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The Baum–Connes Conjecture for Groupoids

J. L. Tu

Abstract

This survey paper is a self-contained overview on the Baum–Connes conjecture for locally compact groupoids.

Introduction

Let G be a locally compact, σ-compact, Hausdorff groupoid with Haar system, and $C_r^*(G)$ its reduced C^*-algebra. The Baum–Connes conjecture states that a certain map

$$\mu_r\colon K_j^{\mathrm{top}}(G) \to K_j(C_r^*(G)) \qquad (j = 0, 1)$$

is an isomorphism. The conjecture thus proposes a means of calculating K-theory groups of algebras as diverse as C^*-algebras of groups, of a group action on a locally compact space or of a foliation. Some of its important applications are described in [2]. In this survey paper, we shall focus more on aspects of the conjecture which are specific to groupoids. The interested reader may wish to consult [2, 4] for an introduction to the Baum–Connes conjecture for groups.

Assuming the reader is not familiar with groupoids, we start by introducing basic definitions about groupoids and equivariant KK-theory in Sections 1 and 2, and give the definition of the assembly map in Section 3. Sections 4, 5 and 6 review important examples for which the conjecture is known to be true. The end of the paper explores a few links with the coarse analogue of the conjecture.

1 Groupoids

1.1 General definitions

We introduce here a few basic definitions about groupoids. For more details, see [20].

A groupoid is a small category in which all morphisms are invertible. In practice, a groupoid is given by the following data:

- the set of objects $G^{(0)}$, also called the unit space;
- the set of morphisms G;

- an inclusion $i: G^{(0)} \hookrightarrow G$ (in the sequel, $G^{(0)}$ is considered as a subset of G);
- "range" and "source" maps $r, s: G \to G^{(0)}$ such that $r \circ i = s \circ i = \mathrm{Id}$;
- an involution $G \to G$, denoted by $g \mapsto g^{-1}$ such that $r(g) = s(g^{-1})$ for every $g \in G$;
- a partially defined product $G^{(2)} \to G$, denoted by $(g, h) \mapsto gh$, where $G^{(2)} := \{(g, h) \in G \times G |\ s(g) = r(h)\}$ is the set of composable pairs.

It is assumed that

- the product is associative, i.e. if $(g, h) \in G^{(2)}$ and $(h, k) \in G^{(2)}$ then the products $(gh)k$ and $g(hk)$ are defined and equal;
- for all $g \in G$, $i(r(g))g = gi(s(g)) = g$;
- for all $g \in G$, $gg^{-1} = i(r(g))$.

A groupoid is *principal* if $(r, s): G \to G^{(0)} \times G^{(0)}$ is injective.

A topological groupoid is a groupoid such that G and $G^{(0)}$ are topological spaces and all maps appearing in the definition are continuous. The unit space $G^{(0)}$ is then identified with a topological subspace of G by the inclusion i. A locally compact, Hausdorff groupoid is said to be *proper* if $(r, s): G \to G^{(0)} \times G^{(0)}$ is proper. G is called *étale*, or *r-discrete*, if the range map $r: G \to G^{(0)}$ is a local homeomorphism, *i.e.* if every $x \in G$ admits an open neighborhood such that $r(U)$ is an open subset of $G^{(0)}$ and $r: U \to r(U)$ is a homeomorphism. In this case, s is also a local homeomorphism, as well as the composition map $G^{(2)} \to G$ and $G^{(0)}$ is an open subset of G.

Some notations will be used in the sequel: for all x, $y \in G^{(0)}$, let $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$, $G_x^y = G_x \cap G^y$. If A, $B \subset G^{(0)}$, one has similar notations G_A, G^A, G_A^B. Note that G_A^A is a groupoid with space of units A, and G_x^x is a group.

A list of examples follows:

Groups. A group G is a groupoid, with $G^{(0)} = \{1\}$ (the unit element).

Spaces. A space X is a groupoid, letting $G = G^{(0)} = X$, $r = s = \mathrm{Id}_X$.

Equivalence relations. Let $R \subset X \times X$ be an equivalence relation on a set X. Then R is endowed with the structure of a groupoid with unit space X, range and source maps $r(x, y) = x$, $s(x, y) = y$, composition $(x, y)(z, t) = (x, t)$ if $y = z$, and inverse $(x, y)^{-1} = (y, x)$. In particular, $X \times X$ is a groupoid.

Transformation groups. More generally if a group Γ acts on the right on a space X, i.e. there is an anti-homomorphism α from Γ to the group of permutations of X, denoted by $\alpha_\gamma(x) = x\gamma$, then one obtains a groupoid G, denoted by $X \rtimes \Gamma$, as follows: as a set, $G = X \times \Gamma$, $G^{(0)} = X \times \{1\} \simeq X$, $r(x,\gamma) = x$, $s(x,\gamma) = x\gamma$, $(x,\gamma)^{-1} = (x\gamma, \gamma^{-1})$, $(x,\gamma)(x\gamma,\gamma') = (x,\gamma\gamma')$. If X is a topological space, G a topological group and the action is continuous, then $X \rtimes \Gamma$ is a topological groupoid, which is Hausdorff if X and G are. In that case, it is étale if Γ is discrete, principal if the action is free.

