\mathbb{Z}[T_1, \ldots, T_r]$. Then the projection formula gives

$$\begin{aligned} & p_*(f(c_1(p^*\mathcal{L}_1), \ldots, c_1(p^*\mathcal{L}_r)).c_1(\mathcal{O}_E(1))^{e+i} \cap p^*\alpha) \\ = \; & f(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_r)) \cap p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*\alpha) \qquad (2) \\ = \; & f(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_r)) \cap s_i(\mathcal{E}) \cap \alpha. \end{aligned}$$

Denoting $\xi = c_1\mathcal{O}_E(1)$, formula (1) becomes

$$\prod_1^r (c_1(p^*\mathcal{L}_\rho) + \xi) = 0$$

or

$$\xi^r + \widetilde{\sigma}_1\xi^{r-1} + \cdots + \widetilde{\sigma}_r = 0 \tag{3}$$

where $\widetilde{\sigma}_i$ denotes the i–th symmetric polynomial in the $c_1(p^*\mathcal{L}_\rho)$. Multiplying with ξ^{i-1} and putting $e = r - 1$, we get the equations

$$\xi^{e+i} + \widetilde{\sigma}_1\xi^{e+i-1} + \cdots + \widetilde{\sigma}_r\xi^{i-1} = 0$$

for $i \geq 1$. This operator equation means that

$$p_*(\xi^{e+i} \cap p^*\alpha) + p_*(\widetilde{\sigma}_1\xi^{e+i-1} \cap p^*\alpha) + \cdots = 0$$

for any $i \geq 1$, and by formula (2) that

$$s_i(\mathcal{E}) \cap \alpha + \sigma_1.s_{i-1}(\mathcal{E}) \cap \alpha + \cdots + \sigma_r s_{i-r}(\mathcal{E}) \cap \alpha = 0$$

for any $i \geq 1$ and $\alpha \in A_*(X)$, where now σ_i is the i–th symmetric function of the $\gamma_\rho = c_1(\mathcal{L}_\rho)$. Let $s_i = s_i(\mathcal{E})$. The last equations are just the equations of the identity

$$(1 + \sigma_1 t + \cdots + \sigma_r t^r)(1 + s_1 t + s_2 t^2 + \cdots) = 1$$

because $s_j(\mathcal{E}) = 0$ for $-r < j < 0$. Therefore, $c_i(\mathcal{E}) = \sigma_i$ and we have

$$\begin{aligned} & 1 + c_1(\mathcal{E})t + \cdots + c_r(\mathcal{E})t^r + \ldots \\ = \; & 1 + \sigma_1 t + \ldots \sigma_r t^r = \prod_1^r (1 + \gamma_\rho t) \end{aligned}$$

which is the formula of the proposition. □

As a corollary of 10.10 and 10.12 we get the

10.13. Theorem: *(Splitting principle)*
Let $\mathbb{B}$ be a finite set of vector bundles on X. There is a proper and flat morphism $Y \xrightarrow{f} X$ of fixed relative dimension such that $A_*(X) \xrightarrow{f^*} A_*(Y)$ is injective, and such that any

$f^*\mathcal{E}, \mathcal{E} \in \mathbb{B}$, has a complete filtration with invertible quotients and such that the Chern polynomial

$$\begin{aligned} c_t(f^*\mathcal{E}) &= 1 + c_1(f^*\mathcal{E})t + c_2(f^*\mathcal{E})t^2 + \cdots \\ &= (1+\gamma_1 t)\cdots(1+\gamma_r t)\,. \end{aligned}$$

The classes $\gamma_i = c_1(\mathcal{L}_i)$ are called Chern roots of $\mathcal{E}$ and $Y \xrightarrow{f} X$ is called a splitting morphism of the bundles in $\mathbb{B}$.

10.14. Proof of proposition 10.7, (a) and (f). Let $Y \xrightarrow{f} X$ be a splitting morphism for E. Then $c_i(f^*\mathcal{E}) = 0$ for $i > r$. The pull–back formula says that $f^*(c_i(\mathcal{E}) \cap \alpha) = c_1(f^*\mathcal{E}) \cap f^*\alpha$ for any α. Because f^* is injective, the result (a) follows. Let now $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of locally free sheaves with corresponding bundles spaces. By the splitting principle we may assume then $Y \xrightarrow{f} X$ is a splitting morphism for the three bundles.

If $\mathcal{L}'_i$ and $\mathcal{L}''_j$ are the invertible quotients for filtrations of $f^*\mathcal{E}'$ and $f^*\mathcal{E}''$ respectively, we can construct a filtration of $f^*\mathcal{E}$ whose quotients are all the sheaves $\mathcal{L}'_i$ and $\mathcal{L}''_j$ together. Then the formula in 10.13 becomes

$$c_t(f^*\mathcal{E}) = \prod_i (1+\gamma'_i t) \prod_j (1+\gamma''_j t) = c_t(f^*\mathcal{E}')c_t(f^*\mathcal{E}'').$$

This is equivalent to

$$c_k(f^*\mathcal{E}) = \sum_{i+j=k} c_i(f^*\mathcal{E}')c_j(f^*\mathcal{E}'').$$

Again the pull–back formula and the injectivity of f^* on the Chow groups imply Whitney's formula.

10.14.1. Corollary 1: *If $\mathcal{E}$ has a nowhere vanishing section, then $c_r(\mathcal{E}) = 0$.*

Proof. The assumption implies that there is an exact sequence $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{F} \to 0$ where $\mathcal{F}$ is locally free of rank $r-1$. Because the Chern polynomial of $\mathcal{O}$ is the constant 1, $c_t(\mathcal{E}) = c_t(\mathcal{F})$ and so $c_r(\mathcal{E}) = c_r(\mathcal{F}) = 0$. □

The next corollary states that the construction of $P(E)$ is just to provide a first root of the Chern polynomial $c_t(\mathcal{E})$ which is $c_1\mathcal{O}_E(1)$. It could be used to define the Chern classes by the following formula.

10.15. Corollary 2: *Let E be a rank r vector bundle on X and let $\xi = c_1\mathcal{O}_E(1)$ be the class of the tautological bundle on $P(E) \xrightarrow{p} X$. Then*

$$\xi^r + c_1(p^*\mathcal{E})\xi^{r-1} + \cdots + c_r(p^*\mathcal{E}) = 0$$

on $A_*(P(E))$.

Proof. Over $P(E)$ we have the exact sequence

$$0 \to \mathcal{O}_E(-1) \to p^*\mathcal{E} \to \mathcal{E}'' \to 0.$$

By Whitney's formula

$$(1 + c_1(\mathcal{O}_E(-1))t)c_t(\mathcal{E}'') = c_t(\mathcal{E})$$

or

$$(1 - \xi t)(1 + a_1 t + \cdots + a_{r-1}t^{r-1}) = 1 + c_1 t + \cdot + c_r t^r$$

where $a_i = c_i(\mathcal{E}'')$ and $c_j = c_j(p^*\mathcal{E})$. Avoiding the substitution $t = \xi^{-1}$, we use the identities

$$c_i = a_i - \xi a_{i-1}$$

to derive the relation

$$\xi^r + c_1\xi^{r-1} + \cdot + c_r = 0. \quad \square$$

Remark*:* The pull–back formula $c_i(p^*\mathcal{E}) \cap p^*\alpha = p^*(c_i(\mathcal{E}) \cap \alpha)$ and the injectivity of p^* imply that the classes $c_i(\mathcal{E})$ are determined by the formula of Corollary 2. Moreover, if $Y \xrightarrow{f} X$ is any splitting morphism for a finite set $\mathbb{B}$ of vector bundles, any polynomial formula between the Chern classes $c_i(f^*\mathcal{E}), \mathcal{E} \in \mathbb{B}$, turns into a formula between the classes $c_i(\mathcal{E})$ with the same terms.

10.16. Remark: The splitting of $f^*\mathcal{E}$ can alternatively be obtained by the flag bundle $F(\mathcal{E}) \xrightarrow{f} X$ on which $f^*\mathcal{E}$ contains the universal or tautological flag

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_r = f^*\mathcal{E}\,,$$

see 10.10 and Section 14.

10.17. Remark: For any locally free sheaf $\mathcal{E}$ there is also a locally trivial fibration $\mathrm{Sp}(\mathcal{E}) \xrightarrow{g} X$ which factors through $\mathrm{Fl}(\mathcal{E}) \xrightarrow{f} X$ such that $A_*(X) \hookrightarrow A_*(\mathrm{Fl}(\mathcal{E})) \xrightarrow{\approx} A_*(\mathrm{Sp}(\mathcal{E}))$ and such that $g^*\mathcal{E} \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$ splits into a direct sum of line bundles, see [13]. This can be used to construct a flat (not necessarily proper) morphism $Z \xrightarrow{g} X$ such that $A_*(X) \overset{g^*}{\hookrightarrow} A_*(Z)$ is injective and $f^*\mathcal{E}$ splits for finitely many $\mathcal{E}$. Moreover, if $Y \xrightarrow{f} X$ is a morphism as in 10.13, one can construct $Z \xrightarrow{h} Y$ such that $g = f \circ h$ has this property. Therefore, in the following applications one might assume that the pulled back bundles all split into direct sums of line bundles. However, the line bundles of the complete flags provide the same formulas for the Chern classes.

10.18. Chern classes of a dual bundle.

Let $\mathcal{E}$ be locally free on X. Then $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$ for any i.

Proof. Let $Y \xrightarrow{f} X$ be a splitting morphism for $\mathcal{E}$ such that $c_t(f^*\mathcal{E}) = (1+\gamma_1 t)\cdot\ldots\cdot(1+\gamma_r t)$ with $\gamma_1 = c_1(\mathcal{L}_i)$. It is easy to see that a filtration of $f^*\mathcal{E}$ with quotients $\mathcal{L}_i$ yields a dual filtration with quotients $\mathcal{L}_{r-i}^\vee$. Therefore $f^*\mathcal{E}^\vee$ has the Chern roots $-\gamma_1, \ldots, -\gamma_r$. This implies the identities. $\square$

10.19. Chern classes of tensor products.

Let $\mathcal{E}$ and $\mathcal{F}$ be locally free of ranks r and s and let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$ be the Chern roots of $\mathcal{E}$ and $\mathcal{F}$ respectively. Then $\mathcal{E} \otimes \mathcal{F}$ has the Chern roots $\alpha_i + \beta_j$ and

$$c_t(f^*(\mathcal{E} \otimes \mathcal{F})) = \prod_{i,j} (1 + (\alpha_i + \beta_j)t)$$

where f is a splitting morphism. Computing the symmetric polynomials of the roots $\alpha_i + \beta_j$ one arrives at a formula

$$c_t(\mathcal{E} \otimes \mathcal{F}) = P_{r,s}(c_1(\mathcal{E}), \ldots, c_r(\mathcal{E}); c_1(\mathcal{F}), \ldots, c_s(\mathcal{F}))$$

where $P_{r,s}$ is a polynomial.

Proof. Let f be a splitting morphism for both $\mathcal{E}$ and $\mathcal{F}$. If $\mathcal{L}_i$ and $\mathcal{L}'_j$ are quotients of filtrations of $f^*\mathcal{E}$ nd $f^*\mathcal{F}$, one can construct a filtrations of $f^*\mathcal{E} \otimes f^*\mathcal{F} = f^*(\mathcal{E} \otimes \mathcal{F})$ with quotients $\mathcal{L}_i \otimes \mathcal{L}'_j$. Since $c_1(\mathcal{L}_i \otimes \mathcal{L}'_j) = c_1(\mathcal{L}_i) + c_1(\mathcal{L}'_j)$, this proves the formula. $\square$

For small ranks the formula for the Chern classes can be derived very quickly by using the elementary symmetric functions of the roots. For example, in case $r = 2 = s$ we get

$$\begin{aligned}
c_1(\mathcal{E} \otimes \mathcal{F}) &= 2c_1(\mathcal{E}) + 2c_1(\mathcal{F}) \\
c_2(\mathcal{E} \otimes \mathcal{F}) &= 2c_2(\mathcal{E}) + 2c_2(\mathcal{F}) + c_1(\mathcal{E})^2 + 3c_1(\mathcal{E})x_1(\mathcal{F}) + c_1(\mathcal{F})^2 \\
c_3(\mathcal{E} \otimes \mathcal{F}) &= 2c_1(\mathcal{E})c_2(\mathcal{E}) + c_1(\mathcal{E})^2c_1(\mathcal{F}) + 2c_2(\mathcal{E})c_1(\mathcal{F}) + 2c_1(\mathcal{E})c_2(\mathcal{F}) + c_1(\mathcal{E})c_1(\mathcal{F})^2 \\
&\quad +2c_1(\mathcal{F})c_2(\mathcal{F}) \\
c_4(\mathcal{E} \otimes \mathcal{F}) &= c_2(\mathcal{E})^2 + c_1(\mathcal{E})c_1(\mathcal{F})(c_2(\mathcal{E}) + c_2(\mathcal{F})) + c_2(\mathcal{E})c_1(\mathcal{F}) - 2c_2(\mathcal{E})c_2(\mathcal{F}) \\
&\quad +c_1(\mathcal{E}^2c_2(\mathcal{F}) + c_2(\mathcal{F})^2.
\end{aligned}$$

For the tensor product of a line bundle $\mathcal{L}$ with a rank r bundle $\mathcal{E}$ we get

$$c_k(\mathcal{E} \otimes \mathcal{L}) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(\mathcal{E})c_1(\mathcal{L})^{k-i} \ .$$

10.20. Chern classes of wedge products.

Let $\mathcal{E}$ be locally free of rank r on X with Chern roots $\alpha_1, \ldots, \alpha_r$. Then

$$c_t(\Lambda^p\mathcal{E}) = \prod_{i_1 < \cdots < i_p} (1 + (\alpha_{i_1} + \ldots + \alpha_{i_p})t) \ .$$

In particular $c_1(\Lambda^r\mathcal{E}) = c_1(\mathcal{E})$. The formula for $\Lambda^p\mathcal{E}$ could also be derived inductively if an exact sequence (e.g. on $P(E)$) $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ is specified. Then there are exact sequences

$$0 \to \Lambda^{p-1}\mathcal{F} \otimes \mathcal{L} \to \Lambda^p\mathcal{E} \to \Lambda^p\mathcal{F} \to 0$$

and one can use Whitney's formula.

10.21. Chern classes of symmetric products.

Let $\mathcal{E}$ be locally free of rank r on X with Chern roots $\alpha_1, \ldots, \alpha_r$. Starting with a filtration of $f^*\mathcal{E}$ one can prove that $S^p\mathcal{E}$ has the Chern roots $p_1\alpha_1 + \ldots + p_r\alpha_r$, where $p_1, \ldots, p_r$ are natural numbers subject to $p_1 + \ldots + p_r = p$.

E.g. for $r = 2$ and $p = 2$ we have

$$\begin{aligned} c_1(S^2\mathcal{E}) &= 3c_1(\mathcal{E}) \\ c_2(S^2\mathcal{E}) &= 4c_2(\mathcal{E}) + 2c_1(\mathcal{E})^2 \\ c_3(S^2\mathcal{E}) &= 4c_1(\mathcal{E})c_2(\mathcal{E}). \end{aligned}$$

10.22. Chern characters

Let $\mathcal{E}$ be locally free of rank r on X with Chern roots $\alpha_1, \ldots, \alpha_r$. The power series $\exp(\alpha_i)$ is finite as an operator on $A_*(Y)$ where $Y \xrightarrow{f} X$ is a splitting morphism. The sum

$$\mathrm{ch}(\mathcal{E}) = \sum_{i=1}^{r} \exp(\alpha_i)$$

becomes a polynomial in the Chern classes $c_i = c_i(\mathcal{E})$ because it is symmetric in the α_i. It is called the Chern character. Its first terms are

$$\mathrm{ch}(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \ldots.$$

Its n–th term p_n may be computed inductively by the Newton formulas

$$p_\nu - c_1 p_{\nu-1} \pm \ldots + (-1)^{\nu-1} c_{\nu-1} p_1 + (-1)^\nu \nu c_\nu = 0.$$

The Chern character has the formal advantage that for an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of locally free sheaves we have

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}(\mathcal{E}') + \mathrm{ch}(\mathcal{E}'')$$

and for a tensor product

$$\mathrm{ch}(\mathcal{E} \otimes \mathcal{F}) = \mathrm{ch}(\mathcal{E}).\mathrm{ch}(\mathcal{F}).$$

10.23. Todd classes.

In a similar way the Todd class of a locally free sheaf $\mathcal{E}$ of rank r had been introduced. It is defined by

$$\mathrm{td}(\mathcal{E}) = \prod_{i=1}^{r} Q(\alpha_i) \ ,$$

where $Q(x)$ is the power series

$$Q(x) = x(1 - \exp(-x))^{-1} = 1 + \frac{1}{2}x + \sum_{n \geq 2} (-1)^{n-1} \frac{B_n}{(2n)!} x^{2n}.$$

The coefficients contain the well known Bernoulli numbers. The first terms of $\mathrm{td}(\mathcal{E})$ are

$$\mathrm{td}(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \ldots$$

Similarly to the Chern character the Todd class is multiplicative on exact sequences, i.e.

$$\mathrm{td}(\mathcal{E}) = \mathrm{td}(\mathcal{E}')\mathrm{td}(\mathcal{E}'')$$

for any exact sequence of locally free sheaves as above.

Exercise: (Borel–Serre) Let $\mathcal{E}$ be locally free of rank r on a scheme X. Then

$$\sum_{p=0}^{r}(-1)^p \mathrm{ch}(\Lambda^p\mathcal{E}^\vee) = c_r(\mathcal{E})\mathrm{td}(\mathcal{E})^{-1}$$

11. Chow groups of vector bundles and projective bundles

Let $\mathcal{E}$ be a locally free sheaf of rank $r = e+1$ on an algebraic scheme X and let $E \xrightarrow{\pi} X$ denote its bundle space and $P(E) \xrightarrow{p} X$ its associated projective bundle. It had been shown in 5.4 and 10.3 that

$$A_k(X) \overset{\pi^*}{\twoheadrightarrow} A_{k+r}(E) \quad \text{is surjective}$$

and that

$$A_k(X) \overset{p}{\rightarrowtail} A_{k+e}(P(E)) \quad \text{is injective.}$$

We are now in position to prove that π^* is bijective and to compute $A_k(P(E))$ in terms of the groups $A_j(X)$ and the Chern classes of E.

11.1. Theorem: *With the above notation*

(1) π^* *is an isomorphism for any* k.
(2) *For any* k *the homomorphism*

$$\bigoplus_{0\le i\le e} A_{k-e+i}(X) \xrightarrow{\theta_E} A_k(P(E))$$

defined by

$$\theta_E(\alpha_{k-e},\dots,\alpha_k) = \sum_{0\le i\le e} c_1(\mathcal{O}_E(1))^i \cap p^*\alpha_{k-e+i}$$

is an isomorphism.

Proof. a) Surjectivity of θ_E. As in the proof of 5.4 we can reduce this case to the situation where X is affine and $\mathcal{E}$ is trivial by induction on the dimension. (If $U \subset X$ is open and affine, consider the exact sequence 4.5 given by $Y = X \smallsetminus U$). Let $\mathcal{E} = \mathcal{F} \oplus \mathcal{O}$ and consider the inclusions

$$P(\mathcal{F}) \overset{i}{\hookrightarrow} P(\mathcal{E}) \quad \text{and} \quad F = P(\mathcal{E}) \smallsetminus P(\mathcal{F}) \overset{j}{\hookrightarrow} P(\mathcal{E}).$$

Here $P(\mathcal{F})$ is the relative hyperplane at infinity and F is its affine complement. We are given the commutative diagram

$$\begin{tikzcd} P(\mathcal{F}) \arrow[r, hook, "i"] \arrow[dr, "q"'] & P(\mathcal{E}) \arrow[d, "p"] & F \arrow[l, hook', "j"'] \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_k(P(\mathcal{F})) \arrow[r, "i_*"] & A_k(P(\mathcal{E})) \arrow[r, "j^*"] & A_k(F) \arrow[r] & 0 \\ & A_{k-e}(X)\,. \arrow[u, "p^*"] \arrow[ur, "\pi^*"'] & & \end{tikzcd}$$

It follows from the definition of the projective bundles that $\mathcal{O}_F(1) \cong i^*\mathcal{O}_E(1)$ on $P(\mathcal{F})$. Moreover, the summand $\mathcal{O}$ of $\mathcal{E}$ induces a section $\mathcal{O}_{P(\mathcal{E})} \to \mathcal{O}_E(1)$ which vanishes exactly on $P(\mathcal{F})$ and has $\mathcal{O}_F(1)$ as its cokernel.

