NOTES ON LOGOI

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To the memory of Jon Beck

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Introduction

The notion of logos was introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. The following notes are a collection of assertions on logoi, many of which have not yet been formally proved. The unproven statements can be regarded as open problems or conjectures. Our goal is to show that category theory can be extended to logoi, and that the extension is natural. The extended theory has applications to homotopy theory, homotopical algebra, higher category theory and higher topos theory. Part of the material is taken from a book under preparation [J2], other parts from the talks I gave on the subject during the last decade, and the rest is taken from the literature. A first draft of the notes was written in 2004 in view of its publication in the Proceedings of the Conference on higher categories held at the IMA in Minneapolis. An expanded version was used in a course given at the Fields Institute in January 2007. The latest version was used in a course given at the CRM in Barcelona in February 2008.

A Kan complex and the nerve of a category are basic examples of logoi. A logos is sometime called a *weak Kan complex* in the literature [KP]. We have introduced the term *quasi-categories* to stress the analogy with categories. We now prefer using the term *logos* because it is shorter.

Logoi abound. The coherent nerve of a category enriched over Kan complexes is a logos. The left (and right) barycentric localisations of a model category is a logos. A logos can be large. For example, the coherent nerve of the category of Kan complexes is a large logos **U**. The fact that category theory can be extended to logoi is not obvious a priori, but can discovered by working on the subject. It is essentially an experimental fact.

Logoi are examples of $(\infty, 1)$-categories in the sense of Baez and Dolan. Other examples are simplicial categories, Segal categories and complete Segal spaces (here called Rezk categories). Simplicial categories were introduced by Dwyer and Kan in their work on simplicial localisation. Segal categories were introduced by Schwnzel and Vogt under the name of Δ-categories [ScVo] and were rediscovered by Hirschowitz and Simpson in their work on higher stacks. Complete Segal spaces were introduced by Rezk in his work on homotopy theories. To each of these examples is associated a model category and the four model categories are Quillen equivalent. The equivalence between simplicial categories, Segal categories and complete Segal spaces was established by Bergner [B2]. The equivalence between logoi, Segal categories and complete Segal spaces was established by Tierney and the author [JT2]. The equivalence between simplicial categories and logoi was established by Lurie [Lu1] and independantly by the author [J4]. Many aspects of category theory were extended to Segal categories by Hirschowitz, Simpson, Toen and Vezzosi. For examples, a notion of Segal topos was introduced by Simpson and a notion of homotopy topos by Toen and Vezzosi along ideas of Rezk. The development of derived algebraic geometry by Toen and Vezzosi [TV1] and independantly by Lurie [Lu1] is one of the main incentive for developing the theory of logoi. A notion of stable Segal category was introduced by Hirschowitz, Simpson and Toen. Lurie has recently formulated his work on ∞-topoi (here called *homotopoi*) in the language of logoi [Lu1]. In doing so, he has extended a considerable amount of category theory to logoi. Our notes may serve as an introduction to his work [Lu2][Lu3] [Lu4] [Lu5]. Many ideas introduced in the notes are due to Charles Rezk. The notion of reduced category is inspired by his notion of complete Segal space, and the notion of *homotopos* is a reformulation of his notion of homotopy topos. The observation that every diagram in a homotopos is a descent diagram is due to him.

Remark: the list $(\infty, 1)$-categories given above is not exhaustive and our account of the history of the subject is incomplete. The notion of A_∞-space introduced by Stasheff is a seminal idea in the whole subject. A theory of A_∞-categories was developped by Batanin [Bat1]. Many aspects of category theory were extended to simpicial categories by Bousfield, Dwyer and Kan, and also by Cordier and Porter [CP2]. The theory of homotopical categories by Dwyer, Hirschhorn, Kan and Smith is closely related to that of logoi [DHKS].

The theory of logoi depends on homotopical algebra for its formulation. A basic result states that the category of simplicial sets $\mathbf{S}$ admits a Quillen model structure in which the fibrant objects are the logoi (and the cofibration are the monomorphisms). This defines the *model structure for logoi.* The classical model structure on $\mathbf{S}$ is a Bousfield localisation of this model structure. Many aspects of category theory can be formulated in the language of homotopical algebra. For example, the category of small categories $\mathbf{Cat}$ admits a *natural* model structure in which the weak equivalences are the equivalence of categories. The corresponding notion of homotopy limit is closely related to the notion of pseudo-limit defined by category theorists.

Many aspects of homotopical algebra become simpler and more conceptual when formulated in the language of logoi. This is true for example of the theory of homotopy limits and colimits which is equivalent to the theory of limits and colimits in a logos. We hope that this reformulation of homotopical algebra will greatly simplify and clarify the proofs in this subject. This may not be entirely clear at present because the theory of logoi is presently in its infancy. A mathematical theory is a kind of social construction, and the complexity of a proof depends on the degree of maturity of the subject. What is considered to be "obvious" is the result of an implicit agreement between the experts based on their knowledge and experience.

The logos $\mathbf{U}$ has many properties in common with the category of sets. For example, it is cocomplete and freely generated by one object (its terminal object). It is the archetype of a *homotopos*, A *prestack* on a simplicial set A is defined to be a map $A^o \to \mathbf{U}$. A general homotopos is a left exact reflection of a logos of prestacks. Homotopoi can be characterized abstractly by a system of axioms similar to those of Giraud for a Grothendieck topos. They also admit an elegant characterization due to Lurie in terms of a descent property discovered by Rezk.

All the machinery of universal algebra can be extended to homotopy theory via the theory of logoi. An *algebraic theory* is defined to be a small logos with finite products T, and a *model* of T to be a map $T \to \mathbf{U}$ which preserves finite products. The models $T \to \mathbf{U}$ form a large logos $Mod(T)$ which is complete an cocomplete. A *variety of homotopy algebras* is a logos equivalent to a logos $Mod(T)$ for some algebraic theory T. The homotopy varieties can be characterized by system of axioms closely related to those of Rosicky [Ros]. The notion of algebraic structure was extended by Ehresman to include *essentially algebraic structures* defined by a *limit sketch*. The classical theory of limit sketches and of essentially algebraic structures are easily extended to logoi. For example, the notions of groupoid and of category are essentially algebraic. A *category object* in a finitely complete logos X is defined to be a simplicial object $C : \Delta^o \to X$ satisfying the *Segal condition*. The theory of limit sketches is a natural framework for studying homotopy coherent algebraic structures in general and higher weak categories in particular. The logos of models of a limit sketch is locally presentable and conversely, every locally presentable logos is equivalent to the logos of models of a limit sketch. The theory of accessible categories and of locally presentable categories was extended to logoi by Lurie.

A *para-variety* is defined to be a left exact reflection of a variety of homotopy algebras. For example, a homotopos is a para-variety. The logoi of spectra and of ring spectra are also examples. Para-varieties can be characterized by a system of axioms closely related to those of Vitale [Vi].

Factorisation systems are playing an important role in the theory of logoi. We introduce a general notion of homotopy factorisation system in a model category with examples in **Cat** and in the model category for logoi. The theory of Dwyer-Kan localisations can be described in the language of homotopy factorisation systems. This is true also on the theory of prestacks.

The theory of logoi can analyse phenomena which belong properly to homotopy theory. The notion of stable logos is an example. The notion of meta-stable logos introduced in the notes is another. We give a proof that the logos of parametrized spectra is a homotopos (joint work with Georg Biedermann). We sketch a new proof of the stabilisation hypothesis of Breen-Baez-Dolan [Si2]. We give a characterisation of homotopy varieties which improves a result of Rosicky.

There are important differences between category theory and the theory of logoi. One difference lies in the fact that the diagonal of an object in a logos is not necessarly monic. The notion of equivalence relation is affected accordingly. Every groupoid in the logos $\mathbf{U}$ is an *equivalence groupoid*. This is true in particular if the groupoid is a group. The classyfying space of a group G is the quotient of the terminal object 1 by G,

$$BG = 1/G.$$

Of course, this sounds like a familiar idea in homotopy theory, since $BG = E/G$, where E is a contractible space on which G is acting freely. The fact that every groupoid in $\mathbf{U}$ is an equivalence groupoid has interesting consequences. In algebra, important mental simplications can be obtained by taking a quotient by a congruence. For example, we may wish to identify two objects of a category when these objects are isomorphic. But the quotient does not exist as a category, unless there is a way to identify the objects coherently. However, the quotient category always exists when C is a category object in $\mathbf{U}$: if $J(C)$ denotes the groupoid of isomorphisms of C, then the quotient C' is constructed by a pushout square of categories,

$$\begin{tikzcd} J(C) \arrow[r] \arrow[d] & C \arrow[d] \\ BJ(C) \arrow[r] & C', \end{tikzcd}$$

where $BJ(C)$ is the quotient of C_0 by the groupoid $J(C)$. The quotient category C' satisfies the *Rezk condition*: every isomorphism of C' is a unit; we shall say that it is *reduced*. Moreover, the canonical functor $C \to C'$ is an equivalence of categories! An important simplification is obtained by working with reduced categories, since a functor between reduced categories $f : C \to D$ is an equivalence iff it is an isomorphism! The notion of reduced category object is essentially algebraic. It turns out that the logos of reduced category objects in $\mathbf{U}$ is equivalent to $\mathbf{U}_1$, the coherent nerve of $\mathbf{QCat}$. This follows from the Quillen equivalence between the model category for logoi and the model category for Rezk categories citeJT2. Hence a logos is essentially the same thing as a reduced category object in $\mathbf{U}$.

In the last sections we venture a few steps in the theory of (∞, n)-categories for every $n \geq 1$. There is a notion of n-fold category object for every $n \geq 1$. The logos of n-fold category objects in $\mathbf{U}$ is denoted by $Cat^n(\mathbf{U})$. By definition, we have

$$Cat^{n+1}(\mathbf{U}) = Cat(Cat^n(\mathbf{U})).$$

There is also a notion of n-category object for every $n \geq 1$. The logos $Cat_n(\mathbf{U})$ of n-category objects in $\mathbf{U}$ is a full sub-logos of $Cat^n(\mathbf{U})$. A n-category C is *reduced* if every invertible cell of C is a unit. The notion of reduced n-category object is essentially algebraic. The logos of reduced n-category objects in $\mathbf{U}$ is denoted by $\mathbf{U}_n$. The logos $\mathbf{U}_n$ is locally presentable, since the notion of reduced n-category object is essentially algebraic. It follows that $\mathbf{U}_n$ is the homotopy localisation of a combinatorial model category. For example, it can be represented by a regular Cisinski model $(\hat{A}, W)$. Such a representation is determined by a map $r : A \to \mathbf{U}_n$ whose left Kan extension $r_! : \hat{A} \to \mathbf{U}_n$ induces an equivalence between the homotopy localisation of $(\hat{A}, W)$ and $\mathbf{U}_n$. The class W is also determined by r, since a map $f : X \to Y$ in $\hat{A}$ belongs to W iff the morphism $r_!(f) : r_!X \to r_!Y$ is invertible in $\mathbf{U}_n$. The notion of *n-logos* is obtained by taking A to be a certain full subcategory Θ_n of the category of strict n-categories and by taking r to be the inclusion $\Theta_n \subset \mathbf{U}_n$. In this case W the class of *weak categorical n-equivalences $Wcat_n$*. The model category $(\hat{\Theta}_n, Wcat_n)$ is cartesian closed and its subcategory of fibrant objects $\mathbf{QCat_n}$ has the structure of a simplicial category enriched over Kan complexes. The coherent nerve of $\mathbf{QCat_n}$ is equivalent to $\mathbf{U}_n$.

Note: The category Θ_n was first defined by the author as the opposite of the category of finite n-disks $\mathcal{D}_n$. It follows from this definition that the topos $\hat{\Theta}_n$ is classifying n-disks and that the geometric realisation functor $\hat{\Theta}_n \to \mathbf{Top}$ introduced by the author preserves finite limits (where **Top** is the category of compactly generated topological spaces). See [Ber] for a proof of these results. It was conjectured (jointly by Batanin, Street and the author) that Θ_n is isomorphic to a category T_n^* introduced by Batanin in his theory of higher operads [Bat3]. The conjecture was proved by Makkai and Zawadowski in [MZ] and by Berger in [Ber]. It shows that Θ_n is a full subcategory of the category of strict n-categories.

Note: It is conjectured by Cisinski and the author that the localiser $Wcat_n$ is generated by a certain set of *spine inclusions* $S[t] \subseteq \Theta[t]$.

We close this introduction with a few general remarks on the notion of weak higher category. There are essentially three approaches for defining this notion: operadic, Segalian and Kanian. In the first approach, a weak higher category is viewed as an algebraic structure defined by a system of operations satisfying certain coherence conditions which are themselve expressed by higher operations, possibly at infinitum. The first algebraic definition of a weak higher groupoid is due to Grothendieck in his "Pursuing Stacks" [Gro] [**?**]. The first general definition of a weak higher category by Baez and Dolan is using operads. The definition by Batanin is using the higher operads introduced for this purpose. The Segalian approach has its origin in the work of Graeme Segal on infinite loop spaces [S1]. A homotopy coherent algebraic structure is defined to be a commutative diagram of spaces satisfying certain exactness conditions, called the *Segal conditions*. The spaces can be simplicial sets, and more generally the objects of a Quillen model category. The approach has the immense advantage of pushing the coherence conditions out of the way. The notions of Segal category, of Segal space and of Rezk category (ie complete Segal space) are explicitly Segalian. The Kanian approach has its origin in the work of Boardman and Vogt after the work of Kan on simplicial homotopy theory. The notion of logos is *Kanian*, since it is defined by cell filling conditions (the Boardman conditions). In the Kanian approach, a weak higher groupoid is the same thing as a Kan complex. We are liberated from the need to represent a homotopy type by an algebraic structure, since it can be represented by itself! Of course, it is instructive to model homotopy types algebraically, and it is the purpose of algebraic topology to study spaces from an algebraic point of view. For example, a 2-type can be modeled by a categorical group and a simply-connected 3-type by a braided categorical group. In these examples, the homotopy type is fully described by the algebraic model, Partial models are also important as in rational homotopy theory. The different approaches to higher categories are not in conflict but complementary. The Kanian approach is heuristically stronger and more effective at the foundational level. It suggests that a weak higher category is the combinatorial representation of a space of a new kind, possibly a higher moduli stack. The nature of these spaces is presently unclear, but like categories, they should admit *irreversible paths*. Grothendieck topoi are not general enough, even in their higher incarnations, the homotopoi. For example, I do not know how to associate a higher topos to a 2-category. For this we need a notion of 2-*prestack*. But this notion depends on what we choose to be the archetype of an $(\infty, 2)$-*topos*. The idea that there is a connection between the notion of weak category and that of space is very potent. It was a guiding principle, a *fil d'Ariane*, in the Pursuing Stacks of Grothendieck. It has inspired the notion of braided monoidal category and many conjectures by Baez and Dolan. It suggests that the category of weak categories has properties similar to that of spaces, for example, that it should be cartesian closed. It suggests the existence of classifying higher categories, in analogy with classifying spaces. Classifying spaces are often equipped with a natural algebraic structure. Operads were originally introduced for studying these structures and the corresponding algebra of operations in (co)homology. Many new invariants of topology, like the Jones polynomial, have not yet been explained within the classical setting of algebraic topology. Topological quantum field theory is pushing for an extension of algebraic topology and the operadic approach to higher categories may find its full meaning in the extension.

Note: A theory of higher operads based on cartesian monads was developed by Leinster. A more general theory is been developed by Batanin and Weber. The Segalian approach in homotopy universal algebra was developed by Badzioch [Bad2]. A notion of higher category based on the notion of complicial set is been developed by Street and Verity. A notion of multi-logoi (or colored quasi-operads) is been developed by Moerdijk and Weiss.

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I am indebted to Jon Beck for guiding my first steps in homotopy theory more than thirty years ago. Jon was deeply aware of the unity between homotopy theory and category theory and he contributed to both fields. He had the dream of using simplicial sets for the foundation of mathematics (including computer science and calculus!). I began to read Boardmann and Vogt after attending the beautiful talk

that Jon gave on their work at the University of Durham in July 1977. I dedicate these notes to his memory.

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1. Elementary aspects

In this section we define the notion of logos and describe some of its basic properties. We introduce the notion of equivalence between logoi.

1.1. For terminology and notation about categories and simplicial sets, see appendix 47 and 49. We denote the category of simplicial sets by $\mathbf{S}$ and the category of small categories by $\mathbf{Cat}$.

1.2. The category Δ is a full subcategory of $\mathbf{Cat}$. Recall that the *nerve* of a small category C is the simplicial set NC obtained by putting

$$(NC)_n = \mathbf{Cat}([n], C)$$

for every $n \geq 0$. The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ is fully faithful. We shall regard it as an inclusion $N : \mathbf{Cat} \subset \mathbf{S}$ by adopting the same notation for a category and its nerve. The nerve functor has a left adjoint

$$\tau_1 : \mathbf{S} \to \mathbf{Cat}$$

which associates to a simplicial set X its *fundamental category* $\tau_1 X$. The classical fundamental groupoid $\pi_1 X$ is obtained by formally inverting the arrows of $\tau_1 X$. If X is a simplicial set, the canonical map $X \to N\tau_1 X$ is denoted as a map $X \to \tau_1 X$.

1.3. Recall that a simplicial set X is said to be a *Kan complex* if it satisfies the *Kan condition*: every horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$,

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "\forall"] \arrow[d, hook] & X \\
\Delta[n]. \arrow[ur, "\exists"'] &
\end{tikzcd}$$

The singular complex of a space and the nerve of a groupoid are examples. We shall denote by $\mathbf{Kan}$ the full subcategory of $\mathbf{S}$ spanned by the Kan complexes. If X is a Kan complex, then so is the simplicial set X^A for any simplicial set A. It follows that the category $\mathbf{Kan}$ is cartesian closed. A simplicial set X is (isomorphic to the nerve of) a groupoid iff every horn $\Lambda^k[n] \to X$ has a *unique* filler.

1.4. Let us say that a horn $\Lambda^k[n]$ is *inner* if $0 < k < n$. We shall say that a simplicial set X is a *logos* if it satisfies the *Boardman condition*: every inner horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$. A Kan complex and the nerve of a category are examples. We may say that a vertex of a logos is an *object* of this logos and that an arrow is a *morphism*. We shall denote by $\mathbf{Log}$ the full subcategory of $\mathbf{S}$ spanned by the logoi. We may say that a map between logoi $f : X \to Y$ is a *functor*. If X is a logos then so is the simplicial set X^A for any simplicial set A. Hence the category $\mathbf{Log}$ is cartesian closed. A simplicial set X is (isomorphic to the nerve of) a category iff every inner horn $\Lambda^k[n] \to X$ has a *unique* filler.

1.5. A logos can be large. A logos X is *locally small* if the simplicial set X is locally small (this means that the vertex map $X_n \to X_0^{n+1}$ has small fibers for every $n \geq 0$). Most logoi considered in these notes are small or locally small.

1.6. [J2] The notion of logos has many equivalent descriptions. Recall that a map of simplicial sets is a called a *trivial fibration* if it has the right lifting property with respect to the inclusion $\partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$. Let us denote by $I[n]$ the simplicial subset of $\Delta[n]$ generated by the edges $(i, i+1)$ for $0 \leq i \leq n-1$ (by convention, $I[0] = \Delta[0]$). The simplicial set $I[n]$ is a chain of n arrows and we shall say that it is the *spine* of $\Delta[n]$. Notice that $I[2] = \Lambda^1[2]$ and that $X^{I[2]} = X^I \times_{s=t} X^I$. A simplicial set X is a logos iff the projection

$$X^{\Delta[2]} \to X^{I[2]}$$

defined from the inclusion $I[2] \subset \Delta[2]$ is a trivial fibration iff the projection $X^{\Delta[n]} \to X^{I[n]}$ defined from the inclusion $I[n] \subset \Delta[n]$ is a trivial fibration for every $n \geq 0$.

1.7. If X is a simplicial set, we shall denote by $X(a,b)$ the fiber at $(a,b) \in X_0 \times X_0$ of the projection

$$(s,t) : X^I \to X^{\{0,1\}} = X \times X$$

defined by the inclusion $\{0,1\} \subset I$. A vertex of $X(a,b)$ is an arrow $a \to b$ in X. If X is a logos, then the simplicial set $X(a,b)$ is a Kan complex for every pair (a,b). Moreover, the projection $X^{\Delta[2]} \to X^I \times_{s=t} X^I$ defined from the inclusion $I[2] \subset \Delta[2]$ has a section, since it is a trivial fibration by 1.6. If we compose this section with the map $X^{d_1} : X^{\Delta[2]} \to X^I$, we obtain a "composition law"

$$X^I \times_{s=t} X^I \to X^I$$

well defined up to homotopy. It induces a "composition law"

$$X(b,c) \times X(a,b) \to X(a,c)$$

for each triple $(a,b,c) \in X_0 \times X_0 \times X_0$.

1.8. The fundamental category $\tau_1 X$ of a simplicial set X has a simple construction when X is a logos. In this case we have

$$\tau_1 X = hoX,$$

where hoX is the *homotopy category* of X introduced by Boardman and Vogt in [BV]. By construction, $(hoX)(a,b) = \pi_0 X(a,b)$ and the composition law

$$hoX(b,c) \times hoX(a,b) \to hoX(a,c)$$

is induced by the "composition law" of 1.7. If $f, g : a \to b$ are two arrows in X, we shall say that a 2-simplex $u : \Delta[2] \to X$ with boundary $\partial u = (1_b, g, f)$,

$$\begin{tikzcd} & b \arrow[dr, "1_b"] & \\ a \arrow[ur, "f"] \arrow[rr, "g"'] & & b, \end{tikzcd}$$

is a *right homotopy* between f and g and we shall write $u : f \Rightarrow_R g$. Dually, we shall say that a 2-simplex $v : \Delta[2] \to X$ with boundary $\partial v = (g, f, 1_a)$,

$$\begin{tikzcd} & a \arrow[dr, "g"] & \\ a \arrow[ur, "1_a"] \arrow[rr, "f"'] & & b. \end{tikzcd}$$

is a *left homotopy* between f and g and we shall write $v : f \Rightarrow_L g$. Two arrows $f, g : a \to b$ in a logos X are homotopic in $X(a,b)$ iff there exists a right homotopy $u : f \Rightarrow_R g$ iff there exists a left homotopy $v : f \Rightarrow_L g$. Let us denote by $[f] : a \to b$ the homotopy class of an arrow $f : a \to b$. The composite of a class $[f] : a \to b$ with a class $[g] : b \to c$ is the class $[wd_1] : a \to c$, where w is any 2-simplex $\Delta[2] \to X$ filling the horn $(g, \star, f) : \Lambda^1[2] \to X$,

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "wd_1"'] & & c. \end{tikzcd}$$

1.9. There is an analogy between Kan complexes and groupoids. The nerve of) a category is a Kan complex iff the category is a groupoid. Hence the following commutative square is a pullback,

$$\begin{tikzcd} \mathbf{Gpd} \arrow[r, "in"] \arrow[d, "in"'] & \mathbf{Kan} \arrow[d, "in"] \\ \mathbf{Cat} \arrow[r, "in"] & \mathbf{Log}, \end{tikzcd}$$

where **Gpd** denotes the category of small groupoids and where the horizontal inclusions are induced by the nerve functor. The inclusion $\mathbf{Gpd} \subset \mathbf{Kan}$ has a left adjoint $\pi_1 : \mathbf{Kan} \to \mathbf{Gpd}$ and the inclusion $\mathbf{Cat} \subset \mathbf{Log}$ has a left adjoint $\tau_1 : \mathbf{Log} \to \mathbf{Cat}$. Moreover, the following square commutes up to a natural isomorphism,

$$\begin{tikzcd} \mathbf{Gpd} \arrow[d, "in"'] & \mathbf{Kan} \arrow[l, "\pi_1"'] \arrow[d, "in"] \\ \mathbf{Cat} & \mathbf{Log} \arrow[l, "\tau_1"] \end{tikzcd}$$

1.10. We say that two vertices of a simplicial set X are *isomorphic* if they are isomorphic in the category $\tau_1 X$. We shall say that an arrow in X is *invertible*, or that it is an *isomorphism*, if its image by the canonical map $X \to \tau_1 X$ is invertible in the category $\tau_1 X$. When X is a logos, two objects $a, b \in X$ are isomorphic iff there exists an isomorphism $f : a \to b$. In this case, there exists an arrow $g : b \to a$ together with two homotopies $gf \Rightarrow 1_a$ and $fg \Rightarrow 1_b$. A logos X is a Kan complex iff the category hoX is a groupoid [J1]. Let J be the groupoid generated by one isomorphism $0 \to 1$. Then an arrow $f : a \to b$ in a logos X is invertible iff the map $f : I \to X$ can be extended along the inclusion $I \subset J$. The inclusion functor $\mathbf{Gpd} \subset \mathbf{Cat}$ has a right adjoint $J : \mathbf{Cat} \to \mathbf{Gpd}$, where $J(C)$ is the groupoid of isomorphisms of a category C. Similarly, the inclusion functor $\mathbf{Kan} \subset \mathbf{Log}$ has a right adjoint $J : \mathbf{Log} \to \mathbf{Kan}$ by [J1]. The simplicial set $J(X)$ is the largest Kan subcomplex of a logos X. It is constructed by the following pullback square

$$\begin{tikzcd} J(X) \arrow[r] \arrow[d, hook] & J(hoX) \arrow[d, hook] \\ X \arrow[r, "h"] & hoX, \end{tikzcd}$$

where h is the canonical map. Moreover, the following square commutes up to a natural isomorphism,

$$\begin{tikzcd} \mathbf{Gpd} & \mathbf{Kan} \arrow[l, "\pi_1"'] \\ \mathbf{Cat} \arrow[u, "J"] & \mathbf{Log}. \arrow[l, "\tau_1"] \arrow[u, "J"'] \end{tikzcd}$$

1.11. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ preserves finite products by a result of Gabriel and Zisman. For any pair (X, Y) of simplicial sets, let us put

$$\tau_1(X, Y) = \tau_1(Y^X).$$

If we apply the functor τ_1 to the composition map $Z^Y \times Y^X \to Z^X$ we obtain a composition law

$$\tau_1(Y, Z) \times \tau_1(X, Y) \to \tau_1(X, Z)$$

for a 2-category $\mathbf{S}^{\tau_1}$, where we put $\mathbf{S}^{\tau_1}(X, Y) = \tau_1(X, Y)$. By definition, a 1-cell of $\mathbf{S}^{\tau_1}$ is a map of simplicial sets $f : X \to Y$, and a 2-cell $f \to g : X \to Y$ is a morphism of the category $\tau_1(X, Y)$; we shall say that it is a *natural transformation* $f \to g$. Recall that a *homotopy* between two maps $f, g : X \to Y$ is an arrow $\alpha : f \to g$ in the simplicial set Y^X; it can be represented as a map $X \times I \to Y$ or as a map $X \to Y^I$. To a homotopy $\alpha : f \to g$ is associated a natural transformation $[\alpha] : f \to g$. When Y is a logos, a natural transformation $[\alpha] : f \to g$ is invertible in $\tau_1(X, Y)$ iff the arrow $\alpha(a) : f(a) \to g(a)$ is invertible in Y for every vertex $a \in X$.

1.12. We call a map of simplicial sets $X \to Y$ a *categorical equivalence* if it is an equivalence in the 2-category $\mathbf{S}^{\tau_1}$. For example, a trivial fibration (as defined in 49.4) is a categorical equivalence. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a categorical equivalence to an equivalence of categories. If X and Y are logoi, we say that a categorical equivalence $X \to Y$ is an *equivalence of logoi*, or just an *equivalence* if the context is clear. A map between logoi $f : X \to Y$ is an equivalence iff there exists a map $g : Y \to X$ and two isomorphisms $gf \to 1_X$ and $fg \to 1_Y$.

1.13. We say that a map of simplicial sets $u : A \to B$ is *essentially surjective* if the functor $\tau_1 A \to \tau_1 B$ is essentially surjective. We say that a map between logoi $f : X \to Y$ is *fully faithful* if the map $X(a, b) \to Y(fa, fb)$ induced by f is a weak homotopy equivalence for every pair $a, b \in X_0$. A map between logoi is an equivalence iff it is fully faithful and essentially surjective.

2. The model structure for logoi

The category of simplicial sets admits a model structure in which the fibrant objects are the logoi. We compare it with the classical model structure the category of simplicial sets and with natural model structure on the category of small categories.

2.1. Recall that a map of simplicial sets $f : X \to Y$ is said to be a *Kan fibration* if it has the right lifting property with respect to the inclusion $\Lambda^k[n] \subset \Delta[n]$ for every $n > 0$ and $k \in [n]$. Recall that a map of simplicial sets is said to be *anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ $(n > 0, k \in [n])$ [GZ]. The category $\mathbf{S}$ admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of anodyne maps and $\mathcal{B}$ is the class of Kan fibrations.

2.2. Let **Top** be the category of compactly generated topological spaces. We recall that the singular complex functor $r^! : \mathbf{Top} \to \mathbf{S}$ has a left adjoint $r_!$ which associates to a simplicial set its *geometric realisation*. A map of simplicial sets $u : A \to B$ is said to be a *weak homotopy equivalence* if the map $r_!(u) : r_!A \to r_!B$ is a homotopy equivalence of topological spaces. The notion of weak homotopy equivalence in $\mathbf{S}$ can be defined combinatorially by using Kan complexes instead of geometric realisation. To see this, we recall the construction of the homotopy category $\mathbf{S}^{\pi_0}$ by Gabriel and Zisman [GZ]. The functor $\pi_0 : \mathbf{S} \to \mathbf{Set}$ preserves finite products. For any pair (A, B) of simplicial sets, let us put

$$\pi_0(A, B) = \pi_0(B^A).$$

If we apply the functor π_0 to the composition map $C^B \times B^A \to C^A$ we obtain a composition law $\pi_0(B, C) \times \pi_0(A, B) \to \pi_0(A, C)$ for a category $\mathbf{S}^{\pi_0}$, where we put $\mathbf{S}^{\pi_0}(A, B) = \pi_0(A, B)$. A map of simplicial sets is called a *simplicial homotopy equivalence* if it is invertible in the category $\mathbf{S}^{\pi_0}$. A map of simplicial sets $u : A \to B$ is a weak homotopy equivalence iff the map

$$\pi_0(u, X) : \pi_0(B, X) \to \pi_0(A, X)$$

is bijective for every Kan complex X. Every simplicial homotopy equivalence is a weak homotopy equivalence and the converse holds for a map between Kan complexes.

2.3. The category $\mathbf{S}$ admits a Quillen model structure in which a weak equivalence is a weak homotopy equivalence and a cofibration is a monomorphism [Q]. The fibrant objects are the Kan complexes. The model structure is cartesian closed and proper. We shall say that it is the *classical model structure* on $\mathbf{S}$ and we shall denote it shortly by $(\mathbf{S}, \mathit{Who})$, where *Who* denotes the class of weak homotopy equivalences. The fibrations are the Kan fibrations. A map is an acyclic cofibration iff it is anodyne.

2.4. The model structure $(\mathbf{S}, \mathit{Who})$ is the Cisinski structure on $\mathbf{S}$ whose fibrant objects are the Kan complexes.

2.5. We shall say that a functor $p : X \to Y$ between two categories is an *iso-fibration* if for every object $x \in X$ and every isomorphism $g \in Y$ with target $p(x)$, there exists an isomorphism $f \in X$ with target x such that $p(f) = g$. This notion is self dual:a functor $p : X \to Y$ is an iso-fibration iff the opposite functor $p^o : X^o \to Y^o$ is. The category **Cat** admits a model structure in which a weak equivalence is an equivalence of categories and a fibration is an iso-fibration [JT1]. The model structure is cartesian closed and proper. We shall say that it is the *natural* model structure on **Cat** and we shall denote it shortly by $(\mathbf{Cat}, Eq)$, where Eq denotes the class of equivalences between categories. A functor $u : A \to B$ is a cofibration iff the map $Ob(u) : ObA \to ObB$ is monic. Every object is fibrant and cofibrant. A functor is an acyclic fibration iff it is fully faithful and surjective on objects.

2.6. For any simplicial set A, let us denote by $\tau_0 A$ the set of isomorphism classes of objects of the category $\tau_1 A$. The functor $\tau_0 : \mathbf{S} \to \mathbf{Set}$ preserves finite products, since the functor τ_1 preserves finite products. For any pair (A, B) of simplicial sets, let us put

$$\tau_0(A, B) = \tau_0(B^A).$$

If we apply the functor τ_0 to the the composition map $C^B \times B^A \to C^A$ we obtain the composition law $\tau_0(B, C) \times \tau_0(A, B) \to \tau_0(A, C)$ of a category $\mathbf{S}^{\tau_0}$, where we put $\mathbf{S}^{\tau_0}(A, B) = \tau_0(A, B)$. A map of simplicial sets is a categorical equivalence iff it is invertible in the category $\mathbf{S}^{\tau_0}$. We shall say that a map of simplicial sets $u : A \to B$ is a *weak categorical equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every logos X. A map $u : A \to B$ is a weak categorical equivalence iff the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is an equivalence of categories for every logos X.

2.7. The category $\mathbf{S}$ admits a model structure in which a weak equivalence is a weak categorical equivalence and a cofibration is a monomorphism [J2]. The fibrant objects are the logoi. The model structure is cartesian closed and left proper. We shall say that it is the *model structure for logoi* and we denote it shortly by $(\mathbf{S}, Wcat)$, where *Who* denotes the class of weak categorical equivalences. A fibration is called a *pseudo-fibration* The functor $X \mapsto X^o$ is an automorphism of the model structure $(\mathbf{S}, Wcat)$. . .

2.8. The model structure $(\mathbf{S}, Wcat)$ is the Cisinski structure on $\mathbf{S}$ whose fibrant objects are the logoi.

2.9. The pair of adjoint functors

$$\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$$

is a Quillen adjunction between the model categories $(\mathbf{S}, Wcat)$ and $(\mathbf{Cat}, Eq)$. A functor $u : A \to B$ in $\mathbf{Cat}$ is an equivalence (resp. an iso-fibration) iff the map $Nu : NA \to NB$ is a (weak) categorical equivalence (resp. a pseudo-fibration).

2.10. The classical model structure on $\mathbf{S}$ is a Bousfield localisation of the model structure for logoi. Hence a weak categorical equivalence is a weak homotopy equivalence and the converse holds for a map between Kan complexes. A Kan fibration is a pseudo-fibration and the converse holds for a map between Kan complexes. A simplicial set A is weakly categorically equivalent to a Kan complex iff its fundamental category $\tau_1 A$ is a groupoid.

2.11. We say that a map of simplicial sets is *mid anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 < k < n$. Every mid anodyne map is a weak categorical equivalence, monic and biunivoque (ie bijective on vertices). We do not have an example of a monic biunivoque weak categorical equivalence which is not mid anodyne.

2.12. We shall say that a map of simplicial sets is a *mid fibration* if it has the right lifting propery with respect to the inclusion $\Lambda^k[n] \subset \Delta[n]$ for every $0 < k < n$. A simplicial set X is a logos iff the map $X \to 1$ is a mid fibration. If X is a logos and C is a category, then every map $X \to C$ is a mid fibration. In particular, every functor in **Cat** is a mid fibration. The category $\mathbf{S}$ admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of mid anodyne maps and $\mathcal{B}$ is the class of mid fibrations.

2.13. Recall that a *reflexive graph* is a 1-truncated simplicial set. If G is a reflexive graph, then the canonical map $G \to \tau_1 G$ is mid anodyne. It is thus a weak categorical equivalence.

2.14. A pseudo-fibration is a mid fibration. Conversely, a mid fibration between logoi $p : X \to Y$ is a pseudo-fibration iff the following equivalent conditions are satisfied:

- the functor $ho(p) : hoX \to hoY$ is an isofibration;
- for every object $x \in X$ and every isomorphism $g \in Y$ with target $p(x)$, there exists an isomorphism $f \in X$ with target x such that $p(f) = g$;
- p has the right lifting property with respect to the inclusion $\{1\} \subset J$

2.15. Let J be the groupoid generated by one isomorphism $0 \to 1$. Then a map between logoi $p : X \to Y$ is a pseudo-fibration iff the map

$$\langle j_0, p \rangle : X^J \to Y^J \times_Y X$$

obtained from the square

$$\begin{tikzcd} X^J \arrow[r, "X^{j_0}"] \arrow[d, "p^I"'] & X \arrow[d, "p"] \\ Y^I \arrow[r, "Y^{j_0}"] & Y, \end{tikzcd}$$

is a trivial fibration, where j_0 denotes the inclusion $\{0\} \subset J$.

2.16. Consider the functor $k : \Delta \to \mathbf{S}$ defined by putting $k[n] = \Delta'[n]$ for every $n \geq 0$, where $\Delta'[n]$ denotes the (nerve of) the groupoid freely generated by the category $[n]$. If $X \in \mathbf{S}$, let us put

$$k^!(X)_n = \mathbf{S}(\Delta'[n], X).$$

The functor $k^! : \mathbf{S} \to \mathbf{S}$ has a left adjoint $k_!$. The pair of adjoint functors

$$k_! : (\mathbf{S}, Who) \leftrightarrow (\mathbf{S}, Wcat) : k^!$$

is a Quillen adjunction and a homotopy coreflection (this means that the left derived functor of $k_!$ is fully faithful). If X is a logos, then the canonical map $k^!(X) \to X$ factors through the inclusion $J(X) \subseteq X$ and the induced map $k^!(X) \to J(X)$ is a trivial fibration.

2.17. Recall that a simplicial set is said to be *finite* if it has a finite number of non-degenerate cells. We say that a Kan complex X is *homotopy finite* if there exists a finite simplicial set K together with a weak homotopy equivalence $K \to X$. We say that a simplicial set A is *homotopy finite* if there exists a homotopy finite Kan complex X together with a weak homotopy equivalence $A \to X$. A finite group is homotopy finite iff it is the trivial group.

2.18. If X is a logos and A is a simplicial set, we shall say that a weak categorical equivalence $A \to X$ is a *strong presentation* of X. The strong presentation is *finite* if A is finite. We say that a logos which admits a finite strong presentation is *essentially finite*. Recall that a *reflexive graph* is a simplicial set of dimension ≤ 1. If A is a reflexive graph, then the canonical map $A \to \tau_1 A$ is mid anodyne; hence the free category $\tau_1 A$ is strongly presented by A.

2.19. We say that a simplicial set A is *essentially finite* if there exists an essentially finite logos X together with a weak categorical equivalence $A \to X$. A finite simplicial set is essentially finite and an essentially finite simplicial set is homotopy finite, but the converse is not necessary true. For example, the monoid freely generated by one idempotent is homotopy finite (it is contractible) but not essentially finite.

2.20. Let *Split* be the category with two objects 0 and 1 and two arrows $s : 0 \to 1$ and $r : 1 \to 0$ such that $rs = id$. If K is the simplicial set defined by the pushout square

$$\begin{tikzcd} \Delta[1] \arrow[r, "d_1"] \arrow[d] & \Delta[2] \arrow[d] \\ 1 \arrow[r] & K \end{tikzcd}$$

then the obvious map $K \to \mathit{Split}$ is mid anodyne. Hence the category *Split* is essentially finite. Observe that *Split* contains the monoid freely generated by one idempotent as a full subcategory. Hence a full subcategory of an essentially finite category is not always essentially finite.

2.21. Let us say that a class of monomorphisms $\mathcal{A} \subseteq \mathbf{S}$ has the *right cancellation property* if the implication

$$vu \in \mathcal{A} \quad \text{and} \quad u \in \mathcal{A} \quad \Rightarrow \quad v \in \mathcal{A}$$

is true for any pair of monomorphisms $u : A \to B$ and $v : B \to C$. Let $\mathcal{A} \subseteq \mathbf{S}$ be a saturated class of monomorphisms having the right cancellation property. Let $I[n] \subseteq \Delta[n]$ be the spine of $\Delta[n]$. If the inclusion $I[n] \subseteq \Delta[n]$ belongs to $\mathcal{A}$ for every $n \geq 0$, then every mid anodyne map belongs to $\mathcal{A}$ [JT2].

2.22. (Cisinski) The localizer *Wcat* in the category $\mathbf{S}$ is generated by the set of spine inclusions $I[n] \subseteq \Delta[n]$ $(n \geq 0)$. Let us give a proof. Let $W \subseteq \mathbf{S}$ be a localizer which contains the inclusions $I[n] \subseteq \Delta[n]$ for $n \geq 0$. We shall denote by $\mathit{Fib}(W)$ the class of fibrant objects of the model structure defined by W. The result will be proved by 51.10 if we show that every object of $\mathit{Fib}(W)$ is a logos. If $\mathcal{C}$ is the class of monomorphisms, then the class $W \cap \mathcal{C}$ is saturated, since W is a localizer. But $W \cap \mathcal{C}$ has the right cancellation property, since W satisfies "3 for 2". Thus, $W \cap \mathcal{C}$ contains the mid anodyne maps by 2.21. It thus contains the inclusion $\Lambda^k[n] \subset \Delta[n]$ for every $0 < k < n$. This shows that every object of $\mathit{Fib}(W)$ is a logos.

3. Equivalence with simplicial categories

Simplicial categories were introduced by Dwyer and Kan in their work on simplicial localisation. The category of simplicial categories admits a Quillen model structure, called the Bergner-Dwyer-Kan model structure. The coherent nerve of a fibrant simplicial category to a logos. The coherent nerve functor induces a Quillen equivalence between simplicial categories and logoi.

3.1. Recall that a *simplicial category* is a category enriched over simplicial sets. A *simplicial functor* between simplicial categories is a strong functor, that is, an enriched functor. We denote by **SCat** the category of simplicial categories and simplicial functors. An ordinary category can be viewed a simplicial category with discrete hom. The inclusion functor $\mathbf{Cat} \subset \mathbf{SCat}$ has a left adjoint

$$ho : \mathbf{SCat} \to \mathbf{Cat}$$

which associates to a simplicial category X its *homotopy category* hoX. By construction, we have $(hoX)(a,b) = \pi_0 X(a,b)$ for every pair of objects $a, b \in X$. A simplicial functor $f : X \to Y$ is said to be *homotopy fully faithful* if the map $X(a,b) \to Y(fa, fb)$ is a weak homotopy equivalence for every pair of objects $a, b \in X$. A simplicial functor $f : X \to Y$ is said to be *homotopy essentially surjective* if the functor $ho(f) : hoX \to hoY$ is essentially surjective. A simplicial functor $f : X \to Y$ is called a *Dwyer-Kan equivalence* if it is homotopy fully faithful and homotopy essentially surjective. We shall say that a simplicial functor $f : X \to Y$ is *trivial fibration* iff the map $Ob(f) : ObX \to ObY$ is surjective and the map $X(a,b) \to Y(fa, fb)$ is a trivial fibration for every pair of objects $a, b \in X$. It was proved by Bergner in [B1] that the category **SCat** admits a Quillen model structure in which the weak equivalences are the Dwyer-Kan equivalences and the acyclic fibrations are the trivial fibrations. The model structure is left proper and the fibrant objects are the categories enriched over Kan complexes. A simplicial functor $f : X \to Y$ is fibration iff it is a *Dwyer-Kan fibration*, that is, the map $X(a,b) \to Y(fa, fb)$ is a Kan fibration for every pair of objects $a, b \in X$, and the functor $ho(f)$ is an iso-fibration. We say that it is the *Bergner model structure* or the *model structure for simplicial categories* We shall denote it by $(\mathbf{SCat}, DK)$, where DK denotes the class of Dwyer-Kan equivalences.

3.2. Recall that a *reflexive graph* is a 1-truncated simplicial set. Let **Grph** be the category of reflexive graphs. The obvious forgetful functor $U : \mathbf{Cat} \to \mathbf{Grph}$ has a left adjoint F. The composite $C = FU$ is a comonad on **Cat**. It follows that for any small category A, the sequence of categories $C_n A = C^{n+1}(A)$ $(n \geq 0)$ has the structure of a simplicial object $C_*(A)$ in **Cat**. The simplicial set $n \mapsto Ob(C_n A)$ is constant with value $Ob(A)$. It follows that $C_*(A)$ can be viewed as a simplicial category instead of a simplicial object in **Cat**. This defines a functor

$$C_* : \mathbf{Cat} \to \mathbf{SCat}.$$

If A is a category then the augmentation $C_*(A) \to A$ is a cofibrant replacement of A in the model category **SCat**. If X is a simplicial category, then a simplicial functor $C_*(A) \to X$ is said to be a *homotopy coherent diagram* $A \to X$. This notion was introduced by Vogt in [V].

3.3. The simplicial category $C_\star[n]$ has the following description. The objects of $C_\star[n]$ are the elements of $[n]$. If $i, j \in [n]$ and $i > j$, then $C_\star[n](i,j) = \emptyset$; if $i \leq j$, then the simplicial set $C_\star[n](i,j)$ is (the nerve of) the poset of subsets $S \subseteq [i.j]$ such that $\{i, j\} \subseteq S$. If $i \leq j \leq k$, the composition operation

$$C_\star[n](j,k) \times C_\star[n](i,j) \to C_\star[n](i,k)$$

is the union $(T, S) \mapsto T \cup S$.

3.4. The *coherent nerve* of a simplicial category X is the simplicial set $C^!X$ obtained by putting

$$(C^!X)_n = \mathbf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. This notion was introduced by Cordier in [C]. The simplicial set $C^!(X)$ is a logos when X is enriched over Kan complexes [**?**]. The functor $C^! : \mathbf{SCat} \to \mathbf{S}$ has a left adjoint $C_!$ and we have $C_!A = C_\star A$ when A is a category [J4]. Thus, a homotopy coherent diagram $A \to X$ with values in a simplicial category X is the same thing as a map of simplicial sets $A \to C^!X$.

3.5. The pair of adjoint functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category $(\mathbf{S}, Wcat)$ and the model category $(\mathbf{SCat}, DK)$ [Lu1][J4].

3.6. A simplicial category can be large. For example, the *logos of Kan complexes* $\mathbf{U}$ is defined to be the coherent nerve of the simplicial category **Kan**. The logos $\mathbf{U}$ is large but locally small. It plays an important role in the theory of logoi, where it is the analog of the category of sets. It is the archetype of a *homotopos*, also called an *∞-topos*.

3.7. The category **Log** becomes enriched over Kan complexes if we put

$$Hom(X, Y) = J(Y^X)$$

for $X, Y \in \mathbf{Log}$. For example, the *logos of small logoi* $\mathbf{U}_1$ is defined to be the coherent nerve of the simplicial category **Log**. The logos $\mathbf{U}_1$ is large but locally small. It plays an important role in the theory of logoi where it is the analog of the category of small categories.

4. Equivalence with Rezk categories

Rezk categories are the fibrant objects of a model structure on the category of simplicial spaces. They were introduced by Charles Rezk under the name of complete Segal spaces. The first row of a Rezk category is a logos. The functor induces a Quillen equivalence between Rezk categories and logoi.

4.1. Recal that a *bisimplicial set* is defined to be a contravariant functor $\Delta \times \Delta \to$ **Set** and that a *simplicial space* to be a contravariant functor $\Delta \to \mathbf{S}$. We can regard a simplicial space X as a bisimplicial set by putting $X_{mn} = (X_m)_n$ for every $m, n \geq 0$. Conversely, we can regard a bisimplicial set X as a simplicial space by putting $X_m = X_{m\star}$ for every $m \geq 0$. We denote the category of bisimplicial sets by $\mathbf{S}^{(2)}$. The *box product* $A \Box B$ of two simplicial sets A and B is the bisimplicial set $A \Box B$ obtained by putting

$$(A \Box B)_{mn} = A_m \times B_n$$

for every $m, n \geq 0$. This defines a functor of two variables $\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}^{(2)}$. The box product funtor $\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}^{(2)}$ is divisible on each side. This means that the functor $A \Box (-) : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $A \backslash (-) : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set A, and that the functor $(-) \Box B : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $(-)/B : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set B. For any pair of simplicial spaces X and Y, let us put

$$Hom(X, Y) = (Y^X)_0$$

This defines an enrichment of the category $\mathbf{S}^{(2)}$ over the category $\mathbf{S}$. For any simplicial set A we have $A\backslash X = Hom(A\Box 1, X)$.

4.2. We recall that the category of simplicial spaces $[\Delta^o, \mathbf{S}]$ admits a Reedy model structure in which the weak equivalences are the term-wise weak homotopy equivalences and the cofibrations are the monomorphisms. The model structure is simplicial if we put $Hom(X,Y) = (Y^X)_0$. It is cartesian closed and proper.

4.3. Let $I[n] \subseteq \Delta[n]$ be the n-chain. For any simplicial space X we have a canonical bijection

$$I[n]\backslash X = X_1 \times_{\partial_0=\partial_1} X_1 \times \cdots \times_{\partial_0=\partial_1} X_1,$$

where the successive fiber products are calculated by using the face maps $\partial_0, \partial_1 : X_1 \to X_0$. We say that a simplicial space X satisfies the *Segal condition* if the map

$$\Delta[n]\backslash X \longrightarrow I[n]\backslash X$$

obtained from the inclusion $I[n] \subseteq \Delta[n]$ is a weak homotopy equivalence for every $n \geq 2$ (the condition is trivially satisfied if $n < 2$). A *Segal space* is a Reedy fibrant simplicial space which satisfies the Segal condition.

4.4. The Reedy model structure on the category $[\Delta^o, \mathbf{S}]$ admits a Bousfield localisation with respect to the set of maps $I[n]\Box 1 \to \Delta[n]\Box 1$ for $n \geq 0$. The fibrant objects of the local model structure are the Segal spaces. The local model structure is simplicial, cartesian closed and left proper. We say that it is the *model structure for Segal spaces.*

4.5. Let J be the groupoid generated by one isomorphism $0 \to 1$. We regard J as a simplicial set via the nerve functor. A Segal space X is said to be *complete*, if it satisfies the *Rezk condition*: the map

$$1\backslash X \longrightarrow J\backslash X$$

obtained from the map $J \to 1$ is a weak homotopy equivalence. We shall say that a complete Segal space is a *Rezk category.*

4.6. The model structure for Segal spaces admits a Bousfield localisation with respect to the map $J\Box 1 \to 1\Box 1$. The fibrant objects of the local model structure are the Rezk categories. The local model structure is simplicial, cartesian closed and left proper. We say that it is the *model structure for Rezk categories.*

4.7. The *first row* of a simplicial space X is the simplicial set $r(X)$ obtained by putting $rw(X)_n = X_{n0}$ for every $n \geq 0$. The functor $rw : \mathbf{S}^{(2)} \to \mathbf{S}$ has a left adjoint c obtained by putting $c(A) = A\Box 1$ for every simplicial set A. The pair of adjoint functors

$$c : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : rw$$

is a Quillen equivalence between the model category for logoi and the model category for Rezk categories [JT2].

4.8. Consider the functor $t^! : \mathbf{S} \to \mathbf{S}^{(2)}$ defined by putting

$$t^!(X)_{mn} = \mathbf{S}(\Delta[m] \times \Delta'[n], X)$$

for every $X \in \mathbf{S}$ and every $m, n \geq 0$, where $\Delta'[n]$ denotes the (nerve of the) groupoid freely generated by the category $[n]$. The functor $t^!$ has a left adjoint $t_!$ and the pair

$$t_! : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!$$

is a Quillen equivalence between the model category for Rezk categories and the model category for logoi [JT2].

5. Equivalence with Segal categories

Segal categories were introduced by Hirschowitz and Simpson in their work on higher stacks. They are fibrant objects in a model structure on the category of precategory. The first row of a fibrant Segal category is a logos. The functor induces a Quillen equivalence between Segal categories and logoi.

5.1. A simplicial space $X : \Delta^o \to \mathbf{S}$ is called a *precategory* if the simplicial set X_0 is discrete. We shall denote by $\mathbf{PCat}$ the full subcategory of $\mathbf{S}^{(2)}$ spanned by the precategories. The category $\mathbf{PCat}$ is a presheaf category and the inclusion functor $p^* : \mathbf{PCat} \subset \mathbf{S}^{(2)}$ has a left adjoint $p_!$ and a right adjoint p_*.

5.2. If X is a precategory and $n \geq 1$, then the vertex map $v_n : X_n \to X_0^{n+1}$ takes its values in a discrete simplicial set. We thus have a decomposition

$$X_n = \bigsqcup_{a \in X_0^{[n]_0}} X(a),$$

where $X(a) = X(a_0, \ldots, a_n)$ denotes the fiber of v_n at $a = (a_0, \cdots, a_n)$. A precategory X satisfies the Segal condition iff he canonical map

$$X(a_0, a_1, \ldots, a_n) \to X(a_0, a_1) \times \cdots \times X(a_{n-1}, a_n)$$

is a weak homotopy equivalence for every $a \in X_0^{[n]_0}$ and $n \geq 2$. A precategory which satisfies the Segal condition is called a *Segal category.*

5.3. If C is a small category, then the bisimplicial set $N(C) = C \Box 1$ is a Segal category. The functor $N : \mathbf{Cat} \to \mathbf{PCat}$ has a left adjoint

$$\tau_1 : \mathbf{PCat} \to \mathbf{Cat}$$

which associates to a precategory X its *fundamental category* $\tau_1 X$. A map of precategories $f : X \to Y$ is said to be *essentially surjective* if the functor $\tau_1(f) : \tau_1 X \to \tau_1 Y$ is essentially surjective. A map of precategories $f : X \to Y$ is said to be *fully faithful* if the map

$$X(a, b) \to Y(fa, fb)$$

is a weak homotopy equivalence for every pair $a, b \in X_0$. We say that $f : X \to Y$ is a *categorical equivalence* if it is fully faithful and essentially surjective.

5.4. In [HS], Hirschowitz and Simpson construct a completion functor

$$S : \mathbf{PCat} \to \mathbf{PCat}$$

which associates to a precategory X a Segal category $S(X)$ "generated" by X. A map of precategories $f : X \to Y$ is called a *weak categorical equivalence* if the map $S(f) : S(X) \to S(Y)$ is a categorical equivalence. The category $\mathbf{PCat}$ admits a left proper model structure in which a a weak equivalence is a weak categorical equivalence and a cofibration is a monomorphism. We say that it is the *Hirschowitz-Simpson model structure* or the *model structure for Segal categories*. The model structure is cartesian closed [P].

5.5. We recall that the category of simplicial spaces $[\Delta^o, \mathbf{S}]$ admits a Reedy model structure in which the weak equivalences are the term-wise weak homotopy equivalences and the cofibrations are the monomorphisms. A precategory is fibrant in the Hirschowitz-Simpson model structure iff it is a Reedy fibrant Segal category [B3].

5.6. The *first row* of a precategory X is the simplicial set $r(X)$ obtained by putting $r(X)_n = X_{n0}$ for every $n \geq 0$. The functor $r : \mathbf{PCat} \to \mathbf{S}$ has a left adjoint h obtained by putting $h(A) = A \Box 1$ for every simplicial set A. It was conjectured in [T1] (and proved in [JT2]) that the pair of adjoint functors

$$h : \mathbf{S} \leftrightarrow \mathbf{PCat} : r$$

is a Quillen equivalence between the model category for logoi and the model category for Segal categories.

5.7. The *diagonal* $d^*(X)$ of a precategory X is defined to be the diagonal of the bisimplicial set X. The functor $d^* : \mathbf{PCat} \to \mathbf{S}$ admits a right adjoint d_* and the pair of adjoint functors

$$d^* : \mathbf{PCat} \leftrightarrow \mathbf{S} : d_*$$

is a Quillen equivalence between the model category for Segal categories and the model category for logoi [JT2].

6. Minimal logoi

The theory of minimal Kan complexes can be extended to logoi. Every logos has a minimal model which is unique up to isomorphism. A category is minimal iff it is skeletal.

6.1. Recall that a sub-Kan complex S of a Kan complex X is said to be a *(sub)model* of X if the inclusion $S \subseteq X$ is a homotopy equivalence. Recall that a Kan complex is said to be *minimal* if it has no proper (sub)model. Every Kan complex has a minimal model and that any two minimal models are isomorphic. Two Kan complexes are homotopy equivalent iff their minimal models are isomorphic.

6.2. We shall say that a subcategory S of a category C is a *model* of C if the inclusion $S \subseteq C$ is an equivalence. We say that a category C is *skeletal* iff it has no proper model.

6.3. A subcategory S of a category C is a model of C iff it is full and

$$\forall a \in ObC \quad \exists b \in ObS \quad a \simeq b,$$

where $a \simeq b$ means that a and b are isomorphic objects. A category C is skeletal iff

$$\forall a, b \in ObC \quad a \simeq b \quad \Rightarrow \quad a = b$$

6.4. Let $f : C \to D$ be an equivalence of categories. If C is skeletal, then f is monic on objects and morphisms. If D is skeletal, then f is surjective on objects and morphisms. If C and D are skeletal, then f is an isomorphism.

6.5. Every category has a skeletal model and any two skeletal models are isomorphic. Two categories are equivalent iff their skeletal models are isomorphic.

6.6. (Definition) If X is a logos, we shall say that a sub-logos $S \subseteq X$ is a *(sub)model* of X if the inclusion $S \subseteq X$ is an equivalence. We say that a logos is *minimal* or *skeletal* if it has no proper (sub)model.

6.7. (Lemma) Let $S \subseteq X$ be model of a logos X. Then the inclusion $u : S \subseteq X$ admits a retraction $r : X \to S$ and there exists an isomorphism $\alpha : ur \simeq 1_X$ such that $\alpha \circ u = 1_u$.

6.8. (Notationj) If X be a simplicial set and $n \geq 0$, consider the projection

$$\partial : X^{\Delta[n]} \to X^{\partial\Delta[n]}$$

defined by the inclusion $\partial\Delta[n] \subset \Delta[n]$. Its fiber at a vertex $x \in X^{\partial\Delta[n]}$ is a simplicial set $X\langle x\rangle$. If $n = 1$ we have $x = (a, b) \in X_0 \times X_0$ and $X\langle x\rangle = X(a, b)$. The simplicial set $X\langle x\rangle$ is a Kan complex when X is a logos and $n > 0$. If $n > 0$, we say that two simplices $a, b : \Delta[n] \to X$ are *homotopic with fixed boundary*, and we write $a \simeq b$, if we have $\partial a = \partial b$ and a and b are homotopic in the simplicial set $X(\partial a) = X(\partial b)$. If $a, b \in X_0$, we shall write $a \simeq b$ to indicate that the vertices a and b are isomorphic.

6.9. (Proposition) If S is a simplicial subset of a simplicial set X, then for every simplex $x \in X_n$ we shall write $\partial x \in S$ to indicate that the map $\partial x : \partial\Delta[n] \to X$ factors through the inclusion $S \subseteq X$. If X is a logos, then the simplicial subset S is a model of X iff

$$\forall n \geq 0 \quad \forall a \in X_n \quad \big(\partial a \in S \quad \Rightarrow \quad \exists b \in S \quad a \simeq b\big).$$

A logos X is a minimal iff

$$\forall n \geq 0 \quad \forall a, b \in X_n \quad \big(a \simeq b \quad \Rightarrow \quad a = b\big).$$

6.10. Let $f : X \to Y$ be an equivalence of logoi. If X is minimal, then f is monic. If Y is minimal, then f is a trivial fibration. If X and Y are minimal, then f is an isomorphism.

6.11. Every logos has a minimal model and any two minimal models are isomorphic. Two logoi are equivalent iff their minimal models are isomorphic.

7. Discrete fibrations and covering maps

We introduce a notion of discrete fibration between simplicial sets. It extends the notion of covering space map and the notion of discrete fibration between categories. The results of this section are taken from [J2].

7.1. Recall that a functor $p : E \to C$ between small categories is said to be a *discrete fibration*, but we shall say a *discrete right fibration*, if for every object $x \in E$ and every arrow $g \in C$ with target $p(x)$, there exists a unique arrow $f \in E$ with target x such that $p(f) = g$. For example, if $el(F)$ denotes the category of elements of a presheaf $F \in \hat{C}$, then the natural projection $el(F) \to C$ is a discrete right fibration. The functor $F \mapsto el(F)$ induces an equivalence between the category of presheaves $\hat{C}$ and the full subcategory of $\mathbf{Cat}/C$ spanned by the discrete right fibrations $E \to C$. Recall that a functor $u : A \to B$ is said to be *final*, but we shall say *0-final*, if the category $b\backslash A$ defined by the pullback square

$$\begin{tikzcd} b\backslash A \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ b\backslash B \arrow[r] & B \end{tikzcd}$$

is connected for every object $b \in B$. The category $\mathbf{Cat}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-final functors and $\mathcal{B}$ is the class of discrete right fibrations.

7.2. A functor $p : E \to C$ is a discrete right fibration iff it is right orthogonal to the inclusion $\{n\} \subseteq \Delta[n]$ for every $n \geq 0$. We shall say that a map of simplicial sets a *discrete right fibration* if it is right orthogonal to the inclusion $\{n\} \subseteq \Delta[n]$ for every $n \geq 0$. We shall say that a map of simplicial sets $u : A \to B$ is *0-final* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is 0-final. The category $\mathbf{S}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-final maps and $\mathcal{B}$ is the class of discrete right fibrations.

7.3. For any simplicial set B, the functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ induces an equivalence between the full subcategory of $\mathbf{S}/B$ spanned by the discrete right fibrations with target B and the full subcategory of $\mathbf{Cat}/B$ spanned by the discrete right fibrations with target $\tau_1 B$. The inverse equivalence associates to a discrete right fibration with target $\tau_1 B$ its base change along the canonical map $B \to \tau_1 B$.