Fundamental groupoid. Let X be a topological space, and G be the set of equivalence classes of paths $\varphi\colon [0,1] \to X$ where φ and ψ are identified if and only if they are homotopic with fixed endpoints. $G^{(0)} \simeq X$ is the set of equivalence classes of constant paths on X. If φ is a path on X and $g = [\varphi]$ denotes its class in G, then $r(g) = \varphi(1)$, $s(g) = \varphi(0)$, $g^{-1} = [\varphi^{-1}]$, where $\varphi^{-1}(t) = \varphi(1-t)$, and $[\varphi][\psi] = [\varphi * \psi]$, where $\varphi * \psi(t) = \varphi(2t)$ for $t \in [0,1/2]$ and $\varphi * \psi(t) = \psi(2t-1)$ for $t \in [1/2,1]$. G is called the fundamental groupoid of X.

Foliations. Let (V,F) be a foliation. The holonomy groupoid G is the set of equivalence classes of paths whose support is contained in one leaf, where two paths are identified if (they have the same endpoints and) they define the same holonomy element. Composition and inverse are defined in the same way as for the fundamental groupoid. The space of units of G is V; if V is of dimension n and the foliation of codimension q, then G is a differentiable groupoid of dimension $2n-q$. It is not Hausdorff in general. If T is a transversal that meets all leaves of the foliation, then the restriction of the holonomy groupoid to T is an étale groupoid equivalent to G.

1.2 Actions of a groupoid on a space

Let G be a groupoid, and Z a set. A right *action* of G on Z is given by

(i) a map $p\colon Z \to G^{(0)}$, called the source map;

(ii) a map $Z \times_{G^{(0)}} G = \{(z,g) \in Z \times G \mid p(z) = r(g)\} \to Z$, denoted by $(z,g) \mapsto zg$, such that $p(zg) = s(g)$, $zp(z) = z$ and $(zg)h = z(gh)$ whenever the products are defined.

A space endowed with an action of G is called a G-space. One obtains a groupoid denoted by $Z \rtimes G$, with underlying set $Z \times G$, unit space $Z \simeq \{(z,p(z)) \mid z \in Z\}$, source and range maps $s(z,g) = zg$, $r(z,g) = z$, inverse $(z,g)^{-1} = (zg, g^{-1})$ and product $(z,g)(zg,h) = (z,gh)$. Note that $Z \rtimes G$ is étale if G is.

The action is said to be free if $Z \rtimes G$ is principal. In the case of a continuous action, if Z and G are locally compact Hausdorff, the action is said to be proper if $Z \rtimes G$ is proper. Z is then said to be a proper G-space.

1.3 Actions of a groupoid on a C^*-algebra

Let X be a locally compact, Hausdorff space. A $C(X)$-algebra is a C^*-algebra endowed with a $*$-homomorphism $\theta: C_0(X) \to Z(M(A))$ (center of the multiplier algebra of A) such that $\theta(C_0(X))A = A$. A continuous field of C^*-algebras over X is a $C(X)$-algebra, but the converse does not hold in general.

If $p: Y \to X$ is a map between two locally compact, Hausdorff spaces and A is a $C(X)$-algebra, then $p^*A := A \otimes_{C_0(X)} C_0(Y)$ is a $C(Y)$-algebra. If $x \in X$, the fiber A_x of A over x is defined by i_x^*A where $i_x: \{x\} \to X$ is the canonical inclusion.

Let G be a locally compact Hausdorff groupoid, and A a $C(G^{(0)})$-algebra. An *action* of G on A [17, 18] is given by an isomorphism of $C(G)$-algebras $\alpha: s^*A \to r^*A$ such that the morphisms $\alpha_g: A_{s(g)} \to A_{r(g)}$ satisfy the relation $\alpha_g \circ \alpha_h = \alpha_{gh}$. A C^*-algebra endowed with an action of G is called a G-algebra. Note that if Z is a G-space, then a $Z \rtimes G$-algebra is at the same time a G-algebra and a $C(Z)$-algebra, such that the two structures are compatible. A G-algebra A is called *proper* if there exists a proper G-space Z such that A is a $Z \rtimes G$-algebra. Of course, a G-space Z is proper if and only if $C_0(Z)$ is a proper G-algebra.

1.4 Actions of a groupoid on a Hilbert module

Let Y be a locally compact Hausdorff space, D a $C(Y)$-algebra and $\mathcal{E}$ a D-Hilbert module. For all $y \in Y$, let $\mathcal{E}_y = \mathcal{E} \otimes_D D_y$ be the fiber of $\mathcal{E}$ over y. If $\mathcal{E}'$ is another D-module, $V_y \in \mathcal{L}(\mathcal{E}_y, \mathcal{E}'_y)$ denotes $V \otimes_D 1_{D_y}$. If $p: Y' \to Y$ is a continuous map and Y' is locally compact Hausdorff, then $p^*\mathcal{E}$ denotes the p^*D-module $\mathcal{E} \otimes_D p^*D$.

