FOUNDATION OF RELATIVE NON-ABELIAN HOMOLOGICAL ALGEBRA

Tamar Janelidze

Thesis Presented for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mathematics and Applied Mathematics

UNIVERSITY OF CAPE TOWN

August 2009

Supervisors: Prof. H.-P. Künzi, Prof. W. P. Tholen, Prof. G. Janelidze

Abstract

We consider pairs $(\mathbf{C}, \mathbf{E})$, where $\mathbf{C}$ is a pointed category and $\mathbf{E}$ a class of regular/normal epimorphisms in $\mathbf{C}$, satisfying various exactness properties. The purpose of this thesis is:

1. To introduce and study suitable notions of a relative homological and a relative semi-abelian category. In the "absolute case", where $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$, the pair $(\mathbf{C}, \mathbf{E})$ is relative homological/semi-abelian if and only if $\mathbf{C}$ is homological/semi-abelian; that is, we obtain known concepts. Accordingly we extend known analysis of the axiom systems, and in particular show that suitable lists of "old style" and "new style" axioms are equivalent; this requires developing a relative version of what is usually called the calculus of relations. We then present various non-absolute examples, where these results can be applied.

2. To formulate and prove relative versions of classical homological lemmas; this includes Five Lemma, Nine Lemma, and Snake Lemma.

Contents

Introduction

The title of the thesis ("Foundation of relative non-abelian homological algebra") is suggested by classical work of S. Eilenberg and J. C. Moore [14]. Relative homological algebra in abelian categories also appears in the first two books in homological algebra, namely in [12] and [30], and in a number of papers of many students and followers of Samuel Eilenberg and Saunders Mac Lane. The term non-abelian has several meanings; here it means "suitable for non-abelian groups, or rings, or algebras". Accordingly, the non-abelian categories of our interest include semi-abelian categories in the sense of G. Janelidze, L. Márki, and W. Tholen [23], and, more generally, homological categories in the sense of F. Borceux and D. Bourn [3] and protomodular categories in the sense of D. Bourn [6]. On the other hand, the term relative refers, just as in the abelian case, to a distinguished class $\mathbf{E}$ of regular/normal epimorphisms in the ground category $\mathbf{C}$ - in contrast to the absolute case, where the role of $\mathbf{E}$ is played by the class of all regular epimorphisms in $\mathbf{C}$. And in fact various axioms we impose on $(\mathbf{C}, \mathbf{E})$ make the ground category $\mathbf{C}$ semi-abelian or homological only in the absolute case. In particular, we do not exclude the trivial case of $\mathbf{C}$ being an arbitrary pointed category and $\mathbf{E}$ the class of isomorphisms in $\mathbf{C}$. In the abelian case this approach goes back to N. Yoneda [34], whose quasi-abelian categories can in fact be defined as pairs $(\mathbf{C}, \mathbf{E})$ where $\mathbf{C}$ is an additive category in which the short exact sequences $K \to A \to B$ with $A \to B$ in $\mathbf{E}$ have the same properties as all short exact sequences in an abelian category.

The purpose of the thesis is two-fold:

1. Detailed study of axiom systems for relative semi-abelian and relative homological categories; the aim was to obtain the relative versions of the results of [23] for semi-abelian categories and of homological categories. For, we develop the calculus of $\mathbf{E}$-relations which

easily follows its well-known "absolute version", in which **E** is the class of all regular epimorphisms in **C** (see e.g. [10]). Those results of [23], symbolically expressed as OLD=NEW, actually have a long history behind them, which begins with Mac Lane's famous "Duality for Groups" [29]: After observing that several basic concepts in group theory can be described in abstract categorical terms, making some of them dual to each other, Mac Lane proposes a number of fundamental categorical constructions to be used in developing categorical group theory. He then says: "A further development can be made by introducing additional carefully chosen dual axioms. This will be done below only in the more symmetrical abelian case" The "careful choice" took more than fifteen years of many researchers, who arrived to a "very non-dual" list of non-abelian axioms, which produced categorical versions of many known results, especially in homological algebra and Kurosh-Amitsur radical theory; at the same time it seemed to be too technical, and eventually was nearly ignored and/or forgotten. The development of topos theory in the sixties/seventies strongly supports a new approach in categorical algebra that arrives to Barr exact categories [1]. A Barr exact category is abelian whenever it is additive; yet, every variety of universal algebras is Barr exact, making the old and new approaches seemingly incomparable. The new concept of a protomodular category, due to Bourn (see [6]), which turned out to be the "missing link", is introduced only in 1990, i.e. after twenty years. And the main conclusion of [23] is that a pointed category satisfies the old forgotten axioms if and only if is Barr exact and Bourn protomodular, and has finite coproducts. Such categories were called semi-abelian for two reasons: (a) a category **C** is abelian if and only if both **C** and its dual category $\mathbf{C}^{\mathrm{op}}$ are semi-abelian; (b) a Barr exact category is semi-abelian if and only if it admits semidirect products in the sense of D. Bourn and G. Janelidze [8], and then it is abelian if and only if its semidirect products are (direct) products. Accordingly, the present work in fact deals with a number of categorical axioms and exactness properties studied by various authors for many years, and it involves relativisation of both old and new axioms systems.

2. The study of relative versions of the so-called classical homological lemmas, especially the Five Lemma, Nine Lemma, and Snake Lemma. The absolute versions are well-known in the abelian case, and in the non-abelian case given in [3]. The proof of the Snake Lemma that we give involves partial composition of internal relations in **C** and goes back at least to S. Mac Lane [28] (see also e.g. [10] for the so-called calculus of relations in regular categories).

The thesis consists of the following chapters:

Chapter 1: We begin with recalling the relevant properties of regular and normal epimorphisms, and then give a brief overview of regular, Barr exact [1], Bourn protomodular [6], homological [3], and semi-abelian [23] categories (see also [2]).

Chapter 2: We develop what we call a relative calculus of relations. That is: for a pair $(\mathbf{C}, \mathbf{E})$, in which $\mathbf{C}$ is a pointed category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ satisfying certain conditions, we study the relations $(R, r_1, r_2) : A \to B$ in $\mathbf{C}$ having the morphisms r_1 and r_2 in $\mathbf{E}$. Our calculus of $\mathbf{E}$-relations extends its well-known "absolute version", in which $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$ (see e.g. [10]).

Chapter 3: In the first section we introduce a notion of *incomplete relative homological category* in such a way that we have:

(a) "Trivial Case": ($\mathbf{C}$, Isomorphisms in $\mathbf{C}$) is an incomplete relative homological category for every pointed category $\mathbf{C}$;

(b) "Absolute Case": ($\mathbf{C}$, Regular epimorphisms in $\mathbf{C}$) is an incomplete relative homological category if and only if $\mathbf{C}$ is a homological category in the sense of [3].

In the second section we consider the special case of $\mathbf{C}$ being finitely complete and cocomplete, and define a *relative homologcial category* accordingly. In the third section we consider various examples.

Chapter 4: We extend Five Lemma, Nine Lemma, and Snake Lemma to the context of incomplete relative homological categories. The proofs follow the proofs given in [3], although the proof of Snake Lemma (as already mentioned) substantially uses the calculus of $\mathbf{E}$-relations described in Chapter 2.

Chapter 5: In the first section we introduce a notion of *incomplete relative semi-abelian category* in such a way that we have:

(a) "Trivial Case": ($\mathbf{C}$, Isomorphisms in $\mathbf{C}$) is an incomplete relative semi-abelian category for every pointed category $\mathbf{C}$;

(b) "Absolute Case": (**C**, Regular epimorphisms in **C**) is an incomplete relative semi-abelian category if and only if **C** is a semi-abelian category in the sense of [23];

(c) A relative semi-abelian category $(\mathbf{C}, \mathbf{E})$ is an incomplete relative homological category in which: (i) every equivalence **E**-relation is **E**-effective (i.e. every equivalence **E**-relation is the kernel pair of some morphism in **E**); (ii) if a morphism $f : A \to B$ is in **E** then the coproduct $\mathrm{Ker}(f) + B$ exists in **C**.

And, we prove the (incomplete) relative version of the main result of [23], which asserts that the "old-style" axioms and the "new-style" axioms for the semi-abelian categories are equivalent. For, we again use the calculus of **E**-relations described in Chapter 2. In the second section we consider the special case of **C** being finitely complete and cocomplete, and define a *relative semi-abelian category* accordingly. In the third section we consider various examples.

The results obtained appear as the three papers [24], [25], [26], and the fourth paper [27] is submitted for the publication.

Chapter 1

Preliminaries

1.1 Regular and normal epimorphisms

Definition 1.1.1. *A morphism $f : A \to B$ in a category $\mathbf{C}$ is said to be a regular epimorphism, if it is the coequalizer of some pair of parallel morphisms in $\mathbf{C}$.*

Proposition 1.1.2. *In a category $\mathbf{C}$ with kernel pairs, every regular epimorphism is the coequalizer of its kernel pair.* $\square$

Proposition 1.1.3. *Let $\mathbf{C}$ be a category with pullbacks. The composite gf of regular epimorphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$ is a regular epimorphism whenever f is a pullback stable epimorphism. In particular, the class of pullback stable regular epimorphisms in $\mathbf{C}$ is closed under composition.*

Proof. Let $f : A \to B$ and $g : B \to C$ be regular epimorphisms in $\mathbf{C}$, and let (f_1, f_2), (g_1, g_2), and (h_1, h_2) be the kernel pairs of f, g, and gf respectively. To prove that gf is a regular epimorphism it suffices to prove that gf is the coequalizer of h_1 and h_2. For, consider the commutative diagram

$$\begin{tikzcd}
 & A\times_C A \arrow[r, dotted, "\bar{f}"] \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] & B\times_C B \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] & \\
A\times_B A \arrow[ur, dotted, "\bar{h}"] \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A \arrow[r, "f"] \arrow[d, "h'"'] & B \arrow[r, "g"] \arrow[dl, dotted, "b"'] & C \arrow[dll, dotted, "c"] \\
 & C' & &
\end{tikzcd}$$

in which:

- $\bar{f} = \langle fh_1, fh_2\rangle$ is the canonical morphism, i.e. since (g_1, g_2) is the kernel pair of g and $gfh_1 = gfh_2$, there exists a unique morphism $\bar{f} : A\times_C A \to B\times_C B$ with $g_1\bar{f} = fh_1$ and $g_2\bar{f} = fh_2$;

- $h' : A \to C'$ is any morphism with $h'h_1 = h'h_2$;

- $\bar{h} = \langle f_1, f_2\rangle$ is the canonical morphism, i.e. since (h_1, h_2) is the kernel pair of gf and $gff_1 = gff_2$, there exists a unique morphism $\bar{h} : A \times_B A \to A \times_C A$ with $h_1\bar{h} = f_1$ and $h_2\bar{h} = f_2$, yielding $h'f_1 = h'h_1\bar{h} = h'h_2\bar{h} = h'f_2$;

- Since f is the coequalizer of f_1 and f_2, and $h'f_1 = h'f_2$, there exists a unique morphism $b : B \to C'$ with $bf = h'$.

We have to show that there exists a unique morphism $c : C \to C'$ with $cgf = h'$ (since gf is an epimorphism we do not need to prove the uniqueness of c), but since g is the coequalizer of g_1 and g_2, it is sufficient to show that $bg_1 = bg_2$.

It is easy to see that the morphism $\bar{f} : A\times_C A \to B\times_C B$ is the composite of the canonical morphisms $\langle fh_1, h_2\rangle : A\times_C A \to B\times_C A$ and $\langle \pi_1, f\pi_2\rangle : B\times_C A \to B\times_C B$, where $(B\times_C A, \pi_1, \pi_2)$ is the pullback of g and gf. Then, since f is a pullback stable epimorphism, we obtain that the morphisms $\langle fh_1, h_2\rangle$ and $\langle \pi_1, f\pi_2\rangle$ are epimorphisms, and therefore their composite $\bar{f}$ also is an epimorphism. We have:

$$bg_1\bar{f} = bfh_1 = h'h_1 = h'h_2 = bfh_2 = bg_2\bar{f},$$

and since $\bar{f}$ is an epimorphism we conclude that $bg_1 = bg_2$, as desired. $\square$

Proposition 1.1.4. *Let* $\mathbf{C}$ *be a category with pullbacks. If the composite* gf *of* $f : A \to B$ *and* $g : B \to C$ *in* $\mathbf{C}$ *is a regular epimorphism, then so is the morphism* g *if any one of the following conditions hold:*

(i) f *is an epimorphism;*

(ii) gf *is a pullback stable epimorphism.*

Proof. Under the condition (i), consider the following diagram

$$\begin{tikzcd}
A\times_C A \arrow[r, dotted, "\bar{f}"] \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] & B\times_C B \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] & \\
A \arrow[r, "f"'] & B \arrow[r, "g"] \arrow[dr, "g'"'] & C \arrow[d, dotted, "c"] \\
& & C'
\end{tikzcd}$$

where (g_1, g_2) is the kernel pair of g, (h_1, h_2) is the kernel pair of gf, $\bar{f}$ is the canonical morphism, and $g' : B \to C'$ is any morphism with $g'g_1 = g'g$. To prove that g is a regular epimorphism it suffices to prove that g is the coequalizer of g_1 and g_2; hence, we need prove the existance of a unique morphism $c : C \to C'$ such that $cg = g'$. Since gf is the coequalizer of h_1 and h_2, the equalities

$$(g'f)h_1 = g'g_1\bar{f} = g'g_2\bar{f} = (g'f)h_2$$

imply the existance of a unique morphism $c : C \to C'$ with $c(gf) = g'f$. Since f is an epimorphism, the last equality implies $cg = g'$; and, since gf is an epimorphism, such c is unique.

Next, suppose condition (ii) holds instead of condition (i). Consider the pullback diagram

$$\begin{tikzcd}
B\times_C A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "gf"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

we have:

- Since π_2 can be obtained as a pullback of the pullback of g along g, it is a split epimorphism, and hence a pullback stable regular epimorphism;

- Since gf is a regular epimorphism and π_2 is a pullback stable regular epimorphism, by Proposition 1.1.3, $g\pi_1 = gf\pi_2$ is a regular epimorphism;

- Since gf is a pullback stable epimorphism, π_1 also is an epimorphism.

Since π_1 is an epimorphism and $g\pi_1$ is a regular epimorphism, the first part of the proof implies that g is an epimorphism, as desired. $\square$

Definition 1.1.5. *A morphism $f : A \to B$ in a category $\mathbf{C}$ is said to be a strong epimorphism, if for every commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "m"'] & D \end{tikzcd} \tag{1.1}$$

where m is a monomorphism, there exists a unique morphism $\beta : B \to C$ with $\beta f = g$ and $m\beta = h$.

As easily follows from Definition 1.1.5, if a category $\mathbf{C}$ has equalizers then every regular epimorphism is strong. Indeed, if $f : A \to B$ is a regular epimorphism in $\mathbf{C}$, then it is the coequalizer of some pair of parallel morphisms (f_1, f_2). For any commutative diagram (1.1) with a monomorphism m, we have $gf_1 = gf_2$, therefore there exists a unique morphism $\beta : B \to C$ with $\beta f = g$; and $m\beta = h$ since $m\beta f = hf$ and f is an epimorphism.

Definition 1.1.6. *A morphism $f : A \to B$ in a category $\mathbf{C}$ is said to be a normal epimorphism, if it the cokernel of some morphism in $\mathbf{C}$.*

Proposition 1.1.7. *In a category $\mathbf{C}$ with kernels, every normal epimorphism is the cokernel of its kernel.* $\square$

Using the same arguments as in the proofs of Proposition 1.1.3 and Proposition 1.1.4 we can prove the following:

Proposition 1.1.8. *Let $\mathbf{C}$ be a category with pullbacks. The composite gf of normal epimorphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$ is a normal epimorphism whenever f is a pullback stable epimorphism. In particular, the class of pullback stable normal epimorphisms in $\mathbf{C}$ is closed under composition.* $\square$

Proposition 1.1.9. *Let $\mathbf{C}$ be a category with pullbacks. If the composite gf of $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$ is a normal epimorphism, then so is the morphism g if any one of the following conditions hold:*

(i) f is an epimorphism;

(ii) gf is a pullback stable epimorphism. $\square$

1.2 Regular and Barr exact categories

Definition 1.2.1. *A category* $\mathbf{C}$ *is said to be regular (see e.g. [2]), if:*

(a) $\mathbf{C}$ *has finite limits;*

(b) $\mathbf{C}$ *has a pullback stable (regular epi, mono)-factorization system.*

If a morphism f in any category $\mathbf{C}$ factors as $f = me$ in which e is a regular epimorphism and m is a monomorphism, then e is the coequalizer of the kernelpair of f, provided that the latter exists. And, conversely, if e is the coequalizer of the kernelpair of f and m is the morphism with $f = me$, then m is a monomorphism if the regular epimorphisms in $\mathbf{C}$ are pullback stable. Therefore, we have:

Proposition 1.2.2. *Let* $\mathbf{C}$ *be a category with finite limits. The following conditions are equivalent:*

(i) $\mathbf{C}$ *has a pullback stable (regular epi, mono)-factorization system.*

(ii) *(a)* $\mathbf{C}$ *has coequlizers of kernel pairs;*

(b) Regular epimorphisms in $\mathbf{C}$ *are pullback stable.* □

Regular categories admit a good calculus of relations (see e.g. [10]): Recall, that a relation R from an object A to an object B in $\mathbf{C}$, written as $R : A \to B$, is a subobject $\langle r_1, r_2 \rangle : R \to A \times B$, where $\langle r_1, r_2 \rangle$ is the canonical morphism from R to the product $A \times B$ induced by $r_1 : R \to A$ and $r_2 : R \to B$; as subobjects, the relations from A to B form an ordered set with finite meets. Equivalently, we can define a relation $R : A \to B$ as a triple (R, r_1, r_2) in which R is an object in $\mathbf{C}$ and r_1 and r_2 are jointly monic morphisms. For the given relations $(R, r_1, r_2) : A \to B$ and $(S, s_1, s_2) : B \to C$ the composite SR is the relation from A to C defined as the mono part of the (regular epi, mono)-factorization of the morphism $\langle r_1 p_1, s_2 p_2 \rangle : R \times_B S \to A \times C$, where $(R \times_B S, p_1, p_2)$ is the pullback of r_2

and s_1:

$$
\begin{tikzcd}
 & & R \times_B S \arrow[dl, "p_1"'] \arrow[d, dotted, "e"] \arrow[dr, "p_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & SR \arrow[dll, dotted, "t_1"] \arrow[drr, dotted, "t_2"'] \arrow[dd, bend right, "m"'] & S \arrow[dl, "s_1"] \arrow[dr, "s_2"] & \\
A & & B & & C \\
 & & A \times C \arrow[ull, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}
$$

That is, the composite of the relations $R : A \to B$ and $S : B \to C$ is the subobject $SR \to A \times C$, i.e. it is the triple (SR, t_1, t_2) where $t_1 = \pi_1 m$ and $t_2 = \pi_2 m$ and π_1 and π_2 are the first and the second product projections of $A \times C$ respectively; regularity of $\mathbf{C}$ implies that such a composition is associative.

Recall, that a relation $R : A \to A$ in a regular category $\mathbf{C}$ is said to be an equivalence relation if it is reflexive, symmetric, and transitive, i.e. $1_A \leq R$, $R^\circ \leq R$, and $RR \leq R$. It is easy to see that for a given morphism $f : A \to B$, the kernelpair of f is an equivalence relation from A to A (for more details about the relations in a regular category see also [11]).

Theorem 1.2.3. *For a regular category the following conditions are equivalent, and define a Mal'tsev category:*

(a) For equivalence relations R and S on an object A, the relation SR is an equivalence relation.

(b) For such equivalence relations we have $SR = RS$.

(c) Every relation $R : A \to B$ is difunctional; that is, $RR^\circ R = R$.

(d) Every reflexive relation is an equivalence relation.

(e) Every reflexive relation is transitive.

(e) Every reflexive relation is symmetric. □

For the proof, see Theorem 3.6 of [10] (see also Theorem 1 of [15]).

Definition 1.2.4. *A category $\mathbf{C}$ is said to be Barr-exact [1], if:*

(a) **C** *is a regular category;*

(b) Every equivalence relation in **C** *is effective, i.e. it is the kernel pair of some morphism in* **C**.

Theorem 1.2.5 (Theorem 5.7 of [10]). *A regular category* **C** *is an exact Mal'tsev category if and only if, given regular epimorphisms* $r : A \to B$ *and* $s : A \to C$ *with a common domain, their pushout*

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "r"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] & D \end{tikzcd}$$

exists in **C**, *and moreover, the canonical morphism* $\langle r, s\rangle : A \to B \times_D C$ *is a regular epimorphism.* □

1.3 Protomodular categories

Let $\mathbf{C}$ be a category and B an object in $\mathbf{C}$. Recall that $\mathrm{Pt}_{\mathbf{C}}(B) = \mathrm{Pt}(\mathbf{C} \downarrow B)$ is a category, whose objects are triples (A, f, g), in which A is an object in $\mathbf{C}$, and $f : A \to B$ and $g : B \to A$ are morphisms in $\mathbf{C}$ satisfying $fg = 1_B$. A morphism $\alpha : (A, f, g) \to (A', f', g')$ in $\mathrm{Pt}_{\mathbf{C}}(B)$ is defined as a morphism $\alpha : A \to A'$ in $\mathbf{C}$ such that $f'\alpha = f$ and $\alpha g = g'$. Note that if $\mathbf{C}$ has pullbacks, then every morphism $v : B \to B'$ in $\mathbf{C}$ induces the pullback functor $v^* : \mathrm{Pt}_{\mathbf{C}}(B') \to \mathrm{Pt}_{\mathbf{C}}(B)$ which pulls back f' of (A', f', g') along v.

Definition 1.3.1. *A category* **C** *is said to be protomodular (in the sense of D. Bourn [6]), if the following conditions hold:*

(a) **C** *has pullbacks;*

(b) For every morphism $v : B \to B'$ *in* **C**, *the pullback functor* $v^* : \mathrm{Pt}_{\mathbf{C}}(B') \to \mathrm{Pt}_{\mathbf{C}}(B)$ *reflects isomorphisms.*

It is easy to see that if $\mathbf{C}$ has a zero object 0, then in Definition 1.3.1(b) it suffices to consider the morphism $0_{B'} : 0 \to B'$ instead of an arbitrary morphism $v : B \to B'$. Indeed: in the presence of a zero object $\mathbf{C}$, the category $\mathrm{Pt}_{\mathbf{C}}(0)$ is isomorphic to the category $\mathbf{C}$, and since $0_{B'} = v0_B$, the reflection of isomorphisms $0^*_{B'} = 0^*_B v^*$ implies the same for v^*.

Since pulling back $f' : A' \to B'$ along $0_{B'}$ is taking the kernel of f, we obtain the following

Corollary 1.3.2. *If* $\mathbf{C}$ *is a category with pullbacks and a zero object, then* $\mathbf{C}$ *is protomodular if and only if for every object* B *in* $\mathbf{C}$*, the kernel functor* $\ker_B : \mathrm{Pt}_{\mathbf{C}}(B) \to \mathbf{C}$ *reflects isomorphisms.*

This proves the following

Proposition 1.3.3. *Let* $\mathbf{C}$ *be a category with pullbacks and a zero object. The following conditions are equivalent:*

(i) $\mathbf{C}$ *is protomodular.*

(ii) The Split Short Five Lemma holds true in $\mathbf{C}$*, that is: for every commutative diagram*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, "v"] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B' \end{tikzcd} \tag{3.1}$$

with $k = \ker(f)$*,* $k' = \ker(f')$*, and* f *and* f' *split epimorphisms,* w *is an isomorphisms if* u *and* v *are isomorphisms.*

Remark 1.3.4. *If a protomodular category is also regular, then the Split Short Five Lemma is equivalent to the Regular Short Five Lemma, which states: given the commutative diagram (3.1) with* $k = \ker(f)$ *and* $k' = \ker(f')$*, if* f *and* f' *are regular epimorphisms and* u *and* v *are isomorphisms, then* w *is an isomorphism [6].*

Proposition 1.3.5. *If* $\mathbf{C}$ *is a pointed protomodular category with pullbacks, then:*

(i) Every regular epimorphism in $\mathbf{C}$ *is a normal epimorphism.*

(ii) Every split epimorphism in $\mathbf{C}$ *is a normal epimorphism.*

Proof.

(i): Let $f : A \to B$ be a regular epimorphism in $\mathbf{C}$, $q = \mathrm{coker}(\ker(f))$, and let (f_1, f_2) and (q_1, q_2) be the kernel pairs of f and q respectively. It is a well known fact, that in this situation the morphisms f_1 and q_1 (and also f_2 and q_2) are split epimorphisms, and $\mathrm{Ker}(q_1) \approx \mathrm{Ker}(f_1)$. This gives us a commutative diagram

$$\begin{tikzcd} \mathrm{Ker}(q_1) \arrow[r, "k'"] \arrow[d, dotted, "\approx"'] & A \times_Q A \arrow[r, "q_1"] \arrow[d, "h"] & A \arrow[d, equal] \\ \mathrm{Ker}(f_1) \arrow[r, "k"'] & A \times_B A \arrow[r, "f_1"'] & A \end{tikzcd}$$

in which: $Q = \mathrm{Coker}(\mathrm{Ker}(f))$, $k = \ker(f_1)$, $k' = \ker(q_1)$, h is the canonical morphism between the pullbacks, and f_1 and q_1 are split epimorphisms. Hence, by protomodularity we obtain that h is an isomorphism. Since f and q are regular epimorphisms and they have isomorphic kernel pairs, we conclude that f is a normal epimorphism, as desired.

Since every split epimorphism is a regular epimorphism, (ii) follows directly from (i). $\square$

Theorem 1.3.6 (Proposition 3.1.19 of [3]). *Any finitely complete protomodular category* **C** *is a Mal'tsev category.* $\square$

This statement was in fact first proved in [7].

1.4 Homological categories

Homological categories, according to [3], provide the most convenient setting for proving non-abelian versions of various *standard homological lemmas*, such as the Five Lemma, the 3×3-Lemma, and the Snake Lemma. We recall:

Definition 1.4.1 (Definition 4.1.1 of [3]). *A category* **C** *is homological when*

(a) **C** *is pointed;*

(b) **C** *is regular;*

(c) **C** *is protomodular.*

Or, equivalently, a category **C** is homological if and only if the following conditions hold:

(a) **C** has finite limits;

(b) **C** has a zero object;

(c) **C** has coequalizers of kernel pairs;

(d) Regular epimorphisms in **C** are pullback stable;

(e) The (Split) Short Five Lemma holds in **C**.

As usually, a sequence of morphisms

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots \tag{4.1}$$

in a homological category $\mathbf{C}$ is said to be exact at A_i, if the mono part of the (regular epi, mono)-factorization of f_{i-1} is the kernel of f_i. And, (4.1) is said to be an exact sequence, if it is exact at A_i for each i (unless the sequence either begins or ends with A_i).

Proposition 1.4.2 (Lemma 4.1.6 of [3])**.** *In a pointed protomodular category* $\mathbf{C}$*, in particular in a homological category, the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if $f = \ker(g)$ *and* g *is a regular epimorphism.*

Proposition 1.4.3 (Proposition 4.1.9 of [3])**.** *Let* $\mathbf{C}$ *be a homological category.*

(i) *The sequence*

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact if and only if f *is a monomorphism.*

(ii) *The sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if $f = \ker(g)$*.*

(iii) *The sequence*

$$A \xrightarrow{f} B \longrightarrow 0$$

is exact if and only if f *is a regular epimorphism.*

(iv) *The sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if $g = \operatorname{coker}(f)$*.*

We now recall the above mentioned homological lemmas involving exact sequences (see again [3]):

Theorem 1.4.4 (The Five Lemma)**.** *Let* $\mathbf{C}$ *be a homological category. If in a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"'] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "k'"'] & E'
\end{tikzcd}$$

the two rows are exact sequences, and the morphisms α, β, δ, and ϵ are isomorphisms, then γ is also an isomorphism.

Theorem 1.4.5 (The Nine Lemma). *Let $\mathbf{C}$ be a homological category. If in a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the three columns and the middle row are exact sequences, then the first row is an exact sequence if and only if the last row is an exact sequence.

Theorem 1.4.6 (The Snake Lemma). *Let $\mathbf{C}$ be a homological category. If in a commutative diagram*

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z &
\end{tikzcd}$$

the two rows are exact sequences, then there exists a morphism $d : \mathrm{Ker}(w) \to \mathrm{Coker}(u)$, such that the sequence

$$\mathrm{Ker}(u) \longrightarrow \mathrm{Ker}(v) \longrightarrow \mathrm{Ker}(w) \xrightarrow{d} \mathrm{Coker}(u) \longrightarrow \mathrm{Coker}(v) \longrightarrow \mathrm{Coker}(w)$$

where the unlabeled arrows are the canonical morphisms, is exact.

1.5 Semi-abelian categories

The notion of an abelian category was introduced by S. Mac Lane in 1950 in his paper "Duality for groups" [29]; it was however more restrictive than the one used today, which was given by D. A. Buchsbaum in “Exact Categories and Duality” [9] in 1955 (under the name “exact category”). Let us recall that a category **C** is said to be abelian ([9], and, see also [16], [18]) if the following conditions hold:

(a) **C** has a zero object;

(b) **C** has binary products and binary coproducts;

(c) Every morphism has a kernel and a cokernel;

(d) Every monomorphism is a kernel, every epimorphism is a cokernel.