Now π^* is surjective by 5.4.

Claim: $c_1(\mathcal{O}_E(1)) \cap p^*\alpha = i_*q^*\alpha$ for any $\alpha \in A_*(X)$.

Proof of the claim: We may assume that $\alpha = [V]$ is the class of a subvariety of X of dimension k. Then $p^*\alpha = [p^{-1}V]$ of dimension $k+e$. Now $\mathcal{O}_E(1) \cong \mathcal{O}_{P(\mathcal{E})}(P(\mathcal{F}))$ because $\mathcal{O}_E(1)$ has a section which has $P(\mathcal{F})$ as its zero scheme. Therefore

$$c_1(\mathcal{O}_E(1)) \cap [p^{-1}V] = [C]$$

where $\mathcal{O}_{P(\mathcal{E})}(P(\mathcal{F})) \mid p^{-1}V \cong \mathcal{O}_{p^{-1}V}(C)$. But $P(\mathcal{F})$ is effective and $p^{-1}V \not\subset P(\mathcal{F})$. Therefore, $C \sim p^{-1}V \cap P(\mathcal{F}) = q^{-1}V$ and then

$$[C] = i_*[q^{-1}V] = i_*q^*[V].$$

This ends the proof of the claim. Let now $\beta \in A_k(P(\mathcal{E}))$. There is an element $\alpha = \alpha_{k-e} \in A_{k-e}(X)$ with $j^*\beta = \pi^*\alpha = j^*p^*\alpha$ or $j^*(\beta - p^*\alpha) = 0$. We may assume that θ_F is surjective by induction on the rank. Hence, there are classes

$$\alpha_{k-e+1}, \ldots, \alpha_k$$

in the Chow groups of the same index respectively such that

$$\beta = p^*\alpha + i_* \sum_{0 \le \nu < e} c_1(\mathcal{O}_F(1))^\nu \cap q^*\alpha_{k-e+\nu+1}.$$

Now $\mathcal{O}_F(1) = i^*\mathcal{O}_E(1)$ and the projection formula together with the claim imply

$$\beta = p^*\alpha + \sum_{0 \le \nu < e} c_1(\mathcal{O}_E(1))^{\nu+1} \cap p^*\alpha_{k-e+1+\nu}.$$

Replacing $\mu = \nu + 1$ we have $\theta_E(\alpha_{k-e}, \ldots, \alpha_k) = \beta$.

b) Injectivity of θ_E. Let $\beta = \theta_E(\alpha_{k-e}, \ldots, \alpha_k) = 0$, and let l be the largest index with $\alpha_l \neq 0, k - e \le l \le k$. Then β intersected with $c_1(\mathcal{O}_E(1))$ gives

$$\begin{aligned} 0 &= p_*(c_1(\mathcal{O}_E(1))^{k-l} \cap \beta) \\ &= p_*(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^{k-l+i} \cap p^*\alpha_{k-e+i}) \\ &= \alpha_l \end{aligned}$$

because $s_j(\mathcal{E}) \cap \alpha$ for $j < 0$ and $s_0(\mathcal{E}) \cap \alpha = \alpha$ for any α and $k - l + i = e + (k-e) - l + i$.

c) Injectivity of π^*. We may assume that the vector bundle F is the complement $P(\mathcal{E}) \smallsetminus P(\mathcal{F})$, see 5.3. Then we can use the previous diagram. Let $\pi^*\alpha = 0$. Then $j^*p^*\alpha = 0$ and

$$p^*\alpha = i_*(\sum_{0 \le \nu < e} c_1(\mathcal{O}_F(1))^\nu \cap q^*\alpha_{\nu})$$

where $\alpha \in A_{k-e}(X)$ and $(\alpha_{k-e+1}, \ldots, \alpha_k)$ is given by the surjectivity of θ_F. Then again

$$\begin{aligned} p^*\alpha &= i_* \sum_{0 \le \nu < e} c_1(i^*\mathcal{O}_E(1))^\nu \cap q^*\alpha_{k-e+1+\nu} \\ &= \sum_{0 \le \nu < e} c_1(\mathcal{O}_E(1))^\nu \cap i_*q^*\alpha_{k-e+1+\nu} \\ &= \sum_{0 \le \nu < e} c_1(\mathcal{O}_E(1))^{\nu+1} \cap p^*\alpha_{k-e+1+\nu}\,. \end{aligned}$$

The injectivity of θ_E implies now that $\alpha = 0$ and $\alpha_{k-e+1+\nu} = 0$. $\square$

Remark: It will be shown in 13.8 that by θ_E already the Chow ring of $P(\mathcal{E})$ is determined.

11.2. Gysin homomorphism of a vector bundle. Let $\mathcal{E}$ be locally free of rank r and now $\mathcal{F} = \mathcal{E} \oplus \mathcal{O}$ with inclusions

$$P(\mathcal{E}) \stackrel{i}{\hookrightarrow} P(\mathcal{F}) \stackrel{j}{\hookleftarrow} E\ ,$$

where E is the bundle space of $\mathcal{E}$. We are given the diagram

$$\begin{tikzcd} A_k(P(\mathcal{E})) \arrow[r, "i_*"] & A_k(P(\mathcal{F})) \arrow[r, "j^*"] & A_k(E) \arrow[r] & 0 \\ & A_{k-r}(X)\,. \arrow[u, "q^*"] \arrow[ur, "\pi^*", "\approx"'] & & \end{tikzcd}$$

We let s^* denote the inverse operator of π^*. We are going to give a formula for s^* in terms of the Chern class $c_r(\mathcal{Q})$ where $\mathcal{Q}$ is the tautological quotient in $0 \to \mathcal{O}_{\mathcal{F}}(-1) \to q^*\mathcal{F} \to \mathcal{Q} \to 0$ on $P(\mathcal{F})$.

11.2.1. Proposition: *(Gysin formula) For any $\beta \in A_k(E)$*

$$s^*(\beta) = q_*(c_r(\mathcal{Q}) \cap \bar{\beta})$$

where $j^*\bar{\beta} = \beta$.

Proof. We can write $\bar{\beta} = q^*\gamma + i_*\delta$ with $\pi^*\gamma = \beta$ by the exactness of the diagram and bijectivity of π^*. We are going to show that

$$\pi^* q_*(c_r(\mathcal{Q}) \cap \bar{\beta}) = j^*\bar{\beta} = \beta$$

which proves the formula.

(a) The Chern polynomial of $\mathcal{Q}$ satisfies $c_t(q^*\mathcal{E}) = c_t(q^*\mathcal{F}) = c_t(\mathcal{Q})(1 - c_1(\mathcal{O}_F(1))t)$ and this implies that

$$c_r(\mathcal{Q}) = \sum_{\nu=0}^{r} c_{r-\nu}(q^*\mathcal{E}) c_1(\mathcal{O}_F(1))^\nu.$$

Then

$$\begin{aligned} q_*(c_r(\mathcal{Q}) \cap q^*\gamma) &= q_*(\sum_{\nu=0}^{r} c_{r-\nu}(q^*\mathcal{E}) c_1(\mathcal{O}_F(1))^\nu \cap q^*\gamma) \\ &= \sum_{\nu=0}^{r} c_{r-\nu}(\mathcal{E}) \cap q_*(c_1(\mathcal{O}_F(1))^\nu \cap q^*\gamma) \\ &= s_0(\mathcal{E}) \cap \gamma = \gamma \end{aligned}$$

by 10.2, (a).

(b) Because $q^*\mathcal{F} = q^*\mathcal{E} \oplus \mathcal{O}_{P(\mathcal{F})}$, the sheaf $\mathcal{Q}$ has a section $\mathcal{O}_{P(\mathcal{F})} \xrightarrow{\sigma} \mathcal{Q}$. This is nowhere vanishing on $P(\mathcal{E})$ by its definition: At a point $\langle\zeta\rangle \in P(E_x \oplus k)$ we have the exact sequence

$$0 \to \langle\zeta\rangle \to E_x \bigoplus k \to \mathcal{Q}(\langle\zeta\rangle) \to 0.$$

If $\sigma_{<\zeta>}(1) = 0$, then $\langle(0,1)\rangle = \langle\zeta\rangle$ and $\langle\zeta\rangle \notin P(\mathcal{E})$. Now the restriction of σ to $P(\mathcal{E})$ defines a subbundle and a quotient bundle

$$0 \to \mathcal{O}_{P(\mathcal{E})} \to i^*\mathcal{Q} \to \mathcal{Q}'' \to 0$$

of rank $r-1$. This implies that $c_r(\mathcal{Q}'')=0$ and $c_r(i^*\mathcal{Q})=0$. This implies

$$c_r(\mathcal{Q})\cap i_*\delta = i_*(c_r(i^*Q)\cap\delta)=0$$

by the projection formula.

(c) Finally, (a) and (b) yield

$$\pi^*q_*(c_r(\mathcal{Q})\cap\bar{\beta}) = \pi^*q_*(c_r(Q)\cap(q^*\gamma+i_*\delta)) = \pi^*\gamma=\beta. \quad \square$$

12. Normal cones

Cones over a scheme X are defined as spectra of graded $\mathcal{O}_X$–algebras $\mathcal{S}^\bullet = \underset{n\geq 0}{\oplus}\mathcal{S}^n$ for which we suppose that $\mathcal{O}_X\to\mathcal{S}^0$ is surjective, $\mathcal{S}^1$ is coherent as $\mathcal{O}_X$–module and $\mathcal{S}^\bullet$ is locally generated by $\mathcal{S}^1$. Then

$$C=\operatorname{Spec}(\mathcal{S}^\bullet)\rightleftarrows X$$

and

$$P(C)=\operatorname{Proj}(\mathcal{S}^\bullet)\to X$$

are called the (affine) cone respectively the projective cone of the graded algebra, the morphism to X being induced by $\mathcal{O}_X\to\mathcal{S}^0$, see [9], §7. If $\mathcal{O}_X\cong\mathcal{S}^0$, then C has a section which is defined by the surjection $\mathcal{S}^\bullet\to\mathcal{S}^0$. If $\mathcal{S}^1$ is locally generated by sections $s_1,\dots,s_N$ of $\mathcal{S}^1|U$ then there is an exact sequence

$$0\to\mathcal{A}\to\mathcal{O}_X[T_1,\dots,T_N]\to\bigoplus_{n\geq 0}\mathcal{S}^n\to 0$$

over U and $C|U$ is defined by the graded ideal sheaf $\mathcal{A}$, and we obtain embeddings

$$C|U\subset U\underset{k}{\times}\mathbb{A}^N \quad\text{and}\quad P(C)|U\subset U\underset{k}{\times}\mathbb{P}_{N-1}\,.$$

12.1. Normal cones and blow up

Let $X\hookrightarrow Y$ be a closed subscheme of an algebraic scheme Y with ideal sheaf $\mathcal{I}$. Then we obtain the following cones:

$C_XY:$	$=$	$\operatorname{Spec}(\underset{n\geq 0}{\oplus}\mathcal{I}^n/\mathcal{I}^{n+1})$	normal cone
$PC_XY:$	$=$	$\operatorname{Proj}(\underset{n\geq 0}{\oplus}\mathcal{I}^n/\mathcal{I}^{n+1})$	projective normal cone
$N_XY:$	$=$	$\operatorname{Spec}(\underset{n\geq 0}{\oplus}S^n(\mathcal{I}/\mathcal{I}^2)=\mathbb{V}(\mathcal{I}/\mathcal{I}^2)$	normal fibration
$PN_XY:$	$=$	$\operatorname{Proj}(\underset{n\geq 0}{\oplus}S^n(\mathcal{I}/\mathcal{I}^2)=\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$	projective normal fibration
$B_XY:$	$=$	$\operatorname{Proj}(\underset{n\geq 0}{\oplus}\mathcal{I}^n)$	blow up of Y along X.

Here S^n denotes the n–th symmetric product. There are natural surjections $S^n(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}^n/\mathcal{I}^{n+1}$ and the induced diagrams

$$\begin{tikzcd} \oplus S^n(\mathcal{I}/\mathcal{I}^2) \arrow[rr, two heads] \arrow[dr, dashed, leftrightarrow] & & \oplus \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[dl, dashed, leftrightarrow] \\ & \mathcal{O}_Y/\mathcal{I} & \end{tikzcd} \qquad \begin{tikzcd} C_XY \arrow[rr, hook] \arrow[dr, dashed, leftrightarrow, "s"'] & & N_XY \arrow[dl, dashed, leftrightarrow, "0"] \\ & X\,. & \end{tikzcd}$$

The section s is called the zero section of the cone. There is also the induced embedding $PC_XY \hookrightarrow PN_XY$ over X. Likewise we have the diagrams

$$\begin{tikzcd} \oplus\mathcal{I}^n \arrow[r, two heads] \arrow[d, dashed, leftrightarrow] & \oplus\mathcal{I}^n/\mathcal{I}^{n+1} \arrow[d, dashed, leftrightarrow] \\ \mathcal{O}_Y \arrow[r, two heads] & \mathcal{O}_Y/\mathcal{I} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} PC_XY \arrow[r, hook] \arrow[d] & B_XY \arrow[d] \\ X \arrow[r, hook] & Y\,. \end{tikzcd}$$

such that PC_XY is the exceptional divisor of the blow up.

12.2. Regular embeddings: In general the normal fibration is not a vector bundle. However, if $X \hookrightarrow Y$ is a regular embedding of codimension d, i.e. $\mathcal{I}$ is locally generated by a regular sequence $f_1, \ldots, f_d \in \Gamma(U, \mathcal{I})$, then the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free and $N_XY = \mathbb{V}(\mathcal{I}/\mathcal{I}^2)$ is the normal bundle. In that case $S^n(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{I}^n/\mathcal{I}^{n+1}$ for any n and

$$C_XY = N_XY.$$

Moreover, if $f_1, \ldots, f_d$ is a regular system for $\mathcal{I}|U,$ then $B_XY|U \subset U \times \mathbb{P}_{d-1}$ is defined by the equations $x_if_j - x_jf_i$.

12.3. Examples: 1) Let $Y \subset \mathbb{A}^N$ be an affine hypersurface with equation $f = f_m + f_{m+1} + \ldots$ where f_m is the leading term of the polynomial, and let $X = \{0\}$ be the origin. Then C_XY can be embedded into $\{0\} \times \mathbb{A}^N = \mathbb{A}^N$ and it is nothing but the zero scheme $Z(f_m)$ of the homogeneous leading term. In particular, if Y is the cuspidal cubic with $f = y^2 - x^3$, then the normal cone is the double line $Z(y^2) \subset \{0\} \times \mathbb{A}^2 \cong \mathbb{A}^2$ and the normal fibration is $N_XY = \{0\} \times \mathbb{A}^2$.

2) If $D \overset{i}{\hookrightarrow} Y$ is an effective divisor, then $C_DY = N_DY$ is the bundle space of the invertible sheaf $i^*\mathcal{O}_Y(D) = \mathcal{O}_D(D)$.

12.4. Lemma: *If Y is purely k–dimensional then also C_XY is purely k–dimensional.*

Proof. Identifying Y with $Y \times \{0\}$ in $Y \times \mathbb{A}^1$ we consider the blow up $B_X(Y \times \mathbb{A}^1)$ with exceptional divisor $PC_X(Y \times \mathbb{A}^1)$ which is the projective completion of C_XY. Because X is nowhere dense in $Y \times \mathbb{A}^1$, the blow up is birational to $Y \times \mathbb{A}^1$ and has pure dimension $k+1$. Then C_XY has pure dimension k as an open dense set of the exceptional Cartier divisor. $\square$

12.5. Normal cone of a pull–back: Given a pull back diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

we obtain the diagrams

$$\begin{tikzcd} C_{X'}Y' \arrow[r] \arrow[d] & C_XY \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} N_{X'}Y' \arrow[r] \arrow[d] & N_XY \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

and from the first morphism $C_{X'}Y' \to X' \times_X C_XY =: g^*C_XY$. This is an embedding by the following argument. When $\mathcal{I}$ respectively $\mathcal{J}$ denote the ideal sheaves of X respectively X', then $\mathcal{J}$ is the image of $f^*\mathcal{I} \to \mathcal{O}_{Y'}$, and this induces the surjective homomorphism $\oplus f^*(\mathcal{I}^n/\mathcal{I}^{n+1}) \to \oplus \mathcal{J}^n/\mathcal{J}^{n+1}$, which defines the embedding $C_{X'}Y' \hookrightarrow g^*C_XY$. Similarly, we obtain an embedding $N_{X'}Y' \hookrightarrow X' \times_X N_XY = g^*N_XY$ because of the induced surjection $f^*S^n(\mathcal{I}/\mathcal{I}^2) \to S^n(\mathcal{J}/\mathcal{J}^2)$.

13. Intersection products

Motivation, see also [8]: Let Y be an algebraic scheme and $X \hookrightarrow Y$ be a regularly embedded subscheme of codimension d. If X is globally the intersection of d Cartier divisors, we can define the intersection class $X.V$ with a subvariety V simply as $D_1. \ \dots \ .D_d.V$. But even when X and Y are smooth, X need not to be the intersection of d divisors globally. On the other hand $X \cap V$ may have irreducible components of various dimensions. It turned out that an intersection class $X.V$ can be defined with a good general behaviour and producing most of the specific classical intersection results by using the normal cone of $X \cap V$ in V. One should note that the normal cone functions as essential leading "part" of a subvariety and thus may be used to define intersection multiplicities. The problem that it is not contained in the ambient scheme can be settled by embedding the cone $C_{X\cap V}V$ into the bundle N_XY and intersecting it with the zero section via the Gysin isomorphism. This yields the expected dimension $k-d$ for the resulting class $X.V$.

The intersection class $V.W$ of two subvarieties of dimensions k and l is then obtained by

$$V.W = \triangle.(V \times W)$$

where $\triangle \subset Y \times Y$ is the diagonal. This corresponds to the set theoretic identity $V \cap W = \triangle \cap (V \times W)$.

13.1. Intersection with a regular embedded subscheme.

Let $X \overset{i}{\hookrightarrow} Y$ be a regularly embedded closed subscheme of codimension d and $V \subset Y$ a k–dimensional subvariety. Then, according to 12.5, there are embeddings

$$C_{X\cap V}V \subset j^*C_XY = j^*N_XY = N_X(Y) \mid X \cap V \subset N_XY\,,$$

where j denotes the embedding $X \cap V \subset X$. Here $C_{X\cap V}V$ has pure dimension k and $C_XY = N_XY$ is a vector bundle over X. Then the Gysin homomorphism $A_k(N_XY) \underset{s^*}{\longrightarrow} A_{k-d}(X)$ is an isomorphism and we obtain a class

$$X.V = s^*[C_{X\cap V}V]$$

from the fundamental class of the normal cone $C_{X\cap V}V$. The class $X.V$ is in the image of $A_{k-d}(X \cap V)$ because there is the commutative diagram

$$\begin{tikzcd}
A_k(N_X(Y) \mid Y \cap V) \arrow[r] \arrow[d, "s^*"] & A_k(N_XY) \arrow[d, "s^*"] \\
A_{k-d}(X \cap V) \arrow[r] & A_{k-d}(X)
\end{tikzcd}$$

of Gysin homomorphisms. This intersection defines a homomorphism i^* in the diagram

$$\begin{tikzcd}
Z_k(Y) \arrow[r] \arrow[d] \arrow[drr, "i^*"] & Z_k(N_XY) \arrow[r] & A_k(N_XY) \arrow[d, "\approx"'] \\
A_k(Y) \arrow[rr, "i^*"] & & A_{k-d}(X)
\end{tikzcd}$$

Using the "deformation to the normal cone", it is proved in [7], 5.2, that i^* passes through $A_k(Y)$. Then $X.V = i^*V$ and $X.\alpha = i^*\alpha$ by definition. The homomorphism i^* is also called the Gysin homomorphism of X.