7.4. Dually, a functor $p : E \to C$ is said to be a *discrete opfibration*, but we shall say a *discrete left fibration*, if for every object $x \in E$ and every arrow $g \in C$ with source $p(x)$, there exists a unique arrow $f \in E$ with source x such that $p(f) = g$. A functor $p : E \to C$ is a discrete left fibration iff the opposite functor $p^o : E^o \to B^o$ is a discrete right fibration. Recall that a functor $u : A \to B$ is said to be *initial*, but we shall say *0-initial*, if the category A/b defined by the pullback square

$$\begin{tikzcd} A/b \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ B/b \arrow[r] & B \end{tikzcd}$$

is connected for every object $b \in B$. The category $\mathbf{Cat}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-initial functors and $\mathcal{B}$ is the class of discrete left fibrations.

7.5. A functor $p : E \to C$ is a discrete left fibration iff it is right orthogonal to the inclusion $\{0\} \subseteq \Delta[n]$ for every $n \geq 0$. We say that a map of simplicial sets is a *discrete left fibration* if it is right orthogonal to the inclusion $\{0\} \subseteq \Delta[n]$ for every $n \geq 0$. We say that a map of simplicial sets $u : A \to B$ is *0-initial* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is 0-initial. The category $\mathbf{S}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-initial maps and $\mathcal{B}$ is the class of discrete left fibrations.

7.6. For any simplicial set B, the functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ induces an equivalence between the full subcategory of $\mathbf{S}/B$ spanned by the discrete left fibrations with target B and the full subcategory of $\mathbf{Cat}/B$ spanned by the discrete left fibrations with target $\tau_1 B$. The inverse equivalence associates to a discrete left fibration with target $\tau_1 B$ its base change along the canonical map $B \to \tau_1 B$.

7.7. We say that functor $p : E \to C$ is a *0-covering* if it is both a discrete fibration and a discrete opfibration. For example, if F is a presheaf on C, then the natural projection $el(F) \to C$ is a 0-covering iff the functor F takes every arrow in C to a bijection. If $c : C \to \pi_1 C$ is the canonical functor, then the functor $F \mapsto el(Fc)$ induces an equivalence between the category of presheaves on $\pi_1 C$ and the full subcategory of $\mathbf{Cat}/C$ spanned by the 0-coverings $E \to C$. We say that a functor $u : A \to B$ is *0-connected* if the functor $\pi_1(u) : \pi_1 A \to \pi_1 B$ is essentially surjective and full. The category $\mathbf{Cat}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-connected functors and $\mathcal{B}$ is the class of 0-coverings.

7.8. We say that a map of simplicial sets $E \to B$ is a *0-covering* if it is a discrete left fibration and a discrete right fibration. A map is a 0-covering if it is right orthogonal to every map $\Delta[m] \to \Delta[n]$ in Δ. Recall that a map of simplicial sets is said to be *0-connected* if its homotopy fibers are connected. A map $u : A \to B$ is 0-connected iff the functor $\pi_1(u) : \pi_1 A \to \pi_1 B$ is 0-connected. The category $\mathbf{S}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-connected maps and $\mathcal{B}$ is the class of 0-coverings.

7.9. If B is a simplicial set, then the functor $\pi_1 : \mathbf{S} \to \mathbf{Gpd}$ induces an equivalence between the category of 0-coverings of B and the category of 0-coverings of $\pi_1 B$. The inverse equivalence associates to a 0-covering with target $\pi_1 B$ its base change along the canonical map $B \to \pi_1 B$.

8. Left and right fibrations

We introduce the notions of left fibration and of right fibration. We also introduce the notions of initial map and of final map. The right fibrations with a fixed codomain B are the *prestacks* over B. The results of the section are taken from [J2].

8.1. Recall [GZ] that a map of simplicial sets is said to be a *Kan fibration* if it has the right lifting property with respect to every horn inclusion $h^k_n : \Lambda^k[n] \subset \Delta[n]$ ($n > 0$ and $k \in [n]$). Recall that a map of simplicial sets is said to be *anodyne* if it belongs to the saturated class generated by the inclusions h^k_n. A map is anodyne iff it is an acyclic cofibration in the model category $(\mathbf{S}, Who)$. Hence the category $\mathbf{S}$ admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of anodyne maps and $\mathcal{B}$ is the class of Kan fibrations.

8.2. We say that a map of simplicial sets is a *right fibration* if it has the right lifting property with respect to the horn inclusions $h_n^k : \Lambda^k[n] \subset \Delta[n]$ with $0 < k \leq n$. Dually, we say that a map is a *left fibration* if it has the right lifting property with respect to the inclusions h_n^k with $0 \leq k < n$. A map $p : X \to Y$ is a left fibration iff the opposite map $p^o : X^o \to Y^o$ is a right fibration. A map is a Kan fibration iff it is both a left and a right fibration.

8.3. Our terminology is consistent with 7.2: every discrete right (resp. left) fibration is a right (resp. left) fibration.

8.4. The fibers of a right (resp. left) fibration are Kan complexes. Every right (resp. left) fibration is a pseudo-fibration.

8.5. A functor $p : E \to B$ is a right fibration iff it is 1-fibration.

8.6. A map of simplicial sets $f : X \to Y$ is a right fibration iff the map

$$\langle i_1, f\rangle : X^I \to Y^I \times_Y X$$

obtained from the square

$$\begin{tikzcd} X^I \arrow[r, "X^{i_1}"] \arrow[d, "f^I"'] & X \arrow[d, "f"] \\ Y^I \arrow[r, "Y^{i_1}"] & Y \end{tikzcd}$$

is a trivial fibration, where i_1 denotes the inclusion $\{1\} \subset I$. Dually, a map $f : X \to Y$ is a left fibration iff the map $\langle i_0, f\rangle$ is a trivial fibration, where i_0 denotes the inclusion $\{0\} \subset I$.

8.7. A right fibration is discrete iff it is right orthogonal the inclusion $h_n^k : \Lambda^k[n] \subset \Delta[n]$ for every $0 < k \leq n$. A functor $A \to B$ in **Cat** is a right fibration iff it is a Grothendieck fibration whose fibers are groupoids.

8.8. We say that a map of simplicial sets is *right anodyne* if it belongs to the saturated class generated by the inclusions $h_n^k : \Lambda^k[n] \subset \Delta[n]$ with $0 < k \leq n$. Dually, we say that a map is *left anodyne* if it belongs to the saturated class generated by the inclusions h_n^k with $0 \leq k < n$. A map of simplicial sets $u : A \to B$ is left anodyne iff the opposite map $u^o : A^o \to B^o$ is right anodyne. The category **S** admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of right (resp. left) anodyne maps and $\mathcal{B}$ is the class of right (resp. left) fibrations.

8.9. If the composite of two monomorphisms $u : A \to B$ and $v : B \to C$. is left (resp. right) anodyne and u is left (resp. right) anodyne, then v is left (resp. right) anodyne.

8.10. Let $\mathcal{E}$ be a category equipped with a class $\mathcal{W}$ of "weak equivalences" satisfying "three-for-two". We say that a class of maps $\mathcal{M} \subseteq \mathcal{E}$ is *invariant under weak equivalences* if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & A' \arrow[d, "u'"] \\ B \arrow[r] & B' \end{tikzcd}$$

in which the horizontal maps are weak equivalences, $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$.

8.11. We say that a map of simplicial sets $u : A \to B$ is *final*

if it admits a factorisation $u = wi : A \to B' \to B$ with i a right anodyne map and w a weak categorical equivalence. The class of final maps is invariant under weak categorical equivalences. A monomorphism is final iff it is right anodyne. The base change of a final map along a left fibration is final. A map $u : A \to B$ is final iff the simplicial set $L \times_B A$ is weakly contractible for every left fibration $L \to B$. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a left anodyne map $1 \to Lb$ followed by a left fibration $Lb \to B$. Then a map $u : A \to B$ is final iff the simplicial set $Lb \times_B A$ is weakly contractible for every vertex $b : 1 \to B$. When B is a logos, we can take $Lb = b\backslash B$ (see 9.9) and a map $u : A \to B$ is final iff the simplicial set $b\backslash A$ defined by the pullback square

$$\begin{tikzcd} b\backslash A \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ b\backslash B \arrow[r] & B \end{tikzcd}$$

is weakly contractible for every object $b \in B$.

8.12. Dually, we say that a map of simplicial sets $u : A \to B$ is *initial* if the opposite map $u^o : A^o \to B^o$ is final. A map $u : A \to B$ is initial iff it admits a factorisation $u = wi : A \to B' \to B$ with i a left anodyne map and w a weak categorical equivalence. The class of initial maps is invariant under weak categorical equivalences. A monomorphism is initial iff it is left anodyne. The base change of an initial map along a right fibration is initial. A map $u : A \to B$ is initial iff the simplicial set $R \times_B A$ is weakly contractible for every right fibration $R \to B$. For each vertex $b \in B$, let us choose a factorisation $1 \to Rb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map $u : A \to B$ is initial iff the simplicial set $Rb \times_B A$ is weakly contractible for every vertex $b : 1 \to B$. When B is a logos, we can take $Rb = B/b$ (see 9.8) and a map $u : A \to B$ is initial iff the simplicial set $b\backslash A$ defined by the pullback square

$$\begin{tikzcd} A/b \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ B/b \arrow[r] & B \end{tikzcd}$$

is weakly contractible for every object $b \in B$.

8.13. The base change of a weak categorical equivalence along a left or a right fibration is a weak categorical equivalence.

8.14. If $f : X \to Y$ is a right fibration, then so is the map

$$\langle u, f \rangle : X^B \to Y^B \times_{Y^A} X^A$$

obtained from the square

$$\begin{tikzcd} X^B \arrow[r] \arrow[d] & X^A \arrow[d] \\ Y^B \arrow[r] & Y^A, \end{tikzcd}$$

for any monomorphism of simplicial sets $u : A \to B$. Moreover, the map $\langle u, f\rangle$ is a trivial fibration if u is right anodyne. There are dual results for left fibrations and left anodyne maps.

8.15. To every left fibration $X \to B$ we can associate a functor

$$D(X) : \tau_1 B \to Ho(\mathbf{S}, Who)$$

called the *homotopy diagram* of X. To see, we first observe that the category $\mathbf{S}/B$ is enriched over $\mathbf{S}$; let us denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. The simplicial set $[X, Y]$ is a Kan complex when the structure map $Y \to B$ is a left or a right fibration. For every vertex $b \in B_0$, the map $b : 1 \to B$ is an object of $\mathbf{S}/B$ and the simplicial set $[b, X]$ is the fiber $X(b)$ of X at b. Let us put $D(X)(b) = [b, X]$. let us see that this defines a functor

$$D(X) : \tau_1 B \to Ho(\mathbf{S}, Who)$$

called the *homotopy diagram* of X. If $f : a \to b$ is an arrow in B, then the map $f : I \to B$ is an object of $\mathbf{S}/B$. From the inclusion $i_0 : \{0\} \to I$ we obtain a map $i_0 : a \to f$ and the inclusion $i_1 : \{1\} \to I$ a map $i_1 : b \to f$. We thus have a diagram of simplicial sets

$$[a, X] \xleftarrow{p_0} [f, X] \xrightarrow{p_1} [b, X],$$

where $p_0 = [i_0, X]$ and $p_1 = [i_1, X]$. The map p_0 is a trivial fibration by 8.14, since the structure map $X \to B$ is a left fibration and i_0 is left anodyne. It thus admits a section s_0. By composing p_1 with s_0 we obtain a map

$$f_! : X(a) \to X(b)$$

well defined up to homotopy. The homotopy class of f only depends on the homotopy class of f. Moreover, if $g : b \to c$, then the map $g_! f_!$ is homotopic to the map $(gf)_!$. This defines the functor $D(X)$ if we put $D(X)(a) = X(a) = [a, X]$ and $D(X)(f) = f_!$. Dually, to a right fibration $X \to B$ we associate a functor

$$D(X) : \tau_1 B^o \to Ho(\mathbf{S}, Who)$$

called the *(contravariant) homotopy diagram* of X. If $f : a \to b$ is an arrow in B, then the inclusion $i_1 : a \to f$ is right anodyne. It follows that the map p_1 in the diagram

$$[a, X] \xleftarrow{p_0} [f, X] \xrightarrow{p_1} [b, X],$$

is a trivial fibration. It thus admits a section s_1. By composing p_0 with s_1 we obtain a map

$$f^* : X(b) \to X(a)$$

well defined up to homotopy. This defines the functor $D(X)$ if we put $D(X)(a) = X(a) = [a, X]$ and $D(X)(f) = f^*$.

9. Join and slice

For any object b of a category C there is a category C/b of objects of C *over* b. Similarly, for any vertex b of a simplicial set X there is a simplicial set X/b. More generally, we construct a simplicial set X/b for any map of simplicial sets $b : B \to X$. The construction uses the join of simplicial sets. The results of this section are taken from [J1] and [J2].

9.1. The *join* of two categories A and B is the category $C = A \star B$ obtained as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for any pair of objects $x, y \in Ob(A) \sqcup Ob(B)$ we have

$$C(x,y) = \begin{cases} A(x,y) & \text{if } x \in A \text{ and } y \in A \\ B(x,y) & \text{if } x \in B \text{ and } y \in B \\ 1 & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition of arrows is obvious. Notice that the category $A \star B$ is a poset if A and B are posets: it is the *ordinal sum* of the posets A and B. The operation $(A, B) \mapsto A \star B$ is functorial and coherently associative. It defines a monoidal structure on **Cat**, with the empty category as the unit object. The monoidal category $(\mathbf{Cat}, \star)$ is not symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

The category $1 \star A$ is called the *projective cone with base* A and the category $A \star 1$ the *inductive cone with cobase* A. The object 1 is terminal in $A \star 1$ and initial in $1 \star A$. The category $A \star B$ is equipped with a natural augmentation $A \star B \to I$ obtained by joining the functors $A \to 1$ and $B \to 1$. The resulting functor $\star : \mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}/I$ is right adjoint to the functor

$$i^* : \mathbf{Cat}/I \to \mathbf{Cat} \times \mathbf{Cat},$$

where i denotes the inclusion $\{0, 1\} = \partial I \subset I,$.

9.2. The monoidal category $(\mathbf{Cat}, \star)$ is not closed. But for any category $B \in \mathbf{Cat}$, the functor

$$(-) \star B : \mathbf{Cat} \to B \backslash \mathbf{Cat}$$

which associates to $A \in \mathbf{Cat}$ the inclusion $B \subseteq A \star B$ has a right adjoint. The right adjoint takes a functor $b : B \to X$ to a category that we shall denote by X/b. We shall say that X/b is the *lower slice* of X by b. For any category A, there is a bijection between the functors $A \to X/b$ and the functors $A \star B \to X$ which extend b along the inclusion $B \subseteq A \star B$,

$$\begin{tikzcd} B \arrow[d] \arrow[dr, "b"] & \\ A \star B \arrow[r] & X. \end{tikzcd}$$

In particular, an object $1 \to X/b$ is a functor $c : 1 \star B \to X$ which extends b; it is a *projective cone with base* b in X.

9.3. Dually, the functor $A \star (-) : \mathbf{Cat} \to A \backslash \mathbf{Cat}$ has a right adjoint which takes a functor $a : A \to X$ to a category that we shall denote $a \backslash X$. We shall say that $a \backslash X$ is the *upper slice* of X by a. An object $1 \to a \backslash X$ is a functor $c : A \star 1 \to C$ which extends a; it is an *inductive cone with cobase* a.

9.4. We shall denote by Δ_+ the category of all finite ordinals and order preserving maps, *including* the empty ordinal 0. We shall denote the ordinal n by n, so that we have $n = [n-1]$ for $n \geq 1$. We may occasionally denote the ordinal 0 by $[-1]$. Notice the isomorphism of categories $1 \star \Delta = \Delta_+$. The ordinal sum $(m, n) \mapsto m + n$ is functorial with respect to order preserving maps. This defines a monoidal structure on Δ_+,

$$+ : \Delta_+ \times \Delta_+ \to \Delta_+,$$

with 0 as the unit object.

9.5. Recall that an *augmented simplicial set* is defined to be a contravariant functor $\Delta_+ \to \mathbf{Set}$. We shall denote by $\mathbf{S}_+$ the category of augmented simplicial sets. By a general procedure due to Brian Day [Da], the monoidal structure of Δ_+ can be extended to $\mathbf{S}_+$ as a closed monoidal structure

$$\star : \mathbf{S}_+ \times \mathbf{S}_+ \to \mathbf{S}_+$$

with $0 = y(0)$ as the unit object. We call $X \star Y$ the *join* of the augmented simplicial sets X and Y. We have

$$(X \star Y)(n) = \bigsqcup_{i+j=n} X(i) \times Y(j)$$

for every $n \geq 0$.

9.6. From the inclusion $t : \Delta \subset \Delta_+$ we obtain a pair of adjoint functors

$$t^* : \mathbf{S}_+ \leftrightarrow \mathbf{S} : t_*.$$

The functor t^* removes the augmentation of an augmented simplicial set. The functor t_* gives a simplicial set A the trivial augmentation $A_0 \to 1$. Notice that $t_*(\emptyset) = 0 = y(0)$, where y is the Yoneda map $\Delta_+ \to \mathbf{S}_+$. The functor t_* is fully faithful and we shall regard it as an inclusion $t_* : \mathbf{S} \subset \mathbf{S}_+$. The operation $\star$ on $\mathbf{S}_+$ induces a monoidal structure on $\mathbf{S}$,

$$\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}.$$

By definition, $t_*(A \star B) = t_*(A) \star t_*(B)$ for any pair $A, B \in \mathbf{S}$. We call $A \star B$ the *join* of the simplicial sets A and B. It follows from the formula above, that we have

$$(A \star B)_n = A_n \sqcup B_n \sqcup \bigsqcup_{i+1+j=n} A_i \times A_j.$$

for every $n \geq 0$. Notice that we have

$$A \star \emptyset = A = \emptyset \star A$$

for any simplicial set A, since $t_*(\emptyset) = 0$ is the unit object for the operation $\star$ on $\mathbf{S}_+$. Hence the empty simplicial set is the unit object for the join operation on $\mathbf{S}$. The monoidal category $(\mathbf{S}, \star)$ is not symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

For every pair $m, n \geq 0$ we have

$$\Delta[m] \star \Delta[n] = \Delta[m + 1 + n]$$

since we have $[m] + [n] = [m + n + 1]$. In particular,

$$1 \star 1 = \Delta[0] \star \Delta[0] = \Delta[1] = I.$$

The simplicial set $1 \star A$ is the *projective cone with base A* A and the simplicial set $A \star 1$ the *inductive cone with cobase A*.

9.7. If A and B are simplicial sets, then the join of the maps $A \to 1$ and $B \to 1$ is a canonical map $A \star B \to I$. The resulting functor

$$\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$$

is right adjoint to the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S} = \mathbf{S}/\partial I$, where i denotes the inclusion $\{0,1\} = \partial I \subset I$. This gives another description of the join operation for simplicial sets. It follows from this description that we have

$$A \star B = (A \star 1) \times_I (1 \star B).$$

9.8. The monoidal category $(\mathbf{S}, \star)$ is not closed. But for any simplicial set B, the functor

$$(-) \star B : \mathbf{S} \to B \backslash \mathbf{S}$$

which associates to a simplicial set A the inclusion $B \subseteq A \star B$ has a right adjoint. The right adjoint takes a map of simplicial set $b : B \to X$ to a simplicial set that we shall denote by X/b. We shall say that it is the *lower slice* of X by b. For any simplicial set A, there is a bijection between the maps $A \to X/b$ and the maps $A \star B \to X$ which extend b along the inclusion $B \subseteq A \star B$,

$$\begin{tikzcd} B \arrow[d] \arrow[dr, "b"] & \\ A \star B \arrow[r] & X. \end{tikzcd}$$

In particular, a vertex $1 \to X/b$ is a map $c : 1 \star B \to X$ which extends the map b; it a *projective cone with base* b in X. The simplicial set X/b is a logos when X is a logos. If $B = 1$ and $b \in X_0$, then a simplex $\Delta[n] \to X/b$ is a map $x : \Delta[n+1] \to X$ such that $x(n+1) = b$.

9.9. Dually, for any simplicial set A, the functor $A \star (-) : \mathbf{S} \to A \backslash \mathbf{S}$ has a right adjoint which takes a map $a : A \to X$ to a simplicial set that we shall denote $a \backslash X$. We shall say that it is the *upper slice* of X by a. A vertex $1 \to a \backslash X$ is a map $c : A \star 1 \to X$ which extends the map a; it is an *inductive cone with cobase* a in X. The simplicial set $a \backslash X$ is a logos when X is a logos. If $A = 1$ and $a \in X_0$, then a simplex $\Delta[n] \to a \backslash X$ is a map $x : \Delta[n+1] \to X$ such that $x(0) = a$.

9.10. If A, B and X are simplicial sets, we obtain a natural inclusion $A \star B \subseteq A \star X \star B$ by joining the maps $1_A : A \to A$, $\emptyset \to X$ and $1_B : B \to B$. The functor

$$A \star (-) \star B : \mathbf{S} \to (A \star B) \backslash \mathbf{S}$$

which associates to X the inclusion $A \star B \subseteq A \star X \star B$ has a right adjoint for any pair A and B. The right adjoint takes a map of simplicial sets $f : A \star B \to X$ to a simplicial set that we shall denote $Fact(f, X)$. . By construction, a vertex $1 \to Fact(f, X)$ is a map $g : A \star 1 \star B \to X$ which extends f. When $A = B = 1$, it is a *factorisation* of the arrow $f : I \to X$. If f is an arrow $a \to b$ then $Fact(f, X) = f \backslash (X/b) = (a \backslash X)/f$.

9.11. Recall that a model structure on a category $\mathcal{E}$ induces a model structure on the slice category $\mathcal{E}/B$ for each object $B \in \mathcal{E}$. In particular, we have a model category $(B \backslash \mathbf{S}, Wcat)$ for each simplicial set B. The pair of adjoint functors $X \mapsto X \star B$ and $(X, b) \mapsto X/b$ is a Quillen pair between the model categories $(\mathbf{S}, Wcat)$ and $(B \backslash \mathbf{S}, Wcat)$.

9.12. If $u : A \to B$ and $v : S \to T$ are two maps in **S**, we shall denote by $u \star' v$ the map

$$(A \star T) \sqcup_{A \star S} (B \star S) \to B \star T$$

obtained from the commutative square

$$\begin{tikzcd} A \star S \arrow[r, "u \star S"] \arrow[d, "A \star v"'] & B \star S \arrow[d, "B \star v"] \\ A \star T \arrow[r, "u \star T"] & B \star T. \end{tikzcd}$$

If u is an inclusion $A \subseteq B$ and v an inclusion $S \subseteq T$, then the map $u \star' v$ is the inclusion

$$(A \star T) \cup (B \star S) \subseteq B \star T.$$

If $u : A \to B$ and $v : S \to T$ are monomorphisms of simplicial sets, then

- $u \star' v$ is mid anodyne if u is right anodyne or v left anodyne;
- $u \star' v$ is left anodyne if u is anodyne;
- $u \star' v$ is right anodyne if v is anodyne.

9.13. [J1] [J2] (Lemma) Suppose that we have a commutative square

$$\begin{tikzcd} (\{0\} \star T) \cup (I \star S) \arrow[r, "u"] \arrow[d] & X \arrow[d, "p"] \\ I \star T \arrow[r, "v"] & Y, \end{tikzcd}$$

where p is a mid fibration between logoi. If the arrow $u(I) \in X$ is invertible, then the square has a diagonal filler.

9.14. [J1] [J2] Suppose that we have a commutative square

$$\begin{tikzcd} \Lambda^0[n] \arrow[r, "x"] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r] & Y, \end{tikzcd}$$

in which p is a mid fibration between logoi. If $n > 1$ and the arrow $x(0,1) \in X$ is invertible, then the square has a diagonal filler. This follows from the lemma above if we use the decompositions $\Delta[n] = I \star \Delta[n-2]$ and $\Lambda^0[n] = (\{0\} \star \Delta[n-2]) \cup (I \star \partial\Delta[n-2])$.

9.15. [J1] [J2] A logos X is a Kan complex iff the category hoX is a groupoid. This follows from the result above.

9.16. The simplicial set X/b depends functorially on the map $b : B \to X$. More precisely, to every commutative diagram

$$\begin{tikzcd} B \arrow[d, "b"'] & A \arrow[l, "u"'] \arrow[d, "a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

we can associate a map

$$f/u : X/b \to Y/a.$$

By definition. if $x : \Delta[n] \to X/b$, then the simplex $(f/u)(x) : \Delta[n] \to Y/a$ is obtained by composing the maps

$$\Delta[n] \star A \xrightarrow{\Delta[n]\star u} \Delta[n] \star B \xrightarrow{x} X \xrightarrow{f} Y.$$

9.17. A map $u : (M,p) \to (N,q)$ in the category $\mathbf{S}/B$ is a contravariant equivalence iff the map $1_X/u : dq\backslash X \to dp\backslash X$ is an equivalence of logoi for any map $d : B \to X$ with values in a logos X. In particular, a map $u : A \to B$ is final iff the map $1_X/u : d\backslash X \to du\backslash X$ is an equivalence of logoi for any map $d : B \to X$ with values in a logos X.

9.18. For any chain of three maps

$$S \xrightarrow{s} T \xrightarrow{t} X \xrightarrow{f} Y$$

we shall denote by $\langle s,t,f\rangle$ the map

$$X/t \to Y/ft \times_{Y/fts} X/ts$$

obtained from the commutative square

$$\begin{tikzcd} X/t \arrow[r] \arrow[d] & X/ts \arrow[d] \\ Y/ft \arrow[r] & Y/fts, \end{tikzcd}$$

Let us suppose that s is monic. Then the map $\langle s,t,f\rangle$ is a right fibration when f is a mid fibration, a Kan fibration when f is a left fibration, and it is a trivial fibration in each of the following cases:

- f is a trivial fibration;
- f is a right fibration and s is anodyne:
- f is a mid fibration and s is left anodyne.

9.19. We now consider another notion of join. The *fat join* of two simplicial sets A and B is the simplicial set $A \diamond B$ defined by the pushout square

$$\begin{tikzcd} (A\times 0\times B)\sqcup(A\times 1\times B) \arrow[r] \arrow[d] & A\sqcup B \arrow[d] \\ A\times I\times B \arrow[r] & A\diamond B. \end{tikzcd}$$

We have $A \sqcup B \subseteq A \diamond B$ and there is a canonical map $A \diamond B \to I$. This defines a continuous functor $\diamond : \mathbf{S}\times\mathbf{S} \to \mathbf{S}/I$ and we have

$$X \diamond Y = (X \diamond 1) \times_I (1 \diamond Y).$$

For a fixed $B \in \mathbf{S}$, the functor $(-)\diamond B : \mathbf{S} \to B\backslash\mathbf{S}$ which takes a simplicial set A to the inclusion $B \subseteq A \diamond B$ has a right adjoint. The right adjoint takes a map $b : B \to X$ to a simplicial set that we shall denote by $X/\!\!/b$; it is the *fat lower slice* of X by b. If $b \in X_0$, it is the fiber of the target map $X^I \to X$ at b. The simplicial set $X/\!\!/b$ is a logos when X is a logos. Dually, there is also a *fat upper slice* $a\backslash\!\!\backslash X$ for any map of simplicial sets $a : A \to X$. The simplicial set $a\backslash\!\!\backslash X$ is a logos when X is a logos.

9.20. For any pair of simplicial sets A and B, there is a unique map

$$\theta_{AB} : A \diamond B \to A \star B$$

which fills diagonally the square

$$\begin{tikzcd} A \sqcup B \arrow[r] \arrow[d] & A \star B \arrow[d] \\ A \diamond B \arrow[r] & I. \end{tikzcd}$$

It is weak categorical equivalence. By adjointness, we obtain a map

$$\rho(b) : X/b \to X/\!\!/b$$

for any simplicial set X and any map $b : B \to X$. The map $\rho(b)$ is an equivalence of logoi when X is a logos.

9.21. The pair of adjoint functors $X \mapsto X \diamond B$ and $(X, b) \mapsto X/\!\!/b$ is a Quillen adjoint pair between the model categories $(\mathbf{S}, Wcat)$ and $(B\backslash\mathbf{S}, Wcat)$.

10. Initial and terminal objects

In this section, we introduce inital and terminal objects. The results of the section are taken from [J1] and [J2].

10.1. We say that an object a in a logos X is *terminal* if it satisfies the following equivalent conditions:

- the simplicial set $X(x, a)$ is contractible for every $x \in X_0$;
- every simplical sphere $x : \partial\Delta[n] \to X$ with target $x(n) = a$ can be filled;
- the projection $X/a \to X$(resp. $X/\!\!/a \to X$) is a weak categorical equivalence;
- the projection $X/a \to X$ (resp. $X/\!\!/a \to X$) is a trivial fibration.

10.2. Dually, we say that an object a in a logos X is *initial* if it satisfies the following equivalent conditions:

- the simplicial set $X(a, x)$ is contractible for every $x \in X_0$;
- every simplical sphere $x : \partial\Delta[n] \to X$ with source $x(0) = a$ can be filled;
- the projection $a\backslash X \to X$ is a weak categorical equivalence (resp. a trivial fibration);
- the projection $a\backslash\!\!\backslash X \to X$ is a weak categorical equivalence (resp. a trivial fibration).

10.3. The full simplicial subset spanned by the terminal (resp. initial) objects of a logos is a contractible Kan complex when non-empty.

10.4. More generally, we say that a vertex $a \in A$ in a simplicial set A is *terminal* if the map $a : 1 \to A$ is final (or equivalently right anodyne). The notion of terminal vertex is invariant under weak categorical equivalence. More precisely, if $u : A \to B$ is a weak categorical equivalence, then a vertex $a \in A$ is terminal in A iff the vertex $u(a)$ is terminal in B. Dually, we say that a vertex $a \in A$ in a simplicial set A is *initial* if the opposite vertex $a^o \in A^o$ is terminal. A vertex $a \in A$ in a simplicial set A is initial iff the map $a : 1 \to A$ is initial (or equivalently left anodyne).

10.5. If A is a simplicial set, then a vertex $a \in A$ which is terminal in A is also terminal in the category $\tau_1 A$. The converse is true when A admits a terminal vertex.

10.6. If A is a simplicial set, then the vertex $1_a \in A/a$ is terminal in A/a for any vertex $a \in A$. Similarly for the vertex $1_a \in A/\!\!/a$.

10.7. If B is a simplicial set, then a vertex $b \in B$ is terminal iff the inclusion $E(b) \subseteq E$ is a weak homotopy equivalence for every left fibration $p : E \to B$, where $E(b) = p^{-1}(b)$. Recall from 12.1 that the category $\mathbf{S}/B$ is enriched over $\mathbf{S}$. For any object E of $\mathbf{S}/B$, let us denote by $\Gamma_B(E)$ the simplicial set $[B, E]$ of global sections of E. Then a vertex $b \in B$ is terminal iff the canonical projection $\Gamma_B(E) \to E(b)$ is a homotopy equivalence for every right fibration $E \to B$,

10.8. Recall that a *pointed category* is a category enriched over the category of pointed sets. A category C is pointed iff the projection $C^I \to C \times C$ admits a section $z : C \times C \to C^I$. The section is unique when it exists. In this case it associates to a pair of objects $a, b \in C$ the *null arrow* $0 : a \to b$. In a pointed category, an object is initial iff it is terminal. Recall that a *null object* in a category C is an object $0 \in C$ which is both initial and final. A category C with a null object $0 \in C$ is pointed; the null arrow $0 = a \to b$ between two objects of C is obtained by composing the arrows $a \to 0 \to b$.

10.9. We say that a logos X is *pointed* if the projection $X^I \to X \times X$ admits a section $X \times X \to X^I$. The section is homotopy unique when it exists. In this case it associates to a pair of objects $a, b \in X$ a *null arrow* $0 : a \to b$. The homotopy category of a pointed logos X is pointed. In a pointed logos, an object is initial iff it is terminal. A *null object* in a logos X is defined to be an object $0 \in X$ which is both initial and terminal. A logos with null objects is pointed; the null arrow $0 = a \to b$ between two objects of X is obtained by composing the arrows $a \to 0 \to b$.

11. Homotopy factorisation systems

The notion of homotopy factorisation system was introduced by Bousfield in his work on localisation. We introduce a more general notion and give examples. Most results of the section are taken from [J2].

11.1. Let $\mathcal{E}$ be a category equipped with a class of maps $\mathcal{W}$ satisfying "three-for-two". We shall say that a class of maps $\mathcal{M} \subseteq \mathcal{E}$ is *invariant under weak equivalences* if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & A' \arrow[d, "u'"] \\ B \arrow[r] & B' \end{tikzcd}$$

in which the horizontal maps are in $\mathcal{W}$, we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$.

11.2. We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ has the *right cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \ \Rightarrow \ v \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$. Dually, we shall say that $\mathcal{M}$ has the *left cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \ \Rightarrow \ u \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$.

11.3. If $\mathcal{E}$ is a Quillen model category, we shall denote by $\mathcal{E}_f$ (resp. $\mathcal{E}_c$) the full subcategory of fibrant (resp. cofibrant) objects of $\mathcal{E}$ and we shall put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. For any class of maps $\mathcal{M} \subseteq \mathcal{E}$ we shall put

$$\mathcal{M}_f = \mathcal{M} \cap \mathcal{E}_f, \qquad \mathcal{M}_c = \mathcal{M} \cap \mathcal{E}_c \quad \text{and} \quad \mathcal{M}_{fc} = \mathcal{M} \cap \mathcal{E}_{fc}.$$

11.4. Let $\mathcal{E}$ be a model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. We say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a *homotopy factorisation system* if the following conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are invariant under weak equivalences;
- the pair $(\mathcal{A}_{fc} \cap \mathcal{C}, \mathcal{B}_{fc} \cap \mathcal{F})$ is a weak factorisation system in $\mathcal{E}_{fc}$;
- the class $\mathcal{A}$ has the right cancellation property;
- the class $\mathcal{B}$ has the left cancellation property.

The last two conditions are equivalent in the presence of the others. The class $\mathcal{A}$ is said to be the *left class* of the system and $\mathcal{B}$ to be the *right class*. We say that a system $(\mathcal{A}, \mathcal{B})$ is uniform if the pair $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak factorisation system.

11.5. The notions of homotopy factorisation systems and of factorisation systems coincide if the model structure is discrete (ie when $\mathcal{W}$ is the class of isomorphisms). The pairs $(\mathcal{E}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{E})$ are trivial examples of homotopy factorisation systems.

11.6. A homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ is determined by each of the following 24 classes,

$$
\begin{array}{cccc}
\mathcal{A} & \mathcal{A}_c & \mathcal{A}_f & \mathcal{A}_{fc} \\[2em]
\mathcal{A} \cap \mathcal{C} & \mathcal{A}_c \cap \mathcal{C} & \mathcal{A}_f \cap \mathcal{C} & \mathcal{A}_{fc} \cap \mathcal{C} \\[2em]
\mathcal{A} \cap \mathcal{F} & \mathcal{A}_c \cap \mathcal{F} & \mathcal{A}_f \cap \mathcal{F} & \mathcal{A}_{fc} \cap \mathcal{F} \\[2em]
\mathcal{B} & \mathcal{B}_c & \mathcal{B}_f & \mathcal{B}_{fc} \\[2em]
\mathcal{B} \cap \mathcal{C} & \mathcal{B}_c \cap \mathcal{C} & \mathcal{B}_f \cap \mathcal{C} & \mathcal{B}_{fc} \cap \mathcal{C} \\[2em]
\mathcal{B} \cap \mathcal{F} & \mathcal{B}_c \cap \mathcal{F} & \mathcal{B}_f \cap \mathcal{F} & \mathcal{B}_{fc} \cap \mathcal{F}.
\end{array}
$$

This property is useful in specifying a homotopy factorisation system.

11.7. Every homotopy factorisation system in a proper model category is uniform. This is true in particular for the homotopy factorisations systems in the model categories $(\mathbf{S}, Who)$ and $(\mathbf{Cat}, Eq)$.

11.8. If $\mathcal{E}$ is a model category we shall denote by $Ho(\mathcal{M})$ the image of a class of maps $\mathcal{M} \subseteq \mathcal{E}$ by the canonical functor $\mathcal{E} \to Ho(\mathcal{E})$. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system $\mathcal{E}$, then the pair $(Ho(\mathcal{A}), Ho(\mathcal{B}))$ is a weak factorisation system in $Ho(\mathcal{E})$. Notice that the pair $(Ho(\mathcal{A}), Ho(\mathcal{B}))$ is not a factorisation system in general. The class $Ho(\mathcal{A})$ has the right cancellation property and the class $Ho(\mathcal{B})$ the left cancellation property. The system $(\mathcal{A}, \mathcal{B})$ is determined by the system $(Ho(\mathcal{A}), Ho(\mathcal{B}))$.

11.9. The intersection of the classes of a homotopy factorisation system is the class of weak equivalences. Each class of a homotopy factorisation system is closed under composition and retracts. The left class is closed under homotopy cobase change and the right class is closed under homotopy base change.

11.10. Let $(\mathcal{A}, \mathcal{B})$ be a homotopy factorisation system in a model category $\mathcal{E}$. Then we have $u \pitchfork p$ for every $u \in \mathcal{A}_c \cap \mathcal{C}$ and $p \in \mathcal{B}_f \cap \mathcal{F}$. If $A \in \mathcal{E}_c$ and $X \in \mathcal{E}_f$, then every map $f : A \to X$ admits a factorisation $f = pu$ with $u \in \mathcal{A}_c \cap \mathcal{C}$ and $p \in \mathcal{B}_c \cap \mathcal{F}$.

11.11. If $\mathcal{E}$ is a model category, then so is the category $\mathcal{E}/C$ for any object $C \in \mathcal{E}$. If $\mathcal{M}$ is a class of maps in $\mathcal{E}$, let us denote by $\mathcal{M}_C$ the class of maps in $\mathcal{E}/C$ whose underlying map belongs to $\mathcal{M}$. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and C is fibrant, then the pair $(\mathcal{A}_C, \mathcal{B}_C)$ is a homotopy factorisation system in $\mathcal{E}/C$. This true without restriction on C when the system $(\mathcal{A}, \mathcal{B})$ is uniform.

11.12. Dually, if $\mathcal{E}$ is a model category, then so is the category $C\backslash\mathcal{E}$ for any object $C \in \mathcal{E}$. If $\mathcal{M}$ is a class of maps in $\mathcal{E}$, let us denote by ${}_C\mathcal{M}$ the class of maps in $C\backslash\mathcal{E}$ whose underlying map belongs to $\mathcal{M}$. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and C is cofibrant, then the pair $({}_C\mathcal{A}, {}_C\mathcal{B})$ is a homotopy factorisation system in $C\backslash\mathcal{E}$. This is true without restriction C when the system $(\mathcal{A}, \mathcal{B})$ is uniform.

11.13. The model category $(\mathbf{Cat}, Eq)$ admits a (uniform) homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective functors and $\mathcal{B}$ the class of fully faithful functors.

11.14. We call a functor $u : A \to B$ a *localisation* (resp. *iterated localisation*) iff it admits a factorisation $u = wu' : A \to B' \to B$ with u' a strict localisation (resp. iterated strict localisation) and w an equivalence of categories. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the the class of iterated localisations and $\mathcal{B}$ is the class of conservative functors.

11.15. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ the class of 0-final functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a discrete right fibration. Dually, the model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}', \mathcal{B}')$ in which $\mathcal{A}'$ is the class of 0-initial functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a discrete left fibration.

11.16. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ the class of 1-final functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a 1-fibration.

11.17. Recall that a functor $u : A \to B$ is said to be *0-connected* if the functor $\pi_1(u) : \pi_1 A \to \pi_1 B$ is essentially surjective and full . The category $\mathbf{Cat}$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-connected functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a 0-covering,

11.18. We say that a map of simplicial sets

is *homotopy monic* if its homotopy fibers are empty or contractible. We say that a map of simplicial sets is *homotopy surjective* if its homotopy fibers are non-empty. A map $u : A \to B$ is homotopy surjective iff the map $\pi_0(u) : \pi_0 A \to \pi_0 B$ is surjective. The model category $(\mathbf{S}, \mathit{Who})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of homotopy surjections and $\mathcal{B}$ the class of homotopy monomorphisms.

11.19. Recall from 1.13 that a map of simplicial sets $u : A \to B$ is said to be *essentially surjective* if the map $\tau_0(u) : \tau_0(A) \to \tau_0(B)$ is surjective. The model category $(\mathbf{S}, \mathit{Wcat})$ admits a (non-uniform) homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective maps. A map in the class $\mathcal{B}$ is said to be *fully faithful*. A map between logoi $f : X \to Y$ is fully faithful iff the map $X(a, b) \to Y(fa, fb)$ induced by f is a weak homotopy equivalence for every pair $a, b \in X_0$.

11.20. We say that a map of simplicial sets $u : A \to B$ is *conservative* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is conservative. The model category $(\mathbf{S}, Wcat)$ admits a (non-uniform) homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps. A map in the class $\mathcal{A}$ is an *iterated homotopy localisation*. See 19.3 for this notion.

11.21. The model category $(\mathbf{S}, Wcat)$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of final maps. A map $p : X \to Y$ belongs to $\mathcal{B}$ iff it admits a factorisation $p'w : X \to X' \to Y$ with p' a right fibration and w a weak categorical equivalence. The intersection $\mathcal{B} \cap \mathcal{F}$ is the class of right fibrations and the intersection $\mathcal{A} \cap \mathcal{C}$ the class of right anodyne maps. Dually, the model category $(\mathbf{S}, Wcat)$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of initial maps.

11.22. The model category $(\mathbf{S}, Wcat)$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of weak homotopy equivalences. A map $p : X \to Y$ belongs to $\mathcal{B}$ iff it admits a factorisation $p'w : X \to X' \to Y$ with p' a Kan fibration and w a weak categorical equivalence. The intersection $\mathcal{B} \cap \mathcal{F}$ is the class of Kan fibrations and the intersection $\mathcal{A} \cap \mathcal{C}$ the class of anodyne maps.

11.23. Let $(\mathcal{A}, \mathcal{B})$ be a homotopy factorisation system in a model category $\mathcal{E}$. Suppose that we have a commutative cube

$$\begin{tikzcd}
A_0 \arrow[rr] \arrow[dr] \arrow[dd] & & C_0 \arrow[dr] \arrow[dd] & \\
& B_0 \arrow[rr] \arrow[dd] & & D_0 \arrow[dd] \\
A_1 \arrow[rr] \arrow[dr] & & C_1 \arrow[dr] & \\
& B_1 \arrow[rr] & & D_1.
\end{tikzcd}$$

in which the top and the bottom faces are homotopy cocartesian. If the arrows $A_0 \to A_1$, $B_0 \to B_1$ and $C_0 \to C_1$ belong to $\mathcal{A}$, then so does the arrow $D_0 \to D_1$.

11.24. [JT3] If $n \geq -1$, we shall say that a simplicial set X is a n-*object* if we have $\pi_i(X, x) = 1$ for every $x \in X$ and every $i > n$. If $n = -1$, this means that X is contractible or empty. If $n = 0$, this means that X is is homotopically equivalent to a discrete simplicial set. A Kan complex X is a n-object iff every simplicial sphere $\partial\Delta[m] \to X$ with $m > n + 1$ has a filler. We say that a map of simplicial sets $f : X \to Y$ is a n-*cover* if its homotopy fibers are n-objects. If $n = -1$, this means that f is homotopy monic. A Kan fibration is a n-cover iff it has the right lifting property with respect to the inclusion $\partial\Delta[m] \subset \Delta[m]$ for every $m > n + 1$. We shall say that a simplicial set X is n-*connected* if $X \neq \emptyset$ and we have $\pi_i(X, x) = 1$ for every $x \in X$ and every $i \leq n$. If $n = -1$, this means that $X \neq \emptyset$. If $n = 0$, this means that X is connected. We shall say that a map $f : X \to Y$ is n-*connected* if its homotopy fibers are n-connected. If $n = -1$, this means that f is homotopy surjective. A map $f : X \to Y$ is n-connected iff the map $\pi_i(X, x) \to \pi_i(Y, fx)$ induced by f is bijective for every $0 \leq i \leq n$ and $x \in X$ and a surjection for

$i = n+1$. If $\mathcal{A}_n$ is the class of n-connected maps and $\mathcal{B}_n$ the class of n-covers, then the pair $(\mathcal{A}_n, \mathcal{B}_n)$ is a uniform homotopy factorisation system on the model category $(\mathbf{S}, \mathit{Who})$. We say that it is the *n-factorisation system* on $(\mathbf{S}, \mathit{Who})$.

11.25. A simplicial set X is a n-object iff the diagonal map $X \to X \times X$ is $(n-1)$-cover (if $n = 0$ this means that the diagonal is homotopy monic). A simplicial set X is a n-connected iff it is non-empty and the diagonal $X \to X \times X$ is a $(n-1)$-connected. (if $n = 0$ this means that the diagonal is homotopy surjective).

11.26. The model category $(\mathbf{S}, \mathit{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of n-connected maps. The intersection $\mathcal{B} \cap \mathcal{F}$ is the class of Kan n-covers.

11.27. If $n \geq -1$, we shall say that a right fibration $f : X \to Y$ is a *right n-fibration* if its fibers are n-objects. If $n = -1$, this means that f is fully faithful. If $n = 0$, this means that f is fiberwise homotopy equivalent to a right covering. A right fibration is a right n-fibration iff it has the right lifting property with respect to the inclusion $\partial\Delta[m] \subset \Delta[m]$ for every $m > n+1$. The model category $(\mathbf{S}, \mathit{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which the intersection $\mathcal{B} \cap \mathcal{F}$ is the class of right n-fibrations. We say that a map in the class $\mathcal{A}$ is *n-final*. A map between logoi $u : A \to B$ is n-final iff the simplicial set $b\backslash A$ is n-connected for every object $b \in B$. indexAfibration!right n-fibration—textbf

11.28. Let $F : \mathcal{E} \leftrightarrow \mathcal{E}' : G$ be a Quillen pair between two model categories. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and $(\mathcal{A}', \mathcal{B}')$ a homotopy factorisation system in $\mathcal{E}'$, then the conditions $F(\mathcal{A}_c) \subseteq \mathcal{A}'_c$ and $G(\mathcal{B}'_f) \subseteq \mathcal{B}_f$ are equivalent. If the pair (F, G) is a Quillen equivalence, then the conditions $\mathcal{A}_c = F^{-1}(\mathcal{A})_c$ and $\mathcal{B}'_f = G^{-1}(\mathcal{B})_f$ are equivalent. In this case we shall say that $(\mathcal{A}', \mathcal{B}')$ is obtained by *transporting* $(\mathcal{A}, \mathcal{B})$ *across* the Quillen equivalence. Every homotopy factorisation system can be transported across a Quillen equivalence.

11.29. We shall say that a simplicial functor $f : X \to Y$ in $\mathbf{SCat}$ is *conservative* if the functor $ho(f) : hoX \to hoY$ is conservative. The Bergner model category $\mathbf{SCat}$ admits a (non-uniform) homotopy factorisation system in which the right class is the class of conservative functors. A map in the left class is an *iterated Dwyer-Kan localisation*. We saw in 3.5 that the adjoint pair of functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category for logoi and the model category for simplicial categories. A map of simplicial sets $X \to Y$ is a homotopy localisation iff the functor $C_!(f) : C_!(X) \to C_!(Y)$ is a Dwyer-Kan localisation.

12. The contravariant model structure

In this section we introduce the covariant and the contravariant model structures on the category $\mathbf{S}/B$ for any simplicial set B. In the covariant structure, the fibrant objects are the left fibrations $X \to B$, and in the contravariant structure they are the right fibrations $X \to B$. The results of this section are taken from [J2].

12.1. The category $\mathbf{S}/B$ is enriched over $\mathbf{S}$ for any simplicial set B. We shall denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. If we apply the functor π_0 to the composition map $[Y, Z] \times [X, Y] \to [X, Z]$ we obtain a composition law

$$\pi_0[Y, Z] \times \pi_0[X, Y] \to \pi_0[X, Z]$$

for a category $(\mathbf{S}/B)^{\pi_0}$ if we put

$$(\mathbf{S}/B)^{\pi_0}(X, Y) = \pi_0[X, Y].$$

We shall say that a map in $\mathbf{S}/B$ is a *fibrewise homotopy equivalence* if the map is invertible in the category $(\mathbf{S}/B)^{\pi_0}$.

12.2. Let $\mathbf{R}(B)$ be the full subcategory of $\mathbf{S}/B$ spanned by the right fibrations $X \to B$. If $X \in \mathbf{R}(B)$, then the simplicial set $[A, X]$ is a Kan complex for every object $A \in \mathbf{S}/B$. In particular, the fiber $[b, X] = X(b)$ is a Kan complex for every vertex $b : 1 \to B$. A map $u : X \to Y$ in $\mathbf{R}(B)$ is a fibrewise homotopy equivalence iff the induced map between the fibers $X(b) \to Y(b)$ is a homotopy equivalence for every vertex $b \in B$.

12.3. We shall say that a map $u : M \to N$ in $\mathbf{S}/B$ is a *contravariant equivalence* if the map

$$\pi_0[u, X] : \pi_0[N, X] \to \pi_0[N, X]$$

is bijective for every object $X \in \mathbf{R}(B)$. A fibrewise homotopy equivalence in $\mathbf{S}/B$ is a contravariant equivalence and the converse holds for a map in $\mathbf{R}(B)$. A final map $M \to N$ in $\mathbf{S}/B$ is a contravariant equivalence and the converse holds if $N \in \mathbf{R}(B)$. A map $u : X \to Y$ in $\mathbf{S}/B$ is a contravariant equivalence iff its base change $L \times_B u : L \times_B X \to L \times_B Y$ along any left fibration $L \to B$ is a weak homotopy equivalence. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a left anodyne map $1 \to Lb$ followed by a left fibration $Lb \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant equivalence iff the map $Lb \times_B u : Lb \times_B M \to Lb \times_B N$ is a weak homotopy equivalence for every vertex $b \in B$. When B is a logos, we can take $Lb = b\backslash B$. In which case a map $u : M \to N$ is a contravariant equivalence iff the map $b\backslash u = b\backslash M \to b\backslash N$ is a weak homotopy equivalence for every object $b \in B$.

12.4. For any simplicial set B, the category $\mathbf{S}/B$ admits a model structure in which the weak equivalences are the contravariant equivalences and the cofibrations are the monomorphisms, We shall say that it is the *contravariant model structure* in $\mathbf{S}/B$. The fibrations are called *contravariant fibrations* and the fibrant objects are the right fibrations $X \to B$. The model structure is simplicial and we shall denote it shortly by $(\mathbf{S}/B, Wcont(B))$, or more simply by $(\mathbf{S}/B, Wcont)$, where $Wcont(B)$ denotes the class of contravariant equivalences in $\mathbf{S}/B$.

12.5. Every contravariant fibration in $\mathbf{S}/B$ is a right fibration and the converse holds for a map in $\mathbf{R}(B)$.

12.6. The contravariant model structure $(\mathbf{S}/B, Wcont)$ is a Bousfield localisation of the model structure $(\mathbf{S}/B, Wcat)$ induced by the model structure $(\mathbf{S}, Wcat)$ on $\mathbf{S}/B$. It follows that a weak categorical equivalence in $\mathbf{S}/B$ is a contravariant equivalence and that the converse holds for a map in $\mathbf{R}(B)$. Every contravariant fibration in $\mathbf{S}/B$ is a pseudo-fibration and the converse holds for a map in $\mathbf{R}(B)$.

12.7. A map $u : (M, p) \to (N, q)$ in $\mathbf{S}/B$ is a contravariant equivalence iff the map $bq\backslash X \to bp\backslash X$ induced by u is an equivalence of logoi of any map $b : B \to X$ with values in a logos X.

12.8. Dually, we say that a map $u : M \to N$ in $\mathbf{S}/B$ is a *covariant equivalence* if the opposite map $u^o : M^o \to N^o$ in $\mathbf{S}/B^o$ is a contravariant equivalence. Let $\mathbf{L}(B)$ be the full subcategory of $\mathbf{S}/B$ spanned by the left fibrations $X \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a covariant equivalence iff the map

$$\pi_0[u, X] : \pi_0[N, X] \to \pi_0[N, X]$$

is bijective for every object $X \in \mathbf{L}(B)$. A fibrewise homotopy equivalence in $\mathbf{S}/B$ is a covariant equivalence and the converse holds for a map in $\mathbf{L}(B)$. An initial map $M \to N$ in $\mathbf{S}/B$ is a covariant equivalence and the converse holds if $N \in \mathbf{N}(B)$. A map $u : M \to N$ in $\mathbf{S}/B$ is a covariant equivalence iff its base change $R \times_B u : R \times_B M \to R \times_B N$ along any right fibration $R \to B$ is a weak homotopy equivalence. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a covariant equivalence iff the map $Rb \times_B u : Rb \times_B X \to Rb \times_B Y$ is a weak homotopy equivalence for every vertex $b \in B$. When B is a logos, we can take $Rb = B/b$. In this case a map $u : M \to N$ is a covariant equivalence iff the map $u/b = M/b \to N/b$ is a weak homotopy equivalence for every object $b \in B$.

12.9. Dually, for any simplicial set B, the category $\mathbf{S}/B$ admits a model structure in which the weak equivalences are the covariant equivalences and the cofibrations are the monomorphisms, We shall say that it is the *covariant model structure* in $\mathbf{S}/B$. The fibrations are called *covariant fibrations* and fibrant objects are the left fibrations $X \to B$. The model structure is simplicial and we shall denote it shortly by $(\mathbf{S}/B, Wcov(B))$, or more simply by $(\mathbf{S}/B, Wcov)$, where $Wcov(B)$ denotes the class of covariant equivalences in $\mathbf{S}/B$.

12.10. Every covariant fibration in $\mathbf{S}/B$ is a left fibration and the converse holds for a map in $\mathbf{L}(B)$.

12.11. The covariant and the contravariant model structures on $\mathbf{S}/B$ are Cisinski structures. The covariant structure is determined by the left fibrations $X \to B$ and the contravariant structure by the right fibrations $X \to B$.

12.12. For any simplicial set B, we shall put

$$\mathcal{P}(B) = Ho(\mathbf{S}/B, Wcont) \quad \text{and} \quad \mathcal{Q}(B) = Ho(\mathbf{S}/B, Wcov).$$

The functor $X \mapsto X^o$ induces an isomorphism of model categories

$$(\mathbf{S}/B, Wcont) \simeq (\mathbf{S}/B^o, Wcov),$$

hence also of categories $\mathcal{P}(B) \simeq \mathcal{Q}(B^o)$.

12.13. The base change of a contravariant equivalence in $\mathbf{S}/B$ along a left fibration $A \to B$ is a contravariant equivalence in $\mathbf{S}/A$. Dually, the base change of a covariant equivalence in $\mathbf{S}/B$ along a right fibration $A \to B$ is a covariant equivalence in $\mathbf{S}/B$.

12.14. When the category $\tau_1 B$ is a groupoid, the two classes $Wcont(B)$ and $Wcov(B)$ coincide with the class of weak homotopy equivalences in $\mathbf{S}/B$. In particular, the model categories $(\mathbf{S}, Wcont)$, $(\mathbf{S}, Wcov)$ and $(\mathbf{S}, Who)$, coincide. Thus,

$$\mathcal{Q}(1) = \mathcal{P}(1) = Ho(\mathbf{S}, Who).$$

12.15. If $X, Y \in \mathbf{S}/B$, let us put

$$\langle X \mid Y \rangle = X \times_B Y.$$

This defines a functor of two variables

$$\langle - \mid - \rangle : \mathbf{S}/B \times \mathbf{S}/B \to \mathbf{S}.$$

If $X \in \mathbf{L}(B)$, then the functor $\langle X \mid - \rangle$ is a left Quillen functor between the model categories $(\mathbf{S}/B, Wcont)$ and $(\mathbf{S}, Who)$. Dually, if $Y \in \mathbf{R}(B)$, then the functor $\langle - \mid Y \rangle$ is a left Quillen functor between the model categories $(\mathbf{S}/B, Wcov)$ and $(\mathbf{S}, Who)$. It follows that the functor $\langle - \mid - \rangle$ induces a functor of two variables,

$$\langle - \mid - \rangle : \mathcal{Q}(B) \times \mathcal{P}(B) \to Ho(\mathbf{S}, Who).$$

A morphism $v : Y \to Y'$ in $\mathcal{P}(B)$ is invertible iff the morphism

$$\langle X|v \rangle : \langle X \mid Y \rangle \to \langle X \mid Y' \rangle$$

is invertible for every $X \in \mathcal{Q}(B)$. Dually, a morphism $u : X \to X'$ in $\mathcal{Q}(B)$ is invertible iff the morphism

$$\langle u|Y \rangle : \langle X \mid Y \rangle \to \langle X' \mid Y \rangle$$

is invertible for every $Y \in \mathcal{P}(B)$.

12.16. We say that an object $X \to B$ in $\mathbf{S}/B$ is *finite* if X is a finite simplicial set. We shall say that a right fibration $X \to B$ is *finitely generated* if it is isomorphic to a finite object in the homotopy category $\mathcal{P}(B)$. A right fibration $X \to B$ is finitely generated iff there exists a final map $F \to X$ with codomain a finite object of $\mathbf{S}/B$. The base change $u^*(X) \to A$ of a finitely generated right fibration $X \to B$ along a weak categorical equivalence is finitely generated.

12.17. We shall say that a map $f : A \to B$ in $\mathbf{S}^I$ is a *contravariant equivalence*

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "a"'] & B_0 \arrow[d, "b"] \\ A_1 \arrow[r, "f_1"] & B_1 \end{tikzcd}$$

if f_1 is a weak categorical equivalence and the map $(f_1)_!(A_0) \to B_1$ induced by f_0 is a contravariant equivalence in $\mathbf{S}/B_1$. The category $\mathbf{S}^I$ admits a cartesian closed model structure in which the weak equivalences are contravariant equivalences and the cofibrations are the monomorphisms. We shall denote it shortly by $(\mathbf{S}^I, Wcont)$. The fibrant objects are the right fibrations between logoi. The target functor

$$t : \mathbf{S}^I \to \mathbf{S}$$

is a Grothendieck bifibration and both a left and a right Quillen functor between the model categories $(\mathbf{S}^I, Wcont)$ and $(\mathbf{S}, Wcat)$. It gives the model category $(\mathbf{S}^I, Wcont)$ the structure of a bifibered model category over the model category $(\mathbf{S}, Wcat)$. We shall say that it is the *fibered model category for right fibrations*. It induces the contravariant model structure on each fiber $\mathbf{S}/B$. See 51.32 for the notion of bifibered model category. There is a dual *fibered model category for left fibrations* $(\mathbf{S}^I, Wcov)$

12.18. The model category $(\mathbf{S}/B, Wcont)$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of weak homotopy equivalences in $\mathbf{S}/B$. A contravariant fibration belongs to $\mathcal{B}$ iff it is a Kan fibration. It follows from 19.13 that a map $X \to Y$ in $\mathbf{R}(B)$ belongs to $\mathcal{B}$ iff the following square of fibers

$$\begin{tikzcd} X(b) \arrow[r, "u^*"] \arrow[d] & X(a) \arrow[d] \\ Y(b) \arrow[r, "u^*"] & Y(a) \end{tikzcd}$$

is homotopy cartesian in $(\mathbf{S}, Who)$ for every arrow $u : a \to b$ in B. We shall say that a map in $\mathcal{B}$ is *term-wise cartesian*.

13. Distributors, cylinders, correspondances and mediators

In this section we introduce the five notions in the title. We show that these notions are Quillen equivalent. It follows that the simplicial presheaves on a fibrant simplicial category are Quillen equivalent to the right fibrations on the coherent nerve of this category.

13.1. Recall that the category **SCat** of small simplicial categories and simplicial functors is cartesian closed. If A and B are small simplicial category, we shall denote by $[A, B]$ the simplicial category of simplicial functors $A \to B$. If A is a small simplicial category, we shall denote by $[A, \mathbf{S}]$ the large simplicial category of simplicial functors $A \to \mathbf{S}$. A simplicial functor $u : A \to B$ induces a functor

$$u^* = [u, \mathbf{S}] : [B, \mathbf{S}] \to [A, \mathbf{S}]$$

with a left adjoint $u_!$ and a right adjoint u_*.

13.2. Recall that the category $[A, \mathbf{S}]$ admits a model structure, called the *projective model structure*, in which the weak equivalences are the term-wise weak homotopy equivalences and the fibrations are the term-wise Kan fibrations [Hi]. We shall denote this model structure by $[A, \mathbf{S}]^{proj}$. If $u : A \to B$ is a simplicial functor, then the pair

$$u_! : [A, \mathbf{S}] \to [B, \mathbf{S}] : u^*$$

is a Quillen adjunction with respect to the projective model structures on these categories. And the pair is a Quillen equivalence when u is a Dwyer-Kan equivalence [Hi].

13.3. Recall that the category $[A, \mathbf{S}]$ admits a model structure, called the *injective model structure* in which the weak equivalences are the term-wise weak homotopy equivalences and the cofibrations are the term-wise cofibrations [Hi]. We shall denote this model structure by $[A, \mathbf{S}]^{inj}$. The identity functor

$$[A, \mathbf{S}]^{proj} \to [A, \mathbf{S}]^{inj}.$$

is the left adjoint in a Quillen equivalence between the projective and the inductive model structures. If $u : A \to B$ is a simplicial functor, then the pair

$$u^* : [B, \mathbf{S}] \to [A, \mathbf{S}] : u_*$$

is a Quillen adjunction with respect to the injective model structures on these categories. And the pair is a Quillen equivalence when u is a Dwyer-Kan equivalence [Hi].