Categories of abelian groups and of modules are abelian categories, which is certainly not the case for the categories of groups, rings or algebras over rings; the easiest way to see this is just to note that not all of their monomorphisms are normal. The semi-abelian categories, introduced by G. Janelidze, L. Marki, and W. Tholen in 1999 (published in 2002; see [23]), play, however, the same role for groups, rings, and algebras, as the abelian categories do for the abelian groups and modules. We recall:

Definition 1.5.1. *A category* **C** *is said to be semi-abelian, if:*

(a) **C** *has a zero object and coproducts;*

(b) **C** *is Barr-exact;*

(c) **C** *is protomodular.*

That is, a category **C** is semi-abelian, if it satisfies the following

Condition 1.5.2 (“New-style axioms”)**.**

(a) **C** *has a zero object, finite limits, and coproducts;*

(b) **C** *has coequalizers of kernel pairs;*

(c) The regular epimorphisms in **C** *are pullback stable;*

(d) The Split Short Five Lemma holds in **C***;*

(e) All equivalence relations $r_1, r_2 : R \to A$ *in* **C** *are effective equivalence relations.*

Conditions 1.5.2(a)-1.5.2(e) are regarded as the "new-style axioms" for a semi-abelian category. As proved in [23], these conditions are equivalent to the "old-style axioms" involving normal monomorphisms and normal epimorphisms:

Condition 1.5.3 ("Old-style axioms")**.**

(a) $\mathbf{C}$ *has a zero object, finite limits, and coproducts;*

(b) $\mathbf{C}$ *has cokernels of kernels, and every morphism with a zero kernel is a monomorphism;*

(c) The normal epimorphisms in $\mathbf{C}$ *are pullback stable;*

(d) (Hofmann's axiom) If in a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

f and f' are normal epimorphisms, w is a monomorphism, v is a normal monomorphism, and $\ker(f') \leq w$, then w is a normal monomorphism;

(e) For every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "m'"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

with f and f' normal epimorphisms and m and m' monomorphisms, if m is a normal monomorphism then m' also is a normal monomorphism.

Protomodularity in terms of the old-style axioms is the "Hofmann's axiom" [19], while the Barr's exactness condition is Condition 1.5.3(e).

Remark 1.5.4. *Using the notion of a homological category, a semi-abelian category can be defined as a homological category with coproducts in which every equivalence relation is effective (see Proposition 5.1.2 of [3]). Therefore, one can obtain the "old-style axioms" also for the homological categories, i.e. define the homological categories using normal epimorphisms and the Hofmann's axiom. This will be done in a more general setting in Chapter 4.*

Chapter 2

Calculus of E-relations

2.1 Category of E-relations

Throughout this chapter we assume that $(\mathbf{C}, \mathbf{E})$ is a pair in which $\mathbf{C}$ is a category and $\mathbf{E}$ is a class of regular epimorphisms in $\mathbf{C}$ containing all isomorphisms and satisfying the following

Condition 2.1.1.

(a) The class $\mathbf{E}$ *is closed under composition;*

(b) If $f \in \mathbf{E}$ *and* $gf \in \mathbf{E}$ *then* $g \in \mathbf{E}$*;*

(c) A diagram of the form

$$\begin{tikzcd}
& A' \arrow[d, "f'"] \arrow[ddr, bend left, "g'"] & \\
A \arrow[r, "f"'] \arrow[drr, bend right, "g"'] & B & \\
& & B
\end{tikzcd}$$

has a limit in $\mathbf{C}$ *provided* f, g, f', *and* g' *are in* $\mathbf{E}$, *and either (i)* $f = g$ *and* $f' = g'$, *or (ii)* (f, g) *and* (f', g') *are reflexive pairs (that is,* $fh = 1_B = gh$ *and* $f'h' = 1_B = g'h'$ *for some* h *and* h'*), and* f *and* g *are jointly monic.*

(d) If

$$\begin{tikzcd}
A \times_B A' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A' \arrow[d, "f'"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is a pullback and f and f' are in $\mathbf{E}$*, then π_1 and π_2 are also in* $\mathbf{E}$*;*

(e) If $h_1 : H \to A$ and $h_2 : H \to B$ are jointly monic morphisms in $\mathbf{C}$ *and if $\alpha : A \to C$ and $\beta : B \to D$ are morphisms in* $\mathbf{E}$*, then there exists a morphism $h : H \to X$ in* $\mathbf{E}$ *and jointly monic morphisms $x_1 : X \to C$ and $x_2 : X \to D$ in* $\mathbf{C}$ *making the diagram*

$$\begin{tikzcd}
& & H \arrow[dl, "h_1"'] \arrow[d, dotted, "h"] \arrow[dr, "h_2"] & & \\
& A \arrow[dl, "\alpha"'] & X \arrow[dll, dotted, "x_1"] \arrow[drr, dotted, "x_2"'] & B \arrow[dr, "\beta"] & \\
C & & & & D
\end{tikzcd}$$

commutative.

Remark 2.1.2. *If the morphisms $f : A \to B$ and $f' : A' \to B$ are in* $\mathbf{E}$*, then the pullback $(A \times_B A', \pi_1, \pi_2)$ of f and f' exists in* $\mathbf{C}$ *by Condition 2.1.1(c), and π_1 and π_2 are in* $\mathbf{E}$ *by Condition 2.1.1(d). Therefore, the kernel pair of f (and f') also exists in* $\mathbf{C}$*.*

The two basic examples of a pair $(\mathbf{C}, \mathbf{E})$ satisfying Condition 2.1.1 are:

1. "Trivial case": $\mathbf{C}$ is a category and $\mathbf{E}$ is the class of all isomorphisms in $\mathbf{C}$.

2. "Absolute case": $\mathbf{C}$ is a regular category and $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$.

Proposition 2.1.3. *The factorization in Condition 2.1.1(e) is functorial. That is, if*

$$\begin{tikzcd}
& & H \arrow[dl, "h_1"'] \arrow[d, dotted, "h"] \arrow[dr, "h_2"] \arrow[ddd, bend right, "q"'] & & \\
& A \arrow[dl, "\alpha"'] \arrow[ddd, "k"'] & X \arrow[dll, dotted, "x_1"] \arrow[drr, dotted, "x_2"'] \arrow[ddd, dashed, "x"] & B \arrow[dr, "\beta"] \arrow[ddd, "l"] & \\
C \arrow[ddd, "m"'] & & & & D \arrow[ddd, "n"] \\
& & H' \arrow[dl, "h'_1"'] \arrow[d, dotted, "h'"'] \arrow[dr, "h'_2"] & & \\
& A' \arrow[dl, "\alpha'"'] & X' \arrow[dll, dotted, "x'_1"] \arrow[drr, dotted, "x'_2"'] & B' \arrow[dr, "\beta'"] & \\
C' & & & & D'
\end{tikzcd}$$

(1.1)

is a commutative diagram in $\mathbf{C}$*, in which:*

- (h_1, h_2), (x_1, x_2), (h'_1, h'_2)*, and* (x'_1, x'_2) *are jointly monic pairs;*

- α, β, h, α', β'*, and* h' *are morphisms in* $\mathbf{E}$*;*

- m, k, q, l*, and* n *are any morphisms making the diagram (1.1) commutative;*

then there exists a morphism $x : X \to X'$ *for which the diagram (1.1) is still commutative.*

Proof. Since h is in $\mathbf{E}$, the kernel pair (u_1, u_2) of h exists by Remark 2.1.2. Since x'_1 and x'_2 are jointly monic and the equalities

$$x'_1 h' q u_1 = m\alpha h_1 u_1 = m x_1 h u_1 = m x_1 h u_2 = m\alpha h_1 u_2 = x'_1 h' q u_2,$$

$$x'_2 h' q u_1 = n\beta h_2 u_1 = n x_2 h u_1 = n x_2 h u_2 = n\beta h_2 u_2 = x'_2 h' q u_2$$

hold, we conclude that $h' q u_1 = h' q u_2$. Since h is the coequalizer of u_1 and u_2, the last equality implies the existence of a unique morphism $x : X \to X'$ with $h'q = xh$. It remains to prove that $x'_1 x = m x_1$ and $x'_2 x = n x_2$; however, since h is an epimorphism, the latter follows from following equalities:

$$x'_1 x h = x'_1 h' q = m x_1 h,$$

$$x'_2 x h = x'_2 h' q = n x_2 h.$$

□

Proposition 2.1.4. *Let*

$$\begin{tikzcd} & & A \arrow[dl, "f"'] \arrow[dr, "f'"] & & \\ & B \arrow[dl, "g"'] \arrow[drrr, "h"' near end] & & B' \arrow[dlll, "g'" near end] \arrow[dr, "h'"] & \\ C & & & & D \end{tikzcd} \tag{1.2}$$

be a commutative diagram in $\mathbf{C}$*. If* f *and* f' *are in* $\mathbf{E}$ *and* (g, h) *and* (g', h') *are jointly monic pairs, then there exists a unique isomorphism* $\beta : B \to B'$ *with* $g'\beta = g$, $\beta f = f'$*, and* $h'\beta = h$*.*

Proof. Since f and f' are in $\mathbf{E}$, the kernel pairs of f and f' exist by Remark 2.1.2; moreover, they coincide since (g, h) and (g', h') are jointly monic pairs and the diagram (1.2) is commutative. Since every regular epimorphism is the coequalizer of its kernel pair, we conclude that there exists a unique isomorphism $\beta : B \to B'$ with $\beta f = f'$, and since f and f' are epimorphisms we obtain $g'\beta = g$ and $h'\beta = h$. $\square$

Remark 2.1.5. *As follows from Proposition 2.1.4, the factorization in Condition 2.1.1(e) is unique up to an isomorphism.*

Proposition 2.1.6. *If a morphism f in $\mathbf{C}$ factors as $f = em$ in which e is in $\mathbf{E}$ and m is a monomorphism, then it also factors (essentially uniquely) as $f = m'e'$ in which m' is a monomorphism and e' is in $\mathbf{E}$.*

Proof. Under the assumptions of Condition 2.1.1(e), take $h_1 = h_2 = m$ and $\alpha = \beta = e$, then we obtain the desired factorization of f. $\square$

Definition 2.1.7. *An $\mathbf{E}$-relation R from an object A to an object B in $\mathbf{C}$, written as $R : A \to B$, is a triple $R = (R, r_1, r_2)$ in which $r_1 : R \to A$ and $r_2 : R \to B$ are jointly monic morphisms in $\mathbf{E}$.*

Let $(R, r_1, r_2) = R : A \to B$ and $(S, s_1, s_2) = S : B \to C$ be the $\mathbf{E}$-relations in $\mathbf{C}$ and let (P, p_1, p_2) be the pullback of s_1 and r_2; by Remark 2.1.2 this pullback does exist and p_1 and p_2 are in $\mathbf{E}$. Since p_1 and p_2 are jointly monic and r_1 and s_2 are in $\mathbf{E}$, using Condition 2.1.1(e) we obtain the commutative diagram

$$\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[d, dotted, "e"] \arrow[dr, "p_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"'] & T \arrow[dll, dotted, "t_1"] \arrow[drr, dotted, "t_2"'] & S \arrow[dl, "s_1"] \arrow[dr, "s_2"] & \\
A & & B & & C
\end{tikzcd} \tag{1.3}$$

in which e is in $\mathbf{E}$, t_1 and t_2 are jointly monic, and such factorization ($t_1 e = r_1 p_1$ and $t_2 e = s_2 p_2$) is unique up to an isomorphism by Remark 2.1.5. Moreover, since r_1, p_1, s_2, and p_2 are in $\mathbf{E}$, the morphisms t_1 and t_2 are also in $\mathbf{E}$ by Conditions 2.1.1(a) and 2.1.1(b). Accordingly, we introduce:

Definition 2.1.8. *If $R : A \to B$ and $S : B \to C$ are the* $\mathbf{E}$*-relations in* $\mathbf{C}$*, then their composite $SR : A \to C$ is the* $\mathbf{E}$*-relation (T, t_1, t_2) in which T, t_1, and t_2 are defined as in the diagram (1.3).*

Proposition 2.1.9. *The composition of* $\mathbf{E}$*-relations in* $\mathbf{C}$ *is associative (if we identify isomorphic relations).*

Proof. Let $R : A \to B$, $S : B \to C$, and $T : C \to D$ be the $\mathbf{E}$-relations in $\mathbf{C}$. Consider the commutative diagram

$$
\begin{tikzcd}
 & & & X \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dd, dotted, "e"'] \arrow[dddr, dotted, "x"] & & & \\
 & & P \arrow[dddl, "p_1"'] \arrow[dddd, "e_1"] \arrow[dddr, "p_2"] & & Q \arrow[dddl, "q_1"'] \arrow[dddr, "q_2"] & & \\
 & & & X' \arrow[ddddlll, dotted, bend right, "x'_1"'] \arrow[ddddrrr, dotted, bend left, "x'_2"] & & & \\
 & & & & Z \arrow[ddll, bend right, "z_1"'] \arrow[dr, "z_2"] \arrow[dd, "e_2"] & & \\
 & R \arrow[ddl, "r_1"'] \arrow[ddr, "r_2"] & & S \arrow[ddl, "s_1"'] \arrow[ddr, "s_2"] & & T \arrow[ddl, "t_1"'] \arrow[ddr, "t_2"] & \\
 & & SR \arrow[dll, "p'_1"'] \arrow[drr, "p'_2"] & & T(SR) \arrow[dllll, "z'_1"] \arrow[drr, "z'_2"] & & \\
A & & B & & C & & D
\end{tikzcd}
\tag{1.4}
$$

in which:

- (P, p_1, p_2) is the pullback of s_1 and r_2, (Q, q_1, q_2) is the pullback of t_1 and s_2, and (X, x_1, x_2) is the pullback of q_1 and p_2; these pullbacks do exist and the morphisms p_1, p_2, q_1, q_2, x_1, and x_2 are in $\mathbf{E}$ by Remark 2.1.2.

- (SR, p'_1, p'_2) is the composite of the $\mathbf{E}$-relations $R : A \to B$ and $S : B \to C$, and $e_1 : P \to SR$ is the canonical morphism, i.e. e_1 is the morphism in $\mathbf{E}$ with $p'_1 e_1 = r_1 p_1$ and $p'_2 e_1 = s_2 p_2$.

- (Z, z_1, z_2) is the pullback of t_1 and p'_2, this pullback does exist and the morphisms z_1 and z_2 are in $\mathbf{E}$ by Remark 2.1.2; since $p'_2 e_1 x_1 = t_1 q_2 x_2$, there exists a unique morphism $x : X \to Z$ with $z_2 x = q_2 x_2$ and $z_1 x = e_1 x_1$.

- $(T(SR), z_1', z_2')$ is the composite of the **E**-relations $SR : A \to C$ and $T : C \to D$, and $e_2 : Z \to T(SR)$ is the canonical morphism, i.e. e_2 is the morphism in **E** with $z_1'e_2 = p_1'z_1$ and $z_2'e_2 = t_2z_2$.

- Since x_1 and x_2 are jointly monic and r_1p_1 and t_2q_2 are in **E**, by Condition 2.1.1(e) there exists a morphism $e : X \to X'$ in **E** and jointly monic morphisms $x_1' : X' \to A$ and $x_2' : X' \to D$ for which $r_1p_1x_1 = x_1'e$ and $t_2q_2x_2 = x_2'e$.

We first prove that the square $e_1x_1 = z_1x$ in the diagram (1.4) is the pullback of e_1 and z_1. For, consider the commutative diagram

$$
\begin{tikzcd}
 & & & Y \arrow[dddlll, bend right, "y_1"'] \arrow[dll, dotted, "\bar{y}"'] \arrow[ddl, dotted, "y"] \arrow[dddl, bend left, "y_2"] \\
 & X \arrow[dl, "x_1"] \arrow[dr, "x_2"] \arrow[ddr, "x"] & & \\
P \arrow[ddr, "p_2"] \arrow[ddd, "e_1"'] & & Q \arrow[ddl, "q_1"] \arrow[ddr, "q_2"] & \\
 & & Z \arrow[dr, "z_2"] \arrow[ddll, bend right, "z_1"'] & \\
 & S \arrow[ddr, "s_2"] & & T \arrow[ddl, "t_1"] \\
SR \arrow[drr, "p_2'"'] & & & \\
 & & C &
\end{tikzcd}
$$

which is a part of the diagram (1.4) with the new arrows y_1, y_2, y, and $\bar{y}$ defined as follows:

- $y_1 : Y \to P$ and $y_2 : Y \to Z$ are any two morphisms with $e_1y_1 = z_1y_2$.

- Since (Q, q_1, q_2) is the pullback of s_2 and t_1, and $t_1z_2y_2 = p_2'z_1y_2 = p_2'e_1y_1 = s_2p_2y_1$, there exists a unique morphism $y : Y \to Q$ with $z_2y_2 = q_2y$ and $q_1y = p_2y_1$.

- Since (X, x_1, x_2) is the pullback of q_1 and p_2 and $q_1y = p_2y_1$, there exists a unique morphism $\bar{y} : Y \to X$ with $x_2\bar{y} = y$ and $x_1\bar{y} = y_1$.

Since z_1 and z_2 are jointly monic and the equalities

$$z_1x\bar{y} = e_1x_1\bar{y} = e_1y_1 = z_1y_2,$$

$$z_2 x\bar{y} = q_2 x_2 \bar{y} = q_2 y = z_2 y_2$$

hold, we conclude that $x\bar{y} = y_2$. That is, there exists a morphism $\bar{y} : Y \to X$ with $x_1\bar{y} = y_1$ and $x\bar{y} = y_2$, and since x_1 and x_2 are jointly monic, such $\bar{y}$ is unique, proving that (X, x, x_1) is the pullback of e_1 and z_1.

After this, since e_1 is in $\mathbf{E}$, the morphism $x : X \to Z$ is in $\mathbf{E}$ by Remark 2.1.2, therefore, the composite $e_2 x$ is also in $\mathbf{E}$ by Condition 2.1.1(a). We obtain: $r_1 p_1 x_1 = x_1' e$ and $t_2 q_2 x_2 = x_2' e$ in which $e \in \mathbf{E}$ and x_1' and x_2' are jointly monic morphisms; and, $z_1' e_2 x = r_1 p_1 x_1$ and $z_2' e_2 x = t_2 q_2 x_2$ in which $e_2 x \in \mathbf{E}$ and z_1' and z_2' are jointly monic morphisms. Therefore, by Proposition 2.1.4 we have $X' \approx (TS)R$. Similarly we can prove that $X' \approx T(SR)$. Hence, $T(RS) \approx (TR)S$, as desired. $\square$

Remark 2.1.10. *As follows from the proof of Proposition 2.1.9, to construct the composite of the* $\mathbf{E}$*-relations* $(R, r_1, r_2) : A \to B$*,* $(S, s_1, s_2) : B \to C$*, and* $(T, t_1, t_2) : C \to D$*, we simply take the pullbacks* (P, p_1, p_2)*,* (Q, q_1, q_2)*, and then the composite* $(X', x_1', x_2') : A \to D$ *will be the* $\mathbf{E}$*-relation obtained from the following factorization:*

$$\begin{tikzcd}
 & & & X \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dd, dotted, "e"] & & & \\
 & & P \arrow[ddl, "p_1"] \arrow[ddr, "p_2"'] & & Q \arrow[ddl, "q_1"] \arrow[ddr, "q_2"'] & & \\
 & & & X' \arrow[ddlll, dotted, bend right, "x_1'"'] \arrow[ddrrr, dotted, bend left, "x_2'"] & & & \\
 & R \arrow[dl, "r_1"] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & & T \arrow[dl, "t_1"'] \arrow[dr, "t_2"'] & \\
A & & B & & C & & D
\end{tikzcd} \tag{1.5}$$

Using the induction principle, we can compose any finite number of the $\mathbf{E}$*-relations accordingly.*

For each $\mathbf{E}$-relation $R : A \to B$ in $\mathbf{C}$ there is an opposite $\mathbf{E}$-relation $R^\circ : B \to A$ given by the triple (R, r_2, r_1), and we have:

Proposition 2.1.11. *If* $(R, r_1, r_2) : A \to B$ *and* $(S, s_1, s_2) : B \to C$ *are the* $\mathbf{E}$*-relations in* $\mathbf{C}$*, then:*

(i) $(R^\circ)^\circ = R$*.*

(ii) $(SR)^\circ = R^\circ S^\circ$. $\square$

The objects of $\mathbf{C}$ and the $\mathbf{E}$-relations between them form a category $\mathrm{Rel}(\mathbf{C}, \mathbf{E})$, in which the identity $\mathbf{E}$-relation on an object A is the $\mathbf{E}$-relation $(A, 1_A, 1_A) : A \to A$. It is in fact an order-enriched category with $(R, r_1, r_2) \leq (R', r'_1, r'_2)$ if and only if there exists a morphism $r : R \to R'$ with $r'_1 r = r_1$ and $r'_2 r = r_2$ (the relevant properties will be given in the next section).

2.2 Properties of the E-relations

Proposition 2.2.1. *Let $(R, r_1, r_2) : A \to B$, $(R', r'_1, r'_2) : A \to B$, $(S, s_1, s_2) : B \to C$, and $(S', s'_1, s'_2) : B \to C$ be the $\mathbf{E}$-relations in $\mathbf{C}$. We have:*

(i) If $R \leq R'$ then $R^\circ \leq R'^\circ$.

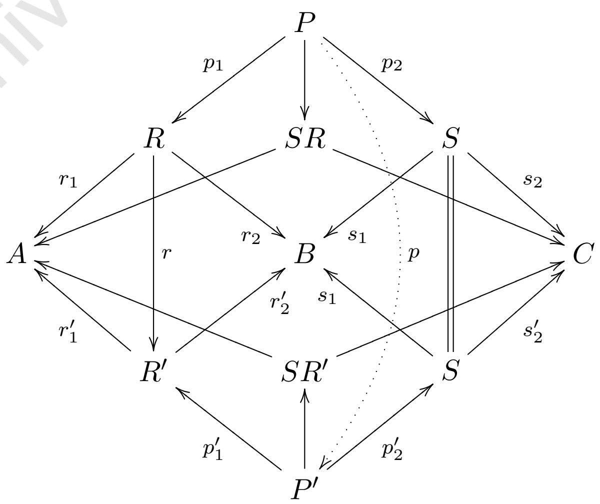
(ii) If $R \leq R'$ then $SR \leq SR'$.

(iii) If $R \leq R'$ and $S \leq S'$ then $SR \leq S'R'$.

Proof.

(i) is obvious.

(ii): If $R \leq R'$ then there exists a morphism $r : R \to R'$ with $r'_1 r = r_1$ and $r'_2 r = r_2$. Let (P, p_1, p_2) be the pullback of r_2 and s_1 and let (P', p'_1, p'_2) be the pullback of r'_2 and s_1. Consider the commutative diagram:



As follows from Proposition 2.1.3, since the pairs of morphisms (p_1, p_2) and (p'_1, p'_2) are jointly monic, and the morphisms r_1, s_2, r'_1, and s'_2 are in $\mathbf{E}$, in order to prove that $SR \leq SR'$ it suffices to prove that there exists a morphism $p : P \to P'$ for which $p'_1 p = rp_1$ and $p'_2 p = p_2$. However, since $r'_2 r p_1 = r_2 p_1 = s_1 p_2$, the latter follows from the fact that the square $s_1 p'_2 = r'_2 p'_1$ is the pullback of r'_2 and s_1.

(iii): If $R \leq R'$ and $S \leq S'$ then by (ii) we have $SR \leq SR'$ and $SR' \leq S'R'$; therefore, $SR \leq S'R'$. $\square$

Remark 2.2.2. *Any morphism $f : A \to B$ in $\mathbf{E}$ can be considered as an $\mathbf{E}$-relation $(A, 1_A, f)$ from A to B. The opposite $\mathbf{E}$-relation f° from B to A will then be the triple $(A, f, 1_A)$.*

Proposition 2.2.3. *Let $(R, r_1, r_2) : A \to B$ be an $\mathbf{E}$-relation in $\mathbf{C}$. If $RR^\circ \leq 1_B$ then $r_1 : R \to A$ is an isomorphism.*

Proof. Let $(R, r_1, r_2) : A \to B$ be an $\mathbf{E}$-relation in $\mathbf{C}$ and let $RR^\circ \leq 1_B$. Consider the commutative diagram

$$
\begin{tikzcd}
 & & X \arrow[ddl, bend right, "u"'] \arrow[ddr, bend left, "v"] \arrow[d, dashed, "\langle u,v\rangle"'] & & \\
 & & P \arrow[dl, "p_1"'] \arrow[d, "e"] \arrow[dr, "p_2"] & & \\
 & R \arrow[ddl, bend right, "r_2"'] \arrow[ddr, "r_1"'] & RR^\circ \arrow[ddll, "p'_1"'] \arrow[d, "f"] \arrow[ddrr, "p'_2"] & R \arrow[ddl, "r_1"] \arrow[ddr, bend left, "r_2"] & \\
 & & B \arrow[dll, "1_B"] \arrow[drr, "1_B"'] & & \\
B & & A & & B
\end{tikzcd}
$$

in which:

- (P, p_1, p_2) is the kernel pair of $r_1 : R \to A$.

- (RR°, p'_1, p'_2) is the composite of the $\mathbf{E}$-relations R° and R, and $e : P \to R^\circ R$ is the canonical morphism; since $RR^\circ \leq 1_B$, there exists a morphism $f : RR^\circ \to B$ with $1_B f = p'_1$ and $1_B f = p'_2$.

- $u, v : X \to R$ are any two morphisms with $r_1 u = r_1 v$, and $\langle u, v\rangle : X \to P$ is the unique morphism with $p_1\langle u, v\rangle = u$ and $p_2\langle u, v\rangle = v$.

We have:

$$fe\langle u, v\rangle = p_1' e\langle u, v\rangle = r_2 p_1\langle u, v\rangle = r_2 u,$$

$$fe\langle u, v\rangle = p_2' e\langle u, v\rangle = r_2 p_2\langle u, v\rangle = r_2 v,$$

yielding $r_2 u = r_2 v$. Since u and v are any two morphisms with $r_1 u = r_1 v$ and since r_1 and r_2 are jointly monic, we obtain $u = v$. Therefore, r_1 is a monomorphism, and since r_1 is in $\mathbf{E}$, we conclude that r_1 is an isomorphism, as desired. □

Similarly we can prove that if $R^\circ R \leq 1_A$ then r_2 is an isomorphism.

Proposition 2.2.4. *If $(R, r_1, r_2) : A \to B$ is an $\mathbf{E}$-relation in $\mathbf{C}$ then $R = r_2 {r_1}^\circ$.*

Proof. Let (R, r_1, r_2) be an $\mathbf{E}$-relation from A to B. As follows from Remark 2.2.2, ${r_1}^\circ$ is the $\mathbf{E}$-relation from A to R and r_2 is the $\mathbf{E}$-relation from R to B. Since the pullback of an identity morphism is again an identity, and since $\mathbf{E}$ contains all isomorphisms, the composite $r_2 {r_1}^\circ : A \to B$ is the $\mathbf{E}$-relation obtained from the following factorization:

$$\begin{tikzcd}
 & & R \arrow[dl, "1_R"'] \arrow[d, dotted, "1_R"] \arrow[dr, "1_R"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "1_R"] & R \arrow[dll, dotted, "r_1"] \arrow[drr, dotted, "r_2"'] & R \arrow[dl, "1_R"'] \arrow[dr, "r_2"] & \\
A & & R & & B
\end{tikzcd}$$

That is, $r_2 {r_1}^\circ$ is the $\mathbf{E}$-relation (R, r_1, r_2) from A to B, proving the desired. □

Proposition 2.2.5. *If $f : A \to B$ and $g : C \to B$ are the morphisms in $\mathbf{E}$, then the $\mathbf{E}$-relation $g^\circ f$ from A to C in $\mathbf{C}$ is given by the pullback $(A \times_B C, p_1, p_2)$ of f along g.*

Proof. Let $f : A \to B$ and $g : C \to B$ be the morphisms in $\mathbf{E}$, and let (P, p_1, p_2) be the pullback of f along g; by Remark 2.1.2, the morphisms p_1 and p_2 are in $\mathbf{E}$. As follows from Remark 2.2.2, f is the $\mathbf{E}$-relation from A to B and g° is the $\mathbf{E}$-relation from B to C. Since $\mathbf{E}$ contains all isomorphisms, the composite $g^\circ f : A \to C$ is the $\mathbf{E}$-relation obtained from the following factorization:

$$\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[d, dotted, "1_P"] \arrow[dr, "p_2"] & & \\
 & A \arrow[dl, "1_A"'] \arrow[dr, "f"] & P \arrow[dll, dotted, "p_1"] \arrow[drr, dotted, "p_2"'] & C \arrow[dl, "g"'] \arrow[dr, "1_C"] & \\
A & & B & & C
\end{tikzcd}$$

That is, (P, p_1, p_2) is the **E**-relation $g^\circ f$ from A to C, proving the desired. $\square$

Remark 2.2.6. *As follows from Proposition 2.2.5, if $f : A \to B$ is a morphism in $\mathbf{E}$, then the $\mathbf{E}$-relation $f^\circ f : A \to A$ is given by the pullback $(A \times_B A, f_1, f_2)$ of f with itself. That is, $f^\circ f = (A \times_B A, f_1, f_2)$ is the kernel pair of f, and therefore $1_A \leq f^\circ f$.*

Proposition 2.2.7. *If a morphism $f : A \to B$ is in $\mathbf{E}$, then $ff^\circ = 1_B$.*

Proof. Let $f : A \to B$ be a morphism in **E**. As follows from Remark 2.2.2, f is the **E**-relation from A to B and f° is the **E**-relation from B to A. Since **E** contains all isomorphisms and f is in **E**, the composite ff° is the **E**-relation obtained from the following factorization:

$$\begin{tikzcd}
 & & A \arrow[dl, "1_A"'] \arrow[d, dotted, "f"] \arrow[dr, "1_A"] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "1_A"] & B \arrow[dll, dotted, "1_B"] \arrow[drr, dotted, "1_B"'] & A \arrow[dl, "1_A"'] \arrow[dr, "f"] & \\
B & & A & & B
\end{tikzcd}$$

That is, $f^\circ f$ is the identity **E**-relation $(B, 1_B, 1_B)$ from B to B, as desired. $\square$

Remark 2.2.8. *It follows from Proposition 2.2.7 that for every morphism $f : A \to B$ in $\mathbf{E}$ the following equalities*

$$ff^\circ f = f,$$

$$f^\circ ff^\circ = f^\circ$$

hold.

Theorem 2.2.9. *Let*

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{2.1}$$

be a diagram in $\mathbf{C}$. *If the morphisms* f, g, h, *and* k *are in* $\mathbf{E}$, *then:*

(i) $kh^\circ \leq g^\circ f$ *if and only if the diagram (2.1) commutes.*

(ii) $kh^\circ = g^\circ f$ *if and only if the diagram (2.1) commutes and the canonical morphism* $\langle h, k\rangle : D \to A \times_B C$ *is in* $\mathbf{E}$.