Let B be a G-algebra and $\mathcal{E}$ be a B-module. Using the action of G on B, $r^*\mathcal{E}$ and $s^*\mathcal{E}$ are endowed with structures of s^*B-module. An *action* [17, 18] of G on $\mathcal{E}$ is given by a unitary $V \in \mathcal{L}(s^*\mathcal{E}, r^*\mathcal{E})$ such that the unitaries $V_g \in \mathcal{L}(\mathcal{E}_{s(g)}, \mathcal{E}_{r(g)})$ $(g \in G)$ satisfy the relation $V_g V_h = V_{gh}$ whenever $(g, h) \in G^{(2)}$.

1.5 Haar systems

Definition 1.1 *[20] Let G be a locally compact, σ-compact, Hausdorff groupoid. A Haar system on G is a family of positive measures $\lambda = \{\lambda^x \mid x \in G^{(0)}\}$ such that*

(i) $\forall x \in G^{(0)}$, $\mathrm{supp}\,(\lambda^x) = G^x$;

(ii) $\forall x \in G^{(0)}$, $\forall \varphi \in C_c(G)$,

$$\lambda(\varphi): x \mapsto \int_{g \in G^x} \varphi(g)\, \mathrm{d}\lambda^x(g) \in C_c(G^{(0)});$$

(iii) $\forall x, y \in G^{(0)}$, $\forall g \in G_x^y$, $\forall \varphi \in C_c(G)$,

$$\int_{h \in G^x} \varphi(gh)\, \mathrm{d}\lambda^x(h) = \int_{h \in G^y} \varphi(h)\, \mathrm{d}\lambda^y(h).$$

It is clear that étale groupoids are endowed with a Haar system, λ^x being the counting measure on G^x ($x \in G^{(0)}$). If a locally compact group Γ acts on a locally compact Hausdorff space X then, using a Haar measure on $\Gamma \simeq G^x$, one obtains a Haar system on the groupoid $X \rtimes \Gamma$. More generally, if a groupoid G satisfying the properties of Definition 1.1 acts on a space Z, then the crossed-product $Z \rtimes G$ has a Haar system.

1.6 Groupoid C^*-algebras

For a finite groupoid, the set of formal sums $\sum_{g\in G} a_g u_g$ ($a_g \in \mathbb{C}$) is endowed with a product and an adjoint

$$\begin{aligned}(\sum_{g\in G} a_g u_g)(\sum_{h\in G} b_h u_h) &= \sum_{(g,h)\in G^{(2)}} a_g b_h u_{gh}\\ (\sum_{g\in G} a_g u_g)^* &= \sum_{g\in G} \bar{a}_g u_{g^{-1}}.\end{aligned}$$

More generally, if G is a locally compact, σ-compact Hausdorff groupoid with Haar system, $C_c(G)$ is a $*$-algebra with

$$\begin{aligned}\varphi * \psi(g) &= \int_{h\in G^{r(g)}} \varphi(h)\psi(h^{-1}g)\, \mathrm{d}\lambda^{r(g)}(h)\\ \varphi^*(g) &= \overline{\varphi(g^{-1})}\end{aligned}$$

It is endowed with the norm $||\varphi||_1 = \sup(||\varphi||_{L^1}, ||\varphi^*||_{L^1})$ where

$$||\varphi||_{L^1} = \sup_{x\in G^{(0)}} \int_{g\in G^x} |\varphi(g)|\, \mathrm{d}\lambda^x(g).$$

Let $C^*(G)$ be the enveloping C^*-algebra [20]. It is represented (using left convolution) in the $C_0(G^{(0)})$-Hilbert module $L^2(G) = (L^2(G_x))_{x\in G^{(0)}}$; the closure of its image in $\mathcal{L}(L^2(G))$ is called the reduced C^*-algebra of G and is denoted by $C_r^*(G)$.

Similarly, if G acts on a C^*-algebra B, one constructs crossed-products $B \rtimes G$ and $B \rtimes_r G$.

1.7 Cutoff functions

Let G be a locally compact, σ-compact Hausdorff groupoid with Haar system. Then G is proper if and only if there exists a cutoff function for G [23, Propositions 6.10, 6.11], i.e. a continuous function $c\colon G^{(0)} \to \mathbb{R}_+$ such that

(i) For every $K \subset G^{(0)}$ compact, $\mathrm{supp}(c) \cap s(G^K)$ is compact.

(ii) For all $x \in G^{(0)}$, $\int_{g\in G^x} c(s(g))\, \mathrm{d}\lambda^x(g) = 1$.

Suppose in addition that $G^{(0)}/G$ is compact (note that properness implies that $G^{(0)}/G$ is locally compact, σ-compact Hausdorff). Then the first condition just means that $\mathrm{supp}(c)$ is compact; the function

$$g \mapsto \sqrt{c(r(g))c(s(g))}$$

in continuous with compact support, hence defines an element in $C^*(G) = C^*_r(G)$. One checks that it is a projection, which is unique up to homotopy. Hence, it defines a canonical element $\lambda_G \in K_0(C^*(G))$.