13.1.1. Remark: If D is an effective divisor, then the newly defined intersection $D.\alpha$ coincides with the definition of $D.\alpha$ in 9.3, see also 13.6.

13.1.2. Remark: The Gysin formula 11.2.1 for the inverse of the homomorphism π^* of a vector bundle can be interpreted as the homomorphism s^* of any section, see [7], Corollary 6.5. Let $E \xrightarrow{\pi} X$ be the projection and let $X \xrightarrow{s} E$ be any section. This is a regular embedding of codimension r, $r = \text{rank}(E)$. By $s^*\alpha = X.\alpha$ we obtain a homomorphism

$$A_k(E) \xrightarrow{s^*} A_{k-r}(X)$$

which turns out to be the inverse of π^*.

13.1.3. Remark: The definition of the Gysin operation $X.V$ in 13.1 can be generalised to morphisms $V \xrightarrow{f} Y$. Given such a morphism from a k–dimensional variety, let $W \xrightarrow{g} X$ be the fibre of f or the pull–back of V. Then $W \subset V$ and the normal cone C_WV is contained in g^*N_XY. We thus get a class

$$X._fV = s^*[C_WV] \in A_{k-d}(W)$$

13.1.4. Remark: For a regular embedding as above and any class $\alpha \in A_*(X)$ we have the self-intersection formula $i^*i_*\alpha = c_d(N_XY) \cap \alpha$.

13.2. Refined Gysin homomorphisms.

Let $X \overset{i}{\hookrightarrow} Y$ be a regularly embedded closed subscheme as above, let $Y' \xrightarrow{f} Y$ be a morphism and let

$$\begin{tikzcd} X' \arrow[r, hook, "i'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

be the pull–back diagram. Then, i' is also a regular embedding of codimension d, and there is an embedding $C_{X'}Y' \hookrightarrow g^*N_XY$, 12.5. We obtain the diagram

$$\begin{tikzcd} Z_k(Y') \arrow[r, "\sigma"] \arrow[dd] \arrow[ddrr, "i^!"] & Z_k(C_{X'}Y') \arrow[r] & Z_k(g^*N_XY) \arrow[d] \\ & & A_k(g^*N_XY) \arrow[d, "\approx"'] \\ A_k(Y') \arrow[rr, "i^!"] & & A_{k-d}(X') \end{tikzcd}$$

where σ is defined by $[V] \to [C_{V\cap X'}V]$ with $C_{V\cap X'}V \subset C_{X'}Y' \mid V \cap X'$, and in which $i^!$ also passes through $A_k(Y')$. For a class $\alpha \in A_k(Y')$ we put

$$X._Y\alpha = i^!\alpha\,.$$

For more details and functorial properties see [7], §6. More generally, let $X \xrightarrow{f} Y$ be a morphism from an arbitrary scheme to a smooth variety Y of dimension n and let $X' \xrightarrow{p} X$ and $Y' \xrightarrow{q} Y$ be schemes over X and Y. Then we have the pull–back diagram

$$\begin{tikzcd} X' \times_Y Y' \arrow[r, hook] \arrow[d] & X' \times Y' \arrow[d, "p\times q"] \\ X \arrow[r, hook, "\gamma_f"'] & X \times Y \end{tikzcd}$$

with the graph morphism γ_f being regular of codimension n. By the above, for any classes $x \in A_k(X')$ and $y \in A_l(Y')$ we are given a class

$$x._f y = \gamma_f^!(x \times y) \in A_{k+l-n}(X' \times_Y Y')\,.$$

In the special case where $X' = X$ and X is purely m–dimensional, we obtain the class

$$f^! y = [X]._f y$$

and a homomorphism

$$A_k(Y') \xrightarrow{f^!} A_{k+m-n}(X \times_Y Y').$$

This is also called a refined Gysin homomorphism. See [7], Definition 8.1.2.

13.3. Intersection pairing on smooth varieties.

In the following Y will be a nonsingular variety of dimension n. Then the diagonal embedding $Y \overset{\delta}{\hookrightarrow} Y \times Y$ is regular of codimension n. Combining the Gysin homomorphism

δ^* with the Künneth homomorphism $\times$ we obtain the pairing

$$A_k(Y) \otimes A_l(Y) \xrightarrow{\times} A_{k+l}(Y \times_k Y) \xrightarrow{\delta^*} A_{k+l-n}(Y)$$

denoted

$$x \otimes y \mapsto x.y = x \cap y.$$

In case of (smooth) varieties of dimension n one puts

$$A^p(Y) = A_{n-p}(Y),$$

indexing codimensions. Then the pairing reads

$$A^p(Y) \otimes A^q(Y) \to A^{p+q}(Y).$$

The graded group $A^*(Y) = \bigoplus_{p \geq 0} A^p(Y)$ becomes a graded ring under the intersection pairing, as follows easily from the functorial properties of the pairing, see [7], 8.3 . $A^*(Y)$ is called the Chow ring of Y. Note that the fundamental class $[Y]$ now serves as the unit element.

13.4. Cap product. Let $X \xrightarrow{f} Y$ be a morphism from an algebraic scheme to the smooth variety Y of dimension n. Then the graph morphism

$$X \overset{\gamma_f}{\hookrightarrow} X \times_k Y$$

is a regular embedding of codimension n. As in the previous case we obtain the pairing

$$A_k(Y) \otimes A_l(X) \to A_{k+l}(X \times_k Y) \xrightarrow{\gamma_f^*} A_{k+l-n}(X)$$

denoted

$$y \otimes x \mapsto x._f y = f^* y \cap x.$$

This can also be written as a cap product

$$A^p(Y) \otimes A_q(X) \xrightarrow{\cap} A_{q-p}(X)$$

and turns $A_*(X)$ into a graded $A^*(Y)$–module.

If X is also smooth, this becomes

$$A^p(Y) \otimes A^q(X) \xrightarrow{\cap} A^{p+q}(X).$$

In this case we obtain a homomorphism

$$A^p(Y) \xrightarrow{f^*} A^p(X)$$

by $y \mapsto f^* y \cap [X]$. One writes again simply $f^* y$ for $f^* y \cap [X]$.

13.5. Projection formula: Let $X \xrightarrow{f} Y$ be a **proper** morphism of **smooth** varieties. Then for $y \in A^*(Y)$ and $x \in A^*(X)$

$$f_*(f^* y \cap x) = y \cap f_* x.$$

For a proof see [7], 8.3.

13.6. Chern operators as classes.

If $\mathcal{L} = \mathcal{O}_X(D)$ is an invertible sheaf on a **smooth** variety, then we have the class $[\mathrm{cyc}(D)] \in A^1(X)$. Then

$$D.\alpha = c_1(\mathcal{L}) \cap \alpha = [\mathrm{cyc}(D)] \cap \alpha = [\mathrm{cyc}(D)].\alpha, \qquad (*)$$

where the first two intersections are those of 8.2 and 8.3 and the third is the new one. In particular,

$$c_1(\mathcal{L}) \cap [X] = [\mathrm{cyc}(D)].$$

For a proof we may assume $\alpha = [V] \in A_k(X)$ and D effective and irreducible and in addition $D \cap V \underset{\neq}{\subset} V$. Then $D \cap V$ is an effective divisor in V and we have $D.[V] = [D \cap V] \in A_{k-1}(V)$ and $\mathrm{cyc}(D) = D$.

The definition of $[D].[V]$ is now given by the normal cone $C = C_{V \cap D}(V \times D) \subset N_X(X \times X)$. Now $N_X(X \times X)$ is the tangent bundle TX and we put $T := TX \mid V \cap D$ such that we have $C \subset T$ over $V \cap D$. Then $[D].[V]$ is the class in $A_{k-1}(V \cap D)$ corresponding to $[C] \in A_{k+n-1}(T)$ via the Gysin isomorphism of T. Because C and T both have dimension $k + n - 1$, they are equal and hence $[C] = [T]$ is the fundamental class. Then also $[D].[V]$ is the fundamental class $[D \cap V] = D.[V]$.

The coincidence $(*)$ can be generalized to Segre and Chern operators of vector bundles. Let $\mathcal{E}$ be locally free on X of rank $r = e + 1$ and let $P(\mathcal{E}) \xrightarrow{p} X$ be its projective bundle with the operator

$$\zeta = c_1(\mathcal{O}_E(1)).$$

Let H be a divisor with $\mathcal{O}_E(1) = \mathcal{O}_{P(\mathcal{E})}(H)$. By definition $s_i(\mathcal{E}) \cap \alpha = p_*(\zeta^{e+i} \cap p^*\alpha)$ and we get

$$\begin{aligned}
(s_i(\mathcal{E}) \cap [X]).\alpha &= p_*(\zeta^{e+i} \cap p^*[X]).\alpha \\
&= p_*([\mathrm{cyc}(H)]^{e+i}.p^*[X]).\alpha \\
&= p_*([\mathrm{cyc}(H)]^{e+i}).[X].\alpha \quad \text{(projection formula)} \\
&= p_*([\mathrm{cyc}(H)]^{e+i}).p^*\alpha) \\
&= p_*(\zeta^{e+i} \cap p^*\alpha) \\
&= s_i(\mathcal{E}) \cap \alpha\ .
\end{aligned}$$

Because the Chern operators are polynomials in the Segre operators, we also have

$$(c_i(\mathcal{E}) \cap [X]).\alpha = c_i(\mathcal{E}) \cap \alpha$$

for any class $\alpha \in A^*(X)$. Therefore the class $c_i(\mathcal{E}) \cap [X] \in A^i(X)$ determines the operator $c_i(\mathcal{E})$ and both will be identified later by abuse of notation. At the moment we put

$$\bar{c}_i(\mathcal{E}) = c_i(\mathcal{E}) \cap [X] \in A^i(X).$$

Then the pull–back formula $p^*(c_1(\mathcal{E}) \cap \alpha = c_i(p^*(\mathcal{E}) \cap p^*\alpha$ immediately implies that

$$p^*\bar{c}_i(\mathcal{E}) = \bar{c}_i(p^*\mathcal{E}).$$

13.7. Remark: Let $X \xrightarrow{f} Y$ be any morphism with Y a smooth variety and let $\mathcal{E}$ be locally free on Y. Then for any classes $x \in A_*(X)$ and $y \in A^*(Y)$ there is the formula

$$f^*(c_i(\mathcal{E}) \cap y) \cap x = f^*y \cap (c_i(f^*\mathcal{E}) \cap x),$$

see [7], Example 8.1.6 . This implies

$$f^*\bar{c}_i(\mathcal{E}) = \bar{c}_i(f^*\mathcal{E})$$

for the fundamental classes $x = [X]$ and $y = [Y]$, when X is pure–dimensional.

13.8. Chow ring of $P(\mathcal{E})$.

Let $\mathcal{E}$ be locally free of rank $e+1$ and X be smooth of dimension n. The result on the groups $A_k(P(\mathcal{E}))$ in 11.1 can now be seen as the determination of the Chow ring $A^*(P(\mathcal{E}))$. The isomorphism θ_E in 11.1 can now be written as

$$\theta_E(\alpha_{k-e}, \ldots, \alpha_k) = \sum_{0 \le i \le e} \zeta^i \cap p^*\alpha_{k-e+i} = \sum_{0 \le i \le e} \alpha_{k-e+i}\zeta^i\,,$$

where $\zeta = c_1\mathcal{O}_E(1)$, and where we identify α_μ with $p^*\alpha_\mu$ because p^* is a monomorphism $A^{n-\mu}(X) \to A^{n-\mu}(P(\mathcal{E}))$. Therefore, we consider the homomorphism

$$\begin{array}{ccc} A^*(X)[t] & \xrightarrow{\zeta^*} & A^*(P(\mathcal{E})) \\ \sum \gamma_\nu t^\nu & \longmapsto & \sum \gamma_\nu \zeta^\nu \end{array}$$

of graded rings with $\gamma_\nu \in A^{n-\nu}(X)$. This is surjective because θ_E is surjective. In order to determine the kernel, we recall the relation

$$\zeta^r + \bar{c}_1(p^*\mathcal{E})\zeta^{r-1} + \cdots + \bar{c}_r(p^*\mathcal{E}) = 0$$

from 10.15. This means that

$$t^r + \bar{c}_1(\mathcal{E})t^{r-1} + \cdots + \bar{c}_r(\mathcal{E}) \qquad (*)$$

is in the kernel of ζ^*. The basis theorem 11.1 tells us that the homomorphism

$$A^j(X) \oplus A^{j-1}(X)t \oplus \cdots \oplus A^{j-e}(X)t^e \longrightarrow A^jP(\mathcal{E})$$

is an isomorphism for any $j \le e$. This implies that $1, \zeta, \ldots, \zeta^e$ are free over $A^*(X)$ and that the relation $(*)$ is of minimal degree. Because the corresponding polynomial in t is monic, the kernel of ζ^* can be reduced modulo that polynomial, proving that

$$A^*(X)[t]/(t^r + \bar{c}_1(\mathcal{E})t^{r-1} + \cdots + \bar{c}_r(\mathcal{E})) \cong A^*(P(\mathcal{E})).$$

13.9. Chow ring of $X \times \mathbb{P}_n$. This product is the projective bundle of the trivial sheaf $\mathcal{O}^{n+1}$ with Chern classes $c_i(\mathcal{E}) = 0$. Then

$$\begin{aligned} A^*(X \times \mathbb{P}_n) &\cong A^*(X)[t]/(t^{n+1}) \cong A^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[t]/(t^{n+1}) \\ &\cong A^*(X) \otimes_{\mathbb{Z}} A^*(\mathbb{P}_n)\,. \end{aligned}$$

It is also easy to verify that this isomorphism is the Künneth homomorphism. In particular

$$\begin{aligned} A^*(\mathbb{P}_m \times \mathbb{P}_n) &\cong A^*(\mathbb{P}_m) \otimes_{\mathbb{Z}} A^*(\mathbb{P}_n) \cong \mathbb{Z}[s]/(s^{m+1}) \otimes_{\mathbb{Z}} \mathbb{Z}[t]/(t^{n+1}) \\ &\cong \mathbb{Z}[s,t]/(s^{m+1}, t^{n+1})\,. \end{aligned}$$

Remark: In general $A^*(X \times Y)$ is not isomorphic to $A^*(X) \otimes A^*(Y)$.

13.10. Chow ring of a Hirzebruch surface.

Let $\Sigma_n = P(\mathcal{O} \oplus \mathcal{O}(n))$ over $\mathbb{P}_1$, let $\xi \in A^1(\mathbb{P}_1) \cong \mathbb{Z}$ be the class of a point an let $\eta \in A^1(\Sigma_n)$ be the class of the tautological line bundle $\mathcal{O}_E(1)$ on Σ_n, which can be represented by a horizontal divisor (relative hyperplane) H in Σ_n. Here $\mathcal{O} \oplus \mathcal{O}(n)$ has the Chern polynomial $1 + n\xi t$ with $c_1(\mathcal{O} \oplus \mathcal{O}(n)) = c_1(\mathcal{O}(n)) = n\xi$. Now the relation of 13.8 is

$$\eta^2 + n\xi\eta = 0,$$

where we identify ξ with $p^*\xi$. Note that also $\xi^2 = 0$. The Chowring is now

$$A^*(\Sigma_n) \cong A^*(\mathbb{P}_1)[t]/(t^2 + n\xi t) \cong \mathbb{Z}[s,t]/(s^2, t^2 + snt),$$

where $\xi \leftrightarrow \bar{s}$ under $A^1(\mathbb{P}_1) \cong \mathbb{Z}$. This example is already demonstrating the **global flavour** of the intersection pairing, because the number n distinguishes the different surfaces Σ_n. However, the Chow groups are the same for all n:

$$\begin{array}{rcl} A^0(\Sigma_n) & \cong & \mathbb{Z} \\ A^1(\Sigma_n) & \cong & \mathbb{Z} \oplus \mathbb{Z} \\ A^2(\Sigma_n) & \cong & \mathbb{Z}\,. \end{array}$$

13.11. Chow ring of $P(\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_k))$ over $\mathbb{P}_m$.

Let again $\xi \in A^1(\mathbb{P}_m)$ be the generating class of $A^*(\mathbb{P}_m) \cong \mathbb{Z}[s]/(s^{m+1})$. Then the sheaf $\mathcal{E}$ has Chern polynomial

$$(1 + n_1\xi t)(1 + n_2\xi t) \cdots (1 + n_k\xi t),$$

where $\xi^{m+1} = 0$. In case $k \le m$ we obtain

$$1 + a_1\xi t + a_2\xi^2 t^2 + \cdots + a_k\xi^k t^k,$$

where a_ν is the ν–th elementary symmetric function evaluated at $(n_1, \ldots, n_k)$. Then the Chowring of this projective bundle is

$$\mathbb{Z}[s,t]/(s^{m+1}, t^k + a_1 s t^{k-1} + a_2 s^2 t^{k-2} + \cdots a_k s^k)$$

13.12. Intersection multiplicities.

Let Y be a smooth n–dimensional variety and $V, W \subset Y$ be closed subschemes of pure dimension k, l. Then any irreducible component Z of $V \cap W$ has dimension $\ge k + l - n$. When $\dim Z$ equals $k + l - n$, Z is called proper. The class

$$V.W = [V].[W] \in A_{k+l-n}(V \cap W) \subset A_{k+l-n}(Y)$$

can then be written as a sum

$$V.W = \sum a_Z [Z]$$

over all $(k+l-n)$–dimensional subvarieties of $V \cap W$. For proper Z the coefficient a_Z is called the intersection multiplicity of V and W along Z and denoted by

$$i(Z, V.W, Y).$$

The following is proved in [7], 8.2 .

13.12.1. Proposition: *Let Z be a proper component of $V \cap W$. Then*

(a) $1 \leq i(Z, V.W, Y) \leq l(\mathcal{O}_{Z,V\cap W})$
(b) *If $\mathcal{O}_{Z,V\cap W}$ is Cohen–Macaulay, then $i(Z, V.W, Y) = l(\mathcal{O}_{Z,V\cap W})$*
(c) *If V and W are varieties, then $i(Z, V.W, Y) = 1$ if and only if the maximal ideal $\mathfrak{m}_{Z,Y}$ is the sum of the prime ideals $Ker(\mathcal{O}_{Z,Y} \to \mathcal{O}_{Z,V})$ and $Ker(\mathcal{O}_{Z,Y} \to \mathcal{O}_{Z,W})$. In this case $\mathcal{O}_{Z,V}$ and $\mathcal{O}_{Z,W}$ are regular.*

13.12.2. Remark: If $Z \subset V \cap W$ is proper, then

$$i(Z, V.W, Y) = i(Z, \Delta_{Y}.(V \times W), Y \times Y).$$

By this formula the properties of the intersection multiplicities are reduced to the properties of the multiplicities

$$i(Z, X.V, Y)$$

for a regular embedding $X \hookrightarrow Y$, see [7], §7.

13.13. Intersections of several subschemes.

Let $V_1, \ldots, V_r$ be pure–dimensional closed subschemes of a smooth variety Y, and let Z be a proper irreducible component of $V_1 \cap \ldots \cap V_r$, $\dim Z = \sum \dim V_i - (r-1)\dim Y$. Then

$$i(Z, V_1 \cdot \ldots \cdot V_r, Y)$$

is the coefficient of Z in $A_{\dim Z}(V_1 \cap \ldots \cap V_r)$. Proposition 13.12.1 extends to this case.