Proof. Consider the diagram (2.1) in which the morphisms f, g, h, and k are in $\mathbf{E}$. By Proposition 2.2.5, we have $g^\circ f = (A \times_B C, p_1, p_2)$; and, the composite kh° is the $\mathbf{E}$-relation (X, x_1, x_2) from A to C obtained from the following factorization:

$$\begin{tikzcd} & & D \arrow[dl, "1_D"'] \arrow[d, dotted, "e"] \arrow[dr, "1_D"] & & \\ & D \arrow[dl, "h"'] \arrow[dr, "1_D"] & X \arrow[dll, dotted, "x_1"] \arrow[drr, dotted, "x_2"'] & D \arrow[dl, "1_D"'] \arrow[dr, "k"] & \\ A & & D & & C \end{tikzcd}$$

(i): Let $kh^\circ \leq g^\circ f$; that is, there exists a morphism $x : X \to A \times_B C$ with $p_1 x = x_1$ and $p_2 x = x_2$. To prove that the diagram (2.1) is commutative, it suffices to prove that there exists a morphism $d : D \to A \times_B C$ with $p_1 d = h$ and $p_2 d = k$. For, consider the commutative diagram

$$\begin{tikzcd} & D \arrow[d, "e"] \arrow[dddl, bend right, "h"'] \arrow[dddr, bend left, "k"] & \\ & X \arrow[d, dashed, "x"] \arrow[ddl, bend right, "x_1"'] \arrow[ddr, bend left, "x_2"] & \\ & A \times_B C \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ A \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & B & \end{tikzcd} \tag{2.2}$$

and take $d = xe$; then $p_1 d = p_1 xe = x_1 e = h$ and $p_2 d = p_2 xe = x_2 e = k$, as desired.

Conversely, suppose the diagram (2.1) is commutative, i.e. $fh = gk$. To prove $kh^\circ \leq g^\circ f$, we need to show that there exists a morphism $x : X \to A \times_B C$ with $p_1 x = x_1$ and $p_2 x = x_2$. For, consider the diagram (2.2); since e is in $\mathbf{E}$ and $fx_1 e = fh = gk = gx_2 e$ we conclude that $fx_1 = gx_2$. Therefore, since $(A \times_B C, p_1, p_2)$ is the pullback of f and g, there exists a unique morphism $x : X \to A \times_B C$ with $p_1 x = x_1$ and $p_2 x = x_2$, as desired.

(ii): Let $kh^\circ = g^\circ f$. As follows from (i), the diagram (2.1) is commutative; therefore the diagram (2.2) is also commutative and since $g^\circ f = (A \times_B C, p_1, p_2)$ and $kh^\circ = (X, x_1, x_2)$, we conclude that $x : X \to A \times_B C$ is an isomorphism. Since $\langle h, k \rangle = xe$ and e is in $\mathbf{E}$, by Condition 2.1.1(a), the morphism $\langle h, k \rangle$ is also in $\mathbf{E}$.

Conversely, suppose the diagram (2.1) is commutative and the canonical morphism $\langle h, k \rangle : D \to A \times_B C$ is in $\mathbf{E}$. As follows from (i), $kh^\circ \leq g^\circ f$ and therefore, there exists a morphism $x : X \to A \times_B C$ with $p_1 x = x_1$ and $p_2 x = x_2$. To prove that $kh^\circ = g^\circ f$ it suffices to prove that x is an isomorphism. For, consider the commutative diagram (2.2). Since $p_1 xe = x_1 e$ and $p_2 xe = x_2 e$ we conclude that $\langle h, k \rangle = xe$, and since $\langle h, k \rangle$ and e are in $\mathbf{E}$, the morphism x is also in $\mathbf{E}$ by Condition 2.1.1(b). Moreover, since p_1 and p_2 are jointly monic and $p_1 x = x_1$ and $p_2 x = x_2$, x is a monomorphism. Therefore, since every morphism in $\mathbf{E}$ is a normal epimorphism, we conclude that x is an isomorphism, as desired. $\square$

2.3 Equivalence E-relations

Definition 2.3.1. *An* $\mathbf{E}$*-relation* $R : A \to A$ *in* $\mathbf{C}$ *is said to be*

(a) a reflexive $\mathbf{E}$*-relation if* $1_A \leq R$*;*

(b) a symmetric $\mathbf{E}$*-relation if* $R^\circ \leq R$ *(so that* $R^\circ = R$*);*

(c) a transitive $\mathbf{E}$*-relation if* $RR \leq R$*;*

(d) an equivalence $\mathbf{E}$*-relation if it is reflexive, symmetric, and transitive.*

As follows from Definition 2.3.1, if R is a reflexive and a transitive $\mathbf{E}$-relation then $RR = R$; indeed, since R is reflexive we have $R \leq RR$, which together with transitivity gives $RR = R$.

Proposition 2.3.2. *The composite of reflexive* $\mathbf{E}$*-relations in* $\mathbf{C}$ *is a reflexive* $\mathbf{E}$*-relation.*

Proof. If $R : A \to A$ and $S : A \to A$ are reflexive $\mathbf{E}$-relations in $\mathbf{C}$, then $1 \leq R$ and $1 \leq S$. Therefore, $1 \leq SR$ by Proposition 2.2.1(iii), proving that $SR : A \to A$ is a reflexive $\mathbf{E}$-relation. $\square$

Proposition 2.3.3. *Let $R : A \to A$ and $S : A \to A$ be equivalence $\mathbf{E}$-relations in $\mathbf{C}$. If the composite SR is an equivalence $\mathbf{E}$-relation, then $SR = S \vee R$ (i.e. SR is the smallest equivalence $\mathbf{E}$-relation containing both S and R).*

Proof. Let $T : A \to A$ be an equivalence $\mathbf{E}$-relation with $R \leq T$ and $S \leq T$. Since T is an equivalence $\mathbf{E}$-relation, we have $TT \leq T$. Therefore, $SR \leq T$ by Proposition 2.2.1(iii), proving the desired. $\square$

Proposition 2.3.4. *If a morphism $f : A \to B$ is in $\mathbf{E}$, then the kernel pair $(A \times_B A, f_1, f_2)$ of f is an equivalence $\mathbf{E}$-relation in $\mathbf{C}$.*

Proof. If $f : A \to B$ is a morphism in $\mathbf{E}$, then by Remark 2.2.6 the kernel pair of f is the $\mathbf{E}$-relation $f^\circ f : A \to A$ and we have $1_A \leq f^\circ f$, therefore, $f^\circ f$ is a reflexive $\mathbf{E}$-relation. Moreover, it is symmetric since $(f^\circ f)^\circ = f^\circ f$ by Proposition 2.1.11, and it is transitive since $f^\circ f f^\circ f = f^\circ f$ by Remark 2.2.8. $\square$

Definition 2.3.5. *An $\mathbf{E}$-relation $R : A \to B$ in $\mathbf{C}$ is said to be difunctional if $RR^\circ R = R$.*

Theorem 2.3.6. *If $(R, r_1, r_2) : A \to A$ and $(S, s_1, s_2) : A \to A$ are equivalence $\mathbf{E}$-relations in $\mathbf{C}$ then the following conditions are equivalent:*

(a) *$SR : A \to A$ is an equivalence $\mathbf{E}$-relation.*

(b) *$SR = RS$.*

(c) *Every $\mathbf{E}$-relation is difunctional.*

(d) *Every reflexive $\mathbf{E}$-relation is an equivalence $\mathbf{E}$-relation.*

(e) *Every reflexive $\mathbf{E}$-relation is symmetric.*

(f) *Every reflexive $\mathbf{E}$-relation is transitive.*

Proof.

(a)⇒(b): Let $SR : A \to A$ be an equivalence $\mathbf{E}$-relation in $\mathbf{C}$. Since R, S, and SR are symmetric $\mathbf{E}$-relations, we have: $R = R^\circ$, $S = S^\circ$, and $SR = (SR)^\circ$. Therefore, $SR = R^\circ S^\circ = RS$, as desired.

(b)⇒(a): Let $SR = RS$. We have:

- $1 \leq R$ and $1 \leq S$ since R and S are reflexive.
- $R^\circ \leq R$ and $S^\circ \leq S$ since R are S are symmetric.
- $RR \leq R$ and $SS \leq S$ since R and S are transitive.

Using Proposition 2.2.1(iii), we obtain:

- $1 \leq SR$, therefore SR is reflexive.
- $S^\circ R^\circ \leq SR$; since $(SR)^\circ = (RS)^\circ = S^\circ R^\circ$, we conclude that $(SR)^\circ \leq SR$, therefore, SR is symmetric.
- $SSRR \leq SR$; since $SRSR = SSRR$, we conclude that $SRSR \leq SR$, therefore, SR is transitive.

That is, SR is a reflexive, symmetric, and a transitive $\mathbf{E}$-relation, proving that SR is an equivalence $\mathbf{E}$-relation.

(b)⇒(c): Let $(U, u_1, u_2) : X \to Y$ be an arbitrary $\mathbf{E}$-relation in $\mathbf{C}$. By Proposition 2.2.4, $U = u_2{u_1}^\circ$; therefore, to prove that the $\mathbf{E}$-relation $U : X \to Y$ is difunctional, i.e. $UU^\circ U = U$, it suffices to prove $u_2{u_1}^\circ u_1{u_2}^\circ u_2{u_1}^\circ = u_2{u_1}^\circ$. Since u_1 and u_2 are in $\mathbf{E}$, by Remark 2.2.6, the $\mathbf{E}$-relations ${u_1}^\circ u_1 : U \to U$ and ${u_2}^\circ u_2 : U \to U$ are the kernel pairs of u_1 and u_2 respectively, therefore, they are the equivalence $\mathbf{E}$-relations by Proposition 2.3.4. Hence, by (b), ${u_1}^\circ u_1{u_2}^\circ u_2 = {u_2}^\circ u_2{u_1}^\circ u_1$, and multiplying the last equality on the left by u_2 and on the right by ${u_1}^\circ$, using Proposition 2.2.7 we obtain $u_2{u_1}^\circ u_1{u_2}^\circ u_2{u_1}^\circ = u_2{u_1}^\circ$, as desired.

(c)⇒(d): Let $(U, u_1, u_2) : X \to X$ be a reflexive $\mathbf{E}$-relation in $\mathbf{C}$. U is symmetric since $U^\circ = 1_X U^\circ 1_X \leq UU^\circ U = U$, and U is transitive since $UU = U1_X U \leq UU^\circ U = U$. Therefore, U is an equivalence $\mathbf{E}$-relation in $\mathbf{C}$.

(d)⇒(e) is obvious.

(e)⇒(c): Let $(U, u_1, u_2) : X \to Y$ be an arbitrary $\mathbf{E}$-relation in $\mathbf{C}$. The proof is essentially the same as the proof of (b)⇒(c): here ${u_1}^\circ u_1{u_2}^\circ u_2 = {u_2}^\circ u_2{u_1}^\circ u_1$ since the composite of the equivalence **E**-relations $u_1{}^\circ u_1 : U \to U$ and $u_2{}^\circ u_2 : U \to U$ is reflexive by Proposition 2.3.2, therefore, by (e) the composite $u_1{}^\circ u_1 u_2{}^\circ u_2$ is also symmetric.

(c)⇒(a): If $R : A \to A$ and $S : A \to A$ are equivalence **E** relations in **C**, then their composite $SR : A \to A$ is a reflexive **E**-relation by Proposition 2.3.2. Since (c) implies (d), we conclude that SR is an equivalence **E**-relation.

(c)⇒(f): Since (c) implies (d), and (d) implies (f), we conclude that (c) implies (f).

(f)⇒(c): Let $(U, u_1, u_2) : X \to Y$ be an arbitrary **E**-relation in **C**. As stated in the proof of (b)⇒(c), to prove that the **E**-relation $U : X \to Y$ is difunctional, it suffices to prove $u_2 u_1{}^\circ u_1 u_2{}^\circ u_2 u_1{}^\circ = u_2 u_1{}^\circ$. Since the kernel pairs of u_1 and u_2 are the equivalence **E**-relations $u_1{}^\circ u_1 : U \to U$ and $u_2{}^\circ u_2 : U \to U$ respectively, by Proposition 2.3.2 their composite $u_2{}^\circ u_2 u_1{}^\circ u_1 : U \to U$ is a reflexive **E**-relation; therefore, $u_2{}^\circ u_2 u_1{}^\circ u_1$ is transitive by (f), and we have $u_2{}^\circ u_2 u_1{}^\circ u_1 u_2{}^\circ u_2 u_1{}^\circ u_1 = u_2{}^\circ u_2 u_1{}^\circ u_1$. Multiplying the last equality on the left by u_2 and on the right by $u_1{}^\circ$, using Proposition 2.2.7 we obtain $u_2 u_1{}^\circ u_1 u_2{}^\circ u_2 u_1{}^\circ = u_2 u_1{}^\circ$, as desired. □

Remark 2.3.7. *Theorem 2.3.6 is the relative version of Theorem 1.2.3.*

Consider the following

Condition 2.3.8. *(a)* **C** *is pointed;*

(b) If $f : A \to B$ is in **E** *then the kernel of f exists in* **C***;*

(c) If in a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\
K \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B
\end{tikzcd}$$

$k = \ker(f)$, $k' = \ker(f')$, and f and f' are in **E***, then w is an isomorphism.*

Remark 2.3.9. *Condition 2.3.8(c) is the relative version of the Short Five Lemma. Accordingly, we will say that Condition 2.3.8(c) is the* **E***-Short Five Lemma.*

Theorem 2.3.10. *If* $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 2.3.8, then every reflexive* **E***-relation in* **C** *is transitive.*

Proof. Let $(R, r_1, r_2) : A \to A$ be a reflexive $\mathbf{E}$-relation in $\mathbf{C}$. We have:

- The pullback (P, p_1, p_2) of r_2 and r_1 exists in $\mathbf{C}$ and the morphisms p_1 and p_2 are in $\mathbf{E}$ by Remark 2.1.2.

- Since p_2 is in $\mathbf{E}$, the kernel $k : K \to P$ of p_2 exists by Condition 2.3.8(b).

- Since r_1, r_2, p_1, and p_2 are in $\mathbf{E}$, the composites r_1p_1 and r_2p_2 are also in $\mathbf{E}$ by Condition 2.1.1(a), and therefore the limit of the diagram

$$\begin{tikzcd} & P \arrow[d, "r_1p_1"'] \arrow[dr, bend left, "r_2p_2"] & \\ R \arrow[r, "r_1"'] \arrow[rr, bend right, "r_2"', shift right=6] & A & \\ & & A \end{tikzcd} \tag{3.1}$$

exists by Condition 2.1.1(c).

Since R is a reflexive $\mathbf{E}$-relation there exists a morphism $\alpha : A \to R$ with $r_1\alpha = r_2\alpha = 1_A$. Consider the commutative diagram:

$$\begin{tikzcd} R \arrow[dr, dashed, "s"] \arrow[drr, bend left, "1_R"] \arrow[ddr, bend right, "\alpha r_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "r_1"] \\ & R \arrow[r, "r_2"'] & A \end{tikzcd}$$

Since the square $r_2p_1 = r_1p_2$ is a pullback and $r_2\alpha r_1 = r_1$, there exists a unique morphism $s : R \to P$ with $p_1s = \alpha r_1$ and $p_2s = 1_R$, yielding that p_2 is a split epimorphism. Next, let

(Q, q_1, q_2) be the limit of the diagram (3.1) and consider the commutative diagram:

$$\begin{tikzcd}
R \arrow[dr, dotted, "t"] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "1_R"'] & & & \\
& Q \arrow[r, "q_2"] \arrow[d, "q_1"'] & P \arrow[d, "r_1 p_1"'] \arrow[ddr, bend left, "r_2 p_2"] & \\
& R \arrow[r, "r_1"'] \arrow[drr, bend right, "r_2"'] & A & \\
& & & A
\end{tikzcd}$$

Since $r_1 p_1 s = r_1 \alpha r_1 = r_1$ and $r_2 p_2 s = r_2$, there exists a unique morphism $t : R \to Q$ with $q_1 t = 1_R$ and $q_2 t = s$. We have $p_2 q_2 t = p_2 s = 1_R$, therefore the composite $p_2 q_2$ is a split epimorphism. Furthermore, since $r_2 p_1 k = r_1 p_2 k = 0 = r_2 p_2 k$ and (Q, q_1, q_2) is the limit of the diagram (3.1), there exists a unique morphism $f : K \to Q$ making the diagram

$$\begin{tikzcd}
K \arrow[dr, dotted, "f"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "p_1 k"'] & & & \\
& Q \arrow[r, "q_2"] \arrow[d, "q_1"'] & P \arrow[d, "r_1 p_1"'] \arrow[ddr, bend left, "r_2 p_2"] & \\
& R \arrow[r, "r_1"'] \arrow[drr, bend right, "r_2"'] & A & \\
& & & A
\end{tikzcd}$$

commutative.

Since r_1 and r_2 are jointly monic and (Q, q_1, q_2) is the limit of the diagram (3.1), we conclude that q_2 is a monomorphism. Therefore, since p_2 is in $\mathbf{E}$ by Proposition 2.1.6 we obtain the factorization $p_2 q_2 = m_1 e_1$ in which e_1 is in $\mathbf{E}$ and m_1 is a monomorphism. Since $p_2 q_2$ is a split epimorphism, it is a strong epimorphism and therefore m_1 is an isomorphism. Hence, since $\mathbf{E}$ contains all isomorphism, $p_2 q_2$ is in $\mathbf{E}$ by Condition 2.1.1(a). We have

$p_2q_2f = p_2k = 0$, let us prove that $f = \ker(p_2q_2)$. For, consider the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\bar{x}"] \arrow[ddr, dashed, bend right, "\bar{x}"'] \arrow[drr, bend left, "x"] & & & \\
& K \arrow[r, "f"] \arrow[d, equal] & Q \arrow[r, "q"] \arrow[d, "q_2"'] & R \arrow[d, equal] \\
& K \arrow[r, "k"'] & P \arrow[r, "p_2"'] & R
\end{tikzcd}$$

in which $q = p_2q_2$ and $x : X \to Q$ is any morphism with $qx = 0$. Since $p_2q_2x = qx = 0$ and $k = \ker(p_2)$, there exists a unique morphism $\bar{x} : X \to K$ with $k\bar{x} = q_2f$. Since q_2 is a monomorphism and $q_2f\bar{x} = q_2x$, we conclude $f\bar{x} = x$, and since k is a monomorphism such $\bar{x}$ is unique, proving $f = \ker(p_2q_2)$. Since p_2 and p_2q_2 are in $\mathbf{E}$, by the $\mathbf{E}$-Short Five Lemma we conclude that q_2 is an isomorphism. Finally, consider the commutative diagram

$$\begin{tikzcd}
E \arrow[dr, shift left, "e_2"] \arrow[dr, shift right, "e_1"'] & & & \\
& P \arrow[r, "e"] \arrow[d, "q_1q_2^{-1}"'] & RR \arrow[dl, dashed, "r"] \arrow[d, "t_1"] \arrow[ddr, bend left, "t_2"] & \\
& R \arrow[r, "r_1"'] \arrow[drr, bend right, "r_2"'] & A & \\
& & & A
\end{tikzcd}$$

where $e : P \to RR$ is the canonical morphism, i.e. it is the morphism in $\mathbf{E}$ for which $t_1e = r_1p_1$ and $t_2e = r_2p_2$, and (E, e_1, e_2) is the kernel pair of e which does exist by Remark 2.1.2. Since r_1 and r_2 are jointly monic we conclude that $q_1q_2^{-1}e_1 = q_1q_2^{-1}e_2$, therefore, since e is in $\mathbf{E}$ (i.e. it is is a regular epimorphism and therefore it is the coequalizer of its kernel pair), there exists a morphism $r : RR \to R$ with $re = q_1q_2^{-1}$. Moreover, since e is an epimorphism, $r_1re = r_1q_1q_2^{-1} = t_1e$ and $r_2re = r_2q_1q_2^{-1} = t_2e$ we obtain $r_1r = t_1$ and $r_2r = t_2$. That is, there exists a morphism $r : RR \to R$ with $r_1r = t_1$ and $r_2r = t_2$, proving that R is a transitive $\mathbf{E}$-relation. $\square$

Theorem 2.3.10 together with Theorem 2.3.6 gives

Corollary 2.3.11. *If* $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 2.3.8, then every reflexive* $\mathbf{E}$*-relation in* $\mathbf{C}$ *is an equivalence* $\mathbf{E}$*-relation.* $\square$

Theorem 2.3.12. *Let*

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "r"'] \arrow[dr, "t"] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] & D \end{tikzcd}$$

be a commutative diagram in $\mathbf{C}$ *with* r*,* s*, and* t *in* $\mathbf{E}$ *and let* (R, r_1, r_2)*,* (S, s_1, s_2)*, and* (T, t_1, t_2) *be the kernel pairs of* r*,* s*, and* t *respectively (they do exist by Remark 2.1.2). If* $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 2.3.8, then the following conditions are equivalent:*

(i) $\langle r, s\rangle : A \to B \times_D C$ *is in* $\mathbf{E}$ *(by Remark 2.1.2 this pullback does exist).*

(ii) $SR = T$*.*

(iii) $RS = T$*.*

Proof. As follows from Proposition 2.3.4, (R, r_1, r_2), (S, s_1, s_2), and (T, t_1, t_2) are the equivalence $\mathbf{E}$-relations in $\mathbf{C}$ and by Remark 2.2.6 we have $r^\circ r = R$, $s^\circ s = S$, and $t^\circ t = T$. Since R and S are the equivalence $\mathbf{E}$-relations, the composite SR is a reflexive $\mathbf{E}$-relation by Proposition 2.3.2, therefore, SR is an equivalence $\mathbf{E}$-relation by Corollary 2.3.11. Then, $SR = RS$ by Theorem 2.3.6, proving (ii)⇔(iii).

(i)⇒(ii): Suppose $\langle r, s\rangle : A \to B \times_D C$ is a morphism in $\mathbf{E}$. Since $ur = t = vs$ and morphisms r, s, and t are in $\mathbf{E}$, the morphisms u and v are also in $\mathbf{E}$ by Conditions 2.1.1(a) and 2.1.1(b). Then, by Theorem 2.2.9 we obtain $sr^\circ = v^\circ u$. Multiplying the last equality on the left by s° and on the right by r we obtain $s^\circ sr^\circ r = s^\circ v^\circ ur$. Since $s^\circ v^\circ = (vs)^\circ = t^\circ$ and $ur = t$, the last equality implies $SR = T$, as desired.

(ii)⇒(i): Let $SR = T$, that is $s^\circ sr^\circ r = s^\circ v^\circ ur$. Multiplying the last equality on the left by s and on the right by r°, we obtain $ss^\circ sr^\circ rr^\circ = ss^\circ v^\circ urr^\circ$. Since r and s are in $\mathbf{E}$, $rr^\circ = 1_B$ and $ss^\circ = 1_C$ by Proposition 2.2.7, therefore $sr^\circ = v^\circ u$. After that, Theorem 2.2.9 implies that $\langle r, s\rangle$ is in $\mathbf{E}$, proving the desired. □

Chapter 3

Relative homological categories

3.1 Axioms for incomplete relative homological categories

Throughout this section we assume that $(\mathbf{C}, \mathbf{E})$ is a pair in which $\mathbf{C}$ is a pointed category and $\mathbf{E}$ is a class of epimorphisms in $\mathbf{C}$ containing all isomorphisms. Consider the following

Condition 3.1.1. *(a) The class* $\mathbf{E}$ *is closed under composition;*

(b) If $f \in \mathbf{E}$ *and* $gf \in \mathbf{E}$ *then* $g \in \mathbf{E}$*;*

(c) If $f : A \to B$ *is in* $\mathbf{E}$ *then* $\ker(f)$ *and* $\operatorname{coker}(\ker(f))$ *exist in* $\mathbf{C}$*;*

(d) A diagram of the form

$$\begin{tikzcd}
& A' \arrow[d, "f'"] \arrow[ddr, bend left, "g'"] & \\
A \arrow[r, "f"'] \arrow[drr, bend right, "g"'] & B & \\
& & B
\end{tikzcd}$$

has a limit in $\mathbf{C}$ *provided* f *and* g *are in* $\mathbf{E}$*, and either (i)* $f = g$ *and* $f' = g'$*, or (ii)* f' *and* g' *are in* $\mathbf{E}$*,* (f, g) *and* (f', g') *are reflexive pairs, and* f *and* g *are jointly monic.*

(e) If

$$\begin{tikzcd}
A \times_B A' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A' \arrow[d, "f'"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is a pullback and f is in $\mathbf{E}$, then π_2 is also in $\mathbf{E}$;

(f) If $h_1 : H \to A$ and $h_2 : H \to B$ are jointly monic morphisms in $\mathbf{C}$ and if $\alpha : A \to C$ and $\beta : B \to D$ are morphisms in $\mathbf{E}$, then there exists a morphism $h : H \to X$ in $\mathbf{E}$ and jointly monic morphisms $x_1 : X \to C$ and $x_2 : X \to D$ in $\mathbf{C}$ making the diagram

$$\begin{tikzcd}
 & & H \arrow[dl, "h_1"'] \arrow[d, dotted, "h"] \arrow[dr, "h_2"] & & \\
 & A \arrow[dl, "\alpha"'] & X \arrow[dll, dotted, "x_1"] \arrow[drr, dotted, "x_2"'] & B \arrow[dr, "\beta"] & \\
C & & & & D
\end{tikzcd}$$

commutative.

Remark 3.1.2. *Comparing Condition 2.1.1 and Condition 3.1.1, we have:*

- *Conditions 3.1.1(a), 3.1.1(b), and 3.1.1(f) are the same as Conditions 2.1.1(a), 2.1.1(b), and 2.1.1(e) respectively.*
- *In Condition 3.1.1(d) we do not require all of the four morphisms f, g, f', and g' to be in $\mathbf{E}$ as we did in Condition 2.1.1(c); accordingly, in Condition 3.1.1(e) we do not require for both of the morphisms f and f' to be in $\mathbf{E}$ as we did in Condition 2.1.1(d).*

Lemma 3.1.3. *Let $(\mathbf{C}, \mathbf{E})$ be a pair satisfying Conditions 3.1.1(a)-3.1.1(c) and 3.1.1(f) and suppose every morphism in $\mathbf{E}$ is a normal epimorphism. Consider the commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd} \tag{1.1}$$

(i) If $\alpha : A \to A'$ and $\beta : B \to B'$ are in $\mathbf{E}$ and if $f : A \to B$ factors as $f = me$ in which e is in $\mathbf{E}$ and m is a monomorphism, then $f' : A' \to B'$ also factors as $f' = m'e'$ in which e' is in $\mathbf{E}$ and m' is monomorphism.

(ii) If $\alpha : A \to A'$ and $\beta : B \to B'$ are monomorphisms and if $f' : A' \to B'$ factors as $f' = m'e'$ in which e' is in $\mathbf{E}$ and m' is a monomorphism, then $f : A \to B$ also factors as $f = me$ in which e is in $\mathbf{E}$ and m is a monomorphism.

Proof.

(i): Consider the commutative diagram (1.1) and suppose α and β are in $\mathbf{E}$ and $f = me$ in which $e : A \to C$ is in $\mathbf{E}$ and $m : C \to B$ is a monomorphism. Since β is in $\mathbf{E}$ and m is a monomorphism, by Proposition 2.1.6 there exists a morphism $\gamma : C \to C'$ in $\mathbf{E}$ and a monomorphism $m' : C' \to B'$ such that $\beta m = m'\gamma$. Consider the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma e"'] & A' \arrow[d, "f'"] \arrow[dl, dotted, "e'"'] \\ C' \arrow[r, "m'"'] & B' \end{tikzcd}$$

Since α is in $\mathbf{E}$ and m' is a monomorphism, Condition 3.1.1(c) and the fact that every morphism in $\mathbf{E}$ is a normal epimorphism, imply the existence of a unique morphism $e' : A' \to C'$ with $e'\alpha = \gamma e$ and $m'e' = f'$. Since α, e, and γ are in $\mathbf{E}$, the morphism e' is also in $\mathbf{E}$ by Conditions 3.1.1(a) and 3.1.1(b). Hence, $f' = m'e'$ in which e' is in $\mathbf{E}$ and m' is a monomorphism, as desired.

(ii): Consider the commutative diagram (1.1) and suppose α and β are monomorphisms and $f' = m'e'$ in which $e' : A' \to C'$ is in $\mathbf{E}$ and $m' : C' \to B'$ is a monomorphism. Since α is a monomorphism and e' is in $\mathbf{E}$, by Proposition 2.1.6 there exists a morphism $e : A \to C$ in $\mathbf{E}$ and a monomorphism $\gamma : C \to C'$ such that $e'\alpha = \gamma e$. Consider the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & C \arrow[d, "m'\gamma"] \arrow[dl, dotted, "m"'] \\ B \arrow[r, "\beta"'] & B' \end{tikzcd}$$

Since e is in $\mathbf{E}$ and β is a monomorphism, Condition 3.1.1(c) and the fact that every morphism in $\mathbf{E}$ is a normal epimorphism, imply the existence of a unique morphism $m : C \to B$ with $me = f$ and $\beta m = m'\gamma$, m is a monomorphism since so is $m'\gamma$. Hence, $f = me$ in which e is in $\mathbf{E}$ and m is a monomorphism, as desired. $\square$

Definition 3.1.4. *The pair* $(\mathbf{C}, \mathbf{E})$ *is said to be an incomplete relative homological category if:*

(a) Condition 3.1.1 holds in $\mathbf{C}$*;*

(b) Every morphism in $\mathbf{E}$ *is a normal epimorphism;*

(c) *The* **E**-*Short Five Lemma holds in* **C**, *i.e. in every commutative diagram of the form*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, equal] \\ K \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

with f and f' in **E** *and with $k = \ker(f)$ and $k' = \ker(f')$, the morphism w is an isomorphism;*

(d) *If in a commutative diagram*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f, f', and u are in **E**, *$k = \ker(f)$ and $k' = \ker(f')$, then there exists a morphism $e : A \to M$ in* **E** *and a monomorphism $m : M \to A'$ in* **C** *such that $w = me$.*

We will also say that the pair $(\mathbf{C}, \mathbf{E})$ is an incomplete relative weakly homological category whenever it satisfies Conditions 3.1.1(a)-3.1.1(e) and conditions (a)-(c) of Definition 3.1.4.