13.4. If A and B are small simplicial categories, we shall say that a simplicial functor $F : A^o \times B \to \mathbf{S}$ is a *simplicial distributor* , or an S-*distributor*, from A to B and we shall write $F : A \Rightarrow B$. The S-distributors $A \Rightarrow B$ are the objects of a simplicial category $SDist(A, B) = [A^o \times B, \mathbf{S}]$.

13.5. A *simplicial distributor*, or S-*distributor*, is defined to be an object $C \to I$ of the category $\mathbf{SCat}/I$, where the category $I = [n]$ is regarded as a simplicial category. The *base* of a distributor $p : C \to I$ is the cosieve $C(1) = p^{-1}(1)$ and its *cobase* is the sieve $C(0) = p^{-1}(0)$. If i denotes the inclusion $\{0, 1\} \subset I$, then the pullback functor

$$i^* : \mathbf{SCat}/I \to \mathbf{SCat} \times \mathbf{SCat}$$

has left adjoint $i_!$ and a right adjoint i_*. The functor i^* is a Grothendieck bifibration, since it is an isofibration and the functor $i_!$ is fully faithful. It s fiber at (A, B) is the category $SCyl(A, B)$ of simplicial distributors with cobase A and base B. To every distributor $C \in SCyl(A, B)$ we can associate a distributor $D(C) \in SDist(A, B)$ by putting $D(C)(a, b) = C(a, b)$ for every pair of objects $a \in A$ and $b \in B$. The resulting functor

$$D : SCyl(A, B) \to SDist(A, B)$$

is an equivalence of categories. The inverse equivalence associates to a S-distributor $F : A \Rightarrow B$ its *collage cylinder* $col(F) = A \star_F B$.

13.6. We shall say that a full simplicial subset $S \subseteq X$ of a simplicial set X is a *sieve* if the implication target$(f) \in S \Rightarrow$ source$(f) \in S$ is true for every arrow $f \in X$. If $h : X \to \tau_1 X$ is the canonical map, then the map $S \mapsto h^{-1}(S)$ induces a bijection between the sieves in the category $\tau_1 X$ and the sieves in X. For any sieve $S \subseteq X$ there exists a unique map $g : X \to I$ such that $S = g^{-1}(0)$. This defines a bijection between the sieves in X and the maps $X \to I$. Dually, we shall say that a full simplicial subset $S \subseteq X$ is a *cosieve* if the implication source$(f) \in S \Rightarrow$ target$(f) \in S$ is true for every arrow $f \in X$. A simplicial subset $S \subseteq X$ is a cosieve iff the opposite subset $S^o \subseteq X^o$ is a sieve. For any cosieve $S \subseteq X$ there exists a unique map $g : X \to I$ such that $S = g^{-1}(1)$. The cosieve $g^{-1}(1)$ and the sieve $g^{-1}(0)$ are said to be *complementary*. Complementation defines a bijection between the sieves and the cosieves of X.

13.7. We shall say that an object $p : C \to I$ of the category $\mathbf{S}/I$ is a *cylinder*. The *base* of the cylinder is the cosieve $C(1) = p^{-1}(1)$ and its *cobase* is the sieve $C(0) = p^{-1}(0)$. If $C(0) = 1$ we say that C is a *projective cone*, and if $C(1) = 1$ we say that it is an *inductive cone*. If $C(0) = C(1) = 1$, we say that C is a *spindle*. If i denotes the inclusion $\partial I \subset I$, then the functor

$$i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$$

has left adjoint $i_!$ and a right adjoint i_*. The functor i^* is a Grothendieck bifibration, since the functor $i_!$ is fully faithful (hence also the functor i_*). Its fiber at (A, B) is the category $Cyl(A, B)$ of cylinders with cobase A and base B. The initial object of this category is the cylinder $A \sqcup B$ and its terminal object is the cylinder $A \star B$.

An object $q : X \to A \star B$ of the category $\mathbf{S}/A \star B$ belongs to $Cyl(A,B)$ iff the map $q^{-1}(A \sqcup B) \to A \sqcup B$ induces by q is an isomorphism. Hence the canonical functors

$$Cyl(A,B) \subset \mathbf{S}/A \star B, \quad Cyl(A,B) \subset A \sqcup B \backslash \mathbf{S}, \quad Cyl(A,B) \subset A \sqcup B \backslash \mathbf{S}/A \star B$$

are fully faithful. The cobase change $(u,v)_!(X)$ of a cylinder $X \in Cyl(A,B)$ along a pair of maps $u : A \to C$ and $v : B \to D$ is calculated by the following pushout square of simplicial sets,

$$\begin{tikzcd} A \sqcup B \arrow[r, "u \sqcup v"] \arrow[d] & C \sqcup D \arrow[d, "p"] \\ X \arrow[r] & (u,v)_!(X). \end{tikzcd}$$

The base change $(u,v)^*(Y)$ of a cylinder $Y \in Cyl(C,D)$ along (u,v) is calculated by the following pullback square of simplicial sets,

$$\begin{tikzcd} (u,v)^*(Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ A \star B \arrow[r, "u \star v"] & C \star D. \end{tikzcd}$$

13.8. Let $\mathbf{S}^{(2)} = [\Delta^o \times \Delta^o, \mathbf{Set}]$ be the category of bisimplicial sets. If $A, B \in \mathbf{S}$, let us put

$$(A \Box B)_{mn} = A_m \times B_n$$

for $m, n \geq 0$. If X is a bisimplicial set, a map $X \to A \Box 1$ is called a *column augmentation* of X and a map $X \to 1 \Box B$ is called a *row augmentation*. We shall say that a map $X \to A \Box B$ is a *biaugmentation* of X or that it is a *correspondence* $A \Rightarrow B$. The correspondences $A \to B$ form a category

$$Cor(A,B) = \mathbf{S}^{(2)}/A \Box B.$$

The simplicial set $\Delta[m] \star \Delta[n]$ has the structure of a cylinder for every $m, n \geq 0$. To every cylinder $C \in \mathbf{S}/I$ we can associate a correspondance $cor(C) \to C(0) \Box C(1)$ by by putting

$$cor(C)_{mn} = Hom(\Delta[m] \star \Delta[n], C)$$

for every $m, n \geq 0$. The structure map $cor(C) \to C(0) \Box C(1)$ is defined from the inclusions $\Delta[m] \sqcup \Delta[n] \subseteq \Delta[m] \star \Delta[n]$. The induced functor

$$cor : Cyl(A,B) \to Cor(A,B).$$

is an equivalence of categories. See[Gon].

13.9. The model structure $(\mathbf{S}, Wcat)$ induces a cartesian closed model structure $(\mathbf{S}/I, Wcat)$ on the category $\mathbf{S}/I$. The resulting model category is bifibered by the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$ over the model category

$$(\mathbf{S}, Wcat) \times (\mathbf{S}, Wcat) = (\mathbf{S} \times \mathbf{S}, Wcat \times Wcat).$$

It thus induces a model structure $(Cyl(A,B), Wcat)$ on the category $Cyl(A,B)$ for each pair (A,B). We conjecture that a cylinder $X \in Cyl(A,B)$ is fibrant in the model category $(Cyl(A,B), Wcat)$ iff its structure map $X \to A \star B$ is a mid fibration. We conjecture that a map between fibrant cylinders in $Cyl(A,B)$ is a fibration iff it is a mid fibration.

13.10. The *opposite* of a cylinder $C \in Cyl(A, B)$ is a cylinder $C^o \in Cyl(B^o, A^o)$.
The functor

$$(-)^o : Cyl(A, B) \to Cyl(B^o, A^o)$$

is isomorphism between the model categories $(Cyl(A, B), Wcat)$ and $(Cyl(B^o, A^o), Wcat)$.

13.11. The cobase change $(u, v)_!(X)$ of a cylinder $X \in Cyl(A, B)$ along a pair of maps $u : A \to C$ and $v : B \to D$ is calculated by the following pushout square of simplicial sets,

$$\begin{tikzcd} A \sqcup B \arrow[r, "u \sqcup v"] \arrow[d] & C \sqcup D \arrow[d, "p"] \\ X \arrow[r] & (u, v)_!(X). \end{tikzcd}$$

The base change $(u, v)^*(Y)$ of a cylinder $Y \in Cyl(C, D)$ along (u, v) is calculated by the following pullback square of simplicial sets,

$$\begin{tikzcd} (u, v)^*(Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ A \star B \arrow[r, "u \star v"] & C \star D. \end{tikzcd}$$

The adjoint pair

$$(u, v)_! : Cyl(A, B) \leftrightarrow Cyl(C, D) : (u, v)^*$$

is a Quillen adjunction with respect to the model structures $(Cyl(A, B), Wcat)$ and $(Cyl(C, D), Wcat)$. And it is a Quillen equivalence if u and v are weak categorical equivalences.

13.12. If A and B are simplicial sets, we shall say that a map $X \to B^o \times A$ is a *mediator* and we shall write $X : A \Rightarrow B$. The mediators $A \Rightarrow B$ form a category

$$Med(A, B) = \mathbf{S}/(A^o \times B).$$

A *map* $X \to Y$ between two mediators $X \in Med(A, B)$ and $Y \in Med(C, D)$ is a triple of maps $u : A \to C$, $v : B \to D$ and $f : X \to Y$ fitting in a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ A^o \times B \arrow[r, "u^o \times v"] & C^o \times D. \end{tikzcd}$$

The mediators form a category Med. The functor

$$p : Med \to \mathbf{S} \times \mathbf{S}$$

defined by putting $p(X, A, B) = (A, B)$ is a Grothendieck bifibration and its fiber at (A, B) is the category $Med(A, B)$. We shall say that the map $(f, u, v) : X \to Y$ in Med is an *mediator equivalence* if the maps u and v are weak categorical equivalences and the map $f : X \to Y$ is a covariant equivalence in $Med(C, D)$. The category Med admits a model structure in which the weak equivalences are the mediator equivalences and the cofibrations are the monomorphisms. The model structure is left proper and cartesian closed. We shall denote the resulting model category by $(Med, Wmed)$. The model category is bifibered by the functor p :

$Med \to \mathbf{S} \times \mathbf{S}$ over the model category $(\mathbf{S}, Wcat) \times (\mathbf{S}, Wcat)$. It induces the model structure $(\mathbf{S}/(B^o \times A), Wcov)$ on the fiber $Med(A, B)$. A mediator $X \in Med(A, B)$ is fibrant with respect to this model structure iff its structure map $X \to A^o \times B$ is a left fibration.

13.13. The simplicial set $\rho[n] = \Delta[n]^o \star \Delta[n]$ has the structure of a cylinder for every $n \geq 0$. The *twisted section* of a cylinder C is the simplicial set $\rho^*(C)$ defined by putting

$$\rho^*(C)_n = Hom_I(\rho[n], C)$$

for every $n \geq 0$. The simplicial set $\rho^*(C)$ has the structure of a mediator with a structure map $\rho^*(C) \to C(0)^o \times C(1)$ obtained from the the inclusion $\Delta[n]^o \sqcup \Delta[n] \subset \Delta[n]^o \star \Delta[n]$. The resulting functor $\rho^* : \mathbf{S}/I \to Med$ has a left adjoint $\rho_!$ and a right adjoint ρ_*. The pair of adjoint functors

$$\rho_! : Med \leftrightarrow: \mathbf{S}/I\rho^*$$

is a Quillen equivalence between the model categories $(\mathbf{S}/I, Wcat)$ and $(Med, Wmed)$. The induced pair of adjoint functors

$$\rho^* : Cyl(A, B) \leftrightarrow Med(A, B) : \rho_*$$

is a Quillen equivalence between the model categories $(Cyl(A, B), Wcat)$ and $(\mathbf{S}/(A^o \times B), Wcov)$ for any pair of simplicial sets (A, B). In particular, the pair

$$\rho_! : \mathbf{S}/B \leftrightarrow Cyl(1, B) : \rho^*$$

is a Quillen equivalence between the model categories $(Cyl(1, B), Wcat)$ and $(\mathbf{S}/B, Wcov)$ Hence a map $f : X \to Y$ in $\mathbf{S}/B$ is a covariant equivalence iff the map $\rho_!(f) : \rho_!(X) \to \rho_!(Y)$ is a weak categorical equivalence.

13.14. The Quillen equivalence

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

of 3.5 induces a Quillen equivalence

$$C_! : \mathbf{S}/I \leftrightarrow \mathbf{SCat}/I : C^!,$$

since we have $C^!(I) = I$ and $C_!(I) = I$. The pair $(C_!, C^!)$ also induces a Quillen equivalence

$$C_! : Cyl(A, B) \leftrightarrow SCyl(C_!A, C_!B) : C^!$$

for any pair pair of simplicial sets A and B. By composing the equivalence with the Quillen equivalence

$$\rho_! : \mathbf{S}/A^o \times B \leftrightarrow Cyl(A, B) : \rho^*$$

of 13.13 and the equivalence of categories

$$D : SCyl(C_!A, C_!B) \to SDist(C_!A, C_!B) : col$$

of 13.5, we obtain a a Quillen equivalence

$$\mathbf{S}/A^o \times B \leftrightarrow SDist(C_!A, C_!B)$$

between the model category $(\mathbf{S}/A^o \times B, Wcov)$ and the projective model category $SDist(C_!A, C_!B)$. In particular, this yields a Quillen equivalence

$$\mathbf{S}/B \leftrightarrow [C_!B, \mathbf{S}]$$

between the model category $(\mathbf{S}/B, Wcov)$ and the projective model category $[C_!B, \mathbf{S}]$.

13.15. Dually, the pair $(C_!, C^!)$ induces a Quillen equivalence

$$C_! : Cyl(C^! X, C^! Y) \leftrightarrow SCyl(X, Y) : C^!$$

for any pair of fibrant simplicial categories X and Y. By composing the equivalence with the Quillen equivalence

$$\rho_! : \mathbf{S}/C^! X^o \times C^! Y \leftrightarrow Cyl(C^! X, C^! Y) : \rho^*$$

of 13.13 and the equivalence of categories

$$D : SCyl(X, Y) \to SDist(X, Y) : col$$

of 13.5, we obtain a a Quillen equivalence

$$\mathbf{S}/C^! X^o \times C^! Y \leftrightarrow SDist(X, Y)$$

between the model category $(\mathbf{S}/C^! X^o \times C^! Y, Wcov)$ and the projective model category $SDist(X, Y)$. In particular, this yields a Quillen equivalence

$$\mathbf{S}/C^! Y \leftrightarrow [Y, \mathbf{S}]$$

between the model category $(\mathbf{S}/C^! Y, Wcov)$ and the projective model category $[Y, \mathbf{S}]$.

13.16. If Y is a small simplicial category, let us denote by $[Y, \mathbf{S}]^f$ the category of fibrant objects of the injective model category $[Y, \mathbf{S}]^{inj}$. If Y is enriched over Kan complexes, then the functor

$$[Y, \mathbf{S}] \to \mathbf{S}/C^! Y]$$

defined in induces a Dwyer-Kan equivalence of simplicial categories

$$[Y, \mathbf{S}]^f \to \mathbf{L}(C^! Y).$$

14. Base changes

In this section, we study the base change functors between the contravariant model structures. We introduce the notion of dominant map. The results of the section are taken from [J2].

14.1. A functor $u : A \to B$ between two small categories induces a pair of adjoint functors between the presheaf categories,

$$u_! : \hat{A} \to \hat{B} : u^*.$$

A functor u is called a *Morita equivalence* if the adjoint pair $(u_!, u^*)$ is an equivalence of categories. By a classical result, u is a Morita equivalence iff it is fully faithful and every object $b \in B$ is a retract of an object in the image of u. The notion of Morita equivalence can be decomposed, since the pair $(u_!, u^*)$ is an equivalence iff the functors $u_!$ and u^* are fully faithful. The functor $u_!$ is fully faithful iff the functor u is fully faithful. A functor $u : A \to B$ is said to be *dominant*, but we shall say *0-dominant*, if the functor u^* is fully faithful. A functor $u : A \to B$ is 0-dominant iff the category $Fact(f, A)$ defined by the pullback square

$$\begin{tikzcd} Fact(f,A) \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ Fact(f,B) \arrow[r] & B \end{tikzcd}$$

is connected for every arrow $f \in B$, where $Fact(f, B) = f\backslash(B/b) = (a\backslash B)/f$ is the category of factorisations of the arrow $f : a \to b$. We notice that the functor u is 0-final iff we have $u_!(1) = 1$, where 1 denotes terminal objects.

14.2. For any map of simplicial sets $u : A \to B$, the adjoint pair

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction with respect to the contravariant model structures on these categories. The adjunction is a Quillen equivalence when u is a weak categorical equivalence. It induces an adjoint pair of derived functors

$$\mathcal{P}_!(u) : \mathcal{P}(A) \leftrightarrow \mathcal{P}(B) : \mathcal{P}^*(u),$$

The functor $\mathcal{P}_!(u)$ is directly induced by the functor $u_!$, since $u_!$ takes a covariant equivalence to a covariant equivalence. Hence we have $\mathcal{P}_!(vu) = \mathcal{P}_!(v)\mathcal{P}_!(u)$ for any pair of maps $u : A \to B$ and $v : B \to C$. This defines a functor

$$\mathcal{P}_! : \mathbf{S} \to \mathbf{CAT},$$

where **CAT** is the category of large categories. We shall see in 23.5 that the functor $\mathcal{P}^*(u)$ has a right adjoint $\mathcal{P}_*(u)$.

14.3. Dually, for any map of simplicial sets $u : A \to B$, the adjoint pair

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction with respect to the covariant model structures on these categories, (and it is a Quillen equivalence when u is a weak categorical equivalence). It induces an adjoint pair of derived functors

$$\mathcal{Q}_!(u) : \mathcal{Q}(A) \leftrightarrow \mathcal{Q}(B) : \mathcal{Q}^*(u).$$

The functor $\mathcal{Q}^*(u)$ has a right adjoint $\mathcal{Q}_*(u)$ by 23.5. If $u : A \to B$ and $v : B \to C$, then we have $\mathcal{Q}_!(vu) = \mathcal{Q}_!(v)\mathcal{Q}_!(u)$. This defines a functor

$$\mathcal{Q}_! : \mathbf{S} \to \mathbf{CAT},$$

where **CAT** is the category of large categories.

14.4. A map of simplicial sets $u : A \to B$ is final iff the functor $\mathcal{P}_!(u)$ preserves terminal objects. A map $u : A \to B$ is fully faithful iff the functor $\mathcal{P}_!(u)$ is fully faithful.

14.5. We shall say that a map of simplicial sets $u : A \to B$ is *dominant* if the functor $\mathcal{P}^*(u)$ is fully faithful.

14.6. A map $u : A \to B$ is dominant iff the opposite map $u^o : A^o \to B^o$ is dominant iff the map $X^u : X^B \to X^A$ is fully faithful for every logos X.

14.7. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a fully faithful map to a fully faithful functor, and a dominant map to a 0-dominant functor.

14.8. If B is a logos, then a map of simplicial sets $u : A \to B$ is dominant iff the simplicial set $Fact(f, A)$ defined by the pullback square

$$\begin{tikzcd} Fact(f,A) \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ Fact(f,B) \arrow[r] & B \end{tikzcd}$$

is weakly contractible for every arrow $f \in B$, where $Fact(f, B) = f\backslash(B/b) = (a\backslash B)/f$ is the simplicial set of factorisations of the arrow $f : a \to b$.

14.9. A dominant map is both final and initial. A map of simplicial set $u : A \to B$ is dominant iff its base change any right fibration is final iff its base change any left fibration is initial. The base change of a dominant map along a left or a right fibration is dominant. A (weak) reflection and a (weak) coreflection are dominant. An iterated homotopy localisation is dominant.

14.10. The functor

$$\mathcal{P}_! : \mathbf{S} \to \mathbf{CAT}$$

has the structure of a 2-functor covariant on 2-cells. It follows by adjointness that $\mathcal{P}^*$ has the structure of a contravariant (pseudo) 2-functor,

$$\mathcal{P}^* : \mathbf{S} \to \mathbf{CAT},$$

contravariant on 2-cells. In order to define the natural transformation

$$\mathcal{P}_!(\alpha) : \mathcal{P}_!(u) \to \mathcal{P}_!(v)$$

associated to a 2-cell $\alpha : u \to v : A \to B$ in the category $\mathbf{S}^{\tau_1}$ we can suppose that $\alpha = [h]$, where h is the homotopy between the inclusions $i_0, i_1 : A \to A \times I$. If $X \in \mathbf{S}/A$, then $(i_0)_!(X) = X \times \{0\}$ and $(i_1)_!(X) = X \times \{1\}$. The inclusion $X \times \{1\} \subseteq X \times I$ is a covariant equivalence in $\mathbf{S}/(A \times I)$, since it is right anodyne; it is thus invertible in the category $\mathcal{P}(A \times I)$. The morphism $\mathcal{P}_!([h]) : \mathcal{P}_!(i_0)(X) \to \mathcal{P}_!(i_1)(X)$ is obtained by composing the inclusion $X \times \{0\} \subseteq X \times I$ with the inverse morphism $X \times I \to X \times \{1\}$.

14.11. If (α, β) is an adjunction between two maps $u : A \leftrightarrow B : v$ in the 2-category $\mathbf{S}^{\tau_1}$, then the pair $(\mathcal{P}_!(\alpha), \mathcal{P}_!(\beta))$ is an adjunction $\mathcal{P}_!(u) \vdash \mathcal{P}_!(v)$, and the pair $(\mathcal{P}^*(\beta), \mathcal{P}^*(\alpha))$ is an adjunction $\mathcal{P}^*(u) \vdash \mathcal{P}^*(v)$. We thus have a canonical isomorphism $\mathcal{P}_!(v) \simeq \mathcal{P}^*(u)$,

$$\mathcal{P}_!(u) \vdash \mathcal{P}_!(v) \simeq \mathcal{P}^*(u) \vdash \mathcal{P}^*(v).$$

14.12. Dually,

$$\mathcal{Q}_! : \mathbf{S} \to \mathbf{CAT}$$

is a covariant 2-functor contravariant on 2-cells, and

$$\mathcal{Q}^* : \mathbf{S}^{\tau_1} \to \mathbf{CAT}$$

is a contravariant (pseudo-) 2-functor covariant on 2-cells. If (α, β) is an adjunction between two maps $u : A \leftrightarrow B : v$ in the 2-category $\mathbf{S}^{\tau_1}$, then the pair $(\mathcal{Q}_!(\beta), \mathcal{Q}_!(\alpha))$ is an adjunction $Q_!(v) \vdash Q_!(u)$, and the pair $(\mathcal{Q}^*(\alpha), \mathcal{Q}^*(\beta))$ is an adjunction $Q^*(v) \vdash Q^*(u)$. We thus have a canonical isomorphism $Q_!(u) \simeq \mathcal{Q}^*(v)$,

$$Q_!(v) \vdash Q_!(u) \simeq Q^*(v) \vdash Q^*(u).$$

14.13. The pseudo-2-functor $\mathcal{Q}^*$ induces a functor

$$\mathcal{Q}^* : \tau_1(A,B) \to \mathbf{CAT}\big(\mathcal{Q}(B), \mathcal{Q}(A)\big)$$

for each pair of simplicial sets A and B. In particular, it induces a functor

$$\mathcal{Q}^* : \tau_1(B) \to \mathbf{CAT}\big(\mathcal{Q}(B), \mathcal{Q}(1)\big)$$

for each simplicial set B. If $X \in \mathcal{Q}(B)$ and $b \in B_0$, let us put $D(X)(b) = \mathcal{Q}^*(b)(X)$. This defines a functor

$$D(X) : \tau_1(B) \to \mathcal{Q}(1) = Ho(\mathbf{S}, Who).$$

We shall say that $D(X)$ is the *homotopy diagram* of X. This extends the notion introduced in 8.15. Dually, every object $X \in \mathcal{P}(B)$ has a contravariant homotopy diagram

$$D(X) : \tau_1(B)^o \to \mathcal{P}(1) = Ho(\mathbf{S}, Who).$$

15. Spans and duality

In this section we introduce the notion of span and show that it is Quillen equivalent to that of cylinder. We also introduce the homotopy bicategory of spans Λ. We show that it has the structure of a compact closed symmetric monoidal bicategory.

15.1. A *span* $S : A \Rightarrow B$ between two simplicial sets is a pair of maps

$$\begin{tikzcd} & S \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B. \end{tikzcd}$$

Equivalently, a span $A \Rightarrow B$ is an object of the category

$$Span(A,B) = \mathbf{S}/(A \times B).$$

The terminal object of this category is the span $A \times_s B$ defined by the pair of projections

$$\begin{tikzcd} & A \times B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ A & & B. \end{tikzcd}$$

The *opposite* of a span $(s,t) : S \to A \times B$ is defined to be the span

$$(t^o, s^o) : S^o \to B^o \times A^o.$$

15.2. Let P be the poset of non-empty subsets of $\{0,1\}$. A functor $X : P^o \to \mathbf{S}$ is a span,

$$\begin{tikzcd} & X(01) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X(0) & & X(1). \end{tikzcd}$$

If i_0 denotes the inclusion $\{0\} \subset P$ and i_1 the inclusion $\{1\} \subset P$, then the functor

$$(i_0^*, i_1^*) : [P^o, \mathbf{S}] \to \mathbf{S} \times \mathbf{S}$$

is a Grothendieck bifibration. Its fiber at (A,B) is the category $Span(A,B)$.

15.3. Recall that the *composite* of a span $S : A \to B$ with a span $T : B \Rightarrow C$ is the span $T \circ S = S \times_B T : A \Rightarrow C$, defined by the pullback diagram,

$$\begin{tikzcd} & & S \times_B T \arrow[dl] \arrow[dr] & & \\ & S \arrow[dl, "s"'] \arrow[dr, "t"] & & T \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B & & C. \end{tikzcd}$$

This defines a functor

$$- \circ - : Span(B, C) \times Span(A, B) \to Span(A, C).$$

If $S : A \Rightarrow B$, $T : B \Rightarrow C$ and $U : C \Rightarrow D$, then the canonical isomorphism

$$(U \circ T) \circ S = S \times_B (T \times_C U) \simeq (S \times_B T) \times_C U = U \circ (T \circ S)$$

satisfies the coherence condition of MacLane. The span $(1_A, 1_A) : A \to A \times A$ is a unit $A \to A$ for this composition law. The spans between simplicial sets form a *bicategory Span* whose objects are simplicial sets. The bicategory *Span* is symmetric monoidal. The *tensor product* of two spans $S : A \Rightarrow B$ and $T : C \Rightarrow D$ is their cartesian product in the category $[P^o, \mathbf{S}]$,

$$S \otimes T = S \times T : A \times C \Rightarrow B \times D.$$

15.4. The image of a map $u : [n] \to I$ in Δ is a non-empty subset $Im(u) \subseteq \{0, 1\}$. Consider the functor

$$\sigma : \Delta/I \to \Delta \times P$$

defined by putting $\sigma(u) = ([n], Im(u))$ for a map $u : [n] \to I$. The functor We have $[(\Delta \times P)^o, \mathbf{Set}] = [P^o, \mathbf{S}]$. The functor

$$\sigma^* : [(\Delta \times P)^o, \mathbf{Set}] \to [(\Delta/I)^o, \mathbf{Set}]$$

is a functor $\sigma^* : [P^o, \mathbf{S}] \to \mathbf{S}/I$, since $[(\Delta \times P)^o, \mathbf{Set}] = [P^o, \mathbf{S}]$ and $[(\Delta/I)^o, \mathbf{Set}] = \mathbf{S}/I$. The functor σ^* is cartesian with respect to the fibered model structure on these categories. It thus induces a functor

$$\sigma^* : Span(A, B) \to Cyl(A, B)$$

for each pair of simplicial sets (A, B). We shall say that the cylinder $\sigma^*(S) \in Cyl(A, B)$ is the *realisation* of a span $S \in Span(A, B)$. The simplicial set $\sigma^*(S)$ can be calculated by the following pushout square of simplicial sets,

$$\begin{tikzcd} S \sqcup S \arrow[r, "s \sqcup t"] \arrow[d] & A \sqcup B \arrow[d] \\ I \times S \arrow[r] & \sigma^*(S). \end{tikzcd}$$

Notice that $\sigma^*(A \times_s B) = A \diamond B$. The functor σ^* has a left adjoint $\sigma_!$ and a right adjoint σ_*. If $C \in Cyl(A, B)$, then $\sigma_*(C)$ is the span

$$\begin{tikzcd} & {[I,C]} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B, \end{tikzcd}$$

where $[I, C]$ denotes the simplicial set of global sections of the structure map $C \to I$, where s is defined by the inclusion $\{0\} \subset I$ and t by the inclusion : $\{1\} \subset I$. In particular, $\sigma_*(A \star B) = A \times_s B$.

15.5. We shall say that a map $u : S \to T$ in $Span(A, B)$ is *bivariant equivalence* if the map

$$\sigma^*(u) : \sigma^*(S) \to \sigma^*(T)$$

is a weak categorical equivalence. The category $Span(A, B)$ admits a model structure in which a weak equivalence is a bivariant equivalence and a cofibration is a monomorphism. We shall say that a fibrant object is a *bifibrant span*. We shall denote it shortly by $Span(A, B), Wbiv)$ and put

$$\Lambda(A, B) = Ho(Span(A, B), Wbiv).$$

15.6. The opposition functor

$$(-)^o : Span(A, B) \to Span(B^o, A^o)$$

induces an isomorphism between the model categories $(Span(A, B), Wbiv)$ and $(Span(B^o, A^o), Wbiv)$. It thus induces an isomorphism of categories,

$$(-)^o : \Lambda(A, B) \to \Lambda(B^o, A^o).$$

15.7. For any pair of maps of simplicial sets $u : A \to C$ and $v : B \to D$, the pair of adjoint functor

$$(u \times v)_! : Span(A, B) \leftrightarrow Span(C, D) : (u \times v)^*$$

is a Quillen adjunction with respect to the biviariant model structures on these categories. And it is a Quillen equivalence if u and v are weak categorical equivalences.

15.8. The pair of adjoint functors

$$\sigma^* : Span(A, B) \leftrightarrow Cyl(A, B) : \sigma_*$$

is a Quillen equivalence between the model category $(Span(A, B), Wbiv)$ and the model category $(Cyl(A, B), Wcat)$.

15.9. A map $u : S \to T$ in $Span(A, B)$ is a bivariant equivalence iff the map

$$X \times_A u \times_B Y : X \times_A S \times_B Y \to X \times_A T \times_B Y$$

is a weak homotopy equivalence for every $X \in \mathbf{L}(A)$ and $Y \in \mathbf{R}(B)$. For each vertex $a \in A$, let us choose a factorisation $1 \to La \to A$ of the map $a : 1 \to A$ as a left anodyne map $1 \to La$ followed by a left fibration $La \to A$. Dually, for each vertex $b \in B$, let us choose a factorisation $1 \to Rb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map $u : S \to T$ in $Span(A, B)$ is a bivariant equivalence iff the map

$$La \times_A u \times_B Rb : La \times_A S \times_B Rb \to La \times_A T \times_B Rb$$

is a weak homotopy equivalence for every pair of vertices $(a, b) \in A \times B$. If A and B are logoi, we can take $La = a\backslash A$ and $Rb = B/b$. In this case, a map $u : S \to T$ in $Span(A, B)$ is a bivariant equivalence iff the map

$$a\backslash u/b : a\backslash S/b \to a\backslash T/b$$

is a weak homotopy equivalence for every pair of objects $(a, b) \in A \times B$, where the simplicial set $a\backslash S/b$ is defined by the pullback square

$$\begin{tikzcd} a\backslash S/b \arrow[r] \arrow[d] & S \arrow[d] \\ a\backslash A \times B/b \arrow[r] & A \times B. \end{tikzcd}$$

15.10. For any span $(s, t) : S \to A \times B$, the composite

$$\mathcal{Q}(A) \xrightarrow{\mathcal{Q}^*(s)} \mathcal{Q}(S) \xrightarrow{\mathcal{Q}_!(t)} \mathcal{Q}(B)$$

is a functor $\mathcal{Q}\langle S\rangle : \mathcal{Q}(A) \to \mathcal{Q}(B)$. If $u : S \to T$ is a map of spans,

$$\begin{tikzcd} & S \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "u"] & \\ A & & B \\ & T \arrow[ul, "l"] \arrow[ur, "r"'] & \end{tikzcd},$$

then form the counit $\mathcal{Q}_!(u)\mathcal{Q}^*(u) \to id$, we obtain a natural transformation

$$\mathcal{Q}\langle u\rangle : \mathcal{Q}\langle S\rangle = \mathcal{Q}_!(t)\mathcal{Q}^*(s) = \mathcal{Q}_!(r)\mathcal{Q}_!(u)\mathcal{Q}^*(u)\mathcal{Q}^*(l) \to \mathcal{Q}_!(r)\mathcal{Q}^*(l) = \mathcal{Q}\langle T\rangle.$$

This defines a functor

$$\mathcal{Q}\langle-\rangle : Span(A, B) \to \mathbf{CAT}(\mathcal{Q}(A), \mathcal{Q}(B)).$$

A map $u : S \to T$ in $Span(A, B)$ is a bivariant equivalence iff the natural transformation

$$\mathcal{Q}\langle u\rangle : \mathcal{Q}\langle S\rangle \to \mathcal{Q}\langle T\rangle$$

is invertible.

15.11. Dually, for any span $(s,t) : S \to A \times B$, the composite

$$\mathcal{P}(B) \xrightarrow{\mathcal{P}^*(t)} \mathcal{P}(S) \xrightarrow{P_!(s)} \mathcal{P}(A)$$

is a functor $\mathcal{P}\langle S\rangle : \mathcal{P}(B) \to \mathcal{P}(A)$. To every map $u : S \to T$ in $Span(A,B)$ we can associate a natural transformation

$$\mathcal{P}\langle u\rangle : \mathcal{P}\langle S\rangle \to \mathcal{P}\langle T\rangle.$$

We obtain a functor

$$\mathcal{P}\langle -\rangle : Span(A,B) \to \mathbf{CAT}(\mathcal{P}(B), \mathcal{P}(A)).$$

A map $u : S \to T$ in $Span(A,B)$ is a bivariant equivalence iff the natural transformation $\mathcal{P}\langle u\rangle$ is invertible.

15.12. If $S \in Span(A,B)$, let us denote by $S(a,b)$ the fiber of the map $(s,t)S \to A \times B$ at $(a,b) \in A_0 \times B_0$. The simplicial set $S(a,b)$ is a Kan complex if S is bifibrant. A map between bifibrant spans $u : S \to T$ in $Span(A,B)$ a bivariant equivalence iff the map

$$S(a,b) \to T(a,b)$$

induced by u is a homotopy equivalence for every pair $(a,b) \in A_0 \times B_0$.

15.13. If A and B are logoi, then a span $(s,t) : S \to A \times B$ is bifibrant iff the following conditions are satisfied:

- the source map $s : S \to A$ is a Grothendieck fibration;
- the target map $t : S \to B$ is a Grothendieck opfibration;
- an arrow $f \in S$ is inverted by t iff f is cartesian with respect to s;
- an arrow $f \in S$ is inverted by s iff f is cocartesian with respect to t.

The last two conditions are equivalent in the presence of the first two.

15.14. If A is a logos, then a fibrant replacement of the span $(1_A, 1_A) : A \to A \times A$ is the span $\delta_A = (s,t) : A^I \to A \times A$. If $u : A \to B$ is a map between logoi then a fibrant replacement of the span $(1_A, u) : A \to A \times B$ is the span $P(u) \to A \times B$ defined by the pullback diagram,

$$\begin{tikzcd}
 & & P(u) \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl, "1_A"'] \arrow[dr, "u"] & & B^I \arrow[dl, "s"'] \arrow[dr, "t"] & \\
A & & B & & B.
\end{tikzcd}$$

Dually, a fibrant replacement of the span $(u, 1_A) : A \to B \times A$ is the span $P^*(u) \to B \times A$ defined in the pullback diagram,

$$\begin{tikzcd} & & P^*(u) \arrow[dl] \arrow[dr] & & \\ & B^I \arrow[dl, "s"'] \arrow[dr, "t"] & & A \arrow[dl, "u"'] \arrow[dr, "1_A"] & \\ B & & B & & A. \end{tikzcd}$$

15.15. The category $Span(B, 1)$ is naturally isomorphic to the category $\mathbf{S}/B$ and the model structures $(Span(B, 1), Wbiv)$ and $(\mathbf{S}/B, Wcont)$ coincide. The *inductive mapping cone* of a map of simplicial sets $u : A \to B$ is the simplicial set $C(u)$ defined by the following pushout square,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ A \diamond 1 \arrow[r] & C(u). \end{tikzcd}$$

By construction $C(u) = \sigma^*(u, !)$, where $(u, !)$ is the span $(u, !) : A \to B \times 1$. The resulting functor

$$C : \mathbf{S}/B \to Cyl(B, 1)$$

is the left adjoint in a Quillen equivalence between the model categories $(\mathbf{S}/B, Wcont)$ and $(Cyl(B, 1), Wcat)$. Dually, the category $Span(1, B)$ is naturally isomorphic to the category $\mathbf{S}/B$ and the model structures $(Span(1, B), Wbiv)$ and $(\mathbf{S}/B, Wcov)$ coincide. The *projective mapping cone* of a map of simplicial sets $u : A \to B$ is the simplicial set $C^o(u)$ constructed by the folllowing pushout square,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ 1 \diamond A \arrow[r] & C^o(u). \end{tikzcd}$$

By construction $C^o(u) = \sigma^*(!, u)$, where $(!, u)$ is the span $(!, u) : A \to 1 \times B$. The resulting functor

$$C^o : \mathbf{S}/B \to Cyl(1, B)$$

is the left adjoint in a Quillen equivalence between the model categories $(\mathbf{S}/B, Wcov)$ and $(Cyl(1, B), Wcat)$. The category $Span(1, 1)$ is naturally isomorphic to the category $\mathbf{S}$ and the model structures $(Span(1, 1), Wbiv)$ and $(\mathbf{S}, Who)$ coincide. The (unreduced) *suspension* of a simplicial set A is the simplicial set $\Sigma(A)$ defined by the following pushout square,

$$\begin{tikzcd} A \sqcup A \arrow[r] \arrow[d] & 1 \sqcup 1 \arrow[d] \\ A \times I \arrow[r] & \Sigma(A). \end{tikzcd}$$

The resulting functor

$$\Sigma : \mathbf{S} \to Cyl(1, 1)$$

is the left adjoint in a Quillen equivalence between the model category $(\mathbf{S}, Who)$ and the model category $(Cyl(1,1), Wcat)$.

15.16. It follows from 15.15 that we have two equivalence of categories,

$$\Lambda(B,1) \simeq \mathcal{P}(B) \quad \text{and} \quad \Lambda(1,B) \simeq \mathcal{Q}(B).$$

for any simplicial set B.

15.17. The simplicial set $\delta[n] = \Delta[n] \star \Delta[n]$ has the structure of a cylinder for every $n \geq 0$. The *diagonal* of a cylinder C is the simplicial set $\delta^*(C)$ obtained by putting

$$\delta^*(C)_n = Hom(\delta[n], C)$$

for every $n \geq 0$. The simplicial set $\delta^*(C)$ has the structure of a span $C(0) \Rightarrow C(1)$ with structure map $(s,t) : \delta^*(C) \to C(0) \times C(1)$ defined by the inclusion $\Delta[n] \sqcup \Delta[n] \subset \Delta[n] \star \Delta[n]$. The resulting functor

$$\delta^* : \mathbf{S}/I \to [P^o, \mathbf{S}]$$

is cartesian with respect to the fibered structure on these categories. It has a right adjoint δ_* and the induced pair

$$\delta^* : Cyl(A,B) \to Span(A,B) : \delta_*$$

is a Quillen equivalence between the model categories $(Cyl(A,B), Wcat)$ and $(Span(A,B), Wbiv)$.

15.18. For any pair of simplicial sets A and B we have $\delta^*(A \star B) = A \times_s B$. It follows that we have $\sigma^*\delta^*(A \star B) = A \diamond B$. Hence the map $\theta_{AB} : A \diamond B \to A \star B$ of 9.20 is a map $\theta_{AB} : \sigma^*\delta^*(A \star B) \to A \star B$. There is then a unique natural transformation

$$\theta_C : \sigma^*\delta^*(C) \to C$$

which extends θ_{AB} to every cylinder $C \to I$. The maps θ_C is a weak categorical equivalence for every $C \in \mathbf{S}/I$.

15.19. If A, B and C are simplicial sets and $S \in Span(A,B)$ is a bifibrant span, then the functor

$$(-) \circ S : Span(B,C) \to Span(A,C)$$

is a left Quillen functor. Dually, $T \in Span(A,B)$ is bifibrant, then the functor

$$T \circ (-) : Span(A,B) \to Span(A,C)$$

is a left Quillen functor. Let us denote by $Span_f(A,B)$ the full subcategory of $Span(A,B)$ spanned by the bifibrant spans. The composition functor

$$- \circ - : Span(B,C)_f \times Span(A,B)_f \to Span(A,C),$$

induces a *derived composition functor* .

$$- \bullet - : \Lambda(B,C) \times \Lambda(A,B) \to \Lambda(A,C).$$

The derived composition is coherently associative. A unit $I_A \in \Lambda(A,A)$ for this composition is a fibrant replacement of the span $(1_A, 1_A) : A \to A \times A$. We thus obtain a bicategory Λ called the *homotopy bicategory of spans*.

15.20. The tensor product functor (which is really the cartesian product)

$$Span(A,B) \times Span(C,D) \to Span(A \times C, B \times D)$$

is a left Quillen functor of two variables with respect to the bivariant model structures on these categories. The induced functor

$$\otimes : \Lambda(A,B) \times \Lambda(C,D) \to \Lambda(A \times C, B \times D)$$

defines a symmetric monoidal structure on the bicategory Λ.

15.21. The symmetric monoidal bicategory Λ is compact closed. The dual of a simplicial set A is the opposite simplicial set A^o. To see this, it suffices to consider the case where A is a logos. We shall exibit a pair of spans,

$$\eta_A \in \Lambda(1, A^o \times A) \quad \text{and} \quad \epsilon_A \in \Lambda(A \times A^o, 1),$$

together with a pair of isomorphisms,

$$\alpha_A : I_A \simeq (\epsilon_A \otimes A) \bullet (A \otimes \eta_A) \quad \text{and} \quad \beta_A : I_{A^o} \simeq (A^o \otimes \epsilon_A) \bullet (\eta_A \otimes A^o).$$

The span η_A is constructed below.

15.22. The *twisted diagonal* C^δ of a category C is defined to be the category of elements of the hom functor $C^o \times C \to \mathbf{Set}$. A simplex $[n] \to C^\delta$ is a map $[n]^o \star [n] \to C$. The *twisted diagonal* A^δ of a simplicial set A is the simplicial set defined by putting

$$(A^\delta)_n = \mathbf{S}(\Delta[n]^o \star \Delta[n], A)$$

for every $n \geq 0$. The simplicial set A^δ is equipped with a canonical map $A^\delta \to A^o \times A$ obtained from the the inclusion $\Delta[n]^o \sqcup \Delta[n] \subset \Delta[n]^o \star \Delta[n]$. This gives the simplicial set A^δ the structure of a mediator $A \Rightarrow A$. Notices that A^δ is the twisted section of the cylinder $A \times I$,

$$A^\delta = \rho^*(A \times I).$$

It follows from this formula and 13.13 that the functor $(-)^\delta : \mathbf{S} \to Med$ has a left adjoint $\delta(-)$ and that the pair

$$\delta(-) : Med \leftrightarrow \mathbf{S} : (-)^\delta$$

is a Quillen adjunction between the model categories $(Med, Wmed)$ and $(\mathbf{S}, Wcat)$. Hence the canonical map $A^\delta \to A^o \times A$ is a left fibration when A is a logos. It defines the span $\eta_A \in Span(1, A^o \times A)$. of the duality 15.21.

15.23. The span $\epsilon_A \in Span(A \times A^o, 1)$ of the duality 15.21, is the opposite of the span $\eta_A \in Span(1, A^o \times A)$. Let us describe the isomorphism α_A of the duality. It is easy to see that the simplicial set

$$T(A) = (\epsilon_A \otimes A) \circ (A \otimes \eta_A)$$

is constructed by the following pullback diagram,

$$\begin{tikzcd}
 & & T(A) \arrow[dl] \arrow[dr] & & \\
 & (A^\delta)^o \arrow[dl, "s_A^o"'] \arrow[dr, "t_A^o"] & & A^\delta \arrow[dl, "s_A"'] \arrow[dr, "t_A"] & \\
A & & A^o & & A.
\end{tikzcd}$$

A simplex $\Delta[n] \to T(A)$ is a pair of simplices $x : \Delta[n] \star \Delta[n]^o \to A$ and $y : \Delta[n]^o \star \Delta[n] \to A$ such that $x \mid \Delta[n]^o = y \mid \Delta[n]^o$. The isomorphism α_A of is obtained by composing in $\Lambda(A, A)$ a chain of bivariant equivalences

$$A^I \xleftarrow{q_A} U(A) \xrightarrow{p_A} T(A)$$

in $Span(A, A)$. The simplicial set $U(A)$ is defined by putting

$$U(A)_n = \mathbf{S}(\Delta[n] \star \Delta[n]^o \star \Delta[n], A)$$

for every $n \geq 0$ and the structure map $U(A) \to A \times A$ is obtained from the inclusion $i_n : \Delta[n] \sqcup \Delta[n] \subset \Delta[n] \star \Delta[n]^o \star \Delta[n]$. Let us describe the map $p_A : U(A) \to T(A)$. If $z : \Delta[n] \star \Delta[n]^o \star \Delta[n] \to A$ is a simplex of $U(A)$, then $p_A(z) = (x, y)$, where $x = z \mid \Delta[n] \star \Delta[n]^o$ and $y = z \mid \Delta[n]^o \star \Delta[n]$. Let us describe the map $q_A : U(A) \to A^I$. There is a unique map

$$\rho_n : \Delta[n] \times I \to \Delta[n] \star \Delta[n]^o \star \Delta[n]$$

which extends i_n. Then we have $q_A(x) = x\rho_n$ for every $x \in U(A)_n$. The isomorphism β_A has a similar description.

15.24. It follows from the duality that there is an equivalence of categories

$$\Lambda(A, B) \to \Lambda(1, A^o \times B).$$

for any pair (A, B). The equivalence associates to a fibrant span $S \to A \times B$ the mediator $S' \to A^o \times B$ calculated by following diagram with a pullback square,

$$\begin{tikzcd} & X' \arrow[r] \arrow[dl] \arrow[d] & X \arrow[r] \arrow[d] & B \\ A^o & A^\delta \arrow[l, "s"'] \arrow[r, "t"] & A. & \end{tikzcd}$$

The inverse equivalence associates to a fibrant mediator $M \to A^o \times B$ the span $M' \to A \times B$ calculated by following diagram with a pullback square,

$$\begin{tikzcd} & M' \arrow[r] \arrow[dl] \arrow[d] & M \arrow[r] \arrow[d] & B \\ A & (A^\delta)^o \arrow[l, "s^o"'] \arrow[r, "t^o"] & A^o. & \end{tikzcd}$$

15.25. The *trace* of a span $S \in \Lambda(A, A)$ is defined by putting

$$Tr(S) = \epsilon_{A^o} \bullet (A^o \otimes S) \circ \eta_A.$$

By definition, $Tr(S) \in \Lambda(1, 1)$ is a homotopy type. If A is a logos and S is bifibrant, then the simplicial set $Tr(S)$ is computed by the following pullback square,

$$\begin{tikzcd} Tr(S) \arrow[r] \arrow[d] & A^I \arrow[d, "(t,s)"] \\ S \arrow[r, "(s,t)"] & A \times A \end{tikzcd} .$$

This construction shows that trace of the identity of a Kan complex X is its free loop space X^{S^1}.

15.26. The *scalar product* of two spans $S \in \Lambda(A, B)$ and $T \in \Lambda(B, A)$ is defined by putting

$$\langle S \mid T\rangle = Tr(S \bullet T) \simeq Tr(T \bullet S).$$

If A and B are logoi and the spans S and T are bifibrant, then we have a pullback square,

$$\begin{tikzcd} \langle S \mid T\rangle \arrow[r] \arrow[d] & T \arrow[d, "(t,s)"] \\ S \arrow[r, "(s,t)"] & A \times B \end{tikzcd}.$$

Notice that we have $Tr_A(X) = \langle X \mid I_A\rangle$ for $X \in \Lambda(A, A)$. A map $u : S \to S'$ in $\Lambda(A, B)$ is invertible iff the map

$$\langle u \mid T\rangle : \langle S \mid T\rangle \to \langle S' \mid T\rangle$$

is invertible in $\Lambda(1, 1)$ for every $T \in \Lambda(B, A)$.

16. Yoneda lemma

The Yoneda lemma is playing an important role in category theory. Here we discuss its extension to logoi. As an application, we construct for any logos X a simplicial category $S(X)$ whose coherent nerve is equivalent to X. We describe four forms of Yoneda lemma.

16.1. (Yoneda lemma 1) If B is a logos and $b \in B_0$, then the right fibration $B/b \to B$ is freely generated by the vertex $1_b \in B/b$. More precisely, for any right fibration $p : X \to B$ and any vertex $x \in X(b) = p^{-1}(b)$, there is a map $f : B/b \to X$ in $\mathbf{S}/B$ such that $f(1_b) = x$, and f is homotopy unique. The homotopy uniqueness means that the simplicial set of maps $f : B/b \to X$ such that $f(1_b) = x$ is contractible. More precisely, let us denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. An object $b \in B$ defines a map $b : 1 \to B$ and we have $[b, X] = X(b)$ for every object $X \to B$ of $\mathbf{S}/B$. The map $b' : b \to B/b$ in $\mathbf{S}/B$ obtained by putting $b'(1) = 1_b$ is a contravariant equivalence by 12.8, since the vertex 1_b is terminal in B/b. Hence the map

$$[b', X] : [B/b, X] \to [b, X] = X(b)$$

is a trivial fibration for every $X \in \mathbf{R}(B)$. It follows that the fiber of $[b', X]$ at $x \in X(b)$ is contractible. This shows that the simplicial set of maps $f : B/b \to X$ such that $f(1_b) = x$ is contractible.

16.2. The hypthesis that B is a logos can be removed in the Yoneda lemma. More precisely, if B is a simplicial set and $b \in B_0$, let us choose a factorisation $1 \to Rb \to B$ of the map $b : 1 \to B$ as a right anodyne map $b' : 1 \to Rb$ followed by a right fibration $Rb \to B$. Then the right fibration $Rb \to B$ is freely generated by the vertex $b' \in Rb$. More precisely, the map

$$[b', X] : [Rb, X] \to [b, X] = X(b)$$

is a trivial fibration for every $X \in \mathbf{R}(B)$.

16.3. We shall say that a right fibration $p : E \to B$ is *representable* if the simplicial set E admits a terminal vertex $v \in E$, in which case we shall say that E is *represented* by v. If $b \in B_0$, then a node $v \in E(b)$ represents E iff the map $v : b \to E$ is a contravariant equivalence in $\mathbf{S}/B$. If B is a logos, then the full simplicial subset of E spanned by the nodes which represents E is a contractible Kan complex when non-empty. Hence a representing vertex $v \in E$ is homotopy unique when it exists. If B is a logos, then the right fibration $B/b \to B$ is represented by the vertex $1_b \in B/b$, and similarly for the right fibration $B/\!\!/b \to B$.

16.4. Dually, we shall say that a left fibration $p : E \to B$ is *corepresentable* if the simplicial set E admits an initial vertex $v \in E$, in which case we shall say E is *corepresented* by v. If $b \in B_0$, then a node $v \in E(b)$ corepresents E iff the map $v : b \to E$ is a covariant equivalence in $\mathbf{S}/B$. If B is a logos, then the left fibration $b\backslash \to B$ is corepresented by the vertex $1_b \in b\backslash B$, and similarly for the left fibration $b\backslash\backslash B \to B$.

16.5. If B is a simplicial set and $b \in B_0$, then the fiber at $a \in B_0$ of the projection $B/\!\!/b \to B$ is equal to $B(a, b)$. If B is a logos; then the canonical map

$$[B/\!\!/a, B/\!\!/b] \to B(a, b)$$

is a trivial fibration by Yoneda lemma.

16.6. To every logos B we can attach a simplicial category $S(B)$ whose coherent nerve is equivalent to B. By construction, $ObS(B) = B_0$ and

$$S(B)(a, b) = [B/a, B/b]$$

for every pair $a, b \in B_0$. The category $S(B)$ is enriched over Kan complexes.

16.7. Let $f : a \to b$ be an arrow in a logos X. Then by Yoneda lemma, there is a map $f_! : B/a \to B/b$ in $\mathbf{S}/B$ such that $f_!(1_a) = b$ and f is homotopy unique. We shall say that $f_!$ is the *pushforward map along* f.

16.8. Recall that the *logos of Kan complexes* $\mathbf{U} = \mathbf{U}_0$ is defined to be the coherent nerve of the category **Kan** of Kan complexes. Let us put $\mathbf{U}' = 1\backslash\mathbf{U}$, where 1 denotes the terminal object of the logos $\mathbf{U}$. Then the canonical map $p_U : \mathbf{U}' \to \mathbf{U}$ is a universal left fibration. The universality means that for any left fibration $f : E \to A$ there exists a homotopy pullback square in $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd} E \arrow[r, "g'"] \arrow[d, "f"'] & \mathbf{U}' \arrow[d, "p_U"] \\ A \arrow[r, "g"] & \mathbf{U} \end{tikzcd}$$

and the pair (g, g') is homotopy unique. We shall say that the map g *classifies* the left fibration $E \to A$.

16.9. The *simplicial set of elements* $el(g)$ of a map $g : A \to \mathbf{U}$ is defined by the pullback square

$$\begin{tikzcd} el(g) \arrow[r] \arrow[d] & \mathbf{U}' \arrow[d, "p_U"] \\ A \arrow[r, "g"] & \mathbf{U}. \end{tikzcd}$$

The map $el(g) \to A$ is a left fibration since p_U is a left fibration. Moreover, the simplicial set $el(g)$ is a logos when A is a logos.

16.10. A *prestack* on a simplicial set A is defined to be a map $A^o \to \mathbf{U}$. The prestacks on A form a logos

$$\mathbf{P}(A) = \mathbf{U}^{A^o} = [A^o, \mathbf{U}].$$

The *simplicial set of elements* $El(g)$ of a prestack $g : A^o \to \mathbf{U}$ is defined by putting $El(g) = el(g)^o$. The canonical map $El(g) \to B$ is a right fibration. We shall say that a prestack $g : A^o \to \mathbf{U}$ is *representable* if the right fibration $El(g) \to A$ is representable.

16.11. Recall that the *twisted diagonal* C^δ of a category C is the category of elements of the hom functor $C^o \times C \to \mathbf{Set}$. A logos A has a twisted diagonal A^δ by 40.36 and the canonical map $(s,t) : A^\delta \to A^o \times A$ is a left fibration. Hence there exists a homotopy pullback square,

$$\begin{tikzcd} A^\delta \arrow[r, "hom'_A"] \arrow[d] & \mathbf{U}' \arrow[d] \\ A^o \times A \arrow[r, "hom_A"] & \mathbf{U}. \end{tikzcd}$$

This defines the map $hom_A : A^o \times A \to \mathbf{U}$. The *Yoneda map*,

$$y_A : A \to \mathbf{P}(A)$$

is obtained by transposing the map hom_A. A prestack $g : A^o \to \mathbf{U}$ is representable iff it belongs to the essential image of y_A.

16.12. The left fibration $L_A \to A \times \mathbf{U}^A$ defined by the pullback square

$$\begin{tikzcd} L_A \arrow[r] \arrow[d] & \mathbf{U}' \arrow[d] \\ A \times \mathbf{U}^A \arrow[r, "ev"] & \mathbf{U} \end{tikzcd}$$

is universal, where ev denotes the evaluation map. The universality means that for any simplicial set B and any left fibration $E \to A \times B$, there exists a homotopy pullback square in $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd} E \arrow[r, "g'"] \arrow[d] & L_A \arrow[d] \\ A \times B \arrow[r, "A\times g"] & A \times \mathbf{U}^A \end{tikzcd}$$

and that the pair (g, g') is homotopy unique.

16.13. Dually, the left fibration M_A defined by the pullback square

$$\begin{tikzcd} M_A \arrow[r] \arrow[d] & \mathbf{U}' \arrow[d] \\ A^o \times \mathbf{P}(A) \arrow[r, "ev"] & \mathbf{U} \end{tikzcd}$$

is a universal mediator $A \Rightarrow \mathbf{P}(A)$. More precisely, for any simplicial set B and any fibrant mediator $E : A \Rightarrow B$, there exists a homotopy pullback square in the model category $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd} E \arrow[r, "g'"] \arrow[d] & M_A \arrow[d] \\ A^o \times B \arrow[r, "A^o \times g"] & A^o \times \mathbf{P}(A), \end{tikzcd}$$

and the pair (g, g') is homotopy unique. We shall say that g *classifies* the mediator $E : A \Rightarrow B$ and that $M_A : A \Rightarrow \mathbf{P}(A)$ is a *Yoneda mediator*. A mediator $E : A \Rightarrow B$ is essentially the same thing as a map $B \to \mathbf{P}(A)$.

16.14. (Yoneda lemma 2) The twisted diagonal $A^\delta \to A^o \times A$ is classified by the Yoneda map $y_A : A \to \mathbf{P}(A)$. We have a diagram of homotopy pullback squares in $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd} A^\delta \arrow[r] \arrow[d] & M_A \arrow[r] \arrow[d] & \mathbf{P}(A)^\delta \arrow[d] \\ A^o \times A \arrow[r, "A^o \times y_A"] & A^o \times \mathbf{P}(A) \arrow[r, "y_A^o \times \mathbf{P}(A)"] & \mathbf{P}(A)^o \times \mathbf{P}(A). \end{tikzcd}$$

The composite square shows that the map y_A is fully faithful.

16.15. The Quillen equivalence 13.13 between mediators, and cylinders implies the existence of a universal cylinder $C_A \in Cyl(A, \mathbf{P}(A))$. The cylinder C_A turns out to be a fibrant replacement of the cylinder $Cl(y_A)$ defined by the pushout square of simplicial sets,

$$\begin{tikzcd} A \arrow[r, "y_A"] \arrow[d, "i_1"'] & \mathbf{P}(A) \arrow[d] \\ A \times I \arrow[r] & Cl(y_A) \end{tikzcd}$$

The universality of C_A means that for any simplicial set B and any cylinder $E \in Cyl(A, B)$, there exists a homotopy pullback square in the model category $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd} E \arrow[r, "g'"] \arrow[d] & C_A \arrow[d] \\ A \star B \arrow[r, "1_A \star g"] & A \star \mathbf{P}(A), \end{tikzcd}$$

and the pair (g, g') is homotopy unique. We shall say that g *classifies* the cylinder $E \in Cyl(A, B)$ and that $C_A \in Cyl(A, \mathbf{P}(A))$ is a *Yoneda cylinder*. A cylinder $C : A \Rightarrow B$ is essentially the same thing as a map $B \to \mathbf{P}(A)$.

16.16. (Yoneda lemma 3) The cylinder $A \times I \in Cyl(A, A)$ is classified by the Yoneda map $y_A : A \to \mathbf{P}(A)$. We have a diagram of homotopy pullback squares in $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd}
A \times I \arrow[r] \arrow[d] & C_A \arrow[r] \arrow[d] & \mathbf{P}(A) \times I \arrow[d] \\
A \star A \arrow[r, "A \star y_A"] & A \star \mathbf{P}(A) \arrow[r, "y_A \star \mathbf{P}(A)"] & \mathbf{P}(A) \star \mathbf{P}(A).
\end{tikzcd}$$

16.17. The Quillen equivalence 15.8 between cylinders and spans implies the existence of a universal span $P_A \in Span(A, \mathbf{P}(A))$. The universality of P_A means that for any simplicial set B and any bifibrant span $S : A \Rightarrow B$, there exists a homotopy pullback square in the model category $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd}
S \arrow[r, "g'"] \arrow[d] & P_A \arrow[d] \\
A \times B \arrow[r, "A \times g"] & A \times \mathbf{P}(A),
\end{tikzcd}$$

and the pair (g, g') is homotopy unique. We shall say that g *classifies* the span $S : A \Rightarrow B$ and that $P_A : A \Rightarrow \mathbf{P}(A)$ is a *Yoneda span*. A bifibrant span $S : A \Rightarrow B$ is essentially the same thing as a map $B \to \mathbf{P}(A)$.

16.18. (Yoneda lemma 4) The span $A^I \to A \times A$ is classified by the Yoneda map $y_A : A \to \mathbf{P}(A)$. We have a diagram of homotopy pullback squares in $(\mathbf{S}, Wcat)$,

$$\begin{tikzcd}
A^I \arrow[r] \arrow[d] & P_A \arrow[r] \arrow[d] & \mathbf{P}(A)^I \arrow[d] \\
A \times A \arrow[r, "A \times y_A"] & A \times \mathbf{P}(A) \arrow[r, "y_A \times \mathbf{P}(A)"] & \mathbf{P}(A) \times \mathbf{P}(A).
\end{tikzcd}$$

The composite square shows that the map y_A is fully faithful.