13.14. Bezout's theorem on $\mathbb{P}_n$: Because $A^k(\mathbb{P}_n) = \mathbb{Z}$ with generator $h^k = h \cdot \ldots \cdot h$, h the class of a hyperplane, any class $\alpha \in A^k(\mathbb{P}_n)$ can be given its degree by

$$\alpha = \deg(\alpha) h^k.$$

Bezout's theorem states that for classes $\alpha_i \in A^{di}(\mathbb{P}_n)$ with $d_1 + \ldots + d_r \leq n$

$$\deg(\alpha_1 \cdot \ldots \cdot \alpha_r) = \deg(\alpha_1) \cdot \ldots \cdot \deg(\alpha_r).$$

This follows directly from the structure of the Chow ring and the definition of the degree. In particular, let $V_1, \ldots, V_r$ be **pure**–dimensional subschemes such that each component Z of $V_1 \cap \ldots \cap V_r$ has codimension equal to $\operatorname{codim}(V_1) + \ldots + \operatorname{codim}(V_r)$. Then

$$V_1 \cdot \ldots \cdot V_r = \sum_{\mu=1}^{m} i(Z_\mu, V_1 \cdot \ldots \cdot V_r, \mathbb{P}_n)[Z_\mu],$$

and taking degree of this:

$$\deg(V_1)\cdot\ldots\cdot\deg(V_r)=\sum_{\mu} i(Z_\mu, V_1\cdot\ldots\cdot V_r, \mathbb{P}_n)\deg(Z_\mu).$$

14. Flag varieties and Chern classes

In this section $\mathcal{E}$ will always denote a rank n vector bundle over a scheme X. In the formulas for the Chow rings of the flag varieties, X is always assumed to be smooth. In Lemma 10.10, a scheme $F(\mathcal{E})\xrightarrow{p}X$ over X was constructed such that p is proper and locally trivial with fibres $F\big(\mathcal{E}(x)\big)$, the full flag varieties of the vector spaces $\mathcal{E}(x)$. More generally, for any sequence $0<d_1<\ldots<d_m<n$ of integers there is a flag variety

$$F(\underline{d},\mathcal{E})=F(d_1,\ldots,d_m,\mathcal{E})\xrightarrow{p}X$$

with the following universal property:

(i) there is a flag of subbundles $\mathcal{S}_1\subset\mathcal{S}_2\subset\ldots\subset\mathcal{S}_m\subset p^*\mathcal{E}$ of rank $\mathcal{S}_\mu=d_\mu$.

(ii) For any morphism $Y\xrightarrow{g}X$ and any flag of subbundles

$$\mathcal{F}_1\subset\mathcal{F}_2\subset\ldots\subset\mathcal{F}_m\subset g^*\mathcal{E}$$

of rank $\mathcal{F}_\mu=d_\mu$ there is a unique morphism $Y\xrightarrow{f}F(\underline{d},\mathcal{E})$ such that $p\circ f=g$ and such that there are isomorphisms $\mathcal{F}_\mu\cong f^*\mathcal{S}_\mu$ fitting into the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{F}_1 & \subset & \mathcal{F}_2 & \subset\ldots\subset & \mathcal{F}_m & \subset & g^*\mathcal{E}\\
\approx\downarrow & & \downarrow\approx & & \downarrow\approx & & \downarrow\approx\\
f^*\mathcal{S}_1 & \subset & f^*\mathcal{S}_2 & \subset\ldots\subset & f^*\mathcal{S}_m & \subset & f^*p^*\mathcal{E}\,.
\end{array}$$

(iii) $A_*(X)\xrightarrow{p^*}A_*F(\underline{d},\mathcal{E})$ is injective.

For proofs of (i) and (ii) see [12] or [10]. The proof can be done by induction on m, as in 10.10, starting with a Grassmann bundle $G(d,\mathcal{E})$. For Grassmann bundles, property (iii) is contained in 16.7. It can then be verified for a flag variety by the induction process. There are unique morphisms between the various flag bundles, which are defined by the universal property of the flags, e.g.

$$F(d_1,\ldots,d_m,\mathcal{E})\cong G(d_1,\mathcal{S}_2)\longrightarrow F(d_2,\ldots,d_m,\mathcal{E}).$$

14.1. The **canonical homomorphism**

$$A^*(X)[t_1^1,\ldots,t_{k_1}^1,\ t_1^2,\ldots,t_{k_2}^2,\ldots,t_1^{m+1},\ldots,t_{k_{m+1}}^{m+1}]\longrightarrow A^*F(d_1,\ldots,d_m,\mathcal{E})$$

is defined over the ring homomorphism $A^*(X)\hookrightarrow A^*F(\underline{d},\mathcal{E})$ by the substitutions

$$t_i^\mu\longmapsto c_i(\mathcal{S}_\mu/\mathcal{S}_{\mu-1}),\quad 1\le i\le k_\mu=d_\mu-d_{\mu-1}\,,$$

where $0=\mathcal{S}_0\subset\mathcal{S}_1\subset\ldots\subset\mathcal{S}_m\subset\mathcal{S}_{m+1}=p^*\mathcal{E}$ is the universal flag. Here the indeterminants t_i^μ have weight i as do the Chern classes $c_i(\mathcal{S}_\mu/\mathcal{S}_{\mu-1})$ or $c_i(\mathcal{S}_\mu/\mathcal{S}_{\mu-1})\cap[X]$. Because of the short exact sequences of the quotient bundles, we have the Whitney decomposition of Chern polynomials

$$c(p^*\mathcal{E}) = c(\mathcal{S}_1)c(\mathcal{S}_2/\mathcal{S}_1)\dots c(\mathcal{S}_{m+1}/\mathcal{S}_m)\,.$$

This is an identity of graded polynomials. The corresponding ideal $\mathfrak{a}$ in the polynomial ring is the ideal generated by the homogeneous parts of the equation

$$1 + e_1 + \dots + e_n = (1 + t_1^1 + t_2^1 + \dots)(1 + t_1^2 + t_2^2 + \dots)\dots(1 + t_1^{m+1} + t_2^{m+1} + \dots)$$

with $e_i = c_i(p^*\mathcal{E})$. So the generators of $\mathfrak{a}$ in the different degrees are

$$\begin{array}{l} e_1 - (t_1^1 + t_1^2 + \dots + t_1^{m+1}) \\ e_2 - (t_2^1 + t_2^2 + \dots + t_2^{m+1} + \sum_{\mu,\nu} t_1^\mu t_1^\nu)\,. \\ \vdots \end{array}$$

It follows that there is the induced graded homomorphism

$$A^*(X)[\underline{t^1}, \dots, \underline{t^{m+1}}]/\mathfrak{a} \xrightarrow{\alpha(\underline{d})} A^*F(d_1, \dots, d_m, \mathcal{E}). \tag{CF}$$

14.2. Theorem: *(*A. Grothendieck, [4]*) Let X be smooth. Then the homomorphism $\alpha(\underline{d})$ is an isomorphism.*

Proof. The proof is a slight modification of the proof of A. Grothendieck in [4] by induction on m and n. For that we shall only use the statement for $P(\mathcal{E}) = G(1, \mathcal{E}) = F(1, \mathcal{E})$, see 13.8. We reformulate first the formula for $P(\mathcal{E})$, then prove it for all full flag bundles $F(\mathcal{E}) = F(1, 2, \dots, n-1, \mathcal{E})$ by induction on n and then deduce the general case by descending induction on m.

(1) The formula (CF) is true for projective bundles $P(\mathcal{E})$.

Let $0 \to \mathcal{S} \to p^*\mathcal{E} \to \mathcal{Q} \to 0$ be the tautological sequence on $P(\mathcal{E})$ with $\mathcal{S} = \mathcal{O}_E(-1)$ and let

$$A^*(X)[s, q_1, \dots, q_{n-1}]/\mathfrak{a} \longrightarrow A^*P(\mathcal{E})$$

be defined by the substitutions

$$s \longmapsto c_1(\mathcal{S}) \text{ and } q_\nu \longmapsto c_\nu(\mathcal{Q})\,.$$

Here the generators of $\mathfrak{a}$ correspond to the homogeneous parts of the graded Whitney identity

$$1 + e_1 + \dots + e_n = \big(1 + c_1(\mathcal{S})\big)\big(1 + c_1(\mathcal{Q}) + \dots + c_{n-1}(\mathcal{Q})\big)\,,$$

i.e. $\mathfrak{a}$ is generated by

$$e_1 - s - q_1,\ e_2 - sq_1 - q_2, \dots, e_n - sq_{n-1}\,.$$

By the result13.8, we know that

$$A^*P(\mathcal{E}) = A^*(X)[t]/(t^n + e_1 t^{n-1} + \dots + e_n)$$

where $t \longmapsto c_1\big(\mathcal{O}_E(1)\big) = c_1(\mathcal{S}^*)$. It is now easy to verify that we have an isomorphism

$$A^*(X)[s, q_1, \dots, q_{n-1}]/\mathfrak{a} \longrightarrow A^*(X)[t]/(t^n + e_1 t^{n-1} + \dots + e_n)\,,$$

which is defined by $s \mapsto -t$ and

$$\begin{aligned} q_1 &\mapsto e_1 + t \\ q_2 &\mapsto e_2 + t(e_1 + t) = e_2 + te_1 + t^2 \\ &\vdots \\ q_{n-1} &\mapsto e_{n-1} + te_{n-2} + \cdots + t^{n-2}e_1 + t^{n-1} \,. \end{aligned}$$

(2) The formula (CF) is true for any full flag bundle $F(\mathcal{E}) = F(1, 2, \ldots, n-1, \mathcal{E}) \to X$. This will be proved by induction on $n \geq 2$. When $n = 2$, we have $F(\mathcal{E}) = P(\mathcal{E})$. Let now $n \geq 3$ and let $P(\mathcal{E}) \xrightarrow{p} X$ be the projective bundle with tautological sequence $0 \to \mathcal{L} \to p^*\mathcal{E} \to \mathcal{Q} \to 0$. Then

$$F(\mathcal{Q}) \xrightarrow{q} P(\mathcal{E})$$

is isomorphic to $F(\mathcal{E})$ over X. This follows from the universal properties of the flag varieties regarding the flags $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_{n-1} \subset q^*p^*\mathcal{E}$ and $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_{n-2} \subset q^*\mathcal{Q}$. After identifying, we have the exact diagrams

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{S}_1 \arrow[r] \arrow[d, equal] & \mathcal{S}_\nu \arrow[r] \arrow[d] & \mathcal{U}_{\nu-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q^*\mathcal{L} \arrow[r] & q^*p^*\mathcal{E} \arrow[r] \arrow[d] & q^*\mathcal{Q} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{Q}_{n-\nu} \arrow[r, equal] \arrow[d] & \mathcal{Q}_{n-\nu} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

by which $\mathcal{S}_\nu$, resp. $\mathcal{U}_{\nu-1}$, may be defined by $\mathcal{U}_{\nu-1}$, resp. $\mathcal{S}_\nu$, as inverse image, resp. quotient. Putting $\mathcal{U}_{n-1} = q^*\mathcal{Q}$ and $\mathcal{S}_n = q^*p^*\mathcal{E}$, we have $\mathcal{S}_2/\mathcal{S}_1 \cong \mathcal{U}_1$, $\mathcal{S}_\nu/\mathcal{S}_{\nu-1} \cong \mathcal{U}_{\nu-1}/\mathcal{U}_{\nu-2}$ for $\nu = 3, \ldots, n$ and the Chern class decomposition

$$c(q^*\mathcal{Q}) = c(\mathcal{S}_2/\mathcal{S}_1)c(\mathcal{S}_3/\mathcal{S}_2)\ldots c(\mathcal{S}_n/\mathcal{S}_{n-1}) \,.$$

By induction hypothesis, we have the isomorphism

$$A^*P(\mathcal{E})[u_1, \ldots, u_{n-1}]/\mathfrak{b} \xrightarrow{\approx} A^*F(\mathcal{E}) \,,$$

where $u_\nu \mapsto c_1(\mathcal{U}_\nu/\mathcal{U}_{\nu-1}) = c_1(\mathcal{S}_{\nu+1}/\mathcal{S}_\nu)$ and the ideal $\mathfrak{b}$ is generated by

$$c_i(q^*\mathcal{Q}) - \sigma'_i(u_1, \ldots, u_{n-1})$$

for $i = 1, \ldots, n-1$, where $\sigma'_1, \ldots, \sigma'_{n-1}$ denote the elementary symmetric functions in $n-1$ variables. In order to substitute $A^*P(\mathcal{E})$ by $A^*(X)$, we consider the diagram

$$\begin{tikzcd}
A^*(X)[s_1, \ldots, s_n] \arrow[r, "\alpha"] & A^*F(\mathcal{E}) \\
A^*(X)[s, q_1, \ldots, q_{n-1}] \arrow[r, "\beta"] \arrow[u, "\gamma"] & A^*P(\mathcal{E}) \arrow[u, "q^*"']
\end{tikzcd}$$

in which γ is defined by $s \mapsto s_1$ and $q_i \longmapsto \sigma_i'(s_2, \ldots, s_n)$ and α, resp. β, by $s_\nu \mapsto c_1(\mathcal{S}_\nu/\mathcal{S}_{\nu-1})$, resp. $s \mapsto c_1(\mathcal{L})$, $q_\nu \mapsto c_\nu(\mathcal{Q})$. This diagram is commutative because

$$\begin{aligned} & q^*c_1(\mathcal{L}) = c_1(q^*\mathcal{L}) = c_1(\mathcal{S}_1) \\ \text{and} \qquad & q^*c_i(\mathcal{Q}) = c_i(q^*\mathcal{Q}) = \sigma_i'\big(c_1(\mathcal{U}_1), c_1(\mathcal{U}_2/\mathcal{U}_1), \ldots, c_1(\mathcal{U}_{n-1}/\mathcal{U}_{n-2})\big)\,. \end{aligned}$$

Taking into account the ideals, we attain the induced diagram

$$\begin{tikzcd} A^*(X)[s_1, \ldots, s_n]/\mathfrak{a} \arrow[rr, "\tilde{\alpha}"] & & A^*F(\mathcal{E}) \\ \big(A^*(X)[s, q_1, \ldots, q_{n-1}]/\mathfrak{a}_0\big)[u_1, \ldots, u_{n-1}]/\mathfrak{b} \arrow[u, "\tilde{\gamma}"'] \arrow[r, "\tilde{\beta}"] & A^*P(\mathcal{E})[u_1, \ldots, u_{n-1}]/\mathfrak{b}\,. \arrow[ur, phantom, ""] & \arrow[u, "\tilde{q}^*"'] \end{tikzcd}$$

Here $\tilde{\alpha}$ and $\tilde{\beta}$ are well–defined because of the Whitney relations. A direct check also shows that $\tilde{\gamma}$ is well–defined by recalling that the ideals are defined as follows:

$$\begin{aligned} \mathfrak{a} &= \big(e_1 - \sigma_1(s_1, \ldots, s_n), \ldots, e_n - \sigma_n(s_1, \ldots, s_n)\big) \\ \mathfrak{a}_0 &= \big(e_1 - s - q_1, e_2 - sq_1 - q_2, \ldots, e_n - sq_{n-q}\big) \\ \mathfrak{b} &= \big(q_1 - \sigma_1'(u_1, \ldots, u_{n-1}), \ldots, q_{n-1} - \sigma_{n-1}'(u_1, \ldots, u_{n-1})\big)\,. \end{aligned}$$

Now $\tilde{\beta}$ and $\tilde{q}^*$ are isomorphisms by the above and $\tilde{\gamma}$ is surjective because $s \mapsto s_1$, $u_\nu \mapsto s_{\nu+1}$. It follows that $\tilde{\alpha}$ is an isomorphism.

(3) The formula (CF) is true for arbitrary flag varieties $F(\underline{d}, \mathcal{E}) = F(d_1, \ldots, d_m, \mathcal{E})$. This will be shown by descending induction on $m < n$. If $m = n-1$, then $F(\underline{d}, \mathcal{E})$ is the full flag variety and (CF) is true by (2). If $1 < d_1$, we find that

$$F(1, d_1, \ldots, d_m, \mathcal{E}) \longrightarrow F(d_1, \ldots, d_m, \mathcal{E})$$

is isomorphic to

$$P(\mathcal{S}_1) \longrightarrow F(d_1, \ldots, d_m, \mathcal{E})\,,$$

where $\mathcal{S}_1$ is the first bundle of the flag on $F(\underline{d}, \mathcal{E})$. This follows by comparing the universal properties of both varieties. If $1 = d_1$, we may choose the first d_μ with $\mu < d_\mu$ and consider the variety $F(1, \ldots, \mu - 1, d_\mu - 1, d_\mu, \ldots, d_n, \mathcal{E}) \cong P(\mathcal{S}_\mu/\mathcal{S}_{\mu-1})$ over $F(\underline{d}) = F(1, \ldots, \mu-1, d_\mu, \ldots, d_m, \mathcal{E})$. Therefore, we restrict ourselves to the case $1 < d_1$.

Let the projections be denoted by

$$F(1, \underline{d}) \xrightarrow{q} F(\underline{d}) \xrightarrow{p} X$$

and let $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_m \subset p^*\mathcal{E}$ be the flag on $F(\underline{d})$. On $F(1, \underline{d})$ we have the additional sequence $0 \to \mathcal{L} \to q^*\mathcal{S}_1 \to \mathcal{Q} \to 0$.

For $F(\underline{d})$ we are given the map (CF)

$$A^*(X)[t_1^1, \ldots, t_1^2, \ldots, t_1^{m+1}, \ldots]/\mathfrak{a} \xrightarrow{\beta} A^*F(\underline{d})$$

where $\mathfrak{a}$ is the ideal defined by the Whitney relations. Let $\mathfrak{a} \subset \tilde{\mathfrak{a}}$ be the larger ideal such that $\tilde{\mathfrak{a}}/\mathfrak{a}$ is the kernel. The following commutative diagram can be derived as in (2)

$$\begin{tikzcd}
A^*(X)[s, u_1, \ldots, u_{d_1-1}, t_1^2, \ldots]/\mathfrak{c} \arrow[rr, "\tilde{\alpha}", "\approx"'] & & A^*F(1, \underline{d}) \\
\big(A^*(X)[t_1^1, \ldots, t_{d_1}^1, t_1^2, \ldots]/\mathfrak{a}\big)[u, v_1, \ldots, v_{d_1-1}]/\mathfrak{b} \arrow[u, "\tilde{\gamma}"] \arrow[d, two heads] \arrow[r, "\tilde{\beta}"] & A^*F(\underline{d})[u, v_1, \ldots, v_{d_1-1}]/\mathfrak{b} \arrow[ur, "\approx", "\pi"'] & \\
\big(A^*(X)[t_1^1, \ldots, d_{d_1}^2, t_1^2, \ldots]/\tilde{\mathfrak{a}}\big)[u, v_1, \ldots, v_{d_1-1}]/\mathfrak{b}\,. \arrow[ur, tail, "\tilde{\beta}'"'] & &
\end{tikzcd}$$

In this diagram also $\mathfrak{b}$ and $\mathfrak{c}$ are the Whitney ideals and $\tilde{\alpha}$ and π are the homomorphisms of type (CF) which are isomorphisms by induction hypothesis and (1) because $F(1, \underline{d}) \cong P(\mathcal{S}_1)$. The homomorphisms $\tilde{\beta}$ and $\tilde{\beta}'$ are induced by β. The homomorphism $\tilde{\gamma}$ is induced by the substitutions

$$t_1^1 \longmapsto s + u_1,\ t_\nu^1 \longmapsto s u_{\nu-1} + u_\nu$$

and $u \mapsto s$, $v_\nu \mapsto u_\nu$, according to the decomposition

$$c(q^*\mathcal{S}_1) = \big(1 + c_1(\mathcal{L})\big)\big(1 + q(\mathcal{Q}) + \cdots + c_{d_1-1}(\mathcal{Q})\big)\,.$$

Claim: $\tilde{\gamma}$ is injective.