2 KK_G-theory

In this section, G is a locally compact, σ-compact, Hausdorff groupoid with Haar system (cf. Definition 1.1). Let A and B be two G-algebras. Le Gall [17, 18] constructs a group $KK_G(A, B)$ in the following way: An A, B-Kasparov G-equivariant bimodule consists of a triple $(\mathcal{E}, \varphi, F)$, where $\mathcal{E}$ is a G-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded B-module, $\varphi: A \to \mathcal{L}(\mathcal{E})$ is a G-equivariant $*$-homomorphism, and $F \in \mathcal{L}(\mathcal{E})$ is of degree 1 and satisfies, for all $a \in A$ and $a' \in r^*A$,

(i) $a(F - F^*) \in \mathcal{K}(\mathcal{E})$;

(ii) $a(F^2 - 1) \in \mathcal{K}(\mathcal{E})$;

(iii) $[a, F] \in \mathcal{K}(\mathcal{E})$;

(iv) $a'(V(s^*F)V^* - (r^*F)) \in r^*\mathcal{K}(\mathcal{E})$.

(V denotes the unitary that defines the action of G on $\mathcal{E}$.)

The group $KK_G(A, B)$, as in [13], is defined as the set of homotopy classes of A, B-Kasparov G-equivariant bimodules. If G is a group, Le Gall's KK_G-theory is the same as Kasparov's, and if $G = X \rtimes \Gamma$ is the crossed-product of a space with a locally compact group, then $KK_G(A, B) = \mathcal{R}KK_\Gamma(X; A, B)$. KK_G-theory has essentially the same features as Kasparov's functor, namely:

(i) $KK_G(A, B)$ is covariant (resp. contravariant) with respect to B (resp. A).

(ii) Let $KK^n_G(A, B) = KK_G(A, B \otimes C_0(\mathbb{R}^n))$. One has Bott periodicity $KK^n_G(A, B) = KK^{n+2}_G(A, B)$.

(iii) For every G-algebra D, there is a natural transformation

$$\sigma_{G^{(0)},D}: KK_G(A, B) \to KK_G(A \hat{\otimes}_{C_0(G^{(0)})} D, B \hat{\otimes}_{C_0(G^{(0)})} D).$$

(iv) There is a natural, associative product

$$KK_G(A, D) \times KK_G(D, B) \to KK_G(A, B),$$

compatible with $\sigma_{G^{(0)},\cdot}$ (in the obvious sense). The product of two elements $\alpha \in KK_G(A, D)$ and $\beta \in KK_G(D, B)$ is denoted by $\alpha \otimes_D \beta$.

(v) There are descent morphisms

$$\begin{aligned} j_G\colon KK_G(A,B) &\to KK(A \rtimes G, B \rtimes G) \\ j_{G,\mathrm{red}}\colon KK_G(A,B) &\to KK(A \rtimes_r G, B \rtimes_r G), \end{aligned}$$

compatible with the product.

The existence of a Haar system is needed only for the construction of j_G and $j_{G,\mathrm{red}}$.

It would be desirable, but probably not exceedingly difficult, to write a construction of KK_G for locally compact groupoids which are not necessarily Hausdorff, since holonomy groupoids of foliations are not Hausdorff in general.

3 The Baum–Connes Conjecture

In this section, G is a locally compact, σ-compact, Hausdorff groupoid with Haar system.

3.1 The classifying space for proper actions

A space $\mathbf{E}G$ is called the universal space for proper actions [2] if

(i) $\mathbf{E}G$ is a proper (locally compact σ-compact Hausdorff) G-space.

(ii) For every proper (locally compact σ-compact Hausdorff) G-space Z, there exists a G-map $Z \to \mathbf{E}G$, and such a map is unique up to G-homotopy.

Such a space always exists, and is unique up to G-equivariant homotopy. For instance [23, Proposition 6.15], one can take $\mathbf{E}G$ to be the set of positive measures μ on G, such that $s_*\mu$ is a Dirac measure (on $G^{(0)}$) and $|\mu| \in (1/2, 1]$. Note that this space is second countable if G is.

If G is a group which does not contain any non trivial compact subgroup (e.g. a torsion-free discrete group), then $\mathbf{E}G$ is G-homotopically equivalent to the ordinary classifying space EG.

3.2 Topological K-theory

Let B be a G-algebra. One defines the topological K-theory of G with coefficients in B as the inductive limit [2, Definition 9.1]

$$K_*^{\mathrm{top}}(G;B) = \varinjlim_{\substack{Y \subset \mathbf{E}G \\ Y\ G\text{-compact}}} KK_G^*(C_0(Y), B).$$

When $B = C_0(G^{(0)})$, the topological K-theory group of G is

$$K_*^{\mathrm{top}}(G) = \varinjlim_{\substack{Y \subset \mathbf{E}G \\ Y\ G\text{-compact}}} KK_G^*(C_0(Y), C_0(G^{(0)})).$$

If G is a discrete group without torsion such that BG is compact and $B=\mathbb{C}$, then using $\mathbf{E}G = EG$, one has $K_*^{\mathrm{top}}(G) = KK_G^*(C_0(EG),\mathbb{C}) = KK^*(C_0(EG)\rtimes G,\mathbb{C}) = KK^*(C(BG),\mathbb{C})$, i.e.

$$K_*^{\mathrm{top}}(G) = K_*(BG)$$

is the K-homology of the topological space BG (hence the name "topological K-theory"). It is in principle computable by means of exact sequences, provided a triangulation of BG is known. In contrast, $K_*(C^*(G))$ is a highly non-commutative object.