The two basic examples of a pair $(\mathbf{C}, \mathbf{E})$ satisfying conditions (a)-(d) of Definition 3.1.4 are:

1. "Trivial case": **C** is a pointed category and **E** is the class of all isomorphisms in **C**.

2. "Absolute case": **C** is a homological category and **E** is the class of all regular epimorphisms in **C** (recall that every regular epimorphism in a homological category is a normal epimorphism).

Lemma 3.1.5. *If a pair $(\mathbf{C}, \mathbf{E})$ satisfies Conditions 3.1.1(a)-3.1.1(c) and every morphism in* **E** *is a regular epimorphism, then $(\mathbf{C}, \mathbf{E})$ satisfies conditions (c) and (d) of Definition 3.1.4 if and only if in every commutative diagram of the form*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd} \tag{1.2}$$

with $k = \ker(f)$, $k' = \ker(f')$, and with f, f', and u in **E**, *the morphism w is also in* **E**.

Proof. Suppose $(\mathbf{C}, \mathbf{E})$ satisfies Conditions 3.1.1(a)-3.1.1(c), and conditions (c) and (d) of Definition 3.1.4. Consider the commutative diagram (1.2) with $k = \ker(f)$, $k' = \ker(f')$, and with f, f', and u in $\mathbf{E}$. By condition (d) of Definition 3.1.4 we have $w = me$ in which $e : A \to M$ is in $\mathbf{E}$ and $m : M \to A'$ is a monomorphism. Consider the commutative diagram:

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, "ek"'] & K' \arrow[d, "k'"] \arrow[dl, dotted, "\bar{m}"'] \\
M \arrow[r, "m"'] & A'
\end{tikzcd}$$

Since u is a normal epimorphism and m is a monomorphism, there exists a unique morphism $\bar{m} : K' \to M$ with $\bar{m}u = ek$ and $m\bar{m} = k'$; $\bar{m}$ is a monomorphism since so k'. Since $f'm\bar{m} = 0$, $\bar{m}$ is a monomorphism, and $k' = \ker(f')$, we conclude that $\bar{m} = \ker(f'm)$. By Condition 3.1.1(b), $f'm$ is in $\mathbf{E}$, therefore we can apply the $\mathbf{E}$-Short Five Lemma to the diagram

$$\begin{tikzcd}
K' \arrow[r, "\bar{m}"] \arrow[d, equal] & M \arrow[r, "f'm"] \arrow[d, "m"] & B \arrow[d, equal] \\
K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B
\end{tikzcd}$$

and conclude that m is an isomorphism. Hence, by condition 3.1.1(a) w is in $\mathbf{E}$, as desired.

Conversely, suppose for every commutative diagram (1.2) with $k = \ker(f)$, $k' = \ker(f')$, and with f and f' in $\mathbf{E}$, if u is in $\mathbf{E}$ then w is also in $\mathbf{E}$. It is a well know fact that under the assumptions of condition (c) of Definition 3.1.4, $\ker(w) = 0$; moreover, since $\mathbf{E}$ contains all isomorphisms and f and f' are in $\mathbf{E}$, $w : A \to A'$ is also in $\mathbf{E}$. Since every morphism in $\mathbf{E}$ is a normal epimorphism, we conclude that w is an isomorphism, proving condition (c) of Definition 3.1.4. The proof of condition (d) of Definition 3.1.4 is trivial. $\square$

Assuming that condition (b) of Definition 3.1.4 holds, we can say that the conditions/axioms used here are much weaker than those used by G. Janelidze, L. Márki, and W. Tholen [23]. However, various arguments from [23], used there in the proof of the equivalence of the so-called old and new axioms, can be extended to our context to obtain various reformulations of the conditions (a)-(d) of Definition 3.1.4. Some of them are given in this section.

Condition 3.1.6. *(a) Every morphism in* $\mathbf{E}$ *is a regular epimorphism;*

(b) If $f \in \mathbf{E}$ *then* $\operatorname{coker}(\ker(f)) \in \mathbf{E}$*;*

(c) ("Relative Hofmann's axiom") If in a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

f *and* f' *are in* $\mathbf{E}$*,* w *is a monomorphism,* v *is normal monomorphism, and* $\ker(f') \leq w$*, then* w *is a normal monomorphism.*

Theorem 3.1.7. *If* $(\mathbf{C}, \mathbf{E})$ *is a pair satisfying Condition 3.1.1, then:*

(i) Condition (b) of Definition 3.1.4 implies Conditions 3.1.6(a) and 3.1.6(b).

(ii) Condition (c) of Definition 3.1.4 and Conditions 3.1.6(a) and 3.1.6(b) imply condition (b) of Definition 3.1.4.

Proof.

(i) is obvious.

(ii): Let $f : A \to B$ be a regular epimorphism in $\mathbf{E}$, and let $k = \ker(f)$ and $q = \operatorname{coker}(k)$, they do exist by Condition 3.1.1(c) and q is in $\mathbf{E}$ by Condition 3.1.6(b) . To prove that f is a normal epimorphism it suffices to prove that the canonical morphism $h : \operatorname{Coker}(k) \to B$ is an isomorphism. For, consider the commutative diagram

$$\begin{tikzcd} A \times_{\operatorname{Coker}(k)} A \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] \arrow[d, "\bar{h}"'] & A \arrow[r, "q"] \arrow[dr, "f"] & \operatorname{Coker}(k) \arrow[d, "h"] \\ A \times_B A \arrow[ur, shift left, "f_1"] \arrow[ur, shift right, "f_2"'] & & B \end{tikzcd}$$

in which:

- (f_1, f_2) is the kernel pair of f and (q_1, q_2) is the kernel pair of q, they do exist by Condition 3.1.1(d) and the morphisms f_1, f_2, q_1, and q_2 are in $\mathbf{E}$ by Condition 3.1.1(e).

- $\bar{h} : A \times_{\operatorname{Coker}(k)} A \to A \times_B A$ is the canonical morphism.

Since there are canonical isomorphisms

$$\mathrm{Ker}(q_1) \approx \mathrm{Ker}(q) \approx \mathrm{Ker}(f) \approx \mathrm{Ker}(f_1)$$

(note that these kernels do exist by Condition 3.1.1(c)), we can apply the **E**-Short Five Lemma (condition (c) of Definition 3.1.4) to the diagram

$$\begin{tikzcd}
\mathrm{Ker}(q_1) \arrow[r] \arrow[d, "\approx"'] & A\times_{\mathrm{Coker}(k)}A \arrow[r, "q_1"] \arrow[d, "\bar{h}"'] & A \arrow[d, equal] \\
\mathrm{Ker}(f_1) \arrow[r] & A\times_B A \arrow[r, "f_1"'] & A
\end{tikzcd}$$

This makes $\bar{h}$ an isomorphism; since f and q are regular epimorphisms, the latter implies that h is also an isomorphism. □

Theorem 3.1.8. *If* $(\mathbf{C}, \mathbf{E})$ *is a pair satisfying Condition 3.1.1, then:*

(i) Condition (c) of Definition 3.1.4 implies Condition 3.1.6(c).

(ii) Condition (b) and (d) of Definition 3.1.4 and Condition 3.1.6(c) imply Condition (c) of Definition 3.1.4.

Proof.

(i): According to the assumptions of Condition 3.1.6(c), consider the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, "v"] \\
K \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

in which f and f' are in $\mathbf{E}$, $k' = \ker(f')$, k is a morphism with $wk = k'$, w is a monomorphism, and v is a normal monomorphism. Since $f'k' = f'wk = vfk = 0$ and v is a monomorphism, we obtain $fk = 0$. Letting $\bar{k} : \bar{K} \to A$ to be another morphism with $f\bar{k} = 0$, the equalities $f'w\bar{k} = vf\bar{k} = 0$ imply the existence of a unique morphism $k'' : \bar{K} \to K$ with $w\bar{k} = k'k''$; since w is a monomorphism we conclude that $k = \ker(f)$. Since f is in $\mathbf{E}$, the pullback $(A' \times_{B'} B, \pi_1, \pi_2)$ of f' along v exists by Condition 3.1.1(d), and π_2 is in $\mathbf{E}$ by

Condition 3.1.1(e). After this, applying condition (c) of Definition 3.1.4 to the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "\langle w,f\rangle"'] & B \arrow[d, equal] \\ K \arrow[r, "h'"'] & A'\times_{B'}B \arrow[r, "\pi_2"'] & B, \end{tikzcd}$$

where $\langle w,f\rangle : A \to A'\times_{B'}B$ is the canonical morphism and $h' = \langle w,f\rangle k$, we conclude that $\langle w,f\rangle$ is an isomorphism. Since normal monomorphisms are pullback stable and v is a normal monomorphism, π_1 is also a normal monomorphism and therefore so is w, proving Condition 3.1.6(c).

(ii): We have to show that if in a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, equal] \\ K \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd} \tag{1.3}$$

f and f' are in $\mathbf{E}$, and k and k' are their kernels respectively, then w is an isomorphism.

It is a well known fact that in the situation above the kernel and the cokernel of w is zero. Since $\mathbf{E}$ contains all isomorphisms, by condition (d) of Definition 3.1.4 there exists a factorization $w = me$ in which e is a morphism in $\mathbf{E}$ and m is a monomorphism. Moreover, since w has a zero kernel, e is an isomorphism since every morphism in $\mathbf{E}$ is a normal epimorphism (condition (b) of Definition 3.1.4). Therefore, w is a monomorphism, and applying Condition 3.1.6(c) to the diagram (1.3) we conclude that w is a normal monomorphism. Since w has a zero cokernel, the latter implies that w is an isomorphism, as desired. □

Theorem 3.1.7 together with Theorem 3.1.8 gives:

Corollary 3.1.9. *The following conditions are equivalent:*

(i) The pair $(\mathbf{C},\mathbf{E})$ *is an incomplete relative homological category.*

(ii) The pair $(\mathbf{C},\mathbf{E})$ *satisfies Condition 3.1.1 and:*

(a) Every morphism in $\mathbf{E}$ *is a regular epimorphism;*

(b) If $f \in \mathbf{E}$ *then* $\mathrm{coker}(\ker(f)) \in \mathbf{E}$*;*

(c) The **E**-*Short Five Lemma holds in* **C**;

(d) If in a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f, f', and u are in **E**, *$k = \ker(f)$ and $k' = \ker(f')$, then there exists a morphism $e : A \to M$ in* **E** *and a monomorphism $m : M \to A'$ in* **C** *such that $w = me$.*

(ii) The pair $(\mathbf{C}, \mathbf{E})$ satisfies Condition 3.1.1 and:

(a) Every morphism in **E** *is a normal epimorphism;*

(b) The relative Hofmann's Axiom holds in **C**;

(c) If in a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f, f', and u are in **E**, *$k = \ker(f)$ and $k' = \ker(f')$, then there exists a morphism $e : A \to M$ in* **E** *and a monomorphism $m : M \to A'$ in* **C** *such that $w = me$.*

□

Remark 3.1.10. *Note that since the incomplete relative homological categories satisfy Conditions 2.1.1 and 2.3.8, all the results of Chapter 1 can be applied to them without any restrictions.*

The axioms of incomplete relative homological category $(\mathbf{C}, \mathbf{E})$, (precisely, Condition 3.1.1) are much simplified when the ground category **C** is finitely complete/cocomplete. This special case will be considered in the next section.

3.2 Relative homological categories

Throughout this section we assume that $(\mathbf{C}, \mathbf{E})$ is a pair in which **C** is a pointed, finitely complete category with cokernels, and **E** is a class epimorphisms in **C** containing all isomorphisms. Consider the following

Condition 3.2.1. *(a) The class* $\mathbf{E}$ *is closed under composition;*

(b) If $f \in \mathbf{E}$ *and* $gf \in \mathbf{E}$ *then* $g \in \mathbf{E}$*;*

(c) The class $\mathbf{E}$ *is pullback stable;*

(d) If a morphism f *in* $\mathbf{C}$ *factors as* $f = em$ *in which* e *is in* $\mathbf{E}$ *and* m *is a monomorphism, then it also factors (essentially uniquely) as* $f = m'e'$ *in which* m' *is a monomorphism and* e' *is in* $\mathbf{E}$*.*

Lemma 3.2.2. *Let* $(\mathbf{C}, \mathbf{E})$ *be the pair satisfying Conditions 3.2.1(a) and 3.2.1(c). If* $f : A \to C$ *and* $g : B \to D$ *are morphisms in* $\mathbf{E}$*, then so is the canonical morphism* $f \times g : A \times B \to C \times D$*.*

Proof. Let $f : A \to C$ and $g : B \to D$ be the morphisms in $\mathbf{E}$. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l] \arrow[r] \arrow[d, "f\times 1_B"] & B \arrow[d, equal] \\
C \arrow[d, equal] & C \times B \arrow[l] \arrow[r] \arrow[d, "1_c\times g"] & B \arrow[d, "g"] \\
C & C \times D \arrow[l] \arrow[r] & D
\end{tikzcd} \tag{2.1}$$

in which the unlabeled arrows are the suitable product projections. It is easy to see that the top left and the bottom right squares of the diagram (2.1) are pullbacks. Therefore, since f and g are in $\mathbf{E}$, the morphisms $f \times 1_B$ and $1_c \times g$ are also in $\mathbf{E}$ by Condition 3.2.1(c), and therefore the composite $(1_c \times g)(f \times 1_B)$ is also in $\mathbf{E}$ by Condition 3.2.1(a). Since $f \times g = (1_c \times g)(f \times 1_B)$, we conclude that $f \times g$ is in $\mathbf{E}$, proving the desired. $\square$

Remark 3.2.3. *Note that in Lemma 3.2.2 it is not necessary for the ground category* $\mathbf{C}$ *to be pointed and have all finite limits and cokernels, we only require the existence of products and pullbacks.*

Comparing Conditions 3.1.1 and 3.2.1, we have:

Proposition 3.2.4. *The pair* $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 3.1.1 if and only if it satisfies Condition 3.2.1.*

Proof. Since $\mathbf{C}$ has finite limits and cokernels, Conditions 3.1.1(c) and 3.1.1(d) always hold in $\mathbf{C}$, Conditions 3.1.1(a), 3.1.1(b), and 3.1.1(e) are the same as Conditions 3.2.1(a) 3.2.1(b) and 3.2.1(c) respectively, Condition 3.2.1(d) follows from Condition 3.1.1(f) (see Proposition 2.1.6); therefore we only need to prove that Condition 3.1.1(f) follows from Condition 3.2.1(d).

For, let $h_1 : H \to A$ and $h_2 : H \to B$ be jointly monic morphisms in $\mathbf{C}$ and let $\alpha : A \to C$ and $\beta : B \to D$ be morphisms in $\mathbf{E}$. Since α and β are in $\mathbf{E}$, so is the morphism $\alpha \times \beta : A \times B \to C \times D$ by Lemma 3.2.2, and since h_1 and h_2 are jointly monic, the canonical morphism $\langle h_1, h_2 \rangle : H \to A \times B$ is a monomorphism. Since $\langle h_1, h_2 \rangle$ is a monomorphism and $\alpha \times \beta$ is in $\mathbf{E}$, by Condition 3.2.1(d) there exists a factorization $(\alpha \times \beta)\langle h_1, h_2 \rangle = me$ in which $e : H \to X$ is in $\mathbf{E}$ and $m : X \to C \times D$ is a monomorphism. We obtain the diagram

$$\begin{tikzcd}
 & & H \arrow[dl, "h_1"'] \arrow[d, dotted, "e"] \arrow[dr, "h_2"] & & \\
 & A \arrow[dl, "\alpha"'] & X \arrow[d, "m"] \arrow[dll, dotted, "x_1"] \arrow[drr, dotted, "x_2"'] & B \arrow[dr, "\beta"] & \\
C & & C \times D \arrow[ll, "\pi_1"] \arrow[rr, "\pi_2"'] & & D
\end{tikzcd}$$

in which π_1 and π_2 are the product projections and $x_1 = \pi_1 m$ and $x_2 = \pi_2 m$. We have $x_1 e = \pi_1 m e = \pi_1(\alpha \times \beta)\langle h_1, h_2 \rangle = \alpha h_1$ and $x_2 e = \pi_2 m e = \pi_2(\alpha \times \beta)\langle h_1, h_2 \rangle = \beta h_2$; moreover, since m is a monomorphism and π_1 and π_2 are jointly monic, the morphisms x_1 and x_2 are also jointly monic. Hence, there exists a morphism e in $\mathbf{E}$, and jointly monic morphisms x_1 and x_2 for which $x_1 e = \alpha h_1$ and $x_2 e = \beta h_2$ proving Condition 3.1.1(f). $\square$

Definition 3.2.5. *The pair* $(\mathbf{C}, \mathbf{E})$ *is said to be a relative homological category if:*

(a) Condition 3.2.1 holds in $\mathbf{C}$*;*

(b) Every morphism in $\mathbf{E}$ *is a normal epimorphism;*

(c) The $\mathbf{E}$*-short five lemma holds in* $\mathbf{C}$*;*

(d) If in a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\
K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B
\end{tikzcd}$$

f, f', and u are in $\mathbf{E}$, $k = \ker(f)$ and $k' = \ker(f')$, then there exists a morphism $e : A \to M$ in $\mathbf{E}$ and a monomorphism $m : M \to A'$ in $\mathbf{C}$ such that $w = me$.

We will also say that the pair $(\mathbf{C}, \mathbf{E})$ is a relative weakly homological category whenever it satisfies Condition 3.2.1(a)-3.2.1(c) and conditions (a)-(c) of Definition 3.2.5.

Comparing Definition 3.2.5 and Definition 3.1.4, we have:

Theorem 3.2.6. *If $\mathbf{C}$ is a pointed category with finite limits and cokernels, and $\mathbf{E}$ is a class of epimorphisms in $\mathbf{C}$ containing all isomorphisms, then $(\mathbf{C}, \mathbf{E})$ is a relative homological category if and only if $(\mathbf{C}, \mathbf{E})$ is an incomplete relative homological category.*

Proof. The proof follows directly from Proposition 3.2.4. Indeed: conditions (b), (c), and (d) of Definition 3.2.5 are the same as the conditions (b), (c), and (d) of Definition 3.1.4, and Conditions 3.2.1 and 3.1.1 are equivalent by Proposition 3.2.4. □

As follows from Theorem 3.2.6, Lemma 3.1.5, Theorem 3.1.7, and Theorem 3.1.8 hold true in the relative homological categories, therefore, we have:

Corollary 3.2.7. *The following conditions are equivalent:*

(i) The pair $(\mathbf{C}, \mathbf{E})$ is a relative homological category.

(ii) The pair $(\mathbf{C}, \mathbf{E})$ satisfies Condition 3.2.1 and:

(b) Every morphism in $\mathbf{E}$ is a regular epimorphism;

(c) If $f \in \mathbf{E}$ then $\operatorname{coker}(\ker(f)) \in \mathbf{E}$;

(d) The $\mathbf{E}$-Short Five Lemma holds in $\mathbf{C}$;

(e) If in a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f, f', and u are in $\mathbf{E}$, $k = \ker(f)$ and $k' = \ker(f')$, then there exists a morphism $e : A \to M$ in $\mathbf{E}$ and a monomorphism $m : M \to A'$ in $\mathbf{C}$ such that $w = me$.

(ii) The pair $(\mathbf{C}, \mathbf{E})$ satisfies Condition 3.2.1 and:

(a) *Every morphism in* $\mathbf{E}$ *is a normal epimorphism;*

(b) *The relative Hofmann's Axiom holds in* $\mathbf{C}$*;*

(c) *If in a commutative diagram*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f, f', and u are in $\mathbf{E}$, $k = \ker(f)$ and $k' = \ker(f')$, then there exists a morphism $e : A \to M$ in $\mathbf{E}$ and a monomorphism $m : M \to A'$ in $\mathbf{C}$ such that $w = me$.

□

3.3 Examples

Proposition 3.3.1. *The following conditions are equivalent:*

(i) A pair $(\mathbf{C}, \mathbf{E})$ in which $\mathbf{E}$ is the class of all split epimorphisms in $\mathbf{C}$, is a relative weakly homological category.

(ii) $\mathbf{C}$ is a protomodular category in the sense of D. Bourn [6].

Proof. The implication (i)⇒(ii) follows directly from the definitions.

(ii)⇒(i): The only condition that requires a verification here is condition (b) of Definition 3.2.5; however, it holds by Proposition 1.3.5. □

Proposition 3.3.2. *If $\mathbf{C}$ has coequalizers of kernel pairs and $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$, then the following conditions are equivalent:*

(i) $(\mathbf{C}, \mathbf{E})$ is a relative weakly homological category.

(ii) $(\mathbf{C}, \mathbf{E})$ is a relative homological category.

(iii) $\mathbf{C}$ is a homological category in the sense of F. Borceux and D. Bourn [3].

Proof.

(i)⇒(iii): As follows from (i), the class of all regular epimorphisms in $\mathbf{C}$ is pullback stable. Therefore, since $\mathbf{C}$ has kernel pairs and their coequalizers, $\mathbf{C}$ admits a (regular epi, mono)-factorization system by Proposition 1.2.2. Since $\mathbf{C}$ has all finite limits, the latter implies that $\mathbf{C}$ is a regular category. Furthermore, since $\mathbf{C}$ has pullbacks and a zero object, protomodularity is equivalent to the Split Short Five Lemma by Proposition 1.3.3; and since the $\mathbf{E}$-Short Five Lemma coincides with the Regular Short Five Lemma when $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$, it follows from Remark 1.3.4 and Proposition 1.3.3 that $\mathbf{C}$ is a protomodular category. Therefore, $\mathbf{C}$ is a homological category.

(iii)⇒(ii): Let $\mathbf{C}$ be a homological category and let $\mathbf{E}$ be the class of all regular epimorphisms in $\mathbf{C}$. Conditions 3.2.1(a) and 3.2.1(b), and condition (b) of Definition 3.2.5 hold in $\mathbf{C}$ by Proposition 1.1.3, Proposition 1.1.4, and Proposition 1.3.5 respectively. Since $\mathbf{C}$ has pullback stable (regular epi,mono)-factorization system, Conditions 3.2.1(c) and 3.2.1(d), and condition (d) of Definition 3.2.5 are satisfied, and Condition (c) of Definition 3.2.5 holds since the Regular Short Five Lemma holds in a homological category by Remark 1.3.4. Therefore, $(\mathbf{C}, \mathbf{E})$ is a relative homological category.

Since the implication (ii)⇒(i) is trivial, this completes the proof. □

Example 3.3.3. *Let $(\mathbf{C}, \mathbf{E})$ be a relative weakly homological category and let $(\mathbf{C}', \mathbf{E}')$ be a pair, in which $\mathbf{C}'$ is a category with finite limits and $\mathbf{E}'$ is a class of morphisms in $\mathbf{C}'$ satisfying Conditions 3.2.1(a), 3.2.1(b), and 3.2.1(c). If the functor $F : \mathbf{C} \to \mathbf{C}'$ preserves finite limits, then the pair $(\mathbf{C}, \mathbf{E} \cap F^{-1}(\mathbf{E}'))$, in which $F^{-1}(\mathbf{E}')$ is the class of all morphisms e in $\mathbf{E}$ for which $F(e)$ is in $\mathbf{E}'$, is a relative weakly homological category. In particular we could take $\mathbf{C}'$ to be an arbitrary category with finite limits and $\mathbf{E}' = \mathbf{SplitEpi}$ to be the class of all split epimorphisms in $\mathbf{C}'$. According to the existing literature (see e.g. [33]), an important example is provided by the forgetful functor F from the homological category $\mathbf{C}$ of topological groups to the category $\mathbf{C}'$ of topological spaces; the class $F^{-1}(\mathbf{SplitEpi})$ and the corresponding concept of exactness play a significant role in the cohomology theory of topological groups. This also applies to the classical case of profinite groups, where, however, $F^{-1}(\mathbf{SplitEpi})$ coincides with the class of all normal epimorphisms, as shown in Section I.1.2 of [32]; another such result is used in [31]. The results of [20] also suggest considering the forgetful functor from the category of topological groups to the category of groups. On the other hand one can replace topological groups with more general, so-called protomodular (=semi-abelian), topological algebras, which form a homological category due* *to a result of F. Borceux and M. M. Clementino [4].*

Let us also mention the following “trivial” examples:

Example 3.3.4. *If* $\mathbf{C}$ *is an abelian category, and* $\mathbf{E}$ *is the proper class of epimorphisms in* $\mathbf{C}$ *in the sense of relative homological algebra (see e.g. Chapter IX in [30]) then* $(\mathbf{C}, \mathbf{E})$ *is a relative weakly homological category.*

Example 3.3.5. *A pair* $(\mathbf{C}, \mathbf{E})$*, in which* $\mathbf{E}$ *is the class of all isomorphisms in* $\mathbf{C}$*, always is a relative homological category.*

Example 3.3.6. *A pair* $(\mathbf{C}, \mathbf{E})$*, in which* $\mathbf{E}$ *is the class of all morphisms in* $\mathbf{C}$*, is a relative homological category if and only if* $\mathbf{C}$ *is a trivial category.*

Chapter 4

Homological lemmas in incomplete relative homological categories

4.1 E-exact sequences

Throughout this section we assume that $(\mathbf{C}, \mathbf{E})$ is an incomplete relative homological category.

Definition 4.1.1. *A sequence of morphisms*

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

in $\mathbf{C}$ *is said to be:*

(i) $\mathbf{E}$*-exact at* A_i*, if the morphism* f_{i-1} *admits a factorization* $f_{i-1} = me$*, in which* $e \in \mathbf{E}$ *and* $m = \ker(f_i)$*.*

(ii) an $\mathbf{E}$*-exact sequence, if it is* $\mathbf{E}$*-exact at* A_i *for each* i *(unless the sequence either begins with* A_i *or ends with* A_i*).*

As in the "absolute case" (see Proposition 1.4.2), we have:

Proposition 4.1.2. *(i) The sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \tag{1.1}$$

is $\mathbf{E}$*-exact if and only if* $f = \ker(g)$*.*

(ii) If the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.2}$$

is $\mathbf{E}$*-exact then* $g = \mathrm{coker}(f)$ *and* g *is in* $\mathbf{E}$*.*

(iii) The sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.3}$$

is $\mathbf{E}$ *exact if and only if* $f = \ker(g)$ *and* g *is in* $\mathbf{E}$*.*

Proof.

(i): If (1.1) is an $\mathbf{E}$-exact sequence then $f = me$ in which e is in $\mathbf{E}$ and $m = \ker(g)$. Since m is a monomorphism we have $\ker(f) = \ker(e)$, but $\ker(f) = 0$ since (1.1) is $\mathbf{E}$-exact at A. Since every morphism in $\mathbf{E}$ is a normal epimorphism we conclude that e is an isomorphism, and therefore $f = \ker(g)$. Conversely, suppose $f = \ker(g)$. Then, f is a monomorphism and therefore $\ker(f) = 0$, and since $\mathbf{E}$ contains all isomorphisms we conclude that (1.1) is $\mathbf{E}$-exact at A. Since $f = f1_A$ and $f = \ker(g)$, the $\mathbf{E}$-exactness of (1.1) at B follows again from the fact that $\mathbf{E}$ contains all isomoprhisms.

(ii): If (1.2) is an $\mathbf{E}$-exact sequence then $f = me$ in which e is in $\mathbf{E}$ and $m = \ker(g)$, and $g = m'e'$ in which e' is in $\mathbf{E}$ and $m' = \ker(0)$. Since the kernel of a zero morphism is an isomorphism, we conclude that g is in $\mathbf{E}$. Since every morphism in $\mathbf{E}$ is a normal epimorphism and $m = \ker(g)$, we conclude that $g = \mathrm{coker}(m)$. Since e is an epimorphism, the latter implies that $g = \mathrm{coker}(f)$.

(iii): The proof follows from the proofs of (i) and (ii). □

In the next sections we will often use the following simple fact:

Lemma 4.1.3. *In a pointed category* $\mathbf{C}$ *consider the commutative diagram:*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, "v"] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B' \end{tikzcd} \tag{1.4}$$

(i) If $k' = \ker(f')$ *and* $\ker(v) = 0$ *then* $k = \ker(f)$ *if and only if the left hand square of the diagram (1.4) is a pullback.*

(ii) If $f = \operatorname{coker}(k)$ and $\operatorname{coker}(u) = 0$ then $f' = \operatorname{coker}(k')$ if and only if the right hand square of the diagram (1.4) is a pushout.

Proof. Consider the commutative diagram

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[ddd, "u"'] \arrow[drr, "0_K"] & A \arrow[rr, "f"] \arrow[ddd, "w"] & & B \arrow[ddd, "v"] \\
& & 0 \arrow[ur, "0_B"] \arrow[d, equal] \arrow[ddr, "0_K"] & \\
& & 0 \arrow[dr, "0_{B'}"'] & \\
K' \arrow[r, "k'"'] \arrow[urr, "0_{K'}"] & A' \arrow[rr, "f'"'] & & B'
\end{tikzcd}
\tag{1.5}
$$

consisting of the diagram (1.4).

(i): If $k' = \ker(f')$ and $\ker(v) = 0$, then in the diagram (1.5), $(K', k', 0_{K'})$ and $(0, 1_0, 0_B)$ are the pullbacks. Therefore, $(K, k, 0_K)$ is a pullback, i.e. $k = \ker(f)$, if and only if (K, u, k) is a pullback.

(ii): If $f = \operatorname{coker}(k)$ and $\operatorname{coker}(u) = 0$, then in the diagram (1.5), $(B, f, 0_B)$ and $(0, 0_{K'}, 1_0)$ are the pushouts. Therefore, $(B', f', 0_{B'})$ is a pushout, i.e. $f' = \operatorname{coker}(k')$, if and only if (B', f', v) is a pushout. $\square$

4.2 The Five Lemma

Theorem 4.2.1 (The Five Lemma)**.** *Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative homological category. If in a commutative diagram*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"'] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "k'"'] & E'
\end{tikzcd}
\tag{2.1}
$$

the two rows are $\mathbf{E}$-exact sequences, $\operatorname{coker}(\alpha) = 0$, $\ker(\epsilon) = 0$, and β and δ are isomorphisms, then γ is also an isomorphism.