This follows from the explicit description of the ideals, and is left to be verified by the reader. Because $\tilde{\gamma}$ is also surjective by its definition, $\tilde{\gamma}$ is an isomorphism and then $\tilde{\beta}$ is an isomorphism. It follows that $\mathfrak{a} = \tilde{\mathfrak{a}}$ because $\tilde{\beta}'$ is injective. This proves that β is an isomorphism. $\square$

14.3. Chowring of Grassmann bundles.

Let $\mathcal{E}$ on X be as above and let $G_d(\mathcal{E}) \xrightarrow{p} X$ be the Grassmann bundle with tautological sequence $0 \to \mathcal{S} \to p^*\mathcal{E} \to \mathcal{Q} \to 0$. Then, as a special case,

$$A^*G_d(\mathcal{E}) \cong A^*(X)[s_1, \ldots, s_d,\ q_1, \ldots, q_{n-d}]/\mathfrak{a}\,,$$

where $s_\nu \mapsto c_\nu(\mathcal{S})$ and $q_\nu \mapsto c_\nu(\mathcal{Q})$ and where $\mathfrak{a}$ is defined by the homogeneous components of the Whitney identity

$$1 + e_1 + \cdots + e_n = (1 + s_1 + \cdots + s_d)(1 + q_1 + \cdots + q_{n-d})\,.$$

The q_ν may be eliminated and then

$$A^*G_d(\mathcal{E}) \cong A^*(X)[s_1, \ldots, s_d]/\mathfrak{b}\,,$$

where $\mathfrak{b}$ is a more complicated ideal. In case $G_2(\mathcal{E})$ and $n = 4$ the ideal $\mathfrak{a}$ is

$$\mathfrak{a} = (s_1 + q_1,\ s_2 + s_1 q_1 + q_2,\ s_1 q_2 + s_2 q_1,\ s_2 q_2)$$

and then $\mathfrak{b}$ becomes

$$\mathfrak{b} = (s_1^3 - 2 s_1 s_2,\ s_1^2 s_2 - s_2^2)\,.$$

14.4. Remark: Even so the formulas (CF) present a beautiful description of the Chowrings of the relative flag varieties $F(d_1,\dots,d_m,\mathcal{E})$, and in special cases of the flag varieties $F(d_1,\dots,d_m,E)$ of a vector space, a good concrete geometric interpretation of cycles underlying the Chern classes is lacking. Such a geometric description will be given by the relative or absolute Schubert cycles for the Grassmannians in Section 16. In order to prepare the definition of the relative degeneracy classes, we consider the following identities for the Chern classes of sequences of bundles and subflags of subbundles.

14.5. Varieties of subflags

Let $0 \underset{\neq}{\subset} \mathcal{A}_1 \underset{\neq}{\subset} \mathcal{A}_2 \underset{\neq}{\subset} \dots \underset{\neq}{\subset} \mathcal{A}_d = \mathcal{A}$ be a flag of vector subbundles on an algebraic scheme X and let a_i be the rank of $\mathcal{A}_i$. There is a variety

$$Fl(\underline{\mathcal{A}}) \xrightarrow{p} X$$

over X together with a flag of vector subbundles

$$\mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_d \subset p^*\mathcal{A}$$

such that $\operatorname{rank}\mathcal{D}_i = i$ and such that each $\mathcal{D}_i$ is a subbundle of $p^*\mathcal{A}_i$, satisfying the following universal property. For any morphism $S \xrightarrow{f} X$ and any flag of vector subbundles

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_d \subset f^*\mathcal{A}$$

with $\operatorname{rank}\mathcal{E}_i = i$ and $\mathcal{E}_i \subset f^*\mathcal{A}_i$, there is a unique morphism $S \xrightarrow{\varphi} Fl(\underline{\mathcal{A}})$ over X such that

$$\begin{pmatrix} \mathcal{E}_1 & \subset \dots \subset & \mathcal{E}_d \\ \cap & & \cap \\ f^*\mathcal{A}_1 & \subset \dots \subset & f^*\mathcal{A}_d \end{pmatrix} \cong \varphi^* \begin{pmatrix} \mathcal{D}_1 & \subset \dots \subset & \mathcal{D}_d \\ \cap & & \cap \\ p^*\mathcal{A}_1 & \subset \dots \subset & p^*\mathcal{A}_d \end{pmatrix}.$$

The construction can be done by induction on the length d. For $d = 1$ we can simply define $Fl(\underline{\mathcal{A}}) = P(\mathcal{A}_1)$ with $\mathcal{D}_1 = \mathcal{O}_{A_1}(-1)$. For $d \geq 2$, suppose $F' = Fl(\underline{\mathcal{A}}_{d-1}) \xrightarrow{q} X$ has been constructed with the universal flag

$$\begin{pmatrix} \mathcal{D}'_1 & \subset \cdots \subset & \mathcal{D}'_{d-1} \\ \cap & & \cap \\ q^*\mathcal{A}_1 & \subset \cdots \subset & q^*\mathcal{A}_{d-1} \end{pmatrix}.$$

Then let $F = P(q^*\mathcal{A}_d/\mathcal{D}'_{d-1}) \xrightarrow{\rho} F'$ be the projective bundle and let $p = q \circ \rho$. Let $\mathcal{D}_d$ be the rank-d bundle determined by

$$\mathcal{D}_d/\rho^*\mathcal{D}'_{d-1} = \mathcal{O}_{q^*A_d/D'_{d-1}}(-1) \subset \rho^*(q^*\mathcal{A}_d/\mathcal{D}'_{d-1}),$$

and let $\mathcal{D}_i = \rho^*\mathcal{D}'_i \subset \rho^*q^*\mathcal{A}_i \cong p^*\mathcal{A}_i$ for $i = 1, \dots d-1$. Then we have the diagram

$$\begin{pmatrix} \mathcal{D}_1 & \subset \dots \subset & \mathcal{D}_d \\ \cap & & \cap \\ p^*\mathcal{A}_1 & \subset \dots \subset & p^*\mathcal{A}_d \end{pmatrix}$$

over F. It is now straightforward to verify that $F \xrightarrow{p} X$ together with this flag has the universal property. Then $Fl(\underline{\mathcal{A}}) = F$ is unique up to canonical isomorphisms. Note that the fibres of $Fl(\underline{\mathcal{A}}) \xrightarrow{p} X$ consist of flags of vector subspaces

$$\begin{pmatrix} V_1 & \subset \cdots \subset & V_d \\ \cap & & \cap \\ \mathcal{A}_1(x) & \subset \cdots \subset & \mathcal{A}_d(x) \end{pmatrix}$$

with $\dim V_i = i$.

14.6. Notation: Given two locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ or vector bundles E and F on an algebraic scheme X, we let $c_i(\mathcal{E} - \mathcal{F})$ or $c_i(E - F)$ denote the coefficients of the Chern polynomial

$$c_t(\mathcal{E})c_t(\mathcal{F})^{-1}.$$

They can be computed recursively by the formulas

$$\sum_{i+j=k} c_i(\mathcal{E} - \mathcal{F})c_j(\mathcal{F}) = c_k(\mathcal{E}).$$

14.7. Proposition: *Let $0 \subsetneq \mathcal{A}_1 \subsetneq \ldots \subsetneq \mathcal{A}_d = \mathcal{A}$ be a flag of subbundles on a scheme X of ranks a_i respectively, let $\mathcal{M}_1, \ldots, \mathcal{M}_d$ be any sequence of locally free sheaves on X and let $i_1, \ldots, i_d$ be a sequence of integers ≥ 0. Then*

$$\begin{aligned} & p_*(c_{i_1}(p^*\mathcal{M}_1 - \mathcal{D}_1) \cdot \ldots \cdot c_{i_d}(p^*\mathcal{M}_d - \mathcal{D}_d) \cap p^*\alpha) \\ = \;& c_{i_1-a_1+1}(\mathcal{M}_1 - \mathcal{A}_1) \cdot \ldots \cdot c_{i_d-a_d+d}(\mathcal{M}_d - \mathcal{A}_d) \cap \alpha \end{aligned}$$

for any class $\alpha \in A_*(X)$, where $\mathcal{D}_\nu$ are the sheaves of the universal flag on $Fl(\underline{\mathcal{A}}) \xrightarrow{p} X$.

Proof. Let

$$\begin{tikzcd} F \arrow[rr, "\rho"] \arrow[dr, "p"'] & & F' \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

be the induction diagram of the construction of $\mathrm{Fl}(\underline{\mathcal{A}})$ as above. For any j and any class $\beta \in A_*(F')$ we are going to prove the formula

$$\rho_*(c_j(p^*\mathcal{M}_d - \mathcal{D}_d) \cap \rho^*\beta = c_{j-a_d+d}(q^*\mathcal{M}_d - q^*\mathcal{A}_d) \cap \beta\,.$$

Over $F = P(q^*\mathcal{A}_d/\mathcal{D}'_{d-1})$ we have the exact sequence

$$0 \to \mathcal{D}_d/\mathcal{D}_{d-1} \to p^*\mathcal{A}_d/\mathcal{D}_{d-1} \to \mathcal{Q} \to 0$$

which is the relative Euler sequence of F over F'. By the Whitney product formula we get

$$c_t(p^*\mathcal{A}_d) = c_t(\mathcal{D}_d).c_t(\mathcal{Q})$$

and from that

$$c_t(p^*\mathcal{M}_d) - \mathcal{D}_d) = c_t(p^*\mathcal{M}_d - p^*\mathcal{A}_d).c_t(\mathcal{Q}).$$

Now the projection formula for ρ supplies the following identity

$$\begin{aligned} & \rho_*(c_j(p^*\mathcal{M}_d - \mathcal{D}_d) \cap \rho^*\beta) \\ = \;& \rho_*(\sum_{\mu+\nu=j} c_\mu(p^*\mathcal{M}_d - p^*\mathcal{A}_d)c_\nu(\mathcal{Q}) \cap \rho^*\beta) \\ = \;& \sum_{\mu+\nu=j} c_\mu(q^*\mathcal{M}_d - q^*\mathcal{A}_d) \cap \rho_*(c_\nu(\mathcal{Q}) \cap \rho^*\beta). \end{aligned}$$

By Exercise 10.9 $\rho_*(c_\nu(\mathcal{Q}) \cap \rho^*\beta) = 0$ for $\nu < \operatorname{rank}(\mathcal{Q}) = a_d - d$ and $= \beta$ for $\nu = a_d - d$. This implies formula $(*)$ with $\mu = j - (a_d - d)$. The proof of the proposition follows by induction corresponding to the inductive construction of F:

$$\begin{aligned} & p_*(c_{i_1}(p^*\mathcal{M}_d - \mathcal{D}_d) \cdot \ldots \cdot c_{i_d}(p^*\mathcal{M}_d - \mathcal{D}_d) \cap p^*\alpha) \\ = \;& q_*\rho_*(c_{i_1}(\rho^*q^*\mathcal{M}_1 - \rho^*\mathcal{D}'_1) \cdot \ldots \cdot c_{i_{d-1}}(\rho^*q^*\mathcal{D}_{d-1} - \rho^*\mathcal{D}'_{d-1}).c_{i_d}(\rho^*q^*\mathcal{M}_d - \mathcal{D}_d) \cap \rho^*q^*\alpha) \\ = \;& q_*(c_{i_1}(q^*\mathcal{M}_1 - \mathcal{D}'_1) \cdot \ldots \cdot c_{i_{d-1}}(q^*\mathcal{M}_{d-1} - \mathcal{D}_{d-1}) \cap \rho_*(c_{i_d}(\rho^*q^*\mathcal{M}_d - \mathcal{D}_d) \cap \rho^*q^*\alpha) \\ = \;& q_*(c_{i_1}(q^*\mathcal{M}_1 - \mathcal{D}'_1) \cdot \ldots \cdot c_{i_{d-1}}(q^*\mathcal{M}_{d-1} - \mathcal{D}_{d-1}) \cap c_{i_d-a_d+d}(q^*\mathcal{M}_d - q^*\mathcal{A}_d) \cap q^*\alpha \\ = \;& q_*(c_{i_1}(q^*\mathcal{M}_1 - \mathcal{D}'_1) \cdot \ldots \cdot c_{i_{d-1}}(q^*\mathcal{M}_{d-1} - \mathcal{D}'_{d-1}) \cap q^*(c_{i_d-a_d+d}(\mathcal{M}_d - \mathcal{A}_d) \cap \alpha) \\ = \;& c_{i_1-a_1+1}(\mathcal{M}_1 - \mathcal{A}_1) \cdot \ldots \cdot c_{i_{d-1}-a_{d-1}+d-1}(\mathcal{M}_{d-1}).c_{i_d-a_d+d}(\mathcal{M}_d - \mathcal{A}_d) \cap \alpha. \qquad \square \end{aligned}$$

Before stating the corollary we introduce the following determinants. Let

$$c^1 = \sum_{\nu\geq 0} c^1_\nu t^\nu, \quad \ldots, \quad c^d = \sum_{\nu \geq 0} c^d_\nu t^\nu$$

be (Chern) polynomials and let $\lambda_1, \ldots, \lambda_d$ be integers. The coefficients c^i_ν are supposed to have weight or degree ν. Then let

$$\Delta_{\lambda_1\ldots\lambda_d}(c^1, \ldots, c^d)$$

denote the determinant of the matrix

$$\begin{pmatrix} c^1_{\lambda_1} & c^1_{\lambda_1+1} & \cdots & c^1_{\lambda_1+d-1} \\ c^2_{\lambda_2-1} & c^2_{\lambda_2} & \cdots & c^2_{\lambda_2+d-2} \\ \vdots & \vdots & & \vdots \\ c^d_{\lambda_d-d+1} & & \cdots & c^d_{\lambda_d} \end{pmatrix}.$$

If all c^i are equal to c, we denote the determinant by $\Delta_{\lambda_1\ldots\lambda_d}(c)$ and if also all λ_i are equal to λ, it is denoted by $\Delta^d_\lambda(c)$.

14.8. Corollary: *With the same data as in 14.7 the following formula holds for any sequence $\lambda_1, \ldots, \lambda_d$ of integers and any $\alpha \in A_d(X)$.*

$$\begin{aligned} & p_*(\Delta_{\lambda_1\ldots\lambda_d}(c(p^*\mathcal{M}_1 - \mathcal{D}_1), \ldots, c(p^*\mathcal{M}_d - \mathcal{D}_d)) \cap p^*\alpha) \\ = \;& \Delta_{\mu_1\ldots\mu_d}(c(\mathcal{M}_1 - \mathcal{A}_1), \ldots, c(\mathcal{M}_d - \mathcal{D}_d)) \cap \alpha, \end{aligned}$$

where $\mu_i = \lambda_i - a_i + 1$.

Proof. Insert the formulas of the proposition for each term of the development of the determinant. $\square$

14.9. Proposition: *Let $\mathcal{E}$ be locally free on X of rank $n, d \leq n$, and $G_d(\mathcal{E}) \xrightarrow{p} X$ the Grassmann bundle of d–planes in $\mathcal{E}$, let $\mathcal{S}$ denote the universal subbundle of $p^*\mathcal{E}$. Then for any locally free sheaf $\mathcal{F}$ of rank f on X and all $\alpha \in A_*(X)$, the highest Chern class c_{fd} of $\mathcal{S}^* \otimes p^*\mathcal{F}$ satisfies*

$$p_*(c_{df}(\mathcal{S}^* \otimes p^*\mathcal{F}) \cap p^*\alpha) = \Delta^d_{f+d-n}(c(\mathcal{F} - \mathcal{E})) \cap \alpha.$$

The proof follows from Corollary 14.8 after reducing to the case where $\mathcal{E}$ has a flag of subbundles by the splitting principle, see [7], proposition 14.2.2 .

15. Degeneracy classes

15.1. Regular sections. Let $\mathcal{E}$ be a locally free sheaf of rank e over a purely n–dimensional scheme X and let s be a section of $\mathcal{E}$. The zero scheme $Z(s)$ is then defined by the exact sequence

$$\mathcal{E}^* \xrightarrow{s^\vee} \mathcal{O} \to \mathcal{O}_{Z(s)} \to 0.$$

As a set $Z(s)$ can be described as the set of points $x \in X$ with $s(x) = 0$, where $s(x)$ is the induced homomorphism $k \to \mathcal{E}(x)$. If $\mathcal{E}|U$ is isomorphic to $\mathcal{O}^e_X|U$, then $s|U$ is given by regular functions $f_1, \ldots, f_e \in \mathcal{O}_X(U)$ which are the generators of $Im(s^\vee)$ over U. The section is called regular, if the functions $f_1, \ldots, f_e$ form a regular sequence for every local trivialization. This means that $Z(s)$ is regularly embedded in X of codimension e. Note however, that the morphism $X \xrightarrow{s} E$ induced by s into the bundle space is always a regular embedding of codimension e because it is locally the graph of a morphism $U \to \mathbb{A}^e$.

15.2. The class $\zeta(s)$. Let $\mathcal{E}$ on X and a section s of $\mathcal{E}$ be as above, let E be the bundle space of $\mathcal{E}$, and let s_0 denote the zero section $X \hookrightarrow E$. Then we have the pull–back diagram

$$\begin{tikzcd} Z(s) \arrow[r, hook, "i"] \arrow[d, hook, "i"'] & X \arrow[d, hook, "s"] \\ X \arrow[r, hook, "s_0"'] & E \end{tikzcd}$$

and from that the refined Gysin homomorphism

$$A_k(X) \xrightarrow{s_0^!} A_{k-e}(Z(s))$$

because s_0 is a regular embedding of codimension e. If s is a regular section, then i is a regular embedding of codimension e and we have $s_0^! = i^*$. For the fundamental class $[X]$ we obtain the class

$$\zeta(s) = s_0^![X] \in A_{n-e}(Z(s)).$$

This is defined even if $Z(s)$ is quite irregular. The following proposition describes its plausible properties.

15.3. Proposition: *Let $\mathcal{E}, X, s$ be as above. Then*

(a) $i_*\zeta(s) = c_e(\mathcal{E}) \cap [X]$ *in* $A_{n-e}(X)$.
(b) *Each irreducible component of* $Z(s)$ *has codimension* $\leq e$.
(c) *If* $\operatorname{codim}(Z(s), X) = e$, *then* $\zeta(s)$ *is a "positive" cycle whose support is* $Z(s)$, *i.e.* $\zeta(s) = \sum \mu_i[Z_i]$ *where* Z_i *are the components of* $Z(s)$ *and* $\mu_i > 0$.
(d) *If* s *is a regular section (then* $Z(s)$ *is a locally complete intersection of codimension* e*), then* $\zeta(s) = [Z(s)]$ *is the fundamental class of* $Z(s)$.
(e) *For any morphism* $Y \xrightarrow{f} X$ *from a pure–dimensional scheme, let* $t = f^*s$ *be the induced section of* $f^*\mathcal{E}$ *and* $Z(t) \xrightarrow{g} Z(s)$ *be the restriction of* f. *Then*
  (i) *If* f *is flat, then* $g^*\zeta(s) = \zeta(t)$.
  (ii) *If* f *is proper, and both* X *and* Y *are varieties, then* $g_*\zeta(t) = \deg(Y/X)\zeta(s)$.

For the proof see §6 of [7].

15.4. Remark: One can generalize the notion of a regular embedding to that of a **local complete intersection** morphism. A morphism $Y \xrightarrow{f} X$ is called an l.c.i. morphism of codimension d if it admits a factorization $Y \overset{i}{\hookrightarrow} P \xrightarrow{p} X$ into a regular embedding of some codimension e and a smooth morphism of (constant) fibre dimension n such that $d = e - n$. This number is independent of the factorization (using the fibre product of two P's). If X is smooth, any morphism to X is l.c.i. because then the graph morphism γ_f is a regular embedding into $Y \times X$, and $d = \dim X - \dim Y$. If f is a l.c.i. morphism of codimension d, there is a refined Gysin homomorphism

$$A_k(X') \xrightarrow{f^!} A_{k-d}(Y')$$

for any pull–back diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd} \qquad \text{with factorization} \qquad \begin{tikzcd} Y' \arrow[r, hook] \arrow[d] & P' \arrow[r, "p'"] \arrow[d, dashed] & X' \arrow[d] \\ Y \arrow[r, hook, "i"] & P \arrow[r] & X, \end{tikzcd}$$

defined by $f^! = i^! p'^*$, the composition

$$A_k(X') \to A_{k+n}(P') \to A_{k+n-e}(Y').$$

It is shown in [7], prop. 4.1, that (e), (i) of the above proposition is still valid for a l.c.i. morphism with g^* replaced by $f^!$.