16.19. If X is a small simplicial category, let us denote by $[X, \mathbf{S}]^f$ the full subcategory of fibrant objects of the model category $[X, \mathbf{S}]^{inj}$. Then the evaluation functor $ev : X \times [X, \mathbf{S}] \to \mathbf{S}$ induces a functor $e : X \times [X, \mathbf{S}]^f \to \mathbf{Kan}$. The coherent nerve of this functor is a map of simplicial sets

$$C^! X \times C^! [X, \mathbf{S}]^f \;\to \mathbf{U}.$$

When X is enriched over Kan complexes, the corresponding map

$$C^! [X, \mathbf{S}]^f \to \mathbf{U}^{C^! X}$$

is an equivalence of logoi. It follows by adjointness that for any simplicial set A we have an equivalence of logoi

$$C^! [C_! A, \mathbf{S}]^f \to \mathbf{U}^A.$$

17. Morita equivalence

In this section, we introduce the notion of Morita equivalence between simplicial sets. The category of simplicial sets admits a model structure in which the weak equivalences are the Morita equivalences and the cofibration are the monomorphisms. The fibrant objects are the Karoubi complete logoi. We give an explicit construction of the Karoubi envelope of a logos. The results of the section are taken from [J2].

17.1. Recall that a functor $u : A \to B$ between small categories is said to be a *Morita equivalence* if the base change functor

$$u^* : [B^o, \mathbf{Set}] \to [A^o, \mathbf{Set}]$$

is an equivalence of categories. A functor $u : A \to B$ is a Morita equivalence iff it is fully faithful and every object $b \in B$ is a retract of an object in the image of u.

17.2. Recall an idempotent $e : b \to b$ in a category is said to *split* if there exists a pair of arrows $s : a \to b$ and $r : b \to a$ such that $e = sr$ and $rs = 1_a$. A category C is said to be *Karoubi complete* if every idempotent in C splits.

17.3. The model structure $(\mathbf{Cat}, Eq)$ admits a Bousfield localisation with respect to Morita equivalences. The local model structure is cartesian closed and left proper. We shall denote it shortly by $(\mathbf{Cat}, Meq)$. A category is fibrant iff it is Karoubi complete. We call a fibration a *Morita fibration*. A *Karoubi envelope* $Kar(C)$ of a category C is a fibrant replacement of C in the model structure $(\mathbf{Cat}, Meq)$. The category $Kar(C)$ is well defined up to an equivalence of categories. The envelope is well defined up to an equivalence of logoi.

17.4. We shall denote by $i : C \to \kappa(C)$ the following explicit construction of the Karoubi envelope of a category C. An *object* of the category $\kappa(C)$ is a pair (c, e), where c is an object of C and $e \in C(c, c)$ is an idempotent. An *arrow* $f : (c, e) \to (c', e')$ of $\kappa(C)$ is a morphism $f \in C(c, c')$ such that $fe = f = e'f$. The composite of $f : (c, e) \to (c', e')$ and $g : (c', e') \to (c", e")$ is the arrow $gf : (c, e) \to (c", e")$. The arrow $e : (c, e) \to (c, e)$ is the unit of (c, e). The functor $i : C \to \kappa(C)$ takes an object $c \in C$ to the object $(c, 1_c) \in \kappa(C)$.

17.5. Let *Split* be the category freely generated by two arrows $s : 0 \to 1$ and $r : 1 \to 0$ such that $rs = 1_0$. The monoid $E = Split(1, 1)$ is freely generated by one idempotent $e = sr$ and we have $\kappa(E) = Split$. A functor is a Morita fibration iff it has the right lifting property with respect to the inclusion $E \subset Split$.

17.6. We shall say that a map of simplicial sets $u : A \to B$ is a *Morita equivalence* if the base change functor

$$\mathcal{P}^*(u) : \mathcal{P}(B) \leftrightarrow \mathcal{P}(A)$$

is an equivalence of categories. A map $u : A \to B$ is a Morita equivalence iff it is fully faithful and every object $b \in \tau_1 B$ is a retract of an object in the image of u. Hence a map $u : A \to B$ is a Morita equivalence iff the opposite map $u^o : A^o \to B^o$ is a Morita equivalence. A weak categorical equivalence is a Morita equivalence.

17.7. An *idempotent* in a logos X is defined to be a map $e : E \to X$, where E is the monoid freely generated by one idempotent. We shall say that an idempotent $e : E \to X$ *split* if it can be extended to a map $Split \to X$. We shall say that a logos X is *Karoubi complete* if every idempotent in X splits. If X is Karoubi complete, then so are the logoi X/b and $b\backslash X$ for every object $b \in X$.

17.8. The model category $(\mathbf{S}, Wcat)$ admits a Bousfield localisation with respect to Morita equivalences. The local model structure is cartesian closed and left proper. We shall denote it shortly by $(\mathbf{S}, Wmor)$. A fibration is called a *Morita fibration.* A logos is fibrant iff it is Karoubi complete. The *Karoubi envelope* $Kar(X)$ of a logos X is defined to be a fibrant replacement of X in the model structure $(\mathbf{S}, Wmor)$. The envelope is well defined up to an equivalence of logoi.

17.9. The pair of adjoint functors

$$\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$$

is a Quillen adjunction between the model categories $(\mathbf{S}, Wmor)$ and $(\mathbf{Cat}, Wmor)$. A functor $u : A \to B$ in $\mathbf{Cat}$ is a Morita equivalence (resp. a Morita fibration) iff the map $Nu : NA \to NB$ is a Morita equivalence (resp. a Morita fibration).

17.10. A mid fibration between logoi is a Morita fibration iff it has the right lifting property with respect to the inclusion $E \subset Split$. The base change of a Morita equivalence along a left or a right fibration is a Morita equivalence. Every right (resp. left) fibration is a Morita fibration.

17.11. The canonical map $X \to hoX$ is a Morita fibration for any logos X. It follows that an idempotent $u : E \to X$ splits iff its image $hu : E \to hoX$ splits in hoX. Hence a logos X is Karoubi complete iff every idempotent $u : E \to X$ which splits in hoX splits in X.

17.12. Let E be the monoid freely generated by one idempotent. Then a logos X is is Karoubi complete iff the projection $X^{Split} \to X^E$ defined by the inclusion $E \subset Split$ is a trivial fibration.

17.13. The Karoubi envelope of a logos X has functorial construction $X \to \kappa(X)$. Observe that the functor $\kappa : Cat \to Cat$ has the structure of a monad, with a left adjoint comonad L. To see this, we need the notion of semi-category. By definition, a *semi-category* B is a category without units. More precisely, it is a graph $(s,t) : B_1 \to B_0 \times B_0$ equipped with a composition law $B_1 \times_{s,t} B_1 \to B_1$ which is associative. There is an obvious notion of semi-functor between semi-categories. Let us denote by $\mathbf{sCat}$ the category of small semi-categories and semi-functors. The forgetful functor $U : \mathbf{Cat} \to \mathbf{sCat}$ has a left adjoint F and a right adjoint G. The existence of F is clear by a general result of algebra. If B is a semi-category, then the category $G(B)$ has the following description. An *object* of $G(B)$ is a pair (b, e), where $b \in B_0$ and $e : b \to b$ is an idempotent; an *arrow* $f : (b, e) \to (b', e')$ of $G(B)$ is a morphism $f \in B(b, b')$ such that $fe = f = e'f$. Composition of arrows is obvious. The unit of (b, e) is the morphism $e : (b, e) \to (b, e)$. It is easy to verify that we have $U \vdash G$. By construction, we have $\kappa(C) = GU(C)$ for any category C. It follows that the functor κ has the structure of a monad. Moreover, we have $L \vdash \kappa$, where $L = FU$. The functor L has the structure of a comonad by adjointness. The category $L[n]$ has the following presentation for each $n \geq 0$. It is generated by a chain of arrows

$$0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 2 \longrightarrow \cdots \xrightarrow{f_n} n,$$

and a sequence of idempotents $e_i : i \to i$ $(0 \leq i \leq n)$. In addition to the relation $e_i e_i = e_i$ for each $0 \leq i \leq n$, we have the relation $f_i e_{i-1} = f_i = e_i f_i$ for each $0 < i \leq n$. If A is a simplicial set, let us put

$$\kappa(A)_n = \mathbf{S}(L[n], A)$$

for every $n \geq 0$. This defines a continuous functor $\kappa : \mathbf{S} \to \mathbf{S}$ having the structure of a monad. If X is a logos, then the unit $X \to \kappa(X)$ is a Karoubi envelope of X. A map between logoi $f : X \to Y$ is a Morita equivalence iff the map $\kappa(f) : \kappa(X) \to \kappa(Y)$ is an equivalence of logoi.

17.14. The model category $(\mathbf{S}, Wcat)$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of Morita equivalences. A map $p : X \to Y$ belongs to $\mathcal{B}$ iff it admits a factorisation $p'w : X \to X' \to Y$ with p' a Morita fibration and w a weak categorical equivalence.

18. Adjoint maps

We introduce the notion of adjoint maps between logoi and formulate a necessary an sufficient condition for the existence of adjoints. We also introduce a weaker form of the notion of adjoint for maps between simplicial sets.

18.1. Recall from 1.11 that the category $\mathbf{S}$ has the structure of a 2-category $\mathbf{S}^{\tau_1}$. If $u : A \to B$ and $v : B \to A$ are maps of simplicial sets, an *adjunction* $(\alpha, \beta) : u \dashv v$ between u and v

$$u : A \leftrightarrow B : v$$

is a pair of natural transformations $\alpha : 1_A \to vu$ and $\beta : uv \to 1_B$ satisfying the *adjunction identities*:

$$(\beta \circ u)(u \circ \alpha) = 1_u \quad \text{and} \quad (v \circ \beta)(\alpha \circ v) = 1_v.$$

The map u is the *left adjoint* and the map v the *right adjoint*. The natural transformation α is the *unit* of the adjunction and the natural transformation β is the *counit*. We shall say that a homotopy $\alpha : 1_A \to vu$ is an *adjunction unit* if the natural transformation $[\alpha] : 1_A \to vu$ is the unit of an adjunction $u \dashv v$. Dually, we say that a homotopy $\beta : uv \to 1_B$ is an *adjunction counit* if the natural transformation $[\beta] : uv \to 1_B$ is the counit of an adjunction $u \dashv v$.

18.2. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes an adjunction to an adjunction. A composite of left adjoints $A \to B \to C$ is left adjoint to the composite of the right adjoints $C \to B \to A$.

18.3. An object a in a logos X is initial iff the map $a : 1 \to X$ is left adjoint to the map $X \to 1$.

18.4. A map between logoi $g : Y \to X$ is a right adjoint iff the logos $a\backslash Y$ defined by the pullback square

$$\begin{tikzcd} a\backslash Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ a\backslash X \arrow[r] & X \end{tikzcd}$$

admits an initial object for every object $a \in X$. An object of the logos $a\backslash Y$ is a pair (b, u), where $b \in Y_0$ and $u : a \to f(b)$ is an arrow in X. We shall say that the arrow u is *universal* if the object (b, u) is initial in $a\backslash Y$. If f is a map $X \to Y$, then a homotopy $\alpha : 1_X \to gf$ is an adjunction unit iff the arrow $\alpha(a) : a \to gf(a)$ is universal for every object $a \in X$. Dually, a map between logoi $f : X \to Y$ is a left adjoint iff the logos X/b defined by the pullback square

$$\begin{tikzcd} X/b \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y/b \arrow[r] & Y \end{tikzcd}$$

admits a terminal object for every object $b \in Y$. An object of the logos X/b is a pair (a, v), where $a \in X_0$ and $v : f(a) \to b$ is an arrow in Y; we shall say that the arrow v is *couniversal* if the object (a, v) is terminal in X/b. If g is a map $Y \to X$, then a homotopy $\beta : fg \to 1_Y$ is an adjunction counit iff the arrow $\beta(b) : fg(b) \to b$ is couniversal for every object $b \in Y$.

18.5. The base change of left adjoint between logoi along a right fibration is a left adjoint.

18.6. If $f : X \leftrightarrow Y : g$ is a pair of adjoint maps between logoi, then the right adjoint g is fully faithful iff the counit of the adjunction $\beta : fg \to 1_Y$ is invertible, in which case the left adjoint f is said to be a *reflection* and the map g to be *reflective*. Dually, the left adjoint f is fully faithful iff the unit of the adjunction $\alpha : 1_X \to gf$ is invertible, in which case the right adjoint g is said to be a *coreflection* and the map f to be *coreflective*.

18.7. The base change of a reflective map along a left fibration is reflective. Dually, the base change of a coreflective map along a right fibration is coreflective.

18.8. We shall say that a map of simplicial sets $u : A \to B$ is a *weak left adjoint* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a right adjoint for every logos X. Dually, we shall say that $u : A \to B$ is a *weak right adjoint* if the functor $\tau_1(u, X)$ is a left adjoint for every logos X. A map of simplicial sets $u : A \to B$ is a weak left adjoint iff the opposite map $u^o : A^o \to B^o$ is a weak right adjoint.

18.9. A map between logoi is a weak left adjoint iff it is a left adjoint. The notion of weak left adjoint is invariant under weak categorical equivalences. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a weak left adjoint to a left adjoint.

18.10. Weak left adjoints are closed under composition. The base change of weak left adjoint along a right fibration is a weak left adjoint. A weak left adjoint is an initial map. A vertex $a \in A$ in simplicial set A is initial iff the map $a : 1 \to A$ is a weak left adjoint.

18.11. Let B a simplicial set. For each vertex $b \in B$, let us choose a factorisation $1 \to Rb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map of simplicial sets $u : A \to B$ is a weak left adjoint iff the simplicial set $Rb \times_B A$ admits a terminal vertex for each vertex $b \in B$.

18.12. We say that a map $v : B \to A$ is a *weak reflection* if the functor

$$\tau_1(v, X) : \tau_1(A, X) \to \tau_1(B, X)$$

is coreflective for every logos X. We say that a map of simplicial sets $u : A \to B$ is *weakly reflective* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a coreflection for every logos X. There are dual notions of weak coreflection and of weakly coreflective maps.

18.13. If a map of simplicial sets is both a weak left adjoint and a weak right adjoint, then it is a weak reflection iff it is weak coreflection.

18.14. A weak left adjoint is a weak reflection iff it is dominant iff it is a localisation. Dually, a weak right adjoint is a weak coreflection iff it is dominant iff it is a localisation.

19. Homotopy localisations

In this section we formulate the Dwyer-Kan localisation theory in terms of logoi instead of simplicial categories.

19.1. Recall that a *strict localisation* is a functor $A \to S^{-1}A$ which inverts a set S of arrows universally. A functor $u : A \to B$ is called a *localisation* iff it admits a factorisation $u = wu' : A \to B' \to B$, with u' a strict localisation and w an equivalence of categories. There is also a notion of *iterated localisation* introduced in 11.14. Recall that that the model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of iterated localisations and $\mathcal{B}$ is the class of conservative functors.

19.2. We say that a map of simplicial sets $u : A \to B$ *inverts* a set of arrows $S \subseteq A$ if every arrow in $u(S)$ is invertible in the category $\tau_1(B)$. If u inverts S we shall say that u is a *homotopy localisation with respect to S* if for any logos X, the map $X^u : X^B \to X^A$ induces an equivalence between the logos X^B and the full simplicial subset of X^A spanned by the maps $A \to X$ which inverts S.

19.3. Let S be a set of arrows in a simplicial set A. If J is the groupoid generated by one arrow $0 \to 1$, then the map $A \to A[S^{-1}]$ in the pushout square

$$\begin{tikzcd}
\bigsqcup_S I \arrow[r] \arrow[d] & A \arrow[d] \\
\bigsqcup_S J \arrow[r] & A[S^{-1}]
\end{tikzcd}$$

is a homotopy localisation with respect to S.

19.4. We shall say that a map of simplicial sets $u : A \to B$ *coinverts* an iterated localisation $l : \tau_1 A \to C$ if the functor $\tau_1(u) : \tau_1(A) \to \tau_1(B)$ can be factored through l up to a natural isomorphism. We shall say that a map $u : A \to B$ is a *homotopy localisation* (resp. *iterated homotopy localisation*) if the functor $\tau_1(u) : \tau_1(A) \to \tau_1(B)$ is a localisation (resp. an iterated localisation) and for any logos X, the map $X^u : X^B \to X^A$ induces an equivalence between the logos X^B and the full simplicial subset of X^A spanned by the maps $A \to X$ which inverts $\tau_1(u)$.

19.5. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a homotopy localisation to a localisation and an iterated homotopy localisation to an iterated localisation. An iterated homotopy localisation $u : A \to B$ is a homotopy localisation iff the iterated localisation $\tau_1(u) : \tau_1 A \to \tau_1 B$ is a localisation.

19.6. The model category $(\mathbf{S}, Wcat)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps and $\mathcal{A}$ is the class of iterated homotopy localisations. Hence a monomorphism of simplicial sets is an iterated homotopy localisation iff it has the left lifting property with respect to conservative pseudo-fibrations between logoi.

19.7. An iterated homotopy localisation is essentially surjective and dominant. A weak reflection (resp. coreflection) is a homotopy localisation. The base change of a homotopy localisation along a left or a right fibration is a homotopy localisation. Similarly for the base change of an iterated homotopy localisation.

19.8. Recall from 48.5 that for any category A, the full subcategory of $A\backslash\mathbf{Cat}$ spanned by the iterated strict localisations $A \to C$ is equivalent to a complete lattice $Loc_0(A)$. Its maximum element is defined by the strict localisation $A \to \pi_1 A$ which inverts every arrow in A. If A is a simplicial set, then the full subcategory of the homotopy category $Ho(A\backslash\mathbf{S}, Wcat)$ spanned by the iterated homotopy localisations $A \to C$ is equivalent to a complete lattice $Loc(A)$. Moreover, the functor τ_1 induces an isomorphism of lattices $Loc(A) \simeq Loc_0(\tau_1 A)$. The maximum element of $Loc(A)$ is the fibrant replacement $A \to K$ of A in the model category $(\mathbf{S}, Who)$.

19.9. Suppose that we have a commutative cube of simplicial sets

$$
\begin{tikzcd}
A_0 \arrow[rr] \arrow[dr] \arrow[dd] & & C_0 \arrow[dr] \arrow[dd] & \\
& B_0 \arrow[rr] \arrow[dd] & & D_0 \arrow[dd] \\
A_1 \arrow[rr] \arrow[dr] & & C_1 \arrow[dr] & \\
& B_1 \arrow[rr] & & D_1.
\end{tikzcd}
$$

in which the top and the bottom faces are homotopy cocartesian. If the maps $A_0 \to A_1$, $B_0 \to B_1$ and $C_0 \to C_1$ are homotopy localisations, then so is the maps $D_0 \to D_1$. Similarly for iterated homotopy localisations.

19.10. Every simplicial set X is the homotopy localisation

$$\lambda_X : el(X) \to X$$

of its category of elements $el(X) = \Delta/X$. The map λ_X was defined by Illusie in [Illu]. Let us first describe λ_X in the case where X is (the nerve of) a category C. Recall that an object of $el(C)$ is a chain of arrows $x : [n] \to C$, and that a morphism $\alpha : y \to x$ from $y : [m] \to C$ to $x : [n] \to C$ is a map $u : [m] \to [n]$ in Δ such that $xu = y$. The functor $\lambda_C : el(C) \to C$ is defined by putting $\lambda_C(x) = x(n)$ and $\lambda_C(\alpha) = x(u(m), n)$, where $(u(m), n)$ denotes the unique arrow $u(m) \to n$ in the category $[n]$. The family of maps $\lambda_C : el(C) \to C$, for $C \in \mathbf{Cat}$, is a natural transformation between two endo-functors of $\mathbf{Cat}$. The natural transformation has then a unique extension $\lambda_X : el(X) \to X$ for $X \in \mathbf{S}$. Let us now show that Λ_X is a homotopy localisation. Let Δ' be the subcategory of Δ whose arrows are the maps $u : [m] \to [n]$ with $u(m) = n$. We shall denote by $el'(X)$ the subcategory of $el(X)$ obtained by pulling back Δ' along the projection $el(X) \to \Delta$. The map $\lambda_X : el(X) \to X$ takes every arrow in $el'(X)$ to a unit in X. It thus induces a canonical map

$$w_X : el(X)[el'(X)^{-1}] \to X.$$

The result will be proved if we show that w_X a weak categorical equivalence. We only sketch of the proof. The domain $F(X)$ of w_X is a cocontinuous functor of X. Moreover, the functor F takes a monomorphism to a monomorphism. The result is easy to verify in the case where $X = \Delta[n]$. The result then follows from a formal argument using the the skeleton filtration of X and the cube lemma.

19.11. If $u : A \to B$ is homotopy localisation, then the base change functor $\mathcal{P}^*(u) : \mathcal{P}(B) \to \mathcal{P}(A)$ is fully faithful, since a homotopy localisation is dominant. An object $X \in \mathcal{P}(A)$ belongs to the essential image of the functor $\mathcal{P}^*(u)$ iff its (contravariant) homotopy diagram $D(X) : \tau_1(A)^o \to Ho(\mathbf{S}, Who)$ inverts the localisation $\tau_1(A) \to \tau_1(B)$.

19.12. If $f : a \to b$ is an arrow in a simplicial set A, then the inclusion $i_0 : \{0\} \to I$ induces a map $f' : a \to f$ between the objects $a : 1 \to A$ and $f : I \to A$ of the category $\mathbf{S}/A$. If S is a set of arrows in A, we shall denote by $(\mathbf{S}/A, S \cup Wcont)$. the Bousfield localisation of the model structure $(\mathbf{S}/A, Wcont)$ with respect to the set of maps $\{f' : f \in S\}$. An object $X \in \mathbf{R}(A)$ is fibrant in the localised structure iff the map $f^* : X(b) \to X(a)$ of the contravariant homotopy diagram of X is a weak homotopy equivalence for every arrow $f : a \to b$ in S. If $p : A \to A[S^{-1}]$ is the canonical map, then the pair of adjoint functors

$$p_! : \mathbf{S}/A \leftrightarrow \mathbf{S}/A[S^{-1}] : p^*$$

is a Quillen equivalence between the model category $(\mathbf{S}/A, \Sigma \cup Wcont)$ and the model category $(\mathbf{S}/A[S^{-1}], Wcont)$.

19.13. It follows from 19.12 that a right fibration $X \to B$ is a Kan fibration iff the map $f^* : X(b) \to X(a)$ of the contravariant homotopy diagram of X is a weak homotopy equivalence for every arrow $f : a \to b$ in B.

20. Barycentric localisations

In this section we give an explicit construction of the quasi-localisation of a model category with respect to weak equivalences.

20.1. For every $n \geq 0$, let us denote by $P_0[n]$ the (nerve of) the poset of non-empty subsets of $[n]$ ordered by the inclusion. From a map $f : [m] \to [n]$, we obtain a map $P_0(f) : P_0[m] \to P_0[n]$ by putting $P_0(f)(S) = f(S)$ for every $S \in P_0[m$. This defines a functor $P_0 : \Delta \to \mathbf{S}$. Recall that the *Ex* functor of Daniel Kan

$$Ex : \mathbf{S} \to \mathbf{S}$$

is defined by putting

$$Ex(X)_n = \mathbf{S}(P_0[n], X)$$

for every $X \in \mathbf{S}$ and $n \geq 0$. The functor Ex has a left adjoint

$$B : \mathbf{S} \to \mathbf{S}$$

obtained by taking the left Kan extension of the functor $P_0 : \Delta \to \mathbf{S}$ along the Yoneda functor $\Delta \to \mathbf{S}$. The simplicial set $B(X)$ is called the *barycentric subdivision* of a simplicial set X.

TO BE COMPLETED

21. Limits and colimits

In this section we study the notions of limit and colimit in a logos. We define the notions of cartesian product, of fiber product, of coproduct and of pushout. The notion of limit in a logos subsume the notion of homotopy limits. For example. the loop space of a pointed object is a pullback and its suspension a pushout. We consider various notions of complete and cocomplete logoi. Many results of this section are taken from [J1] and [J2].

21.1. If X is a logos and A is a simplicial set, we say that a map $d : A \to X$ is a *diagram* indexed by A in X . The logos X can be large. The *cardinality* of a diagram $d : A \to X$ is the cardinality of A. A diagram $d : A \to X$ is *small* (resp. *finite*) if A is small (resp. finite).

21.2. Recall that a *projective cone* with base $d : A \to X$ in a logos X is a map $c : 1 \star A \to X$ which extends d along the inclusion $A \subset 1 \star A$. The projective cones with base d are the vertices of a logos X/d by 9.8. We say that a projective cone $c : 1 \star A \to X$ with base d is *exact* if it is a terminal object of the logos X/d. When $c : 1 \star A \to X$ is exact, the vertex $l = c(1) \in X$ is said to be the (*homotopy*) *limit* of the diagram d and we write

$$l = \lim_{a \in A} d(a) = \lim_{A} d.$$

21.3. If $d : A \to X$ is a diagram in a logos X, then the full simplicial subset of X/d spanned by the exact projective cones with base d is a contractible Kan complex when non-empty. It follows that the limit of a diagram is homotopy unique when it exists.

21.4. The notion of limit can also be defined by using fat projective cones $1 \diamond A \to X$ instead of projective cones $1 \star A \to X$. But the canonical map $X/d \to X/\!\!/d$ obtained from the canonical map $1 \diamond A \to 1 \star A$ is an equivalence of logoi by 9.20. It thus induces an equivalence between the Kan complex spanned by the terminal vertices of X/d and the Kan complex spanned by the terminal vertices of $X/\!\!/d$.

21.5. The colimit of a diagram with values in a logos X is defined dually. We recall that an *inductive cone* with *cobase* $d : A \to X$ in a logos X is a map $c : A \star 1 \to X$ which extends d along the inclusion $A \subset A \star 1$. The inductive cones with a fixed cobase d are the objects of a logos $d\backslash X$. We say that an inductive cone $c : 1 \star A \to X$ with cobase d is *coexact* if it is an initial object of the logos $d\backslash X$. When $c : A \star 1 \to X$ is coexact, the vertex $l = c(1) \in X$ is said to be the (*homotopy*) *colimit* of the diagram d and we write

$$l = colim_{a \in A} d(a) = colim_A d.$$

The notion of colimit can also be defined by using fat inductive cones $A \diamond 1 \to X$, but the two notions are equivalent.

21.6. If X is a logos and A is a simplicial set, then the diagonal map $X \to X^A$ has a right (resp. left) adjoint iff every diagram $A \to X$ has a limit (resp. colimit).

21.7. We shall say that a (large) logos X is *complete* if every (small) diagram $A \to X$ has a limit. There is a dual notion of a *cocomplete* logos. We shall say that a large logos is *bicomplete* if it is complete and cocomplete.

21.8. We say that a logos X is *finitely complete* or *cartesian* if every finite diagram $A \to X$ has a limit. There is dual notion of a *finitely cocomplete* or *cocartesian* logos. We shall say that a logos X is *finitely bicomplete* if it is finitely complete and cocomplete.

21.9. The homotopy localisation $L(\mathcal{E})$ of a model category $\mathcal{E}$ is finitely bicomplete, and it is (bi)complete when the category $\mathcal{E}$ is (bi)complete.

21.10. The coherent nerve of the category of Kan complexes is a bicomplete logos $\mathbf{U} = \mathbf{U}_0$. Similarly for the coherent nerve $\mathbf{U}_1$ of the category of small logoi.

21.11. A map between logoi $f : X \to Y$ is said to *preserve* the limit of a diagram $d : A \to X$ if this limit exists and f takes an exact projective cone $c : 1 \star A \to X$ with base d to an exact cone $fc : 1 \star A \to Y$. The map $f : X \to Y$ is said to be *continuous* if it takes every (small) exact projective cone in X to an exact cone. A map between cartesian logoi is said to be *left exact* if it preserves finite limits.

21.12. Dually, a map $f : X \to Y$ is said to *preserve* the the colimit of a diagram $d : A \to X$ if this colimit exists and f takes a coexact inductive cone with cobase d to a coexact cone. A map $f : X \to Y$ is said to be *cocontinuous* if it takes every (small) coexact inductive cone in X to a coexact cone. A map which is both continuous and cocontinuous is said to be *bicontinuous*,

21.13. A right adjoint between logoi is continuous and a left adjoint cocontinuous.

21.14. If X is a logos and S is a discrete simplicial set (ie a set), then a map $x : S \to X$ is the same thing as a family $(x_i \mid i \in S)$ of objects of X. A projective cone $c : 1 \star S \to X$ with base $x : S \to X$ is the same thing as a family of arrows $(p_i : y \to x_i \mid i \in S)$ with domain $y = c(1)$. When c is exact, the object y is said to be the *product* of the family $(x_i : i \in S)$, the arrow $p_i : y \to x_t$ is said to be a *projection* and we write

$$y = \prod_{i \in S} x_i.$$

Dually, an inductive cone $c : S \star 1 \to X$ with cobase x is a family of arrows $(u_i : x_i \to y \mid t \in S)$ with codomain $y = c(1)$. When c is coexact, the object y is called the *coproduct* of the family $(x_i : i \in S)$, the arrow $u_i : x_i \to y$ is said to be a *coprojection* and we write

$$y = \coprod_{i \in S} x_i.$$

21.15. The canonical map $X \to hoX$ preserves products and coproducts.

21.16. We say that a logos X *has finite products* if every finite family of objects of X has a product. A logos with a terminal object and binary products has finite products. We say that a large logos X *has products* if every small family of objects of X has a product. There are dual notions of a logos *with finite coproducts* and of large logos *with coproducts*

21.17. If X is a logos and $b \in X_0$, then an object of the logos X/b is an arrow $a \to b$ in X. The *fiber product* of two arrows $a \to b$ and $c \to b$ in X is defined to be their product as objects of the logos X/b,

$$\begin{tikzcd} a \times_b c \arrow[r] \arrow[d] & c \arrow[d] \\ a \arrow[r] & b. \end{tikzcd}$$

21.18. If X is a logos and $a \in X_0$, then an object of the logos $a\backslash X$ is an arrow $a \to b$ in X. The *amalgameted coproduct* of two arrows $a \to b$ and $a \to c$ in X is defined to be their coproduct as objects of the logos $a\backslash X$,

$$\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] \\ b \arrow[r] & b \sqcup_a c. \end{tikzcd}$$

21.19. A *commutative square* in a logos X is map $I \times I \to X$. The square $I \times I$ is a projective cone $1 \star \Lambda^2[2]$. A commutative square $I \times I \to X$ is said to be *cartesian*, or to be a *pullback* if it is exact as a projective cone. We shall say that a logos X *has pullbacks* if every diagram $\Lambda^2[2] \to X$ has a limit.

21.20. A diagram $d : \Lambda^2[2] \to X$ is the same thing as a pair of arrows $f : a \to b$ and $g : c \to b$ in X. The limit of d is the domain of the fiber product of f and g. A logos X has pullbacks iff the logos X/b has finite products for every object $b \in X$.

21.21. Dually, a commutative square $I \times I \to X$ in a logos X is said to be *cocartesian*, or to be a *pushout*, if it is coexact as an inductive cone. We say that a logos X has *pushouts*) if every diagram $\Lambda^0[2] \to X$ has a colimit. A logos X has pushouts iff the logos $a\backslash X$ has finite coproducts for every object $a \in X$.

21.22. A logos with terminal objects and pullbacks is finitely complete. A map between finitely complete logoi is finitely continuous iff it preserves terminal objects and pullbacks.

21.23. A logos with products and pullbacks is complete. A map between complete logoi is continuous iff it preserves products and pullbacks.

21.24. We say that a logos X is *cartesian closed* if it has finite products and the product map $a \times (-) : X \to X$ has a right adjoint $[a, -] : X \to X$, called the *exponential*, for every object $a \in X$. We say that a logos X is *locally cartesian closed* if the slice logos X/a is cartesian closed for every object $a \in X$.

21.25. The logos $\mathbf{U}$ is locally cartesian closed. The logoi $\mathbf{U}_1$ and $\mathbf{U}_1/I$ are cartesian closed, where $I = \Delta[1]$.

21.26. The *base change* of a morphism $f : a \to b$ in a logos along another morphism $u : a' \to a$ is the morphism f' in a pullback square,

$$\begin{tikzcd} a' \arrow[r] \arrow[d, "f'"'] & a \arrow[d, "f"] \\ b' \arrow[r, "u"] & b. \end{tikzcd}$$

21.27. To every arrow $f : a \to b$ in a logos X we can associate a pushforward map $f_! : X/a \to X/b$ by 16.7. The map $f_!$ is unique up to a unique invertible 2-cell in the 2-category $\mathbf{QCat}$. The logos X has pullbacks iff the pushforward map $f_! : X/a \to X/b$ has a right adjoint

$$f^* : X/b \to X/a$$

for every arrow $f : a \to b$. We shall say that f^* is the *base change map* along f.

21.28. A cartesian logos X is locally cartesian closed iff the base change map

$$f^* : X/b \to X/a$$

has a right adjoint f_* for every arrow $f : a \to b$.

21.29. Let $d : B \to X$ a diagram with values in a logos X and let $u : A \to B$ a map of simplicial sets. If the colimit of the diagrams d and du exist, then there is a canonical morphism

$$colim_A du \to colim_B d$$

in the category hoX. When the map $u : A \to B$ is final, the map $d\backslash X \to du\backslash X$ induced by u is an equivalence of logoi by 9.17. It follows that the colimit of d exists iff the colimit of du exists, in which case the canonical morphism above is invertible and the two colimits are isomorphic.

21.30. Let $d : B \to X$ a diagram with values in a logos X. If $u : (M, p) \to (N, q)$ is a contravariant equivalence in the category $\mathbf{S}/B$, then the map $dq\backslash X \to dp\backslash X$ induced by u is an equivalence of logoi. It follows that the colimit of dp exists iff the colimit of dq exists, in which case the two colimits are naturally isomorphic in the category hoX.

21.31. Let $(A_i \mid i \in S)$ be a family of simplicial sets and let us put

$$A = \bigsqcup_{i \in S} A_i.$$

If X is a logos, then a diagram $d : A \to X$ is the same thing as a family of diagrams $d_i : A_i \to X$ for $i \in S$. If each diagram d_i has a colimit x_i, then the diagram d has a colimit iff the coproduct of the family $(x_i : i \in I)$ exists, in which case we have

$$colim_A d = \coprod_{i \in S} colim_{A_i} d.$$

21.32. Suppose we have a pushout square of simplicial sets

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "v"] & T. \end{tikzcd}$$

with i monic. Let $d : T \to X$ be a diagram with values in a logos X and suppose that each diagram dv, dvi and dj has a colimit. Then the diagram d has a colimit iff the pushout square

$$\begin{tikzcd} colim_A dvi \arrow[r] \arrow[d] & colim_C dj \arrow[d] \\ colim_B dv \arrow[r] & Z \end{tikzcd}$$

exists, in which case $colim_T d = Z$.

21.33. In a logos with finite colimit X, the coproduct of n objects can be computed inductively by taking pushouts starting from the initial object. More generally, the colimit of any finite diagram $d : A \to X$ can be computed inductively by taking pushouts and the initial object. To see this, let us put

$$l_n = colim_{Sk^n A} d \mid Sk^n A$$

for each $n \geq 0$. The object l_0 is the coproduct of the family $d \mid A_0$. If $n > 0$, the object l_n can be constructed from l_{n-1} by taking pushouts. To see this, let us denote by $C_n(A)$ the set of non-degenerate n-simplices of A. We then have a pushout square

$$\begin{tikzcd} C_n(A) \times \partial\Delta[n] \arrow[r] \arrow[d] & Sk^{n-1}A \arrow[d] \\ C_n(A) \times \Delta[n] \arrow[r] & Sk^n A \end{tikzcd}$$

for each $n \geq 1$. The colimit of a simplex $x : \Delta[n] \to X$ is equal to $x(n)$, since n is a terminal object of $\Delta[n]$. Let us denote by $\delta(x)$ the colimit of the simplicial sphere $x \mid \partial\Delta[n]$. There is then a canonical morphism $\delta(x) \to x(n)$, since $\partial\Delta[n] \subset \Delta[n]$. It then follows from 21.32 that we have a pushout square,

$$\begin{tikzcd} \coprod_{x \in C_n(A)} \delta(x) \arrow[r] \arrow[d] & l_{n-1} \arrow[d] \\ \coprod_{x \in C_n(A)} x(n) \arrow[r] & l_n. \end{tikzcd}$$

The construction shows that a logos with initial object and pushouts is finitely cocomplete.

21.34. Recall from 17.7 than an *idempotent* in a logos X is defined to be a map $e : E \to X$, where E is the monoid freely generated by one idempotent. An idempotent $e : E \to X$ splits iff the diagram $e : E \to X$ has a limit iff it has a colimit. A complete logos is Karoubi complete Beware that the simplicial set E is not quasi-finite. Hence a cartesian logos is not necessarly Karoubi complete.

21.35. The Karoubi envelope of a cartesian logos is cartesian. The Karoubi envelope of a logos with finite products has finite products.

21.36. Every cocartesian logos X admits a natural action $(A, x) \mapsto A \cdot x$ by finite simplicial sets. By definition, the object $A \cdot x$ is the colimit of the constant diagram $A \to X$ with value x. The map $x \mapsto A \cdot x$ is obtained by composing the diagonal $\Delta_A : X \to X^A$ with its left adjoint $l_A : X^A \to X$. Equivalently $A \cdot x$ is the colimit of $x : 1 \to X$ *weighted* by $A \to 1$ There is a canonical homotopy equivalence

$$X(A \cdot x, y) \simeq X(x, y)^A$$

for every $y \in X$. Dually, every cartesian logos X admits a contravariant action $(x, A) \mapsto x^A$ by finite simplicial set A. By definition, the object $x^A \in X$ is he limit of the constant diagram $A \to X$ with value x. There is a a canonical homotopy equivalence

$$X(y, x^A) \simeq X(y, x)^A$$

for every $y \in X$. The covariant and contravariant actions are related by the formula $x^A = (A^o \cdot x^o)^o$.

21.37. In the logos $\mathbf{U}_1$, the product map $\times : \mathbf{U}_1 \times \mathbf{U}_1 \to \mathbf{U}_1$ coincides with the action map defined above and the exponential $[-,-] : \mathbf{U}_1^o \times \mathbf{U}_1 \to \mathbf{U}_1$ with the coaction map.

21.38. A cocartesian logos X with a null object 0 admits a natural action by finite pointed simplicial sets. The *smash product* $A \wedge x$ of an object $x \in X$ by a finite pointed simplicial set A is defined by the pushout square,

$$\begin{tikzcd} 1 \cdot x \arrow[r] \arrow[d, "a \cdot x"'] & 1 \cdot 0 \arrow[d] \\ A \cdot x \arrow[r] & A \wedge x, \end{tikzcd}$$

where $a : 1 \to A$ is the base point. This defines a map $A \wedge (-) : X \to X$. There is a natural isomorphism

$$A \wedge (B \wedge x) \simeq (A \wedge B) \wedge x$$

for any pair of finite pointed simplicial sets A and B. There is also a natural isomorphism $S^0 \wedge x \simeq x$, where S^0 is the pointed 0-sphere. The *suspension* of an element $x \in X$, is defined to be the smash product $\Sigma(x) = S^1 \wedge x$, where S^1 is the pointed circle. The *n-fold suspension* is the smash product $S^n \wedge x$. For a fixed object $x \in X$, the map $A \mapsto A \wedge x$ takes an homotopy pushout square of finite pointed simplicial sets to a pushout square in X. For example, it takes the homotopy pushout square

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & S^1 \end{tikzcd}$$

to a pushout square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma(x). \end{tikzcd}$$

$\Sigma^n(x)$

21.39. Dually, let X be a pointed cartesian logos with null object $0 \in X$. The *cotensor* of an element $x \in X$ by a finite pointed simplicial set A is the element $[A, x] \in X$ defined by the pullback square,

$$\begin{tikzcd} {[A,x]} \arrow[r] \arrow[d] & 0 \arrow[d] \\ x^A \arrow[r, "x^a"] & x^1, \end{tikzcd}$$

where $a : 1 \to A$ is the base point. This defines a map $[A, -] : X \to X$. For any pair of pointed simplicial sets A and B, there is a natural isomorphism

$$[A, [B, x]] \simeq [A \wedge B, x]$$

and it is homotopy unique. There is also a natural isomorphism $[S^0, x] \simeq x$. The *loop space* of an element $x \in X$ is defined to be the cotensor $[S^1, x]$. The *n-fold loop* *space* is the cotensor $[S^n, x]$. For a fixed object $x \in X$, the map $A \mapsto A \wedge x$ takes an homotopy pushout square of finite pointed simplicial sets to a pushout square in X. For example, it takes the homotopy pushout square

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & S^1 \end{tikzcd}$$

to a pullback square

$$\begin{tikzcd} \Omega(x) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & x. \end{tikzcd}$$

$\Omega(x)$ $\Omega^n(x)$

21.40. If the logos X is cocartesian and pointed, then its opposite X^o is cartesian and pointed. By duality we have

$$(A \wedge x)^o = [A^o, x^o] \simeq [A, x^o],$$

since the simplicial sets A and A^o are weakly homotopy equivalent. In particular, we have $(\Sigma \cdot x)^o \simeq \Omega(x^o)$. When X is bicartesian, the map $A \wedge (-) : X \to X$ is left adjoint to the map $[A, -] : X \to X$.

21.41. Unless exception, we only consider small ordinals and cardinals. Recall that an ordinal α is said to be a *cardinal* if it is smallest among the ordinals with the same cardinality. Recall that a cardinal α is said to be *regular* if the sum of a family of cardinals $< \alpha$, indexed by a set of cardinality $< \alpha$, is $< \alpha$.

21.42. Let α be a regular cardinal. We say that a diagram $A \to X$ in a logos X is α-*small* if the simplicial set A has cardinality $< \alpha$. We say that a logos X is α-*complete* if every α-small diagram $A \to X$ has a limit. We say that map $X \to Y$ between α-complete logoi is α-*continuous* if it preserves the limit every α-small diagram $K \to X$. There are dual notions of α-cocomplete logos, and of α-*cocontinuous* map. ogos!α-completetextbf ogos!α-cocompletetextbf

21.43. Let α be an infinite regular cardinal. Then a logos with α-products and pullbacks. is α-complete. A map between α-complete logoi is α-continuous iff it preserves α-small products and pullbacks.

21.44. For any simplicial set A, the map

$$\lambda_A : \Delta/A \to A$$

defined in 21.44 is initial, since a localisation is dominant and a dominant map is initial. Hence the limit of a diagram $d : A \to X$ in a logos X is isomorphic to the limit of the composite $d\lambda_A : \Delta/A \to X$. Observe that the projection $q : \Delta/A \to \Delta$ is a discrete fibration. If $d : A \to X$ is a diagram in a logos with products X, then the map $d\lambda_A$ admits a right Kan extension $\Pi_A(d) = \Pi_q(d\lambda_A) : \Delta \to X$ along the projection q. See section 24 for Kan extensions. Moreover, we have

$$\Pi_A(d)(n) = \prod_{a \in A_n} d(a(n))$$

for every $n \geq 0$. The diagram d has a limit iff the diagram $\Pi_A(d)$ has a limit, in which case we have

$$\lim_A d = \lim_\Delta \Pi_A(d).$$

It follows that a logos with products and Δ-indexed limits is complete.

21.45. Dually, for any simplicial set A, the opposite of the map $\lambda_{A^o} : \Delta/A^o \to A^o$ is a final map

$$\lambda_A^o =: A/\Delta^o \to A.$$

Observe that the canonical projection $p : A/\Delta^o \to \Delta^o$ is a discrete opfibration. If $d : A \to X$ is a diagram in a logos with coproducts X, then the map $d\lambda_A^o$ admits a left Kan extension $\Sigma_A(d) = RKan_p : \Delta^o \to X$ along the projection p. We have

$$\Sigma_A(d)_n = \coprod_{a \in A_n} d(a(0))$$

for every $n \geq 0$. The diagram d has a colimit iff the diagram $\Sigma_A(d)$ has a colimit, in which case we have

$$colim_A d = colim_\Delta \Sigma_A(d).$$

It follows that a logos with coproducts and Δ^o-indexed colimits is cocomplete.

22. Grothendieck fibrations

22.1. We first recall the notion of Grothendieck fibration between categories. A morphism $f : a \to b$ in a category E is said to be *cartesian* with respect a functor $p : E \to B$ if for every morphism $g : c \to b$ in E and every factorisation $p(g) = p(f)u : p(c) \to p(a) \to p(b)$ in B, there is a unique morphism $v : c \to a$ in E such that $g = fv$ and $p(v) = u$. A morphism $f : a \to b$ is cartesian with respect to the functor p iff the square of categories

$$\begin{tikzcd}
E/a \arrow[r] \arrow[d] & E/b \arrow[d] \\
B/p(a) \arrow[r] & B/p(b)
\end{tikzcd}$$

is cartesian, where the functor $E/a \to E/b$ (resp. $B/p(a) \to B/p(b)$) is obtained by composing with f (resp. $p(f)$). A functor $p : E \to B$ is called a *Grothendieck fibration over* B if for every object $b \in E$ and every morphism $g \in B$ with target $p(b)$ there exists a cartesian morphism $f \in E$ with target b such that $p(f) = g$. There are dual notions of *cocartesian morphism* and of Grothendieck *opfibration*. A functor $p : E \to B$ is a Grothendieck opfibration iff the opposite functor $p^o : E^o \to B^o$ is a Grothendieck fibration. We shall say that a functor $p : E \to B$ is a *Grothendieck bifibration* if it is both a fibration and an opfibration.

22.2. If X and Y are two Grothendieck fibrations over B, then a functor $X \to Y$ in $\mathbf{Cat}/B$ is said to be *cartesian* if its takes every cartesian morphism in X to a cartesian morphism in Y. There is a dual notion of cocartesian functor between Grothendieck opfibrations over B and a notion of bicartesian functor between Grothendieck bifibrations.

22.3. Observe that a morphism $f : a \to b$ in a category E is cartesian with respect a functor $p : E \to B$ iff every commutative square

$$\begin{tikzcd} \Lambda^2[2] \arrow[r, "x"] \arrow[d] & E \arrow[d, "p"] \\ \Delta[2] \arrow[r] & B \end{tikzcd}$$

with $x(1,2) = f$ has a unique diagonal filler.

22.4. Let $p : E \to B$ be a mid fibration between simplicial sets. We shall say that an arrow $f \in E$ is *cartesian* if every commutative square

$$\begin{tikzcd} \Lambda^n[n] \arrow[r, "x"] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] & B \end{tikzcd}$$

with $n > 1$ and $x(n-1,n) = f$ has a diagonal filler. Equivalently, an arrow $f \in E$ with target $b \in E$ is cartesian with respect to p if the map $E/f \to B/pf \times_{B/pb} E/b$ obtained from the commutative square

$$\begin{tikzcd} E/f \arrow[r] \arrow[d] & E/b \arrow[d] \\ B/pf \arrow[r] & B/pb \end{tikzcd}$$

is a trivial fibration. Every isomorphism in E is cartesian when B is logos by 9.14. We call a map of simplicial sets $p : E \to B$ a *Grothendieck fibration* if it is a mid fibration and for every vertex $b \in E$ and every arrow $g \in B$ with target $p(b)$ there exists a cartesian arrow $f \in E$ with target b such that $p(f) = g$.

22.5. A map $X \to 1$ is a Grothendieck fibration iff X is a logos. A right fibration is a Grothendieck fibration whose fibers are Kan complexes. Every Grothendieck fibration is a pseudo-fibration.

22.6. The class of Grothendieck fibrations is closed under composition and base changes. The base change of a weak left adjoint along a Grothendieck fibration is a weak left adjoint [**?**].

22.7. If $p : E \to B$ is a Grothendieck fibration, then so is the map $p^A : E^A \to B^A$ for any simplicial set A.

22.8. The source map $s : X^I \to X$ a Grothendieck fibration for any logos X. More generally, if a monomorphism of simplicial sets $u : A \to B$ is fully faithful and a weak left adjoint then the map $X^u : X^B \to X^A$ is a Grothendieck fibration for any logos X.

22.9. The target map $t : X^I \to X$ a Grothendieck fibration for any logos with pullbacks X. More generally, if a monomorphism of simplicial sets $u : A \to B$ is fully faithful and a weak right adjoint, then the map $X^u : X^B \to X^A$ is a Grothendieck fibration for any logos with pullbacks X.

22.10. If E is an object of $\mathbf{S}/B$, then for every simplex $u : \Delta[n] \to B$ we have a simplicial set $[u, E]$. If $n = 0$ and $u = b : 1 \to B$, then the simplicial set $[b, E]$ is the fiber $E(b)$ of the structure map $E \to B$ at $b \in B$. For any arrow $f : a \to b$ in B consider the projections $p_0 : [f, E] \to E(a)$ and $p_1 : [f, E] \to E(b)$ respectively defined by the inclusions $\{0\} \subset I$ and $\{1\} \subset I$. If the structure map $E \to B$ is a Grothendieck fibration, then the projection $p_1 : [f, E] \to E(b)$ has a right adjoint $i_1 : E(b) \to [f, E]$ and the composite

$$f^* = p_0 i_1 : E(b) \to E(a)$$

is well defined up to a unique invertible 2-cell. We shall say that f^* is the *base change* along f, or the *pullback* along f . If $t : \Delta[2] \to B$ is a simplex with boundary $\partial t = (g, h, f)$,

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c, \end{tikzcd}$$

then we can define a canonical invertible 2-cell

$$h^* \simeq f^* g^* : E(c) \to E(b) \to E(a).$$

22.11. We shall say that a map $g : X \to Y$ between two Grothendieck fibrations in $\mathbf{S}/B$ is *cartesian* if it takes every cartesian arrow in X to a cartesian arrow in Y. A cartesian map $g : X \to Y$ respects base changes. More precisely, for any arrow $f : a \to b$ in B, the following square commutes up to a canonical invertible 2-cell,

$$\begin{tikzcd} X(b) \arrow[r] \arrow[d, "f^*"'] & Y(b) \arrow[d, "f^*"] \\ X(a) \arrow[r] & Y(a), \end{tikzcd}$$

where the horizontal maps are induced by g.

22.12. Recall that the category $\mathbf{S}/B$ is enriched over $\mathbf{S}$ for any simplicial set B by 12.1. We shall denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. Let $\mathbf{G}(B)$ the full sub-category of $\mathbf{S}/B$ spanned by the Grothendieck fibrations $X \to B$. If $X \in \mathbf{G}(B)$, then the simplicial set $[A, X]$ is a logos for any object $A \in \mathbf{S}/B$. It follows that the category $\mathbf{G}(B)$ is enriched over $\mathbf{QCat}$. By composing this enrichement with the functor $\tau_1 : \mathbf{QCat} \to \mathbf{Cat}$ we obtain a 2-category structure on $\mathbf{G}(B)$.

22.13. Let us denote by $\mathbf{Cart}(B)$ the subcategory of $\mathbf{G}(B)$ whose morphisms are the cartesian maps. The category $\mathbf{R}(B)$ is a full subcategory of $\mathbf{Cart}(B)$, since every right fibration is a Grothendieck fibration and every map in $\mathbf{R}(B)$ is cartesian. The inclusion functor $\mathbf{R}(B) \subseteq \mathbf{Cart}(B)$ has a right adjoint

$$J_B : \mathbf{Cart}(B) \to \mathbf{R}(B).$$

By construction, we have $J_B(X) \subseteq X$ and a simplex $x : \Delta[n] \to X$ belongs to $J_B(X)$ iff the arrow $x(i, j)$ is cartesian for every $0 \leq i < j \leq n$. Notice that a map $g : X \to Y$ in $\mathbf{G}(B)$ is cartesian iff we have $g(J_B(X)) \subseteq J_B(Y)$. The functor J_B

respects base change. More precisely, for any Grothendieck fibration $X \to B$ and any map of simplicial sets $u : A \to B$ we have a canonical isomorphism

$$u^* J_B(X) = J_A(u^* X).$$

In particular, we have $J_B(X)(b) = J(X(b))$ for every vertex $b \in B$.

22.14. Every map between logoi $u : A \to B$ admits a factorisation

$$u = gi : A \to P(u) \to B$$

with g a Grothendieck fibration and i a fully faithful right adjoint [**?**]. The simplicial set $P(u)$ is constructed by the pullback square

$$\begin{tikzcd} P(u) \arrow[r, "q"] \arrow[d, "p"'] & B^I \arrow[d, "t"] \\ A \arrow[r, "u"] & B, \end{tikzcd}$$

where t is the target map. If $s : B^I \to B$ is the source map, then the composite $g = sq : P(u) \to B$ is a Grothendieck fibration. There is a unique map $i : A \to P(u)$ such that $pi = 1_A$ and $qi = \delta u$, where $\delta : B \to B^I$ is the diagonal. We have $g \vdash i$ and the counit of the adjunction is the identity of $gi = 1_X$. Thus, i is fully faithful. If $p : X \to B$ is a Grothendieck fibration, then for every map $f : A \to X$ in $\mathbf{S}/B$ there exists a cartesian map $c : P(u) \to X$ such that $f = ci$. Moreover, c is unique up to a unique invertible 2-cell in the 2-category $\mathbf{G}(B)$.

22.15. There are dual notions of *cocartesian* arrow and of *Grothendieck opfibration*. A map $p : E \to B$ is a Grothendieck opfibration iff the opposite map $p^o : E \to B$ is a Grothendieck fibration. We shall say that a map is a *Grothendieck bifibration* if it is both a Grothendieck fibration and a Grothendieck opfibration.

22.16. A Kan fibration is a Grothendieck bifibration whose fibers are Kan complexes.

22.17. If a logos X is bicomplete, then the map $X^u : X^B \to X^A$ is a Grothendieck bifibration for any fully faithful monomorphism of simplicial sets $u : A \to B$.

22.18. If $p : E \to B$ is a Grothendieck opfibration, then the projection $p_0 : [f, E] \to E(a)$ has a left adjoint $i_0 : E(a) \to [f, E]$ and the composite

$$f_! = p_1 i_0 : E(a) \to E(b)$$

is well defined up to a unique invertible 2-cell. We shall say that $f_!$ is the *cobase change* along f, or the *pushforward* along f. The map $f_!$ is well defined of to a unique invertible 2-cell. If $p : E \to B$ is a Grothendieck bifibration, the map $f_!$ is left adjoint to the map f^*.

23. Proper and smooth maps

The notions of proper and of smooth functors were introduced by Grothendieck in 31.30. We extend these notions to maps of simplicial sets. The results of the section are taken from [J2].

23.1. We shall say that a map of simplicial sets $u : A \to B$ is *proper* if the pullback functor $u^* : \mathbf{S}/B \to \mathbf{S}/A$ takes a right anodyne map to a right anodyne map. A map of simplicial sets $u : A \to B$ is proper iff the inclusion $u^{-1}(b(n)) \subseteq b^*(E)$ is right anodyne for every simplex $b : \Delta[n] \to B$.

23.2. A Grothendieck opfibration is proper. In particular, a left fibration is proper. The class of proper maps is closed under composition and base changes. A projection $A \times B \to B$ is proper.

23.3. The pullback functor $u^* : \mathbf{S}/B \to \mathbf{S}/A$ has a right adjoint u_* for any map of simplicial sets $u : A \to B$. When u is proper, the pair of adjoint functors

$$u^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/A : u_*.$$

is a Quillen pair with respect to the contravariant model structures on these categories. The functor u^* takes a contravariant equivalence to a contravariant equivalence and we obtain an adjoint pair of derived functors

$$\mathcal{P}^*(u) : \mathcal{P}(B) \leftrightarrow \mathcal{P}(A) : \mathcal{P}_*(u).$$

23.4. Dually, we shall say that a map of simplicial sets $p : E \to B$ is *smooth* if the functor $p^* : \mathbf{S}/B \to \mathbf{S}/E$ takes a left anodyne map to a left anodyne map. A map p is smooth iff the opposite map $p^o : E^o \to B^o$ is proper.

23.5. The functor $\mathcal{P}^*(u)$ admits a right adjoint $\mathcal{P}_*(u)$ for any map of simplicial sets $u : A \to B$. To see this, it suffices by Morita equivalence to consider the case where A and B are logoi. By 24.10, we have a factorisation $u = pi : A \to C \to B$, with i a left adjoint and p a Grothendieck opfibration. We then have $\mathcal{P}^*(u) \simeq \mathcal{P}^*(i)\mathcal{P}^*(p)$. Hence it suffices to prove the result when u is a Grothendieck opfibration and when u is a left adjoint. The first case is clear by 23.3, since a Grothendieck opfibration is proper by 23.2. If $v : B \to A$ is right adjoint to u, then we have $\mathcal{P}^*(u) \vdash \mathcal{P}^*(v)$ by 14.11.

23.6. Suppose that we have a commutative square of simplicial sets

$$\begin{tikzcd} F \arrow[r, "v"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r, "u"] & B \end{tikzcd}$$

Then the following square commutes,

$$\begin{tikzcd} \mathcal{P}(F) \arrow[r, "\mathcal{P}_!(v)"] \arrow[d, "\mathcal{P}_!(q)"'] & \mathcal{P}(E) \arrow[d, "\mathcal{P}_!(p)"] \\ \mathcal{P}(A) \arrow[r, "\mathcal{P}_!(u)"] & \mathcal{P}(B). \end{tikzcd}$$

From the adjunctions $\mathcal{P}_!(p) \vdash \mathcal{P}^*(p)$ and $\mathcal{P}_!(q) \vdash \mathcal{P}^*(q)$ we can define a canonical natural transformation

$$\alpha : \mathcal{P}_!(v)\mathcal{P}^*(q) \to \mathcal{P}^*(p)\mathcal{P}_!(u).$$

We shall say that the *Beck-Chevalley law holds* if α is invertible. This means that the following square commutes up to a canonical isomorphism,

$$\begin{tikzcd}
\mathcal{P}(F) \arrow[r, "\mathcal{P}_!(v)"] & \mathcal{P}(E) \\
\mathcal{P}(A) \arrow[u, "\mathcal{P}^*(q)"] \arrow[r, "\mathcal{P}_!(u)"] & \mathcal{P}(B). \arrow[u, "\mathcal{P}^*(p)"']
\end{tikzcd}$$

Equivalently, this means that the following square of right adjoints commutes up to a canonical isomorphism,

$$\begin{tikzcd}
\mathcal{P}(F) \arrow[d, "\mathcal{P}_*(q)"'] & \mathcal{P}(E) \arrow[l, "\mathcal{P}^*(v)"'] \arrow[d, "\mathcal{P}_*(p)"] \\
\mathcal{P}(A) & \mathcal{P}(B). \arrow[l, "\mathcal{P}^*(u)"']
\end{tikzcd}$$

23.7. (Proper or smooth base change) [J2] Suppose that we have a cartesian square of simplicial sets,

$$\begin{tikzcd}
F \arrow[r, "v"] \arrow[d, "q"'] & E \arrow[d, "p"] \\
A \arrow[r, "u"] & B.
\end{tikzcd}$$

Then the Beck-Chevalley law holds if p is proper *or* if u is smooth.

24. Kan extensions

We introduce the notion of Kan extension for maps between logoi. The results of the section are taken from [J2].

24.1. Let $\mathcal{C}$ be a 2-category. Let us call a 1-cell of $\mathcal{C}$ a *map*. The *left Kan extension* of a map $f : A \to X$ along a map $u : A \to B$ is a pair (g, α), where $g : B \to X$ is a map and $\alpha : f \to gu$ is a 2-cell, which reflects the map f along the functor

$$\mathcal{C}(u, X) : \mathcal{C}(B, X) \to \mathcal{C}(A, X).$$

This means that for any map $g' : B \to X$ and any 2-cell $\alpha' : f \to g'u$, there is a unique 2-cell $\beta : g \to g'$ such that $(\beta \circ u)\alpha = \alpha'$. The pair (g, α) is unique up to a unique invertible 2-cell when it exists, in which case we shall put $g = \Sigma_u(f)$. Dually, the *right Kan extension* of a map $f : A \to X$ along a map $u : A \to B$ is a pair (g, β), where $g : B \to X$ is a map and $\beta : gu \to f$ is a 2-cell, which coreflects the map f along the functor $\mathcal{C}(X, u)$. This means that for any map $g' : B \to X$ and any 2-cell $\alpha' : g'u \to f$, there is a unique 2-cell $\beta : g' \to g$ such that $\alpha(\beta \circ u) = \alpha'$. The pair (g, β) is unique up to a unique invertible 2-cell when it exists, in which case we shall put $g = \Pi_u(f)$.

24.2. If $u : A \leftrightarrow B : v$ is an adjoint pair in a 2-category $\mathcal{C}$, then we have $\mathcal{C}(X, v) \vdash \mathcal{C}(X, u)$ for any object X. Hence we have $fv = \Sigma_u(f)$ for every $f : A \to X$ and we have $gu = \Pi_v(g)$ for every map $g : B \to X$.

24.3. The category $\mathbf{S}$ has the structure of a 2-category $(= \mathbf{S}^{\tau_1})$. Hence there is a notion of Kan extension for maps of simplicial sets. We will only consider Kan extension of maps with values in a logos. If X is a logos, we shall denote by $\Sigma_u(f)$ the *left Kan extension* of a map $f : A \to X$ along a map of simplicial sets $u : A \to B$. Dually, we shall denote by $\Pi_u(f)$ the *right Kan extension* of a map $f : A \to X$ along $u : A \to B$. By duality we have

$$\Pi_u(f)^o = \Sigma_{u^o}(f^o).$$

24.4. If X is a cocomplete logos and $u : A \to B$ is a map between (small) simplicial sets, then every map $f : A \to X$ has a left Kan extension $\Sigma_u(f) : B \to X$ and the map $X^u : X^B \to X^A$ has a left adjoint

$$\Sigma_u : X^A \to X^B.$$

Dually, if X is a complete logos, then every map $f : A \to X$ has a right Kan extension $\Pi_u(f) : B \to X$ and the map X^u has a right adjoint

$$\Pi_u : X^A \to X^B.$$

24.5. If $u : A \to B$ is a map of simplicial sets, then the colimit of a diagram $d : A \to X$ is isomorphic to the colimit of its left Kan extension $\Sigma_u(d) : B \to X$, when they exist. Dually, the limit of a diagram $d : A \to X$ is isomorphic to the limit of its right Kan extension $\Pi_u(d) : B \to X$, when they exist.