Proof. Since the first and the second rows of the diagram (2.1) are $\mathbf{E}$-exact at D and D' respectively, there exists the factorizations $h = h_2h_1$ and $h' = h_2'h_1'$ in which h_1 and h_1' are morphisms in $\mathbf{E}$ and $h_2 = \ker(k)$ and $h_2' = \ker(k')$. Since $h_2' = \ker(k')$ and $k'\delta h_2 = \epsilon k h_2 = 0$, there exists a unique morphism $\bar{\gamma} : \bar{C} \to \bar{C}'$ with $\delta h_2 = h_2'\bar{\gamma}$. We obtain the commutative diagram:

$$\begin{tikzcd}
\bar{C} \arrow[r, "h_2"] \arrow[d, dotted, "\bar{\gamma}"'] & D \arrow[r, "k"] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"] \\
\bar{C}' \arrow[r, "h_2'"'] & D' \arrow[r, "k'"'] & E'
\end{tikzcd}$$

Since $h_2 = \ker(k)$, $h_2' = \ker(k')$, and $\ker(\epsilon) = 0$, the square $\delta h_2 = h_2'\bar{\gamma}$ is a pullback by Lemma 4.1.3(i). Therefore, since δ is an isomorphism, so is $\bar{\gamma}$.

Since the first row of the diagram (2.1) is $\mathbf{E}$-exact at C, there exists a factorization $g = g_2 g_1$ in which g_1 is in $\mathbf{E}$ and $g_2 = \ker(h)$, and since h_2 is a monomorphism we conclude that $g_2 = \ker(h_1)$. Moreover, since the first row of the diagram (2.1) is $\mathbf{E}$-exact at B, and g_2 is a monomorphism and g_1 is a normal epimorphism, we conclude that $g_1 = \operatorname{coker}(f)$. Similarly, the $\mathbf{E}$-exactness of the second row of the diagram (2.1) at C' and B' implies the existence of the factorization $g' = g_2' g_1'$, in which $g_1' = \operatorname{coker}(f')$ and $g_2' = \ker(h_2')$. We obtain the commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g_1"] \arrow[d, "\beta"'] & \bar{B} \arrow[d, "\bar{\beta}"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g_1'"'] & \bar{B}'
\end{tikzcd}$$

Since $g_1 = \operatorname{coker}(f)$, $g_1' = \operatorname{coker}(f')$, and $\operatorname{coker}(\alpha) = 0$, the square $g_1'\beta = \bar{\beta} g_1$ is a pushout by Lemma 4.1.3(ii). Therefore, since β is an isomorphism, so is $\bar{\beta}$.

Finally, we obtain the commutative diagram

$$\begin{tikzcd}
\bar{B} \arrow[r, "g_2"] \arrow[d, "\bar{\beta}"'] & C \arrow[r, "h_1"] \arrow[d, "\gamma"'] & \bar{C} \arrow[d, "\bar{\gamma}"] \\
\bar{B}' \arrow[r, "g_2'"'] & C' \arrow[r, "h_1'"'] & \bar{C}'
\end{tikzcd}$$

in which $g_2 = \ker(h_1)$, $g_2' = \ker(h_1')$, the morphisms h_1 and h_1' are in $\mathbf{E}$, and $\bar{\beta}$ and $\bar{\gamma}$ are isomorphisms. Then, by the $\mathbf{E}$-Short Five Lemma the morphism γ is also an isomorphism, proving the desired. $\square$

4.3 The Nine Lemma

Theorem 4.3.1 (The Nine Lemma)**.** *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative homological category. If in a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f'"'] \arrow[d, "u'"'] & Y' \arrow[r, "g'"'] \arrow[d, "v'"'] & Z' \arrow[r] \arrow[d, "w'"] & 0 \\
0 \arrow[r] & X'' \arrow[r, "f''"'] \arrow[d] & Y'' \arrow[r, "g''"'] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3.1}$$

the three columns and the middle row are $\mathbf{E}$*-exact sequences, then the first row is an* $\mathbf{E}$*-exact sequence if and only if the last row is an* $\mathbf{E}$*-exact sequence.*

Proof. Let us first assume that the last row of the diagram (3.1) is an $\mathbf{E}$-exact sequence. As follows from Proposition 4.1.2(iii), since the three columns and the second and the third rows of the diagram (3.1) are $\mathbf{E}$-exact sequences, the morphisms u', v', w', g' and g'' are in $\mathbf{E}$, and $u = \ker(u')$, $v = \ker(v')$, $w = \ker(w')$, $f' = \ker(g')$, and $f'' = \ker(g'')$. And, to prove that the first row of the diagram (3.1) is an $\mathbf{E}$-exact sequence it suffices to prove that $f = \ker(g)$ and $g \in \mathbf{E}$.

Since w is a monomorphism and $wgf = g'f'u = 0$, we have $gf = 0$. We first prove that $f = \ker(g)$. It easily follows from Lemma 4.1.3 that the square $f'u = vf$ is a pullback. Indeed, since f'' is a monomorphism and $u = \ker(u')$ and $v = \ker(v')$, we can apply Lemma 4.1.3 to the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & X' \arrow[r, "u'"] \arrow[d, "f'"'] & X'' \arrow[d, "f''"] \\
Y \arrow[r, "v"'] & Y' \arrow[r, "v'"'] & Z'
\end{tikzcd}$$

and conclude that $f'u = vf$ is the pullback of v and f'. Let $\bar{f} : \bar{X} \to Y$ be any morphism with $g\bar{f} = 0$ and consider the following diagram:

$$\begin{tikzcd}
\bar{X} \arrow[dr, dotted, "\bar{x}"] \arrow[ddr, dotted, bend right, "\bar{x}'"'] \arrow[drr, bend left, "\bar{f}"] & & & \\
& X \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "(*)"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[d, "w"] \\
& X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'
\end{tikzcd}$$

Since $f' = \ker(g')$ and $g'v\bar{f} = g'f'u = 0$, there exists a unique morphism $\bar{x}' : \bar{X} \to X'$ with $f'\bar{x}' = v\bar{f}$. Since $(*)$ is a pullback, the latter implies the existence of a unique morphism $\bar{x} : \bar{X} \to X$ with $f\bar{x} = \bar{f}$ and $u\bar{x} = \bar{x}'$. Since f is a monomorphism, a morphism $\bar{x}$ with $f\bar{x} = f$ is unique, proving that $f = \ker(g)$.

It remains to prove that g is in $\mathbf{E}$. For, let $(Y''\times_{Z''}Z', \pi_1, \pi_2)$ be the pullback of g'' and w', since g'' is in $\mathbf{E}$ this pullback does exist by Condition 3.1.1(d), and π_2 is in $\mathbf{E}$ by Condition 3.1.1(e). Consider the commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "u'"'] & Y' \arrow[r, "g'"] \arrow[d, "\langle v', g' \rangle"] & Z' \arrow[d, equal] \\
X'' \arrow[r, "\langle f'', 0 \rangle"'] \arrow[d, equal] & Y''\times_{Z''}Z' \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & Z' \arrow[d, "w'"] \\
X'' \arrow[r, "f''"'] & Y'' \arrow[r, "g''"'] & Z''
\end{tikzcd}$$

in which $\langle v', g' \rangle$ and $\langle f'', 0 \rangle$ are the canonical morphisms. Since $f'' = \ker(g'')$ and $(Y''\times_{Z''}Z', \pi_1, \pi_2)$ a pullback of g'' and w', we conclude that $\langle f'', 0 \rangle = \ker(\pi_2)$. Then, since $f' = \ker(g')$ and $u' \in \mathbf{E}$, the morphism $\langle v', g' \rangle$ is also in $\mathbf{E}$ by Lemma 3.1.5.

Next, consider the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "g"'] & Y' \arrow[r, "v'"] \arrow[d, "\langle v', g' \rangle"] & Y'' \arrow[d, equal] \\
Z \arrow[r, "\langle 0, w \rangle"'] & Y''\times_{Z''}Z \arrow[r, "\pi_1"'] & Y''
\end{tikzcd} \tag{3.2}$$

in which $\langle 0, w \rangle$ is the canonical morphism. Since $(Y''\times_{Z''}Z, \pi_1, \pi_2)$ is the pullback of g'' and w', and $w = \ker(w')$, we conclude that $\langle 0, w \rangle = \ker(\pi_1)$. Sine $v = \ker(v')$, we can apply Lemma 4.1.3 to the diagram (3.2) and conclude that g is in $\mathbf{E}$, as desired.

Let us now assume that the first row of the diagram (3.1) is **E**-exact and prove that the last row of (3.1) is an **E**-exact sequence, for, it suffices to prove that $f'' = \ker(g'')$ and $g'' \in \mathbf{E}$.

Since $g''v' = w'g'$ and g',w', and v', are in **E**, the morphism g'' is also in **E** by Conditions 3.1.1(a) and 3.1.1(b); hence, we only need to prove that $f'' = \ker(g'')$. Since u' is an epimorphism and $g''f''u' = w'g'f' = 0$, we conclude that $g''f'' = 0$. Since g'' is in **E**, $k = \ker(g'')$ exists by Condition 3.1.1(c); consider the commutative diagram:

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dd, "u'"'] \arrow[dr, dotted, "k_2"] & & Y' \arrow[r, "g'"] \arrow[dd, "v'"] & Z' \arrow[dd, "w'"] \\
& K \arrow[dr, "k"] & & \\
X'' \arrow[rr, "f''"'] \arrow[ur, dotted, "k_1"] & & Y'' \arrow[r, "g''"'] & Z''
\end{tikzcd}$$

We have:

- Since $g''f'' = 0$ and $k = \ker(g'')$, there exists a unique morphism $k_1 : X'' \to K$ with $kk_1 = f''$.

- Since $g''v'f' = g''f''u' = 0$ and $k = \ker(g'')$, there exists a unique morphism $k_2 : X' \to K$ with $kk_2 = v'f'$.

Since g'' is in **E** and $k = \ker(g'')$, the sequence

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"'] & Y'' \arrow[r, "g''"'] & Z'' \arrow[r] & 0
\end{tikzcd}$$

is **E**-exact. Therefore, we obtain the commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "f"] & X' \arrow[r, "k_2"] \arrow[d, "f'"] & K \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & Y \arrow[r, "v"'] \arrow[d, "g"] & Y' \arrow[r, "v'"'] \arrow[d, "g'"] & Y'' \arrow[r] \arrow[d, "g''"] & 0 \\
0 \arrow[r] & Z \arrow[r, "w"'] \arrow[d] & Z' \arrow[r, "w'"'] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

in which the three columns, the second and the third rows are **E**-exact sequences. Then, by the first part of the proof, the sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] & X' \arrow[r, "k_2"] & K \arrow[r] & 0 \end{tikzcd}$$

also is **E**-exact. Finally, we obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & X' \arrow[r, "u'"] \arrow[d, equal] & X'' \arrow[r] \arrow[d, "k_1"] & 0 \\ 0 \arrow[r] & X \arrow[r, "u"'] & X' \arrow[r, "k_2"'] & K \arrow[r] & 0 \end{tikzcd}$$

in which both rows are **E**-exact sequences; since every morphism in **E** is a normal epimorphism, we conclude that k_1 is an isomorphism. Thus, $f'' = \ker(g'')$, as desired. $\square$

4.4 The Snake Lemma

Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative homological category. In Chapter 2 we defined the composition of **E**-relations in **C**. In this chapter we will also need to compose certain relations in **C**:

Let $R = (R, r_1, r_2) : A \to B$ be a relation from A to B, i.e. a pair of jointly monic morphisms $r_1 : R \to A$ and $r_2 : R \to B$ with the same domain, and let $S = (S, s_1, s_2) : B \to C$ be a relation from B to C. If the pullback $(R \times_B S, \pi_1, \pi_2)$ of r_2 and s_1 exists in **C**, and if there exists a morphism $e : R \times_B S \to T$ in **E** and a jointly monic pair of morphisms $t_1 : T \to A$ and $t_2 : T \to C$ in **C** making the diagram

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\pi_1"'] \arrow[d, dashed, "e"] \arrow[dr, "\pi_2"] & & \\ & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & T \arrow[dll, dotted, "t_1"] \arrow[drr, dotted, "t_2"'] & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ A & & B & & C \end{tikzcd} \tag{4.1}$$

commutative, then we will say that $(T, t_1, t_2) : A \to C$ is the composite of $(R, r_1, r_2) : A \to B$ and $(S, s_1, s_2) : B \to C$. One can similarly define partial composition for three or more relations satisfying a suitable associativity condition. Omitting details, let us just mention that, say, a composite $RR'R''$ might exist even if neither RR' nor $R'R''$ does (in particular, this extends the composition of **E**-relations considered in Chapter 1 (see Definition 2.1.8)).

Convention 4.4.1. *We will say that a relation $R = (R, r_1, r_2) : A \to B$ is a morphism in $\mathbf{C}$ if r_1 is an isomorphism.*

Theorem 4.4.2 (Snake Lemma)**.** *Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative homological category. Consider the commutative diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & K_u \arrow[r, "f_K"] \arrow[d, "k_u"'] & K_v \arrow[r, "g_K"] \arrow[d, "k_v"'] & K_w \arrow[d, "k_w"] \arrow[dddll, dotted, "d"] & \\
 & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f'"] \arrow[d, "q_u"'] & Y' \arrow[r, "g'"] \arrow[d, "q_v"'] & Z' \arrow[d, "q_w"] & \\
 & Q_u \arrow[r, "{f'}_Q"'] \arrow[d] & Q_v \arrow[r, "{g'}_Q"'] \arrow[d] & Q_w \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{4.2}
$$

in which all columns, the second and the third rows are $\mathbf{E}$-exact sequences. If the morphism g' factors as $g' = g'_2 g'_1$ in which g'_1 is in $\mathbf{E}$ and g'_2 is a monomorphism, then:

(a) The composite $q_u {f'}^{\circ} v g^{\circ} k_w : K_w \to Q_u$ is a morphism in $\mathbf{C}$.

(b) The sequence

$$K_u \longrightarrow K_v \longrightarrow K_w \xrightarrow{d} Q_u \longrightarrow Q_v \longrightarrow Q_w \tag{4.3}$$

where $d = q_u {f'}^{\circ} v g^{\circ} k_w$, is $\mathbf{E}$-exact.

Proof. Under the assumptions of the theorem, consider the commutative diagram (4.2). Since the three columns of the diagram (4.2) are $\mathbf{E}$-exact sequences, by Proposition 4.1.2 the morphisms k_u, k_v, k_w, and q_u, q_v, and q_w are the kernels and the cokernels of u, v, and w respectively. Since the second and the third rows of the diagram (4.2) are **E**-exact, again by Proposition 4.1.2, $g = \text{coker}(f)$, g is in **E**, and $f' = \ker(g')$. Consider the following commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_u \arrow[rr, "f_K"] \arrow[dd, "k_u"'] & & K_v \arrow[rr, "g_K"] \arrow[dd, "k_v"'] \arrow[dr, dotted, "h"] & & K_w \arrow[dd, "k_w"] \arrow[dddddllll, dotted, bend left, "d"] & \\
 & & & & Y \times_Z K_w \arrow[dl, dashed, "\pi_1"'] \arrow[ur, dashed, "\pi_2"] \arrow[dddlll, dotted, bend left, "\varphi"] & & \\
 & X \arrow[rr, "f"] \arrow[dr, "f_1"'] \arrow[dd, "u"'] & & Y \arrow[rr, "g"] \arrow[dd, "v"] & & Z \arrow[r] \arrow[dd, "w"] & 0 \\
 & & X'' \arrow[ur, "f_2"'] \arrow[dl, dotted, "u'"] \arrow[uurr, dotted, bend left, "\theta"] & & & & \\
0 \arrow[r] & X' \arrow[rr, "f'"'] \arrow[d, "q_u"'] & & Y' \arrow[rr, "g'"'] \arrow[d, "q_v"'] & & Z' \arrow[d, "q_w"] & \\
 & Q_u \arrow[rr, "f_Q{}'"'] \arrow[d] & & Q_v \arrow[rr, "g_Q{}'"'] \arrow[d] & & Q_w \arrow[d] & \\
 & 0 & & 0 & & 0 &
\end{tikzcd}
\tag{4.4}
$$

in which:

(i) All the horizontal and the vertical arrows are as in the diagram (4.2).

(ii) $f = f_2 f_1$ where $f_1 : X \to X''$ is a morphism in **E** and $f_2 : X'' \to Y$ is the kernel of g (such factorization of f does exist in **C** since the second row of the diagram (4.2) is **E**-exact).

(iii) $(Y \times_Z K_w, \pi_1, \pi_2)$ is the pullback of g and k_w, by Condition 3.1.1(d) this pullback does exist in **C**. Since g is in **E**, the morphism π_2 is also in **E** by Condition 3.1.1(e), and since k_w is a normal monomorphism so is π_1.

(iv) Since $gk_v = k_w g_K$, (iii) implies the existence of a unique morphism $h : K_v \to Y \times_Z K_w$ with $\pi_1 h = k_v$ and $\pi_2 h = g_K$. Since k_v and π_1 are normal monomorphisms, so is h.

(v) Since $gf_2 = 0 = k_w 0$, (iii) implies the existence of a unique morphism $\theta : X'' \to Y \times_Z K_w$ with $\pi_1 \theta = f_2$ and $\pi_2 \theta = 0$. Therefore, since $f_2 = \ker(g)$ and π_2 is in $\mathbf{E}$, we conclude that $\pi_2 = \mathrm{coker}(\theta)$.

(vi) Since f_1 is in $\mathbf{E}$, it is an epimorphism, and therefore the equalities $g'vf_2f_1 = g'vf = = g'f'u = 0$ imply $g'vf_2 = 0$. Since $f' = \ker(g')$ the latter implies the existence of a unique morphism $u' : X'' \to X'$ with $f'u' = vf_2$. Since $vf_2f_1 = f'u'f_1$ and f_1 is an epimorphism, we conclude that $u'f_1 = u$. Moreover, since $q_u = \mathrm{coker}(u)$ and f_1 is an epimorphism, we obtain $q_u = \mathrm{coker}(u')$.

(vii) Since $f' = \ker(g')$ and $g'v\pi_1 = wg\pi_1 = wk_w\pi_2 = 0$, there exists a unique morphism $\varphi : Y \times_Z K_w \to X'$ with $f'\varphi = v\pi_1$. It follows that $(Y \times_Z K_w, \varphi, \pi_1)$ is the pullback of f' and φ. Indeed: Let $h_1 : H \to X'$ and $h_2 : H \to Y$ be any morphisms with $f'h_1 = vh_2$. Consider the diagram:

$$\begin{tikzcd}
H \arrow[dr, dotted, "h^*"] \arrow[drr, dotted, bend left, "h_3"] \arrow[ddr, bend left, "h_2"'] \arrow[ddd, bend right, "h_1"'] & & \\
& Y \times_Z K_w \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[ddl, "\varphi"'] \arrow[dr, phantom, "(*)"] & K_w \arrow[d, "k_w"] \\
X'' \arrow[r, "f_2"'] \arrow[d, "u'"'] & Y \arrow[r, "g"'] \arrow[d, "v"] & Z \arrow[d, "w"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'
\end{tikzcd}$$

Since $k_w = \ker(w)$ and $wgh_2 = g'vh_2 = g'f'h_1 = 0$, there exists a unique morphism $h_3 : H \to K_w$ with $k_w h_3 = gh_2$; since $(*)$ is a pullback, the last equality implies that there exists a unique morphism $h^* : H \to Y \times_Z K_w$ with $\pi_2 h^* = h_3$ and $\pi_1 h^* = h_2$. Since f' is a monomorphism, the equalities $f'\varphi h^* = v\pi_1 h^* = vh_2 = f'h_1$ imply $\varphi h^* = h_1$. Hence, there exists a unique morphism $h^* : H \to Y \times_Z K_w$ with $\pi_1 h^* = h_2$

and $\varphi h^* = h_1$ (uniqueness follows from the fact that π_1 is a monomorphism), proving that the square $f'\varphi = v\pi_1$ is the pullback of f' and φ.

(viii) $h = \ker(\varphi)$. Indeed: since f' is a monomorphism, the equalities $f'\varphi h = v\pi_1 h = vk_v = 0$ imply $\varphi h = 0$. Let $h' : H' \to Y\times_Z K_w$ be another morphism with $\varphi h' = 0$. Consider the diagram:

$$\begin{tikzcd}
 & H' \arrow[dl, dotted, "h''"'] \arrow[d, "h'"] & \\
K_v \arrow[r, "h"] \arrow[dr, "k_v"'] & Y\times_Z K_w \arrow[r, "\varphi"] \arrow[d, "\pi_1"] & X' \arrow[d, "f'"] \\
 & Y \arrow[r, "v"'] & Y'
\end{tikzcd}$$

Since $k_v = \ker(v)$ and $v\pi_1 h' = f'\varphi h' = 0$, there exists a unique morphism $h'' : H' \to K_v$ with $k_v h'' = \pi_1 h'$. Since π_1 is a monomorphism and $\pi_1 hh'' = k_v h'' = \pi_1 h'$, we conclude that $hh'' = h'$. Since h is a monomorphism, this means that $h = \ker(\varphi)$.

(ix) Since f' is a monomorphism, the equalities $f'\varphi\theta = v\pi_1\theta = vf_2 = f'u'$ imply $\varphi\theta = u'$; then, since $q_u = \operatorname{coker}(u')$, we obtain $q_u\varphi\theta = q_u u' = 0$. Since $\pi_2 = \operatorname{coker}(\theta)$, the last equality implies the existence of a unique morphism $d : K_w \to Q_u$ with $d\pi_2 = q_u\varphi$.

(a): Since $(Y \times_Z K_w, \pi_1, \pi_2)$ is the pullback of g and k_w, π_1 is a monomorphism, and $(Y \times_Z K_w, \pi_1, \varphi)$ is the pullback of f' and v (see (vii) above), we obtain the commutative diagram

$$
\begin{tikzcd}
&&&&& P \arrow[dl, "1_P"'] \arrow[dr, "1_P"] \\
&&&& P \arrow[dl, "1_P"'] \arrow[dr, "1_P"] && P \arrow[dl, "1_P"'] \arrow[dr, "1_P"] \\
&&& P \arrow[dl, "1_P"'] \arrow[dr, "\pi_1"] && P \arrow[dl, "\pi_1"'] \arrow[dr, "1_P"] && P \arrow[dl, "1_P"'] \arrow[dr, "\varphi"] \\
&& P \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] && Y \arrow[dl, "1_Y"'] \arrow[dr, "1_Y"] && P \arrow[dl, "\pi_1"'] \arrow[dr, "\varphi"] && X' \arrow[dl, "1_{X'}"'] \arrow[dr, "1_{X'}"] \\
& K_w \arrow[dl, "1_{K_w}"'] \arrow[dr, "k_w"] && Y \arrow[dl, "g"'] \arrow[dr, "1_Y"] && Y \arrow[dl, "1_Y"'] \arrow[dr, "v"] && X' \arrow[dl, "f'"'] \arrow[dr, "1_{X'}"] && X' \arrow[dl, "1_{X'}"'] \arrow[dr, "q_u"] \\
K_w && Z && Y && Y' && X' && Q_u
\end{tikzcd}
$$

where $P = Y \times_Z K_w$, and all the diamond parts are pullbacks. Since π_2 and q_u are in $\mathbf{E}$, by Condition 3.1.1(f) we have the factorization (unique up to an isomorphism)

$$
\begin{tikzcd}
&& Y \times_Z K_w \arrow[dl, "1_{Y \times_Z K_w}"'] \arrow[d, dotted, "r"] \arrow[dr, "\varphi"] \\
& Y \times_Z K_w \arrow[dl, "\pi_2"'] & R \arrow[dll, dotted, "r_1"] \arrow[drr, dotted, "r_2"'] & X' \arrow[dr, "q_u"] \\
K_w &&&& Q_u
\end{tikzcd}
\tag{4.5}
$$

where $r : Y \times_Z K_w \to R$ is a morphism in $\mathbf{E}$ and $r_1 : R \to K_w$ and $r_2 : R \to Q_u$ are jointly monic morphisms in $\mathbf{C}$. As follows from the definition of composition of relations, (R, r_1, r_2) is the composite relation $q_u f'^{\circ} v g^{\circ} k_w$ from K_w to Q_u (Note, that since the pullback $(Y \times_Z K_w, \pi_1, \pi_2)$ of k_w and g, and the pullback $(Y \times_Z K_w, \pi_1, \varphi)$ of v and f' exists in $\mathbf{C}$, the composite relations $g^{\circ} k_w : K_w \to Y$ and $f'^{\circ} v : Y \to X'$ exist. Moreover, since π_2 and q_u are in $\mathbf{E}$, the composite $q_u(f'^{\circ} v)(g^{\circ} k_w)$ of the three relations $g^{\circ} k_w$, $f'^{\circ} v$, and q_u also exists and we have $q_u(f'^{\circ} v)(g^{\circ} k_w) = q_u f'^{\circ} v g^{\circ} k_w$).

Since the morphism $d : K_w \to Q_u$ is such that $q_u \varphi = d\pi_2$ (see (ix) above), we obtain the following factorization:

$$\begin{tikzcd}
& & Y \times_Z K_w \arrow[dl, "1_{Y\times_Z K_w}"'] \arrow[d, dotted, "\pi_2"] \arrow[dr, "\varphi"] & & \\
& Y \times_Z K_w \arrow[dl, "\pi_2"'] & K_w \arrow[ll, dotted, "1_{K_w}"] \arrow[drr, dotted, "d"] & X' \arrow[dr, "q_u"] & \\
K_w & & & & Q_u
\end{tikzcd} \tag{4.6}$$

Comparing the diagrams (4.5) and (4.6), we conclude that the relation $(K_w, 1_{K_w}, d)$ can be identified with the relation (R, r_1, r_2). Therefore, r_1 is an isomorphism, as desired.

(ii): To prove that the sequence (4.3) is $\mathbf{E}$-exact, we need to prove that it is $\mathbf{E}$-exact at K_v, K_w, Q_u, and Q_v.

$\mathbf{E}$-*exactness at* K_v: It follows from the fact that the first column of the diagram (4.4) is $\mathbf{E}$-exact at X', that the kernel of u' exists in $\mathbf{C}$. Indeed, consider the commutative diagram

$$\begin{tikzcd}
& X'' \arrow[dr, "u'"] \arrow[dd, dotted, bend right, "u'_1"] & & \\
X \arrow[ur, "f_1"] \arrow[rr, "u"'] \arrow[dr, "u_1"'] & & X' \arrow[r, "q_u"'] & Q_u \\
& K_{q_u} \arrow[ur, "u_2"'] & &
\end{tikzcd} \tag{4.7}$$

in which $u = u_2 u_1$ is the factorization of u with $u_2 = \ker(q_u)$ and $u_1 \in \mathbf{E}$ (which does exists since the first column of the diagram (4.4) is $\mathbf{E}$ exact at X'), and u'_1 is the induced morphism ($q_u u' = 0$ by (vi)). Since f_1 and u_1 are in $\mathbf{E}$, u'_1 is also in $\mathbf{E}$ by Condition 3.1.1(b), and therefore the kernel of u'_1 exists by Condition 3.1.1(c). Since u_2 is a monomorphism we conclude that $\mathrm{Ker}(u') \approx \mathrm{Ker}(u'_1)$.

Consider the following part of the diagram (4.4)

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
 & K_u \arrow[rr, "f_K"] \arrow[dd, "k_u"'] \arrow[dr, dotted, "e"'] & & K_v \arrow[r, "g_K"] \arrow[dd, "k_v"] & K_w \arrow[dd, "k_w"] & \\
 & & K_{u'} \arrow[ur, dotted, "m"'] \arrow[dd, "k_{u'}"] & & & \\
 & X \arrow[rr, "f"] \arrow[dr, "f_1"'] \arrow[dd, "u"'] & & Y \arrow[r, "g"] \arrow[dd, "v"] & Z \arrow[r] \arrow[dd, "w"] & 0 \\
 & & X'' \arrow[ur, "f_2"'] \arrow[dl, "u'"] & & & \\
0 \arrow[r] & X' \arrow[rr, "f'"'] & & Y' \arrow[r, "g'"'] & Z' &
\end{tikzcd}$$

in which:

- $k_{u'} = \ker(u')$.
- Since $k_v = \ker(v)$ and $vf_2k_{u'} = f'u'k_{u'} = 0$, there exists a unique morphism $m : K_{u'} \to K_v$ with $k_v m = f_2 k_{u'}$.
- Since $k_{u'} = \ker(u')$ and $u' f_1 k_u = u k_u = 0$, there exists a unique morphism $e : K_u \to K_{u'}$ with $k_{u'} e = f_1 k_u$.
- Since k_v is a monomorphism and $k_v m e = f_2 k_{u'} e = f_2 f_1 k_u = f k_u = k_v f_K$, we conclude that $me = f_K$.

The **E**-exactness at K_v will be proved if we show that $e \in \mathbf{E}$ and $m = \ker(g_K)$. The latter, however, easily follows from Lemma 4.1.3(i). Indeed: since the kernels of $1_{X'}$ and f' are zeros, by Lemma 4.1.3(i) the squares $f_1 k_u = k_{u'} e$ and $f_2 k_{u'} = k_v m$ are pullbacks. Therefore, since f_1 is in **E** the morphism e is also is in **E** by Condition 3.1.1(e); and since k_w is a monomorphism and the kernel of a monomorphism is zero, by the same Lemma 4.1.3 we obtain $m = \ker(g_K)$.

E-*exactness at* K_w: Consider the commutative diagram (4.4), we have: $dg_K = d\pi_2 h = = q_u \varphi h = 0$ (by (iv), (ix), and (viii)). To prove that the sequence (4.3) is **E**-exact at K_w, it suffices to prove that the kernel of d exists in **C** and that the induced morphism from K_v to the kernel of d is in **E**.