15.5. Degeneracy classes of homomorphisms

Let $\mathcal{E} \xrightarrow{\sigma} \mathcal{F}$ be a homomorphism of locally free sheaves of ranks e and f on a purely n–dimensional scheme X. Its zero–scheme $Z(\sigma)$ can be defined as the zero–scheme of the corresponding section of $\mathcal{E}^* \otimes \mathcal{F}$. The zero scheme of the induced homomorphism

$$\Lambda^{k+1}\mathcal{E} \xrightarrow{\Lambda^{k+1}\sigma} \Lambda^{k+1}\mathcal{F}$$

is denoted by

$$D_k(\sigma) = Z(\Lambda^{k+1}\sigma).$$

For any point let $\sigma(x)$ be the induced homomorphism of the vector spaces $\mathcal{E}(x), \mathcal{F}(x)$. Then as a set

$$D_k(\sigma) = \{x \in X \mid \operatorname{rank} \sigma(x) \le k\}.$$

We are going to define the more refined degeneracy loci of σ with respect to a flag of subbundles

$$0 \underset{\neq}{\subset} \mathcal{A}_1 \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} \mathcal{A}_d \subset \mathcal{E}.$$

Let

$$\Omega(\underline{\mathcal{A}}, \sigma) := \{x \in X \mid \ \dim \operatorname{Ker}(\sigma(x)) \cap \mathcal{A}_i(x) \ge i \text{ for any } i\}.$$

If σ_i denotes the restriction of σ to $\mathcal{A}_i$, we have

$$\Omega(\underline{\mathcal{A}}, \sigma) = \bigcap_i Z(\Lambda^{a_i - i+1}\sigma_i),$$

where a_i is the rank of $\mathcal{A}_i$. This defines the scheme structure of $\Omega(\underline{\mathcal{A}}, \sigma)$. The scheme $\Omega(\underline{\mathcal{A}}, \sigma)$ may be quite arbitrary, neither equi–dimensional nor reduced. Its expected dimension is $m = \dim Fl(\underline{\mathcal{A}}) - df$. We are going to replace it by a class $\omega(\underline{\mathcal{A}}, \sigma) \in A_m\big(\Omega(\underline{\mathcal{A}}, \sigma)\big)$. For that the scheme $\Omega(\underline{\mathcal{A}}), \sigma)$ can be related to the zero scheme of the induced homomorphism $\mathcal{D}_d \to p^*\mathcal{F}$ on the flag variety $\mathrm{Fl}(\underline{\mathcal{A}}) \xrightarrow{p} X$ as follows. Let

$$\begin{array}{ccc} \mathcal{D}_1 & \subset \cdots \subset & \mathcal{D}_d \\ \cap & & \cap \\ p^*\mathcal{A}_1 & \subset \cdots \subset & p^*\mathcal{A}_d \end{array}$$

be the universal flag of subbundles and let s_σ be the section of $\mathcal{D}_d^* \otimes p^*\mathcal{F}$ corresponding to the composition $\mathcal{D}_d \hookrightarrow p^*\mathcal{E} \xrightarrow{p^*\sigma} p^*\mathcal{F}$. Then p maps $Z(s_\sigma)$ onto $\Omega(\underline{\mathcal{A}}, \sigma)$, i.e. we have a diagram

$$\begin{array}{ccc} \mathrm{Fl}(\underline{\mathcal{A}}) & \xrightarrow{p} & X \\ \cup & & \cup \\ Z(s_\sigma) & \xrightarrow{q} & \Omega(\underline{\mathcal{A}}, \sigma) \end{array}$$

with q proper. This follows from the definition of the universal flag: If $p(y) = x$ and $y \in Z(s_\sigma)$, then $\mathcal{D}_i(y) \subset \mathcal{A}_i(x)$ is contained in the kernel of $\sigma_i(x)$, and conversely, if the kernel of each $\sigma_i(x)$ contains an i–dimensional subspace V_i, we obtain a flag

$$\begin{array}{ccc} V_1 & \subset \ldots \subset & V_d \\ \cap & & \cap \\ \mathcal{A}_1(x) & \subset \ldots \subset & \mathcal{A}_d(x) \end{array}$$

which is a point y of the flag variety over x which belongs to $Z(s_\sigma)$. Now the class

$$\zeta(s_\sigma) \in A_m(Z(s_\sigma))$$

with

$$m = \dim \mathrm{Fl}(\underline{\mathcal{A}}) - df = n + \sum_1^d (a_i - i) - df$$

is well–defined as the class of the section s_σ of $\mathcal{D}_d^* \otimes p^*\mathcal{F}$ by 15.2. We denote its proper image as

$$\omega(\underline{\mathcal{A}}, \sigma) = q_*\zeta(s_\sigma) \in A_m(\Omega(\underline{\mathcal{A}}, \sigma)).$$

This class has similar natural properties as $\zeta(s)$ even so the scheme $\Omega(\underline{\mathcal{A}}, \sigma)$ may have a complicated structure in general.

15.6. Proposition: *Let $\mathcal{E} \xrightarrow{\sigma} \mathcal{F}$ be a homomorphism of locally free sheaves of ranks e and f on a purely n–dimensional scheme X, and let $0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_d \subset \mathcal{E}$ be a flag of subbundles of ranks $0 < a_1 < \cdots < a_d$. Let*

$$\lambda_i = f - a_i + i \quad \textit{and} \quad h = \sum_i \lambda_i = df - \sum_i (a_i - i)$$

and suppose that $\lambda_d \geq 0$. Let $m = n - h = \dim Fl(\underline{\mathcal{A}}) - df$, such that $\omega(\underline{\mathcal{A}}, \sigma) \in A_m(\Omega(\underline{\mathcal{A}}, \sigma))$. Let i denote the inclusion of $\Omega(\underline{\mathcal{A}}, \sigma)$ into X. Then

(a) $i_*\omega(\underline{\mathcal{A}}, \sigma) = \Delta_{\lambda_1 \ldots \lambda_d}(c(\mathcal{F} - \mathcal{A}_1), \ldots, c(\mathcal{F} - \mathcal{A}_d)) \cap [X]$.

(b) *Each component of $\Omega(\underline{\mathcal{A}}, \sigma)$ has codimension $\leq h$.*

(c) *If* codim $\Omega(\underline{\mathcal{A}}, \sigma) = h$*, then $\Omega(\underline{\mathcal{A}}, \sigma)$ is pure–dimensional and $\omega(\underline{\mathcal{A}}, \sigma)$ is a positive cycle with support $\Omega(\underline{\mathcal{A}}, \sigma)$.*

(c') *If* codim $\Omega(\underline{\mathcal{A}}, \sigma) = h$ *and X is Cohen–Macaulay, then $\Omega(\underline{\mathcal{A}}, \sigma)$ is Cohen Macaulay and $\omega(\underline{\mathcal{A}}, \sigma) = [\Omega(\underline{\mathcal{A}}, \sigma)]$.*

(d) *The formation of ω commutes with Gysin maps and proper push–forwards: Let $X' \xrightarrow{f} X$ be a morphism and let $\mathcal{E}' \xrightarrow{\sigma'} \mathcal{F}'$ and $\mathcal{A}'_i$ be the pull–backs of the corresponding objects on X, suppose that also X' is pure–dimensional and let $\Omega(\underline{\mathcal{A}}', \sigma') \xrightarrow{g} \Omega(\underline{\mathcal{A}}, \sigma)$ denote the restriction of f. Then*
  (i) *If f is flat of constant relative dimension, then $g^*\omega(\underline{\mathcal{A}}, \sigma) = \omega(\underline{\mathcal{A}}', \sigma')$.*
  (ii) *If f is proper and X', X are varieties, then*

$$g_*\omega(\underline{\mathcal{A}}', \sigma') = \deg(X'/X)\omega(\underline{\mathcal{A}}, \sigma).$$

Proof. We only sketch a proof. For (d) we are given the pull–back diagram

$$\begin{array}{ccc} \mathrm{Fl}(\underline{\mathcal{A}}') & \xrightarrow{\widetilde{f}} & \mathrm{Fl}(\underline{\mathcal{A}}) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} Z(s_{\sigma'}) & \xrightarrow{\widetilde{g}} & Z(s_\sigma) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ \Omega(\underline{\mathcal{A}}', \sigma') & \xrightarrow{g} & \Omega(\underline{\mathcal{A}}, \sigma) \end{array}$$

where $\widetilde{f}^*(s_\sigma) = s_{\sigma'}$ and $q, q', g, \widetilde{g}$ are the restrictions of $p, p', f, \widetilde{f}$. By 15.3 we have

$$\widetilde{g}^*\zeta(s_\sigma) = \zeta(s_{\sigma'})$$

in case (i). Then, using 7.17,

$$\begin{aligned} \omega(\underline{\mathcal{A}}', \sigma') &= q'_*\zeta(s_{\sigma'}) \\ &= q'_*\widetilde{g}^*\zeta(s_\sigma) \\ &= g^*q_*\zeta(s_\sigma) = g^*\omega(\underline{\mathcal{A}}, \sigma). \end{aligned}$$

In case (ii) we have

$$\begin{aligned} g_*\omega(\underline{\mathcal{A}'}, \sigma') &= g_* q'_* \zeta(s_{\sigma'}) \\ &= q_* \widetilde{g}_* \zeta(s_{\sigma'}) \\ &= q_* \zeta(s_\sigma) = \omega(\underline{\mathcal{A}}, \sigma). \end{aligned}$$

This proves (d). For (a) we use the formula

$$i_*\zeta(s_\sigma) = c_{df}(\mathcal{D}_d^* \otimes p^*\mathcal{F}) \cap [\mathrm{Fl}(\underline{\mathcal{A}})]$$

of 15.3, which is a class in $A_m(\mathrm{Fl}(\underline{\mathcal{A}}))$. By the appendix A.9.1 in [7]

$$c_{df}(\mathcal{D}_d^* \otimes p^*\mathcal{F}) = \Delta_{f,\ldots,f}(c(p^*\mathcal{F} - \mathcal{D}_1), \ldots, c(p^*\mathcal{F} - \mathcal{D}_d)),$$

using Chern roots. Now corollary 14.8 yields

$$\begin{aligned} i_*\omega(\underline{\mathcal{A}}, \sigma) &= i_* q_* \zeta(s_\sigma) \\ &= p_* i_* \zeta(s_\sigma) \\ &= p_* \Delta_{f,\ldots,f}(c(p^*\mathcal{F} - \mathcal{D}_1), \ldots, c(p^*\mathcal{F} - \mathcal{D}_d)) \cap [\mathrm{Fl}(\underline{\mathcal{A}})] \\ &= \Delta_{\lambda_1,\ldots,\lambda_d}(c(\mathcal{F} - \mathcal{A}_1), \ldots, c(\mathcal{F} - \mathcal{A}_d)) \cap [X]. \end{aligned}$$

This proves (a). For (b), (c), (c') we consider first the special case where $X = \mathbb{A}^{ef}, \mathcal{E}$ and $\mathcal{F}$ are trivial and σ is the (universal) homomorphism given by the coordinate functions x_{ij}, and we let $\mathcal{A}_i$ be the trivial subbundle of $\mathcal{E}$ spanned by the first a_i standard basis sections of $\mathcal{E} = \mathcal{O}_X^e$, for given $0 < a_1 < \cdots < a_d \leq e$. Then $\Omega = \Omega(\underline{\mathcal{A}}, \sigma)$ is the scheme

$$\Omega = \bigcap_{k=1}^{d} Z_k$$

where Z_k is the zero scheme of the $(a_k - k + 1)$–minors of the first a_k columns of (x_{ij}). In this situation one can prove that

- $\Omega(\underline{\mathcal{A}}, \sigma)$ is irreducible of codimension h
- $\Omega(\underline{\mathcal{A}}, \sigma)$ is Cohen–Macaulay
- s_σ is a regular section of $\mathcal{D}_d^* \otimes p^*\mathcal{F}$,
- codim $Z(s_\sigma) = d \cdot f$ in the non–singular variety $\mathrm{Fl}(\underline{\mathcal{A}})$
- $Z(s_\sigma) \to \Omega(\underline{\mathcal{A}}, \sigma)$ is birational

see [7], appendix A.7 and [14], Ch. II.

Now (b) is satisfied and (c) and (c') follow immediately from 15.3, (d), because q is birational and

$$\omega(\underline{\mathcal{A}}, \sigma) = q_*\zeta(s_\sigma) = q_*[Z(s_\sigma)] = [\Omega(\underline{\mathcal{A}}, \sigma)].$$

For (b), (c), (c') in the general situation we may replace X by one of its open affine subsets on which $\mathcal{E}, \mathcal{F}$ and the $\mathcal{A}_i$ are trivial, because the statement (b) is local and in case (c), (c') the subvariety Ω is purely m–dimensional such that $A_m(\Omega) \cong A_m(\Omega \cap U)$ for an open subset. Then σ is a matrix (f_{ij}) of regular functions and defines a morphism $X \xrightarrow{\varphi} \mathbb{A}^{ef}$ such that $\mathcal{E} \xrightarrow{\sigma} \mathcal{F}$, any $\mathcal{A}_i$ and $\Omega = \Omega(\underline{\mathcal{A}}, \sigma)$ is the pull–back of the corresponding generic objects on $\mathbb{A}^{ef}$ denoted by a tilde. It follows that $\Omega = f^{-1}\widetilde{\Omega}$ has codimension $\leq h = \mathrm{codim}\widetilde{\Omega}$, which proves (b). If $\mathrm{codim}\Omega = h$ and X is Cohen–Macaulay, then the local rings of Ω are also Cohen–Macaulay, see e.g. [6], prop. 18.13.

In order to obtain (c) and (c') we consider the diagram

$$\begin{tikzcd}
\mathrm{Fl}(\underline{\mathcal{A}}) \arrow[d, "\widetilde{\varphi}"] & Z(s_\sigma) \arrow[l, hook'] \arrow[r, two heads, "q"] \arrow[d, "\widetilde{\varphi}"] & \Omega \arrow[r, hook] \arrow[d, "\varphi"] & X \arrow[d, "\varphi"] \\
\mathrm{Fl}(\underline{\mathcal{A}}) & Z(s_{\widetilde{\sigma}}) \arrow[l, hook'] \arrow[r, two heads, "\widetilde{q}"] & \widetilde{\Omega} \arrow[r, hook] & \mathbb{A}^{ef}
\end{tikzcd}$$

with $\widetilde{\sigma} = (x_{ij})$. Because φ is a l.c.i. morphism, it follows from 15.4 that

$$\zeta(s_\sigma) = \varphi^!\zeta(s_{\widetilde{\sigma}}).$$

Now $q_*\varphi^! = \varphi^!\widetilde{q}_*$, see [7], thm. 6.2 and prop. 6.6 . Therefore,

$$\omega(\underline{\mathcal{A}}, \sigma) = \varphi^!\omega(\widetilde{\underline{\mathcal{A}}}, \widetilde{\sigma}) = \varphi^![\widetilde{\Omega}].$$

¿From this (c) and (c') follow with $[\Omega] = \varphi^![\widetilde{\Omega}]$ in case (c'), using remarks on the intersection multiplicities in [7], §7 and example 14.3.1 . $\square$

15.7. One can obtain the class $\omega(\underline{\mathcal{A}}, \sigma)$ alternatively from the universal space of homomorphisms as follows. Let $H = \mathrm{Hom}(E, F)$ with projection $H \xrightarrow{p} X$. On H there is a universal or tautological homomorphism $p^*\mathcal{E} \xrightarrow{u} p^*\mathcal{F}$. It is easy to prove that $\Omega(p^*\underline{\mathcal{A}}, u)$ has codimension h. Now $\mathcal{E} \xrightarrow{\sigma} \mathcal{F}$ determines a section t_σ of H such that $t_\sigma^* u = \sigma$, and we have the pull–back diagram

$$\begin{tikzcd}
\Omega(\underline{\mathcal{A}}, \sigma) \arrow[r, hook] \arrow[d, hook] & \Omega(p^*\underline{\mathcal{A}}, u) \arrow[d, hook] \\
X \arrow[r, hook, "t_\sigma"'] & H\,.
\end{tikzcd}$$

Then $\Omega(\underline{\mathcal{A}}, \sigma) = t_\sigma^![\Omega(p^*\underline{\mathcal{A}}, u)]$ by (c) and (d) of the proposition.

15.8. Specialized degeneracy loci of sections.

Let $\mathcal{E}$ be a rank r locally free sheaf on an n–dimensional variety X and let $s_1, \ldots, s_N$ be sections of $\mathcal{E}$, $2r \leq N$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a **partition**, i.e. a sequence of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$, and suppose $r \geq \lambda_1$. Let

$$\Omega_\lambda = \{x \in X \mid \dim \mathrm{Span}(s_1(x), \ldots, s_{r+i-\lambda_i}(x)) \leq r - \lambda_i \quad \text{for all } i\}.$$

Then Ω_λ is a closed subscheme of codimension $\leq h = \sum \lambda_i$ and there is a class

$$\omega_\lambda \in A_{n-h}(\Omega_\lambda)$$

with

$$i_*\omega_\lambda = \Delta_\lambda(c(\mathcal{E})) \cap [X].$$

If codim $\Omega_\lambda = h$ and X is Cohen–Macaulay, then $\omega_\lambda = [\Omega_\lambda]$. All this is a special case of proposition 15.6: The sections correspond to a homomorphism $\mathcal{O}_X^N \xrightarrow{\sigma} \mathcal{E}$ and one can consider the flag $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_r$ where $\mathcal{A}_i$ is spanned by the first standard basis sections $e_1, \ldots, e_{a_i}$ of $\mathcal{O}_X^N$ with $a_i = r + i - \lambda_i$. Then

$$\Omega_\lambda = \Omega(\underline{\mathcal{A}}, \sigma).$$

If $\lambda = (p, 0, \dots, 0)$, then

$$\Omega_\lambda = \{x \in X \mid \dim \operatorname{Span}(s_1(x), \dots, s_{r+1-p}(x)) \leq r - p)$$

and then the above determinant formula becomes

$$i_*\omega_\lambda = c_p(\mathcal{E}) \cap [X].$$

If $\lambda = (1, \dots, 1, 0, \dots, 0)$ with p times a 1, then

$$\Omega_\lambda = \{x \in X \mid \dim \operatorname{Span}(s_1(x), \dots, s_{r+\nu}(x)) \leq r - 1 \quad \text{for all } 0 \leq \nu < p\}$$

In that case we obtain

$$i_*\omega_\lambda = (-1)^p s_p(\mathcal{E}) \cap [X]$$

where s_p denotes the p–th Segre class.

Remark: It is shown in [7], example 14.3.2, that codim $\Omega_\lambda = h$ and $\omega_\lambda = [\Omega_\lambda]$, if $\mathcal{E}$ is globally generated and $s_1, \dots, s_N$ are generic sections.