3.3 The assembly map

If Y is a G-compact proper G-space, there are morphisms [2, Definition 3.8], [23, Definition 5.1]

$$KK_G^*(C_0(Y),B) \xrightarrow{j_G} KK_*(C^*(Y\rtimes G), B\rtimes G) \overset{\lambda_{Y\rtimes G}\otimes\cdot}{\longrightarrow} K_*(B\rtimes G)$$

$(*=0,1)$. Passing to the inductive limit yields the assembly map (with coefficient B)

$$\mu\colon K_*^{\mathrm{top}}(G;B)\to K_*(B\rtimes G).$$

Similarly, there is a map $\mu_r\colon K_*^{\mathrm{top}}(G;B)\to K_*(B\rtimes_r G)$. The *Baum–Connes conjecture* with coefficients states that μ_r is an isomorphism. For $B=C_0(G^{(0)})$, one has assembly maps without coefficients $\mu\colon K_*^{\mathrm{top}}(G)\to K_*(C^*(G))$ and $\mu_r\colon K_*^{\mathrm{top}}(G)\to K_*(C_r^*(G))$. As seen above, the topological K-theory group is in general easier to compute than the K-theory group of $C_r^*(G)$, hence knowing that the assembly map is an isomorphism for a given groupoid G provides a means of computing $K_*(C_r^*(G))$.

4 The dual Dirac method

In this section, G is a locally compact, σ-compact Hausdorff groupoid with Haar system.

Proposition 4.1 *Suppose there exist a proper G-algebra A and elements*

$$\eta\in KK_G(C_0(G^{(0)}),A),\qquad D\in KK_G(A,C_0(G^{(0)})),$$

$$\gamma\in KK_G(C_0(G^{(0)}),C_0(G^{(0)}))$$

such that $\eta\otimes_A D=\gamma$ and $p^*\gamma = 1\in KK_{\mathbf{E}G\rtimes G}(C_0(\mathbf{E}G),C_0(\mathbf{E}G))$, where $p\colon \mathbf{E}G\to G^{(0)}$ is the source map for the action of G on $\mathbf{E}G$. Then the element γ is unique, in the sense that if γ', η', D' and A' satisfy the same hypotheses, then $\gamma=\gamma'$.

Proof. Since $p^*\gamma' = 1$ and since, by the universal property of $\mathbf{E}G$, A is a $\mathbf{E}G \rtimes G$-algebra, one has $\gamma' \otimes_{C_0(G^{(0)})} \eta = \eta$. Multiplying on the right by D, we obtain $\gamma' \otimes_{C_0(G^{(0)})} \gamma = \gamma$. By symmetry, we also have $\gamma \otimes_{C_0(G^{(0)})} \gamma' = \gamma'$. Since the Kasparov product over $C_0(G^{(0)})$ is commutative, it follows that $\gamma = \gamma'$. □

If for a given groupoid G the condition of Proposition 4.1 is satisfied, we shall say that G has a γ element. This is the same element as the one Kasparov constructed for every connected locally compact group [13, Theorem 5.7].

Theorem 4.2 *[23, 25] G has a γ element, then the Baum–Connes maps with coefficients μ and μ_r are split injective.*

Proof. (Sketch.) Suppose for simplicity that $\mathbf{E}G$ is G-compact. One constructs a map $\kappa\colon K_*(B \rtimes G) \to K_*^{\mathrm{top}}(G; B)$ as follows:

$$K_*(B\rtimes G) \overset{\cdot\otimes j_G(\sigma_B(\eta))}{\longrightarrow} K_*((A\otimes_{C_0(G^{(0)})}B)\rtimes G) \simeq KK_{\mathbf{E}G\rtimes G}(C_0(\mathbf{E}G), A\otimes_{C_0(G^{(0)})}B)$$

$$\overset{\cdot\otimes_A D}{\longrightarrow} KK_G(C_0(\mathbf{E}G), B) = K_*^{\mathrm{top}}(G; B).$$

The isomorphism above is the generalized Green–Julg isomorphism, i.e. the Baum–Connes isomorphism for the proper groupoid $\mathbf{E}G \rtimes G$. One then proves that κ is a left inverse for μ. Similarly, μ_r admits a left inverse. The proof for general $\mathbf{E}G$ is similar; see [23, Proposition 5.23] or [25, Theorem 2.2] for details. □

In fact, a weaker assumption than the existence of γ suffices to ensure injectivity of μ_r:

Theorem 4.3 *[23, Théorème 5.24] Let G be a locally compact, σ-compact groupoid with Haar system. Suppose that for every G-compact subset Y of $\mathbf{E}G$ there exist a G-compact subset Y' of $\mathbf{E}G$, a $Y' \rtimes G$-algebra A_Y and elements*

$$\eta_Y \in KK_G(C_0(G^{(0)}), A_Y), \qquad D_Y \in KK_G(A_Y, C_0(G^{(0)}))$$

such that the element $\gamma_Y = \eta_Y \otimes_{A_Y} D_Y$ of $KK_G(C_0(G^{(0)}), C_0(G^{(0)}))$ satisfies $p_Y^*(\gamma_Y) = 1$ in $KK_{Y\rtimes G}(C_0(Y), C_0(Y))$ where $p_Y\colon Y \to G^{(0)}$ is the source map of the action of G on Y.