24.6. If $u : A \to B$ and $v : B \to C$ are maps of simplicial sets, then we have a canonical isomorphism

$$\Sigma_v \circ \Sigma_u = \Sigma_{vu} : X^A \to X^C$$

for any cocomplete logos X. Dually, we have a canonical isomorphism

$$\Pi_v \circ \Pi_u = \Pi_{vu} : X^A \to X^C$$

for any complete logos X.

24.7. Let X be a bicomplete logos. If $u : A \leftrightarrow B : v$ is an adjunction between two maps of simplicial sets, then we have three adjunctions and two isomorphisms,

$$\Sigma_v \vdash \Sigma_u = X^v \vdash X^u = \Pi_v \vdash \Pi_u.$$

24.8. Every map between logoi $u : A \to B$ admits a factorisation

$$u = qi : A \to P \to B$$

with q a Grothendieck opfibration and i a fully faithful left adjoint (a coreflection) by 24.10. If $p : P \to A$ is the righ adjoint of i, then we have $X^p = \Sigma_i$ for any cocomplete logos X, since we have $X^p \vdash X^i$. Thus

$$\Sigma_u = \Sigma_q \circ \Sigma_i = \Sigma_q \circ X^p.$$

24.9. If $u : A \to B$ is a map between (small) simplicial sets, we shall denote the map $\mathbf{U}^{u^o}$ by u^*, the map Σ_{u^o} by $u_!$ and the map Π_{u^o} by u_*. We have $u_! \vdash u^* \vdash u_*$,

$$u_! : \mathbf{P}(A) \leftrightarrow \mathbf{P}(B) : u^* : \mathbf{P}(B) \leftrightarrow \mathbf{P}(A) : u_*.$$

Notice the equality $(vu)^* = u^*v^*$ and the isomorphisms $(vu)_! \simeq v_!u_!$ and $(vu)_* \simeq v_*u_*$. for a pair of maps $u : A \to B$ and $v : B \to C$. More generally, if X is a complete logos anhd $u : A \to B$ is a map between (small) simplicial sets, we may denote the map X^{u^o} by u^*, the map Σ_{u^o} by $u_!$ and the map Π_{u^o} by u_*.

24.10. If $u : A \leftrightarrow B : v$ is an adjunction between two maps of simplicial sets, then we have three adjunctions and two isomorphisms,

$$u_! \vdash v_! = u^* \vdash v^* = u_* \vdash v_*.$$

24.11. Suppose that we have commutative square of simplicial sets,

$$\begin{tikzcd} F \arrow[r, "v"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r, "u"] & B. \end{tikzcd}$$

If X is a cocomplete logos. then from the commutative square

$$\begin{tikzcd} X^F & & X^E \arrow[ll, "X^v"'] \\ X^A \arrow[u, "X^q"] & & X^B. \arrow[ll, "X^u"'] \arrow[u, "X^p"'] \end{tikzcd}$$

then from the adjunctions $\Sigma_u \vdash X^u$ and $\Sigma_v \vdash X^v$, we can define a natural transformation

$$\alpha : \Sigma_v X^q \to X^p \Sigma_u.$$

We shall say that the *Beck-Chevalley law holds* if α is invertible. Dually, if X is complete, then from the adjunctions $X^p \vdash \Pi_p$ and $X^q \vdash \Pi_q$ we obtain natural transformation

$$\beta : X^u \Pi_p \to \Pi_q X^v.$$

We shall say that the *Beck-Chevalley law holds* if β is invertible. When X is bicomplete, the transformation β is the right transpose of α. Thus, β is invertible iff α is invertible. Hence the Beck-Chevalley law holds in the first sense iff it holds in the second sense. The Beck-Chevalley law holds in the first sense if the square $pv = uq$ is cartesian and u is a smooth map. The Beck-Chevalley law holds in the second sense if the square $pv = uq$ is cartesian and p is a proper map.

24.12. Suppose that we have commutative square of simplicial sets,

$$\begin{tikzcd} F \arrow[r, "v"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r, "u"] & B. \end{tikzcd}$$

If X is a complete logos. then from the commutative square

$$\begin{tikzcd} X^{F^o} & & X^{E^o} \arrow[ll, "v^*"'] \\ X^{A^o} \arrow[u, "q^*"] & & X^{B^o}. \arrow[ll, "u^*"'] \arrow[u, "p^*"'] \end{tikzcd}$$

and the adjunctions $p^* \vdash p_*$ and $q^* \vdash q_*$, we obtain natural transformation

$$\alpha : u^* p_* \to q_* v^*.$$

We shall say that the *Beck-Chevalley law holds* if α is invertible. The Beck-Chevalley law holds if the square $pv = uq$ is cartesian and p is a proper map. Dually, if X is a cocomplete logos, then from the adjunctions $u_! \vdash u^*$ and $v_! \vdash v^*$, we obtain natural transformation

$$\beta : v_! q^* \to p^* u_!.$$

We shall say that the *Beck-Chevalley law holds* if β is invertible. The Beck-Chevalley law holds if the square $pv = uq$ is cartesian and u is a smooth map. When X is bicomplete, the transformation β is the left transpose of α. Thus, β is invertible iff α is invertible. Hence the Beck-Chevalley law holds in the first sense iff it holds in the second sense.

24.13. If $p : E \to B$ is a proper map and $E(b)$ is the fiber of p at $b \in B_0$, then the Beck-Chevalley law holds for the square

$$\begin{tikzcd} E(b) \arrow[r, "v"] \arrow[d] & E \arrow[d, "p"] \\ 1 \arrow[r, "b"] & B. \end{tikzcd}$$

This means that if X is a complete logos, then we have

$$p_*(f)(b) = \varprojlim_{x \in E(b)} f(x)$$

for any map $f : E^o \to X$. Dually, If $p : E \to B$ is a smooth map and X is a cocomplete logos, then we have

$$p_!(f)(b) = \varinjlim_{x \in E(b)} f(x)$$

for any map $f : E^o \to X$.

24.14. It follows from 24.13 that if $p : E \to B$ is a smooth map and X is a complete logos, then we have

$$\Pi_p(f)(b) = \varprojlim_{x \in E(b)} f(x),$$

for every map $f : E \to X$ and every $b \in B_0$. Dually, if $p : E \to B$ is a proper map and X is a cocomplete logos, then we have

$$\Sigma_p(f)(b) = \varinjlim_{x \in E(b)} f(x)$$

for every map $f : E \to X$ and every $b \in B_0$.

24.15. If X is a cocomplete logos and B is a logos, let us compute the left Kan extension of a map $f : A \to X$ along a map $u : A \to B$. We shall apply the Beck-Chevalley law to the pullback square

$$\begin{tikzcd} A/b \arrow[r] \arrow[d] & B/b \arrow[d, "p"] \\ A \arrow[r, "u"] & B \end{tikzcd}$$

The value of $\Sigma_u(f)$ at $b : 1 \to B$ is obtained by composing the maps

$$X^A \xrightarrow{\Sigma_u} X^B \xrightarrow{X^b} X.$$

If $t : 1 \to B/b$ is the terminal vertex, then we have $X^b = X^t X^p$, since we haved $b = pt$. The map $t : 1 \to B/b$ is right adjoint to the map $r : B/b \to 1$. It follows that $X^t = \Sigma_r$. Thus,

$$X^b \Sigma_u = X^t X^p \Sigma_u = \Sigma_r X^p \Sigma_u.$$

The projection p is smooth since a right fibration is smooth. Hence the following square commutes up to a natural isomorphism by 24.12,

$$\begin{tikzcd} X^{A/b} \arrow[d, "\Sigma_v"'] & X^A \arrow[l, "X^q"'] \arrow[d, "\Sigma_u"] \\ X^{B/b} & X^B. \arrow[l, "X^p"'] \end{tikzcd}$$

Thus,

$$\Sigma_r X^p \Sigma_u \simeq \Sigma_r \Sigma_v X^q \simeq \Sigma_{rv} X^q.$$

But Σ_{rv} is the colimit map

$$\varinjlim : X^{A/b} \to X,$$

since rv is the map $A/b \to 1$. Hence the square

$$\begin{tikzcd} X^{A/b} \arrow[d, "\varinjlim"'] & X^A \arrow[l, "X^q"'] \arrow[d, "\Sigma_u"] \\ X & X^B \arrow[l, "X^b"'] \end{tikzcd}$$

commutes up to a canonical isomorphism. This yields Kan's formula

$$\Sigma_u(f)(b) = \varinjlim_{u(a) \to b} f(a).$$

24.16. Dually, if X is a complete logos and B is a logos, then the right Kan extension of a map $f : A \to X$ along a map $u : A \to B$ is computed by Kan's formula

$$\Pi_u(f)(b) = \varprojlim_{b \to u(a)} f(a),$$

where the limit is taken over the simplicial set $b\backslash A$ defined by the pullback square

$$\begin{tikzcd} b\backslash A \arrow[r] \arrow[d] & b\backslash B \arrow[d, "p"] \\ A \arrow[r, "u"] & B. \end{tikzcd}$$

24.17. A map of simplicial sets $u : A \to B$ is fully faithful iff the map $\Sigma_u : X^A \to X^B$ is fully faithful for every cocomplete logos X.

24.18. Let X be a cocomplete logos. For any span $(s,t) : S \to A \times B$, the composite

$$X^A \xrightarrow{X^s} X^S \xrightarrow{\Sigma_t} X^B$$

is a cocontinuous map

$$X\langle S\rangle : X^A \to X^B.$$

If $u : S \to T$ is a map in $Span(A,B)$, then from the commutative diagram

$$\begin{tikzcd} & S \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "u"] & \\ A & & B \\ & T \arrow[ul, "l"] \arrow[ur, "r"'] & \end{tikzcd}$$

and the counit $\Sigma_u \circ X^u \to id$, we can define a 2-cell,

$$X\langle u\rangle : X\langle S\rangle = \Sigma_t \circ X^s = \Sigma_r \circ \Sigma_u \circ X^u \circ X^l \to \Sigma_r \circ X^l = X\langle T\rangle.$$

This defines a functor

$$X\langle - \rangle : Span(A,B) \to \tau_1(X^A, X^B).$$

A map $u : S \to T$ in $Span(A,B)$ is a bivariant equivalence if the 2-cell

$$X\langle u\rangle : X\langle S\rangle \to X\langle T\rangle.$$

is invertible for any cocomplete logos X iff the 2-cell

$$\mathbf{U}\langle u\rangle : \mathbf{U}\langle S\rangle \to \mathbf{U}\langle T\rangle.$$

is invertible. We thus obtain a functor

$$X\langle - \rangle : \Lambda(A,B) \to \tau_1(X^A, X^B).$$

24.19. If $S \in Span(A,B)$ and $T \in Span(B,C)$ are bifibrant spans, then we have a canonical isomorphism

$$X\langle T \circ S\rangle \simeq X\langle T\rangle \circ X\langle S\rangle$$

for any cocomplete logos X. To see this, it suffices to consider the case where A, B and C are logoi. We have a pullback diagram,

$$\begin{tikzcd} & & T \circ S \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ & S \arrow[dl, "s"'] \arrow[dr, "t"] & & T \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B & & C. \end{tikzcd}$$

The map $s : T \to B$ is smooth, since it is Grothendieck fibration by 15.13. It then follows from 24.12 that the Beck-Chevalley law holds for the square in the following diagram,

$$\begin{tikzcd}
& & X^{T\circ S} \arrow[dr, "\Sigma_q"] & & \\
& X^S \arrow[ur, "X^p"] \arrow[dr, "\Sigma_t"] & & X^T \arrow[dr, "\Sigma_t"] & \\
X^A \arrow[ur, "X^s"] & & X^B \arrow[ur, "X^s"] & & X^C.
\end{tikzcd}$$

Thus,

$$X\langle T\circ S\rangle = \Sigma_r\Sigma_q X^p X^s \simeq \Sigma_t X^p \Sigma_t X^s = X\langle T\rangle \circ X\langle S\rangle.$$

We have defined a (pseudo) functor

$$X\langle - \rangle : \Lambda \to \mathbf{CQ},$$

where $\mathbf{CQ}$ the 2-category of cocomplete logoi and cocontinuous maps.

24.20. If A is a (small) simplicial set, then the endo-functor $X \mapsto X^A$ of $\mathbf{CQ}$ is right adjoint to the endo-functor $X \mapsto X^{A^o}$. More precisely, for any pair of cocomplete logoi X and Y, we have a natural equivalence of categories

$$\mathbf{CQ}(X^{A^o}, Y) \simeq \mathbf{CQ}(X, Y^A).$$

The unit of the adjunction is the map $X\langle \eta_A\rangle : X \to X^{A^o\times A}$ and the counit is the map $X\langle \epsilon_A\rangle : X^{A\times A^o} \to X$. It folllows from this adjunction that the logos X^{A^o} can be regarded as the *tensor product* $A\otimes X$ of X by A. More precisely, the map

$$c_A : A\times X \to X^{A^o}$$

which corresponds to the map $X\langle\eta_A\rangle : X \to X^{A^o\times A}$ by the exponential adjointness is cocontinuous in the second variable and universal with respect to that property. This means that for any cocomplete logos Y and any map $f : A\times X \to Y$ cocontinuous in the second variable, there exists a cocontinuous map $g : X^{A^o} \to Y$ together with an isomorphism $\alpha : f \simeq gc_A$ and moreover that the pair (f,α) is unique up to unique isomorphism. Notice that we have $c_A(a,x)(b^o) = Hom_A(b,a)\cdot x$ for every $a,b\in A$ and $x\in X$. The 2-category $\mathbf{CQ}$ becomes tensored over the category Λ^{rev} if we put $A\otimes X = X^{A^o}$ and

$$\langle S\rangle \otimes X = X\langle S^o\rangle : A\otimes X \to B\otimes X$$

for $S \in Span(B,A)$. In particular, we have $A\otimes \mathbf{U} = \mathbf{P}(A)$.

24.21. The counit of the adjunction $(-)^{A^o} \vdash (-)^A$ described above is the *trace map*

$$Tr_A = X\langle\epsilon_A\rangle : X^{A\times A^o} \to X.$$

In category theory, the trace of a functor $f : A\times A^o \to Y$ is called the *coend*

$$coend_A(f) = \int^{a\in A} f(a,a).$$

We shall use the same notation for the trace of a map $f : A\times A^o \to Y$. Notice that

$$Tr_A(f) = Tr_{A^o}({}^t f),$$

where ${}^t f : A^o\times A \to Y$ is the transpose of f. The inverse of the equivalence

$$\mathbf{CQ}(X^{A^o}, Y) \simeq \mathbf{CQ}(X, Y^A)$$

associates to a map $f : X \times A \to Y$ cocontinuous in the first variable the map $g : X^{A^o} \to Y$ obtained by putting

$$g(z) = \int^{a \in A} f(z(a), a).$$

for every $z \in X^{A^o}$.

24.22. If X is a complete logos, the *cotrace map*

$$Tr^o_A : X^{A^o \times A} \to X$$

is defined to be the opposite of the trace map $Tr_A : (X^o)^{A \times A^o} \to X^o$. In category theory, the cotrace of a functor $f : A^o \times A \to X$ is the *end*

$$end_A(f) = \int_{a \in A} f(a, a),$$

and we shall use the same notation. Notice that

$$Tr^o_A(f) = Tr^o_{A^o}({}^t f),$$

where ${}^t f : A \times A^o \to X$ is the transpose of f.

24.23. If X is a logos, then the contravariant functor $A \mapsto ho(A, X) = ho(X^A)$ is a kind of cohomology theory with values in **Cat**. When X is bicomplete, the map $ho(u, X) : ho(B, X) \to ho(A, X)$ has a left adjoint $ho(\Sigma_u)$ and a right adjoint $ho(\Pi_u)$ for any map $u : A \to B$. If we restrict the functor $A \mapsto ho(A, X)$ to the subcategory $\mathbf{Cat} \subset \mathbf{S}$, we obtain a *homotopy theory* in the sense of Heller, also called a *derivateur* by Grothendieck [Malt1] Most derivateurs occuring naturally in mathematics can be represented by bicomplete logoi.

25. The logos U

The logos **U** is cocomplete and freely generated by its terminal objects. More generally, the logos of prestacks on a simplicial set A is cocomplete and freely generated by A. A cocomplete logos is equivalent to a logos of prestacks iff it is cogenerated by a small set of atoms.

25.1. Recall that the logos $\mathbf{U} = \mathbf{U}_0$ is defined to be the coherent nerve of the category **Kan** of Kan complexes. The logos **U** is bicomplete and freely generated by the object $1 \in \mathbf{U}$ as a cocomplete logos. More precisely, if S and T are cocomplete logoi, let us denote by $CC(S, T)$ the full simplicial subset of T^S spanned by the cocontinuous maps $S \to T$. Then the evaluation map

$$ev : CC(\mathbf{U}, X) \to X$$

defined by putting $ev(f) = f(1)$ is an equivalence for any cocomplete logos X. The map ev is actually a trivial fibration. If s is a section of ev, then the map

$$\cdot : \mathbf{U} \times X \to X$$

defined by putting $k \cdot x = s(x)(k)$ is cocontinuous in each variable and we have $1 \cdot x = x$ for every $x \in X$.

25.2. The Yoneda map $y_A : A \to \mathbf{P}(A)$ exibits the logos $\mathbf{P}(A)$ as the free completion of A under colimits. More precisely, the map

$$y_A^* : CC(\mathbf{P}(A), X) \to X^A$$

induced by the map $y_A : A \to \mathbf{P}(A)$ is an equivalence for any cocomplete logos X. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathbf{P}(A) \to X$ along y_A :. The value of $g_!$ on a prestack $k \in \mathbf{P}(A)$ is the colimit of the map $g : A \to X$ weighted by the diagram $El(k) \to A$. In other words, we have

$$g_!(k) = \varinjlim_{El(k)} g.$$

Compare with Dugger [Du].

25.3. The left Kan extension of the Yoneda map $y_A : A \to \mathbf{P}(A)$ along itself is the identity of $\mathbf{P}(A)$. It follows that we have

$$k = \varinjlim_{El(k)} y_A$$

for every object $k \in \mathbf{P}(A)$.

25.4. If $f : A \to X$ is a map between small logoi, we shall say that the map

$$f^! = f^* y_X : X \to \mathbf{P}(A)$$

is the *probe map* associated to f. The map $f^!$ can be defined under the weaker assumption that X is locally small.

25.5. For example, if f is the map $\Delta \to \mathbf{U}_1$ obtained by applying the coherent nerve functor to the inclusion $\Delta \to \mathbf{QCat}$, then the probe map

$$f^! : \mathbf{U}_1 \to \mathbf{P}(\Delta)$$

associates to an object $C \in \mathbf{U}_1$ its *nerve* $N(C) : \Delta^o \to \mathbf{U}$. By construction, we have

$$N(C)_n = J(C^{\Delta[n]})$$

for every $n \geq 0$.

25.6. If X is locally small and cocomplete, then the left Kan extension of a map $f : A \to X$ along the Yoneda map $y_A : A \to \mathbf{P}(A)$ is left adjoint to the probe map $f^!$,

$$f_! : \mathbf{P}(A) \leftrightarrow X : f^!.$$

25.7. For any simplicial set A, the logos $\mathbf{P}(A)$ is the homotopy localisation of the model category $(\mathbf{S}/A, Wcont)$. More precisely, we saw in 21.44 that the map $\lambda_A : \Delta/A \to A$ is a homotopy localisation. The left Kan extension of the composite

$$y_A \lambda_A : \Delta/A \to \mathbf{P}(A)$$

along the inclusion $\Delta/A \to \mathbf{S}/A$ induces an equivalence of logoi

$$L(\mathbf{S}/A, Wcont) \to \mathbf{P}(A).$$

The inverse equivalence associates to a prestack $f : A \to \mathbf{U}$ the right fibration $El(f) \to A$.

25.8. If $f : A \to B$ is a map between small logoi, then the map $f^* = \mathbf{U}^{f^o}$ is the probe of the composite $y_B f : A \to B \to \mathbf{P}(B)$ and we have

$$f_! : \mathbf{P}(A) \leftrightarrow \mathbf{P}(B) : f^*.$$

25.9. It follows from Yoneda lemma that the logos of elements $El(g)$ of a prestack $g \in \mathbf{P}(A)$, is equivalent to the logos A/g defined by the pullback square

$$\begin{tikzcd} A/g \arrow[r, "q"] \arrow[d] & \mathbf{P}(A)/g \arrow[d] \\ A \arrow[r, "y_A"] & \mathbf{P}(A), \end{tikzcd}$$

The adjoint pair

$$q_! : \mathbf{P}(A/g) \leftrightarrow \mathbf{P}(A)/g : q^!$$

obtained from the map q is an equivalence of logoi.

25.10. Let X be a locally small logos. If A is a small simplicial set, we shall say that a map $f : A \to X$ is *dense* if the probe map $f^! : X \to \mathbf{P}(A)$ is fully faithful. We shall say that a small full subcategory $A \subseteq X$ is *dense* if the inclusion $i : A \subseteq X$ is dense; we shall say that a set of objects $S \subseteq X$ is *dense* if the full sub logos spanned by S is dense.

25.11. Let X be a locally small logos. If A is a small simplicial set, we shall say that a map $f : A \to X$ is *separating* if the probe map $f^! : X \to \mathbf{P}(A)$ is conservative; we shall say that a set of objects $S \subseteq X$ is *separating* if the inclusion $S \subseteq X$ is separating. Every dense map is separating.

25.12. For example, the Yoneda map $y_A : A \to \mathbf{P}(A)$ is dense, since the map $(y_A)_!$ is the identity. In particular, the map $1 : 1 \to \mathbf{U}$ is dense. The map $f : \Delta \to \mathbf{U}_1$ defined in **??** is dense; this means that the nerve map

$$N : \mathbf{U}_1 \to \mathbf{P}(\Delta)$$

is fully faithful.

25.13. A map of simplicial sets $u : A \to B$ is dominant iff the map $y_B u : A \to \mathbf{P}(B)$ is dense.

25.14. Let X be a locally small logos. If A is a simplicial set, then a map $f : A \to X$ is dense iff the counit of the adjunction

$$f_! : \mathbf{P}(A) \leftrightarrow X : f^!$$

is invertible. The value of this counit at $x \in X$ is the canonical morphism

$$\varinjlim_{A/x} f \to x$$

where the diagram $A/x \to A$ is defined by the pullback square

$$\begin{tikzcd} A/x \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ X/x \arrow[r] & X. \end{tikzcd}$$

25.15. If X is a logos, we shall say that a simplicial subset $A \subseteq X$ is *replete* if every object of X isomorphic to an object in A belongs to A. If X is cocomplete, let us denote by $\overline{A}$ the smallest full simplicial subset of X which is replete, which is closed closed under colimits and which contains A. We shall say that A is *cogenerating* if we have $\overline{A} = X$.

25.16. A cogenerating simplicial subset of a cocomplete logos is separating.

25.17. Let X be a (locally small) cocomplete logos. We shall say that an object $x \in X$ is *atomic* if the map $hom_X(x, -) : X \to \mathbf{U}$ is cocontinuous.

25.18. Any complete or cocmplete logos is Karoubi complete. In particular, the logos $\mathbf{P}(A)$ is Karoubi complete for any simplicial set A. Moreover, the Yoneda map $y_A : A \to \mathbf{P}(A)$ admits an extension $y'_A : \kappa(A) \to \mathbf{P}(A)$ to the Karoubi envelope of A and this xtension is homotopy unique. The map y'_A is fully faithful and it induces an equivalence between $\kappa(A)$ and the full simplicial subset of $\mathbf{P}(A)$ spanned by atomic objects.

25.19. If $g : A^o \to \mathbf{U}$ is a prestack on a small logos A, then the atoms of the logos $\mathbf{P}(A)/g$ are the morphisms $a \to g$, with $a \in \kappa(A)$.

25.20. Let X be a cocomplete (locally small) logos and $A \subseteq X$ be a small full simplicial subset spanned by atomic objects. Then the left Kan extension

$$i_! : \mathbf{P}(A) \to X.$$

of the inclusion $i : A \subseteq X$ along the map $y_A : A \to \mathbf{P}(A)$ is fully faithful. Moreover $i_!$ is an equivalence iff A cogenerates X iff A separates X.

25.21. A cocomplete logos X is equivalent to a logos of prestacks iff it is cogenerated by a small set of atoms.

25.22. If A is a simplicial set, we say that a pre-stack $g \in \mathbf{P}(A)$ is *finitely presented* if it is the colimit of a finite diagram of representable prestacks.

25.23. If A is a simplicial set, we say that a pre-stack $g \in \mathbf{P}(A)$ is *finitely presentable* if it is the colimit of a finite diagram of representable prestacks. We shall denote by $\mathbf{P}^f(A)$ the full simplicial subset of $\mathbf{P}(A)$ spanned by the finitely presentable prestacks. The Yoneda map $y_A : A \to \mathbf{P}(A)$ induces a map $A \to \mathbf{P}^f(A)$ (also denoted y_A).

25.24. We conjecture that an atomic prestack is finitely presentable iff it is representable.

25.25. For any (small) simplicial set A, the map $y_A : A \to \mathbf{P}^f(A)$ exhibits the logos $\mathbf{P}^f(A)$ as the free completion of A under finite colimits. More precisely, if S and T are cocartesian logoi, let us denote by $C^f(S,T)$ the full simplicial subset of T^S spanned by the finitely cocontinuous maps $S \to T$. Then the map

$$y_A^* : C^f(\mathbf{P}^f(A), X) \to X^A$$

induced by the map $y_A : A \to \mathbf{P}^f(A)$ is an equivalence for any cocartesian logos X. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathbf{P}^f(A) \to X$ along the map $y_A : A \to \mathbf{P}^f(A)$.

25.26. A logos A is cocartesian iff the map $y_A : A \to \mathbf{P}^f(A)$ has a left adjoint.

25.27. For any prestack $g \in \mathbf{P}(A)$, the equivalence $\mathbf{P}(A/g) \simeq \mathbf{P}(A)/g$ of 25.19 induces an equivalence

$$\mathbf{P}^f(A/g) \simeq \mathbf{P}^f(A)/g,$$

25.28. If $u : A \to B$ is a map of simplicial sets, then the map $u_! : \mathbf{P}(A) \to \mathbf{P}(B)$ takes finitely presentable prestacks to a finitely presentable prestacks. We obtain a square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "y_A"'] & B \arrow[d, "y_B"] \\ \mathbf{P}^f(A) \arrow[r, "u_!"] & \mathbf{P}^f(B) \end{tikzcd}$$

which commutes up to a canonical isomorphism. The induced map $u_! : \mathbf{P}^f(A) \to \mathbf{P}^f(B)$ is fully faithful iff u is fully faithful. We conjecture that $u_!$ is an equivalence iff u is a weak categorical equivalence.

25.29. If A is a simplicial set, we say that a pre-stack $g \in \mathbf{P}(A)$ is *α-presentabe* if it is the colimit of a diagram of cardinality $< \alpha$ of representable prestacks. We shall denote by $\mathbf{P}_\alpha(A)$ the full simplicial subset of $\mathbf{P}(A)$ spanned by the α-presentable prestacks. The Yoneda map $y_A : A \to \mathbf{P}(A)$ induces a map $A \to \mathbf{P}^f(A)$ (also denoted y_A).

25.30. If For any (small) simplicial set A the map $y_A : A \to \mathbf{P}^f(A)$ exhibits the logos $\mathbf{P}_\alpha(A)$ as the free completion of A under α-colimits. More precisely, if S and T are α-cocomplete logoi, let us denote by $C_\alpha(S,T)$ the full simplicial subset of T^S spanned by the α-cocontinuous maps $S \to T$. Then the map

$$y_A^* : C_\alpha(\mathbf{P}_\alpha(A), X) \to X^A$$

induced by the map $y_A : A \to \mathbf{P}_\alpha(A)$ is an equivalence for any α-cocomplete logos X. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathbf{P}_\alpha(A) \to X$ along the map $y_A : A \to \mathbf{P}_\alpha(A)$.

25.31. A logos A is α-cocomplete iff the map $y_A : A \to \mathbf{P}_\alpha(A)$ has a left adjoint.

25.32. For any prestack $g \in \mathbf{P}(A)$, the equivalence $\mathbf{P}(A/g) \simeq \mathbf{P}(A)/g$ of 25.19 induces an equivalence

$$\mathbf{P}_\alpha(A/g) \simeq \mathbf{P}_\alpha(A)/g,$$

25.33. The correspondance $f \mapsto el(f)$ of 16.8 between the maps $B \to \mathbf{U}$ and the left fibrations in $\mathbf{S}/B$ can be used for translating the properties of the former to the latter. For example, a map $f : B \to \mathbf{U}$ is said to be *coexact* on an inductive cone $c : K \star 1 \to B$ if the inductive cone $fc : K \star 1 \to \mathbf{U}$ is coexact. We shall say a left fibration $q : X \to B$ is *coexact* on $c : K \star 1 \to B$ if it is classified by a map $f : B \to \mathbf{U}$ wjhich is coexact on c. It is easy to see that $q : X \to B$ is coexact on c iff the inclusion $(ci)^*(X) \subseteq c^*(X)$ is a weak homotopy equivalence, where i denotes the inclusion $K \subseteq K \star 1$. Dually, a map $f : B \to \mathbf{U}$ is said to be *exact on a cone* $c : 1 \star K \to B$ if the composite $fc : 1 \star K \to \mathbf{U}$ is exact. A left fibration $q : X \to B$ is *exact* on $c : 1 \star K \to B$ iff the map $[i, X] : [1 \star K, X] \to [K, X]$ obtained from the inclusion $i : K \subseteq 1 \star K$ is a weak homotopy equivalence.

25.34. If B is a logos, we shall say that a left fibration $p : E \to B$ *preserves the colimit* of a diagram $d : K \to B$ if this colimit exists and p is coexact on the coexact inductive cone which extends d. Dually, we shall say that p *preserves the limit* of d if this limit exists and p is exact on the exact projective cone which extends d. For example, for any map $a : A \to B$, the left fibration $a\backslash B \to B$ preserves the limit of any diagram $K \to B$. The notion of a right fibration $p : E \to B$ preserving the limit or the colimit of a diagram $d : K \to B$ is defined dually.

26. Factorisation systems in logoi

In this section, we introduce the notion of factorisation system in a logos. It is closely related to the notion of homotopy factorisation system in a model category introduced in section 11.

26.1. We first define the orthogonality relation $u \perp f$ between the arrows of a logos X. If $u : a \to b$ and $f : x \to y$ are two arrows in X, then an arrow $s \in X^I(u, f)$ in the logos X^I is a a commutative square $s : I \times I \to X$,

$$\begin{tikzcd} a \arrow[r] \arrow[d, "u"'] & x \arrow[d, "f"] \\ b \arrow[r] & y, \end{tikzcd}$$

such that $s|\{0\} \times I = u$ and $s|\{1\} \times I = f$. A *diagonal filler* for s is a map $I \star I \to X$ which extends s along the inclusion $I \times I \subset I \star I$. The projection $q : X^{I \star I} \to X^{I \times I}$ defined by the inclusion $I \times I \subset I \star I$ is a Kan fibration. We shall say that u is *left orthogonal* to f, or that f is *right orthogonal* to u, and we shall write $u \perp f$, if the fiber of q at s is contractible for every commutative square $s \in X^I(u, f)$. An arrow $f \in X$ is invertible iff we have $f \perp f$.

26.2. When X has a terminal object 1, then an arrow $x \to 1$ is right orthogonal to an arrow $u : a \to b$ iff the map

$$X(u, x) : X(b, x) \to X(a, x)$$

induced by u is a homotopy equivalence. In this case we shall say that x is *right orthogonal* to the arrow u, or that x *local* with respect to u, and we shall write $u \perp x$.

26.3. If $h : X \to hoX$ is the canonical map, then the relation $u \perp f$ between the arrows of X implies the relation $h(u) \pitchfork h(f)$ in hoX. However, if $h(u) = h(u')$ and $h(f) = h(f')$, then the relations $u \perp f$ and $u' \perp f'$ are equivalent. Hence the relation $u \perp f$ only depends on the homotopy classes of u and f. If A and B are two sets of arrows in X, we shall write $A \perp B$ to indicate the we have $u \perp f$ for every $u \in A$ and $f \in B$. We shall put

$$A^\perp = \{f \in X_1 : \forall u \in A,\ u \perp f\}, \qquad {}^\perp A = \{u \in X_1 : \forall f \in A,\ u \perp f\}.$$

The set $A^\perp$ contains the isomorphisms, it is closed under composition and it has the left cancellation property. It is closed under retracts in the logos X^I. And it is closed under base changes when they exist. This means that the implication $f \in A^{\perp} \Rightarrow f' \in A^{\perp}$ is true for any pullback square

$$\begin{tikzcd} x' \arrow[r] \arrow[d, "f'"'] & x \arrow[d, "f"] \\ y' \arrow[r] & y \end{tikzcd}$$

in X.

26.4. Let X be a (large or small) logos. We shall say that a pair (A, B) of class of arrows in X is a *factorisation system* if the following two conditions are satisfied:

- $A^{\perp} = B$ and $A = {}^{\perp}B$;
- every arrow $f \in X$ admits a factorisation $f = pu$ (in hoX) with $u \in A$ and $p \in B$.

We say that A is the *left class* and that B is the *right class* of the factorisation system.

26.5. If X is a logos, then the image by the canonical map $h : X \to hoX$ of a factorisation system (A, B) is a weak factorisation system $(h(A), h(B))$ on the category hoX. Moreover, we have $A = h^{-1}h(A)$ and $B = h^{-1}h(B)$. Conversely, if (C, D) is a weak factorisation system on the category $ho(X)$, then the pair $(h^{-1}(C), h^{-1}(D))$ is a factorisation system on X iff we have $h^{-1}(C) \perp h^{-1}(D)$.

26.6. The left class A of a factorisation system (A, B) in a logos has the right cancellation property and the right class B the left cancellation property. Each class is closed under composition and retracts. The class A is closed under cobase changes when they exist. and the class B under base changes when they exist.

26.7. The intersection $A \cap B$ of the classes of a factorisation system (A, B) on a logos X is the class of isomorphisms in X. Let us denote by A' the 1-full simplicial subset of X spanned by A. The simplicial set A' is a logos by **??**, since we have $A = h^{-1}h(A)$ and $h(A)$ is a subcategory of hoX. We shall say that it is the *sub-logos spanned* by A. If B' is the sub-logos spanned by B, then we have $A' \cap B' = J(X)$, where $J(X)$ is the largest sub Kan complex of X.

26.8. Let (A, B) be a factorisation system in a logos X. Then the full sub-logos of X^I spanned by the elements in B is reflective; it is thus closed under limits. Dually, the full sub-logos of X^I spanned by the elements in A is coreflective; it is thus closed under colimits.

26.9. Let (A, B) be a factorisation system in a logos X. Then the full sub-logos of X^I spanned by the elements in B is reflective. Hence this sub-logos is closed under limits. Dually, the full sub=logos of X^I spanned by the elements in A is coreflective.

26.10. L'et (A, B) be a factorisation system in a logos X. If $p : E \to X$ is a left or a right fibration, then the pair $(p^{-1}(A), p^{-1}(B))$ is a factorisation system in E; we shall say that the system $(p^{-1}(A), p^{-1}(B))$ is obtained by *lifting* the system (A, B) to E along p. In particular, every factorisation system on X can lifted to X/b (resp. $b\backslash X$) for any vertex $b \in X$.

26.11. A factorisation system (A, B) on a logos X induces a factorisation system (A_S, B_S) on the logos X^S for any simplicial set S. By definition, a natural transformation $\alpha : f \to g : S \to X$ belongs to A_S (resp. B_S) iff the arrow $\alpha(s) : f(s) \to g(s)$ belongs to A (resp. B) for every vertex $s \in S$. We shall say that the system (A_S, B_S) is *induced* by the system (A, B).

26.12. Let $p : \mathcal{E} \to L(\mathcal{E})$ be the homotopy localisation of a model category. If (A, B) is a factorisation system in $L(\mathcal{E})$, then the pair $(p^{-1}(A), p^{-1}(B)$ is a homotopy factorisation system in $\mathcal{E}$, and this defines a bijection between the factorisation systems in $L(\mathcal{E})$ and the homotopy factorisation systems in $\mathcal{E}$.

26.13. If A is the class of essentially surjective maps in the logos $\mathbf{U}_1$ and B is the class of fully faithful maps, then the pair (A, B) is a factorisation system. If A is the class of final maps in $\mathbf{U}_1$ and B is the class of right fibrations then the pair (A, B) is a factorisation system. If B is the class of conservative maps in $\mathbf{U}_1$ and A is the class of iterated homotopy localisations, then the pair (A, B) is a factorisation system. If A is the class of weak homotopy equivalences in $\mathbf{U}_1$ and B is the class of Kan fibrations then the pair (A, B) is a factorisation system.

26.14. Let $p : X \to Y$ be a Grothendieck fibration between logoi. If $A \subseteq X$ is the set of arrows inverted by p and $B \subseteq X$ is the set of cartesian arrows, then the pair (A, B) is a factorisation system on X.

26.15. If X is a logos with pullbacks then the target functor $t : X^I \to X$ is a Grothendieck fibration. It thus admits a factorisation system (A, B) in which B is the class of pullback squares. An arrow $u : a \to b$ in X^I belongs to A iff the arrow u_1 in the square

$$\begin{tikzcd} a_0 \arrow[r, "u_0"] \arrow[d] & b_0 \arrow[d] \\ a_1 \arrow[r, "u_1"] & b_1 \end{tikzcd}$$

is quasi-invertible.

26.16. We say that a factorisation system (A, B) in a logos with finite products X is *stable under finite products* if the class A is closed under products in the category X^I. When X has pullbacks, we say that a factorisation system (A, B) is *stable under base changes* if the class A is closed under base changes. This means that the implication $f \in A \Rightarrow f' \in A$ is true for any pullback square

$$\begin{tikzcd} x' \arrow[r] \arrow[d, "f'"'] & x \arrow[d, "f"] \\ y' \arrow[r] & y. \end{tikzcd}$$

26.17. Every factorisation system in the logos $\mathbf{U}$ is stable under finite products.

26.18. We shall say that an arrow $u : a \to b$ in a logos X is a *monomorphism* or that it is *monic* if the commutative square

$$\begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] & a \arrow[d, "u"] \\ a \arrow[r, "u"] & b \end{tikzcd}$$

is cartesian. Every monomorphism in X is monic in the category hoX but the converse is not necessarly true. A map between Kan complexes $u : A \to B$ is monic in $\mathbf{U}$ iff it is homotopy monic.

26.19. We shall say that an arrow in a cartesian logos X is *surjective*, or that is a *surjection* , if it is left orthogonal to every monomorphism of X. We shall say that a cartesian logos X admits *surjection-mono factorisations* if every arrow $f \in X$ admits a factorisation $f = up$, with u a monomorphism and p a surjection. In this case X admits a factorisation system (A, B), with A the set of surjections and B the set of monomorphisms. If logos X admits surjection-mono factorisations, then so do the logoi $b\backslash X$ and X/b for every vertex $b \in X$, and the logos X^S for every simplicial set S.

26.20. If logos X admits surjection-mono factorisations, then so does the category hoX.

26.21. We say that a cartesian logos X is *regular* if it admits surjection-mono factorisations and system is stable under base changes.

26.22. The logos $\mathbf{U}$ is regular. If a logos X is regular then so are the logoi $b\backslash X$ and X/b for any vertex $b \in X$ and the logos X^A for any simplicial set A.

26.23. Recall that a simplicial set A is said to be a *0-object* if the canonical map $A \to \pi_0(A)$ is a weak homotopy equivalence, If X is a logos, we shall say that an object $a \in X$ is *discrete* or that it is a 0-*object* if the simplicial set $X(x, a)$ is a 0-object for every object $x \in X$. When the product $a \times a$ exists, the object $a \in X$ is a 0-object iff the diagonal $a \to a \times a$ is monic. When the exponential a^{S^1} exists, the object $a \in X$ is a 0-object iff the projection $a^{S^1} \to a$ is invertible. We shall say that an arrow $u : a \to b$ in X is a *0-cover* if it is a 0-object of the slice logos X/b. An arrow $u : a \to b$ is a 0-cover iff the map $X(x, u) : X(x, a) \to X(x, b)$ is a 0-cover for every node $x \in X$. We shall say that an arrow $u : a \to b$ in X is *0-connected* if it is left orthogonal to every 0-cover in X. We shall say that a logos X admits *0-factorisations* if every arrow $f \in X$ admits a factorisation $f = pu$ with u a 0-connected arrow and p a 0-cover. In this case X admits a factorisation system (A, B) with A the set of 0-connected maps and B the set of 0-covers. If a logos X admits 0-factorisations, then so do the logoi $b\backslash X$ and X/b for every vertex $b \in X$, and the logos X^S for every simplicial set S.

26.24. There is a notion of n-cover and of n-connected arrow in every logos for every $n \geq -1$. If X is a logos, we shall say that a vertex $a \in X$ is a *n-object* if the simplicial set $X(x, a)$ is a n-object for every vertex $x \in X$. If $n = -1$, this means that $X(x, a)$ is contractible or empty. When the exponential $a^{S^{n+1}}$ exists, then a is a n-object iff the projection $a^{S^{n+1}} \to a$ is invertible. We shall say that an arrow $u : a \to b$ is a *n-cover* if it is a n-object of the slice logos X/b. If $n \geq 0$ and the product $a \times a$ exists, the vertex a is a n-object iff the diagonal $a \to a \times a$ is a $(n-1)$-cover. We shall say that an arrow in a logos X is *n-connected* if it is left orthogonal to every n-cover. We shall say that a logos X admits *n-factorisations* if every arrow $f \in X$ admits a factorisation $f = pu$ with u a n-connected map and p a n-cover. In this case X admits a factorisation system (A, B) with A the set of n-connected maps and B the class of n-covers. If $n = -1$, this is the surjection-mono factorisation system. If X admits k-factorisations for every $-1 \leq k \leq n$, then we have a sequence of inclusions

$$A_{-1} \supseteq A_0 \supseteq A_1 \supseteq A_2 \cdots \supseteq A_n$$

$$B_{-1} \subseteq B_0 \subseteq B_1 \subseteq B_2 \cdots \subseteq B_n,$$

where (A_k, B_k) denotes the k-factorisation system in X.

26.25. The logos $\mathbf{U}$ admits n-factorisations for every $n \geq -1$ and the system is stable under base change.

26.26. If a logos X admits n-factorisations, then so do the logoi $b\backslash X$ and X/b for every vertex $b \in X$, and the logos X^S for every simplicial set S.

26.27. Suppose that X admits k-factorisations for every $0 \leq k \leq n$. If $k > 0$, we shall say that a k-cover $f : x \to y$ in X is an *Eilenberg-MacLane k-gerb* and f is $(k-1)$-connected. A *Postnikov tower* (of height n) for an arrow $f : a \to b$ is a factorisation of length $n+1$ of f

$$a \xleftarrow{p_0} x_0 \xleftarrow{p_1} x_1 \xleftarrow{p_2} \cdots \xleftarrow{p_n} x_n \xleftarrow{q_n} b,$$

where p_0 is a 0-cover, where p_k is an *EM* k-gerb for every $1 \leq k \leq n$ and where q_n is n-connected. The tower can be augmented by further factoring p_0 as a surjection followed by a monomorphism. Every arrow in X admits a Postnikov tower of height n and the tower is unique up to a homotopy unique isomorphism in the logos $X^{\Delta[n+1]}$.

26.28. We shall say that a factorisation system (A, B) in a logos X is *generated* by a set Σ of arrows in X if we have $B = \Sigma^{\perp}$. Let X be a cartesian closed logos. We shall say that a factorisation system (A, B) in X is *multiplicatively generated* by a set of arrows Σ if it is generated by the set

$$\Sigma' = \bigcup_{a \in X_0} a \times \Sigma.$$

A multiplicatively generated system is stable under products. For example, in the logos $\mathbf{U}$, the n-factorisations system is multiplicatively generated by the map $S^{n+1} \to 1$. In the logos $\mathbf{U}_1$, the system of essentially surjective maps and fully faithful maps is multiplicatively generated by the inclusion $\partial I \subset I$. The system of final maps and right fibrations is multiplicatively generated by the inclusion $\{1\} \subset I$. The dual system of initial maps and left fibrations is multiplicatively generated by the inclusion $\{0\} \subset I$. The system of iterated homotopy localisations and conservative maps is multiplicatively generated by the map $I \to 1$ (or by the inclusion $I \subset J$, where J is the groupoid generated by one isomorphism $0 \to 1$). The system of weak homotopy equivalences and Kan fibrations is multiplicatively generated by the pair of inclusions $\{0\} \subset I$ and $\{1\} \subset I$.

27. n-OBJECTS

27.1. Recall that a simplicial set X is said to be a *n-object*, where $n \geq 0$, if we have $\pi_i(X, x) = 1$ for every $i > n$ and $x \in X$. A Kan complex X is a n-object iff every sphere $\partial\Delta[m] \to X$ of dimension $m - 1 > n$ can be filled. We shall say that a map of simplicial sets $u : A \to B$ is a *weak homotopy n-equivalence* if the map $\pi_0(u) : \pi_0(A) \to \pi_0(B)$ is bijective as well as the maps $\pi_i(u, a) : \pi_i(A, a) \to \pi_i(B, u(a))$ for every $1 \leq i \leq n$ and $a \in A$. The model category $(\mathbf{S}, \mathit{Who})$ admits a Bousfield localisation with respect to the class of weak homotopy n-equivalences. We shall denote the local model structure shortly by $(\mathbf{S}, \mathit{Who}[n])$, where $\mathit{Who}[n]$ denotes the class of weak homotopy n-equivalences. Its fibrant objects are the Kan n-objects.

27.2. Recal that a simplicial set X is said to be a (-1)-*object* if it is contractible or empty (ie if the map $X \to \exists X$ is a weak homotopy equivalence, where $\exists X \subseteq 1$ denotes the image of the map $X \to 1$). A Kan complex X is a (-1)-object iff every sphere $\partial\Delta[m] \to X$ with $m > 0$ can be filled. We shall say that a map of simplicial sets $u : A \to B$ is a (-1)-*equivalence* if it induces a bijection $\exists A \to \exists B$. The model category $(\mathbf{S}, \mathit{Who})$ admits a Bousfield localisation with respect to the class of weak homotopy (-1)-equivalences. We shall denote the local model structure shortly by $(\mathbf{S}, \mathit{Who}[-1])$, where $\mathit{Who}[-1]$ denotes the class of weak homotopy (-1)-equivalences. Its fibrant objects are the Kan (-1)-objects.

27.3. Recall that a simplicial set X is said to be a (-2)-*object* if it is contractible. Every map of simplicial sets is by definition a (-2)-*equivalence*. The model category $(\mathbf{S}, \mathit{Who})$ admits a Bousfield localisation with respect to the class of (-2)-equivalences (ie of all maps). The local model can be denoted by by $(\mathbf{S}, \mathit{Who}[-2])$, where $\mathit{Who}[-2]$ denotes the class of all maps. Its fibrant objects are the contractible Kan complexes.

27.4. The *homotopy n-type* of a simplicial set A is defined to be a fibrant replacement of $A \to \pi_{[n]}(A)$ of A in the model category $(\mathbf{S}, \mathit{Who}[n])$.

27.5. If $n \geq -2$, we shall denote by $\mathbf{U}[n]$ the coherent nerve of the category of Kan n-objects. It is the full simplicial subset of $\mathbf{U}$ spanned by these objects. We have an infinite sequence of logoi,

$$\mathbf{U}[-2] \longrightarrow \mathbf{U}[-1] \longrightarrow \mathbf{U}[0] \longrightarrow \mathbf{U}[1] \longrightarrow \mathbf{U}[2] \longrightarrow \cdots .$$

The logos $\mathbf{U}[-2]$ is equivalent to the terminal logos 1. The logos $\mathbf{U}[-1]$ is equivalent to the poset $\{0, 1\}$ and the logos $\mathbf{U}[0]$ to the category of sets. The logos $\mathbf{U}[1]$ is equivalent to the coherent nerve of the category of groupoids. Each logos $\mathbf{U}[n]$ is bicomplete and locally cartesian closed. The inclusion $\mathbf{U}[n] \to \mathbf{U}$ is reflective and its left adjoint is the map

$$\pi_{[n]} : \mathbf{U} \to \mathbf{U}[n]$$

which associates to a Kan complex its homotopy n-type. The map $\pi_{[n]}$ preserves finite products.

27.6. We shall say that a map $f : X \to Y$ in $\mathbf{S}/B$ is a *fibrewise homotopy n-equivalence* if the map $X(b) \to Y(b)$ induced by f between the homotopy fibers of X and Y is a weak homotopy n-equivalence for every vertex $b \in B$. The model category $(\mathbf{S}/B, Who)$ admits a Bousfield localisation with respect to the fibrewise homotopy n-equivalences. We shall denotes the local model structure shortly by $(\mathbf{S}/B, Who_B[n])$, where $Who_B[n]$ denotes the class of fibrewise homotopy n-equivalences in $\mathbf{S}/B$. Its fibrant objects are the Kan n-covers $X \to B$.

27.7. If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint functors

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction between the model category $(\mathbf{S}/A, Who_A[n])$ and the model category $(\mathbf{S}/B, Who_B[n])$. Moreover, it is a Quillen equivalence when u is a weak homotopy $(n+1)$-equivalence. This is true in particular when u is the canonical map $A \to \pi_{[n+1]}A$.

28. Truncated logoi

28.1. We shall say that a logos X is *1-truncated* if the canonical map $X \to \tau_1 X$ is a weak categorical equivalence. A logos X is 1-truncated iff the following equivalent conditions are satisfied:

- the simplicial set $X(a, b)$ is a 0-object for every pair $a, b \in X_0$.
- every simplicial sphere $\partial\Delta[m] \to X$ with $m > 2$ can be filled.

A Kan complex is 1-truncated iff it is a 1-object.

28.2. A category C is equivalent to a poset iff the set $C(a, b)$ has at most one element for every pair of objects $a, b \in C$. We say that a logos X is *0-truncated* if it is 1-truncated and the category $\tau_1 X$ is equivalent to a poset. A logos X is 0-truncated iff the following equivalent conditions are satisfied:

- the simplicial set $X(a, b)$ is empty or contractible for every pair $a, b \in X_0$;
- every simplicial sphere $\partial\Delta[m] \to X$ with $m > 1$ can be filled.

A Kan complex is 0-truncated iff it is a 0-object.

28.3. For any $n \geq 2$, we say that a logos X is *n-truncated* if the simplicial set $X(a, b)$ is a $(n-1)$-object for every pair $a, b \in X_0$. A logos X is n-truncated iff every simplicial sphere $\partial\Delta[m] \to X$ with $m > n+1$ can be filled. A Kan complex is n-truncated iff it is a n-object.

28.4. The logos $\mathbf{U}[n]$ is $(n+1)$ truncated for every $n \geq -1$.

28.5. We shall say that a map of simplicial sets $u : A \to B$ is a *weak categorical n-equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every n-truncated logos X. The model structure $(\mathbf{S}, Wcat)$ admits a Bousfield localisation with respect to the class $Wcat[n]$ of weak categorical n-equivalences. The fibrant objects are the n-truncated logoi. The localised model structure is cartesian closed and left proper. We shall denote it shortly by $(\mathbf{S}, Wcat[n])$.

28.6. If $n \geq 0$, then a map between logoi $f : X \to Y$ is a categorical n-equivalence iff it is essentially surjective and the map $X(a, b) \to Y(fa, fb)$ induced by f is a homotopy $(n-1)$-equivalence for every pair of objects $a, b \in X$. A map of simplicial sets $uj : A \to B$ is a weak categorical 1-equivalence iff the functor $\tau_1(u) : \tau_1 \to \tau_1 B$ is an equivalence of categories. A map of simplicial sets $u : A \to B$ is a weak categorical 0-equivalence iff it induces an isomorphism between the poset reflections of A and B.

28.7. The *categorical n-truncation* of a simplicial set A is defined to be a fibrant replacement of $A \to \tau_{[n]}(A)$ of A in the model category $(\mathbf{S}, Wcat[n])$. The fundamental category $\tau_1 A$ is a categorical 1-truncation of A. The poset reflection of A is a categorical 0-truncation of A.

28.8. If $n \geq 0$, we shall denote by $\mathbf{U}_1[n]$ the coherent nerve of the (simplicial) category of n-truncated logoi. It is the full simplicial subset of $\mathbf{U}_1$ spanned by the n-truncated logoi. We have an infinite sequence of logoi,

$$\mathbf{U}_1[0] \longrightarrow \mathbf{U}_1[1] \longrightarrow \mathbf{U}_1[2] \longrightarrow \mathbf{U}_1[3] \longrightarrow \cdots$$

The logos $\mathbf{U}_1[0]$ is equivalent to the category of posets and the logos $\mathbf{U}_1[1]$ to the coherent nerve of **Cat**. We have $\mathbf{U}[n] = \mathbf{U} \cap \mathbf{U}_1[n]$ for every $n \geq 0$. The inclusion $\mathbf{U}_1[n] \to \mathbf{U}_1$ is reflective and its left adjoint is the map

$$\tau_{[n]} : \mathbf{U}_1 \to \mathbf{U}_1[n]$$

which associates to a logos its categorical n-truncation. The map $\tau_{[n]}$ preserves finite products. The logos $\mathbf{U}_1[n]$ is is cartesian closed and $(n+1)$-truncated.

28.9. We right fibration $X \to B$ is said to be *n-truncated* if its fibers are n-objects. The model category $(\mathbf{S}/B, Wcont)$ admits a Bousfield localisation in which the fibrant objects are the right n-fibrations $X \to B$. The weak equivalences of the localised structure are called *contravariant n-equivalences*. The localised model structure is simplicial. We shall denotes it by $(\mathbf{S}/B, Wcont[n])$,

28.10. A map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant n-equivalence if the map

$$\pi_0[u, X] : \pi_0[M, X] \to \pi_0[N, X]$$

is bijective for every right n-fibration $X \to B$.

28.11. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a left anodyne map $1 \to Lb$ followed by a left fibration $Lb \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant n-equivalence iff the map $Lb \times_B u : Lb \times_B M \to Lb \times_B N$ is a homotopy n-equivalence for every vertex $b \in B$. When B is a logos, we can take $Lb = b\backslash B$. In this case a map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant n-equivalence iff the map $b\backslash u = b\backslash M \to b\backslash N$ is a homotopy n-equivalence for every object $b \in B$.

28.12. If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint functors

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction between the model category $(\mathbf{S}/A, Wcont[n])$ and the model category $(\mathbf{S}/B, Wcont[n])$. Moreover, it is a Quillen equivalence when u is a categorical $(n+1)$-equivalence. This is true in particular when u is the canonical map $A \to \tau_{[n+1]} A$.

28.13. Dually, we say that a map $u : M \to N$ in $\mathbf{S}/B$ is a *covariant n-equivalence* if the map $u^o : M^o \to N^o$ is a contravariant n-equivalence in $\mathbf{S}/B^o$. The model category $(\mathbf{S}/B, \mathit{Wcov})$ admits a Bousfield localisation with respect to the class of covariant n-equivalences for any $n \geq 0$. A fibrant object of this model category is a left n-fibration $X \to B$. The localised model structure is simplicial. We shall denote it by $(\mathbf{S}/B, \mathit{Wcov}[n])$,

29. Accessible logoi

29.1. A shall say that a (small) category C is *directed* if the colimit map

$$\varinjlim_{C} : \mathbf{Set}^C \to \mathbf{Set}$$

preserves finite limits. We shall say that C is *filtered* if C^o is directed.

29.2. If A is a logos, we say that a diagram $d : K \to A$ is *bounded above* if d admits an extension $K \star 1 \to A$. Dually, we say that the diagram is *bounded below* if d admits an extension $1 \star K \to A$.

29.3. A logos A is directed iff every finite diagram $K \to A$ is bounded above iff every simplicial sphere $\partial\Delta[n] \to A$ is bounded above iff the simplicial set $Ex(A)$ is a contractible Kan complex

29.4. If A is a directed logos, then so is the logos $d\backslash A$ for any finite diagram $d : K \to A$ and the map $d\backslash A \to A$ is final.

29.5. A logos A is directed iff it is non-empty and the simplicial set $d\backslash A$ is weakly contractible for any diagram $d : \Lambda^0[2] \to A$.

29.6. [Lu1] For every directed logos A there exists a directed category C together with a final map $C \to A$. Moreover, C can be chosen to be a poset [SGA].

29.7. We say that a diagram $d : A \to X$ in a logos X is *directed* if A is directed, in which case we shall say that the colimit of d is *directed* if it exists. We say that a logos X *has* directed colimits if every (small) directed diagram $A \to X$ has a colimit. We shall say that a map between logoi is *inductive* if it preserves directed colimits.

29.8. A logos with directed colimits is Karoubi complete. A cocartesian logos with directed colimits is cocomplete.

29.9. If A is a simplicial set, we shall say that a prestack $g \in \mathbf{P}(A)$ is *directed*, (or that it is an *ind-object*) if the simplicial set A/g (or $El(g)$) is directed. We shall denote by $\mathbf{Ind}(A)$ the full simplicial subset of $\mathbf{P}(A)$ spanned by the ind-objects. The Yoneda map $y_A : A \to \mathbf{P}(A)$ induces a map $A \to \mathbf{Ind}(A)$ (also denoted y_A). The map $y_A : A \to \mathbf{Ind}(A)$ exhibits the logos $\mathbf{Ind}(A)$ as the free completion of A under directed colimits. More precisely, if S and T are logoi with directed colimits, let us denote by $Ind(S,T)$ the full simplicial subset of T^S spanned by the inductive maps $S \to T$. Then the map

$$y_A^* : Ind(\mathbf{Ind}(A), X) \to X^A$$

induced by the map $y_A : A \to \mathbf{Ind}(A)$ is an equivalence for any logos with directed colimits X. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathbf{Ind}(A) \to X$ along the map $y_A : A \to \mathbf{Ind}(A)$.

29.10. If A is a simplicial set, then we have a decomposition

$$\mathbf{P}(A) \simeq \mathbf{Ind}(\mathbf{P}^f(A)).$$

29.11. We shall say that a logos X is *finitary accessible* if it is equivalent to a logos $\mathbf{Ind}(A)$ for a small logos A.

29.12. Let X be a (locally small) logos with directed colimits. We say that an object $a \in X$ is *compact* if the map

$$hom_X(a, -) : X \to \mathbf{U}$$

is inductive.

29.13. A finite colimit of compact objects is compact. A retract of a compact object is compact.

29.14. A pre-stack $g \in \mathbf{P}(A)$ is compact iff it is a retract of a finitely presented pre-stack. Not every compact object of $\mathbf{U} = \mathbf{P}(1)$ is finitely presented. A pre-stack $g \in \mathbf{Ind}(A)$ is compact iff it it is a retract of a representable prestack.

29.15. Let X be a logos with directed colimits and let $K \subseteq X$ be a small full sub logos of compact objects. Then the left Kan extension

$$i_! : \mathbf{Ind}(K) \to X$$

of the inclusion $i : K \subseteq X$ is fully faithful. Moreover, $i_!$ is an equivalence if every object of X is a colimit of a directed diagram of compact objects.

29.16. A locally small logos X is *finitary accessible* iff the following conditions are satisfied:

- X has directed colimits;
- every object of X is the colimit of a directed diagram of compact objects;
- the full sub-logos of compact objects of X is essentially small.

29.17. Let α be a regular cardinal. We shall say that a simplicial set A is *α-directed* if the colimit map

$$\varinjlim_A : \mathbf{U}^A \to \mathbf{U}$$

is α-continuous.

29.18. The notion of α-directed simplicial set is invariant under Morita equivalence. A simplicial set A is α-directed iff the canonical map $A \to \mathbf{P}_\alpha(A)$ is final. A logos A is α-directed iff every diagram $K \to A$ of cardinality $< \alpha$ is bounded above. A simplicial set with a terminal vertex is α-directed. An α-cocomplete logos is α-directed.

29.19. Let α be a regular cardinal. We shall say that a diagram $d : A \to X$ is *α-directed* if A is α-directed, in which case we shall say that the colimit of d is *α-directed* when it exists. We shall say that a logos X *has* α-directed colimits if every (small) α-directed diagram $A \to X$ has a colimit. We shall say that a map between logoi is *α-inductive* if it preserves α-directed colimits.

29.20. If an α-cocomplete logos has α-directed colimits, then it is cocomplete.

29.21. If A is a simplicial set, we shall say that a prestack $g \in \mathbf{P}(A)$ is *α-directed*, if the simplicial set A/g (or $El(g)$) is α-directed. We shall denote by $\mathbf{Ind}^\alpha(A)$ the full simplicial subset of $\mathbf{P}(A)$ spanned by the α-directed prestacks. The Yoneda map $y_A : A \to \mathbf{P}(A)$ induces a map $A \to \mathbf{Ind}^\alpha(A)$ (also denoted y_A). The map $y_A : A \to \mathbf{Ind}^\alpha(A)$ exhibits the logos $\mathbf{Ind}^\alpha(A)$ as the free completion of A under α-directed colimits. More precisely, if S and T are logoi with directed colimits, let us denote by $Ind^\alpha(S, T)$ the full simplicial subset of T^S spanned by the inductive maps $S \to T$. Then the map

$$y_A^* : Ind(\mathbf{Ind}^\alpha(A), X) \to X^A$$

induced by the map $y_A : A \to \mathbf{Ind}^\alpha(A)$ is an equivalence for any logos with α-directed colimits X. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathbf{Ind}^\alpha(A) \to X$ along the map $y_A : A \to \mathbf{Ind}^\alpha(A)$.