15.9. Thom–Porteous–Formula.

Let as before $\mathcal{E} \xrightarrow{\sigma} \mathcal{F}$ be a homomorphism of locally free sheaves of ranks e and f on a scheme of pure dimension n, and let

$$0 \leq k \leq e, f \quad \text{and} \quad m = n - (e-k)(f-k) \geq 0,\ d = e - k.$$

The k–th degeneracy locus had been defined as

$$D_k(\sigma) = Z(\Lambda^{k+1}\sigma) = \{x \in X \mid \operatorname{rank} \sigma(x) \leq k\}.$$

We are going to define a class of $D_k(\sigma)$ using the Grassmann bundle $G_d(\mathcal{E}) \xrightarrow{p} X$. If $\mathcal{S}$ denotes the universal subbundle of $p^*\mathcal{E}$, we obtain an induced homomorphism $\mathcal{S} \hookrightarrow p^*\mathcal{E} \to p^*\mathcal{F}$ and by that a section s_σ of $\mathcal{S}^* \otimes p^*\mathcal{F}$ over the Grassmann bundle. Then we have the diagram

$$\begin{array}{ccc} G_d(\mathcal{E}) & \overset{p}{\twoheadrightarrow} & X \\ \cup & & \cup \\ Z(s_\sigma) & \overset{q}{\twoheadrightarrow} & D_k(\sigma) \end{array}$$

and the class $\zeta(s_\sigma)$ belongs to $A_m(Z(s_\sigma))$ because $\dim G_d(\mathcal{E}) = m + df$. Now the class

$$\vartheta_k(\sigma) = q_*\zeta(s_\sigma) \in A_m(D_k(\sigma))$$

is well-defined. The properties of these classes are analogous to those of the classes $\omega(\underline{\mathcal{A}}, \sigma)$. We have

(a) $i_*\vartheta_k(\sigma) = \Delta_{f-k}^{e-k}(c(\mathcal{F} - \mathcal{E})) \cap [X]$

(b) $\operatorname{codim} D_k(\sigma) \leq (e-k)(f-k)$

(c) If $\operatorname{codim} D_k(\sigma) = (e-k)(f-k)$, then $\vartheta_k(\sigma)$ is a positive cycle with support $D_k(\sigma)$.

(c') If $\operatorname{codim} D_k(\sigma) = (e-k)(f-k)$ and X is Cohen–Macaulay, then $D_k(\sigma)$ is Cohen–Macaulay and $\vartheta_k(\sigma) = [D_k(\sigma)]$.

(d) The formation of ϑ_k commutes with Gysin maps as in 15.6.

Remark: If $D_k(\sigma)$ has codimension $(e-k)(f-k)$, we have the fundamental cycle $[D_k(\sigma)] = \sum m_i[D_i]$ and $\vartheta_k(\sigma) = \sum e_i D_i$ with $0 < e_i \leq m_i$. Then depth $D_k(\sigma) \leq (e-k)(f-k)$ iff $\vartheta_k(\sigma) = [D_k(\sigma)]$.

The proofs of these properties are analogous or special cases of those for the classes $\omega(\underline{\mathcal{A}}, \sigma)$: (d) with same proof. (a) follows from 15.3 and 14.9:

$$\begin{aligned} i_*\zeta(s_\sigma) &= c_{df}(S^\vee \otimes p^*\mathcal{F}) \cap [G_d(\mathcal{E})] \\ i_*\vartheta_k(\sigma) &= i_*q_*\zeta(S_\sigma) = p_*i_*\zeta(s_\sigma) \\ &= p_*c_{df}(S^\vee \otimes p^*\mathcal{F}) \cap [G_d(\mathcal{E})] \\ &= \Delta_{f-k}^{e-k}(c(\mathcal{F} - \mathcal{E})) \cap [X], \end{aligned}$$

which is the Thom–Porteous formula. If $\mathcal{E}$ contains a flag $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_d = \mathcal{E}$ of ranks $a_i = k + i$, then $D_k(\sigma) = \Omega(\underline{\mathcal{A}}, \sigma)$ and (b), (c), (c') follow from 15.6 by comparing $\mathrm{Fl}(\underline{\mathcal{A}}) \to G_d(\mathcal{E})$. If there is no such flag on X, one can use the splitting principle for a proper and flat morphism $Y \xrightarrow{f} X$ such that $f^!\vartheta_k(\sigma) = \vartheta_k(f^*\sigma)$ by (d).

Remark on dual classes: If $\mathcal{F}^* \xrightarrow{\sigma^\vee} \mathcal{E}^*$ is the dual of σ, then $\vartheta_k(\sigma^\vee) = \vartheta_k(\sigma)$ and we have the formula

$$\Delta_{e-k}^{f-k}(c(\mathcal{E}^* - \mathcal{F}^*)) \cap [X] = \Delta_{f-k}^{e-k}(c(\mathcal{F} - \mathcal{E})) \cap [X].$$

15.10. Dependency loci of sections

Let $\mathcal{E}$ be locally free of rank r on an n–dimensional variety X, let $k \leq r$ and let $s_1, \ldots, s_{r-k+1}$ be sections of $\mathcal{E}$. The dependency locus is defined by

$$D(s_1, \ldots, s_{r-k+1}) = \{x \in X \mid s_1(x), \ldots, s_{r-k+1}(x) \quad \text{linearly dependent in } \mathcal{E}(x)\}.$$

If $\mathcal{O}_X^{r-k+1} \xrightarrow{\sigma} \mathcal{E}$ is the homomorphism defined by the sections, then

$$D(s_1, \ldots, s_{r-k+1}) = D_{r-k}(\sigma).$$

By the previous result there is a class

$$\vartheta(s_1, \ldots, s_{r-k+1}) \in A_{n-k}(D(s_1, \ldots, s_{r-k+1})),$$

where now $m = n((r-k+1)-(r-k))(r-(r-k)) = n-k$. In this case $\mathrm{codim} D(s_1, \ldots, s_{r-k+1}) \leq k$ and the Thom–Porteous formula reduces to

$$i_*\vartheta(s_1, \ldots, s_{r-k+1}) = c_k(\mathcal{E}) \cap [X]$$

because of the identity $\Delta_k^1(c(\mathcal{E})) = c_k(\mathcal{E})$.

Note that $D(s_1, \ldots, s_{r-k+1})$ can have codimension ≤ 1 and may be empty if the sections are independent everywhere. In that case $c_k(\mathcal{E}) = 0$. If the codimension is k and X is Cohen–Macaulay, then $\vartheta(\underline{s}) = [D(\underline{s})]$.

15.11. Geometric definition of Chern classes:

The last result leads to the following geometric construction of the Chern classes. Let $\mathcal{E}$ be locally free of rank r on a quasi–projective variety of dimension n over an algebraically closed field. Then there is an invertible sheaf $\mathcal{L}$ such that $\mathcal{E} \otimes \mathcal{L}$ is generated by sections and admits $r+1$ sections $s_1, \dots, s_{r+1}$. For $k \leq r$ let

$$D_k = \{x \in X \mid s_1(x), \dots, s_{r-k+1}(x) \quad \text{linearly dependent } \}.$$

We may assume that D_k has pure dimension k for any k or is empty, after choosing a suitable $\mathcal{L}$. Then

$$[D_k] = c_k(\mathcal{E} \otimes \mathcal{L}) \cap [X].$$

Now the Chern classes are determined by the formula

$$c_k(\mathcal{E}) = \sum_{i=0}^{k} (-1)^{k-1} \binom{r-i}{k-i} c_1(\mathcal{L})^{k-i} c_i(\mathcal{E} \otimes \mathcal{L}).$$

15.12. Giambelli formula. For the degeneracy locus of the universal matrix of size $m \times n, m \leq n$, the Thom–Porteous formula implies degree–formulas. Let $\mathbb{P}_{mn-1} = \mathbb{P}\mathrm{Hom}(k^m, k^n), k$ algebraically closed, and let $\mathcal{O}^m \xrightarrow{\sigma} \mathcal{O}(1)^n$ be the tautological homomorphism, given by the homogeneous coordinates of $\mathbb{P}_{mn-1}$. Let

$$V_k(m,n) = D_k(\sigma)$$

in this case. Here $V_k(m,n)$ has the expected codimension $(m-k)(n-k)$ and is Cohen–Macaulay. Therefore, its class in $A_{k(m+n-k)-1}(\mathbb{P}_{mn-1}) \cong \mathbb{Z}$ is

$$\begin{aligned} [V_k(m,n)] &= \Delta_{n-k}^{m-k}(c(\mathcal{O}(1)^n - \mathcal{O}^m) \cap [\mathbb{P}_{mn-1}] \\ &= \Delta_{n-k}^{m-k}(c(\mathcal{O}(1)^n) \cap [\mathbb{P}_{mn-1}]. \end{aligned}$$

The computation of this number gives

$$\deg(V_k(m,n) = \prod_{i=0}^{m-k-1} \frac{(n+1)!}{(m-1-i)!(n-k+i)!} \cdot 1!2! \cdot \ldots \cdot (m-k-1)!\ .$$

15.13. Degeneracy loci of morphisms.

The formula for the degeneracy classes of homomorphism between vector bundles can be applied to the tangent maps of morphisms. Let $X \xrightarrow{f} Y$ be a morphism between smooth varieties of dimensions m and n and let $\tau = Tf$ be the induced homomorphism $\mathcal{T}X \to f^*\mathcal{T}Y$. Define $S_k(f) := D_k(\tau)$ for $k \leq m, n$. Then $\mathrm{codim} S_k(f) \leq (m-k)(n-k)$ and there is the class

$$\sigma_k(f) \in A_N(S_k(f)),\ N = m - (m-k)(n-k).$$

with

$$i_*\sigma_k(f) = \Delta_{n-k}^{m-k}(c(f^*\mathcal{T}Y - \mathcal{T}X)) \cap [X].$$

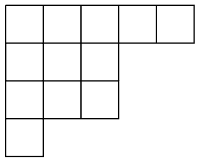
If $\mathrm{codim} S_k(f) = (m-k)(n-k)$, then $S_k(f)$ is Cohen–Macaulay and $\sigma_k(f) = [S_k(f)]$.

16. Intersections on Grassmannians

In this section the intersections of the determinantal classes of the previous section will be studied. They are defined as

$$\Delta_\lambda(c) = \Delta_{\lambda_1,\dots,\lambda_d}(c,\dots,c) = \det(c_{\lambda_i+j-i})$$

for a power series $c = 1 + c_1 t + c_2 t^2 + \cdots$ (of Chern classes) and are called **Schur polynomials**. We suppose that $\lambda_1 \geq \ldots \geq \lambda_d \geq 0$. Such tuples $\lambda = (\lambda_1,\dots,\lambda_d)$ will be called **partitions** of $|\lambda| = \lambda_1 + \cdots + \lambda_d$, or simply partitions. A partition corresponds to a Young diagram consisting of λ_i boxes in the i–th row. The following is the Young diagram of the partition $(5,3,3,1)$.



The conjugate partition λ^c is defined by the transposed Young diagram of λ. Thus $(4,3,3,1,1)$ is the conjugate of $(5,3,3,1)$.

If $c_1, c_2, \ldots$ and $s_1, s_2, \ldots$ are series of commuting variables related by

$$(1 + c_1 t^2 + c_2 t^2 + \cdots)(1 - s_1 t + s_2 t^2 \mp \cdots) = 1,$$

and if λ and μ are conjugate partitions, then

$$\boxed{\Delta_\lambda(c) = \Delta_\mu(s)}, \qquad (SI)$$

see [7], appendix A.9.2. For example, $c_i = c_i(E)$ and $s_i = s_i(E^*)$ the Chern and Segre classes of a vector bundle and its dual, or $c_i = c_i(F - E)$ and $s_i = c_i(E^* - F^*)$. Special cases of (SI) are:

(a) $\lambda = (1,\dots,1)$ and $\mu = (d)$ with

$$s_d = \det\begin{pmatrix} c_1 & c_2 & \dots & c_d \\ 1 & c_1 & \ddots & \vdots \\ 0 & 1 & \ddots & c_2 \\ \vdots & \ddots & \ddots & c_1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(b) $\lambda = \underbrace{(e,\dots,e)}_{d}$ and $\underbrace{(d,\dots,d)}_{e}$ with

$$\Delta_e^d(c) = \Delta_d^e(s) = (-1)^{de}\Delta_d^e(c^{-1}).$$

16.1. The Littlewood–Richardson rule for determinants.

Let $\lambda = (\lambda_1,\dots,\lambda_d)$ and $\mu = (\mu_1,\dots,\mu_e)$ be two partitions and (c_ν) a series of commuting variables. Then

$$\Delta_\lambda(c)\cdot\Delta_\mu(c) = \sum_\rho N_{\lambda\mu\rho}\Delta_\rho(c)$$

where the sum is over all partitions ρ with $|\rho| = |\lambda| + |\mu|$ which arise as strict expansions from λ and μ by the following recipe, and where the coefficients are the number of Young tableaus arising in the construction and defining the same partition ρ, see [7], Lemma 14.5.3 :

Let λ be given and let $\mu = (m)$. The partition $\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{d+1})$ is called a simple m–expansion if

$$\widetilde{\lambda}_1 \geq \lambda_1 \geq \widetilde{\lambda}_2 \geq \cdots \geq \lambda_d \geq \widetilde{\lambda}_{d+1} \geq 0$$

and $|\widetilde{\lambda}| = |\lambda| + m$. If $\mu = (\mu_1, \ldots, \mu_e)$ is arbitrary, a μ–expansion $\widetilde{\lambda}$ of λ is obtained with a Young–diagram $\widetilde{Y}$ as follows. Construct Y_1 from Y of λ as a simple μ_1–expansion, and insert the integer 1 into each of the new μ_1 boxes, then construct a Young diagram Y_2 from Y_1 by a simple μ_2–expansion with new entry 2 etc. to obtain a Young diagram $Y_e = \widetilde{Y}$ with entries in the new boxes. A Young diagram with entries is called a Young tableau. The resulting partition $\widetilde{\lambda}$ is called strict if, when the integers in the new boxes are listed from right to left, starting with the top row and going down, for any $1 \leq t \leq |\mu|$ and each $1 \leq k \leq e-1$ the integer k occurs at least as many times as the next integer $k+1$ among the first t integers in the list. The Littlewood–Richardson rule states that the number $N_{\lambda\mu\rho}$ is the number of different Young tableaus occuring as strict μ–expansions of λ, which define the same Young diagram ρ. If $\mu = (m)$, the Littlewood–Richardson rule becomes

$$\Delta_\lambda c_m = \sum_\rho \Delta_\rho$$

with the sum over all simple m–expansions of λ. This formula is called Pieri's formula.

16.1.1. Lemma: *Let c and s be related as above.*

(i) *If $s_i = 0$ for $i > d$, then $\Delta_\lambda(c) = 0$ for any partition λ with $\lambda_{d+1} > 0$*
(ii) *If $c_i = 0$ for $i > k$, then $\Delta_\lambda(c) = 0$ if $\lambda_1 > k$.*

Proof. Let μ be the conjugate of λ. If $\lambda_{d+1} > 0$, then $\mu_1 > d$ and then the first row of the matrix of the determinant $\Delta_\lambda(c) = \Delta_\mu(s)$ vanishes. This proves (i). Case (ii) is dual to (i). In that case $\mu_{k+1} > 0$ and by (i) $\Delta_\mu(s) = 0$. □

16.1.2. Corollary: *Let c and s be related as above and let $s_i = 0$ for $i > d$. Then for all partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ of length d,*

$$\Delta_\lambda(c)\Delta_\mu(c) = \sum_\rho N_{\lambda\mu\rho}\Delta_\rho(c)$$

where the sum is over all $\rho = (\rho_1, \ldots, \rho_d)$ of length d with $|\rho| = |\lambda| + |\mu|$, and which are strict μ–expansions of λ.

16.2. Chern class rules for Grassmann bundles.

Let now again $\mathcal{E}$ be a locally free sheaf of rank n on a scheme X and let $d \le n$. Denote $G = G_d(\mathcal{E})$ the Grassmann bundle of d–planes in the fibres of $\mathcal{E}$ with projection $G \xrightarrow{p} X$. On G we have the exact sequence

$$0 \to \mathcal{S} \to p^*\mathcal{E} \to \mathcal{Q} \to 0.$$

We let $c_i = c_i(\mathcal{Q} - p^*\mathcal{E})$ such that the corresponding s_i in the formula (SI) are

$$s_i = c_i(p^*\mathcal{E}^* - \mathcal{Q}^*) = c_i(\mathcal{S}^*) = (-1)^i c_i(\mathcal{S}).$$

Because $s_i = 0$ for $i > d$, the Littlewood–Richardson rule for

$$\Delta_\lambda = \Delta_\lambda(c) = \Delta_{\lambda_1,\dots,\lambda_d}(c(\mathcal{Q} - p^*\mathcal{E}))$$

becomes

$$\Delta_\lambda \Delta_\mu = \sum_\rho N_{\lambda\mu\rho}\Delta_\rho$$

where all the partitions have length $\le d$. The following proposition is the key for the intersection theory on Grassmannians and Grassmann bundles. Note that Δ_λ and each summand of this determinant is an operator $A_k(G) \to A_{k-|\lambda|}(G)$.

16.3. Proposition*:* (Duality) *With the above notation let λ and μ be partitions of length d with $|\lambda| + |\mu| \le d(n-d)$. Then for any $\alpha \in A_*(X)$,*

$$p_*(\Delta_\lambda\Delta_\mu \cap p^*\alpha) = \begin{cases} \alpha & \textit{if } \lambda_i + \mu_{d-i+1} = n-d \textit{ for } 1 \le i \le d \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Because p^* and p_* are compatible with inclusions, we may assume that $\alpha = [V]$ for a subvariety of dimension k and in addition that $V = X$ is a variety. Then $p^*[X] = [G]$. If $|\lambda| + |\mu| < d(n-d)$, then $p_*(\Delta_\lambda\Delta_\mu \cap [G])$ is in $A_{k+(n-d)-|\lambda|-|\mu|}(X) = 0$. Therefore, we can assume that $|\lambda| + |\mu| = d(n-d)$. Now for the highest degree we can replace X by an open affine subset and thus $\mathcal{E}$ can be assumed trivial. In that situation $c_i = 0$ for $i > n-d$ and then $\Delta_\rho = 0$ in the Littlewood–Richardson formula if $\rho_1 > n-d$, see 16.1.1, (ii). Now it is easy to combine that

$$\Delta_\lambda\Delta_\mu = \begin{cases} \Delta_{\rho_0} & \text{if } \lambda_i + \mu_{d-i+1} = n-d \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

where $\rho_0 = (n-d, \dots, n-d)$ of length d. Now we prove that

$$p_*(\Delta_{\rho_0} \cap [G]) = [X]$$

as follows. Because $\mathcal{E}$ is trivial of rank n, we can choose a flag $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_d \subset \mathcal{E}$ of trivial subbundles of rank $\mathcal{A}_i = i$. Let σ denote the homomorphism $p^*\mathcal{E} \to \mathcal{Q}$. Then

$$\begin{aligned} \Omega = \Omega(p^*\underline{\mathcal{A}}, \sigma) &= \{L \in G \mid \dim L \cap \mathcal{A}_i(x) \ge i \text{ for } 1 \le i \le d,\ x = p(L)\} \\ &\cong G_d(\mathcal{A}_d) \cong X, \end{aligned}$$

so that p maps Ω isomorphically onto X. Since we may also assume that X is smooth, the formula 15.6, (a), (c'), implies

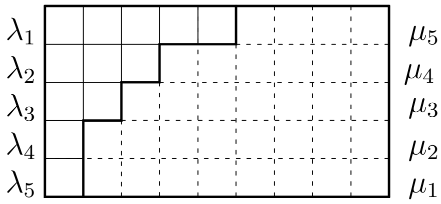
$$[\Omega] = \Delta_{\rho_0}(c(Q)) \cap [G]$$

and so $p_*(\Delta_\lambda\Delta_\mu \cap [G]) = [X]$ in the case $\lambda + \mu = \rho_0$. $\square$

16.3.1. Duality in Grassmannians. In the case of the absolute Grassmannian $G_d(E)$ of an n–dimensional vector space, the formula becomes

$$\Delta_\lambda\Delta_\mu \cap [G_d(E)] = \begin{cases} 1 & \text{if } \lambda_i + \mu_{d-i+1} = n - d \text{ for } 1 \leq i \leq d \\ 0 & \text{otherwise} \end{cases}$$

if $|\lambda| + |\mu| = d(n-d)$. For the proof note that $p_* \neq 0$ only on $A_0 G_d(E)$ and that $p^*\{pt\} = G_d(E)$. The duality condition for the partitions λ and μ means that μ in revised order fills the Young diagram of λ to a rectangle of size $d \times (n-d)$, e.g.