Then μ and μ_r are injective.

It is known that for a discrete group G, injectivity of μ implies the Novikov conjecture on higher signatures. Proofs of the injectivity of μ based on Theorem 4.2 are constructive, in the sense that they require explicit constructions of a C^*-algebra A and KK_G-elements as in Proposition 4.1. In general, one uses the existence of an action of the groupoid on some space with particular geometric properties, like negative curvature. Examples will be shown in the next sections.

Theorem 4.4 *[23, Proposition 5.23][25, Theorem 2.2] If G has a γ element, and if $\gamma = 1 \in KK_G(C_0(G^{(0)}), C_0(G^{(0)}))$, then the Baum–Connes maps with coefficients μ and μ_r are isomorphisms, and G is K-amenable.*

(For a definition and properties of K-amenability for groups, see [3, 11, 19]; for groupoids, see [24].) In particular, if $\gamma = 1$ then $C^*(G)$ and $C^*_r(G)$ have the same K-theory. Consequently, for a group having property (T), if a γ element exists then it cannot be equal to 1, and thus the Dual Dirac method as formulated above won't be helpful. Read [4, 9] for a discussion along these lines. V. Lafforgue [16] circumvents that difficulty by constructing a bivariant K-theory for Banach algebras. He proves in particular that discrete cocompact subgroups of $\mathrm{Sp}(n,1)$ satisfies the Baum–Connes conjecture (without coefficients).

5 Non-positive curvature

It is known that a (locally compact σ-compact) group G has an element γ in each of the following cases:

(i) G acts properly by isometries on a simply connected complete Riemannian manifold with non-positive curvature [13, Theorem 5.3];

(ii) G is connected [13, Theorem 5.7];

(iii) $G = GL_n(K)$ where K is a local field [14].

Kasparov and Skandalis [15] introduced the concept of "bolic" space, which generalizes non-positively curved simply connected complete Riemannian manifolds, Euclidean buildings and hyperbolic spaces in the sense of Gromov. The precise definition is

Definition 5.1 *[15] A metric space (X, d) is said to be δ-bolic if*
(B1) $\forall r > 0$, $\exists R > 0$ such that for every quadruple x, y, z, t of X satisfying $d(x,y) + d(z,t) \le r$ and $d(x,z) + d(y,t) \ge R$ we have $d(x,t) + d(y,z) \le d(x,z) + d(y,t) + 2\delta$;
(B2) There exists a map $m\colon X \times X \to X$ such that for all x, y, $z \in X$ we have $2d(m(x,y), z) \le (2d(x,z)^2 + 2d(y,z)^2 - d(x,y)^2)^{1/2} + 4\delta$.

They showed:

Theorem 5.2 *[15] If G is a locally compact group acting properly by isometries on a discrete, bolic metric space with bounded geometry, then G satisfies the conditions of Theorem 4.3. In particular, the Baum–Connes assembly map for G is injective.*

In the theorem above, a metric space Z is said to be of bounded geometry if for every $r > 0$, there exists $N(r) > 0$ such that every ball in Z of radius r has at most $N(r)$ elements. In particular, for a finitely generated, discrete hyperbolic group in the sense of Gromov, μ is injective and the Novikov conjecture holds.

In [23] is introduced the concept of bolic foliation [23, Définition 1.15]. It is required that holonomy coverings of leaves constitute a family of δ-bolic spaces, with δ independent of the leaf, and that in condition (B1), for each $r > 0$ a common $R > 0$ can be chosen. Using the fact that the holonomy groupoid of a foliation is equivalent to an étale one, it is proven [23, Théorème 5.25] that a bolic foliation (V, F) with V compact and whose holonomy groupoid is Hausdorff satisfies the conditions of Theorem 4.3, hence its assembly map μ_r is injective.

6 Amenable groupoids

A locally compact, σ-compact groupoid with Haar system G is said to be (topologically) amenable [1] if for every $\varepsilon > 0$ and every compact sets $K \subset G$, and $C \subset G^{(0)}$ there exists $\xi \in C_c(G)_+$ such that

(i) for all $x \in C$, $\int_{G^x} \xi(\gamma)\, \mathrm{d}\lambda^x(\gamma) = 1$;

(ii) for all $g \in K$, $\int_{\gamma \in G^{s(g)}} |\xi(\gamma) - \xi(g\gamma)|\, \mathrm{d}\lambda^x(\gamma) < \varepsilon$.