It easily follows from Lemma 3.1.3 that there exists a factorization $d = d_2d_1$ where d_2 is a monomorphism and d_1 is in $\mathbf{E}$. Indeed: since the second column of the Diagram (4.4) is $\mathbf{E}$-exact at Y', there exists a factorization $v = v_2v_1$, where $v_2 = \ker(q_v)$ (i.e. v_2 is a monomorphism) and v_1 is in $\mathbf{E}$. Since $\pi_1 : Y \times_Z K_w \to Y$ and $f' : X' \to Y'$ are monomorphisms, we can apply Lemma 3.1.3(ii) to the diagram

$$\begin{tikzcd}
Y \times_Z K_w \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "v"'] & Y'
\end{tikzcd}$$

(where $\varphi : Y \times_Z K_w \to X'$ is defined as in (vii)) and conclude that $\varphi = \varphi_2\varphi_1$ were φ_2 is a monomorphism and φ_1 is in $\mathbf{E}$. Then, since $\pi_2 : Y \times_Z K_w \to K_w$ and $q_u : X' \to Q_u$ are in $\mathbf{E}$, applying Lemma 3.1.3(i) to the diagram

$$\begin{tikzcd}
Y \times_Z K_w \arrow[r, "\varphi"] \arrow[d, "\pi_2"'] & X' \arrow[d, "q_u"] \\
K_w \arrow[r, "d"'] & Q_u
\end{tikzcd}$$

we obtain the desired factorization of d. Since d_1 is in $\mathbf{E}$, the kernel of d_1 exists by Condition 3.1.1(c); moreover, since d_2 is a monomorphism and $d = d_2d_1$, we conclude that the kernel of d also exists (precisely, it is the kernel of d_1). Let $k_d : K_d \to K_w$ be the kernel of d, since $dg_K = 0$ there exists a unique morphism $e_d : K_v \to K_d$ with $k_de_d = g_K$; it remains to prove that e_d is in $\mathbf{E}$. For, consider the commutative diagram

$$\begin{tikzcd}
 & K_v \arrow[rr, "e_d"] \arrow[d, "h"'] & & K_d \arrow[d, "k_d"] \arrow[dl, dotted, "h'"'] \\
X'' \arrow[r, "\theta"] \arrow[dd, "u'_1"'] & Y \times_Z K_w \arrow[rr, "\pi_2"] \arrow[dr, dotted, "s"] \arrow[dd, "\varphi"'] & & K_w \arrow[dd, "d"] \\
 & & X' \times_{Q_u} K_w \arrow[ur, "\pi'_2"'] \arrow[dl, "\pi'_1"] & \\
K_{q_u} \arrow[r, "u_2"'] \arrow[urr, dotted, "\theta'"] & X' \arrow[rr, "q_u"'] & & Q_u
\end{tikzcd}$$

in which:

- The morphisms $\theta : X'' \to Y \times_Z K_w$ and $h : K_v \to Y \times_Z K_w$ are defined as in (v) and (iv) respectively.

- The morphisms $u_1' : X'' \to K_{q_u}$ and $u_2 : K_{q_u} \to X'$ are as in the diagram (4.7).

- $(X' \times_{Q_u} K_w, \pi_1', \pi_2')$ is the pullback of q_u and d (note that by Condition 3.1.1(d) the pullback of q_u and d does exist in $\mathbf{C}$ since q_u is in $\mathbf{E}$), and $s = \langle \varphi, \pi_2 \rangle$, $\theta' = \langle u_2, 0 \rangle$, and $h' = \langle 0, k_d \rangle$ are the canonical morphisms; since q_u is in $\mathbf{E}$, the morphism π_2' is also in $\mathbf{E}$ by Condition 3.1.1(e).

- Since $(X' \times_{Q_u} K_w, \pi_1', \pi_2')$ is the pullback of q_u and d, and $u_2 = \ker(q_u)$ and $k_d = \ker(d)$, we conclude that $\theta' = \ker(\pi_2')$ and $h' = \ker(\pi_1')$.

Since $\theta = \ker(\pi_2)$, $\theta' = \ker(\pi_2')$, and the morphisms π_2, π_2' u_1' are in $\mathbf{E}$, the morphism $s : Y \times_Z K_w \to X' \times_{Q_u} K_w$ is also in $\mathbf{E}$ by Lemma 3.1.5. And, since $h = \ker(\varphi)$ and $h' = \ker(\pi_1')$, the square $sh = h'e_d$ is the pullback of s and h' by Lemma 4.1.3(i). Therefore, since s is in $\mathbf{E}$, the morphism e_d is also in $\mathbf{E}$ by Condition 2.1.1(e), as desired.

$\mathbf{E}$-*exactness at* Q_u: Consider the commutative diagram (4.4), we have: $f_Q' d\pi_2 = f_Q' q_u \varphi = = q_v f' \varphi = q_v v \pi_1 = 0$ (by (ix) and (vii)), and since π_2 is an epimorphism we conclude that $f_Q' d = 0$. To prove that the sequence (4.3) is $\mathbf{E}$-exact at Q_u, it suffices to prove that the kernel of f_Q' exists in $\mathbf{C}$ and that the induced morphism from K_w to the kernel of f_Q' is in $\mathbf{E}$.

It easily follows from Lemma 3.1.3(i) that there exists a factorization $f_Q' = f_{Q_2}' f_{Q_1}'$ where f_{Q_2}' is a monomorphism and f_{Q_1}' is in $\mathbf{E}$. Indeed: since q_u and q_v are in $\mathbf{E}$, f' is a monomorphism and $\mathbf{E}$ contains all isomorphisms, we can apply Lemma 3.1.3(i) to the diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "q_u"'] & Y' \arrow[d, "q_v"] \\
Q_u \arrow[r, "f_Q'"'] & Q_v
\end{tikzcd}$$

and obtain the desired factorization of f_Q'. Since f_{Q_1}' is in $\mathbf{E}$, the kernel of f_{Q_1}' exists by Condition 3.1.1(c), and therefore, since f_{Q_2}' is a monomorphism we conclude that the kernel of f_Q' also exists (precisely, it is the kernel of f_{Q_1}'). Let $k_{f_Q'} : K_{f_Q'} \to Q_u$ be the kernel of f_Q', since $f_Q' d = 0$ there exists a unique morphism $e_{f_Q'} : K_w \to K_{f_Q'}$ with $e_{f_Q'} k_{f_Q'} = d$; it remains to prove that $e_{f_Q'}$ is in $\mathbf{E}$. Since q_u is in $\mathbf{E}$, the pullback $(K_{f_Q'} \times_{Q_u} X', p_1, p_2)$ of $k_{f_Q'}$ and

q_u exists by Condition 3.1.1(d) and p_1 is in $\mathbf{E}$ by Condition 3.1.1(e); therefore, we have the commutative diagram

$$\begin{tikzcd}
 & & Y'' \arrow[d, "v_2"] \\
K_{f'_Q} \times_{Q_u} X' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[rru, dotted, bend left, "p"] & X' \arrow[r, "f'"] \arrow[d, "q_u"] & Y' \arrow[d, "q_v"] \\
K_{f'_Q} \arrow[r, "k_{f'_Q}"'] & Q_u \arrow[r, "f'_Q"'] & Q_v
\end{tikzcd}$$

in which $v_2 = \ker(q_v)$ (recall, that since the second column of the diagram (4.4) is $\mathbf{E}$-exact at Y', there exists a factorization $v = v_2v_1$ such that $v_1 \in \mathbf{E}$ and $v_2 = \ker(q_v)$); and, since $q_vf'p_2 = f'_Qk_{f'_Q}p_1 = 0$ and $v_2 = \ker(q_v)$, there exists a unique morphism $p : K_{f'_Q} \times_{Q_u} X' \to Y''$ with $v_2p = f'p_2$. Let us first prove that the square $f'p_2 = v_2p$ is the pullback of f' and v_2. For, consider the commutative diagram

$$\begin{tikzcd}
A \arrow[dr, dotted, "a"] \arrow[ddr, bend right, "a_1"] \arrow[rrd, bend left, "a_2"] \arrow[ddd, dotted, bend right, "a_3"'] & & \\
 & K_{f'_Q} \times_{Q_u} X' \arrow[r, "p"] \arrow[d, "p_2"] \arrow[ddl, "p_1"] & Y'' \arrow[d, "v_2"] \\
 & X' \arrow[r, "f'"] \arrow[d, "q_u"] & Y' \arrow[d, "q_v"] \\
K_{f'_Q} \arrow[r, "k_{f'_Q}"'] & Q_u \arrow[r, "f'_Q"'] & Q_v
\end{tikzcd}$$

in which $a_1 : A \to X'$ and $a_2 : A \to Y''$ are any morphisms with $f'a_1 = v_2a_2$. Since $k_{f'_Q} = \ker(f'_Q)$ and $f'_Qq_ua_1 = q_vf'a_1 = q_vv_2a_2 = 0$, there exists a unique morphism $a_3 : A \to K_{f'_Q}$ with $k_{f'_Q}a_3 = q_ua_1$. Then, since $(K_{f'_Q} \times_{Q_u} X', p_1, p_2)$ is the pullback of $k_{f'_Q}$ and q_u, there exists a unique morphism $a : A \to K_{f'_Q} \times_{Q_u} X'$ with $p_2a = a_1$ and $p_1a = a_3$. Moreover, $pa = a_2$ since $v_2pa = f'p_2a = f'a_1 = v_2a_2$ and v_2 is a monomorphism. That is, there exists a morphism $a : A \to K_{f'_Q} \times_{Q_u} X'$ such that $p_2a = a_1$ and $pa = a_2$, and, such a is unique since p_2 is a monomorphism, proving that $(K_{f'_Q} \times_{Q_u} X', p_2, p)$ is the pullback of f' and v_2.

Next, consider the commutative diagram:

$$\begin{tikzcd}
Y \arrow[d, "v_1"] \arrow[dd, bend right=60, "v"'] & Y \times_Z K_w \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dotted, "\psi"] \arrow[dd, bend right=60, "\varphi"'] & K_w \arrow[d, "e_{f'_Q}"] \arrow[dd, bend left=60, "d"] \\
Y'' \arrow[d, "v_2"] & K_{f'_Q} \times_{Q_u} X' \arrow[l, "p"] \arrow[r, "p_1"'] \arrow[d, "p_2"] & K_{f'_Q} \arrow[d, "k_{f'_Q}"] \\
Y' & X' \arrow[l, "f'"] \arrow[r, "q_u"'] & Q_u
\end{tikzcd}$$

Since $(K_{f'_Q} \times_{Q_u} X', p_2, p)$ is the pullback and $f'\varphi = v\pi_1$, there exists a unique morphism $\psi : Y \times_Z K_w \to K_{f'_Q} \times_{Q_u} X'$ with $p_2\psi = \varphi$ and $p\psi = v_1\pi_1$. We have:

$$k_{f'_Q} p_1 \psi = q_u p_2 \psi = q_u \varphi = d\pi_2 = k_{f'_Q} e_{f'_Q} \pi_2$$

and since $k_{f'_Q}$ is a monomorphism we conclude that $p_1\psi = e_{f'_Q}\pi_2$. Since $f'\varphi = v\pi_1$ and $f'p_2 = v_2 p$ are the pullback squares (see (vii)), we conclude that $(Y \times_Z K_w, \pi_1, \psi)$ is the pullback of p and v_1. Therefore, since v_1 is in $\mathbf{E}$, the morphism ψ is also in $\mathbf{E}$ by Condition 3.1.1(e). That is, in the equality $p_1\psi = e_{f'_Q}\pi_2$ the morphisms π_2, ψ, and p_1 are in $\mathbf{E}$, therefore $e_{f'_Q}$ is also in $\mathbf{E}$ by Conditions 3.1.1(a) and 3.1.1(b), as desired.

$\mathbf{E}$-*exactness at* Q_v: Consider the commutative diagram (4.4), we have: $g'_Q f'_Q q_u = q_w g' f' = 0$, and since q_u is an epimorphism we conclude that $g'_Q f'_Q = 0$. To prove that the sequence (4.3) is $\mathbf{E}$-exact at Q_v, it suffices to prove that the kernel of g'_Q exists in $\mathbf{C}$ and that the induced morphism from Q_u to the kernel of g'_Q is in $\mathbf{E}$.

It easily follows from Lemma 3.1.3(i) that $g'_Q = g'_{Q_2} g'_{Q_1}$ in which g'_{Q_1} is a morphism in $\mathbf{E}$ and g'_{Q_2} is a monomorphism. Indeed: according to the assumptions of the theorem, we have $g' = g'_2 g'_1$ were g'_1 is a morphism in $\mathbf{E}$ and g'_2 is a monomorphism, therefore, since q_v and q_w are in $\mathbf{E}$ we can apply Lemma 3.1.3(i) to the diagram

$$\begin{tikzcd}
Y' \arrow[r, "g'"] \arrow[d, "q_v"'] & Z' \arrow[d, "q_w"] \\
Q_v \arrow[r, "g'_Q"'] & Q_w
\end{tikzcd}$$

and obtain the desired factorization of g'_Q. Since g'_{Q_1} is in $\mathbf{E}$, the kernel of g'_{Q_1} exists by Condition 3.1.1(c), and therefore, since g'_{Q_2} is a monomorphism we conclude that the kernel of g'_Q also exists (precisely, it is the kernel of g'_{Q_1}). Let $k_{g'_Q} : K_{g'_Q} \to Q_v$ be the kernel of g'_Q, since $g'_Q f'_Q = 0$ there exists a unique morphism $e_{g'_Q} : Q_u \to K_{g'_Q}$ with $k_{g'_Q} e_{g'_Q} = f'_Q$; it remains to prove that $e_{g'_Q}$ is in $\mathbf{E}$. For, consider the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "u"'] & & Y \arrow[rrrr, "g"] \arrow[dr, "v_1"] \arrow[dd, "v"] & & & & Z \arrow[dl, "w_1"'] \arrow[dd, "w"] \\
& & & \bar{Y} \arrow[rr, dotted, "\bar{y}"] \arrow[dl, "v_2"] & & \bar{Z} \arrow[dr, "w_2"] \arrow[dl, dotted, "\bar{z}"'] & \\
X' \arrow[rr, "f'"] \arrow[dr, dotted, "e_2"] \arrow[dd, "q_u"'] & & Y' \arrow[rrrr, "g'"] \arrow[drr, dotted, "e_1"] \arrow[dd, "q_v"] & & & & Z' \arrow[dd, "q_w"] \\
& K_{g'_Q} \arrow[rrr, dotted, "f''"] \arrow[dr, "k_{g'_Q}"] & & & Q_v \times_{Q_w} Z' \arrow[urr, "p'_2"'] \arrow[dll, "p'_1"] & & \\
Q_u \arrow[ur, "e_{g'_Q}"] \arrow[rr, "f'_Q"'] & & Q_v \arrow[rrrr, "g'_Q"'] & & & & Q_w
\end{tikzcd}$$

in which:

- Since the second and the third columns of the diagram (4.4) are $\mathbf{E}$-exact at Y' and Z' respectively, we have the factorizations $v = v_2 v_1$ and $w = w_2 w_1$, where $v_1, w_1 \in \mathbf{E}$, $v_2 = \ker(q_v)$, and $w_2 = \ker(q_w)$.

- $(Q_v \times_{Q_w} Z', p'_1, p'_2)$ is the pullback of g'_Q and q_w (note that by Condition 3.1.1(d) the pullback of g'_Q and q_w does exist since q_w is in $\mathbf{E}$), and $e_1 = \langle q_v, g' \rangle$, $f'' = \langle k_{g'_Q}, 0 \rangle$, and $\bar{z} = \langle 0, w_2 \rangle$ are the canonical morphisms; since q_w is in $\mathbf{E}$, the morphism p'_1 is also in $\mathbf{E}$ by Condition 3.1.1(e).

- Since $(Q_v \times_{Q_w} Z', p'_1, p'_2)$ is the pullback of g'_Q and q_w, and, $k_{g'_Q} = \ker(g'_Q)$ and $w_2 =$ $= \ker(q_w)$, we conclude that $f'' = \ker(p'_2)$ and $\bar{z} = \ker(p'_1)$.

- Since $f'' = \ker(p'_2)$ and $p'_2 e_1 f' = g' f' = 0$ there exists a unique morphism $e_2 : X'' \to K_{g'_Q}$ with $f'' e_2 = e_1 f'$. Since $k_{g'_Q}$ is a monomorphism and $k_{g'_Q} e_2 = p'_1 f'' e_2 = p'_1 e_1 f' =$ $= q_v f' = f'_Q q_u = k_{g'_Q} e_{g'_Q} q_u$, we conclude that $e_{g'_Q} q_u = e_2$.

- Since $\bar{z} = \ker(p'_1)$ and $p'_1 e_1 v_2 = q_v v_2 = 0$, there exists a unique morphism $\bar{y} : \bar{Y} \to \bar{Z}$ with $e_1 v_2 = \bar{z} \bar{y}$. Since $w_2 \bar{y} v_1 = p'_2 \bar{z} \bar{y} v_1 = p'_2 e_1 v_2 v_1 = g' v_2 v_1$ and v_1 is an epimorphism, we conclude that $w_2\bar{y} = g'v_2$. Therefore, since $w_2\bar{y}v_1 = g'v_2v_1 = g'v = wg = w_2w_1g$ and w_2 is a monomorphism we obtain $\bar{y}v_1 = w_1g$.

Since v_1, g, and w_1 are in **E**, and $\bar{y}v_1 = w_1g$, the morphism $\bar{y}$ is also in **E** by Conditions 3.1.1(a) and 3.1.1(b). Therefore, since $v_2 = \ker(q_v)$, $\bar{z} = \ker(p'_1)$, and $\bar{y}$, q_v, and p'_1 are in **E**, by Lemma 3.1.5 the morphism e_1 is also in **E**. Then, since $f' = \ker(g')$, $f'' = \ker(p'_2)$, the square $f''e_2 = e_1f'$ is a pullback by Lemma 4.1.3(i); therefore, since e_1 is in **E**, the morphism e_2 is also in **E** by Condition 3.1.1(e). Finally, since $e_{g'_Q}q_u = e_2$ and e_2 and q_u are in **E**, the morphism $e_{g'_Q}$ is also in **E** by Condition 3.1.1(b), as desired. □

Chapter 5

Relative semi-abelian categories

5.1 Axioms for incomplete relative semi-abelian categories

Definition 5.1.1. *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative homological category. An equivalence* $\mathbf{E}$*-relation* (R, r_1, r_2) *in* $\mathbf{C}$ *is said to be* $\mathbf{E}$*-effective, if it is the kernel pair of some morphism in* $\mathbf{E}$*.*

Definition 5.1.2. *Let* $\mathbf{C}$ *be a pointed category and let* $\mathbf{E}$ *be a class of epimorphisms in* $\mathbf{C}$ *containing all isomorphisms. The pair* $(\mathbf{C}, \mathbf{E})$ *is said to be an incomplete relative semi-abelian category if:*

(a) $(\mathbf{C}, \mathbf{E})$ *is an incomplete relative homological category;*

(b) If $f : A \to B$ *is in* $\mathbf{E}$ *then the coproduct* $\mathrm{Ker}(f) + B$ *exists in* $\mathbf{C}$*;*

(c) Every equivalence $\mathbf{E}$*-relation in* $\mathbf{C}$ *is* $\mathbf{E}$*-effective.*

As follows from Definition 5.1.2, the two basic examples of an incomplete relative semi-abelian category are:

1. "Trivial case": $\mathbf{C}$ is a pointed category and $\mathbf{E}$ is the class of all isomorphisms in $\mathbf{C}$.

2. "Absolute case": $\mathbf{C}$ is a semi-abelian category [23] and $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$.

As proved in [23], the so called "old" and the "new" axioms that characterize semi-abelian categories are equivalent. Below, we consider the relative versions of the "old" and the "new" axioms, and show that also in the (incomplete) relative case these two sets of axioms are equivalent.

Theorem 5.1.3. *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative semi-abelian category. If* $f : A \to B$ *is a split epimorphism in* $\mathbf{E}$ *with* $fg = 1_B$ *and* $K = \mathrm{Ker}(f)$ *then the canonical morphism* $[k, g] : K + B \to A$ *is an extremal epimorphism.*

Proof. Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative semi-abelian category, and let $f : A \to B$ be a morphism in $\mathbf{E}$ with $k = \ker(f)$. Note that since f is in $\mathbf{E}$, the kernel K of f and the coproduct $K + B$ exist in $\mathbf{C}$ by Condition 3.1.1(c) and condition (b) of Definition 5.1.2. It follows from Condition 3.1.1(f) and the $\mathbf{E}$-Short Five Lemma that $[k, g]$ is an extremal epimorphism. Indeed, let $[k, g] = mx$ were $m : X \to A$ a monomorphism. Consider the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\iota_1"'] \arrow[rrr, bend left=40, "k"] & K + B \arrow[r, "x"] \arrow[rr, bend left, "{[k,g]}"] & X \arrow[r, "m"'] & A \arrow[r, "f"'] & B \\
K' \arrow[u, dashed, "\bar{k}"] \arrow[urr, "k'"'] & & & &
\end{tikzcd}$$

in which $\iota_1 : K \to K + B$ is the first coproduct injection and $k' : K' \to X$ is any morphism with $fmk' = 0$. Since $k = \ker(f)$, the latter implies the existence of a unique morphism $\bar{k} : K' \to K$ with $k\bar{k} = mk'$; therefore, since $k\bar{k} = [k, g]\iota_1\bar{k} = mx\iota_1\bar{k}$ and m is a monomorphism we obtain $x\iota_1\bar{k} = k'$. Since $(fm)(x\iota_1) = 0$, and for any other morphism $k' : K' \to X$ with $(fm)k' = 0$ there exists a unique morphism $\bar{k} : K' \to K$ with $x\iota_1\bar{k} = k'$, we conclude that $x\iota_1 = \ker(fm)$. Consider the commutative diagram:

$$\begin{tikzcd}
K \arrow[r, "x\iota_1"] \arrow[dd, equal] & X \arrow[rr, "fm"] \arrow[dd, "m"'] \arrow[dr, dotted, "e_1"] & & B \arrow[dd, equal] \arrow[ll, bend right, "x\iota_2"'] \\
& & X_1 \arrow[dr, dotted, "m_1"] & \\
K \arrow[r, "k"'] & A \arrow[rr, "f"'] & & B
\end{tikzcd} \tag{1.1}$$

Since m is a monomorphism and f is in $\mathbf{E}$, by Condition 3.1.1(f) there exists a factorization $fm = m_1e_1$ in which $e_1 : X \to X_1$ is a morphism in $\mathbf{E}$ and $m_1 : X_1 \to B$ is a monomorphism; $\iota_2 : B \to K + B$ is the second coproduct injection. Since m_1 is a monomorphism and

$m_1 e_1 x \iota_2 = fmx\iota_2 = f[k,g]\iota_2 = fg = 1_B$, we conclude that m_1 is isomorphism. Therefore, fm is in $\mathbf{E}$ and we can apply the $\mathbf{E}$-Short Five Lemma to the diagram (1.1), yielding that m is an isomorphism, as desired. $\square$

Theorem 5.1.4. *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative semi-abelian category. If* $f : A \to B$ *is a split epimorphism in* $\mathbf{E}$ *and* $g : B \to A$ *is a morphism with* $fg = 1_B$*, then* $\ker(f)$ *and* g *are jointly extremal epic.*

Proof. Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative semi-abelian category, and let $f : A \to B$ be a split epimorphism in $\mathbf{E}$ with $fg = 1_B$, and $k = \ker(f)$. Let $m : M \to A$ be a monomorphism with $m\bar{k} = k$ and $m\bar{g} = g$, and consider the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[dr, "\bar{k}"'] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[dl, "\bar{g}"] \\
& M \arrow[u, "m"] \arrow[r, dashed, "\bar{f}"'] & \bar{B} \arrow[u, dashed, "\bar{m}"']
\end{tikzcd}$$

in which $\bar{m} : \bar{B} \to B$ is a monomorphism, $\bar{f} : M \to \bar{B}$ is a morphism in $\mathbf{E}$, and $fm = \bar{m}\bar{f}$; such factorization does exist since m is a monomorphism and f is in $\mathbf{E}$. The equalities $\bar{m}\bar{f}\bar{g} = fm\bar{g} = fg = 1_B$ imply that $\bar{m}$ is a split epimorphism, therefore it is an isomorphism, yielding that fm is in $\mathbf{E}$. Since $fm\bar{k} = 0$, m is a monomorphism, and $k = \ker(f)$, we conclude that $\bar{k} = \ker(fm)$. Therefore, we can apply the $\mathbf{E}$-Short Five Lemma to the diagram

$$\begin{tikzcd}
K \arrow[r, "\bar{k}"] \arrow[d, equal] & M \arrow[r, "fm"] \arrow[d, "m"'] & B \arrow[d, equal] \\
K \arrow[r, "k"'] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

yielding that m is an isomorphism, as desired. $\square$

Remark 5.1.5. *Note that in the proofs of Theorem 5.1.3 and Theorem 5.1.4 we did not use all the axioms of incomplete relative semi-abelian category. Precisely, Theorem 5.1.4 holds true in* $\mathbf{C}$ *whenever the pair* $(\mathbf{C}, \mathbf{E})$ *in which* $\mathbf{C}$ *is a pointed category and* $\mathbf{E}$ *is a class of epimorphisms in* $\mathbf{C}$ *containing all isomorphisms, satisfies Conditions 3.1.1(a), 3.1.1(b), 3.1.1(c), 3.1.1(f), and the* $\mathbf{E}$*-Short Five Lemma; and Theorem 5.1.3 holds true in* $\mathbf{C}$ *if in addition it satisfies condition (b) of Definition 5.1.2.*

Theorem 5.1.6. *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative homological category in which every equivalence* $\mathbf{E}$*-relations is* $\mathbf{E}$*-effective (i.e. satisfies condition (c) of Definition 5.1.2). If* $r : A \to B$ *and* $s : A \to C$ *are in* $\mathbf{E}$*, then there exists morphisms* $u : B \to D$ *and* $v : C \to D$ *in* $\mathbf{E}$ *such that the diagram*

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "r"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] & D \end{tikzcd} \tag{1.2}$$

commutes and the canonical morphism $\langle r, s\rangle : A \to B \times_D C$ *is in* $\mathbf{E}$.

Proof. Let $r : A \to B$ and $s : A \to C$ be the morphisms in $\mathbf{E}$, and let (R, r_1, r_2) and (S, s_1, s_2) be the kernel pairs of r and s respectively; they do exist by Condition 3.1.1(d), and R and S are the equivalence $\mathbf{E}$-relations by Proposition 2.3.4. Moreover, since every morphism in $\mathbf{E}$ is a regular epimorphism, r and s are the coequalizers of their kernelpairs. Let $(SR, t_1, t_2) : A \to A$ be the composite of the $\mathbf{E}$-relation of R and S. Then, SR is a reflexive $\mathbf{E}$-relation by Proposition 2.3.2, moreover, SR is an equivalence $\mathbf{E}$-relation by Corollary 2.3.11. Since every equivalence $\mathbf{E}$-relation is $\mathbf{E}$-effective, (SR, t_1, t_2) is the kernel pair of some morphism $t : A \to D$ in $\mathbf{E}$; and since every morphism in $\mathbf{E}$ is a regular epimorphism, we conclude that t is the coequalizer of t_1 and t_2.

Consider the commutative diagram

$$\begin{tikzcd} SR \arrow[dr, shift left, "t_2"] \arrow[dr, shift right, "t_1"'] & R \arrow[l, dotted, "x"'] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & \\ S \arrow[u, dotted, "y"] \arrow[r, shift left, "s_2"] \arrow[r, shift right, "s_1"'] & A \arrow[r, "s"] \arrow[d, "r"'] \arrow[dr, "t"] & C \arrow[d, dotted, "v"] \\ & B \arrow[r, dotted, "u"'] & D \end{tikzcd}$$

in which the dotted arrows are defined as follows:

- Since $1 \leq S$ we have $R \leq SR$ by Proposition 2.2.1(iii), therefore, there exists a unique morphism $x : R \to SR$ with $t_1 x = r_1$ and $t_2 x = r_2$.

- Since $1 \leq R$ we have $S \leq SR$ by Proposition 2.2.1(iii), therefore, there exists a unique morphism $y : S \to SR$ with $t_1 y = s_1$ and $t_2 y = s_2$.

- Since r is the coequalizer of r_1 and r_2, and $tr_1 = tt_1x = tt_2x = tr_2$, there exists a unique morphism $u : B \to D$ with $ur = t$; since t and r are in $\mathbf{E}$, the morphism u is also in $\mathbf{E}$ by Condition 3.1.1(b).

- Since s is the coequalizer of s_1 and s_2, and $ts_1 = tt_1y = tt_2y = ts_2$, there exists a unique morphism $v : C \to D$ with $vs = t$; since t and s are in $\mathbf{E}$, the morphism v is also in $\mathbf{E}$ by Condition 3.1.1(b).

It is left to prove that the canonical morphism $\langle r, s\rangle : A \to B \times_D C$ is in $\mathbf{E}$. The latter, however, follows directly from Theorem 2.3.12 since the morphisms r, s, and t are in $\mathbf{E}$, and the kernel pair of t is (SR, t_1, t_2). $\square$

Remark 5.1.7. *In the exact Mal'cev category $\mathbf{C}$, the diagram (1.2) with r and s regular epimorphisms, is a pushout by Theorem 1.2.5; the same is true if $(\mathbf{C}, \mathbf{E})$ is a "relative semi-abelian category" (see Theorem 5.2.3 below). In the incomplete relative semi-abelian category $(\mathbf{C}, \mathbf{E})$, however, the diagram (1.2) is not necessarily a pushout since $\mathbf{C}$ does not have all kernel pairs.*

Theorem 5.1.8. *Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative homological category. If $(\mathbf{C}, \mathbf{E})$ satisfies condition (b) of Definition 5.1.2 then the following conditions are equivalent:*

(i) Every equivalence $\mathbf{E}$-relation in $\mathbf{C}$ is $\mathbf{E}$-effective, i.e. $(\mathbf{C}, \mathbf{E})$ is an incomplete relative semi-abelian category.

(ii) For every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ A' & \xrightarrow[f']{} & B' \end{array} \qquad (1.3)$$

with f and f' in $\mathbf{E}$ and m and m' monomorphisms, if m is a normal monomorphism, $\operatorname{coker}(m)$ *exists in $\mathbf{C}$ and it is in $\mathbf{E}$, then m' is also a normal monomorphism and* $\operatorname{coker}(m')$ *exists and is it in $\mathbf{E}$.*

Proof.