16.4. Giambelli's formula for relative Schubert varieties

Let $\mathcal{E}$ and X be as above in 16.2, let $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_d \subset \mathcal{E}$ be a flag of subbundles of ranks $0 < a_1 < \ldots < a_d \leq n$, let $\lambda_i = n - d + i - a_i$ and let

$$\Omega(\underline{\mathcal{A}}) = \Omega(p^*\underline{\mathcal{A}}, \sigma) \subset G_d(\mathcal{E})$$

be the degeneracy locus of the canonical homomorphism $p^*\mathcal{E} \xrightarrow{\sigma} \mathcal{Q}$ on the Grassmann bundle. By its definition it can be described as

$$\Omega(\underline{\mathcal{A}}) = \{L \in G_d(\mathcal{E}) \mid \dim L \cap \mathcal{A}_i(x) \geq i \text{ for } 1 \leq i \leq d, x = p(L)\}.$$

By 15.6 the corresponding class is given by

$$\omega(\underline{\mathcal{A}}) = \omega(p^*\underline{\mathcal{A}}, \sigma) = \Delta_{\lambda_1\ldots\lambda_d}(c(\mathcal{Q} - p^*\mathcal{A}_1), \ldots, c(\mathcal{Q} - p^*\mathcal{A}_d)) \cap [G_d(\mathcal{E})].$$

16.4.1. Lemma*:*

(i) *If $c_i(\mathcal{E} - \mathcal{A}_j) = 0$ for $i > 0$ and all j, then $\omega(\underline{\mathcal{A}}) = \Delta_\lambda \cap [G_d(\mathcal{E})]$.*
(ii) *If X is pure-dimensional, then $\Omega(\underline{\mathcal{A}})$ has pure codimension $|\lambda|$ in $G_d(\mathcal{E})$ and*

$$[\Omega(\underline{\mathcal{A}})] = \Delta_{\lambda_1\ldots\lambda_d}(c(\mathcal{Q} - p^*\mathcal{A}_1), \ldots, c(\mathcal{Q} - p^*\mathcal{A}_d)) \cap [G_d(\mathcal{E})].$$

Proof. The assumption (i) implies that

$$c(\mathcal{Q} - p^*\mathcal{A}_j) = c(\mathcal{Q} - p^*\mathcal{E})c(p^*\mathcal{E} - p^*\mathcal{A}_j) = c(\mathcal{Q} - p^*\mathcal{E}).$$

For (ii) we show that $\omega(\underline{\mathcal{A}})$ equals the fundamental class $[\Omega(\underline{\mathcal{A}})]$. This can be verified locally w.r.t. X. Therefore, we may assume that X is smooth and affine and $\mathcal{E}$ is trivial. Then $\Omega(\underline{\mathcal{A}})$ is Cohen–Macaulay and $\omega(\underline{\mathcal{A}}) = [\Omega(\underline{\mathcal{A}})]$, by 15.6, (c'). $\square$

16.4.2. Remark: Note, that even when X is smooth, the class $\omega(\underline{\mathcal{A}})$ need not equal $\Delta_\lambda \cap [G_d(\mathcal{E})]$ because there are global obstructions arising from the classes $c_i(\mathcal{Q} - p^*\mathcal{A}_j)$. We have, however, the special cases

$$\Delta_\lambda = c_q(\mathcal{Q} - p^*\mathcal{E}) \quad \text{for} \quad (\lambda_1, \ldots, \lambda_d) = (q, 0, \ldots, 0)$$

and

$$\Delta_\lambda = (-1)^q c_q(S) \quad \text{for} \quad (\lambda_1, \ldots, \lambda_d) = (1, \ldots, 1,\ 0, \ldots, 0)\,,$$

where $\lambda_1 = \ldots = \lambda_q = 1$ and $\lambda_{q+1} = 0$. In the smooth case, all the $\Omega(\underline{\mathcal{A}})$ are Cohen–Macaulay and the ring structure of $A^*G_d(\mathcal{E})$, see 14.1, may be described by the intersections of the classes $[\Omega(\underline{\mathcal{A}})]$.

16.5. Schubert varieties.

The varieties $\Omega(\underline{\mathcal{A}})$ of the previous proposition are the relative versions of the classical Schubert varieties $\Omega(a)$ in the usual Grassmannians $G_d(E)$ of a vector space E of dimension n. In that case the varieties $\Omega(\underline{\mathcal{A}})$ are defined by

$$\Omega(\underline{A}) = \{U \in G_d(E) \mid \dim U \cap A_i \geq i, \quad 1 \leq i \leq d\},$$

where $A_1 \subset A_2 \subset \cdots \subset A_d \subset E$ is a flag of vector subspaces of dimensions $0 < a_1 < \ldots < a_d \leq n$. Because now the Chern classes $c_i(\mathcal{E} - \mathcal{A}_j)$ disappear, Giambelli's formula states that each $\Omega(\underline{A})$ is irreducible and Cohen–Macaulay of codimension $|\lambda|$ and that

$$\omega(\underline{A}) = [\Omega(\underline{A})] = \Delta_\lambda \cap G_d(E)\,,$$

where as before $\lambda_i = n - d + i - a_i$ such that $n - d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0$. We can also write

$$\omega_\lambda = \omega(a) = \omega(\underline{A})$$

because the classes do not depend on the choice of the flag.

If $e_1, \ldots, e_n$ is a basis of E, the spaces A_i may be chosen as the spans of the first a_i vectors and we put $\Omega(a) = \Omega(\underline{A})$ in that case. In the following we suppose that such a basis is given, and we consider the intersection of $\Omega(a)$ with a suitable standard affine chart of $G_d(E)$ defined as follows. Let $k^n \to k^d$ be the projection $\pi(a)$ given by $a_1 < a_2 < \cdots < a_d$ such that for any linear map $k^d \xrightarrow{u} k^n$, $\det(\pi(a) \circ u)$ is the d–minor determined by the columns of u with indices $a_1 < \cdots < a_d$. Then

$$G(a) = \{Im(u) \mid \det(\pi(a) \circ u) \neq 0\}.$$

Let then

$$\Omega^0(a) = \Omega(a) \cap G(a).$$

It is easy to check that $\Omega^0(a)$ is isomorphic to a linear subspace of $\mathrm{Hom}(k^d, k^{n-d})$ of dimension $\sum(a_i - i)$ by presenting a subspace $U \in G(a)$ as the span of the unique $d \times n$–matrix, whose columns with index $a_1, \ldots, a_d$ form the unit matrix. Such an $\Omega^0(a)$ is called a **Schubert cell**.

Example: $d = 4$, $n = 12$, $(a_1, a_2, a_3, a_4) = (2, 5, 7, 10)$. Any $U \in \Omega^0(a)$ is the span of the rows of a unique matrix

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & * & 0 & * & * & 1 & 0 & 0 \end{pmatrix}.$$

$G(a) \cong \mathrm{Hom}(k^4, k^8)$ has the coordinate functions x_{ij}, $j \neq a_i$. The condition $\dim U \cap A_i \geq i$ means that the entries in the ith row after $x_{ia_i} = 1$ are zero, whereas the rest of the entries are free. For $U \in \Omega^0(a)$ we have $\dim U \cap A_i = i$. The free entries in that case fill a linear subspace of $\mathrm{Hom}(k^4, k^8)$ of dimension $(a_1 - 1) + (a_2 - 2) + (a_3 - 3) + (a_4 - 4) = 14$.

In general, a unique matrix for $U \in \Omega^0(a)$ is obtained by putting $x_{ia_i} = 1$ and $x_{ij} = 0$ in the hook determined by x_{ia_i} as in the above matrix.

Moreover, by this consideration we find that

$$\Omega(a) \smallsetminus \Omega^0(a) = \bigcup_{\substack{b \leq a \\ |b| = |a| - 1}} \Omega(b) \text{ with } \Omega^0(b) \cap \Omega^0(b') = \emptyset \text{ for } b \neq b'.$$

It follows that each $\Omega(a)$ is cellular, see 6.5, and that $A_*(\Omega(a))$ is generated by the classes $[\Omega(b)]$ with $b \leq a$. If $a_i = n - d + i$, then

$$\Omega(a) = G_d(E),$$

because for any $U \in G_d(E)$ we have $\dim U \cap A_i \geq d + a_i - n = i$. In particular $A_*(G_d(E))$ is generated by the classes $[\Omega(a)]$, see 6.5.

16.5.1. The Schubert varieties of $G_2(k^4)$ are indexed by pairs (a_1, a_2) with $0 < a_1 < a_2 \leq 4$.

We find

$$\begin{array}{lcll} \Omega(1,2) & = & \{U \in G \mid U = A_2\} & \text{a point} \\ \Omega(1,3) & = & \{U \in G \mid A_1 \subset U \subset A_2\} & \text{a projective line} \\ \Omega(1,4) & = & \{U \in G \mid A_1 \subset U\} & \\ \Omega(2,3) & = & \{U \in G \mid U \subset A_2\} & \\ \Omega(2,3) & = & \{U \in G \mid \dim U \cap A_1 \geq 1\} & \\ \Omega(3,4) & = & G_2(k^4)\,. & \end{array}$$

In the projective interpretation these are the varieties described in 6.6. In the following we determine their intersections and the Chow ring of $G_2(k^4)$.

16.6. Chow groups of Schubert varieties.

With the same notation as in 16.5 Giambelli's formulas read

$$\omega_\lambda = \omega(a) = [\Omega(a)] = \Delta_\lambda \cap [G_d(E)],$$

where now $\Delta_\lambda = \Delta_\lambda\big(c(\mathcal{Q})\big)$ and where as before the partition λ and the dimension tuple a are related by $\lambda_i = n - d + i - a_i$. Note that $\Delta_\lambda = c_q(\mathcal{Q})$ for $\lambda = (q, 0, \ldots, 0)$ and

$\Delta_\lambda = (-1)^q c_q(\mathcal{S})$ for $\lambda = (1,\dots,1,0,\dots,0)$ such that the Chern classes $c_q(\mathcal{Q})$ resp. $c_q(\mathcal{S})$ are represented by the Schubert varieties $\Omega(n-d+1-q,\ n-d+2,\dots,n)$, resp. $\Omega(n-d,\ n-d+1,\dots,n-d+q-1,\ n-d+q+1,\dots,n)$. For any index k we consider the set of partitions

$$\Lambda_k = \{\lambda \mid n-d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0,\ |\lambda| = d(n-d)-k\}$$

and the homomorphism

$$\mathbb{Z}^{\Lambda_k} \xrightarrow{\Theta} A_k(G_d(E))$$

defined by

$$(\alpha_\lambda) \mapsto \Sigma\alpha_\lambda\Delta_\lambda \cap [G_d(E)] = \Sigma\alpha_\lambda[\Omega(a)].$$

If $b_i \leq a_i$ for all i, we have $\Omega(b) \subset \Omega(a)$. Therefore, if

$$\Lambda_k(a) = \{\mu \in \Lambda_k \mid \mu_i \geq \lambda_i\},$$

we have the restriction

$$\mathbb{Z}^{\Lambda_k(a)} \xrightarrow{\Theta(a)} A_k(\Omega(a)) \to A_k(G_d(E))$$

of the homomorphism Θ, given by

$$(\alpha_\mu) \mapsto \Sigma\alpha_\mu[\Omega(b)],$$

where μ and b are related by the same formula $\mu_i = (n-d)+i-b_i$.

16.6.1. Proposition: *For any k and any $a = (a_1,\dots,a_d)$, $0 < a_1 < \cdots < a_d \leq n$, the homomorphism*

$$\mathbb{Z}^{\Lambda_k(a)} \xrightarrow{\Theta(a)} A_k(\Omega(a))$$

is an isomorphism. In particular

$$\mathbb{Z}^{\Lambda_k} \xrightarrow{\Theta} A_k(G_d(E))$$

is an isomorphism.

Proof. The injectivity of Θ will be shown in the proof of the more general relative version 16.7 of this proposition using 16.3. Then also the restrictions $\Theta(a)$ are injective. The surjectivity of each $\Theta(a)$ follows from the remarks in 16.5 or directly from the exact sequences

$$A_k(\Omega(b) \setminus \Omega^0(b)) \to A_k(\Omega(b)) \to A_k(\Omega^0(b)) \to 0$$

by induction on $|b|$ for $b \leq a$. $\square$

16.6.2. Remark: The isomorphisms Θ can, of course, also be derived from the isomorphism

$$A^*G_d(E) = \mathbb{Z}[s_1,\dots,s_d,\ q_1,\dots,q_{n-d}]/\mathfrak{a}$$

in 14.3 using

$$\omega_\lambda = c_i(\mathcal{Q}) \quad \text{for } \lambda = (i,0,\dots,0) \quad \text{and} \quad \omega_\lambda = (-1)^i c_i(\mathcal{S}) \quad \text{for } \lambda = (\underbrace{1,\dots,1}_{i},\ 0,\dots,0).$$

The ring structure is then described by the Littlewood–Richardson rule. Moreover, the q_i or the s_i may be replaced by each other and eliminated from the formula by the relations defined by the graded formula

$$(1 + s_1 + \cdots + s_d)\,(1 + q_1 + \cdots + q_{n-d}) = 1\,.$$

Example: $\dim E = 4$ and $d = 2$. Then

$$\omega_{10} = \omega(2,4) \quad \text{and} \quad \omega_{20} = \omega(1,4)$$

generate $A^*G_2(E)$, and we have the graded identity

$$(1 - \omega_{10} + \omega_{11})\,(1 + \omega_{10} + \omega_{20}) = 1 \quad \text{and} \quad \omega_{21} = \det\left(\begin{smallmatrix} \omega_{20} & 0 \\ 1 & \omega_{10} \end{smallmatrix}\right) = \omega_{10}\omega_{20}.$$

Explicitly:

$$\begin{array}{lcl} \omega_{10}^2 & = & \omega_{20} + \omega_{11} \quad \text{with} \quad \omega_{20}\omega_{11} = 0 \\ \omega_{10}\omega_{20} & = & \omega_{21} = \omega_{10}\omega_{11} \\ \omega_{10}^3 & = & \omega_{21} + \omega_{10}\omega_{11} = 2\omega_{21} \\ \omega_{10}\omega_{21} & = & \omega_{20}^2 = \omega_{22} = \omega_{11}^2 \\ \omega_{10}^4 & = & 2\omega_{10}\omega_{21} = 2\omega_{22}\,. \end{array}$$

An analogous result for a Grassmann bundle $G_d(\mathcal{E})$ over an arbitrary scheme X can be obtained using the determinants Δ_λ, whereas the relative Schubert classes $\omega(\underline{\mathcal{A}})$ may be more complicated, see 16.4.

16.7. Theorem: *Let $\mathcal{E}$ be a locally free sheaf of rank n on an algebraic scheme X. Then for each $k \geq 0$ there is an isomorphism*

$$\bigoplus_\lambda A_{k+|\lambda|-d(n-d)}(X) \xrightarrow[\approx]{\Theta} A_k(G_d(\mathcal{E}))$$

where the sum is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $n - d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0$ and

$$\Theta(\alpha_\lambda) = \sum_\lambda \Delta_\lambda \cap p^*\alpha_\lambda\,,$$

whereas in 16.2 $\Delta_\lambda = \Delta_{\lambda_1\ldots\lambda_d}\big(c(\mathcal{Q} - p^*\mathcal{E})\big)$.

Proof. The injectivity of Θ follows from the formula in 16.3. Let $\Theta(\alpha_\lambda) = 0$ and assume that $(\alpha_\lambda) \neq 0$. Choose any $\bar\lambda$ with $\alpha_{\bar\lambda} \neq 0$ and $|\bar\lambda|$ maximal and let μ be complementary, $\mu_i + \bar\lambda_{d-i+1} = n - d$. Then

$$p_*(\Delta_\mu \Delta_\lambda \cap p^*\alpha_\lambda) = \begin{cases} \alpha_{\bar\lambda} & \lambda = \bar\lambda \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum\limits_\lambda \Delta_\mu\Delta_\lambda \cap p^*\alpha_\lambda = 0$, we get $\alpha_{\bar\lambda} = 0$, a contradiction.

In order to show that Θ is also surjective, we may assume that X is irreducible and affine, and $\mathcal{E}$ is trivial on X as in the case $d = 1$, see 11.1. Now we can reduce the proof to the absolute case because $\mathcal{E} \cong \mathcal{O}_X^n$ and therefore

$$G_d(\mathcal{E}) \cong X \times G_d(k^n).$$

For any sequence $0 < a_1 < \cdots < a_d \leq n$ we let $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_d \subset \mathcal{E}$ be the flag of trivial subbundles defined by the first a_i basis sections of $\mathcal{O}_X^n$ and we have

$$\Omega(\underline{\mathcal{A}}), \sigma) \cong X \times \Omega(a)$$

with the Giambelli's formulas

$$\Delta_\lambda \cap [G_d(\mathcal{E})] = [X \times \Omega(a)].$$

For any fixed a we have the restrictions

$$\bigoplus_{b \leq a} A_{k-|b|}(X) \xrightarrow{\Theta(a)} A_k(X \times \Omega(a)) \to A_k(G_d(\mathcal{E})),$$

where here $|b|$ denotes $\sum(b_i - i)$.

Using the decompositions of $\Omega(a) \smallsetminus \Omega^0(a)$ described in 16.5 and $A_{l-\dim \Omega^0(b)}(X) \cong A_l(X \times \Omega^0(b))$, the surjectivity of $\Theta(a)$ follows by induction as in 16.6.1. $\square$

16.8. Exercise

Find the Schubert varieties Ω in a flag variety $F = F(d_1, \ldots, d_m, E)$ in analogy to a Grassmannian and prove that the corresponding classes form bases of the groups $A_k(F)$ and $A_k(\Omega)$ as in 16.6.

Try to find a Giambelli formula relating the Schubert classes of F to polynomials in the Chern classes defined by the quotients of the tautological flag.

Script to be continued

References

[1] A. Grothendieck - J.A. Dieudonné, *Eléments de G 'eometrie Algébrique I*, Springer Verlag

[2] A. Grothendieck - J.A. Dieudonné, *Eléments de G 'eometrie Algébrique III*, Publ.Math. 11

[3] A. Grothendieck - J.A. Dieudonné, *Eléments de G 'eometrie Algébrique IV*, Publ.Math. 32

[4] A. Grothendieck, *Sur quelques propriétés fondamentales en théorie des intersections*, Séminaire C. Chevalley E.N.S., 1958

[5] J. Dieudonné, *Cours de géométrie algébrique 1 & 2* , Presses Universitaires de France 1974

[6] D. Eisenbud, *Commutative Algebra*, GTM 150, Springer 1995

[7] W. Fulton, *Intersection Theory*, Springer 1984

[8] W. Fulton, *Introduction to Intersection Theory in Algebraic Geometry*, AMS regional conference series in mathematics, no 54, 1984

[9] R. Hartshorne, *Algebraic Geometry*, Springer 1977

[10] Y. Manin, *Gauge field theory and complex geomtry*, Springer 1988

[11] J.S. Milne, *Algebraic Geometry*, Lecture notes, www.math.lsa.imich.edu/jmilne, 1998

[12] G. Trautmann, *Moduli spaces in Algebraic Geometry, an introduction*, www.mathematik.uni-kl.de/ wwwagag/de/scripts.html

[13] H. Weigand, *Faserbündel-Techniken in der Schnitttheorie geometrischer Quotienten*, Shaker Verlag 2000

[14] E. Arbarello - M. Cornalba - P.A. Griffiths - J. Harris, *Geometry of Algebraic Curves I*, Springer 1985.