Let G be a groupoid. Recall [8] that a negative type function on G is a function $f\colon G \to \mathbb{R}$ such that

(i) $f_{|G^{(0)}} = 0$.

(ii) $\forall g \in G$, $f(g^{-1}) = f(g)$.

(iii) Given $g_1, g_2, \ldots, g_n \in G$ all having the same range and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum \lambda_k = 0$, we have $\displaystyle\sum_{j,k} f(g_j^{-1} g_k)\lambda_j \lambda_k \leq 0$.

Moreover, if G is a locally compact σ-compact groupoid, the following are equivalent (*cf.* [24]):

(i) There exists continuous negative type function $f\colon G \to \mathbb{R}$ such that $(f, r, s)\colon G \to \mathbb{R} \times G^{(0)} \times G^{(0)}$ is proper.

(ii) There is a continuous field of real Hilbert spaces $(H_x)_{x \in G^{(0)}}$ over $G^{(0)}$ with an affine action of G by isometries, such that $(f, r, s)\colon G \to \mathbb{R} \times G^{(0)} \times G^{(0)}$ is proper, where the function f is defined by $f(g) = \|0_{r(g)} - g 0_{s(g)}\|$. Here, 0_x denotes the zero vector of the Hilbert space H_x.

If these properties are satisfied, then G is said to have Haagerup property. Topologically amenable locally compact σ-compact groupoids satisfy Haagerup property [24]. Using the C^*-algebra constructed by Higson and Kasparov [6], it is proven in [24] that:

Theorem 6.1 *If G has Haagerup property, then it has a γ element and $\gamma = 1$. Therefore, G satisfies the Baum–Connes conjecture with coefficients and is K-amenable.*

This theorem is a strengthening of Higson and Kasparov's: in [6] it is proven that second countable, locally compact groups having Haagerup property satisfy the Baum–Connes conjecture. Theorem 6.1 applies in particular to

- topologically amenable groupoids, and in particular amenable groups;
- Coxeter groups;
- $SO(n, 1)$, $SU(n, 1)$. The result for these groups was already known [12, 10].

7 Property A and uniform embedding

Recall that a discrete metric space X is said to be of bounded geometry if for every $r > 0$ there exists $N(r) > 0$ such that every ball in X of radius r has at most $N(r)$ elements. Property A was introduced by Yu [27]. Let X be a metric space with bounded geometry. It is said to have property A if for every $\varepsilon > 0$ and $R > 0$ there exists a family $(\xi_x)_{x \in X}$ of elements of $C_c(X)$ and a real number $S > 0$ such that

(i) for every x, ξ_x is supported in $B(x, S)$;

(ii) $||\xi_x - \xi_y||_{l^1(X)} < \varepsilon$ whenever $d(x, y) \leq R$.

Yu [27] observed that if X has property A, then it is uniformly embedded into Hilbert space in the following sense:

Definition 7.1 *Let X and Y be two metric spaces. A (not necessarily continuous) function $f : X \to Y$ is said to be a uniform embedding if there exist two non-decreasing function ρ_1, $\rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

(i) $\lim_{r \to \infty} \rho_i(r) = +\infty$ *($i = 1, 2$);*

(ii) *for every x, $x' \in X$,*

$$\rho_1(d(f(x), f(x'))) \leq d(x, x') \leq \rho_2(d(f(x), f(x'))).$$

A space X is said to be uniformly embedded into Hilbert space if there exists a uniform embedding $f : X \to l^2(\mathbb{N})$.

Higson and Roe observed that in the case where X is the geometric realization of a finitely generated discrete group Γ,

- Γ has property A if and only if $\beta\Gamma \rtimes \Gamma$ is amenable, where $\beta\Gamma$ is the Stone–Čech compactification of Γ; this is also equivalent to the existence of a compact space X on which Γ acts amenably [7].

- If Γ has property A, then the Baum–Connes map with coefficients μ_r for Γ is injective and $C_r^*(\Gamma)$ is an exact C^*-algebra [5].

It is conjectured that every discrete, countable metric space with bounded geometry has property A. If the conjecture is true, then every discrete group satisfies the Novikov conjecture and its reduced C^*-algebra is exact.

The idea of Higson's proof of the second point is first to take a compact separable space Y on which Γ acts amenably, prove that Γ acts amenably on $\bar{Y}$, the space of probability measures on Y and consider the commutative diagram

$$\begin{tikzcd}
K_*^{\mathrm{top}}(\Gamma; B) \arrow[r, "\mu_r"] \arrow[d, "\simeq"] & K_*(B \rtimes_r \Gamma) \arrow[d] \\
K_*^{\mathrm{top}}(\bar{Y} \rtimes \Gamma; B \otimes C(\bar{Y})) \arrow[r, "\mu_r"] & K_*((B \otimes C(\bar{Y})) \rtimes_r \Gamma)
\end{tikzcd}$$

That the left vertical map is an isomorphism results from the fact that $\bar{Y}$ is F-contractible for every finite subgroup F of Γ. The bottom horizontal map is an isomorphism, thanks to Theorem 6.1. It follows that the top horizontal map is injective.