29.22. If A is a simplicial set, then we have a decomposition

$$\mathbf{P}(A) \simeq \mathbf{Ind}^\alpha(\mathbf{P}_\alpha(A)).$$

29.23. We shall say that a logos X is *α-accessible* if it is equivalent to a logos $\mathbf{Ind}^\alpha(A)$ for a small logos A. We shall say that X is *accessible* if it is equivalent to a logos $\mathbf{Ind}^\alpha(A)$ for a small logos A and some regular ordinal α.

29.24. Let X be a (locally small) logos with α-directed colimits. We say that an object $a \in X$ is *α-compact* if the map

$$hom_X(a, -) : X \to \mathbf{U}$$

is α-inductive.

29.25. An α-colimit of α-compact objects is compact. A pre-stack $g \in \mathbf{Ind}^\alpha(A)$ is α-compact iff it it is a retract of a representable prestack.

29.26. Let X be a logos with α-directed colimits and let $K \subseteq X$ be a small full sub logos of α-compact objects. Then the left Kan extension

$$i_! : \mathbf{Ind}^\alpha(K) \to X$$

of the inclusion $i : K \subseteq X$ is fully faithful. Moreover, $i_!$ is an equivalence if every object of X is a colimit of an α-directed diagram of α-compact objects.

29.27. A locally small logos X is *α-accessible* iff the following conditions are satisfied:

- X has α-directed colimits;
- every object of X is the colimit of an α-directed diagram of α-compact objects;
- the full sub-logos of α-compact objects of X is essentially small.

30. Limit sketches

The notion of limit sketch was introduced by Ehresman. A structure defined by a limit sketch is said to be essentially algebraic.

30.1. Recall that a *projective cone* in a simplicial set A is a map of simplicial sets $c : 1 \star K \to A$; A *limit sketch* is a pair (A, P), where A is a simplicial set and P is a set of projective cones in A. A *model* of (A, P) with values in a logos X is a map $f : A \to X$ which takes every cone $c : 1 \star K \to A$ in P to an exact cone $fc : 1 \star K \to X$. In other words, a model is a diagram $A \to X$ which satisfies the exactness conditions specified by the cones in P. A model of a limit sketch is said to be *essedntially algebraic*. We shall write $f : A/P \to X$ to indicate that a map $f : A \to X$ is a model of (A, P). A model $A/P \to \mathbf{U}$ is called a *homotopy model*, or just a *model* if the context is clear. The models of (A, P) with values in X form a logos $Mod(A/P, X)$; it is the full simplicial subset of X^A spanned by the models $A/P \to X$. We shall write

$$Mod(A/P) = Mod(A/P, \mathbf{U}).$$

The logos $Mod(A/P)$ is bicomplete and the inclusion $Mod(A/P) \subseteq \mathbf{U}^A$ has a left adjoint.

30.2. Recall from 21.42 that the *cardinality* of a diagram $d : K \to A$ in a simplicial set A is defined to be the cardinality of K. We say that a limit sketch (A, P) is *finitary* if every cone in P is finite. We say that an essentially algebraic structure is *finitary* if it can be defined by a finitary limit sketch.

30.3. The notion of *stack* on a topological space is essentially algebraic, but it is not finitary in general.

30.4. Recall that a logos is said to be *cartesian* if it has finite limits. A *cartesian theory* is defined to be a small cartesian logos. A *model* of a cartesian theory T with values in a logos X is a map $f : T \to X$ which preserves finite limits (such a map is said to be left exact). The models of $T \to X$ form a logos $Mod(T, X)$, also denoted $T(X)$. By definition, it is the full simplicial subset of X^T spanned by the models $T \to X$. We shall say that a model $T \to \mathbf{U}$ is a *homotopy model*, or just a *model* if the context is clear. We shall write

$$Mod(T) = Mod(T, \mathbf{U}).$$

We shall say that a model $T \to \mathbf{Set}$ is *discrete*. If S and T are cartesian theories, we shall say that a model $S \to T$ is a *morphism* $S \to T$, or that it is an *interpretation* of S into T. The identity morphism $T \to T$ is the *generic* or *tautological* model of T.

30.5. The logos of models of a cartesian theory T is bicomplete and the inclusion $Mod(T) \subseteq \mathbf{U}^T$ has a left adjoint. If $u : S \to T$ is a morphism of cartesian theories, then the map

$$u^* : Mod(T) \to Mod(S)$$

induced by the map $\mathbf{U}^u : \mathbf{U}^T \to \mathbf{U}^S$ has a left adjoint $u_!$. The pair $(u_!, u^*)$ an equivalence of logoi iff the map $u : S \to T$ is a Morita equivalence.

30.6. It T is a cartesian theory, then the map $hom(a, -) : T \to \mathbf{U}$ is a model for every objects $a \in T$. Hence the Yoneda map $T^o \to \mathbf{U}^T$ induces a map

$$y : T^o \to Mod(T).$$

We shall say that a model $f \in Mod(T)$ is *finitely presentable* if it belongs the the image of y. A model is compact iff it is a retract of a finitely presentable model. The map $y : T^o \to Mod(T)$ preserves finite colimits and it induces an equivalence between T^o and the full sub-logos $Mod(T)^f$ spanned by the finitely presentable models of T. The left Kan extension of the inclusion $Mod^f \subseteq Mod(T)$ along the inclusion $Mod^f \subseteq \mathbf{Ind}(Mod^f)$ is an equivalence of logoi,

$$\mathbf{Ind}(Mod^f) \simeq Mod(T).$$

30.7. If (A, P) is a finitary limit sketch, we shall say that a model $u : A/P \to T$ with values in a cartesian theory T is *universal* if the map

$$u^* : Mod(T, X) \to Mod(A/P, X)$$

induced by u is an equivalence for any cartesian logos X. We shall say that a universal model $u : A/P \to T$ is a *presentation* of the theory T by the sketch (A, P). Every finitary limit sketch (A, P) has a universal model $u : A \to U(A/P)$ with values in a cartesian theory $U(A/P)$. The opposite logos $U(A/P)^o$ is equivalent to the logos of finitely presentable models of (A, P). More precisely. if we compose the Yoneda map $A^o \to \mathbf{U}^A$ with the left adjoint to the inclusion $Mod(A/P) \subseteq \mathbf{U}^A$ we obtain a map

$$y : A^o \to Mod(A/P)$$

which takes a cone in P^o to a coexact cone. We say that a model of (A, P) is *finitely presentable* if it is the colimit of a finite diagram of objects in the image of y. Let us denote by $Mod(A/P)^f$ the full simplicial subset of $Mod(A/P)$ spanned by the finitely presentable models. Then the map $u : A \to U(A/P)$ is the opposite of the map $A^o \to Mod(A/P)^f$ induced by the map $y : A^o \to Mod(A/P)$.

30.8. We shall denote by OB the cartesian theory of a naked (unstructured) object. The opposite logos OB^o is equivalent to the logos $\mathbf{U}^f$ of finite homotopy types.

30.9. A *stable object* or a *spectrum* in a cartesian logos X is defined to be an infinite sequence of pointed objects (x_n) together with an infinite sequence of isomorphisms

$$u_n : x_n \to \Omega(x_{n+1}).$$

Hence the notion of spectrum is defined by a finitary limit sketch (A, P). We shall denote by *Spec* the cartesian theory $U(A/P)$. Fot any cartesian logos X, the logos $Spec(X)$ is the (homotopy) projective limit of the infinite sequence of logoi

$$1\backslash X \xleftarrow{\Omega} 1\backslash X \xleftarrow{\Omega} 1\backslash X \xleftarrow{\Omega} \cdots .$$

The logos $Spec(\mathbf{U})$ is equivalent to the coherent nerve of the (simplicial) category of spectra.

30.10. Let OB' be the cartesian theory of pointed objects and *Spec* be the cartesian theory of spectra defined in 30.9. Consider the interpretation $u : OB' \to Spec$ defined by the pointed object x_0 of the generic spectrum (x_n). The adjoint pair $u_! : Mod^\times(OB') \leftrightarrow Mod(Spec) : u^*$ is the classical adjoint pair

$$\Sigma^\infty : 1\backslash\mathbf{U} \leftrightarrow \mathbf{Spec} : \Omega^\infty,$$

30.11. We denote by **CT** the category of cartesian logoi and morphisms. The category **CT** has the structure of a 2-category induced by the 2-category structure on the category of simplicial sets. If S and T are two cartesian theories then so is the logos $Mod(S,T)$ of morphisms $S \to T$. The 2-category **CT** is symmetric monoidal closed. The *tensor product* $S \odot T$ of two cartesian theories is the target of a map $S \times T \to S \odot T$ left exact in each variable and universal with respect to that property. More precisely, for any cartesian logos X, let us denote by $Mod(S,T;Z)$ the full simplicial subset of $X^{S\times T}$ spanned by the maps $S \times T \to X$ left exact in each variable. Then the map

$$\phi^* : Mod(S \odot T, X) \to Mod(S,T;X)$$

induced by ϕ is an equivalence of logoi. It follows that we have two canonical equivalences of logoi

$$Mod(S \odot T, Z) \simeq Mod(S, Mod(T,X)) \simeq Mod(T, Mod(S,X)).$$

In particular, we have two canonical equivalences

$$Mod(S \odot T) \simeq Mod(S, Mod(T)) \simeq Mod(T, Mod(S)).$$

The unit for the tensor product is the cartesian theoryOB.

30.12. The notion of monomorphism between two objects of a logos is essentially algebraic (and finitary): an arrow $a \to b$ is monic iff the square

$$\begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] & a \arrow[d, "u"] \\ a \arrow[r, "u"] & b \end{tikzcd}$$

is cartesian. The notion of (homotopy) discrete object is essentially algebraic: an object a is discrete iff the diagonal $a \to a \times a$ is monic. This condition is expressed by two exact cones,

$$\begin{tikzcd} & b \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ a & & a, \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] & a \arrow[d, "d"] \\ a \arrow[r, "d"] & b. \end{tikzcd}$$

and two relations $pd = qd = 1_a$. The notion of 0-cover is also essentially algebraic, since an arrow $a \to b$ is a 0-cover iff the diagonal $a \to a \times_b a$ is monic. It follows that the notion of 1-object is essentially algebraic, since an object a is a 1-object iff its diagonal $a \to a \times a$ is a 0-cover. It is easy to see by induction on n that the notions of n-object and of n-cover are essentially algebraic for every $n \geq 0$. We shall denote the cartesian theory of n-objects by $OB[n]$. We have

$$Mod(OB[n])) = \mathbf{U}[n],$$

where $\mathbf{U}[n]$ is the logos of n-objects in $\mathbf{U}$. Let us say that an object of $\mathbf{U}[n]$ is n-*finite* if it is the n-type of a finite homotopy type. It then follows from 30.7 that the logos $OB[n]^o$ is equivalent to the logos $\mathbf{U}[n]^f$ of n-finite n-objects. In particular, the logos $OB[1]^o$ is equivalent to the logos of finitely presentable groupoids, and the logos $OB[0]^o$ to the category of finite sets.

30.13. If X is a cartesian category, a *simplicial object* $C : \Delta^o \to X$ is called a *category* if it satisfies the *Segal condition.* The condition can be expressed in many equivalent ways, for example by demanding that C takes every pushout square of the form

$$\begin{tikzcd} {[0]} \arrow[r, "0"] \arrow[d, "m"'] & {[n]} \arrow[d] \\ {[m]} \arrow[r] & {[m+n]}, \end{tikzcd}$$

to a pullback square in X. The notion of category object is essentially algebraic and finitary. If $C : \Delta^o \to X$ is a category object, we shall say that $C_0 \in X$ is the *object of objects* of C and that C_1 is the *object of arrows.* A category C is a monoid iff $C_0 = 1$. The *source morphism* $s : C_1 \to C_0$ is the image of the arrow $d_1 : [0] \to [1]$, the *target morphism* $t : C_1 \to C_0$ is the image of $d_0 : [0] \to [1]$, and the *unit morphism* $u : C_0 \to C_1$ is the image of $s_0 : [1] \to [0]$. The *multiplication* $C_2 \to C_1$ is image of $d_1 : [1] \to [2]$. If Q is the set of pushout squares involved in the Segal condition, then the pair (Δ^o, Q^o) is a finitary limit sketch. The sketch has a universal model $u : \Delta^o \to Cat$, and this defines the cartesian theory of categories Cat. If X is a cartesian logos, then we have an equivalence of logoi,

$$Cat(X) = Mod(\Delta^o/Q^o, X).$$

30.14. If Cat denotes the cartesian theory of categories then $Cat^2 = Cat \odot Cat$ is the theory of double categories. If X is a cartesian logos, then an object of

$$Cat^2(X) = Cat(Cat(X))$$

is a *double category* in X. By definition, a double simplicial object $\Delta^o \times \Delta^o \to X$ is a double category iff it is a category object in each variable. We shall denote by $Cat^n(X)$ the logos of n-fold categories in X and by Cat^n the cartesian theory of n-fold categories.

30.15. A category C is a monoid iff we have $C_0 = 1$. We shall denote by $CMon$ the cartesian theory of monoids. It follows from the conjecture in 31.15 that $CMon^{\odot n}$ is the cartesian theory of E_n-monoids for every $n \geq 1$. It follows that $CMon^{\odot n} \odot Cat$ is the cartesian theory of E_n-monoidal categories for every $n \geq 1$. In particular, $CMon \odot Cat$ is the cartesian theory of monoidal categories, and $CMon^{\odot 2} \odot Cat$ is the cartesian theory of braided monoidal categories.

30.16. The notion of groupoid is essentially algebraic and finitary. By definition, a category object $C : \Delta^o \to X$ is a *groupoid* if it takes the squares

$$\begin{tikzcd} {[0]} \arrow[r, "d_0"] \arrow[d, "d_0"'] & {[1]} \arrow[d, "d_0"] \\ {[1]} \arrow[r, "d_1"] & {[2]}, \end{tikzcd} \qquad \begin{tikzcd} {[0]} \arrow[r, "d_1"] \arrow[d, "d_1"'] & {[1]} \arrow[d, "d_2"] \\ {[1]} \arrow[r, "d_1"] & {[2]} \end{tikzcd}$$

to pullback squares,

$$\begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\partial_0"'] & C_1 \arrow[d, "t"] \\ C_1 \arrow[r, "t"] & C_0, \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\partial_2"'] & C_1 \arrow[d, "s"] \\ C_1 \arrow[r, "s"] & C_0. \end{tikzcd}$$

(one is enough). We shall denotes the cartesian theory of groupoids by *Gpd* and the logos of groupoid objects in a cartesian logos X by $Gpd(X)$. A groupoid C is a group iff $C_0 = 1$.

30.17. Let *Cat* be the cartesian theory of categories and *Gpd* be the cartesian theory of groupoids. If X is a cartesian logos, then the inclusion $Gpd(X) \subseteq Cat(X)$ has a right adjoint which associates to a category $C \in Cat(X)$ its groupoid of isomorphisms $J(C)$. We have $J(C) = q^*(C)$, where $q : Gpd \to Cat$ is the groupoid of isomorphisms of the generic category. We say that a category C satisfies the *Rezk condition*, or that it is *reduced*, if the map $J(C)_1 \to J(C)_0 = C_0$ is invertible. The notion of a reduced category is essentially algebraic. We shall denote the cartesian theory of reduced categories by *RCat* and the logos of reduced category objects in a cartesian logos X by $RCat(X)$.

30.18. Let $N : \mathbf{U}_1 \to [\Delta^o, \mathbf{U}]$ be the nerve map defined in 25.5. It follows from [JT2] that N is fully faithful and that it induces an equivalence of logoi

$$N : \mathbf{U}_1 \simeq Mod(RCat).$$

30.19. We say that a category object C in a cartesian logos X is *n-truncated* if the map $C_1 \to C_0 \times C_0$ is a $(n-1)$-cover. The notion of n-truncated category is essentially algebraic and finitary. We denotes the cartesian theory of n-truncated categories by $Cat[n]$. If a category object $C \in Cat(X)$ is n-truncated and reduced, then C_k is a n-object for every $k \geq 0$. The notion of n-truncated reduced category is essentially algebraic. We denotes the cartesian theory of n-truncated reduced categories by $RCat[n]$.

30.20. If X is a cartesian logos, then the map $Ob : Gpd(X) \to X$ has a left adjoint $Sk^0 : X \to Gpd(X)$ and a right adjoint $Cosk^0 : X \to Gpd(X)$. The left adjoint associate to an object $b \in X$ the constant simplicial object $Sk^0(b) : \Delta^o \to X$ with value b. The right adjoint associates to b the simplicial object $Cosk^0(b)$ obtained by putting $Cosk^0(b)_n = b^{[n]}$ for each $n \geq 0$. We say that $Cosk^0(b)$ is the *coarse groupoid* of b. More generally, the *equivalence groupoid* $Eq(f)$ of an arrow $f : a \to b$ in X is defined to be the coarse groupoid of the object $f \in X/b$ (or rather its image by the canonical map $X/b \to X$). The *loop group* $\Omega(b)$ of a pointed object $1 \to b$ is the equivalence groupoid of the arrow $1 \to b$. Consider the interpretation $u : Grp \to OB'$ defined by the loop group of the generic pointed object. The map

$$u_! : Mod(Grp) \to Mod(OB')$$

takes a group object G to its (pointed) classifying space BG. It induces an equivalence between $Mod(Grp)$ and the full sub-logos of pointed connected spaces $1\backslash\mathbf{U}$ by a classical result. It is thus fully faithful. Hence the morphism $Grp \to OB'$ is fully faithful. More generally, let *Gpd* be the cartesian theory of groupoids and let

Map be the cartesian theory of maps. Consider the interpretation $v : Gpd \to Map$ defined by the equivalence groupoid of the generic map. Then the map

$$v_! : Mod(Gpd) \to Mod(Map)$$

takes a groupoid C to its classifying space BC equipped with the map $C_0 \to BC$. It induces an equivalence between $Mod(Gpd)$ and the full sub-logos of $\mathbf{U}^I$ spanned by the surjections by a classical result. It is thus fully faithful. Hence the morphism $v : Gpd \to Map$ is fully faithful.

30.21. We shall say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathbf{CT}$ is a *homotopy factorisation system* if the following conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are invariant under categorical equivalences;
- the pair $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak factorisation system in $\mathbf{CT}$, where $\mathcal{C}$ is the class of monomorphism (in $\mathbf{S}$) and $\mathcal{F}$ is the class of pseudo-fibrations;
- the class $\mathcal{A}$ has the right cancellation property;
- the class $\mathcal{B}$ has the left cancellation property.

The last two conditions are equivalent in the presence of the others. The class $\mathcal{A}$ is the *left class* of the system and $\mathcal{B}$ the *right class.*

30.22. The category $\mathbf{CT}$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective morphisms and $\mathcal{B}$ the class of fully faithful morphisms. A morphism $u : S \to T$ is fully faithful, iff the map $u_! : Mod(T) \to Mod(S)$ is fully faithful. If $u : S \to T$ is essentially surjective, then $u^* : Mod(T) \to Mod(S)$ is conservative.

30.23. The category $\mathbf{CT}$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative morphisms. A morphism in the class ($\mathcal{A}$ is said to be an *iterated cartesian localisation.* Let us define the notion of cartesian localisation. We say that a set Σ of arrows in a cartesian logos X is *closed under base change* if the implication $f \in \Sigma \Rightarrow f' \in \Sigma$ is true for any pullback square

$$\begin{tikzcd} x' \arrow[r] \arrow[d, "f'"'] & x \arrow[d, "f"] \\ y' \arrow[r] & y \end{tikzcd}$$

in X. If Σ is closed under base change, then the logos $L(X, S)$ has finite limits and and they are preserved by the canonical map $c : X \to L(X, S)$. We then say that c is a *cartesian localisation.* If a morphism of cartesian theories $u : S \to T$ belongs to ($\mathcal{A}$, then the map $u^* : Mod(T) \to Mod(S)$ is fully faithful.

30.24. It T is a cartesian theory, we shall say that a left fibration $X \to T$ is a *comodel* of T if its classifying map $T \to \mathbf{U}$ is a model. A left fibration $X \to T$ is a comodel iff the logos X has finite limits and the map $X \to T$ is left exact. The simplicial set of elements of a model $f : T \to \mathbf{U}$ is a comodel $el(f) \to T$ and conversely. The notions of model $T \to \mathbf{U}$ and of comodel $X \to T$ are essentially equivalent. The left fibration $a\backslash T \to T$ is a comodel for every object $a \in T$. The coherent nerve of the simplicial category $Comod(T)$ of comodels of T is equivalent to $Mod(T)$. If $p : X \to T$ is a comodel, then a left fibration $g : Y \to X$ is a comodel of X iff the composite $pg : Y \to T$ is a comodel of T. We thus have an equivalence of simplicial categories

$$Comod(X) \simeq Comod(T)/X.$$

If $X = el(f)$, the equivalence induces an equivalence of logoi

$$Mod(el(f)) \simeq Mod(T)/f.$$

We shall say that a morphism of algebraic theories $u : S \to T$ is *coinitial* if the map

$$u_! : Mod(S) \to Mod(T)$$

takes a terminal model to a terminal model. The category **CT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of coinitial morphisms. A morphism $u : S \to T$ belongs to $\mathcal{B}$ iff its admits a factorisation $u = u'w : S \to S' \to T$ with w a categorical equivalence and u' a comodel. See [**?**].

30.25. If T is a cartesian theory, then the model category $(\mathbf{S}/T, Wcov)$ admits a Bousfield localisation in which the (fibrant) local objects are the comodels of T. A map $u : A \to B$ in $\mathbf{S}/T$ is a weak equivalence for the local model structure iff the map

$$\pi_0[u, X] : \pi_0[B, X] \to \pi_0[A, X]$$

is bijective for every comodel $X \to T$.

30.26. If T is a cartesian theory, then so is the logos T/a for any object $a \in T$. The base change map $x \mapsto a \times x$ is a morphism of theories $u : T \to T/a$. The extension $u : T \to T/a$ is obtained by freely adjoining an arrow $1 \to a$ to the theory T.

30.27. Every cartesian theory T has an initial model

$$0_T = Hom(1, -) : T \to \mathbf{U}.$$

We shall say that a morphism of cartesian theories $u : S \to T$ is *tight* if the map

$$u^* : Mod(T) \to Mod(S)$$

takes an initial model to an initial model. A morphism $u : S \to T$ is tight iff the commutative square

$$\begin{tikzcd} 1\backslash S \arrow[r, "1\backslash u"] \arrow[d] & 1\backslash T \arrow[d] \\ S \arrow[r, "u"] & T \end{tikzcd}$$

is a homotopy pullback (in the model category $(\mathbf{S}, Wcat)$). The category **CT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of tight morphisms. We shall say that a morphism in the class $\mathcal{A}$ is *a Henkin extension*. The base change map $a \times - : T \to T/a$ is a Henkin extension for every object $a \in T$.

30.28. If T is a cartesian theory, then every model $f : T \to \mathbf{U}$ admits a factorisation $f = f'e : T \to T[f] \to \mathbf{U}$ with $e : T \to T[f]$ a Henkin extension and $f' : T[f] \to \mathbf{U}$ an initial model, and the factorisation is homotopy unique. Conversely, to every Henkin extension $u : T \to E$ we can associates a model $u^*(0_E) : T \to \mathbf{U}$. This defines a "one-to-one" correspondance between the Henkin extensions $T \to E$ and the models $T \to \mathbf{U}$. It follows that there is an equivalence of logoi

$$f\backslash Mod(T) \simeq Mod(T[f])$$

for any model $f \in Mod(T)$.

30.29. If $u : T \to E$ is a Henkin extension, let us denote by $\Gamma(u)$ the comodel of T defined by the pullback square

$$\begin{tikzcd} \Gamma(u) \arrow[r] \arrow[d] & 1\backslash E \arrow[d] \\ T \arrow[r, "u"] & E \end{tikzcd}$$

This defines a functor Γ which induces a DK-equivalence between the simplicial category of cofibrant Henkin extension of T and the simplicial category of comodels of T (an extension $u : T \to E$ is cofibrant if u is monic).

30.30. We shall say that a small logos T with pullbacks is a *pullback theory*. A *pullback model* of T with values in a logos X is a map $f : T \to X$ which preserves pullbacks. We shall say that a limit sketch (A, C) is a *pullback sketch* if all the projective cones in C are squares $I \times I \to A$. Every pullback sketch (A, C) has a universal model $u : A \to PT(A/C)$ with values in pullback theory $PT(A/C)$.

30.31. For example, the notions of category and of groupoids are defined by a pullback sketch. Similarly for the notion of monomorphism and more generally for the notion of n-cover for every $n \geq 0$. The notion of a reduced category is also defined by a pullback sketch. The notion of a pointed object cannot be defined by a pullback sketch.

30.32. If T is a cartesian theory and b is an object of a cartesian logos X, then a model $T \to X/b$ is called a *parametrized model*, and b is said to be the *parameter space* or the *base* of the model. The composite $T \to X/b \to X$ is a pullback model of T. Conversely, every pullback model $f : T \to X$ admits a factorisation $T \to X/b \to X$, where $T \to X/b$ is a model of T. This defines an equivalence between the pullback models of T and the parametrized models of T. More precisely, let $\mathbf{PT}$ be the category of pullback theories and morphisms the maps preserving pullbacks. Then the forgetful functor $U : \mathbf{CT} \to \mathbf{PT}$ has a left adjoint which associates to a pullback theory V a cartesian theory CV equipped with a pullback preserving map $u : V \to CV$ which is universal with respect to that property. If $T \in \mathbf{CT}$, let us put $PT = CU(T)$. There is then an equivalence between the parametrized model of T, the pullback models of T and the models of PT. Thus, PT is the *cartesian theory of parametrized models of* T. A model of PT in $\mathbf{U}$ is a model

$$T \to \mathbf{U}/K.$$

where K is a Kan complex. Equivalently, a model of PT is a map $K \to Mod(T)$; it is thus a *Kan diagram* of models of T.

30.33. For any sketch (A, P), we can construct another sketch (A', P') whose models are the parametrized models of (A, P). Here is the construction If $u : 1 \star B \to A$ is a projective cone, then $u \star 1 : 1 \star B \star 1 \to A \star 1$ is a projective cone in $A \star 1$. By construction $(A', P') = (A \star 1, P \star 1)$, where $P \star 1 = \{u \star 1 : u \in P\}$.

30.34. If X is a logos and K is a Kan complex, we say that a map $K \to X$ is a *Kan diagram* in X. There is then a logos $D(Kan, X)$ of such diagrams. A *morphism* between two Kan diagrams $d : K \to X$ and $d' : K' \to X$ is a map $u : K \to K'$ together with a homotopy $d \to d'u$. Let us give a global description of the logos $D(Kan, X)$ in the case where X is a small logos. To a Kan diagram $d : K \to X$ we can associate the colimit $L(d)$ of the composite $y_X d : K \to \mathbf{P}(X)$. Then $D(Kan, X)$ is defined to be the full simplicial subset of $\mathbf{P}(X)$ spanned by the objects $L(d)$. The Yoneda map $y : X \to \mathbf{P}(A)$ induces a map $X \to D(Kan, X)$, also denoted y. The logos $D(Kan, X)$ is closed under colimit of Kan diagrams $K \to D(Kan, X)$. The map $y : X \to D(Kan, X)$ exhibits the logos $D(Kan, X)$ as the free cocompletion of X under colimits of Kan diagrams. The colimit map $D(Kan, X) \to \mathbf{U}$ associates to $L(d)$ the domain of the map $d : K \to X$. It is a Grothendieck fibration, and a bifibration when X is has colimits of Kan diagrams.

30.35. If PT is the theory of parametrized models of a cartesian theory T, then there is a pullback preserving map $u : T \to PT$ which is universal. The object $u(1) \in PT$ is the *generic base*. The map u admits a factorisation $u = pv : T \to PT/u(1) \to PT$, where $v : T \to PT/u(1)$ preserves finite limits and p is the projection $PT/u(1) \to PT$. The base $b(f)$ of a model $f \in Mod(PT)$ is obtained by evaluating the map $f : PT \to \mathbf{U}$ at $u(1)$. If we compose the map $v : T \to PT/u(1)$ with the map $PT/u(1) \to \mathbf{U}/b(f)$ induced by f, we obtain a parametrized model of T,

$$f' : T \to \mathbf{U}/b(f).$$

This defines the equivalence of logoi,

$$Mod(PT) \to D(Kan, Mod(T)).$$

30.36. If OB' is the theory of pointed objects, then POB' is the theory of *split morphisms*. The notion of split morphism is described by a sketch $(A, \emptyset)$ where A is the simplicial set $\Delta[2]/\partial_1\Delta[2]$ of 35.6.

30.37. If *Spec* is the theory of spectra described in 30.9, then *PSpec* is the *theory of parametrized spectra*. More precisely, a model of *PSpec* is a stable object of the logos $\mathbf{U}/K$ for some Kan complex $K \in \mathbf{U}$. There is then an equivalence of logoi

$$Mod(PSpec) = D(Kan, \mathbf{Spec}).$$

30.38. If α is a regular cardinal, we say that a limit sketch (A, P) is *α-bounded* if every cone in P has a cardinality $< \alpha$. We say that an essentially algebraic structure is *α-definable* if it can be defined by an α-bounded limit sketch.

30.39. Let α is a regular cardinal. An *α-cartesian theory* is a small α-complete logos T. A *model* of T with values in a logos X is an α-continuous map $f : T \to X$. The models of T with values in X form a logos $Mod^\alpha(T, X)$. By definition, it is the full simplicial subset of X^T spanned by the models $T \to X$. We shall put

$$Mod^\alpha(T) = Mod^\alpha(T, \mathbf{U}).$$

The inclusion $Mod^{\alpha}(T) \subseteq \mathbf{U}^T$ has a left adjoint and the logos $Mod^{\alpha}(T)$ is bicomplete.

30.40. Let α is a regular cardinal. Every α-bounded limit sketch (A, P) has a universal model $u : A \to C_{\alpha}(A/P)$ with values in an α-complete logos $C_{\alpha}(A/P)$. The logos $C_{\alpha}(A/P)$ is small and the universality means that the map

$$u^* : Mod^{\alpha}(C_{\alpha}(A/P), X) \to Mod(A/P, X)$$

induced by u is an equivalence for any α-complete logos X. Let us describe a construction of $C_{\alpha}(A/P)$. If we compose the Yoneda map $A^o \to [A, \mathbf{U}]$ with the left adjoint to the inclusion $Mod(A/P) \subseteq [A, \mathbf{U}]$ we obtain a map

$$y : A^o \to Mod(A/P)$$

which takes a cone in P^o to a coexact cone. We say that a model of (A, P) is *α-presented* if it is a colimit of a diagram $yd : K \to Mod(A/P)$, where $d : K \to A^o$ is a diagram of cardinality $< \alpha$. We denote by $Mod(A/P)(\alpha)$ the full simplicial subset of $Mod(A/P)$ spanned by the α-presented models. The map $A \to Mod(A/P)(\alpha)^o$ induced by the map $y^o : A \to Mod(A/P)^o$ is the universal model $u : A \to C_{\alpha}(A/P)$. If T is an α-cartesian theory, then the Yoneda map $T^o \to Mod^{\alpha}(T)$ induces an equivalence of logoi $T^o \to Mod^{\alpha}(T)(\alpha)$.

31. Universal algebra

An algebraic structure can be defined to be a product preserving Universal algebra studies In this section we extend universal algebra from structured objects in a category to structured objects in a logos.

31.1. Recall that an algebraic theory in the sense of Lawvere is a small category with finite products T [La]. More generally, we shall say that a small logos with finite products T is an *algebraic theory.* A *model* of T with values in a logos X, is a map $f : T \to X$ which preserves finite products. The models $T \to X$ form a logos $Mod(T, X)$, also denoted $T(X)$. By definition, it is the full simplicial subset of X^T spanned by the models $T \to X$. We shall say that a model $T \to \mathbf{U}$ is a *T-algebra* and we shall put

$$Mod(T) = Mod(T, \mathbf{U}).$$

We shall say that a model $T \to \mathbf{Set}$ is *discrete.* If S and T are algebraic theories, we shall say that a model $S \to T$ is a *morphism* $S \to T$ or that it is an *interpretation* of S into T. The identity morphism $T \to T$ is the *generic* or *tautological* model of T.

31.2. If T is an algebraic theory, then so is the category hoT. We shall say that an algebraic theory T is *discrete* if the map $T \to hoT$ is an equivalence of logoi. An algebraic theory T is discrete iff the logos T is 1-truncated. We shall say that a model $T \to \mathbf{U}$ is *discrete* if it takes its values in the 0-objects of $\mathbf{U}$. The logos of discrete models $Mod(T, \mathbf{U}[0])$ is equivalent to the category $Mod(hoT, \mathbf{Set})$.

31.3. The logos of models of an algebraic theory T is bicomplete and the inclusion $Mod(T) \subseteq \mathbf{U}^T$ has a left adjoint. If $u : S \to T$ is a morphism of algebraic theories, then the map

$$u^* : Mod(T) \to Mod(S)$$

induced by the map $\mathbf{U}^u : \mathbf{U}^T \to \mathbf{U}^S$ has a left adjoint $u_!$. Moreover, the pair $(u_!, u^*)$ an equivalence of logoi iff the map $u : S \to T$ is a Morita equivalence.

31.4. It T is a cartesian theory, then the map $hom(a, -) : T \to \mathbf{U}$ is a model for every objects $a \in T$. Hence the Yoneda map $T^o \to \mathbf{U}^T$ induces a map

$$y : T^o \to Mod(T).$$

We shall say that a model $f \in Mod(T)$ is *finitely free* if it belongs the the image of y. The map $y : T^o \to Mod(T)$ preserves finite coproducts and it induces an equivalence between T^o and the full sub-logos $Mod(T)^{ff}$ spanned by the finitely free models of T. We say that a model of T is *finitely presented* if it is a finite colimit of finitely free models.

31.5. We call a projective cone $1 \star K \to A$ in a simplicial set A a *product cone* if the simplicial set K is discrete. We call a limit sketch (A, P) a *product sketch* if every cone in P is a product cone. A model $A/P \to X$ of a product sketch with values in a quasi-category X is called an *algebra*. If $X = \mathbf{U}$, it is called a *homotopy algebra*, and if $X = \mathbf{Set}$, it is called a *discrete algebra*. The models of (A, P) with values in X form a quasi-category $Mod(A/P, X)$. By definition, it is the full simplicial subset of $[A, X]$ spanned by the models $A/P \to X$. We shall put

$$Mod(A/P) = Mod(A/P, \mathbf{U}).$$

The inclusion $Mod(A/P) \subseteq \mathbf{U}^A$ has a left adjoint $r : Ho^A \to Mod(A/P)$. If we compose the Yoneda map $A^o \to [A, \mathbf{U}]$ with r, we obtain a map

$$y : A^o \to Mod(A/P).$$

A model of (A, P) is said to be *free* if it is a coproduct of objects in the image of y.

31.6. Let Γ be the category of finite pointed sets and basepoint preserving maps. If $n_+ = n \sqcup \{\star\}$ is a finite pointed set with base point $\star$, then for each $k \in n$ consider the map $p_k : n_+ \to 1_+$ taking the value 1 at k and $\star$ elsewere. The family of maps $(p_k : k \in n)$ defines a product cone $c_n : 1 \star n \to \Gamma$ for each $n \geq 0$. If C is the set of cones $C = \{c_n : n \geq 0\}$, then (Γ, C) is the product sketch introduced by Segal in [S2]. A model of (Γ, C) is often called a Γ-*space*, or an E_∞-*space*.

31.7. Every finitary product sketch (A, P) has a universal model $u : A \to U(A/P)$ with values in an algebraic theory. The universality means that the map

$$u^* : Mod(U(A/P), X) \to Mod(A/P, X)$$

induced by u is an equivalence for any logos with finite products X. Let us describe a construction of $U(A/P)$. We say that a model of (A, P) is *finitely free* if it is the coproduct of a finite family of models in the image of the map $y : A^o \to Mod(A/P)$. Let us denote by $Mod(A/P)^{ff}$ the full simplicial subset of $Mod(A/P)$ spanned by the finitely free models. Then the map $u : A \to U(A/P)$ is the opposite of the map $A^o \to Mod(A/P)^{ff}$ induced by the map $y : A^o \to Mod(A/P)$.

31.8. We shall denote by O the algebraic theory of a naked (unstructured) object. By definition, O is a logos with finite products freely generated by an object $u \in O$. This means that for any logoi with finite products X, the evalutation map $u^* : Mod(O, X) \to X$ defined by putting $u^*(f) = f(u)$ is an equivalence of logoi. Every logos A generates freely a logos with finite products $i : A \to \sqcap(A)$. The freeness means here that for any logos with finite products X, the map $i^* : Mod(\sqcup(A), X) \to X^A$ induced by i is an equivalence of logoi. The logos $\sqcap(A)$ is a category when A is a category and its construction is classical. We have $\sqcap(A)^o = \sqcup(A^o)$, where $\sqcup(A^o)$ denotes the free completion of A^o under finite coproducts. For example, the category $\sqcup(1)$ can be taken to be the category $\underline{N}$ whose objects are the natural numbers and whose arrows are the maps $\underline{m} \to \underline{n}$, where $\underline{n} = \{1, \cdots, n\}$. The opposite category $\underline{N}^o$ is equivalent to O.

31.9. We shall denote by **AT** the category of algebraic theories and morphisms. The category **AT** has the structure of a 2-category induced by that of the category of simplicial sets. If S and T are two algebraic theories then so is the logos $Mod(S, T)$ of morphisms $S \to T$. The 2-category **AT** is symmetric monoidal closed. The *tensor product* $S \odot T$ of two algebraic theories is defined to be the target of a map $\phi : S \times T \to S \odot T$ which preserves finite products in each variable and which is universal with respect to that property [BV]. More precisely, for any logos with finite product X, let us denote by $Mod(S, T; X)$ the full simplicial subset of $X^{S \times T}$ spanned by the maps $S \times T \to X$ which preserves finite products in each variable. Then the map

$$\phi^* : Mod(S \odot T, X) \to Mod(S, T; X)$$

induced by ϕ is an equivalence of logoi. It follows that we have two canonical equivalence of logoi

$$Mod(S \odot T, X) \simeq Mod(S, Mod(T, X)) \simeq Mod(T, Mod(S, X)).$$

In particular, we have two equivalences of logoi,

$$Mod(S \odot T) \simeq Mod(S, Mod(T)) \simeq Mod(T, Mod(S)).$$

The unit for the tensor product is the theory O described in 31.8.

31.10. The forgetful functor $\mathbf{CT} \to \mathbf{AT}$ has a left adjoint

$$C : \mathbf{AT} \to \mathbf{CT}$$

which associates to an algebraic theory T the cartesian theory CT freely generated by T. The freeness means that for any cartesian logos X, the map

$$i_T^* : Mod_c(CT, X) \to Mod_p(T, X)$$

induced by the canonical map $i_T; T \to CT$ is an equivalence of logos, where Mod_c denotes the left exact maps and Mod_p the product preserving maps. The opposite of the map $i_T; T \to CT$ is equivalent to the inclusion $Mod(T)^f \subseteq Mod(T)^{ff}$.

The functor C preserves the tensor product $\odot$ that we have on each side. This means that the canonical map $C(S \odot T) \to CS \odot CT$ is an equivalence of logos for any pair of algebraic theories S and T. Notices that $C(O) = OB$.

31.11. We shall denote by Map the algebraic theory of maps. In the notation of **??**, we have $Map = \sqcap(I)$, where $I = [1]$. For any algebraic theory T and any logos with finite products X we have two equivalences of logoi

$$Mod(T \odot Map, X) \simeq Mod(T, X^I) \simeq Mod(T, X)^I.$$

This means that $T \odot Map$ is the theory of maps between two models of T.

31.12. A *pointed object* in a logos with terminal object X is defined to be an object of the logos $1\backslash X$. We shall denote by O' the algebraic theory of pointed objects. By definition, O' is freely generated by a morphism $p : 1 \to u$, where 1 denotes a terminal object. It turns out that we have $O' = O/u$, where $u \in O$ is the universal object. The opposite logos $(O')^o$ is equivalent to the category of finite pointed sets. For any algebraic theory T and any logos with finite products X we have two equivalences of logoi

$$Mod(O' \odot T, X) \simeq Mod(T, 1\backslash X) \simeq 1\backslash Mod(T, X).$$

Hence the logoi $O' \odot T$ is the theory of pointed models of T.

31.13. The category **AT** has cartesian products and the forgetful functor $\mathbf{AT} \to \mathbf{QCat}$ preserves them. The simplicial category **AT** is semi-additive in the following sense. Observe that the terminal algebraic theory 1 is also initial, since the logos $Mod(1, T)$ is equivalent to 1 for every $T \in \mathbf{AT}$. If S and T are algebraic theories, consider the maps

$$i_S : S \to S \times T \quad \text{and} \quad i_T : S \to S \times T$$

defined by putting $i_S(x) = (x, 1)$ and $i_T(y) = (1, y)$ for every $x \in S$ and $y \in T$. The maps are turning the product $S \times T$ into a coproduct of S and T, since the map

$$(i_S^*, i_T^*) : Mod(S \times T, X) \to Mod(S, X) \times Mod(T, X)$$

induced by the pair (i_S, i_T) is an equivalence of logoi for any logos with finite products X.

31.14. We shall say that an object v of an algebraic theory T *generates* the theory if every object of T is isomorphic to a power v^n for some $n \geq 0$. An algebraic theory T equipped with a generator $v \in T$ is said to be *unisorted.* If $T = (T, v)$ is a unisorted theory, then the *forgetful map*

$$v^* : Mod(T) \to \mathbf{U}$$

defined by putting $v^*(f) = f(v)$ is conservative.

31.15. The *bar construction* associates to a monoid object M in a category with product $\mathcal{E}$. a simplicial object $BM : \Delta^o \to \mathcal{E}$ called the *nerve* of M. A simplicial object $C : \Delta^o \to \mathcal{E}$ is the nerve of a monoid iff $C_0 = 1$ and the edge map $C_n \to C_1^n$ defined from the inclusions $(i-1, i) \subseteq [n]$ for $1 \leq i \leq n$ is invertible for every $n \geq 1$. A *monoid object* in a logos X can be defined to be a simplicial object $C : \Delta^o \to \mathcal{E}$ with $C_0 = 1$ and such that the edge map $C_n \to C_1^n$ is invertible for every $n \geq 1$. There is then a universal monoid $M : \Delta^o \to Mon$ with values in an algebraic theory Mon. The logos Mon is 1-truncated and the opposite logos Mon^o is equivalent to the category of finitely generated free monoids. We conjecture that the tensor power $Mon^2 = Mon \odot Mon$ is the algebraic theory of *braided monoids.* More generally, we conjecture that $Mon^n = Mon^{\otimes n}$ is the algebraic theory of E_n-monoids for every

$n \geq 0$. The theory Mon is unisorted; from the canonical morphism $u : O \to Mon$ we can define a morphism $u_n : Mon^n \to Mon^{n+1}$ for every $n \geq 0$. The (homotopy) colimit of the infinite sequence ot theories

$$O \xrightarrow{u_0} Mon \xrightarrow{u_1} Mon^2 \xrightarrow{u_2} Mon^3 \xrightarrow{u_3} \cdots$$

is the theory of *symmetric monoids* $SMon = Mon^\infty$, defined by the Segal sketch 31.17.

31.16. The algebraic theory of symmetric monoids $SMon$ has the following explicit description in terms of spans between finite sets. Let us denote by $\underline{N}$ the category of finite cardinals and maps. A (finite) *span* $A \to B$ is a map $(s,t) : S \to A \times B$ in $\underline{N}$,

$$\begin{tikzcd} & S \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B. \end{tikzcd}$$

We shall denote by $Span(A,B)$ the groupoid of isomorphisms of the category $\underline{N}/(A \times B)$ of finite spans $A \to B$. The *composite* of a span $X \in Span(A,B)$ with a span $Y \in Span(B,C)$ is the span $Y \circ X = X \times_B Y \in Span(A,C)$,

$$\begin{tikzcd} & & X \times_B Y \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl, "s"'] \arrow[dr, "t"] & & Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B & & C \end{tikzcd}$$

The composition functor

$$- \circ - : Span(B,C) \times Span(A,B) \to Span(A,C)$$

is coherently associative and this defines a bicategory $Span$. Every bicategory has a coherent nerve, and the coherent nerve of $Span$ is a logos that we shall denote by $SMon$. The bicategory $Span$ has finite cartesian products, where the cartesian product of B and C is the disjoint union $B \sqcup C$, viz the canonical equivalence of groupoids

$$Span(A, B \sqcup C) \simeq Span(A,B) \times Span(A,C).$$

It follows that the logos $SMon$ has finite product. A model $SMon \to \mathbf{U}$ is an E_∞-space. A model $SMon \to \mathbf{U}_1$ is a symmetric monoidal logos.

31.17. The Segal sketch (Γ, C) of 31.17 has the following universal model $u : \Gamma/C \to SMon$ induced by the functor $u : \Gamma \to SMon$ which associates to a pointed map $f : m_+ \to n_+$, the span $u(f)$ described by the diagram

$$\begin{tikzcd} & k \arrow[dl, "l"'] \arrow[dr, "r"] & \\ m & & n, \end{tikzcd}$$

where $k = \{i \in m : f(i) \neq \star\}$, where l is the inclusion $k \subseteq n$ and where r is induced by f.

31.18. We shall say that a monoid M is a group if the map

$$(m, p_1) : M \times M \to M \times M$$

is invertible. We denote by Grp the algebraic theory of groups The logos Grp is 1-truncated and the opposite logos Grp^o is equivalent to the category of finitely generated free groups. The conjecture 31.15 implies that $Grp^n = Grp^{\odot n}$ is the algebraic theory of n-fold loop spaces for every $n \geq 1$. The theory Grp is unisorted; from the canonical morphism $u : O \to Mon$ we can define a morphism $u_n : Grp^n \to Grp^{n+1}$ for every $n \geq 0$. The (homotopy) colimit of the infinite sequence

$$O \xrightarrow{u_0} Grp \xrightarrow{u_1} Grp^2 \xrightarrow{u_2} Grp^3 \xrightarrow{u_3} \cdots$$

is the theory of infinite loop spaces $SGrp = Grp^\infty$. .

31.19. The algebraic theory of E_∞-rig spaces has the following explicit description (a rig is a ring without additive inverse). Let $\underline{N}$ be the category of finite cardinals and maps. We recall from [BJ] and [KJBM] that a a diagram of finite sets

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "l"'] & B \arrow[dr, "r"] & \\ S & & & T \end{tikzcd}$$

defines a polynomial functor $F = r_! p_* l^*$,

$$\underline{N}/S \xrightarrow{l^*} \underline{N}/E \xrightarrow{p_*} \underline{N}/B \xrightarrow{r_!} \underline{N}/T,$$

where l^* is the pullback functor along l, where p_* is the right adjoint to p^* and where $r_!$ is left adjoint to r^*. If we fix S and T, there is a groupoid $Pol(S, T)$ of such diagrams, where a map $F \to F'$ is a pair of isomorphisms (α, β) fitting in a commutative diagram

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] \arrow[dd, "\alpha"] & B \arrow[dr, "t"] \arrow[dd, "\beta"'] & \\ S & & & T \\ & E' \arrow[r, "p'"] \arrow[ul, "s"] & B'. \arrow[ur, "t"'] & \end{tikzcd}$$

There is a natural composition functor

$$- \circ - : Pol(S, T) \times Pol(T, U) \to Pol(S, U)$$

with a natural isomorphism $(G \circ F)(X) \simeq G(F(X))$ for $X \in \underline{N}/S$. The resulting bicategory Pol has cartesian products. Its coherent nerve is an algebraic theory that we shall denote by $SRig$. A model of $SRig$ is an E_∞-rig spaces. Notice that the algebraic theory $SMon$ admits two interpretations in $SRig$. The *additive interpretation* $u : SMon \to SRig$ is induced by the (2)-functor $Span \to Pol$ which takes a span $S : A \to B$ to the polynomial

$$\begin{tikzcd} & S \arrow[r, "1_S"] \arrow[dl, "s"'] & S \arrow[dr, "t"] & \\ A & & & B. \end{tikzcd}$$

The resulting map $u^* : Mod(SRig) \to Mod(SMon)$ takes an E_∞-rig space to its underlying additive structure. The *multiplicative interpretation* $v : SMon \to SRig$ is induced by the (2)-functor $Span \to Pol$ which takes a span $S : A \to B$ to the polynomial

$$\begin{tikzcd} & S \arrow[r, "t"] \arrow[dl, "s"'] & B \arrow[dr, "1_B"] & \\ A & & & B. \end{tikzcd}$$

The resulting map $v^* : Mod(SRig) \to Mod(SMon)$ takes an E_∞-rig space to its underlying mutiplicative structure.

31.20. The notion of homotopy factorisation system in **AT** is defined as in 31.22. The category **AT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective morphisms and $\mathcal{B}$ the class of fully faithful morphisms. If a morphism of algebraic theories $u : S \to T$ is essentially surjective, then the map $u^* : Mod(T) \to Mod(S)$ is conservative.

31.21. A *unisorted theory* can be defined to be a theory T equipped with an essentially surjective morphism $s : O \to T$. More generally, a theory T is *multisorted* if it is equipped with an essentially surjective morphism $s : \sqcap(S) \to T$, where S is the set of sorts. The corresponding *forgetful map*

$$s^* : Mod(T) \to \mathbf{U}^S$$

is conservative.

31.22. The category **AT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative morphisms. A morphism in the class $(\mathcal{A}$ is said to be an *iterated multiplicative localisation*. Let us define the notion of multiplicative localisation. We say that a set Σ of arrows in a logos with finite products X is *stable under finite products* if the implication $f \in \Sigma \Rightarrow a \times f \in \Sigma$ is true for every object $a \in X$. If Σ is stable under finite products, then the logos $L(X, S)$ has finite products and they are preserved by the canonical map $X \to L(X, S)$. We then say that c a *multiplicative localisation*. If a morphism of algebraic theories $u : S \to T$ belongs to $(\mathcal{A}$, then the map $u^* : Mod(T) \to Mod(S)$ is fully faithful.

31.23. The theory of groups Grp is a multiplicative localisation of the theory of monoids Mon. This is because a monoid M is a group iff the map $(m, p_2) : M \times M \to M \times M$ is invertible, where m is the multiplication. More generally, the theory of n-fold loop Grp^n spaces is a multiplicative localisation of the theory of n-fold monoids Mon^n for every $n \geq 0$; the theory of infinite loop spaces $SGrp$ is a multiplicative localisation of the theory of symmetric monoids $SMon$.

31.24. Every algebraic theory T is the multiplicative localisation of a free theory $\sqcap(A)$, for example we can take $A = T$. Moreover, we can suppose that A is a category. It follows that every algebraic theory is a multiplicative localisation of a discrete algebraic theory.

31.25. It T is an algebraic theory, we shall say that a left fibration $X \to T$ is a *comodel* of T if its classifying map $T \to \mathbf{U}$ is a model. A left fibration $X \to T$ is a comodel iff the logos X has a finite products and the map $X \to T$ preserves finite products. The simplicial set of elements of a model $f : T \to \mathbf{U}$ is a comodel $el(f) \to T$ and conversely. The notions of model $T \to \mathbf{U}$ and of comodel $X \to T$ are essentially equivalent. The coherent nerve of the simplicial category of comodels of T is equivalent to $Mod(T)$. If $p : X \to T$ is a comodel, then a left fibration $g : Y \to X$ is a comodel of X iff the composite $pg : Y \to T$ is a comodel of T. We thus have an equivalence of simplicial categories

$$Comod(X) \simeq Comod(T)/X.$$

If $X = el(f)$, the equivalence induces an equivalence of logoi

$$Mod(el(f)) \simeq Mod(T)/f.$$

We shall say that a morphism of algebraic theories $u : S \to T$ is *coinitial* if the map

$$u_! : Mod(S) \to Mod(T)$$

takes a terminal model to a terminal model. The category **AT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of coinitial morphisms. A morphism $u : S \to T$ belongs to $\mathcal{B}$ iff its admits a factorisation $u = u'w : S \to S' \to T$ with w a categorical equivalence and u' a comodel. See [**?**].

31.26. If T is an algebraic theory, then the model category $(\mathbf{S}/T, Wcov)$ admits a Bousfield localisation in which the (fibrant) local objects are the comodels of T. A map $u : A \to B$ in $\mathbf{S}/T$ is a weak equivalence for the local model structure iff the map

$$\pi_0[u, X] : \pi_0[B, X] \to \pi_0[A, X]$$

is bijective for every comodel $X \to T$.

31.27. Every algebraic theory T has an initial model

$$0_T = Hom(1, -) : T \to \mathbf{U}.$$

We shall say that a morphism of algebraic theories $u : S \to T$ is *tight* if the map

$$u^* : Mod(T) \to Mod(S)$$

takes an initial model to an initial model. A morphism $u : S \to T$ is tight iff the commutative square

$$\begin{tikzcd} 1\backslash S \arrow[r, "1\backslash u"] \arrow[d] & 1\backslash T \arrow[d] \\ S \arrow[r, "u"] & T \end{tikzcd}$$

is a homotopy pullback (in the model category $(\mathbf{S}, Wcat)$). The category **AT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of tight morphisms. We shall say that a morphism in the class $\mathcal{A}$ is *a Henkin extension*. See [**?**].

31.28. If T is an algebraic theory, then every model $f : T \to \mathbf{U}$ admits a factorisation $f = f'e : T \to T[f] \to \mathbf{U}$ with $e : T \to T[f]$ a Henkin extension and $f' : T[f] \to \mathbf{U}$ an initial model, and the factorisation is homotopy unique. Conversely, to every Henkin extension $u : T \to E$ we can associates a model $u^*(0_E) : T \to \mathbf{U}$. This defines a "one-to-one" correspondance between the Henkin extensions $T \to E$ and the models $T \to \mathbf{U}$. It follows that there is an equivalence of logoi

$$f \backslash Mod(T) \simeq Mod(T[f])$$

for any model $f \in Mod(T)$.

31.29. A *simplicial algebraic theory* X can be defined to be a small simplicial category with finite products. A *model* of X is a simplicial functor $X \to \mathbf{S}$ which preserves finite products. We shall denote by $Mod(X)$ the full subcategory of $[X, \mathbf{S}]$ spanned by the models $X \to \mathbf{S}$. The category $Mod(X)$ admits a simplicial model structure $Mod(X)^{proj}$ in which the weak equivalences and the fibrations are defined pointwise [Q][B4]. Recall from [Bad1] that a simplicial functor $X \to \mathbf{S}$ is said to be a *homotopy model* if it preserves finite products up to homotopy. The projective model structure $[X, \mathbf{S}]^{proj}$ admits a Bousfield localisation in which the (fibrant) local objects are the fibrant homotopy models [Bad1][B4]. We shall denote the local model structure by $[X, \mathbf{S}]^m$. The inclusion functor $Mod(X)^{proj} \to [X, \mathbf{S}]^m$ is the right adjoint in a Quiilen equivalence of model categories.

31.30. Recall that an object t in a simplicial category X is said to be *homotopy terminal* if the simplicial set $X(x, t)$ is contractible for every object $x \in X$. Recall that the *homotopy product* of two objects a and b in a simplicial category X is an object $a \times b$ equipped with a pair of maps $p_1 : a \times b \to a$ and $p_2 : a \times b \to c$ such that the induced map

$$(X(x, p_1), X(x, p_2)) : X(x, a \times b) \to X(x, a) \times X(x, b)$$

is a weak homotopy equivalence for every object $x \in X$. A simplicial category enriched over Kan complexes X has finite homotopy products iff the logos $C^! X$ has finite products. Conversely, a logos A has finite products iff the simplicial category $C_! A$ has finite homotopy products. We shall say that a small simplicial category with finite homotopy products X is a (generalised) *simplicial algebraic theory*. A *homotopy model* of a simplicial algebraic theory X is a simplicial functor $F : X \to \mathcal{S}$ which preserves finite homotopy products. The projective model structure $[X, \mathbf{S}]^{proj}$ admits a Bousfield localisation in which the (fibrant) local objects are the fibrant homotopy models. Let us denote by $Mod^h(X)$ the full subcategory of fibrant-cofibrant objects of the localised model structure. Then the evaluation functor $ev : X \times [X, \mathbf{S}] \to \mathbf{S}$ induces a functor $e : X \times Mod^h(X) \to \mathbf{Kan}$. The coherent nerve of this functor is a map of simplicial sets

$$C^! X \times C^! Mod^h(X) \to \mathbf{U}.$$

If the simplicial category X is fibrant, the corresponding map $C^! Mod^h(X) \to \mathbf{U}^{C^! X}$ induces an equivalence of logoi,

$$C^! Mod^h(X) \to Mod(C^! X).$$

It follows by adjointness that we have an equivalence of logoi

$$C^! Mod^h(C_! A) \to Mod(A)$$

for any algebraic theory A.

32. Locally presentable logoi

The logos of models of a limit sketch is locally presentable. See Lurie [Lu1] for another treatment of locally presentable logoi.

32.1. Recall that an *inductive cone* in a simplicial set A is a map of simplicial sets $K \star 1 \to A$. A *colimit sketch* is defined to be a pair (A, Q), where A is a simplicial set and Q is a set of inductive cones in A. A *model* of the sketch with values in a logos X is a map $f : A \to X$ which takes every cone $c : K \star 1 \to A$ in Q to a coexact cone $fc : K \star 1 \to X$ in X. We shall write $f : Q\backslash A \to X$ to indicate that a map $f : A \to X$ is a model of (A, Q). The models of (A, Q) with values in a logos X form a logos $Mod(Q\backslash A, X)$. By definition, it is the full simplicial subset of X^A spanned by the models $Q\backslash A \to X$.

32.2. Every colimit sketch (A, Q) has a universal model $u : A \to U(Q\backslash A)$ with values in a (locally small) cocomplete logos $U(Q\backslash A)$. The universality means that the map

$$u^* : CC(U(Q\backslash A), X) \to Mod(Q\backslash A, X)$$

induced by u is an equivalence for any cocomplete logos X, where CC denotes cocontinuous maps. We say that a logos Y is *locally presentable* is if it is equivalent to a logos $U(Q\backslash A)$ for some colimit sketch (A, Q). In this case, we shall say that a the universal model $Q\backslash A \to Y$ is a *presentation* of Y by (A, Q). Every locally presentable logos is bicomplete.

32.3. A colimit sketch (A, Q) is *finitary* if every cone in Q is finite. We say that a logos X is *finitary presentable* if it admits a presentation $Q\backslash A \to X$ by a finitary colimit sketch (A, Q).

32.4. It follows from **??** that the Yoneda map $y_A : A \to \mathbf{P}(A)$ is a presentation of the logos $\mathbf{P}(A)$ by the colimit sketch $(A, \emptyset)$. Hence the logos $\mathbf{P}(A)$ is locally presentable. In particular, the logos $\mathbf{U} = \mathbf{P}(1)$ is locally presentable. The opposite logos $\mathbf{U}^o$ is not locally presentable.

32.5. A logos is locally presentable iff it is equivalent to a logos of models of a limit sketch. More precisely, the opposite of an inductive cone $c : K \star 1 \to A$ is a projective cone $c^o : 1 \star K^o \to A^o$. The opposite of a colimit sketch (A, Q) is defined to be the limit sketch (A^o, Q^o), where $Q^o = \{c^o : c \in Q\}$. If $u : A \to U(Q\backslash A)$ is the canonical map, then for every object $x \in U(Q\backslash A)$ the map

$$\rho(x) = hom(u(-), x) : A^o \to \mathbf{U}$$

is a model of the limit sketch (A^o, Q^o), The resulting map

$$\rho : U(Q\backslash A) \simeq Mod(A^o/Q^o)$$

is an equivalence of logoi. Conversely, if (A, P) is a limit sketch, then the logoi $Mod(A/P)$ and $U(P^o\backslash A^o)$ are equivalent.

32.6. A logos X is finitary presentable iff it is equivalent to the logos of models of a finitary limit sketch.

32.7. If X is locally presentable, then so are the slice logoi $a\backslash X$ and X/a for any object $a \in X$ and the logos X^A for any simplicial set A. More generally, the logos $Mod(A/P, X)$ is locally presentable for any limit sketch (A, P).

32.8. If X is locally presentable and Y is cocomplete (locally small), then every cocontinuous map $X \to Y$ has a right adjoint. In particular, every continuous map $X^o \to \mathbf{U}$ is representable.