(i) $\Rightarrow$ (ii): Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative semi-abelian category. Under the assumptions of (ii), consider the commutative diagram (1.3). Since $f' : A' \to B'$ and $\operatorname{coker}(m)$:

$A' \to \mathrm{Coker}(m)$ are in $\mathbf{E}$, by Theorem 5.1.6 there exists the morphisms $u : \mathrm{Coker}(m) \to C$ and $v : B' \to C$ in $\mathbf{E}$ such that $u\mathrm{coker}(m) = vf'$ and the canonical morphism $\langle \mathrm{coker}(m), f' \rangle :$ $A' \to \mathrm{Coker}(m) \times_C B'$ is in $\mathbf{E}$. We obtain the commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "m"'] & & B \arrow[dd, "m'"] \arrow[dr, dotted, "t"] & \\
& & & K \arrow[dl, "k"] \arrow[ddll, bend left, "k'"] \\
A' \arrow[rr, "f'"] \arrow[dr, "\langle \mathrm{coker}(m), f' \rangle"] \arrow[dd, "\mathrm{coker}(m)"'] & & B' \arrow[dd, "v"] & \\
& \mathrm{Coker}(m) \times_C B' \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & & \\
\mathrm{Coker}(m) \arrow[rr, "u"'] & & C &
\end{tikzcd}$$

in which:

- $(\mathrm{Coker}(m) \times_C B', \pi_1, \pi_2)$ is the pullback of u and v; since u and v are in $\mathbf{E}$, the morphisms π_1 and π_2 are also in $\mathbf{E}$ by Condition 3.1.1(e).

- $k : K \to B'$ is the kernel of v (since v is in $\mathbf{E}$, the kernel of v exists by Condition 3.1.1(c)) and since every morphism in $\mathbf{E}$ is a normal epimorphism we conclude that $v = \mathrm{coker}(k)$. Since $vm'f = vf'm = u\mathrm{coker}(m)m = 0$ and f is an epimorphism, we have $vm' = 0$. Therefore, there exists a unique morphism $t : B \to K$ with $kt = m'$; since m' is a monomorphism, so is t.

- $k' = \langle 0, k \rangle : K \to \mathrm{Coker}(m) \times_C B'$ is the canonical morphism; since $k = \ker(v)$, we conclude that $k' = \ker(\pi_1)$.

To prove that m' is a normal monomorphism and $\mathrm{coker}(m')$ exists and is in $\mathbf{E}$, it suffices to prove that t is an isomorphism, but since t is a monomorphism, we only need to prove that t is in $\mathbf{E}$. Let us first prove that (A, tf, m) is the pullback of k' and $\langle \mathrm{coker}(m), f' \rangle$. We have:

$$\pi_1 \langle \mathrm{coker}(m), f' \rangle m = \mathrm{coker}(m)m = 0 = \pi_1 k' t f,$$

$$\pi_2 \langle \mathrm{coker}(m), f' \rangle m = f'm = m'f = ktf = \pi_2 k' t f,$$

and since π_1 and π_2 are jointly monic we conclude that $k'tf = \langle \mathrm{coker}(m), f'\rangle m$. Let $x_1 : X \to K$ and $x_2 : X \to A'$ be any morphisms with $k'x_1 = \langle \mathrm{coker}(m), f'\rangle x_2$. Consider the commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, dotted, "x"] \arrow[drr, bend left, "x_2"] \arrow[ddr, bend right, "x_1"'] & & & \\
& A \arrow[r, "m"] \arrow[d, "tf"'] & A' \arrow[r, "\mathrm{coker}(m)"] \arrow[d, "\langle \mathrm{coker}(m), f'\rangle"] & \mathrm{Coker}(m) \arrow[d, equal] \\
& K \arrow[r, "k'"'] & \mathrm{Coker}(m) \times_C B' \arrow[r, "\pi_1"'] & \mathrm{Coker}(m)
\end{tikzcd}$$

Since m is a normal monomorphism we have $m = \ker(\mathrm{coker}(m))$; therefore, since $\mathrm{coker}(m)x_2 = \pi_1\langle \mathrm{coker}(m), f'\rangle x_2 = \pi_1 k'x_1 = 0$, there exists a unique morphism $x : X \to A$ with $mx = x_2$. Since k' is a monomorphism and $k'tfx = \langle \mathrm{coker}(m), f'\rangle mx = = \langle \mathrm{coker}(m), f'\rangle x_2 = k'x_1$, we conclude that $tfx = x_1$, proving that the square $k'tf = = \langle \mathrm{coker}(m), f'\rangle m$ is the pullback of k' and $\langle \mathrm{coker}(m), f'\rangle$. Therefore, since the class $\mathbf{E}$ is pullback stable and $\langle \mathrm{coker}(m), f'\rangle$ is in $\mathbf{E}$, the morphism tf is also in $\mathbf{E}$; but then t is also in $\mathbf{E}$ since so is f (by Condition 3.1.1(b)), as desired.

(ii) $\Rightarrow$ (i): Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative homological category satisfying condition (b) of Definition 5.1.2 and let $(R, r_1, r_2) : A \to A$ be an equivalence $\mathbf{E}$-relation in $\mathbf{C}$. Since R is a reflexive $\mathbf{E}$-relation, there exists a morphism $d : A \to R$ such that $r_1 d = 1_A = r_2 d$. Since r_1 is in $\mathbf{E}$, $\ker(r_1)$ and $\mathrm{coker}(\ker(r_1))$ exist by Condition 3.1.1(c), moreover, $\mathrm{coker}(\ker(r_1))$ is in $\mathbf{E}$ by Condition 3.1.6(b). Let $k = \ker(r_1)$ and $m = r_2 k$; since k is a monomorphism so is m. And since $\mathbf{E}$ contains all isomorphisms, we can apply (ii) to the diagram

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d, "k"'] & K \arrow[d, "m"] \\
R \arrow[r, "r_2"'] & A
\end{tikzcd}$$

and conclude that m is a normal monomorphism, and $\mathrm{coker}(m)$ exists and is in $\mathbf{E}$; moreover, since m is a normal monomorphism we have $m = \ker(\mathrm{coker}(m))$. Let $q = \mathrm{coker}(m)$ and consider the commutative diagram

$$\begin{tikzcd}
K + A \arrow[r, "{[k,d]}"] \arrow[d, dotted, "h"'] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[rr, bend left, "qr_1"] \arrow[rr, bend right, "qr_2"'] & A \arrow[r, "q"] & Q \\
L \arrow[ur, "l"'] & & &
\end{tikzcd}$$

in which:

- $[k, d] : K + A \to R$ is the canonical morphism (note that the coproduct $K + A$ of K and A does exist by condition (b) of Definition 5.1.2).

- $l : L \to R$ is the equalizer of qr_1 and qr_2; since q, r_1 and r_2 are in $\mathbf{E}$, the composites qr_1 and qr_2 are also in $\mathbf{E}$, and therefore their equalizer does exist by Condition 3.1.1(d).

- Since $qr_1[k, d] = qr_2[k, d]$ and l is the equalizer of qr_1 and qr_2, there exists a unique morphism $h : K + A \to L$ with $lh = [k, g]$.

As follows from Theorem 5.1.3, the morphism $[k, d]$ is an extremal epimorphism. Therefore, since $[k, d] = hl$ and l is a monomoprhism, we conclude that l is an isomorphism. Since l is the equalizer of qr_1 and qr_2, the latter implies that $qr_1 = qr_2$. We obtain the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[rr, bend left, "m"] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[dr, dotted, "t"'] & A \arrow[r, "q"] & \mathrm{Coker}(m) \\
& & A \times_{\mathrm{Coker}(m)} A \arrow[u, shift left, "q_1"'] \arrow[u, shift right, "q_2"] &
\end{tikzcd}$$

in which:

- (q_1, q_2) is the kernel pair of q; since q is in $\mathbf{E}$, the kernel pair of q does exist by Conditions 3.1.1(d).

- Since $qr_1 = qr_2$, and (q_1, q_2) is the kernel pair of q, there exists a unique morphism $t : R \to A \times_{\mathrm{Coker(m)}} A$ with $q_1 t = r_1$ and $q_2 t = r_2$.

Since q is in $\mathbf{E}$, it remains to prove that t is an isomorphism. The latter, however, easily follows from the $\mathbf{E}$-Short Five Lemma. Indeed, since q is in $\mathbf{E}$ the morphism q_1 is also in $\mathbf{E}$ by Condition 3.1.1(e). Therefore, since there are canonical isomorphisms $\mathrm{Ker}(q_1) \approx$ $\approx \mathrm{Ker}(q) \approx K$, we can apply the $\mathbf{E}$-short five lemma to the diagram:

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, equal] & R \arrow[r, "r_1"] \arrow[d, "t"] & A \arrow[d, equal] \\ K \arrow[r, "\ker(q_1)"'] & A\times_B A \arrow[r, "q_1"'] & A \end{tikzcd}$$

and conclude that t is an isomorphism, as desired. □

From Corollary 3.2.7 and Theorem 5.1.8 we obtain:

Corollary 5.1.9. *Let* $\mathbf{C}$ *be a pointed category and let* $\mathbf{E}$ *be a class of epimorphisms in* $\mathbf{C}$ *containing all isomorphisms. The following conditions are equivalent:*

(i) The pair $(\mathbf{C}, \mathbf{E})$ *is an incomplete relative semi-abelian category.*

(ii) The pair $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 3.1.1 and:*

(a) Every morphism in $\mathbf{E}$ *is a regular epimorphism;*

(b) If $f \in \mathbf{E}$ *then* $\mathrm{coker}(\ker(f)) \in \mathbf{E}$*;*

(c) If $f : A \to B$ *is in* $\mathbf{E}$ *then the coproduct* $\mathrm{Ker}(f) + B$ *exists in* $\mathbf{C}$*;*

(d) The $\mathbf{E}$*-Short Five Lemma holds in* $\mathbf{C}$*;*

(e) If in a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f*,* f'*, and* u *are in* $\mathbf{E}$*,* $k = \ker(f)$ *and* $k' = \ker(f')$*, then there exists a morphism* $e : A \to M$ *in* $\mathbf{E}$ *and a monomorphism* $m : M \to A'$ *in* $\mathbf{C}$ *such that* $w = me$*.*

(f) Every equivalence $\mathbf{E}$*-relation* $\mathbf{C}$ *is* $\mathbf{E}$*-effective equivalence* $\mathbf{E}$*-relation.*

(iii) The pair $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 3.1.1 and:*

(a) Every morphism in $\mathbf{E}$ *is a normal epimorphism;*

(b) If $f : A \to B$ *is in* $\mathbf{E}$ *then the coproduct* $\mathrm{Ker}(f) + B$ *exists in* $\mathbf{C}$*;*

(c) ("Relative Hofmann's axiom") If in a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

f and f' are in $\mathbf{E}$, w is a monomorphism, v is a normal monomorphism, and $\ker(f') \leq w$, then w is a normal monomorphism;

(d) If in a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f, f', and u are in $\mathbf{E}$, $k = \ker(f)$ and $k' = \ker(f')$, then there exists a morphism $e : A \to M$ in $\mathbf{E}$ and a monomorphism $m : M \to A'$ in $\mathbf{C}$ such that $w = me$;

(e) For every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "m'"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

with f and f' in $\mathbf{E}$ and m and m' monomorphisms, if m is a normal monomorphism, $\operatorname{coker}(m)$ exists in $\mathbf{C}$ and it is in $\mathbf{E}$, then m' is also a normal monomorphism and $\operatorname{coker}(m')$ exists and is it in $\mathbf{E}$.

Conditions 5.1.9(ii) and 5.1.9(iii) are to be considered, respectively, as the (incomplete) relative versions of what was called "new style" and "old style" axioms for a semi-abelian category in [23].

5.2 Relative semi-abelian categories

Throughout this section we assume that $(\mathbf{C}, \mathbf{E})$ is a pair in which $\mathbf{C}$ is a pointed category with finite limits and cokernels, and $\mathbf{E}$ is a class epimorphisms in $\mathbf{C}$ containing all isomorphisms.

Definition 5.2.1. *The pair* $(\mathbf{C}, \mathbf{E})$ *is said to be a relative semi-abelian category if:*

(a) $(\mathbf{C}, \mathbf{E})$ *is a relative homological category;*

(b) $\mathbf{C}$ *has coproducts;*

(c) Every equivalence $\mathbf{E}$*-relation in* $\mathbf{C}$ *is* $\mathbf{E}$*-effective.*

Comparing Definition 5.2.1 and Definition 5.1.2, we have:

Theorem 5.2.2. *If* $\mathbf{C}$ *is a pointed category with finite limits, cokernels, and coproducts, and* $\mathbf{E}$ *is a class of epimorphisms in* $\mathbf{C}$ *containing all isomorphisms, then* $(\mathbf{C}, \mathbf{E})$ *is a relative semi-abelian category if and only if* $(\mathbf{C}, \mathbf{E})$ *is an incomplete relative semi-abelian category.*

Proof. The proof follows from Theorem 3.2.6. Indeed, by Theorem 3.2.6, condition (a) of Definition 5.2.1 is equivalent to the condition (a) of Definition 5.1.2. Moreover, since $\mathbf{C}$ has coproducts, condition (b) of Definition 5.2.1 and of Definition 5.1.2 are the same and always hold in $\mathbf{C}$, and, condition (c) of Definition 5.2.1 and of Definition 5.1.2 are the same. $\square$

Therefore, the theorems proved in the previous section hold true in the relative semi-abelian categories.

Theorem 5.2.3. *Let* $\mathbf{C}$ *be a relative homological category in which every equivalence* $\mathbf{E}$*-relation is* $\mathbf{E}$*-effective. If* $r : A \to B$ *and* $s : A \to C$ *are in* $\mathbf{E}$*, then the pushout diagram*

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "r"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] & D \end{tikzcd} \tag{2.1}$$

exists in $\mathbf{C}$*, and the morphisms* u*,* v*, and* $\langle r, s \rangle : A \to B \times_D C$ *are in* $\mathbf{E}$*.*

Proof. By Theorem 5.1.6 there exists the morphisms $u : B \to D$ and $v : C \to D$ in $\mathbf{E}$ such that the diagram (2.1) commutes and the canonical morphism $\langle r, s \rangle : A \to B \times_D C$ is in $\mathbf{E}$. Moreover, if (R, r_1, r_2) and (S, s_1, s_2) are the kernel pairs of r and s, and if $(SR, t_1, t_2) : A \to C$ is the composite of the $\mathbf{E}$-relations R and S, then R, S, and SR are the equivalence $\mathbf{E}$-relations and (SR, t_1, t_2) is the kernel pair of $t = ur = vs$. Therefore, it remains to prove that the square $ur = vs$ is a pushout. For, let $\bar{u} : B \to \bar{D}$ and $\bar{v} : C \to \bar{D}$ be any morphisms with $\bar{u}r = \bar{v}s$, and let (Z, z_1, z_2) be the kernel pair of $\bar{u}r$. Since (Z, z_1, z_2)

is the kernel pair, it is an equivalence relation (note that it may not be an equivalence **E**-relation since the composite $\bar{u}r$ is not necessarily in **E**), yielding $ZZ = Z$, and since $R \leq Z$ and $S \leq Z$, we obtain $SR \leq Z$. Therefore, there exists a morphism $\bar{z} : SR \to Z$ with $z_1\bar{z} = t_1$ and $z_2\bar{z} = t_2$. Consider the commutative diagram:

$$\begin{tikzcd}
SR \arrow[r, shift left, "t_1"] \arrow[r, shift right, "t_2"'] \arrow[d, "\bar{z}"'] & A \arrow[r, "t"] \arrow[d, equal] & D \arrow[d, dotted, "d"] \\
Z \arrow[r, shift left, "z_1"] \arrow[r, shift right, "z_2"'] & A \arrow[r, "\bar{u}r"'] & \bar{D}
\end{tikzcd}$$

Since t is the coequalizer of t_1 and t_2, and since $\bar{u}rt_1 = \bar{u}rz_1\bar{z} = \bar{u}rz_2\bar{z} = \bar{u}rt_2$, there exists a unique morphism $d : D \to \bar{D}$ with $dt = \bar{u}r$. Since $\bar{u}r = \bar{v}s$, the latter implies $dt = \bar{v}s$. We have $dur = \bar{u}r$ and $dvs = \bar{v}s$, and since r and s are epimorphisms, we conclude $du = \bar{u}$ and $dv = \bar{v}$. A morphism $d : D \to \bar{D}$ satisfying the last two equalities is unique since t is an epimorphism, proving that the square $ur = vs$ is a pushout. □

Note that the crucial part in the proof that the square (2.1) is a pushout in Theorem 5.2.3, is that since **C** has all finite limits, we can take the kernel pair of the morphism $\bar{u}r$ which is not in **E**, which does not always exist in the incomplete relative semi-abelian categories.

Using Theorem 5.2.2 and Corollary 5.1.9 we obtain the equivalent definitions of a relative semi-abelian category:

Corollary 5.2.4. *The following conditions are equivalent:*

(i) The pair $(\mathbf{C}, \mathbf{E})$ *is a relative semi-abelian category.*

(ii) The pair $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 3.2.1,* **C** *has coproducts, and:*

(a) Every morphism in **E** *is a regular epimorphism;*

(b) If $f \in \mathbf{E}$ *then* $\mathrm{coker}(\ker(f)) \in \mathbf{E}$*;*

(c) The **E***-Short Five Lemma holds in* **C***;*

(d) If in a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\
K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B
\end{tikzcd}$$

f, f', *and* u *are in* $\mathbf{E}$, $k = \ker(f)$ *and* $k' = \ker(f')$, *then there exists a morphism* $e : A \to M$ *in* $\mathbf{E}$ *and a monomorphism* $m : M \to A'$ *in* $\mathbf{C}$ *such that* $w = me$.

(e) *Every equivalence* $\mathbf{E}$*-relation* $\mathbf{C}$ *is* $\mathbf{E}$*-effective equivalence* $\mathbf{E}$*-relation.*

(iii) *The pair* $(\mathbf{C}, \mathbf{E})$ *satisfies Condition 3.2.1,* $\mathbf{C}$ *has coproducts, and:*

(a) *Every morphism in* $\mathbf{E}$ *is a normal epimorphism;*

(b) *("Relative Hofmann's axiom") If in a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "w"'] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

f *and* f' *are in* $\mathbf{E}$, w *is a monomorphism,* v *is a normal monomorphism, and* $\ker(f') \leq w$, *then* w *is a normal monomorphism;*

(c) *If in a commutative diagram*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

f, f', *and* u *are in* $\mathbf{E}$, $k = \ker(f)$ *and* $k' = \ker(f')$, *then there exists a morphism* $e : A \to M$ *in* $\mathbf{E}$ *and a monomorphism* $m : M \to A'$ *in* $\mathbf{C}$ *such that* $w = me$.

(d) *For every commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "m'"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

with f *and* f' *in* $\mathbf{E}$ *and* m *and* m' *monomorphisms, if* m *is a normal monomorphism with* $\operatorname{coker}(m) \in \mathbf{E}$ *then* m' *also is a normal monomorphism with* $\operatorname{coker}(m') \in \mathbf{E}$.

Conditions 5.2.4(ii) and 5.2.4(iii) are to be considered, respectively, as the relative versions of what was called "new style" and "old style" axioms for a semi-abelian category in [23].

5.3 Examples

Let us first prove the following

Theorem 5.3.1. *Let* $\mathbf{C}$ *be a semi-abelian category and let* $\mathbf{E}$ *be a class of regular epimorphisms in* $\mathbf{C}$ *satisfying the following conditions:*

(i) If f *and* gf *are in* $\mathbf{E}$ *then* g *is also in* $\mathbf{E}$*;*

(ii) The class $\mathbf{E}$ *is pullback stable;*

(iii) If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

is a pushout diagram and f *and* g *are in* $\mathbf{E}$*, then* h *and* k *are also in* $\mathbf{E}$*;*

(iv) If a morphism f *in* $\mathbf{C}$ *factors as* $f = em$ *in which* e *is in* $\mathbf{E}$ *and* m *is a monomorphism, then it also factors (essentially uniquely) as* $f = m'e'$ *in which* m' *is a monomorphism and* e' *is in* $\mathbf{E}$*;*

(v) If in a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\ K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B \end{tikzcd}$$

$f = \mathrm{coker}(k)$, $f' = \mathrm{coker}(k')$, *there exists factorizations* $k = me$ *and* $k' = m'e'$ *were* e *and* e' *are normal epimorphisms and* m *and* m' *are normal monomorphisms, and,* u *and* f *are in* $\mathbf{E}$*, then* w *is also in* $\mathbf{E}$.

If $\bar{\mathbf{E}}$ *is the closure of* $\mathbf{E}$ *under composition, then* $(\mathbf{C}, \bar{\mathbf{E}})$ *is a relative semi-abelian category.*

Proof. By mathematical induction it suffices to consider the case where $\bar{\mathbf{E}}$ is the class of all those morphisms in $\mathbf{C}$ which can be presented as the composite of some composable pair of morphisms in $\mathbf{E}$.

Since $\mathbf{C}$ is a semi-abelian category, the composite of regular epimorphisms is a regular epimorphism by Proposition 1.1.3, and every regular epimorphism is a normal epimorphism by Proposition 1.3.5. Therefore, $\bar{\mathbf{E}}$ is a class of normal epimorphisms in $\mathbf{C}$ which is closed under composition. Moreover, since the Regular Short Five Lemma holds in $\mathbf{C}$ (see Remark 1.3.4), the $\bar{\mathbf{E}}$-Short Five Lemma also holds, again, since the composite of regular epimorphism in $\mathbf{C}$ is a regular epimorphism. Let us prove the rest of the axioms defining a relative semi-abelian category.

(a) If f and gf are in $\bar{\mathbf{E}}$ then g is also in $\bar{\mathbf{E}}$: Let $f = f_2 f_1$ and $gf = h_2 h_1$ where $f_1 : A \to B$, $f_2 : B \to C$, $h_1 : A \to D$, and $h_2 : D \to F$ are the morphisms in $\mathbf{E}$. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[ddr, "h_1"'] & B \arrow[rrr, "f_2"] \arrow[dr, "\bar{h}_1"] & & & C \arrow[dl, "\bar{\bar{h}}_1"'] \arrow[dd, "g"] \\
& & G \arrow[r, "\bar{f}_2"] \arrow[drr, dotted, "k_1"] & H \arrow[dr, dotted, "k_2"] & \\
& D \arrow[ur, "\bar{f}_1"] \arrow[rrr, "h_2"'] & & & F
\end{tikzcd}$$

in which:

- $\bar{h}_1 f_1 = \bar{f}_1 h_1$ is the pushout of h_1 and f_1 and therefore $\bar{h}_1$ and $\bar{f}_1$ are in $\mathbf{E}$ by condition (iii) of Theorem 5.3.1. Since $h_2 h_1 = g f_2 f_1$ there exists a unique morphism $k_1 : G \to F$ with $k_1 \bar{f}_1 = h_2$ and $k_1 \bar{h}_1 = g f_2$, moreover, by condition (i) of Theorem 5.3.1, k_1 is in $\mathbf{E}$ since so are the morphisms $\bar{f}_1$ and h_2.

- $\bar{f}_2 \bar{h}_1 = \bar{\bar{h}}_1 f_2$ is the pushout of $\bar{h}_1$ and f_2, and therefore $\bar{\bar{h}}_1$ and $\bar{f}_2$ are also in $\mathbf{E}$ by condition (iii) of Theorem 5.3.1. Since f_1 is an epimorphism and $k_1 \bar{h}_1 f_1 = k_1 \bar{f}_1 h_1 = = h_2 h_1 = g f_2 f_1$ we conclude that $k_1 \bar{h}_1 = g f_2$, therefore, there exists a unique morphism $k_2 : H \to F$ with $k_2 \bar{f}_2 = k_1$ and $k_2 \bar{\bar{h}}_1 = g$. Moreover, by condition (i) of Theorem 5.3.1. k_2 is in $\mathbf{E}$ since so are the morphisms $\bar{f}_2$ and k_1.

Therefore, since $g = k_2 \bar{\bar{h}}_1$ and $\bar{\bar{h}}_1$ and k_2 are in $\mathbf{E}$, the morphism g is in $\bar{\mathbf{E}}$, as desired.

(b) The pullback stability of $\bar{\mathbf{E}}$ easily follows from the pullback stability of $\mathbf{E}$. Indeed, let $f : A \to C$ be a morphism in $\bar{\mathbf{E}}$, i.e. $f = f_2 f_1$ where $f_1 : A \to B$ and $f_2 : B \to C$ are the morphisms in $\mathbf{E}$, and let $g : D \to C$ be any morphism in $\mathbf{C}$. Consider the commutative diagram

$$
\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[rr, bend left, "p"] & B \times_C D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D \arrow[d, "g"] \\
A \arrow[r, "f_1"'] \arrow[rr, bend right, "f"'] & B \arrow[r, "f_2"'] & C
\end{tikzcd}
$$

in which $(B \times_C D, \pi_1, \pi_2)$ is the pullback of f_2 and g, (P, p_1, p_2) is the pullback of f_1 and π_1, and $p = \pi_2 p_2$. Since the pullback of f along g is (P, p_1, p) and the morphisms f_1 and f_2 are in $\mathbf{E}$, by pullback stability of $\mathbf{E}$ we obtain that the morphisms π_2 and p_2 are in $\mathbf{E}$, therefore p is in $\bar{\mathbf{E}}$, as desired.

(c) If a morphism f in $\mathbf{C}$ factors as $f = em$ in which e is in $\bar{\mathbf{E}}$ and m is a monomorphism, then it also factors as $f = \bar{m}\bar{e}$ in which $\bar{m}$ is a monomorphism and $\bar{e}$ is in $\bar{\mathbf{E}}$: Let $m : A \to B$ be a monomorphism and let $e : B \to D$ be a morphism in $\bar{\mathbf{E}}$, i.e. $e = e_2 e_1$ where the morphisms $e_1 : B \to C$ and $e_2 : C \to D$ are in $\mathbf{E}$. Using condition (iv) of Theorem 5.3.1 we obtain the commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, dotted, "e_1'"'] \arrow[dd, dotted, bend right=60, "\bar{e}"'] & B \arrow[d, "e_1"] \arrow[dd, bend left=60, "e"] \\
C' \arrow[r, "m'"] \arrow[d, dotted, "e_2'"'] & C \arrow[d, "e_2"] \\
D' \arrow[r, dotted, "\bar{m}"'] & D
\end{tikzcd}
$$

in which m' and $\bar{m}$ are monomorphisms, and e_1' and e_2' are in $\mathbf{E}$. Let $\bar{e} = e_2' e_1'$, then $\bar{e}$ is in $\bar{\mathbf{E}}$ since e_1' and e_2' are in $\mathbf{E}$. Therefore, we have $em = \bar{m}\bar{e}$ in which $\bar{e}$ is a morphism in $\bar{\mathbf{E}}$ and $\bar{m}$ is a monomorphism, as desired.

(d) If in a commutative diagram

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\
K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B
\end{tikzcd}
\tag{3.1}
$$

f, f', and u are in $\bar{\mathbf{E}}$, $k = \ker(f)$ and $k' = \ker(f')$, then w is in $\bar{\mathbf{E}}$: Since u is in $\bar{\mathbf{E}}$ we have $u = u_2 u_1$ where the morphisms u_1 and u_2 are in $\mathbf{E}$. Let (Q, q_1, q_2) be the pushout of u_1 and k, and consider the commutative diagram

$$\begin{tikzcd}
K \arrow[rr, "k"] \arrow[dr, "eu_1"] \arrow[dd, "u_1"'] \arrow[ddd, bend right, "u"'] & & A \arrow[r, "f"] \arrow[dd, "q_1"] \arrow[ddd, bend left, "w"] & B \arrow[ddd, equal] \\
& \bar{U} \arrow[dr, "m"] & & \\
U \arrow[ur, "e"] \arrow[rr, "q_2"] \arrow[d, "u_2"'] & & Q \arrow[uur, dotted, "h_1"'] \arrow[d, dotted, "h_2"] & \\
K' \arrow[rr, "k'"'] & & A' \arrow[r, "f'"'] & B
\end{tikzcd}$$

in which $h_1 : Q \to B$ and $h_2 : Q \to A'$ are the canonical morphisms and $q_2 = me$ is the (normal epi, mono)-factorization of q_2 (i.e. $e : U \to \bar{U}$ is a normal epimorphism and $m : \bar{U} \to Q$ is a monomorphism, and, such a factorization does exist by Proposition 1.2.2). (d) will be proved if we show that q_1 and h_2 are in $\mathbf{E}$, the latter, however, follows from condition (v) of Theorem 5.3.1. Indeed, since $f = \mathrm{coker}(k)$ and the square $q_2 u_1 = q_1 k$ is a pushout, we conclude that $h_1 = \mathrm{coker}(q_2)$. Moreover, since k is a normal monomorphism and eu_1 and q_1 are normal epimorphisms, and m is a monomorphism, we conclude that m is a normal monomorphism (see Condition 1.5.3(e)). Then, since u_1 is in $\mathbf{E}$, the morphism q_1 is also in $\mathbf{E}$ by condition (v) of Theorem 5.3.1. Furthermore, since f is in $\mathbf{E}$, the latter implies that h_1 is also in $\mathbf{E}$ by condition (i) of Theorem 5.3.1, and therefore, again by condition (v) of Theorem 5.3.1 we obtain that h_2 is in $\mathbf{E}$, as desired.