That $C^*(\Gamma)$ is exact follows from the fact that it is a subalgebra of $C(\beta\Gamma) \rtimes_r \Gamma$, which is nuclear if Γ acts amenably on $\beta\Gamma$.

A proof along the same lines [22] shows that Γ is uniformly embedded into Hilbert space if and only if $\beta\Gamma \rtimes \Gamma$ has Haagerup property, and that if it is the case, then the Baum–Connes map with coefficients for Γ is injective. We thus have:

$$\begin{tikzcd}
\Gamma \text{ has property A} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \beta\Gamma \rtimes \Gamma \text{ amenable} \arrow[d, Rightarrow] \\
\begin{array}{c}\Gamma \text{ is uniformly embedded} \\ \text{into Hilbert space}\end{array} \arrow[r, Leftrightarrow] & \begin{array}{c}\beta\Gamma \rtimes \Gamma \text{ has} \\ \text{Haagerup property}\end{array} \arrow[d, Rightarrow] \\
 & \mu \text{ injective}
\end{tikzcd}$$

8 Relation to the coarse Baum–Connes conjecture

Let X be a countable metric space with bounded geometry. A subset E of $X \times X$ is called an *entourage* if d is bounded on E, i.e. if there exists $R > 0$ such that

$$\forall (x, y) \in E,\ d(x, y) \leq R.$$

It can be shown that the spectrum of the algebra of bounded functions on $X \times X$ whose support is an entourage can be endowed with a structure of groupoid, extending the one on $X \times X$:

$$G(X) = \bigcup_{E \text{ entourage}} \bar{E} \subset \beta(X \times X),$$

where $\beta(X \times X)$ is the Stone–Čech compactification of $X \times X$ and $\bar{E}$ is the closure of E in $\beta(X \times X)$. The groupoid $G(X)$ is étale, Hausdorff, locally compact, σ-compact, principal. Its unit space is βX, hence it is not second countable.

In the case $X = \Gamma$ (finitely generated discrete group with any word metric), the groupoid $G(X)$ is $\beta\Gamma \rtimes \Gamma$. Higson and Roe's theorem [7] can be generalized as follows:

Theorem 8.1 *[22] Let X be a discrete metric space with bounded geometry.*

- *X has property A if and only if $G(X)$ is amenable;*
- *X is uniformly embedded into Hilbert space if and only if $G(X)$ has Haagerup property.*

A C^*-algebra is associated to (X, d) as follows [21]: $C^*(X)$ is the closure of the set of operators $T = (T_{xy})_{(x,y)\in X^2}$ acting on $l^2(X) \otimes l^2(\mathbb{N})$ such that $\mathrm{supp}(T)$ is an entourage and $T_{xy} \in \mathcal{K}$ for every $(x, y) \in X^2$. In other words, $C^*(X)$ is generated by locally compact operators with bounded propagation.

The coarse homology group of X is defined as

$$KX_*(X) = \lim_{d\to\infty} K_*(P_d(X))$$

where $P_d(X)$ is the Rips' complex of X, i.e. a subset $F \subset X$ spans a simplex in $P_d(X)$ if and only if its diameter is less or equal to d. An assembly map

$$KX_*(X) \xrightarrow{A} K_*(C^*(X))$$

is defined, and conjectured to be an isomorphism for every discrete space with bounded geometry [26].

It turns out [22] that $C^*(X)$ is isomorphic to the reduced crossed-product of $l^\infty(X, \mathcal{K})$ with $G(X)$, and that the coarse assembly map identifies with the Baum–Connes assembly map for the groupoid $G(X)$ with coefficient $l^\infty(X, \mathcal{K})$:

$$\begin{array}{ccc}
KX_*(X) & \xrightarrow{\quad A \quad} & K_*(C^*(X)) \\
\big\downarrow \simeq & & \big\downarrow \simeq \\
K_*^{\mathrm{top}}(G(X); l^\infty(X, \mathcal{K})) & \xrightarrow{\mu_r} & K_*(l^\infty(X, \mathcal{K}) \rtimes_r G(X)),
\end{array}$$

where the right vertical map is an isomorphism at the C^*-algebra level. The coarse Baum–Connes conjecture is thus put inside the framework of the conjecture for groupoids.

Using Theorem 6.1, one gets a new proof of an earlier theorem by Yu:

Theorem 8.2 *[27] If X is uniformly embedded into Hilbert space, then the coarse Baum–Connes map for X is an isomorphism.*

In summary,

$$\begin{tikzcd}
X \text{ has property A} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & G(X) \text{ is amenable} \arrow[d, Rightarrow] \\
\begin{array}{c} X \text{ is uniformly embedded} \\ \text{into Hilbert space} \end{array} \arrow[r, Leftrightarrow] & G(X) \text{ has Haagerup property} \arrow[d, Rightarrow] \\
 & \begin{array}{c} X \text{ satisfies the coarse} \\ \text{Baum–Connes conjecture.} \end{array}
\end{tikzcd}$$

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