32.9. We denote by **LT** the category of locally presentable logoi and cocontinuous maps. The category **LT** has the structure of a 2-category induced by the 2-category structure on the category of (large) simplicial sets. If X and Y are locally presentables logoi, then so is the logos $Map(X, Y)$ of cocontinuous maps $X \to Y$. The 2-category LP is symmetric monoidal closed. The *tensor product* $X \otimes Y$ of two locally presentable logoi is defined to be the target of a map $\phi : X \times Y \to X \otimes Y$ cocontinuous in each variable and universal with respect to that property. More precisely, for any locally presentable logos Z, let us denote by $Map(X, Y; Z)$ the full simplicial subset of $Z^{X\times Y}$ spanned by the maps $X \times Y \to Z$ ccontinuous in each variable. Then the map

$$\phi^* : Map(X \otimes Y, X) \to Map(X, Y; Z)$$

induced by ϕ is an equivalence of logoi. The natural isomorphism $Z^{X\times Y} = (Z^Y)^X$ induces a natural isomorphism $Map(X, Y; Z) = Map(X, Map(Y, Z)$. It follows that we have a canonical equivalence of logoi

$$Map(X \otimes Y, Z) \simeq Map(X, Map(Y, Z)).$$

The unit object for the tensor product is the logos $\mathbf{U}$. The equivalence $\mathbf{U} \otimes X \simeq X$ is induced by the action map $(A, x) \mapsto A \cdot x$ described in 25.1.

32.10. If A is a (small) simplicial set, then the logos $\mathbf{P}(A)$ is locally presentable and freely generated by the Yoneda map $y_A : A \to \mathbf{P}(A)$. It follows that the map

$$y_A^* : Map(\mathbf{P}(A), X) \to X^A$$

is an equivalence of logoi. for any $X \in \mathbf{LP}$. The map $\rho : A \times X \to X^{A^o}$ defined in 24.20 can be extended as a map $\mathbf{P}(A) \times X \to X^{A^o}$ cocontinuous in each variable. The resulting map

$$\mathbf{P}(A) \otimes X \to X^{A^o}$$

is an equivalence of logoi. The *external product* of a pre-stack $f \in \mathbf{P}(A)$ with a pre-stack $g \in \mathbf{P}(B)$ is the prestack $f\square g \in \mathbf{P}(A \times B)$ defined by putting

$$(f\square g)(a, b) = f(a) \times g(b)$$

for every pair of objects $(a, b) \in A \times B$. The map $(f, g) \mapsto f\square g$ is cocontinuous in each variable and the induced map

$$\mathbf{P}(A) \otimes \mathbf{P}(B) \to \mathbf{P}(A \times B).$$

is an equivalence of logoi. The cocontinuous extension of the map $hom_A : A^o \times A \to \mathbf{U}$ is the *trace map*

$$Tr_A : \mathbf{P}(A^o \times A) \to \mathbf{U}$$

defined in 24.21 . The *scalar product* of $f \in \mathbf{P}(A^o)$ and $g \in \mathbf{P}(A)$ is defined by putting

$$\langle f, g\rangle = Tr_A(f \cdot g),$$

where

$$(f \cdot g)(a^o, b) = f(a^o) \cdot g(b).$$

The map

$$\langle - \mid - \rangle : \mathbf{P}(A^o) \times \mathbf{P}(A) \to \mathbf{U}$$

is obtained by composing the canonical map $\mathbf{P}(A^o) \times \mathbf{P}(A) \to \mathbf{P}(A^o \times A)$ with the trace map. The map $\langle f, - \rangle : \mathbf{P}(A) \to \mathbf{U}$ is a cocontinuous extension of the map $f : A \to \mathbf{U}$ and the map $\langle -, g \rangle : \mathbf{P}(A^o) \to \mathbf{U}$ a cocontinuous extension of the map $g : A^o \to \mathbf{U}$. The scalar product defines a perfect duality between $\mathbf{P}(A)$ and $\mathbf{P}(A^o)$. It yields the equivalence above

$$\mathbf{P}(A) \otimes X \simeq Map(\mathbf{P}(A^o), X) = X^{A^o}$$

and the equivalence of 24.20,

$$Map(X^{A^o}, Y) \simeq Map(X, Y^A)$$

for $X.Y \in \mathbf{LP}$.

32.11. For any logos X, the map $hom_X : X^o \times X \to \mathbf{U}$ is continuous in each variable. If X is locally presentable, the opposite map $X \times X^o \to \mathbf{U}^o$ induces an equivalence of logoi

$$X^o \simeq CC(X, \mathbf{U}^o).$$

More generally, it induces two equivalences of logoi

$$\begin{aligned} (X \otimes Y)^o &\simeq CC(X, Y^o) \\ &\simeq CC(Y, X^o) \end{aligned}$$

for any pair of locally presentable logoi X and Y.

32.12. If X is locally presentable and (A, P) is a limit sketch, then we have an equivalence of logoi,

$$Mod(A/P) \otimes X = Mod(A/P, X).$$

The canonical map $\otimes : Mod(A/P) \times X \to Mod(A/P, X)$ can be defined as follows. The inclusion $Mod(A/P, X) \subseteq X^A$ has a left adjoint $r : X^A \to Mod(A/P, X)$. If $f \in Mod(A/P)$ and $x \in X$, then the object $f \otimes x \in Mod(A/P, X)$ is obtained by applying r to the map $a \mapsto f(a) \cdot x$. If $(B.Q)$ is another limit sketch and $X = Mod(B/Q)$ this gives two equivalence of logoi:

$$\begin{aligned} Mod(A/P) \otimes Mod(B/Q) &\simeq Mod(A/P, Mod(B/Q)) \\ &\simeq Mod(B/Q, Mod(A/P)). \end{aligned}$$

32.13. Let Σ be a (small) set of arrows in a locally presentable logos X. Then the pair $({}^\perp(\Sigma^\perp), \Sigma^\perp)$ is a factorisation system. We say that an object $a \in X$ is Σ-*local* if it is right orthogonal to every arrow in Σ (see 26.2). Let us denote by X^Σ the full simplicial subset of X spanned by the Σ-local objects. Then the inclusion $i : X^\Sigma \subseteq X$ has a left adjoint $r : X \to X^\Sigma$ and the logos X^Σ is locally presentable. The map r is cocontinuous and it inverts universally the arrows in Σ. More precisely, if Y is a cocomplete locally small logos and $f : X \to Y$ is a cocontinuous map which inverts the arrows in Σ, then there exists a cocontinuous map $g : X^\Sigma \to X$ together with an invertible 2-cell $\alpha : f \simeq gr$, and the pair (g, α) is unique up to a unique invertible 2-cell. We shall say that the map $r : X \to X^\Sigma$ is a *localisation*. A map $f : X \to Y$ in $\mathbf{LP}$ is a reflection iff it admits a factorisation $wr : X \to X^\Sigma \to Y$, where r a localisation with respect to a (small) set Σ of arrows in X and w is an equivalence of logoi. The 2-category **LP** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of localisations (=reflections) and $\mathcal{B}$ is the class of conservative maps.

32.14. For any cone $c : B \star 1 \to A$, let us denote by $l(c)$ the colimit of the diagram $y_A c i_B : B \to \mathbf{P}(A)$, where y_A is the Yoneda map $A \to \mathbf{P}(A)$ and where i_B is the inclusion $B \subset B \star 1$. There is then canonical arrow $\sigma(c) : l(c) \to yc(1)$ in $\mathbf{P}(A)$. If $f \in \mathbf{P}(A)$, then the cone $fc^o : 1 \star B \to \mathbf{U}$ is exact iff we have $\sigma(c) \perp f$, where $\perp$ is the orthogonality relation defined in 26.2. It follows by dualising that for any limit sketch (A, P), there is a set Σ of arrows in $\mathbf{U}^A$ with the property that a map $f : A \to \mathbf{U}$ is a model of (A, P) iff we have $u \perp f$ for every $u \in \Sigma$. In the notation of 32.13, this means that we have

$$Mod(A/P) = (\mathbf{U}^A)^{\Sigma}.$$

32.15. We shall say that a map $f : X \to Y$ in **LP** is *weakly dense* if Y is cogenerated by $f(X_0)$. A map $f : X \to Y$ is weakly dense iff its right adjoint is conservative. The 2-category **LP** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of weakly dense maps and $\mathcal{B}$ is the class of fully faithful maps.

32.16. Let $f : X \to Y$ be a map in **LP**. If t_X is the terminal object of X, then for every object $x \in X$, the image by f of the morphism $x \to t_X$ is an object $f'(x) : f(x) \to f(t_X)$ of the logos $Y/f(t_X)$. This defines a cocontinuous map $f' : X \to Y/f(t_X)$ and we obtain a factorisation

$$f = pf' : X \to Y/f(t_X) \to Y,$$

where p is the projection $Y/f(t_X) \to Y$. The 2-category **LP** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which a map $f : X \to Y$ belongs to $\mathcal{A}$ iff it preserves terminal objects. A map $f : X \to Y$ belongs to $\mathcal{B}$ iff it is equivalent to a map $Y/b \to Y$ iff it is equivalent to a right fibration.

32.17. Let $f : X \to Y$ be a map in **LP** with right adjoint $g : Y \to X$. If i_Y is the initial object of Y, then for every object $y \in Y$, the image by g of the morphism $i_Y \to y$ is an object $g'(y) : g(i_Y) \to g(y)$ of the logos $g(i_Y)\backslash X$. This defines a continuous map $g' : Y \to g(i_Y)\backslash X$ and we obtain a factorisation $g = pg' : Y \to g(i_Y)\backslash X \to X$, where p is the projection $g(i_Y)\backslash X \to X$. There is a corresponding factorisation the left adjoint f,

$$f = f'i : X \to g(i_Y)\backslash X \to Y,$$

where i is the cobase change map $x \mapsto g(i_Y) \sqcup x$. The map f' takes a morphism $g(i_Y) \to x$ to the object $f'(x)$ defined by the pushout square

$$\begin{tikzcd}
fg(i_Y) \arrow[r] \arrow[d] & f(x) \arrow[d] \\
i_Y \arrow[r] & f'(x),
\end{tikzcd}$$

where the morphism $fg(i_Y) \to i_Y$ is the counit of the adjunction $f \vdash g$. The 2-category **LP** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which a map $f : X \to Y$ belongs to $\mathcal{B}$ iff its right adjoint preserves initial objects. A map $f : X \to Y$ belongs to $\mathcal{A}$ iff it is equivalent to a cobase change map $X \to a\backslash X$ iff its right adjoint is equivalent to a left fibration.

32.18. The category 1 is both terminal and initial in the 2- category **LP**. The cartesian product $X \times Y$ of two locally presentable logoi X and Y is also their coproduct. More precisely, if $i_X : X \to X \times Y$ and $i_Y : Y \to X \times Y$ are the maps defined by putting $i_X(x) = (x, 1)$ and $i_Y(y) = (1, y)$, then the induced map

$$(i_X^*, i_Y^*) : Map(X \times Y, Z) \to Map(X, Z) \times Map(Y, Z)$$

is an equivalence for any cocomplete logos X. More generally,the product

$$\prod_{i \in S} X_i.$$

of a family of locally presentable logoi is also a coproduct,

32.19. Let X be a finitary presentable logos and $f : X \leftrightarrow Y; g$ be a pair of adjoint maps, where Y is a cocomplete logos. Then the right adjoint g is inductive iff the left adjoint f takes a compact object to a compact object.

32.20. If (A, P) is a finitary limit sketch, then the inclusion $i : Mod(A/P) \subseteq \mathbf{U}^A$ is inductive. If X is a finitary presentable logos, then the limit map $X^A \to X$ is inductive for any finite simplicial set A.

32.21. Let X be a cocomplete logos and let $K \subseteq X$ is a small full sub logos of compact objects closed under finite colimits. Then the left Kan extension

$$i_! : \mathbf{Ind}(K) \to X$$

of the inclusion $i : K \subseteq X$ iMoreover, $i_!$ is an equivalence if in addition K cogenerates X.

32.22. A (locally small) cocomplete logos X is finitary presentable iff it is generated by a small set of compact objects. More precisely, if X is finitary presentable, then the full sub logos $k(X)$ of compact objects of X is cocartesian, essentially small, and there is a canonical equivalence of logoi,

$$X \simeq \mathbf{In}(k(X)).$$

32.23. Two finitary presentable logoi are equivalent iff their (full) sub logoi of compact objects are equivalent.

32.24. The category of finitary presentable logoi and inductive maps is cartesian closed. If X and Y are ω-presentables, then the logos of inductive maps $X \to Y$ is equivalent to the logos Y^{kX}.

32.25. If α is a regular cardinal, we say that a colimit sketch (A, Q) is *α-bounded* if the cardinality of every cone in P is $< \alpha$. We say that a logos X *α-presentable* if it admits a presentation $Q\backslash A \to X$ by an α-bounded colimit sketch (A, Q).

33. Varieties of homotopy algebras

We introduce the notion of variety of homotopy algebras. We show that a cocomplete quasi-category is a variety iff it is generated by a set of bicompact objects [Ros]. We show that a cocontinuous map is sifted iff it is finitary and preserves geometric realisation. We obtain a new characterisation of bicompact objects.

33.1. We say that a quasi-category is a *variety of homotopy algebras* if it is equivalent to a quasi-category $Mod^{\times}(T)$ for some (finitary) algebraic theory T.

33.2. If X is a variety of homotopy algebras then so are the slice quasi-categories $a\backslash X$ and X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. More generally, the quasi-category $Mod^\times(T, X)$ is a variety for any algebraic theory T.

33.3. A variety of homotopy algebras X is ω-presentable. Hence the colimit map

$$\varinjlim : \mathbf{U}^A \to \mathbf{U}$$

preserves finite limits for any directed simplicial set A.

33.4. A variety of homotopy algebras is a regular quasi-category. An arbtrary product of surjections is a surjection.

33.5. Let X be a regular quasi-category. We say that an object $a \in X$ is *projective* if every surjection $b \to a$ splits.

33.6. Let X be a regular quasi-category. Then an object $a \in X$ is *projective* iff the map

$$hom_X(a, -) : X \to \mathbf{U}$$

takes a surjection to a surjection.

33.7. A retract of a projective object is projective. A coproduct of projective objects is projective.

33.8. An object of $\mathbf{U}$ is projective iff it is discrete.

33.9. If T is an algebraic theory, then the Yoneda map $y : T^o \to Mod^\times(T)$ induces an equivalence between the Karoubi envelope of T^o and the quasi-category of compact projective models of T.

33.10. Let us denote by **Var** the sub (2-)category of **LP** whose objects are the of varieties and whose morphisms are the cocontinuous maps which preserves compact projective objects. If $u : S \to T$ is a morphism of algebraic theories, then the map $u^* : Mod^\times(T) \to Mod^\times(S)$ has a left adjoint

$$u_! : Mod^\times(S) \to Mod^\times(T)$$

which preserves compact projective objects, This defines a (pseudo-2) functor

$$Mod^\times : \mathbf{AT} \to \mathbf{Var}.$$

If X is a variety, let us denote by $kp(X)$ the full simplicial subset of X spanned by compact projective objects of X, The (pseudo-2) functor $Mod^\times$ has a right adjoint

$$kp^o : \mathbf{Var} \to \mathbf{AT}$$

which associates to X the opposite of $kp(X)$. The (2-)functor kp^o is fully faithful and the unit of the adjunction $Mod^\times \vdash kp^o$ is the Karoubi envelope $T \to \kappa(T)$ for every $T \in \mathbf{AT}$. The essential image of kp^o is the full sub (2-)category of $\mathbf{AT}$ spanned by the Karoubi complete theories. A morphism of algebraic theories $u : S \to T$ is a Morita equivalence iff the adjoint pair

$$u_! : Mod^\times(S) \leftrightarrow Mod^\times(T) : u^*$$

an equivalence of quasi-categories.

33.11. Recall that a category C is said to be *sifted*, but we shall say *0-sifted*, if the colimit functor

$$\varinjlim : \mathbf{Set}^C \to \mathbf{Set}$$

preserves finite products. This notion was introduced by C. Lair in [La] under the name of *categorie tamisante*. We shall say that a simplicial set A is *sifted* if the colimit map

$$\varinjlim : \mathbf{U}^A \to \mathbf{U}$$

preserves finite products.

33.12. The notion of sifted simplicial set is invariant under Morita equivalence. A directed simplicial set is sifted. A sifted simplicial set is weakly contractible. A non-empty simplicial set A is sifted iff the diagonal $A \to A \times A$ is final. A non-empty quasi-category A is sifted iff the simplicial set $a\backslash A \times_A b\backslash A$ defined by the pullback square

$$\begin{tikzcd}
a\backslash A \times_A b\backslash A \arrow[r] \arrow[d] & b\backslash A \arrow[d] \\
a\backslash A \arrow[r] & A
\end{tikzcd}$$

is weakly contractible for any pair of objects $a, b \in A$. A quasi-category A is sifted iff the canonical map $A \to \sqcup(A)$ is final. A quasi-category with finite coproducts is sifted. The category Δ^o is sifted.

33.13. If X is a quasi-category, we shall say that a diagram $d : A \to X$ is *sifted* if the simplicial set A is sifted, in which case we shall say that the colimit of d is *sifted* if it exists. We shall say that X *has* sifted colimits if every (small) sifted diagram $A \to X$ has a colimit. A quasi-category with sifted colimits has directed colimits. We shall say that a map between quasi-categories is *stronly inductive* if it preserves sifted colimits.

33.14. If T is an algebraic theory, then the inclusion $i : Mod^\times(T) \subseteq \mathbf{U}^T$ is reflective and strongly inductive.

33.15. If X is a homotopy variety, then the product map $X^n \to X$ is strongly inductive for every $n \geq 2$.

33.16. A quasi-category with sifted colimits and finite coproducts is cocomplete. A map between cocomplete quasi-category is cocontinuous iff it preserves finite coproducts and sifted colimits.

33.17. Let us sketch the proof of 33.16. If a quasi-category X has finite coproducts and directed colimit, then it has arbitrary coproducts, since we have

$$\bigsqcup_{i \in S} x_i = \varinjlim_{F_{\text{fin}} \subseteq S} \bigsqcup_{i \in F} x_i,$$

where F runs in the poset of finite subsets of S. Hence a quasi-category with sifted colimits and finite coproducts has coproducts. But a quasi-category with coproducts and Δ^o-indexed colimits is cocomplete by 21.45. The proof of the second statement is similar.

33.18. Let X be a (locally small) quasi-category with sifted colimits. We shall say that an object $a \in X$ is *bicompact* if the map

$$hom_X(a, -) : X \to \mathbf{U}$$

preserves sifted colimits. We shall say that a is *adequate* if the map $hom_X(a, -)$ preserves Δ^o-indexed colimits. A bicompact object is compact and adequate.

33.19. A finite colimit of bicompact objects is bicompact. A retract of a bicompact object is bicompact. And similarly for adequate objects.

33.20. A model of an algebraic theory T is bicompact iff it is compact and projective iff it is a retract of a representable.

33.21. Let $f : X \leftrightarrow Y ; g$ be a pair of adjoint maps, where X is a variety and Y is cocomplete. Then g is strongly inductive iff f takes a bicompact object to a bicompact object.

33.22. We shall use the following observations in 33.23. Let N be the category of finite cardinals and maps. If S is a set, let us denote by N/S the full subcategory of $\mathbf{Set}/S$ whose objects are the maps $n \to S$ with $n \in N$. Consider the map $i : S \to Ob(N/S)$ which associates to $s \in S$ the map $s : 1 \to S$. The category N/S has finite coproducts and the map $i : S \to N/S$ exhibits the category N/S as the free cocompletion of the discrete category S under finite coproducts. Hence we have $\sqcup(S) = N/S$ and it follows that the category $\sqcup(S)$ has finite colimits when S is discrete. If $f : S \to T$ is a map between discrete categories, then the functor $f_! : N/S \to N/T$ is a discrete fibration, hence also the functor $\sqcup(f) : \sqcup(S) \to \sqcup(T)$. Recall the Grothendieck construction which associates to a functor $F : B^o \to \mathbf{Cat}$ a category of elements $Tot(F)$ whose objects are the pairs (a, x), with $a \in ObB$ and $x \in ObF(a)$, and whose arrows $(a, x) \to (b, y)$ are the pairs (f, u) with $f : a \to b$ and $u : x \to F(f)(y)$. The natural projection $Tot(F) \to B$ is a Grothendieck fibration. Let us denote by L the set of arrows $(f, u) \in Tot(F)$ with f an identity arrow and by R the set of arrows (f, u) with u an identity arrow. Both sets are closed under composition and every arrow $g \in Tot(F)$ admits a *truly unique* factorisation $g = rl$ with $l \in L$ and $r \in R$. We say that (L, R) is *strict* factorisation system (compare with 26.14). We saw in **??** that if C is a category, then the category $\sqcup(C)$ is obtained by applying the Grothendieck construction to the functor $F : N^o \to Cat$ defined by putting $F(n) = C^n$ for every $n \in N$. In this case we have $R = \sqcup(C_0) \subseteq \sqcup(C)$ and L is the coproduct of the categories C^n for $n \geq 0$.

33.23. The following construction is needed for proving a basic result on strongly inductive maps in 33.24. For any category C, there is a diagram of categories

$$\begin{tikzcd} E \arrow[r, "\beta"] \arrow[d, "q"'] & \sqcap(C) \\ \Delta & \end{tikzcd}$$

with β an initial functor and q is a Grothendieck fibration with cartesian fibers. Before giving the construction, we make a preliminary remark. It follows from 33.22 that the category $\sqcap(C)$ is equipped with a strict factorisation system (L, R) where $L = \sqcap(C_0) \subseteq \sqcap(C)$ and R is the coproduct of the categories C^n for $n \geq 0$.

Let us now describe the construction of the diagram (q, β). Recall from 21.44 that we have a diagram of categories

$$\begin{tikzcd} el(C) \arrow[r, "\theta"] \arrow[d, "p"'] & C \\ \Delta, & \end{tikzcd}$$

where $el(C)$ is the category of elements of (the nerve of) C and where p is the natural projection. The value of θ on a chain $x : [n] \to C$ is defined to be the top object $x(n) \in C$. The category $el(C)$ can be obtained by applying the Grothendieck construction to the functor $[n[\mapsto C_n$, where C_n is the set of chains $[n] \to C$. For each $n \geq 0$, let us put $F([n]) = \sqcap(C_n)$. This defines a functor $F : \Delta^o \to \mathbf{Cat}$. If we apply the Grothendieck construction to the functor F and put $E = Tot(F)$, we obtain a Grothendieck fibration $q : E \to \Delta$ whose fiber at $[n]$ is the category $\sqcap(C_n)$. The category $\sqcap(C_n)$ is cartesian by 33.22 dualised. Let us now define the map $\beta : E \to \sqcap(C)$. We have $el(C) \subset E$ since we have $C_n \subset \sqcap(C_n)$ for each $n \geq 0$. There is then a unique functor $\beta : E \to \sqcap(C)$ which extends the functor $\theta : el(C) \to C$ and which preserves finite products on each fiber of q. The functor β extends each functor $\sqcap(\theta_n) : \sqcap(C_n) \to \sqcap(C_0)$, where $\theta_n : C_n \to C_0$ is the map defined by θ. Thus, β is a discrete opfibration on each fiber of q by 33.22 dualised. We shall use this observation in the proof that the functor θ is initial. In order to show this, it suffices to show by **??** that the category E/b defined by the pullback square

$$\begin{tikzcd} E/b \arrow[r] \arrow[d] & E \arrow[d, "\beta"] \\ \sqcap(C)/b \arrow[r] & \sqcap(C) \end{tikzcd}$$

is weakly contractible for every object $b \in \sqcap(C)$. The object b is defined by a map $n \to C_0$ for some $n \in N$. Let us denote by E_b the fiber at b of the functor β. We have $E_b \subseteq E/b$ and $b \in E_b$, since $\beta(b) = b$. It is easy to see that the object b is initial in E_b. Hence que category E_b is contractible. Let us show that the inclusion $i : E_b \subseteq E/b$ admits a left adjoint $r : E/b \to E_b$. An object of E/b is pair (x, f), where x is an object of $\sqcap(C_k)$ for some $k \geq 0$ and where $f : \beta(x) \to b$ is a morphism in $\sqcap(C)$. The category $\sqcap(C)$ admits a factorisation system (L, R) with $L = \sqcap(C_0)$ by our first remark above. The subcategory R is the disjoint union of the subcategories C^n for $n \geq 0$. There is thus a factorisation $f = gu : \beta(x) \to a \to b$ with $u \in L$ and $g \in R$. We have $g \in C^n$, since $g \in R$ and $b \in C_0^n$. Thus, $a \in C_0^n$ and $g(i) : a(i) \to b(i)$ for every $i \in n$. The functor $\sqcap(C_k) \to \sqcap(C_0)$ induced by β is a discrete opfibration by the made observation above. Hence there is a unique map $v \in \sqcap(C_k)$ with source x such that $\beta(v) = u$. We have $v : x \to x'$ for some object $x' \in \sqcap(C_k)$. For each $i \in n$, let us denote by $\overline{x}(i) : [k+1] \to C$ the chain which extends the chain $x'(i) : [k] \to C$ by grafting the arrow $g(i) : a(i) \to b(i)$ at the top of $x'(i)$. This defines an element $\overline{x} \in \sqcap(C_{k+1}) \cap E_b$. Let us put $r(x, f) = \overline{x} \in E_b$. The inclusion $[k] \subset [k+1]$ induces a map $d : x' \to \overline{x}$ in E and we have $\beta(dv) = \beta(d)\beta(v) = gu = f$. Hence the map $dv : x \to \overline{x}$ defines a map $\eta : (x, f) \to (\overline{x}, 1_b)$ in E/b. This defines a functor $r : E/b \to E_b$ equipped with a natural transformation $\eta : id \to ir$. It is easy to verify that it is the unit of an adjunction $r \vdash i$. Hence the category E/b is weakly contractible, since E_b is contractible. We have completed the proof that the functor β is initial.

33.24. A map $f : X \to Y$ between cocomplete quasi-categories is strongly inductive iff it preserves directed colimits and Δ^o-indexed colimits. The implication ($\Rightarrow$) is obvious. Let us prove the implication ($\Leftarrow$). Let us show that f preserves the colimit of any sifted diagram $d : A \to X$. Let us choose a fibrant replacement $i : A \to A'$ in the model structure for quasi-categories. We can suppose that i is a cofibration. In this case the map $d : A \to X$ admits an extension $d' : A' \to X$, since X is a quasi-category. The map i is final, since a weak categorical equivalence is final by 11.21. Hence the the map f preserves the colimit of d iff it preserves the colimit of d' by 21.29. But A' is sifted, since A is sifted. Hence we can suppose that A is a quasi-category, In this case, the canonical map $A \to \sqcup(A)$ is final by 33.12, since A is sifted. The map $d : A \to X$ admits an extension $d_! : \sqcup(A) \to X$ by **??**, since X is cocomplete. Hence the map f preserves the colimit of d iff it preserves the colimit of $d_!$ by 21.29. The map $\theta_A : el(A) \to A$ is a localisation by **??**. Hence also the map $\lambda = \sqcup(\theta_A) : \sqcup(el(A)) \to \sqcup(A)$. But a localisation is final by 19.7. Hence it suffices to show that f preserves the colimit of $d_!\lambda$ by 21.29. Let us show more generally that f preserves the colimit of any diagram $\sqcup(C) \to X$, where C a category. It follows from 33.23 dualised that there is a diagram of categories

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d, "p"'] & \sqcup(C) \\ \Delta^o & \end{tikzcd}$$

with α a final functor and p a Grothendieck opfibration with cocartesian fibers. The map f preserves the colimit of a diagram $u : \sqcup(C) \to X$ iff it preserves the colimit of the diagram $v = u\alpha : E \to X$, since α is final. But the colimit of v is the same as the colimit of its left Kan extension along $p : E \to \Delta^o$ by 24.5. Hence it show that f preserves left Kan extensions along p, since it preserves Δ^o-indexed colimits by hypothesis. For this, we have to show that the canonical 2-cell

$$\Sigma_p(fv) \to f\Sigma_p(v)$$

is invertible. But for this, it suffices to show that the arrow

$$\Sigma_p(fv)([n]) \to f\Sigma_p(v)([n])$$

is quasi-invertible for every $n \geq 0$ by 1.11. But the left Kan extensions along p are fiberwise by **??**, since p is a Grothendieck opfibration. Hence it suffices to show that the canonical arrow

$$\varinjlim_{x \in E(n)} fv(x) \to f \varinjlim_{x \in E(n)} v(x)$$

is quasi-invertible for every $n \geq 0$, where $E(n)$ denotes the fiber of p over $[n]$. But the category $E(n)$ is directed, since a cocartesian category is directed. This shows that f preserves the left Kan extensions along p. We have proved that f preserves sifted colomits.

33.25. It follows from 33.24 that an object in a cocomplete quasi-category is bicompact iff it is compact and adequate.

33.26. If A is a simplicial set, we shall say that a pre-stack $g \in \mathbf{P}(A)$ is *sifted*, if it is a sifted colimit of representables. We shall denote by $\mathbf{Sft}(A)$ the full simplicial subset of $\mathbf{P}(A)$ spanned by sifted pre-stacks. The quasi-category $\mathbf{Sft}(A)$ is has sifted colimits and the Yoneda map $y_A : A \to \mathbf{P}(A)$ induces a map $y_A : A \to \mathbf{Sft}(A)$. If A is a quasi-category with finite coproducts then a prestack $g \in \mathbf{P}(A)$ is sifted iff the map $g : A^o \to \mathbf{U}$ preserves finite products; hence we have

$$\mathbf{Sft}(A) = Mod^{\times}(A^o)$$

in this case.

33.27. A pre-stack $g \in \mathbf{P}(A)$ is sifted iff its simplicial set of elements $El(g) = A/g$ is sifted iff its cocontinuous extension

$$\langle -, g \rangle : \mathbf{P}(A^o) \to \mathbf{U}$$

defined in 32.10 preserves finite products.

33.28. If A is a simplicial set, then the map $y_A : A \to \mathbf{Sft}(A)$ exhibits the quasi-category $\mathbf{Sft}(A)$ as the free cocompletion of A under sifted colimits. More precisely, if X is a quasi-category with sifted colimits, then the left Kan extension

$$f_! : \mathbf{Sft}(A) \to X$$

of any map $f : A \to X$ along the map $y_A : A \to \mathbf{Sft}(A)$ is strongly inductive. Moreover, any strongly inductive extension of f is canonically isomorphic to $f_!$.

33.29. If A is a quasi-category with finite coproducts, then the map $y_A : A \to Mod^{\times}(A^o)$ exhibits the quasi-category $Mod^{\times}(A^o)$ as the (relatively free) cocompletion of A under colimits. More precisely, if X is a cocomplete quasi-category, then the left Kan extension

$$f_! : Mod^{\times}(A^o) \to X$$

(along y_A) of a map $f : A \to X$ which preserves finite coproducts is cocontinuous. Moreover, any cocontinuous extension of f is canonically isomorphic to $f_!$.

33.30. If A is a simplicial set, then we have a decomposition

$$\mathbf{P}(A) \simeq \mathbf{Sft}(\sqcup(A)).$$

33.31. Let X be a quasi-category with sifted colimits and let $K \subseteq X$ be a small full sub quasi-category of bicompact objects. Then the left Kan extension

$$i_! : \mathbf{Sft}(K) \to X.$$

of the inclusion $i : K \subseteq X$ is strongly inductive and fully faithful. Moreover, if K is closed under finite coproducts, then we have $\mathbf{Sft}(K) = Mod^{\times}(K^o)$ and the map

$$i_! : Mod^{\times}(K^o) \to X$$

is cocontinuous and fully faithful. Moreover, $i_!$ is an equivalence if in addition K separates X, or if X is cocomplete and K generates X.

33.32. Let $\mathcal{E}$ be a (cocomplete locally small) quasi-category and $U_k : \mathcal{E} \to \mathbf{U}$ ($k \in K$) be a family of continuous maps. Let us suppose that U_k is epresented by an object $u_k \in X$. Let E be the closure under finite coproducts of the full sub quasi-category of X spanned by the objects u_k for $k \in K$. The opposite quasi-category $T = E^o$ is the theory of algebraic operation on the family of maps $U = (U_k : k \in K)$ (by Yoneda lemma). The map $G : \mathcal{E}^o \to Mod^\times(T)$ defined by putting $G(x)(y) = hom(y, x)$ for every $y \in E$ has a left adjoint L. We have a commutative diagram,

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "G"] \arrow[dr, "U"'] & & Mod^\times(T) \arrow[dl, "V"] \\ & \mathbf{U}^K, & \end{tikzcd}$$

where V associates to $f : E^o \to \mathbf{U}$ the family $(f(u_k) : k \in K)$. The map G is a coreflection (ie L is fully faithful) if the maps U_k preserve directed colimits and the colimit of Δ^o-indexed diagrams by 33.31. Moreover, G is an equivalence of quasi-categories if in addition U is conservative. Hence the quasi-category $\mathcal{E}$ is a variety in this case.

33.33. A (locally small) cocomplete quasi-category X is a variety iff it is generated by a small set of bicompact objects. More precisely, if X is a variety, then the full sub quasi-category $A = kk(X)$ of bicompact objects of X is closed under finite coproducts, essentially small, and there is a canonical equivalence of quasi-categories,

$$X \simeq \mathbf{Sft}(A) = Mod^\times(A^o).$$

33.34. Two varieties are equivalent iff their (full) sub quasi-categories of bicompact objects are equivalent.

33.35. A prestack $g \in \mathbf{Sft}(A)$ is bicompact iff it is atomic. More generally, if $g \in \mathbf{Sft}(A)$, then the bicompact objects of the quasi-category $\mathbf{Sft}(A)/g$ are the morphisms $k \to g$, with $k \in \kappa(A)$. Moreover, the equivalence $\mathbf{P}(A/g) \simeq \mathbf{P}(A)/g$ of 25.19 induces an equivalence

$$\mathbf{Sft}(A/g) \to \mathbf{Sft}(A)/g.$$

33.36. From a map of simplicial sets $u : A \to B$, we obtain a square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "y_A"'] & B \arrow[d, "y_B"] \\ \mathbf{Sft}(A) \arrow[r, "u_!"] & \mathbf{Sft}(B) \end{tikzcd}$$

which commutes up to a canonical isomorphism. The map $u_!$ is defined to be the left Kan extension of the map $y_B u : A \to \mathbf{Sft}(B)$ along y_A. The map $u_!$ is fully faithful iff u is fully faithful. It is an equivalence iff u is a Morita equivalence.

33.37. The category of varieties and strongly inductive maps is cartesian closed. If X and Y are varieties, then the quasi-category of strongy inductive maps $X \to Y$ is equivalent to the quasi-category Y^{kkX}.

34. Para-varieties

34.1. Recall that a category $\mathcal{E}$ is said to be a *Grothendieck topos*, but we shall say a *1-topos*, if it is a left exact reflection of a presheaf category $[C^o, \mathbf{Set}]$. This means that $\mathcal{E}$ is equivalent to a reflective category of $[C^o, \mathbf{Set}]$, with a reflection functor $[C^o, \mathbf{Set}] \to \mathcal{E}$ which is left exact.

34.2. We call a locally presentable quasi-category X an ∞*-topos* if it is a left exact reflection of a quasi-category of pre-stacks $\mathbf{P}(A)$ for some simplicial set A. If $n \geq 0$ we call a locally presentable quasi-category X a n*-topos* if it is a left exact reflection of a quasi-category of n-pre-stacks $\mathbf{P}(A)(n)$ for some simplicial set A.

34.3. We call a locally presentable quasi-category X a *para-variety* if it is a left exact reflection of a variety of homotopy algebras. We call a locally presentable quasi-category X a *para-*n*-variety* if it is a left exact reflection of a n-variety of homotopy algebras $Prod(T)(n)$ for some algebraic theory T.

34.4. If X is a para-variety, then so are the slice quasi-categories $a\backslash X$ and X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. More generally, the quasi-category $Prod(T, X)$ is a para-variety for any (finitary) algebraic theory T. A similar result is true for para-n-variety,

34.5. A para-variety admits surjection-mono factorisations and the factorisations are stable under base changes. More generally, it admits k-factorisations stable under base changes for every $k \geq -1$. A similar result is true for para-n-variety,

34.6. If X is a para-variety, then the colimit map

$$\varinjlim_A : X^A \to X$$

preserves finite limits (finite products) for any directed (resp. sifted) simplicial set A. A similar result is true for para-n-variety,

35. Stabilisation

35.1. The (homotopy) colimit of an infinite sequence of maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

in the category **LP** is the (homotopy) limit in the category **QCAT** of the corresponding sequence of right adjoints

$$X_0 \xleftarrow{g_0} X_1 \xleftarrow{g_1} X \xleftarrow{g_2} \cdots .$$

An object of this limit L is a pair (x, a), where $x = (x_n)$ is a sequence of objects $x_n \in X_n$ and $a = (a_n)$ is a sequence of equimorphisms $a_n : x_n \simeq g_n(x_{n+1})$. The canonical map $u_0 : X_0 \to L$ has no simple description, but its right adjoint $L \to X$ is the projection $(x, a) \mapsto x_0$. The quasi-category L can also be obtained by localising another locally presentable quasi-category of L' constructed as follows. An object of L' is pair (x, b), where $x = (x_n)$ is a sequence of objects $x_n \in X_n$ and $b = (b_n)$ is a sequence of morphisms $b_n : f_n(x_n) \to x_{n+1}$. The object (x, b) can also be described as a pair $y = (x, a)$, where $x = (x_n)$ is a sequence of objects $x_n \in X_n$ and $a = (a_n)$ is a sequence of morphisms $a_n : x_n) \to g_n(x_{n+1})$. The obvious inclusion $L \subseteq L'$ has a left adjoint $q : L' \to L$ which can be described explicitly by a colimit process using transfinite iteration. If $y = (x, a) \in L'$ let us put $\rho(y) = \rho(x, a) = \rho(x), \rho(a))$, where $\rho(x)_n = g_n(x_{n+1})$ and $g(a)_n = g_n(a_{n+1})$. This defines a map $\rho : L' \to L'$ and the sequence $a_n : x_n \to g_n(x_{n+1})$ defines a morphism $\theta(y) : y \to \rho(y)$ in L' which is natural in y. It is easy to see that we have $\theta \circ \rho = \rho \circ \theta : \rho \to \rho^2$. By iterating ρ transfinitly, we obtain a cocontinuous chain

$$Id \xrightarrow{\theta} \rho \xrightarrow{\theta} \rho^2 \xrightarrow{\theta} \rho^3 \xrightarrow{\theta} \cdots$$

where

$$\rho^\alpha(y) = \varinjlim_{i<\alpha} \rho^i(y).$$

for a limit ordinal α. The chain stabilises enventually and we have

$$q(x) = \varinjlim_{\alpha} \rho^\alpha(x).$$

If directed colimits commute with finite limits in each X_n, then the reflection $q : L' \to L$ is left exact.

35.2. We conjecture that the quasi-category L' is a para-variety (resp. an ∞-topos) if each quasi-category X_n is a para-variety (resp. an ∞-topos) It follows that the quasi-category L is a para-variety (**??**n ∞-topos) in this case.

35.3. Consider the category $End(\mathbf{LP})$ whose objects are the pairs (X, ϕ), where $X \in \mathbf{LP}$ and ϕ is a cocontinuous map $X \to X$, and whose *morphisms* $(X, \phi) \to (Y, \psi)$ are the pairs (f, α), where $f : X \to Y$ is a map and α is an invertible 2-cell $\alpha : f\phi \to \psi f$ in the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

The category $End(\mathbf{LP})$ has the structure of a 2-category, induced by that of $\mathbf{LP}$. Let $Aut(\mathbf{LP})$ be the full sub-category of $End(\mathbf{LP})$ whose objects are the pairs (X, ϕ) with ϕ an equivalence. The inclusion functor $Aut(\mathbf{LP}) \subset End(\mathbf{LP})$ has a (pseudo-) left adjoint

$$S : End(\mathbf{LP}) \to Aut(\mathbf{LP})$$

which associates to an object $(X, \phi) \in End(\mathbf{LP})$, its *stabilisation* $S(X, \phi) = (X', \phi')$. The quasi-category X' is the (homotopy) colimit in $\mathbf{LP}$ of the sequence of quasi-categories

$$X \xrightarrow{\phi} X \xrightarrow{\phi} X \xrightarrow{\phi} \cdots$$

If $\omega : X \to X$ is right adjoint to $\phi : X \to X$, then X' is the (homotopy) limit of the sequence of quasi-categories

$$X \xleftarrow{\omega} X \xleftarrow{\omega} X \xleftarrow{\omega} \cdots$$

An object of X' is an *ω-spectrum*: it is a pair (x, a), where $x = (x_n)$ is a sequence of objects of X and $a = (a_n)$ is a sequence of equimorphisms $a_n : x_n \simeq \omega(x_{n+1})$. The map $\phi' : X' \to X'$ is obtained by putting $\phi'(x, a) = (\sigma(x), \sigma(a))$, where $\sigma(x)_n = x_{n+1}$ and $\sigma(a)_n = a_{n+1}$. Its inverse is the map $\omega' : X' \to X'$ obtained by putting $\omega'(x, a) = (\omega(x), \omega(a))$, where $\omega(x)_n = \omega(x_n)$ and $\omega(a)_n = \omega(a_n)$ for every $n \geq 0$. The canonical map $u : X \to X'$ has no simple description, but its right adjoint $X' \to X$ is the projection $(x, a) \mapsto x_0$. It follows from the conjecture in 35.1 that X' is a para-variety if X is a para-variety.

35.4. If X is a para-variety, then so is the quasi-category $Spec(X)$ of spectra in X. Let us sketch a proof. We can suppose that X is pointed. We then have $Spec(X) = S(X, \Sigma)$, where $\Sigma : X \to X$ is the suspension map. Let us show that $S(X, \Sigma)$ is a para-variety if X is a para-variety (hence proving conjecture 35.2 in this case). A *pre-spectrum* in X is an infinite sequence of pointed objects (x_n) together with an infinite sequence of commutative squares

$$\begin{tikzcd} x_n \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & x_{n+1}, \end{tikzcd}$$

The notion of pre-spectrum is essentially algebraic and finitary. Let us denote by $PSpec$ the algebraic theory of pre-spectra. The quasi-category $PSpec(X) = Mod^{\times}(PSpec, X)$ is a para-variety by 34.4, since X is a para-variety. But the quasi-category $Spec(X)$ is a left exact reflection of $PreSpec(X)$ by 35.1, since directed colimits commute with finite limits in X by 34.6. It is thus a para-variety.

35.5. (Joint work with Georg Biedermann) If X is a para-variety (resp. an ∞-topos), then so is the quasi-category of parametrised spectra in X. Let us sketch the proof. Let us denote by $PSpec$ the cartesian theory of parametrised spectra 30.37. If X is a para-variety (resp. an ∞-topos), let us show that $PSpec(X)$ is a para-variety (resp. an ∞-topos). Let $PPreSpec$ be the cartesian theory of parametrized pre-spectra. An object of $PPreSpec(X)$ is a pre-spectrum in X/b for some object $b \in X$. A pointed object of X/b is an arrow $p : x \to b$ equipped with a section $s : b \to x$. A *pre-spectrum* in X/b is an infinite sequence of pointed objects (x_n, p_n, s_n) together with an infinite sequence of commutative squares

$$\begin{tikzcd} x_n \arrow[r, "p_n"] \arrow[d, "p_n"'] & b \arrow[d, "s_{n+1}"] \\ b \arrow[r, "s_{n+1}"] & x_{n+1}. \end{tikzcd}$$

Clearly, a parametrised pre-spectrum in X is a map $B \to X$, where B is a certain simplicial set. Hence the quasi-category $PPreSpec(X)$ of parametrized pre-spectra in X is of the form X^B for some simplicial set B. It is thus a para-variety (resp. an ∞-topos), since X is a para-variety (resp. an ∞-topos). But the quasi-category $PSpec(X)$ is a left exact reflection of $PPreSpec(X)$ by 35.1, since directed colimits commute with finite limits in X by 34.6. It is thus a para-variety (resp. an ∞-topos).

35.6. Let $PSpec$ be the cartesian theory of parametrised spectra described in 30.37. If X is a para-variety (resp. an ∞-topos), let us show that $PSpec(X)$ is a para-variety (resp. an ∞-topos). Let $PPreSpec$ be the cartesian theory of parametrized pre-spectra. An object of $PPreSpec(X)$ is a pre-spectrum in X/b for some object $b \in X$. A pointed object of X/b is an arrow $p : x \to b$ equipped with a section $s : b \to x$. A *pre-spectrum* in X/b is an infinite sequence of pointed objects (x_n, p_n, s_n)

together with an infinite sequence of commutative squares

$$\begin{tikzcd} x_n \arrow[r, "p_n"] \arrow[d, "p_n"'] & b \arrow[d, "s_{n+1}"] \\ b \arrow[r, "s_{n+1}"] & x_{n+1}. \end{tikzcd}$$

Clearly, a parametrised pre-spectrum in X is a map $B \to X$, where B is a certain simplicial set. Hence the quasi-category $PPreSpec(X)$ of parametrized pre-spectra in X is of the form X^B for some simplicial set B. It is thus a para-variety (resp. an ∞-topos), since X is a para-variety (resp. an ∞-topos). But the quasi-category $PSpec(X)$ is a left exact reflection of $PPreSpec(X)$ by 35.1, since directed colimits commute with finite limits in X by 34.6. It is thus a para-variety (resp. an ∞-topos).

36. Descent theory

36.1. Let X be a cartesian quasi-category. If $C : \Delta^o \to X$ is a category object, we say that a functor $p : E \to C$ in $Cat(X)$ is a *left fibration* if the naturality square

$$\begin{tikzcd} E_1 \arrow[r, "s"] \arrow[d, "p_1"'] & E_0 \arrow[d, "p_0"] \\ C_1 \arrow[r, "s"] & C_0 \end{tikzcd}$$

is cartesian, where s is the source map. The notion of right fibration is defined dually by using the target map. The two notions coincide when C is a groupoid. We shall denote by X^C the full simplicial subset of $Cat(X)/C$ spanned by the left fibrations $E \to C$. If $f : C \to D$ is a functor in $Cat(X)$, then the pullback of a left fibration $E \to D$ along f is a left fibration $f^*(E) \to C$. This defines the *base change map*

$$f^* : X^D \to X^C.$$

36.2. Let X be a cartesian quasi-category. There is then a map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ its equivalence groupoid $Eq(u)$ as defined in 39.1. Let us describe the map explicitely. Let Δ_+ be the category of finite ordinals, empty or not. The category I is isomorphic to the full subcategory of Δ_+ spanned by the ordinals $0 = \emptyset$ and $1 = [0]$. If i is the inclusion $I \subset \Delta_+$, then the functor

$$i^* : [\Delta_+^o, X] \to [I^o, X]$$

has a right adjoint i_* which associates to an arrow $u : a \to b$ an augmented simplicial object $Eq_+(u)$. The groupoid $Eq(u)$ is obtained by restricting $Eq_+(u)$ along the inclusion $\Delta \subset \Delta_+$. If $u : a \to b$ is an arrow in X, then the base change map $u^* : X/b \to X/a$ admits a lifting $\tilde{u}^* : X/b \to X^{Eq(u)}$,

$$\begin{tikzcd} & X^{Eq(u)} \arrow[d, "p"] \\ X/b \arrow[ur, "\tilde{u}^*"] \arrow[r, "u^*"] & X/a, \end{tikzcd}$$

where p is the forgeful map. We shall say that $\tilde{u}^*$ is the *lifted base change map*. It associates to an arrow $e \to b$ the arrow $a \times_b e \to a$

$$\begin{tikzcd} a \times_b e \arrow[r] \arrow[d] & e \arrow[d] \\ a \arrow[r, "u"] & b \end{tikzcd}$$

equipped with a natural action of the groupoid $Eq(u)$. Let us describe $\tilde{u}^*$ explicitly. The map $Eq : X^I \to Gpd(X)$ takes the morphism $(u, 1_b) : u \to 1_b$ of X^I

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "u"'] & b \arrow[d, "1_b"] \\ b \arrow[r, "1_b"] & b \end{tikzcd}$$

to a functor $q : Eq(u) \to Eq(1_b) = Sk^0(b)$. We thus obtain a base change map

$$q^* : X^{Sk^0(b)} \to X^{Eq(u)}$$

by 36.1. But the quasi-category $X^{Sk^0(b)}$ is equivalent to the quasi-category X/b. The map $\tilde{u}^*$ is obtained by composing this equivalence with q^*. An arrow $u : a \to b$ is said to be a *descent morphism* if the lifted base change map $\tilde{u}^*$ is an equivalence of quasi-categories.

36.3. If $u : 1 \to b$ is a pointed object in a cartesian quasi-category X, then the groupoid $Eq(u)$ is the *loop group* $\Omega_u(b)$ defined in 39.1. In this case, the lifted base change map

$$\tilde{u}^* : X/b \to X^{\Omega_u(b)}$$

associates to an arrow $e \to b$ its fiber $e(u) = u^*(e)$ equipped with the natural action (say on the right) of the group $\Omega_u(b)$.

36.4. Every surjection in $\mathbf{U}$ is a descent morphism. This is true more generally for any surjection in a para-variety.

36.5. Let X be a cartesian quasi-category. If A is a simplicial set, then the projection $X^{A\star 1} \to X$ defined from the inclusion $1 \subseteq A \star 1$ is a Grothendieck fibration. The base change of a cone $c : A \star 1 \to X$ along an arrow $u : e \to c(1)$ is a cone $u^*(c) : A \star 1 \to X$ with $u^*(c)(1) = e$. By construction, we have

$$u^*(c)(a) = e \times_{c(1)} c(a)$$

for every $a \in A$. We say that the cone c is *stably coexact* if the cone $u^*(c)$ is coexact for any arrow $u : e \to c(1)$. In this case, the colimit $c(1)$ of the diagram $c \mid A$ is said to be *stable under base change*

36.6. Let X be a cartesian quasi-category and A be a simplicial set. We say that a natural transformation $\alpha : f \to g : A \to X$ is *cartesian* if the naturality square

$$\begin{tikzcd} f(a) \arrow[r, "f(u)"] \arrow[d] & f(b) \arrow[d] \\ g(a) \arrow[r, "g(u)"] & g(b) \end{tikzcd}$$

is cartesian for every arrow $u : a \to b$ in A. This notion only depends on the homotopy class of α in the simplicial set $X^A(f, g)$. It is thus a property of the 2-cell $[\alpha] : f \to g$. The set of cartesian natural transformations is invariant under equimorphism in X^A. Moreover, it is closed under composition, base changes and it has the left cancellation property. We call a cartesian natural transformation $\alpha : f \to g$ a *gluing datum over* $g : A \to X$. We shall denote by $Glue(g)$ the full simplicial subset of X^A/g spanned by the gluing data over g. When the category $\tau_1 A$ is a groupoid, every natural transformation $\alpha : f \to g : A \to X$ is cartesian. Thus, $Glue(g) = X^A/g$ in this case.

36.7. A functor $f : C \to D$ in $Cat(X)$ is a natural transformation $f : C \to D : \Delta^o \to X$. The natural transformation is cartesian iff it is both a left and a right fibration.

36.8. Let X be a cartesian quasi-category. If $u : A \to B$ is a map of simplicial sets, then the map $X^u : X^B \to X^A$ takes a cartesian natural transformation to a cartesian natural transformation. It thus induces a map

$$u^* : Glue(g) \to Glue(gu)$$

for any diagram $g : B \to X$. We call u^* the *restriction along* u. The map u^* is an equivalence of quasi-categories when u is final. In particular, it is an equivalence if u is a weak categorical equivalence. In particualr, if $c : A \star 1 \to X$, then the restriction map

$$Glue(c) \to X/c(1)$$

is an equivalence of quasi-categories, since the inclusion $1 \subseteq A \star 1$ is final. By composing the inverse equivalence with the restriction

$$i^* : Glue(c) \to Glue(ci)$$

along the inclusion $i : A \subseteq A \star 1$ we obtain a map

$$X/c(1) \to Glue(ci)$$

called the *spread map.* We say that a diagram $d : A \to X$ is a *descent diagram* if it has a colimit b and the spread map

$$\sigma : X/b \to Glue(d)$$

is an equivalence of quasi-categories. In which case, the inverse of σ associates to a cartesian morphism $f \to g$ its colimit

$$\varinjlim_{a \in A} f(a) \to \varinjlim_{a \in A} g(a) = b.$$

The colimit of an descent diagram is stable under base change.

36.9. Every diagram in $\mathbf{U}$ is a descent diagram (and every colimit is stable under base change). Let us sketch a proof using the correspondance $f \mapsto El(f)$ of 16.8. If B is a simplicial set, let us denote by $\mathcal{K}(B)$ the full sub-category of $\mathbf{S}/B$ whose objects are the Kan fibrations $X \to B$. The category $\mathcal{K}(B)$ is enriched over Kan complexes. Moreover, if $i : A \to B$ is a weak homotopy equivalence, then the map $i^* : \mathcal{K}(B) \to \mathcal{K}(A)$ is a Dwyer-Kan equivalence. If $g : A \to \mathbf{U}$ is a diagram, let us put $G = El(g)$ and let us choose a weak homotopy equivalence $i : G \subseteq Y$ with Y a Kan complex. Then the object $Y \in \mathbf{U}$ is the colimit of g. A natural transformation $\alpha : f \to g : A \to \mathbf{U}$ is cartesian iff the map $El(\alpha) : El(f) \to El(g) = G$ is a *homotopy covering* in the sense of 11.22. It follows that the quasi-category $Glue(g)$ is equivalent to the coherent nerve of the simplicial category $\mathcal{K}(G)$. Moreover, the spread map $\mathbf{U}/Y \to Glue(g)$ is induced by the functor $i^* : \mathcal{K}(Y) \to \mathcal{K}(G)$. It is thus an equivalence of quasi-categories, since i is a weak homotopy equivalence.

36.10. Let $X \subseteq Y$ be a left exact reflection of a cartesian quasi-category Y. Then a diagram $g : A \to X$ which is a descent diagram in Y is also a descent diagram in X. Let us sketch a proof. Let i be the inclusion $X \subseteq Y$. The composite $ig : A \to Y$ is a descent diagram by assumption. If b is the colimit of the ig, then $r(b)$ is the colimit of g in X. Consider the diagram

$$\begin{tikzcd} X/r(b) \arrow[rr, "\sigma'"] \arrow[d, "i_0"'] & & Glue(g) \arrow[d, "i_1"] \\ Y/r(b) \arrow[r, "p^*"] & Y/b \arrow[r, "\sigma"] & Glue(ig), \end{tikzcd}$$

where i_0 and i_1 are induced by $i : X \subseteq Y$, where σ and σ' are the spread maps, and where p^* is base change along the canonical arrow $p : b \to r(b)$. It is easy to see that the diagram commutes up to a canonical isomorphism. The map $q : Y/b \to X/r(b)$ induced by r is left adjoint to the composite $p^*i_0 : X/r(b) \to Y/r(b) \to Y/b$. Moreover, the counit of the adjunction $q \vdash p^*i_0$ is invertible by the left exactness of r. Thus. p^*i_0 is fully faithful. It follows that $i_1\sigma' = \sigma p^* i_0$ is fully faithful, since σ is an equivalence by assumption. Thus, σ' is fully faithful, since i_1 is fully faithful. It remains to show that σ' is essentially surjective. Let $\alpha : f \to g$ be an object of $Glue(g)$ and $u : a \to b$ be the colimit of α in Y. Then the canonical square

$$\begin{tikzcd} f(a) \arrow[r] \arrow[d, "\alpha(a)"'] & a \arrow[d, "u"] \\ g(a) \arrow[r] & b \end{tikzcd}$$

is a pullback for every $a \in A$, since g is a descent diagram in Y. Hence the square

$$\begin{tikzcd} f(a) \arrow[r] \arrow[d, "\alpha(a)"'] & r(a) \arrow[d, "r(u)"] \\ g(a) \arrow[r] & r(b), \end{tikzcd}$$

is also a pullback in X, since r is left exact. This proves that σ' is essentially surjective.

36.11. Every diagram in an ∞-topos is a descent diagram (and every colimit is stable under base changes).

36.12. Let Y be a cartesian quasi-category, $X \subseteq Y$ be a full reflective sub quasi-category and A be a simplicial set. Suppose that X admits A-indexed colimits and that these colimits are preserved by the inclusion $i : X \subseteq Y$. Then a diagram $g : A \to X$ which is a descent diagram in Y, is a descent diagram in X. Let us sketch a proof. If b is the colimit of g in X, then $i(b)$ is the colimit of ig in Y, since

i preserves A-indexed colimits. Consider the square

$$\begin{tikzcd} X/b \arrow[r] \arrow[d, "i_0"'] & Glue(g) \arrow[d, "i_1"] \\ Y/i(b) \arrow[r] & Glue(ig), \end{tikzcd}$$

where the maps i_0 and i_1 are induced by i and where the horizontal maps are the spread maps. The square commutes up to a canonical isomorphism, since i preserves finite limits. The maps i_0 and i_1 are fully faithful, since i is fully faithful. The bottom spread map is an equivalence, since ig is a descent diagram by assumption. It follows that the top spread map is fully faithful. Hence it suffices to show that it is essentially surjective. Let $\alpha : f \to g$ be an object of $Glue(g)$. If a is the colimit of f in X, then $i(a)$ is the colimit of if in Y, since i preserves A-indexed colimits by assumption. The colimit of α is an arrow $u : a \to b$. The image by the bottom spread map of the object $i(u)$ of $Y/i(b)$ is quasi-isomorphic to the object $i(\alpha) : if \to ig$ of $Glue(ig)$, since ig is a descent diagram by assumption. Hence the image of u by the top spread map is quasi-isomorphic to the object $\alpha : f \to g$ of $Glue(g)$, since i_1 is fully faithful and the square commutes up to a natural isomorphism.

36.13. Every sifted diagram in a para-variety is a descent diagram (and every sifted colimit is stable under base changes). In particular, every groupoid is a descent diagram. Let us sketch a proof. We first consider the case of a sifted diagram $g : A \to Y$ with values in an ω-variety Y. The quasi-category Y is equivalent to a quasi-category $Mod(T)$ for some algebraic theory T by 33.9. The inclusion $i : Mod(T) \subseteq \mathbf{U}^T$ is reflective and it preserves sifted colimits by 33.14. The diagram $ig : A \to \mathbf{U}^T$ is a descent diagram by 36.11. Hence also g by 36.12. We can now consider the case of a sifted diagram $g : A \to X$ with values in a pseudo-ω-variety X. The quasi-category X is a left exact reflection of an ω-variety Y. If i is the inclusion $X \subseteq Y$, then the composite $ig : A \to Y$ is descent diagram in Y by the first part of the proof. It then follows from 36.10 that g is a descent diagram.

37. Exact quasi-categories

37.1. Let X be a cartesian quasi-category. There is then a map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ its equivalence groupoid $Eq(u)$. If a groupoid $C : \Delta^o \to X$ has a colimit BC and $p : C_0 \to BC$ is the canonical morphism, then there is a canonical functor $C \to Eq(p)$. We say that a groupoid C is *effective* if it has a colimit $p : C_0 \to BC$ and the canonical functor $C \to Eq(p)$ is invertible.

37.2. Let X be a cartesian quasi-category. Recall from 36.2 that an arrow $u : a \to b$ in X is said to be a *descent morphism* if the lifted base change map

$$\tilde{u}^* : X/b \to X^{Eq(u)}$$

is an equivalence of quasi-categories.

37.3. Recall that a cartesian quasi-category X is said to be *regular* if it admits surjection-mono factorisations stable under base changes. Recall that a regular category C is said to be *exact*, but we shall say *1-exact*, if every equivalence relation is effective. It follows from this condition that every surjection is a descent morphism.

37.4. We say that a regular quasi-category X is *exact* if it satisfies the following two conditions:

- Every surjection is a descent morphism;
- Every groupoid is effective.

37.5. The quasi-category $\mathbf{U}$ is exact. If a quasi-category X is exact, then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$, the quasi-category X^A for any simplicial set A and the quasi-category $Prod(T, X)$ for any algebraic theory T. A variety of homotopy algebras is exact. A left exact reflection of an exact quasi-category is exact. A para-variety is exact.

37.6. We say that a map $X \to Y$ between regular quasi-categories is *exact* if it is left exact and preserves surjections.

37.7. Let $u : a \to b$ be an arrow in an exact quasi-category X. Then the base change map $u^* : X/b \to X/a$ is exact. Moreover, u^* is conservative if u is surjective.

37.8. An exact quasi-category X admits n-factorisations for every $n \geq 0$. An object a is connected iff the arrows $a \to 1$ and $a \to a \times a$ are surjective. An arrow $a \to b$ is 0-connected iff it is surjective and the diagonal $a \to a \times_b a$ is surjective. If $n > 0$, an arrow $a \to b$ is n-connected iff it is surjective and the diagonal $a \to a \times_b a$ is $(n-1)$-connected. If $a \to e \to b$ is the n-factorisation of an arrow $a \to b$, then $a \to a \times_e a \to a \times_b a$ is the $(n-1)$-factorisation of the arrow $a \to a \times_b a$. An exact map $f : X \to Y$ between exact quasi-categories preserves the n-factorisations for every $n \geq 0$.

37.9. Let $u : a \to b$ be a surjection in an exact quasi-category X. Then the lifted base change map

$$\tilde{u}^* : X/b \to X^{Eq(u)}$$

of 36.2 is an equivalence of quasi-categories. A pointed object $u : 1 \to b$ is connected iff the map u is surjective. In this case the map

$$u^* : X/b \to X^{\Omega_u(b)}$$

defined in 36.3 is an equivalence of quasi-categories.

37.10. Let X be a cartesian quasi-category. We shall say that a functor $f : C \to D$ in $Cat(X)$ is *fully faithful* if the square

$$\begin{tikzcd}
C_1 \arrow[r, "f_1"] \arrow[d, "{(s,t)}"'] & D_1 \arrow[d, "{(s,t)}"] \\
C_0 \times C_0 \arrow[r, "f_0 \times f_0"] & D_0 \times D_0
\end{tikzcd}$$

is cartesian. For example, if $u : a \to b$ is an arrow in X, then the canonical functor $Eq(u) \to b$ is a fully faithful, where b denotes the category $Sk^0(b)$.