(e) For every commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "m'"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

with f and f' in $\bar{\mathbf{E}}$ and m and m' monomorphisms, if m is a normal monomorphism with $\mathrm{coker}(m) \in \bar{\mathbf{E}}$ then m' also is a normal monomorphism with $\mathrm{coker}(m') \in \bar{\mathbf{E}}$: Let $q : A' \to Q$

be the cokernel of m and let (Q', f'', q') be the pushout of f' and q. Since q is in $\bar{\mathbf{E}}$ we have $q = q_2 q_1$ where the morphisms q_1 and q_2 are in $\mathbf{E}$. Consider the commutative diagram

$$\begin{tikzcd}
& A \arrow[rr, "f"] \arrow[d, "m"'] & & B \arrow[d, "m'"] \\
& A' \arrow[rr, "f'"] \arrow[dl, "q_1"'] \arrow[dd, "q"] & & B' \arrow[dl, "q_1'"'] \arrow[dd, "q'"] \\
X \arrow[rr, "x"] \arrow[dr, "q_2"'] & & X' \arrow[dr, dotted, "q_2'"'] & \\
& Q \arrow[rr, "f''"'] & & Q'
\end{tikzcd}$$

in which (X', x, q_1') is the pushout of f' and q_1, and $q_2' : X' \to Q'$ is the canonical morphism. Since (Q', f'', q') and (X', x, q_1') are the pushouts, the square $f''q_2 = q_2'x$ is also a pushout. Since q_1 is in $\mathbf{E}$ and f' is in $\bar{\mathbf{E}}$, it easily follows from condition (iii) of Theorem 5.3.1 that q_1' is in $\mathbf{E}$ and x is in $\bar{\mathbf{E}}$, and, f'' is in $\bar{\mathbf{E}}$ and q_2' is in $\mathbf{E}$; therefore, q' is in $\bar{\mathbf{E}}$. It remains to prove that $q' = \mathrm{coker}(m')$. However, since $\mathbf{C}$ is a semi-abelian category, by Theorem 1.2.5, the canonical morphism $\langle q, f' \rangle : A' \to Q \times_{Q'} B'$, where $(Q \times_{Q'} B', \pi_1, \pi_2)$ is the pullback of f'' and q', is a regular epimorphism; therefore, the proof of $q' = \mathrm{coker}(m')$ follows from the first part of the proof of Theorem 5.1.8.

After this, it follows from Corollary 5.2.4 that $(\mathbf{C}, \bar{\mathbf{E}})$ is a relative semi-abelian category. $\square$

Proposition 5.3.2. *If* $\mathbf{C}$ *is a semi-abelian category and* $\mathbf{E}$ *is the class of all central extensions, in the sense of Huq [21], in* $\mathbf{C}$*; more precisely, if* $\mathbf{E}$ *is the class of normal epimorphisms* $f : A \to B$ *with* $[\mathrm{Ker}(f), A] = 0$*, where* $[\mathrm{Ker}(f), A]$ *denotes the commutator of* $\mathrm{Ker}(f)$ *and* A *in the sense of Huq [21]), then* $(\mathbf{C}, \bar{\mathbf{E}})$*, where* $\bar{\mathbf{E}}$ *is defined as in Theorem 5.3.1, is a relative semi-abelian category.*

Proof. Let $\mathbf{C}$ be a semi-abelian category, let $\mathbf{E}$ be the class of all central extensions in $\mathbf{C}$, and let $\bar{\mathbf{E}}$ be the closure of $\mathbf{E}$ under composition. As follows from Theorem 5.3.1, to prove that $(\mathbf{C}, \bar{\mathbf{E}})$ is a relative semi-abelian category it suffices to prove that $(\mathbf{C}, \mathbf{E})$ satisfies conditions (i)-(v) of Theorem 5.3.1.

Central extensions are pullback stable since they are covering maps in the sence of categorical Galois theory (see e.g. [5], Corollary 6.6.2). Let us prove the rest of the conditions of Theorem 5.3.1.

(a) If f and gf are in $\mathbf{E}$, then g is also in $\mathbf{E}$: Let $f : A \to B$ and $gf : A \to C$ be the central extensions in $\mathbf{C}$ (note that in the proof we will not use the fact that f is a central extension, it suffices for f to be a (pullback stable) regular epimorphism) and let k_f, k_g, and k_{gf} be the kernels of f, g, and gf respectively. We obtain the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Ker}(gf) \arrow[r, dotted, "\bar{f}"] \arrow[d, "k_{gf}"'] & \mathrm{Ker}(g) \arrow[d, "k_g"'] & \\
\mathrm{Ker}(f) \arrow[ur, dotted, "l"] \arrow[r, "k_f"] & A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

in which $l : \mathrm{Ker}(f) \to \mathrm{Ker}(gf)$ and $\bar{f} : \mathrm{Ker}(gf) \to \mathrm{Ker}(g)$ are the canonical morphisms. It easily follows that the square $fk_{gf} = k_g\bar{f}$ is a pullback, therefore $\bar{f}$ is a normal epimorphism (since f is a normal epimorphism and normal epimorphisms in a semi-abelian category are pullback stable) and $l = \ker(\bar{f})$.

Since gf is a central extension, there exists a unique morphism $\varphi_{gf} : \mathrm{Ker}(gf) \times A \to A$ such that the diagram

$$\begin{tikzcd}
\mathrm{Ker}(gf) \arrow[r, "\langle 1{,}0\rangle"] \arrow[dr, "k_{gf}"'] & \mathrm{Ker}(gf) \times A \arrow[d, dotted, "\varphi_{gf}"] & A \arrow[l, "\langle 0{,}1\rangle"'] \arrow[dl, "1_A"] \\
 & A &
\end{tikzcd}$$

commutes; and, to prove that g is a central extension, we need to prove the existence of a unique morphism $\varphi_g : \mathrm{Ker}(g) \times B \to B$ making the diagram

$$\begin{tikzcd}
\mathrm{Ker}(g) \arrow[r, "\langle 1{,}0\rangle"] \arrow[dr, "k_g"'] & \mathrm{Ker}(g) \times B \arrow[d, dotted, "\varphi_g"] & B \arrow[l, "\langle 0{,}1\rangle"'] \arrow[dl, "1_B"] \\
 & B &
\end{tikzcd}$$

commutative. For, consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[rr, "\langle 1,0\rangle"] \arrow[d, "l"'] & & \mathrm{Ker}(f)\times\mathrm{Ker}(f) \arrow[d, "l\times k_f"] & & \mathrm{Ker}(f) \arrow[ll, "\langle 0,1\rangle"'] \arrow[d, "k_f"] \\
\mathrm{Ker}(gf) \arrow[rr, "\langle 1,0\rangle"] \arrow[d, "\bar{f}"'] \arrow[ddr, "k_{gf}"] & & \mathrm{Ker}(gf)\times A \arrow[d, "\bar{f}\times f"] \arrow[ddl, "\varphi_{gf}"'] & & A \arrow[ll, "\langle 0,1\rangle"'] \arrow[d, "f"] \arrow[ddlll, bend left, "1_A"] \\
\mathrm{Ker}(g) \arrow[rr, "\langle 1,0\rangle"] \arrow[ddrr, bend right, "k_g"'] & & \mathrm{Ker}(g)\times B \arrow[dd, dotted, "\varphi_g"'] & & B \arrow[ll, "\langle 0,1\rangle"'] \arrow[ddll, "1_B"] \\
& A \arrow[dr, "f"] & & & \\
& & B & &
\end{tikzcd}$$

Since $l = \ker(\bar{f})$ and $k_f = \ker(f)$, we conclude that $l \times k_f = \ker(\bar{f} \times f)$; moreover, since $\bar{f}$ and f are normal epimorphisms, $\bar{f} \times f$ is a also a normal epimorphism and therefore $\bar{f} \times f = \mathrm{coker}(l \times k_f)$. Since $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are jointly epic and the equalities

$$f\varphi_{gf}(l \times k_f)\langle 1, 0\rangle = f\varphi_{gf}\langle 1, 0\rangle l = fk_{gf}l = k_g\bar{f}l = 0,$$

$$f\varphi_{gf}(l \times k_f)\langle 0, 1\rangle = f\varphi_{gf}\langle 0, 1\rangle k_f = f1_Ak_f = 0$$

hold, we conclude that $f\varphi_{gf}(l \times k_f) = 0$. Therefore, since $\bar{f} \times f = \mathrm{coker}(l \times k_f)$, there exists a unique morphism $\varphi_g : \mathrm{Ker}(g) \times B \to B$ with $\varphi_g(\bar{f} \times f) = f\varphi_{gf}$. It remains to prove that $\varphi_g\langle 1, 0\rangle = k_g$ and $\varphi_g\langle 0, 1\rangle = 1_B$. However, since f and $\bar{f}$ are epimorphisms, the latter follows from the following equalities:

$$\varphi_g\langle 1, 0\rangle\bar{f} = \varphi_g(\bar{f} \times f)\langle 1, 0\rangle = f\varphi_{gf}\langle 1, 0\rangle = fk_{gf} = k_g\bar{f},$$

$$\varphi_g\langle 0, 1\rangle f = \varphi_g(\bar{f} \times f)\langle 0, 1\rangle = f\varphi_{gf}\langle 0, 1\rangle = 1_Bf.$$

(b) If

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\
C \arrow[r, "h"'] & D
\end{tikzcd}$$

is a pushout diagram and f and g are in $\mathbf{E}$, then h and k are also in $\mathbf{E}$: We shall only prove that h is a central extension, as the proof for k being the central extension is similar. Let f and g be central extensions in $\mathbf{C}$ and let $k_f : \mathrm{Ker}(f) \to A$ and $k_h : \mathrm{Ker}(h) \to C$ be the kernels of f and h respectively. We obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, dotted, "l"'] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\
\mathrm{Ker}(h) \arrow[r, "k_h"'] & C \arrow[r, "h"'] & D
\end{tikzcd}$$

where $l : \mathrm{Ker}(f) \to \mathrm{Ker}(h)$ is the canonical morphism. Since f and g are regular epimorphisms, h and k are also regular epimorphisms, and therefore they are normal epimorphisms since $\mathbf{C}$ is a semi-abelian category. Since the canonical morphism $\langle g, f\rangle : A \to C \times_D B$ is a normal epimorphism (by Theorem 1.2.5) and normal epimorphisms are pullback stable, we conclude that l is a normal epimorphism and therefore $l = \mathrm{coker}(k_l)$ where $k_l : \mathrm{Ker}(l) \to \mathrm{Ker}(f)$ is the kernel of l.

Since f is a central extension, there exists a unique morphism $\varphi_f : \mathrm{Ker}(f) \times A \to A$ such that the diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\langle 1{,}0\rangle"] \arrow[dr, "k_f"'] & \mathrm{Ker}(f) \times A \arrow[d, dotted, "\varphi_f"] & A \arrow[l, "\langle 0{,}1\rangle"'] \arrow[dl, "1_A"] \\
& A &
\end{tikzcd}$$

commutes; and to prove that h is a central extension, we need to prove the existence of a unique morphism $\varphi_h : \mathrm{Ker}(h) \times C \to C$ making the diagram

$$\begin{tikzcd}
\mathrm{Ker}(h) \arrow[r, "\langle 1{,}0\rangle"] \arrow[dr, "k_h"'] & \mathrm{Ker}(h) \times C \arrow[d, dotted, "\varphi_h"] & C \arrow[l, "\langle 0{,}1\rangle"'] \arrow[dl, "1_C"] \\
& C &
\end{tikzcd}$$

commutative. For, consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{Ker}(l) \arrow[rr, "\langle 1,0\rangle"] \arrow[d, "k_l"'] & & \mathrm{Ker}(l)\times\mathrm{Ker}(g) \arrow[d, "k_l\times k_g"] & & \mathrm{Ker}(g) \arrow[ll, "\langle 0,1\rangle"'] \arrow[d, "k_g"] \\
\mathrm{Ker}(f) \arrow[rr, "\langle 1,0\rangle"] \arrow[d, "l"'] \arrow[ddr, "k_f"] & & \mathrm{Ker}(f)\times A \arrow[d, "l\times g"] \arrow[ddl, "\varphi_f"'] & & A \arrow[ll, "\langle 0,1\rangle"'] \arrow[d, "g"] \arrow[ddlll, bend left, "1_A"] \\
\mathrm{Ker}(h) \arrow[rr, "\langle 1,0\rangle"] \arrow[ddrr, bend right, "k_h"'] & & \mathrm{Ker}(h)\times C \arrow[dd, dotted, "\varphi_h"'] & & C \arrow[ll, "\langle 0,1\rangle"'] \arrow[ddll, "1_C"] \\
& A \arrow[dr, "g"] & & & \\
& & C & &
\end{tikzcd}$$

Since $k_l = \ker(l)$ and $k_g = \ker(g)$, we conclude that $k_l \times k_g = \ker(l \times g)$; moreover, since l and g are normal epimorphisms, $l \times g$ is a also a normal epimorphism and therefore $l \times g = \mathrm{coker}(k_l \times k_g)$. Since $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are jointly epic and the equalities

$$g\varphi_f(k_l \times k_g)\langle 1, 0\rangle = g\varphi_f\langle 1, 0\rangle k_l = gk_f k_l = k_h l k_l = 0,$$

$$g\varphi_f(k_l \times k_g)\langle 0, 1\rangle = g\varphi_f\langle 0, 1\rangle k_g = g1_A k_g = 0$$

hold, we conclude that $g\varphi_f(k_l \times k_g) = 0$. Therefore, since $l \times g = \mathrm{coker}(k_l \times k_g)$, there exists a unique morphism $\varphi_h : \mathrm{Ker}(h) \times C \to C$ with $\varphi_h(l \times g) = g\varphi_f$. It remains to prove that $\varphi_h\langle 1, 0\rangle = k_h$ and $\varphi_h\langle 0, 1\rangle = 1_C$. However, since l and g are epimorphisms, the latter follows from the following equalities:

$$\varphi_h\langle 1, 0\rangle l = \varphi_h(l \times g)\langle 1, 0\rangle = g\varphi_f\langle 1, 0\rangle = gk_f = k_h l,$$

$$\varphi_h\langle 0, 1\rangle g = \varphi_h(l \times g)\langle 0, 1\rangle = g\varphi_f\langle 0, 1\rangle = g.$$

(c) If a morphism f in $\mathbf{C}$ factors as $f = em$ in which e is in $\mathbf{E}$ and m is a monomorphism, then it also factors as $f = m'e'$ in which m' is a monomorphism and e' is in $\mathbf{E}$: Let $f = em$ in which $m : A \to B$ is a monomorphism and $e : B \to C$ is a central extension in $\mathbf{C}$. Since $\mathbf{C}$ is a semi-abelian category, we have the factorization $f = m'e'$ in which $e' : A \to B'$ is a regular epimorphism and $m' : B' \to C$ is a monomorphism; (c) will be proved if we show that e' is a central extension.

Let $k_e : \mathrm{Ker}(e) \to B$ be the kernel of e and let $k_{e'} : \mathrm{Ker}(e') \to A$ be the kernel of e'. We obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(e') \arrow[r, dotted, "n"] \arrow[d, "k'_e"'] & \mathrm{Ker}(e) \arrow[d, "k_e"] \\
A \arrow[r, "m"] \arrow[d, "e'"'] \arrow[dr, "f"'] & B \arrow[d, "e"] \\
B' \arrow[r, "m'"'] & C
\end{tikzcd}$$

in which $n : \mathrm{Ker}(e') \to \mathrm{Ker}(e)$ is the canonical morphism.

Since e is a central extension, there exists a unique morphism $\varphi_e : \mathrm{Ker}(e) \times B \to B$ such that the diagram

$$\begin{tikzcd}
\mathrm{Ker}(e) \arrow[r, "\langle 1{,}0 \rangle"] \arrow[dr, "k_e"'] & \mathrm{Ker}(e) \times B \arrow[d, dotted, "\varphi_e"] & B \arrow[l, "\langle 0{,}1 \rangle"'] \arrow[dl, "1_B"] \\
& B &
\end{tikzcd}$$

commutes; and, to prove that e' is a central extension, we need to prove the existence of a unique morphism $\varphi_{e'} : \mathrm{Ker}(e') \times A \to A$ making the diagram

$$\begin{tikzcd}
\mathrm{Ker}(e') \arrow[r, "\langle 1{,}0 \rangle"] \arrow[dr, "k_{e'}"'] & \mathrm{Ker}(e') \times A \arrow[d, dotted, "\varphi_{e'}"] & A \arrow[l, "\langle 0{,}1 \rangle"'] \arrow[dl, "1_A"] \\
& A &
\end{tikzcd}$$

commutative. For, consider the following commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Ker}(h_1) \arrow[d, "k_{h_1}"] & & \\
 & & \mathrm{Ker}(e')+A \arrow[d, "h_1"] \arrow[ddl, bend right, "h_2"'] & & \\
\mathrm{Ker}(e') \arrow[urr, "\iota_1"] \arrow[rr, "\langle 1,0\rangle"'] \arrow[dr, "k_{e'}"'] \arrow[dd, "n"'] & & \mathrm{Ker}(e')\times A \arrow[dl, dotted, "\varphi_{e'}"] \arrow[dd, "n\times m"] & & A \arrow[ull, "\iota_2"'] \arrow[ll, "\langle 0,1\rangle"] \arrow[dlll, bend left, "1_A"] \arrow[dd, "m"] \\
 & A \arrow[ddr, bend right, "m"'] & & & \\
\mathrm{Ker}(e) \arrow[rr, "\langle 1,0\rangle"] \arrow[drr, bend right, "k_e"'] & & \mathrm{Ker}(e)\times B \arrow[d, "\varphi_e"] & & B \arrow[ll, "\langle 0,1\rangle"'] \arrow[dll, "1_B"] \\
 & & B & &
\end{tikzcd}$$

in which:

- $\iota_1 : \mathrm{Ker}(e') \to \mathrm{Ker}(e') + A$ and $\iota_2 : A \to \mathrm{Ker}(e') + A$ are the coproduct injections.
- $h_1 = [\langle 1, 0\rangle, \langle 0, 1\rangle] : \mathrm{Ker}(e') + A \to \mathrm{Ker}(e') \times A$ and $h_2 = [k_{e'}, 1_A] : \mathrm{Ker}(e') + A \to A$ are the canonical morphisms; since **C** is a semi-abelian category, h_1 is a normal epimorphism.
- $k_{h_1} : \mathrm{Ker}(h_1) \to \mathrm{Ker}(e') + A$ is the kernel of h_1; since h_1 is a normal epimorphism we conclude that $h_1 = \mathrm{coker}(k_{h_1})$.
- Since m is a monomorphism and $mh_2k_{h_1} = \varphi_e(n \times m)h_1k_{h_1} = 0$ we conclude that $h_2k_{h_1} = 0$.
- Since $h_1 = \mathrm{coker}(k_{h_1})$ and $h_2k_{h_1} = 0$, there exists a unique morphism $\varphi_{e'} : \mathrm{Ker}(e') \times A \to A$ with $\varphi_{e'}h_1 = h_2$.

It remains to prove that $\varphi_{e'}\langle 1, 0\rangle = k_{e'}$ and $\varphi_{e'}\langle 0, 1\rangle = 1_A$. The latter, however, easily follows from the equalities:

$$\varphi_{e'}\langle 1, 0\rangle = \varphi_{e'}h_1\iota_1 = h_2\iota_1 = k_{e'},$$

$$\varphi_{e'}\langle 0,1\rangle = \varphi_{e'}h_1\iota_2 = h_2\iota_2 = 1_A.$$

(d) If in a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, equal] \\
K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B
\end{tikzcd} \tag{3.2}$$

$f = \mathrm{coker}(k)$, $f' = \mathrm{coker}(k')$, there exists factorizations $k = me$ and $k' = m'e'$ were e and e' are normal epimorphisms and m and m' are normal monomorphisms, and, u and f are in $\mathbf{E}$, then w is also in $\mathbf{E}$: Let us prove a more general fact, namely, if a normal epimorphism h is a central extension in $\mathbf{C}$ and $h = gf$ where $f : A \to B$ and $g : B \to C$ are normal epimorphisms, then f is also a central extension. For, let $k_h : \mathrm{Ker}(h) \to A$ and $k_g : \mathrm{Ker}(g) \to B$ be the kernels of h and g respectively. We obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(h) \arrow[r, "k_h"] \arrow[d, dotted, "\bar{f}"'] & A \arrow[r, "h"] \arrow[d, "f"] & C \arrow[d, equal] \\
\mathrm{Ker}(g) \arrow[r, "k_g"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

in which $\bar{f} : \mathrm{Ker}(h) \to \mathrm{Ker}(g)$ is the canonical morphism; the square $k_g\bar{f} = fk_h$ is a pullback by Lemma 4.1.3(i) and therefore $\bar{f}$ and f have isomorphic kernels. Let $k_f : \mathrm{Ker}(f) \to A$ be the kernel of f and let $k_{\bar{f}} : \mathrm{Ker}(f) \to \mathrm{Ker}(h)$ be the kernel of $\bar{f}$.

Since h is a central extension, there exists a unique morphism $\varphi_h : \mathrm{Ker}(h) \times A \to A$ such that the diagram

$$\begin{tikzcd}
\mathrm{Ker}(h) \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "k_h"'] & \mathrm{Ker}(h) \times A \arrow[d, dotted, "\varphi_h"] & A \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "1_A"] \\
 & A &
\end{tikzcd}$$

commutes; and to prove that f is a central extension, we need to prove the existence of a unique morphism $\varphi_f : \mathrm{Ker}(f) \times A \to A$ making the diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "k_f"'] & \mathrm{Ker}(f) \times A \arrow[d, dotted, "\varphi_f"] & A \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "1_A"] \\
 & A &
\end{tikzcd}$$

commutative. However, since the diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\langle 1,0\rangle"] \arrow[d, "k_{\bar{f}}"'] \arrow[ddr, bend left, "k_f"] & \mathrm{Ker}(f)\times A \arrow[d, "k_f\times 1_A"] & A \arrow[l, "\langle 0,1\rangle"'] \arrow[d, equal] \\
\mathrm{Ker}(h) \arrow[r, "\langle 1,0\rangle"'] \arrow[dr, "k_h"'] & \mathrm{Ker}(h)\times A \arrow[d, "\varphi_h"] & A \arrow[l, "\langle 0,1\rangle"] \arrow[dl, "1_A"] \\
& A &
\end{tikzcd}$$

commutes, we can simply take $\varphi_f = \varphi_h(k_f \times 1_A)$; then $\varphi_f\langle 1,0\rangle = \varphi_f(k_f \times 1_A)\langle 1,0\rangle =$ $= k_h k_{\bar{f}} = k_f$ and $\varphi_f\langle 1,0\rangle = 1_A$, proving the desired. $\square$

Remark 5.3.3. *As shown by M. Gran and T. Van der Linden in [17], the class* $\mathbf{E}$ *in Proposition 5.3.2 coincides with the class of central extensions in* $\mathbf{C}$ *in the sense of [22] with respect to its abelianization reflection* $\mathbf{C} \to \mathrm{Ab}(\mathbf{C})$.

Example 5.3.4. *Let* $\mathbf{C}$ *be a homological category and let* $\mathbf{S}$ *be a class of objects in* $\mathbf{C}$ *satisfying the following conditions:*

(i) $\mathbf{S}$ *is closed under subobjects, i.e. if* $m : S \to A$ *is a monomorphism and* A *is in* $\mathbf{S}$*, then* S *also is in* $\mathbf{S}$*;*

(ii) $\mathbf{S}$ *is closed under cokernels, i.e. if* $0 \to A \to B \to C \to 0$ *is a short exact sequence in* $\mathbf{C}$ *and* A *and* B *are in* $\mathbf{S}$*, then* C *also is in* $\mathbf{S}$*;*

(iii) $\mathbf{S}$ *is closed under extensions, i.e. if* $0 \to A \to B \to C \to 0$ *is a short exact sequence in* $\mathbf{C}$ *and* A *and* C *are in* $\mathbf{S}$*, then* B *also is in* $\mathbf{S}$*;*

(iv) Every equivalence relation $(R, r_1, r_2) : A \to A$ *with* $\mathrm{Ker}(r_1) \in \mathbf{S}$ *is an effective equivalence relation.*

As easily follows from Corollary 5.2.2, if $\mathbf{E}$ *is the class of all normal epimorphisms in* $\mathbf{C}$ *whose kernels are in* $\mathbf{S}$*, then* $(\mathbf{C}, \mathbf{E})$ *is a relative semi-abelian category. In particular, we can take:*

(a) $\mathbf{C}$ *to be a semi-abelian category and* $\mathbf{S}$ *to be any class of objects in* $\mathbf{C}$ *satisfying conditions (i)-(iii) (if* $\mathbf{C}$ *were abelian, this would mean that* $\mathbf{S}$ *is a Serre class).*

(b) $\mathbf{C}$ *to be a homological additive category and* $\mathbf{S}$ *to be a class of objects in* $\mathbf{C}$ *satisfying conditions (i)-(iii), such that every monomorphism* $m : S \to A$ *where* S *is in* $\mathbf{S}$ *is a normal monomorphism. For instance, we could take* $\mathbf{C}$ *to be a category of abelian Hausdorf topological groups and* $\mathbf{S}$ *to be the class of all finite abelian Hausdorf topological groups.*

Remark 5.3.5. *Recall that* $(\mathbf{E}, \mathbf{M})$*-normal categories in the sense of M. M. Clementino, D. Dikranjan, and W. Tholen [13] are also a kind of "relative semi-abelian categories". However, that relativization with respect to a factorization system* $(\mathbf{E}, \mathbf{M})$ *is quite different. Indeed, we observe:*

(a) If $\mathbf{C}$ *is an* $(\mathbf{E}, \mathbf{M})$*-normal category in which* $\mathbf{E}$ *is contained in the class of normal epimorphisms of* $\mathbf{C}$*, then* $\mathbf{E}$ *is the class of all normal epimorphisms,* $\mathbf{M}$ *is the class of all monomorphisms, and* $\mathbf{C}$ *is a semi-abelian category.*

(b) The same is true if $(\mathbf{C}, \mathbf{E})$ *is a relative semi-abelian category in which* $(\mathbf{E}, \mathbf{M})$ *forms a proper factorization system for some* $\mathbf{M}$*.*

(c) Therefore, the two relativizations have "trivial intersection", i.e. if $\mathbf{C}$ *is an* $(\mathbf{E}, \mathbf{M})$*-normal category and* $(\mathbf{C}, \mathbf{E})$ *is a relative semi-abelian category at the same time, then again,* $\mathbf{C}$ *simply is a semi-abelian category with* $(\mathbf{E}, \mathbf{M}) = (\text{Normal Epi}, \text{Mono})$*.*

Bibliography

[1] M. Barr, Exact categories, Springer Lect. Notes in Math. 236 (1971) 11-20.

[2] F. Borceux, Handbook of categorical algebra, Cambridge University Press, 1994.

[3] F. Borceux and D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, Mathematics and its Applications, Kluwer (2004).

[4] F. Borceux and M. M. Clementino, Topological semi-abelian algebras, Advances in Mathimatics, 190 (2005) 425-453.

[5] F. Borceux G. Janelidze, Galois Theories, Cambridge Studies in Advanced Mathematics, Vol. 72, Cambridge University Press, 2001.

[6] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, Lecture notes in mathematics 1448, Springer 1991, 43-62.

[7] D. Bourn, Mal'cev categories and fibrations of pointed object, Applied Categorical Structures, 4 (1996) 307-327.

[8] D. Bourn and G. Janelidze, Protomodularity, descent, and semidirect products, Theory and Applications of Categories, 4 (2) (1998) 37-46.

[9] D. A. Buchsbaum, Exact categories and duality, Transactions of American Mathematical Society, 801(955) 1-34.

[10] A. Carboni, G.M. Kelly, and M.C. Pedicchio, Some remarks on Mal'tsev and Goursat categories, Applied Categorical Structures, 1 (1993) 385-421.

[11] A. Carboni, J. Lambek, and M.C. Pedicchio, Diagram chasing in Mal'tsev categories, Journal of Pure and Applied Algebra, 69 (1990) 271-284.

[12] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton 1956.

[13] M. M. Clementino, D. Dikranjan, and W. Tholen, Torsion theories and radicals in normal categories, Journal of Algebra, 305 (2006) 98-129.

[14] S. Eilenberg and J. C. Moore, Foundations of relative homological algebra, Memoirs of the American Mathematical Society, 55 (1965).

[15] T. H. Fay, On categorical conditions for congruences to commute, Algebra Universalis, 8 (1978) 173-179.

[16] P. Freyd, Abelian categories, Harper and Row, 1964.

[17] M. Gran and T. Van der Linden, On the second cohomology group in semi-abelian categories, Journal of Pure and Applied Algebra, 212 (2008) 636-651.

[18] H. Herrlich, G. E. Strecker, Category Theory, Allyn Bacon, Boston, 1973.

[19] F. Hofmann, Uber eine die Kategorie der Gruppen umfassande Kategorie, Sitzungsberichte Bayerische Akad. Wissenschaften, Math. Naturw. Klasse, (1960) 163-204.

[20] G. Hochschild and D. Wigner, Abstractly split group extensions, Pacific Jounal of Mathematics, 68 (2) (1977).

[21] S. A. Huq, Commutator, nilpotency and solvability in categories, Quart. J. Math. Oxford, 19 (2) (1968) 363389.

[22] G. Janelidze and G. M. Kelly, Galois theory and a general notion of central extension, Journal of Pure and Applied Algebra, 97 (1994) 135161.

[23] G. Janelidze, L. Márki, and W. Tholen, Semi-abelian cateogires, Journal of Pure and Applied Algebra, 168 (2002) 367-386.

[24] T. Janelidze, Relative homological categories, Journal of Homotopy and Related Structures, 1 (1) (2006) 185-194.

[25] T. Janelidze, Relative semi-abelian categories, Applied Categorical Structures, 17 (2009) 373-386.

[26] T. Janelidze, Incomplete relative semi-abelian categories, Applied Categorical Structures, accepted and available online from 17 March 2009.

[27] T. Janelidze, Snake Lemma in incomplete relative homological categories, Theory and Application of Categories, accepted.

[28] S. Mac Lane, An algebra of additive relations, Proc. Nat. Acad. Sci. USA, 47 (1961) 1043-1051.

[29] S. Mac Lane, Duality for groups, Bulletin of American Mathematical Society, 56 (1950) 485-516.

[30] S. Mac Lane, Homology, Springer-Verlag, Berlin, Heidelberg, New York, 1963.

[31] C. C. Moore, Group extensions and cohomology for locally compact groups. III, Transactions of the American Mathematical Society, 221 (1) (1976) 1-33.

[32] J.-P. Serre, Galois cohomology, Springer-Verlag, 1997.

[33] D. Wigner, Algebraic cohomology of topological groups, Transactions of the American Mathematical Society, 178 (1973) 83-93.

[34] N. Yoneda, On Ext and exact sequences, J. Fac. Sci. Univ. Tokyo, 1 (8) (1961) 507-576.