37.11. Let X be a regular quasi-category. We say that a functor $f : C \to D$ in $Gpd(X)$ is *essentially surjective* if the morphism tp_1 in the square

$$\begin{tikzcd} & D_1 \times_{D_0} C_0 \arrow[r, "p_2"] \arrow[dl, "tp_1"'] \arrow[d, "p_1"'] & C_0 \arrow[d, "f_0"] \\ D_0 & D_1 \arrow[l, "t"] \arrow[r, "s"] & D_0 \end{tikzcd}$$

is surjective. Let $J : Cat(X) \to Gpd(X)$ be the right adjoint to the inclusion $Gpd(X) \subseteq Cat(X)$. We say that a functor $f : C \to D$ in $Cat(X)$ is *essentially surjective* if the functor $J(f) : J(C) \to J(D)$ is essentially surjective. We say that f is a *weak equivalence* if it is fully faithful and essentially surjective. For example, if $u : a \to b$ is a surjection in X, then the canonical functor $Eq(u) \to b$ is a weak equivalence.

37.12. Let X be a cartesian quasi-category. We say that a functor $f : C \to D$ in $Cat(X)$ is a *Morita equivalence* if the induced map $f^* : X^D \to X^C$ defined in 36.1 is an equivalence of quasi-categories. If X is an exact quasi-category, then every weak equivalence $f : C \to D$ is a Morita equivalence, and the converse is true if C and D are groupoids.

37.13. Let X be an exact quasi-category. Then the map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ its equivalence groupoid $Eq(u)$ has left adjoint $B : Gpd(X) \to X^I$ which associates to a groupoid C its "quotient" or "classifying space" BC equipped with the canonical map $C_0 \to BC$. Let us denote by $Surj(X)$ the full simplicial subset of X^I spanned by the surjections. The map B is fully faithful and its essential image is equal to $Surj(X)$. Hence the adjoint pair $B \vdash Eq$ induces an equivalence of quasi-categories

$$B : Gpd(X) \leftrightarrow Surj(X) : Eq.$$

37.14. The canonical map $Ob : Gpd(X) \to X$ is a Grothendieck fibration; we denote its fiber at $a \in X$ by $Gpd(X, a)$. An object of the quasi-category $Gpd(X, a)$ is a groupoid $C \in Gpd(X)$ with $C_0 = a$. The source map $s : X^I \to X$ is a Grothendieck fibration; its fiber at $a \in X$ is the quasi-category $a\backslash\!\!\backslash X$, which is equivalent to the quasi-category $a\backslash X$. The adjoint pair $B : Gpd(X) \leftrightarrow X^I : Eq$ induces an adjoint pair

$$B : Gpd(X, a) \leftrightarrow a\backslash X : Eq$$

for each object $a \in A$. Let us denote by $Surj(a, X)$ the full simplicial subset of $a\backslash X$ spanned by the surjection $a \to x$. Then the equivalence above induces an equivalence

$$B : Gpd(X, a) \leftrightarrow Surj(a, X) : Eq.$$

A groupoid in $Gpd(X, 1)$ is a group object in X. Hence the adjoint pair above induces an adjoint pair

$$B : Grp(X) \leftrightarrow 1\backslash X : \Omega,$$

where Ω associates to pointed objec $1 \to b$ its "loop group" $\Omega(b)$. An object of $Surj(1, X)$ is a "pointed connected" object of X. The adjoint pair above induces an equivalence of quasi-categories

$$B : Grp(X) \leftrightarrow Surj(1, X) : \Omega.$$

37.15. Let X be an exact quasi-category. If we iterate the adjoint pair $B : Gpd(X) \leftrightarrow X^I : Eq$ we obtain an adjoint pair

$$B^n : Gpd^n(X) \leftrightarrow X^{I^n} : Eq^n$$

for each $n \geq 1$, where $Gpd^n(X)$ is the quasi-category of n-fold groupoids in X. Let us denote by $Surj^n(X)$ the full simplicial subset of X^{I^n} spanned by the cubes of surjections $I^n \to X$. The map B^n is fully faithful and its essential image is equal to $Surj^n(X)$. Hence the adjoint pair $B \vdash Eq$ induces an equivalence of quasi-categories

$$B^n : Gpd^n(X) \leftrightarrow Surj^n(X) : Eq^n.$$

37.16. Let X be a pointed exact quasi-category. Then an object $x \in X$ is connected iff the morphism $0 \to x$ is surjective. More generally, an object $x \in X$ is n-connected iff the morphism $0 \to x$ is $(n-1)$-connected. If $CO(X)$ denotes the quasi-category of connected objects in X, then we have an equivalence of quasi-categories

$$B : Grp(X) \leftrightarrow CO(X) : \Omega$$

by 37.14. Hence the quasi-category $CO(X)$ is exact, since the quasi-category $Grp(X)$ is exact. A morphism in $CO(X)$ is n-connected iff it is $(n+1)$ connected in X. Similarly, a morphism in $CO(X)$ is a n-cover iff it is a $(n+1)$ cover in X. Let us put $CO^{n+1}(X) = CO(CO^n(X))$ for every $n \geq 1$. This defines a decreasing chain

$$X \supseteq CO(X) \supseteq CO^2(X) \supseteq \cdots .$$

An object $x \in X$ belongs to $CO^n(X)$ iff x is $(n-1)$-connected. Let $Grp^n(X)$ be the quasi-category of n-fold groups in X. By iterating the equivalence above we obtain an equivalence

$$B^n : Grp^n(X) \leftrightarrow CO^n(X) : \Omega^n$$

for every $n \geq 0$.

37.17. Let X be a cartesian quasi-category. We say that a groupoid $C : \Delta^o \to X$ is *n-truncated* if the morphism $C_1 \to C_0 \times C_0$ is a $(n-1)$-cover. An object $b \in X$ is a *n-object* iff the groupoid $Sk^0(b)$ is a n-truncated.

37.18. Recall that a cartesian quasi-category X is n-truncated iff every object in X is a $(n-1)$ object.

37.19. If $n \geq 1$, we say that a n-truncated regular quasi-category X is *n-exact* if it satisfies the following two conditions:

- Every surjection is a descent morphism;
- Every $(n-1)$-truncated groupoid is effective.

37.20. The quasi-category $\mathbf{U}(n)$ is n-exact. If a quasi-category X is exact, then the quasi-category $X(n)$ of $(n-1)$-objects in X is n-exact. If a quasi-category X is n-exact, then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$, and the quasi-category $Prod(T, X)$ for any algebraic theory T. A n-variety of homotopy algebras is n-exact. A left exact reflection of a n-exact quasi-category is n-exact. A para-n-variety is n-exact.

37.21. If X is an exact quasi-category, we denote by $X(n)$ the full simplicial subset of X spanned by the $(n-1)$-objects of X. The quasi-category $X(n)$ is n-exact. The inclusion $X(n) \subseteq X$ has a left adjoint

$$\pi_{(n-1)} : X \to X(n)$$

which preserves finite products. We call an arrow $f : a \to b$ in X a *n-equivalence* if the arrow $\pi_{(n)}(f) : \pi_{(n)}(a) \to \pi_{(n)}(b)$ is invertible. Every n-connected arrow is a n-equivalence and every n-equivalence is $(n-1)$-connected. If $f : a \to b$ is an arrow in X, then the map $f^*(n) : (X/b)(n) \to (X/a)(n)$ induced by the map $f^* : X/b \to X/a$ has a left adjoint $f_!(n)$. Moreover, the adjoint pair

$$f_!(n) : (X/a)(n) \leftrightarrow (X/b)(n) : f^*(n)$$

is an equivalence of quasi-categories when f is a n-equivalence. It follows that the quasi-category $(X/b)(n)$ is equivalent to the quasi-category $(X/\pi^{(n)}(b))(n)$ for every object $b \in X$. If $n = 0$, this means that the poset of subobjects of b is isomorphic to the poset of subobjects of $\pi_0(b) = \pi_{(0)}(b)$. If $n = 1$, this means that the (quasi-) category of 0-objects in X/b is equivalent to category of 0-objects in $X/\pi_{(1)}(b)$.

37.22. We say that an exact quasi-category X is *n-generated* if for every object $b \in X$ there exists a surjection $a \to b$ whith a a n-object. The quasi-category $\mathbf{U}$ is 0-generated. The quasi-category $\mathbf{U}^A$ is n-generated for any $(n+1)$-truncated quasi-category A. More generally, the quasi-category $Mod(T)$ is n-generated for any $(n+1)$-truncated algebraic theory T.

37.23. Let X be an exact quasi-category. We say that an arrow in X is *∞-connected* if it is n-connected for every $n \geq 0$. An arrow $f \in X$ is ∞-connected iff it is a n-equivalence for every $n \geq 0$. We say that X is *t-complete* if every ∞-connected arrow is invertible.

37.24. The exact quasi-category $\mathbf{U}$ is t-complete. If an exact quasi-category X is t-complete, then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$, and the quasi-category $Mod(T, X)$ for any algebraic theory T.

37.25. Let X be a t-complete exact quasi-categories. If Y is an exact quasi-category, then an exact map $X \to Y$ is conservative iff the induced map $X(1) \to Y(1)$ is conservative. Every t-complete exact quasi-category X admits a conservative exact map $X \to \mathbf{U}$.

37.26. Let $\mathbf{Ex}$ be the category of exact categories and exact maps. The category $\mathbf{Ex}$ has the structure of a 2-category induced by the 2-category structure of the category of simplicial sets. If $\mathbf{TEx}$ is the full sub-quasi-category of $\mathbf{Ex}$ spanned by the t-complete exact quasi-categories, then the inclusion $\mathbf{TEx} \subset \mathbf{Ex}$ has a left adjoint which associates to an exact quasi-category X its *t-completion* ${}^tX$. The quasi-category ${}^tX$ can be constructed as a localisation $L(X, \Sigma)$, where Σ is the set of ∞-connected arrows in X.

37.27. Let X be an exact quasi-category. If $RCat(X)$ is the quasi-category of reduced categories in X, then the inclusion $RCat(X) \subseteq Cat(X)$ has a left adjoint

$$R : Cat(X) \to RCat(X)$$

which associates to a category $C \in Cat(X)$ its *reduction* $R(C)$. When C is a groupoid, we have $R(C) = B(C)$. In general, we have a a pushout square in $Cat(X)$,

$$\begin{tikzcd} J(C) \arrow[r] \arrow[d] & C \arrow[d] \\ B(J(C)) \arrow[r] & R(C), \end{tikzcd}$$

where $J(C)$ is the groupoid of isomorphisms of a category C. The simplicial object $R(C)$ can be constructed by putting $(RC)_n = B(J(C^{[n]}))$ for every $n \geq 0$, where $C^{[n]}$ is the (internal) category of functor $[n] \to C$. The canonical map $C \to R(C)$ is an equivalence of categories, hence it is also a Morita equivalence. A functor $f : C \to D$ in $Cat(X)$ is an equivalence iff the functor $R(f) : R(C) \to R(D)$ is a isomorphism in $RCat(X)$. If $W \subseteq Cat(X)$ is the set of equivalences, then the induced map

$$L(Cat(X), W) \to RCat(X)$$

is an equivalence of quasi-categories.

37.28. Let X be an exact pointed quasi-category. The quasi-category of n-connected $2n$-objects in X is equivalent to the quasi-category of $(n-1)$-objects in $CO^{n+1}(X)$, which is equivalent to the quasi-category of $(n-1)$-objects in $Grp^{n+1}(X)$. In other words, the quasi-category of n-connected $2n$-objects in X is equivalent to the quasi-category

$$Mod(OB(n-1), Grp^{n+1}(X)) = Mod(OB(n-1) \odot_c Grp^{n+1}, X).$$

The suspension theorem of Freudenthal implies that the cartesian theory $OB(n - 1) \odot_c Grp^{n+1}$ is additive by 40.30. It follows that a n-connected $2n$-object in X has the structure of an infinite loop space.

37.29. Let X be an exact quasi-category. If G is a group object in X, we say that an object $E \in X^G$ with group action $a : G \times E \to E$ is a *G-torsor* if the map $E \to 1$ is surjective and the map $(p_1, a) : G \times E \to E \times E$ is invertible. If $n > 0$, every n-connected $2n$-object $a \in X$ is a torsor over an infinite loop space $J(a)$, called the *Jacobian* of a. Let us sketch the construction of $J(a)$. The object $a' = a^*(a) = a \times a \in X/a$ is pointed by the diagonal $a \to a \times a$. It is a n-connected $2n$-object. Hence it has the structure of an infinite loop space $J(a')$ by 37.28. The map $a \to 1$ is surjective, since a is n-connected. It is thus a descent morphism. The infinite loop space $J(a')$ is equipped with a descent datum, since the structure is canonical. Descent theory implies that there is an infinite loop space $J(a)$ such that $a^*J(a) = J(a')$. From the action of $J(a')$ on a' $(= J(a'))$ we obtain an action of $J(a)$ on a. The object a is a torsor over $J(a)$, since the object a' is a torsor over $J(a')$.

37.30. Let X be an exact quasi-category. An *Eilenberg-MacLane n-space* is defined to be a pointed n-object $a \in A$ which is $(n-1)$-connected. The equivalence $B : Grp(X) \leftrightarrow Surj(1, X) : \Omega$ of 37.14 induces an equivalence

$$K(-,1) : Grp(X)(1) \leftrightarrow EM_1 : \Omega$$

between the category of discrete group objects in X and the quasi-category of Eilenberg-MacLane 1-spaces. If $n > 1$, every Eilenberg-MacLane n-spaces has the structure of an infinite loop space. The equivalence $B^n : Grp^n(X) \leftrightarrow CO^n(1\backslash X) : \Omega^n$ of 37.16 induces an equivalence

$$K(-,n) : Ab(X)(1) \leftrightarrow EM_n : \Omega^n$$

between the category of abelian discrete group objects in X and the quasi-category of Eilenberg-MacLane n-spaces. An *Eilenberg-MacLane n-gerbe* is defined to be a n-object which is $(n-1)$-connected. If $n > 1$, every Eilenberg-MacLane n-gerbe $a \in A$ is naturally a torsor over its Jacobian $J(a)$ which is an Eilenberg-MacLane n-space.

38. Meta-stable quasi-categories

38.1. We say that an exact quasi-category X is *meta-stable* if every object in X is ∞-connected. A cartesian quasi-category X is meta-stable iff if it satisfies the following two conditions:

- Every morphism is a descent morphism;
- Every groupoid is effective.

38.2. The sub-quasi-category of ∞-connected objects in an exact quasi-category is meta-stable. We shall see in 31.30 that the quasi-category of spectra is meta-stable. In a meta-stable quasi-category, every monomorphism is invertible and every morphism is surjective.

38.3. If a quasi-category X is meta-stable then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$, the quasi-category X^A for any simplicial set A, and the quasi-category $Prod(T, X)$ for any algebraic theory T. A left exact reflection of a meta-stable quasi-category is meta-stable.

38.4. Let $u : a \to b$ be an arrow in a meta-stable quasi-category X. Then the lifted base change map

$$\tilde{u}^* : X/b \to X^{Eq(u)}$$

of 36.2 is an equivalence of quasi-categories. In particular, if $u : 1 \to b$ is a pointed object, then the map

$$\tilde{u}^* : X/b \to X^{\Omega_u(b)}$$

defined in 36.3 is an equivalence of quasi-categories.

38.5. Let X be a meta-stable quasi-category. Then the map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ the equivalence groupoid $Eq(u)$ is invertible. We thus have an equivalence of quasi-categories

$$B : Gpd(X) \leftrightarrow X^I : Eq.$$

The equivalence can be iterated as in 37.15. It yields an equivalence of quasi-categories

$$B^n : Gpd^n(X) \leftrightarrow X^{I^n} : Eq^n$$

for each $n \geq 1$.

38.6. Let X be a meta-stable quasi-category. Then the equivalence

$$B : Gpd(X) \leftrightarrow X^I : Eq.$$

induces an equivalence

$$B : Gpd(X, a) \leftrightarrow a\backslash X : Eq$$

for each object $a \in A$, where $Gpd(X, a)$ is the quasi-category of groupoids $C \in Gpd(X)$ with $C_0 = a$. In particular, it induces an equivalence

$$B : Grp(X) \leftrightarrow 1\backslash X : \Omega,$$

where $Grp(X)$ is the quasi-category of groups in X. By iterating, we obtain an equivalence

$$B^n : Grp^n(X) \leftrightarrow 1\backslash X : \Omega^n,$$

for each $n \geq 1$.

38.7. Let **Ex** be the category of exact categories and exact maps. If **MEx** is the full sub-quasi-category of **Ex** spanned by the meta stable quasi-categories, then the inclusion **MEx** $\subset$ **Ex** has a right adjoint which associates to an exact quasi-category X its full sub-quasi-category of meta-stable objects.

39. Fiber sequences

39.1. Let Grp be the theory of groups and $Point = OB'_c$ be the cartesian theory of a pointed object. Consider the interpretation $u : Grp \to Point$ defined by the loop group of the generic pointed object. The map

$$u_! : Mod(Grp) \to Mod(Point) = Mod^{\times}(OB') = 1\backslash \mathbf{U}$$

takes a group G to its classifying space BG. It induces an equivalence between $Mod(Grp)$ and the full sub-quasicategory of pointed connected objects in $1\backslash\mathbf{U}$. It is thus fully faithful. Hence the morphism of theories $u : Grp \to Point$ is fully faithful. More generally, let Gpd be the cartesian theory of groupoids and Map_c be the cartesian theory of maps. Consider the interpretation $v : Gpd \to Map_c$ defined by the equivalence groupoid of the generic map. Then the map

$$v_! : Mod(Gpd) \to Mod(Map_c) = Mod^{\times}(Map) = \mathbf{U}^I$$

takes a groupoid C to its classifying space BC equipped with the map $C_0 \to BC$. It induces an equivalence between $Mod(Gpd)$ and the full sub-quasicategory of $\mathbf{U}^I$ spanned by the surjections It is thus fully faithful. Hence the morphism of cartesian theories $v : Gpd \to Map_c$ is fully faithful.

39.2. Recall that a *null object* in a quasi-category X is an object $0 \in X$ which is both initial and final. The homotopy category of a quasi-category with nul object is pointed. In a quasi-category with nul object X we say that sequence of two arrows $a \to b \to c$ is *null* if its composite is null in hoX. A *fiber sequence* is a null sequence $a \to b \to c$ fitting in a cartesian square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ 0 \arrow[r] & c. \end{tikzcd}$$

The notion of cofiber sequence is defined dually.

39.3. Let X be a cartesian quasi-category. Recall from 39.1 that the equivalence groupoid of a pointed object $u : 1 \to b$ is the *loop group* $\Omega_u(b)$. Consider the map

$$u^* : X/b \to X^{\Omega_u(b)}$$

which associates to an arrow $p : e \to b$ its fiber $u^*(e)$ equipped with a natural action (say on the right) of the group $\Omega_u(b)$. In the special case where $p = u : 1 \to b$, this gives the natural right action of $\Omega_u(b)$ on itself. If $l : e' \to e$ is an arrow in X/b, then the arrow $u^*(l) : u^*(e') \to u^*(e)$ respects the right action by $\Omega_u(b)$. Suppose that we have a base point $v : 1 \to e$ over the base point $u : 1 \to b$. Then the arrow $\partial = u^*(v) : \Omega_u(b) \to e(u)$ respects the right action by $\Omega_u(b)$. The top square of the following commutative diagram is cartesian, since the bottom square and the boundary rectangle are cartesians,

$$\begin{tikzcd}
\Omega_u(b) \arrow[r] \arrow[d, "\partial"'] & 1 \arrow[d, "v"] \\
u^*(e) \arrow[r, "i"] \arrow[d] & e \arrow[d, "p"] \\
1 \arrow[r, "u"] & b.
\end{tikzcd}$$

Hence the arrow $\partial : \Omega_u(b) \to e(u)$ is the fiber at v of the arrow $u^*(e) \to e$. The base point $v : 1 \to e$ lifts naturally as a base point $w : 1 \to u^*(e)$. Let us show that the arrow $\Omega(p) : \Omega_v(e) \to \Omega_u(b)$ is the fiber at w of the arrow ∂. For this, it suffices to show that we have a cartesian square

$$\begin{tikzcd}
\Omega_v(e) \arrow[r, "\Omega(p)"] \arrow[d] & \Omega_u(b) \arrow[d, "\partial"] \\
1 \arrow[r, "w"] & u^*(e)
\end{tikzcd}$$

By working in the quasi-category $Y = X/b$, we can suppose that $b = 1$, since the canonical map $X/b \to X$ preserves pullbacks. For clarity, we shall use a magnifying glass by denoting the objects of $Y = X/b$ by capital letters. The base point $u : 1 \to b$ defines an object $T \in Y$ and the arrow $p : e \to b$ an object $E \in Y$. The base point $v : 1 \to e$ defines a morphism $v : T \to E$. Observe that the image of the projection $p_2 : T \times E \to E$ by the canonical map $Y \to X$ is the arrow $i : u^*(u) \to e$. Similarly, the image of the canonical morphism $j : T \times_E T \to T \times T$ by the map $Y \to X$ is the arrow $\Omega(p) : \Omega(e) \to \Omega(b)$. The square in the NE corner of the following commutative diagram is cartesian,

$$\begin{tikzcd}
T \times_E T \arrow[r, "j"] \arrow[d, "p_1"'] & T \times T \arrow[r, "p_2"] \arrow[d, "T\times v"] & T \arrow[d, "v"] \\
T \arrow[r, "(1_T, v)"] & T \times E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d] \\
& T \arrow[r] & 1.
\end{tikzcd}$$

It follows that the square in the NW corner is cartesian, since the composite of the top squares is cartesian. This shows that the square above is cartesian and hence that the arrow $\Omega(p) : \Omega_v(e) \to \Omega_u(b)$ is the fiber at w of the arrow ∂. We thus obtain a *fiber sequence*

$$\Omega(e) \xrightarrow{\Omega(p)} \Omega(b) \xrightarrow{\partial} f \xrightarrow{i} e \xrightarrow{p} b\ .$$

By iterating, the sequence can be extended to a *long fiber sequence*

$$\cdots \longrightarrow \Omega^2(e) \xrightarrow{\partial} \Omega(f) \xrightarrow{\Omega(i)} \Omega(e) \xrightarrow{\Omega(p)} \Omega(b) \xrightarrow{\partial} f \xrightarrow{i} e \xrightarrow{p} b\ .$$

39.4. The considerations above can be dualised. Let X be a pointed cocartesian quasi-category with nul object $0 \in X$. The *cofiber* of an arrow $u : x \to y$ is the arrow $v : x \to y$ defined by a pushout square

$$\begin{tikzcd} x \arrow[r] \arrow[d, "u"'] & 0 \arrow[d] \\ y \arrow[r, "v"] & z. \end{tikzcd}$$

The suspension $\Sigma(x)$ is the cofiber of the nul arrow $x \to 0$. It follows from the duality that $\Sigma(x)$ has the structure of a cogroup object in X. We obtain the *Puppe cofiber sequence*

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{\partial} \Sigma(x) \xrightarrow{\Sigma(u)} \Sigma(y) \xrightarrow{\Sigma(v)} \Sigma(z) \xrightarrow{\partial} \Sigma^2(z) \longrightarrow \cdots .$$

40. Additive quasi-categories

40.1. A category with finite products is pointed iff its terminal object is initial. Recall that the product of two objects $x \times y$ in a pointed category C is called a *direct sum* $x \oplus y$ if the pair of arrows

$$x \xrightarrow{(1_x,0)} x \times y \xleftarrow{(0,1_y)} y$$

is a coproduct diagram. A pointed category C is said to be *semi-additive* if has finite products and the product $x \times y$ of two objects of C is a direct sum $x \oplus y$. In a semi-additive category, the coproduct x of a family of objects $(x_i : i \in I)$ is denoted as a direct sum

$$x = \bigoplus_{i \in I} x_i.$$

The opposite of a semi-additive category is semi-additive. The set of arrows between two object of a semi-additive category has the structure of a commutative monoid. A semi-additive category C is said to be *additive* if the monoid $C(x, y)$ is a group for any pair of objects $x, y \in C$.

40.2. A quasi-category with finite products is pointed iff its terminal object is initial. We shall say that the product $x \times y$ of two objects in a pointed quasi-category is a *direct sum* $x \oplus y$ if the pair of arrows

$$x \xrightarrow{(1_x,0)} x \times y \xleftarrow{(0,1_y)} y$$

is a coproduct diagram. A pointed quasi-category with finite products X is said to be *semi-additive* if the product $x \times y$ of any two objects is a direct sum $x \oplus y$. In a semi-additive quasi-category, the coproduct x of a family of objects $(x_i : i \in I)$ is denoted as a direct sum

$$x = \bigoplus_{i \in I} x_i.$$

The opposite of a semi-additive quasi-category X is semi-additive. The homotopy category of a semi-additive quasi-category is semi-addditive. A semi-additive quasi-category X is said to be *additive* if the category hoX is additive. The opposite of an additive quasi-category is additive.

40.3. If a quasi-category X is semi-additive (resp. additive), then so is the quasi-category X^A for any simplicial set A and the quasi-category $Mod^{\times}(T, X)$ for any algebraic theory T.

40.4. An additive quasi-category T is *unisorted* iff it is equipped with an essentially surjective morphism $Add \to T$. A unisorted additive quasi-category is essentially the same thing as a ring space.

40.5. Recall that the *fiber* $a \to x$ of an arrow $x \to y$ in a quasi-category with null object 0 is defined by a pullback square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & x \arrow[d] \\ 0 \arrow[r] & y. \end{tikzcd}$$

The cofiber of an arrow is defined dually. An additive quasi-category is cartesian iff every arrow has a fiber. An additive quasi-category is finitely bicomplete iff every arrow has a fiber and a cofiber.

40.6. Let X be an additive quasi-category. To a commutative square in X

$$\begin{tikzcd} a \arrow[r, "v"] \arrow[d, "u"'] & c \arrow[d, "f"] \\ b \arrow[r, "g"] & d \end{tikzcd}$$

corresponds a null sequence

$$a \xrightarrow{(u,v)} b \oplus c \xrightarrow{(-g,f)} d.$$

The square is a pullback iff the sequence is a fiber sequence. The square is a pushout iff the sequence is a cofiber sequence.

40.7. Let X be a cartesian additive quasi-category. Then to each arrow $f : x \to y$ in X we can associate by 39.3 a *long fiber sequence*,

$$\cdots \longrightarrow \Omega^2(y) \xrightarrow{\partial} \Omega(z) \xrightarrow{\Omega(i)} \Omega(x) \xrightarrow{\Omega(f)} \Omega(y) \xrightarrow{\partial} z \xrightarrow{i} x \xrightarrow{f} y\ .$$

where $i : z \to x$ is the fiber of f.

40.8. A map $f : X \to Y$ between semi-additive quasi-categories preserves finite products iff it preserves finite coproducts iff it preserves finite direct sums. Such a map is said to be *additive*. The canonical map $X \to hoX$ is additive for any semi-additive quasi-category X.

40.9. A map between cocomplete additive quasi-categories is cocontinuous iff it preserves direct sums and cofibers. An additive map between cartesian additive quasi-categories is left exact iff it preserves fibers.

40.10. The Karoubi envelope of a semi-additive (resp. additive) quasi-category is semi-additive (resp. additive). If an additive quasi-category X is Karoubi complete, then every idempotent $e : x \to x$ has a fiber $Ker(e)$ and we have a decomposition $x \simeq Ker(e) \oplus Im(e)$, where $x \to Im(e) \to x$ is a splitting of e.

40.11. Let X be a Karoubi complete additive quasi-category. If $C : \Delta^o \to X$ is a simplicial object, then the morphism $C(d_n) : C_{n+1} \to C_n$ is splitted by the morphism $C(s_n) : C_n \to C_{n+1}$ for every $n \geq 0$. We thus obtain a decomposition

$$\delta C_n \oplus C_n \simeq C_{n+1},$$

where $\delta C_n \to C_{n+1}$ is the fiber of $C(d_n) : C_{n+1} \to C_n$. This defines a simplicial object $\delta C : \Delta^o \to X$ called the *first difference* of C. The simplicial object δC is augmented, with the augmentation $\partial_C : \delta C_0 \to C_0$ obtained by composing the canonical morphism $\delta C_0 \to C_1$ with the morphism $C(d_0) : C_1 \to C_0$. This defines an augmented simplicial object $\delta_+ C = (\delta C, \partial_C)$. The resulting map

$$\delta_+ : [\Delta^o, X] \to [\Delta^o_+, X]$$

is an equivalence of quasi-categories. The inverse equivalence associates to an augmented simplicial object $D : \Delta^o_+ \to X$ the simplicial object ΣD obtained by putting

$$(\Sigma D)_n = \bigoplus_{i=0}^{n} D(i)$$

for every $n \geq 0$. Let us describe the simplicial object ΣD more explicitly. Let $\sigma : \Delta_+ \to \Delta$ be the functor defined by putting $\sigma(n) = n + 1 = [n]$. Then the left Kan extension of D along σ is computed by the following formula:

$$\sigma_!(D)_n = \bigoplus_{i=0}^{n+1} D(i).$$

If u denotes the inclusion $\Delta \subset \Delta_+$, then we have $u^*(D)_n = D(n{+}1)$. Let us describe a morphism $\alpha_D : u^*(D) \to \sigma_!(D)$ whose cofiber is a morphism $\sigma_!(D) \to \Sigma D$. From the obvious natural transformation $Id \to \sigma u$, we obtain a natural transformation $\beta : Id \to \sigma_! u_!$. If $\epsilon : u_! u^* \to Id$ is the counit of the adjunction $u_! \vdash u^*$, then we can take

$$\alpha_D = \sigma_!(\epsilon_D)\beta_{u^*(D)} : u^*(D) \to \sigma_! u_! u^*(D) \to \sigma_!(D).$$

40.12. The first difference $C \mapsto \delta C$ can be iterated. Let Ch be the pointed category whose objects are the natural numbers and whose arrows are given by

$$Ch(m,n) = \begin{cases} \{\partial, 0\} & \text{if } m = n+1 \\ \{id, 0\} & \text{if } m = n. \\ \{0\} & \text{otherwise} \end{cases}$$

The only relation is $\partial\partial = 0$. If X is an additive quasi-category, a *chain complex* in X is defined to be a map $Ch \to X$ which send a null arrow to a nul arrow. We shall denote by $Ch(X)$ the full simplicial subset of X^{Ch} spanned by the chain complexes in X. The quasi-category $Ch(X)$ is additive. If $C : \Delta^o \to X$, then the simplicial object $\delta(C)$ is equipped with an augmentation map $\partial : \delta C_0 \to C_0$. The second difference $\delta^2 C = \delta(\delta(C))$ is equipped with an augmentation $\partial : \delta^2 C_0 \to \delta C_0$ and we have $\partial\partial = 0$. By iterating, we obtain a chain complex

$$C_0 \xleftarrow{\partial} \delta C_0 \xleftarrow{\partial} \delta^2 C_0 \xleftarrow{\partial} \cdots$$

It follows from the construction that $\delta^n C_0$ is the fiber of the map

$$(\partial_1, \ldots, \partial_n) : C_n \to \bigoplus_{i=1}^{n} C_{n-1}$$

for every $n > 0$. The map

$$ch : [\Delta^o, X] \to Ch(X)$$

defined by putting $ch(C) = \delta^* C_0$ is an equivalence of quasi-categories; it is the *Dold-Kan correspondance*. The inverse equivalence associates to a chain complex $D \in Ch(X)$ the simplicial object $S(D)$ obtained by putting

$$S(D)_n = \bigoplus_{k=0}^{n} \binom{n}{k} D_k$$

for every $n \geq 0$. The equivalence

$$C_n \simeq S(\delta^* C)_n = \bigoplus_{k=0}^{n} \binom{n}{k} \delta^k C_0$$

is *Newton's formula* of finite differences calculus. Let us describe the simplicial object $S(D)$ more explictly. The binomial coefficient $\binom{n}{k}$ is the number of surjections $[n] \to [k]$. Let $\Delta_m \subset \Delta$ be the subcategory of monomorphisms. Consider the functor $G : \Delta_m \to Ch$ which takes a monomorphism $f : [m] \to [n]$ to the morphism $G(f) : m \to n$ defined by putting

$$G(f) = \begin{cases} \partial & \text{if } n = m+1 \quad \text{and} \quad f = d_0 \\ id & \qquad\text{if } m = n. \\ 0 & \qquad\text{otherwise} \end{cases}$$

Then the map $S(D) : \Delta^o \to X$ is the left Kan extension of the composite $D \circ G : \Delta_m^o \to X$ along the inclusion $\Delta_m \subset \Delta$.

40.13. Let X be an additive cartesian quasi-category. Then a simplicial object $C : \Delta^o \to X$ is a groupoid iff we have $\delta^n C = 0$ for every $n > 1$. The Dold-Kan correspondance associates to a groupoid $C : \Delta^o \to X$ the map $\partial_C : \delta C_0 \to C_0$. It induces an equivalence

$$\partial_- : Grpd(X) \simeq X^I$$

between the quasi-category $Grpd(X)$ of groupoids in X and the quasi-category X^I of maps in X. If C is the equivalence groupoid of an arrow $u : x \to y$, then the arrow ∂_C is the fiber $Ker(u) \to x$. A functor $p : E \to C$ in $Grpd(X)$ is fully faithful iff the square

$$\begin{tikzcd} \delta E_0 \arrow[r, "\partial_E"] \arrow[d, "\delta p_0"'] & E_0 \arrow[d, "p_0"] \\ \delta C_0 \arrow[r, "\partial_C"] & C_0 \end{tikzcd}$$

is cartesian. A functor $p : E \to C$ in is a left (or right) fibration iff the morphism $\delta p_0 : \delta E_0 \to \delta C_0$ is invertible. Hence the Dold-Kan correspondance induces an equivalence between the quasi-category X^C and the quasi-category $Fact(\partial_C, X)$ of factorisations of the arrow ∂_C.

40.14. An additive quasi-category X is exact iff the following five conditions are satisfied:

- X admits surjection-mono factorisations;
- The base change of a surjection is a surjection;
- Every morphism in has a fiber and a cofiber;
- Every morphism is the fiber of its cofiber;
- Every surjection is the cofiber of its fiber.

40.15. Let us sketch a proof of 40.14. ($\Rightarrow$) Every morphism has a fiber, since X is cartesian. Moreover, X admits surjection-mono factorisations stable under base changes, since X is exact. Let $Surj(X)$ be the full sub-quasi-category of X^I spanned by the surjections in X. It follows from 40.13 and 37.13 that the map $Ker : Surj(X) \to X^I$ which associates to a surjection its fiber is an equivalence of quasi-categories. The inverse equivalence $X^I \to Surj(X)$ associates to a morphism its cofiber. Thus, every morphism has a cofiber and is the fiber of its cofiber. Moreover, every surjection is the cofiber of its fiber. ($\Leftarrow$) The quasi-category X is cartesian, since X is additive and every morphism has a fiber. Let us show that every surjection $f : x \to y$ is a descent morphism. For this we have to show that the map $D : X/y \to X^{Eq(f)}$ induced by $f^* : X/y \to X/x$ is an equivalence of quasi-categories. If $i : z \to x$ is the fiber of f, then D associates to a morphism $q : y' \to y$ a factorisation $i = pi' : z \to x' \to x$, where $x' = f^*(y')$ and $i' = (i, 0) : z \to x \times_y y'$. Let $f' : x' \to\!\!\!\to y'$ be the projection. The boundary squares of the following diagram is a pullback,

$$\begin{tikzcd} z \arrow[r] \arrow[d, "i'"'] & 0 \arrow[d] \\ x' \arrow[r, "f'"] \arrow[d, "p"'] & y' \arrow[d, "q"] \\ x \arrow[r, "f"] & y. \end{tikzcd}$$

Hence also the top square, since the boundary square is a pullback. Hence the morphism $i' : z \to x'$ is the fiber of the morphism $f' : x' \to\!\!\!\to y'$. But f' is surjective, since f is surjective. Thus, f' is the cofiber of of i'. The map D has a left adjoint L which associates to a factorisation $i = pi' : z \to x' \to x$ the arrow $q : y' \to y$ in the following diagram of pushout squares,

$$\begin{tikzcd} z \arrow[r] \arrow[d, "i'"'] & 0 \arrow[d] \\ x' \arrow[r, "f'"] \arrow[d, "p"'] & y' \arrow[d, "q"] \\ x \arrow[r, "f"] & y. \end{tikzcd}$$

We have $LD(q) \simeq q$ for every morphism $q : y' \to y$, since f' is the cofiber of of i'. Let us show that the map DL is isomorphic to the identity. The map L associates to a factorisation $i = pi' : z \to x' \to x$ the morphism $q : y' \to y$ in the diagram of pushout squares above. We have to show that the the pushout square

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "p"'] & y' \arrow[d, "q"] \\ x \arrow[r, "f"] & y. \end{tikzcd}$$

is a pullback. By 40.6, it suffices to show that the null sequence

$$x' \xrightarrow{(p,f')} x \oplus y' \xrightarrow{(f,-q)} y$$

is a fiber sequence. But the sequence is a cofiber sequence by 40.6, since the square is a pushout. It is thus a fiber sequence, since every morphism is the fiber of its cofiber. This proves that every surjection is a descent morphism. It remains to show that every groupoid is effective. But this follows from the fact that every morphism is the fiber of its cofiber.

40.16. Let X be an exact additive quasi-category. If a morphism $f : x \to y$ is surjective, then a null sequence $0 = fi : z \to x \to y$ is a fiber sequence iff it is a cofiber sequence.

40.17. Let X be an exact additive quasi-category. An object $a \in X$ is discrete iff $\Omega(a) = 0$. A morphism $u : a \to b$ in X is a 0-cover iff its fiber $Ker(u)$ is discrete. An object $a \in X$ is connected iff the morphism $0 \to a$ is surjective. A morphism $a \to b$ is 0-connected iff it is surjective and the fiber $Ker(u)$ is connected. The suspension $\Sigma : X \to X$ induces an equivalence between X and the full sub-quasi-category of connected objects of X.

40.18. Let X be an exact additive quasi-category. An object $a \in X$ is a n-object iff $\Omega^n(a) = 0$. An arrow $u : a \to b$ in X is a n-cover iff its fiber $Ker(u)$ is a n-object. An object $a \in X$ is n-connected iff it is connected and $\Omega(a)$ is $(n-1)$-connected. A morphism $a \to b$ is n-connected iff it is surjective and its fiber $Ker(u)$ is n-connected.

40.19. If a quasi-category X is additive and exact, then so is the quasi-category X^A for any simplicial set A and the quasi-category $Mod^{\times}(T, X)$ for any algebraic theory T.

40.20. The algebraic theory of symmetric monoids $SMon$ is semi-additive. The direct sum $A \oplus B$ of two finite sets is their disjoint union $A \sqcup B$. This is illustrated by the equivalences of groupoids,

$$\begin{aligned} SMon(A \sqcup B, C) &\simeq SMon(A, C) \times SMon(B, C), \\ SMon(C, A \sqcup B) &\simeq SMon(C, A) \times SMon(C, B) \end{aligned}$$

for any triple of finite sets (A, B, C).

40.21. If an algebraic theory T is semi-additive, then so is the quasi-category $Mod^{\times}(T, X)$ for any quasi-category with finite products X.

40.22. Let us sketch a proof of 40.21. We first verify that the quasi-category $Mod^{\times}(T,X)$ is pointed. The nul object $0 \in T$ is both initial and terminal. Hence the map $0 : 1 \to T$ is both left and right adjoint to the map $p : T \to 1$. Hence the map $0^* : Mod^{\times}(T,X) \to Mod^{\times}(1,X) = 1$ is both left and right adjoint to the map $p^* : 1 = Mod^{\times}(1,X) \to Mod^{\times}(T,X)$. It follows that the object $p^*(1)$ is both initial and terminal in $Mod^{\times}(T,X)$. We have proved that the constant map $T \to X$ with value 1 is both initial and terminal in $Mod^{\times}(T,X)$. Let us now show that the product map $Mod^{\times}(T,X) \times Mod^{\times}(T,X) \to Mod^{\times}(T,X)$ is left adjoint to the diagonal. The 2-category of algebraic theories **AT** is by itself "semi-additive" by **??**. Hence the map $Mod^{\times}(T,X) \times Mod^{\times}(T,X) \to Mod^{\times}(T \times T,X)$ obtained from the inclusions $in_1 : T \to T \times T$ and $in_2 : T \to T \times T$ is an equivalence of quasi-categories. If $\delta : T \to T \times T$ is the diagonal, then

$$\delta^* : Mod^{\times}(T,X) \times Mod^{\times}(T,X) \to Mod^{\times}(T,X)$$

is the product map. If $\sigma : T \times T \to T$ is the product map, then

$$\sigma^* : Mod^{\times}(T,X) \to Mod^{\times}(T,X) \times Mod^{\times}(T,X)$$

is the diagonal. But σ is left adjoint to the diagonal, since the product is a coproduct in T. Hence we have $\delta^* \vdash \sigma^*$ since we have $\sigma \vdash \delta$. This shows that the product map δ^* is left adjoint to the diagonal. A closer examination reveals that δ^* is a direct sum map.

40.23. Let Mon be the algebraic theory of monoids and $SMon$ be the algebraic theory of symmetric monoids. Then the following conditions on a quasi-category with finite products X are equivalent (assuming a conjecture in 31.15):

- X is semi-additive;
- the forgetful map $Mon(X) \to X$ is an equivalence of quasi-categories;
- the forgetful map $SMon(X) \to X$ is an equivalence of quasi-categories.

40.24. Let us sketch a proof of 40.23. (i)⇒(iii) Let X be an additive quasi-category. Let Γ be the category of finite pointed sets and basepoint preserving maps. The category Γ has finite coproducts. It follows from 31.12 that it is freely generated by the pointed object $1_+ = \{1, \star\}$. Hence the forgetful map

$$Map_{\sqcup}(\Gamma, X) \to X$$

is an equivalence of quasi-categories, where $Map_{\sqcup}(\Gamma, X)$ is the quasi-category of maps $\Gamma \to X$ which preserves finite coproducts. It is easy to see that a map $\Gamma \to X$ preserves finite coproducts iff it is a model of the Segal sketch (Γ, C) of 31.17. Thus,

$$Map_{\sqcup}(\Gamma, X) = Mod(\Gamma/C, X).$$

It follows that the forgetful map

$$Mod(\Gamma/C, X) \to X,$$

is an equivalence of quasi-categories. But we have an equivalence of quasi-categories

$$Mod(\Gamma/C, X) \simeq Mod^{\times}(SMon, X)$$

by **??**. It follows that the forgetful map

$$Mod^{\times}(SMon, X) \to X,$$

is an equivalence of quasi-categories. (iii)⇒(i) The quasi-category $SMon(X)$ is semi-additive by 40.21, since $SMon$ is semi-additive by 40.20. Hence the quasi-category X is semi-additive if the forgetful map $SMon(X) \to X$ is an equivalence. (iii)⇒(ii) The canonical morphism $SMon \to SMon \odot Mon$ is an equivalence, since $SMon$ is the (homotopy) colimit of the theories Mon^n for $n \geq 0$ by a conjecture in 31.15. Hence the forgelful map $Mon(SMon(X)) \to SMon(X)$ is an equivalence. This shows that the forgelful map $Mon(X) \to X$ is an equivalence since we have $X \simeq SMon(X)$. (ii)⇒(iii) If the forgetful map $Mon(X) \to X$ is an equivalence, then so is the map forgetful map $Mon^n(X) \to X$ for every $n \geq 0$ if we iterate. Hence also the the forgetful map $SMon(X) \to X$, since the quasi-category $SMon(X)$ is the (homotopy) projective limits of the quasi-categories $Mon^n(X)$ when $n \to \infty$. Hence the forgetful map $SMon(X) \to X$ is an equivalence by a conjecture in 31.15.

40.25. Let Mon be the algebraic theory of monoids and $SMon$ be the theory of symmetric monoids. Then following conditions on an algebraic theory T are equivalent (assuming a conjecture in 31.15):

- T is semi-additive;
- $Mod^{\times}(T)$ is semi-additive;
- the canonical morphism $T \to T \odot Mon$ is an equivalence;
- the canonical morphism $T \to T \odot SMon$ is an equivalence.

40.26. Let us sketch a proof of 40.25. The implication (i)⇒(ii) follows from 40.21. Let us prove the implication (ii)⇒ (i). The Yoneda map $T^o \to Mod^{\times}(T)$ is fully faithful and it preserves finite coproducts. It thus induces an equivalence between the quasi-category T^o and a full additive sub-quasi-category of $Mod^{\times}(T)$. This shows that T^o is semi-additive. It follows that T is semi-additive. (i)⇒(iii) If T is semi-additive, then so is the quasi-category $Mod^{\times}(T, X)$ for any quasi-category with finite product X by 40.21. Hence the forgetful map $Mon(Mod^{\times}(T, X)) \to Mod^{\times}(T, X)$ is an equivalence by 40.23. But we have

$$Mon(Mod^{\times}(T, X)) = Mod^{\times}(Mon, Mod^{\times}(T, X)) = Mod^{\times}(T \odot Mon, X).$$

It follows by Yoneda lemma that the canonical morphism $T \to T \odot Mon$ is an equivalence. (iii)⇒(ii) If the canonical morphism $T \to T \odot Mon$ is an equivalence, then so is the forgetful morphism $Mon(Mod^{\times}(T)) \to Mod^{\times}(T)$. This proves that $Mod^{\times}(T)$ is semi-additive by 40.23. The implications (i)⇒(iii)⇒(ii) are proved similarly.

40.27. The results of 40.21 and 40.25 can be reformulated for cartesian theories instead of algebraic theories. If a cartesian theory T is semi-additive, then so is the quasi-category $Mod(T, X)$ for any cartesian quasi-category X. Moreover, the following conditions on a cartesian theory T are equivalent (assuming a conjecture in 31.15):

- T is semi-additive;
- $Mod(T)$ is semi-additive;
- the canonical morphism $T \to T \odot Mon$ is an equivalence;
- the canonical morphism $T \to T \odot SMon$ is an equivalence.

40.28. A (multiplicative) localisation of a semi-additive (resp. additive) algebraic theory is semi-additive (resp.additive). The algebraic theory of infinite loop spaces *Add* is additive.

40.29. The results of 40.23 and of 40.27 can be reformulated in the additive case. Let Grp be the algebraic theory of groups and Add be the algebraic theory of infinite loop spaces. If X is a quasi-category with finite products, then the following conditions are equivalent:

- X is additive;
- the forgetful map $Grp(X) \to X$ is an equivalence of quasi-categories;
- the forgetful map $Add(X) \to X$ is an equivalence of quasi-categories.

If a cartesian theory T is additive, then so is the quasi-category $Mod(T, X)$ for any cartesian quasi-category X. Moreover, the following conditions on a cartesian theory T are equivalent (assuming a conjecture in 31.15):

- T is additive;
- $Mod(T)$ is additive;
- the canonical morphism $T \to T \odot Grp$ is an equivalence;
- the canonical morphism $T \to T \odot Add$ is an equivalence.

40.30. The suspension theorem of Freudenthal implies that a pointed n-connected space with vanishing homotopy groups in dimension $> 2n$ is naturally a loop space [May2]. The $(n+1)$-fold loop space functor induces an equivalence between the homotopy category of pointed n-connected spaces and the homotopy category of $(n+1)$-fold loop spaces by a classical result [**?**]. The $(n+1)$-fold loop space of a $2n$-object is a $(n-1)$ object. It then follows from Freudenthal theorem that a $(n+1)$-fold loop space with vanishing homotopy groups in dimension $> n-1$ is naturally a $(n+2)$-fold loop space. This means that the forgetful map

$$Grp^{n+2}(\mathbf{U}(n-1)) \to Grp^{n+1}(\mathbf{U}(n-1))$$

is an equivalence of quasi-categories for every $n \geq 1$. It follows that the cartesian theory

$$OB(n-1) \odot Grp^{n+1}$$

is additive for every $n \geq 1$ by 40.29.

40.31. (Generalised Suspension Conjecture) We conjecture that the cartesian theory

$$OB(n) \odot Mon^{n+2}$$

is semi-additive for every $n \geq 0$.

40.32. Let **AT** be the (2-)category of algebraic theories. Then the full sub(2-)category **AAT** of **AT** spanned by the additive algebraic theories is (pseudo) reflective and coreflective. The left adjoint to the inclusion $\mathbf{AAT} \subset \mathbf{AT}$ is the functor $X \mapsto X \odot Add$ and its right adjoint is the functor $X \mapsto Mod^{\times}(Add, X)$.

40.33. Let **CT** be the 2-category of cartesian theories. Then the full sub 2-category **ACT** of **CT** spanned by the additive cartesian theories is (pseudo) reflective and coreflective. The left adjoint to the inclusion $\mathbf{ACT} \subset \mathbf{CT}$ is the functor $T \mapsto T \odot Add$ and its right adjoint is the functor $T \mapsto Mod^{\times}(Add, T)$.

40.34. Let **LP** be the (2-)category of locally representable quasi-categories. Then the full sub(2-)category **ALP** of **LP** spanned by the additive locally presentable quasi-categories is (pseudo) reflective and coreflective. The left adjoint to the inclusion $\mathbf{ALP} \subset \mathbf{LP}$ is the functor $X \mapsto Mod^{\times}(Add, X) \simeq X \otimes \mathbf{Add}$ and its right adjoint is the functor $X \mapsto Map(\mathbf{Add}, X)$, where $\mathbf{Add} = Mod^{\times}(Add)$ is the quasi-category of infinite loop spaces.

40.35. If A is a algebraic theory, then the quasi-category $Mod^{\times}(A, \mathbf{Add})$ is exact and a variety of homotopy algebras. Let us give a proof. The quasi-category $\mathbf{Add}) = Mod^{\times}(Add)$ is obviously a variety. Hence also the quasi-category $Mod^{\times}(A, \mathbf{Add})$ by 33.2. It is thus exact by 38.3.

40.36. If A is an additive quasi-category, then the map $hom_A : A^o \times A \to \mathbf{U}$ admits a factorisation

$$\begin{tikzcd} & \mathbf{Add} \arrow[d, "U"] \\ A^o \times A \arrow[ur, "hom'_A"] \arrow[r, "hom_A"'] & \mathbf{U}, \end{tikzcd}$$

where hom'_A preserves finite products in each variable and where U is the forgetful map. The factorisation is unique up to a unique invertible 2-cell. This defines an "enrichement" of the quasi-category A over the quasi-category **Add**. The *Yoneda map*

$$y_A : A^o \to Mod^{\times}(A, \mathbf{Add})$$

is obtained from hom'_A by exponential adjointness.

40.37. Let A be an additive algebraic theory. We say that a map $f : A \to \mathbf{Add}$ is *representable* if it belongs to the essential image of the Yoneda map $A^o \to Mod^{\times}(A, \mathbf{Add})$.

40.38. Let X be a (locally small) cocomplete additive quasi-category. Then an object $x \in X$ is bicompact iff the map $y(x) : X \to \mathbf{Add}$ is cocontinuous. Let us give a proof. $(\Rightarrow)$ If x is bicompact, then the map $y(x)$ is cocontinuous by 33.16 since it preserves finite coproducts.

40.39. Let A be an additive algebraic theory. Then the following conditions on an object $x \in Mod^{\times}(A, \mathbf{Add})$ are equivalent:

- x is bicompact
- x is compact and projective;
- x is a retract of a representable.

Let us give a proof. The forgetful map $\mathbf{Add} \to \mathbf{U}$ induces an equivalence of quasi-categories

$$Mod^{\times}(A, \mathbf{Add}) \to Mod^{\times}(A, \mathbf{U})$$

by 40.32 since we have $Mod^{\times}(Add, \mathbf{U}) = \mathbf{Add}$. The result then follows from 33.20.

40.40. A cocomplete additive quasi-category X is equivalent to a quasi-category $Mod^{\times}(A, \mathbf{Add})$ for some additive algebraic theory A iff it is generated by a set of bicompact objects.

41. Stable quasi-categories

41.1. Let X be a quasi-category with null object $0 \in X$. Recall that the *loop space* $\Omega(x)$ of an object $x \in X$ is defined to be the fiber of the arrow $0 \to x$. We say that X is *stable* if if every object $x \in X$ has a loop space, and the loop space map $\Omega : X \to X$ is an equivalence of quasi-categories.

41.2. Let X be a stable quasi-category. Then the arrow $x \to 0$ has a cofiber $x \to \Sigma(x)$ for every object $x \in X$. The map $\Sigma : X \to X$ is the inverse of the map $\Omega : X \to X$. We shall put

$$\Omega^{-n} = \Sigma^n \quad \text{and} \quad \Sigma^{-n} = \Omega^n$$

for every $n \geq 0$. The opposite of a stable quasi-category X is stable. The loop space map $\Omega : X^o \to X^o$ is obtained by putting $\Omega(x^o) = \Sigma(x)^o$ for every object $x \in X$.

41.3. We say that a map $f : X \to Y$ between stable quasi-categories is *stable* if it preserves nul objects and the canonical morphism $f \circ \Omega \to \Omega \circ f$ is invertible.

41.4. If T is a stable algebraic theory and X is a pointed cartesian quasi-category, we say that a model $T \to X$ is *stable* if the canonical natural transformation $f \circ \Omega \to \Omega \circ f$ is invertible. We denote by $SProd(T, X)$ the full simplicial subset of $Prod(T, X)$ spanned by the stable models. We shall put

$$SProd(T) = SProd(T, 1\backslash\mathbf{U}).$$

The forgetful map $\mathbf{Spec} \to 1\backslash\mathbf{U}$ induces an equivalence of quasi-categories

$$SProd(T, \mathbf{Spec}) \simeq SProd(T).$$

41.5. The quasi-category $Spec(X)$ is stable for any cartesian quasi-category X. A cartesian quasi-category X is stable iff the forgetful map $Spec(X) \to X$ is an equivalence. The quasi-categories $Spec$ and $Mod(Spec) = \mathbf{Spec}$ are stable.

41.6. Let us give a proof of 41.5, starting with the first statement. We have $Spec(X) = Spec(1\backslash X)$. Hence we can suppose that X is pointed. In this case, the quasi-category $Spec(X)$ is the (homotopy) projective limit of the infinite sequence of quasi-categories

$$X \xleftarrow{\Omega} X \xleftarrow{\Omega} X \xleftarrow{\Omega} \cdots.$$

It follows that the map $\Omega : Spec(X) \to Spec(X)$ is an equivalence. Let us prove the second statement. Clearly, the forgetful map $Spec(X) \to X$ is an equivalence if X is stable, Conversely, if the forgetful map $Spec(X) \to X$ is an equivalence, then X is stable, since $Spec(X)$ is stable. Let us prove the last statement. The quasi-category $\mathbf{Spec} = Spec(\mathbf{U})$ is stable. Hence the quasi-category $Spec$ is pointed, since Yoneda map $Spec^o \to \mathbf{Spec}$ preserves initial objects. Let us show that the map $\Omega : Spec \to Spec$ is invertible. The map $Mod(\Omega, X) : Mod(Spec, X) \to Mod(Spec, X)$ is invertible for any cartesian quasi-category X, since the quasi-category $Mod(Spec, X) = Spec(X)$ is stable. It follows by Yoneda lemma that the map $\Omega : Spec \to Spec$ is invertible.

41.7. A stable quasi-category with finite products X is additive. Let us sketch a proof. The loop space $\Omega(x)$ of an object $x \in X$ has the structure of a group by 39.1. If $Grp(X)$ denotes the quasi-category of group objects in X, then the map $\Omega : X \to X$ can be lifted along the forgetful map $Grp(X) \to X$,

$$\begin{tikzcd} & Grp(X) \arrow[d] \\ X \arrow[ur, "G"] \arrow[r, "\Omega"] & X. \end{tikzcd}$$

But Ω is an equivalence of quasi-categories, since X is stable. It follows that the forgetful map $Grp(X) \to X$ admits a section which preserves finite products. Hence also the forgetful map $Grp^{n+1}(X) \to Grp^n(X)$ for every $n \geq 0$. But the quasi-category $Add(X)$ is the (homotopy) projective limit of the sequence of quasi-categories

$$X \leftarrow Grp(X) \leftarrow Grp^2(X) \leftarrow \cdots$$

by **??**. It follows that the forgetful map $Add(X) \to X$ admits a section which preserves finite products. But the quasi-category $Add(X)$ is additive by 40.29, since Add is additive by 40.28. Hence it suffices to show that a retract of an additive quasi-category is additive (when the retraction and the section preserve finite products). But this follows from 40.23.

41.8. We denote by **SAT** the category of stable algebraic theories and stable morphisms of theories. The category **SAT** has the structure of a 2-category induced by the 2-category structure of the category of simplicial sets. If T is a stable algebraic theory and X is a pointed cartesian quasi-category, we say that a model $T \to X$ is *stable* if the canonical natural transformation $f \circ \Omega \to \Omega \circ f$ is invertible. We denote by $SProd(T, X)$ the full simplicial subset of $Prod(T, X)$ spanned by the stable models. We shall put

$$SProd(T) = SProd(T, 1\backslash \mathbf{U}).$$

The forgetful map $\mathbf{Spec} \to 1\backslash\mathbf{U}$ induces an equivalence of quasi-categories

$$SProd(T, \mathbf{Spec}) \simeq SProd(T).$$

If $u : S \to T$ is a stable morphism of theories, then the map

$$u^* : SProd(T) \to SProd(S)$$

induced by u has a left adjoint $u_!$. The adjoint pair $(u_!, u^*)$ an equivalence iff the map $u : S \to T$ is a Morita equivalence.

41.9. We denote by $SAdd$ the stable algebraic theory freely generated by one object $u \in SAdd$. Every object $f \in SAdd$ is a finite direct sum

$$f = \bigoplus_{i \in F} \Sigma^{n_i}(u)$$

where n_i is an integer. We say that a stable algebraic theory T is *unisorted* if it is equipped with an essentially surjective map $SAdd \to T$. A "ring spectrum" is essentially the same thing as a unisorted stable theory. In other words, a stable algebraic theory T is a "ring spectrum with many objects". A stable model $f : T \to \mathbf{Spec}$ is a left T-module.

41.10. The opposite of a stable algebraic theory is a stable algebraic theory. The stable theory $SAdd^o$ is freely generated by the object $u^o \in SAdd$. Hence the stable morphism $SAdd \to SAdd^o$ which takes u to u^o is an equivalence. The duality takes the object $\Sigma^n(u)$ to the object $\Sigma^{-n}(u)$ for every integer n.

41.11. Every stable algebraic theory T generates freely a cartesian theory $u : T \to T_c$. By definition, T_c is a pointed cartesian theory and $u : T \to T_c$ is a stable morphism which induces an equivalence of quasi-categories

$$Mod(T_c, X) \simeq SProd(T, X)$$

for any pointed cartesian quasi-category X. The cartesian theory T_c is stable. For example, we have $SAdd_c = Spec$.

41.12. The quasi-category of spectra **Spec** is exact. More generally, if T is a stable algebraic theory, then the quasi-category $SProd(T)$ is stable and exact.

41.13. Let us sketch a proof of 41.12. The quasi-category **Spec** is a para-variety by 35.4. It is thus exact by **??**. Let us show that the quasi-category $SProd(T, \mathbf{Spec})$ is stable and exact. It is easy to see that it is stable. Let us show that it is a para-variety. The quasi-category $Prod(T, \mathbf{Spec})$ is a para-variety by 34.4. Hence it suffices to show that the quasi-category $SProd(T, \mathbf{Spec})$ is a left exact reflection of the quasi-category $Prod(T, \mathbf{Spec})$. A model $f : T \to \mathbf{Spec}$ is stable iff the the canonical natural transformation $\alpha : f \to \Omega f \Sigma$ is invertible. By iterating, we obtain an infinite sequence

$$f \xrightarrow{\alpha} \Omega f \Sigma \xrightarrow{\Omega\alpha\Sigma} \Omega^2 f \Sigma^2 \longrightarrow \cdots$$

The colimit $R(f)$ of this sequence is a stable map $T \to \mathbf{Spec}$. This defines a left exact reflection

$$R : Prod(T, \mathbf{Spec}) \to SProd(T, \mathbf{Spec}).$$

Thus, $SProd(T, \mathbf{Spec})$ is a para-variety. Hence it is exact by **??**.

41.14. An additive quasi-category X is stable and exact iff the following two conditions are satisfied:

- Every morphism has a fiber and a cofiber;
- A null sequence $z \to x \to y$ is a fiber sequence iff it is a cofiber sequence.

41.15. Let us sketch a proof of 40.14. ($\Rightarrow$) Every morphism in X has a fiber and a cofiber by 40.14, since X is exact and additive. Let us show that every arrow is surjective. For this it suffices to show that every monomorphism is invertible, since every arrow is right orthogonal to every quasi-isomorphism. If $u : a \to b$ is a monomorphism, then we have a fiber sequence

$$\Omega(a) \xrightarrow{\Omega(u)} \Omega(b) \longrightarrow 0 \longrightarrow a \xrightarrow{u} b .$$

by 40.7. Thus, $\Omega(u)$ invertible, since it is the fiber of a nul morphism, It follows that u is invertible, since the map $\Omega : X \to X$ is an equivalence. We have proved that every arrow is surjective. It then follows from 40.16 that a nul sequence $z \to x \to y$ is a fiber sequence iff it is a cofiber sequence. ($\Leftarrow$) Let us show that X is stable. If $x \in X$, then we have $\Sigma\Omega(x) \simeq x$, since the fiber sequence $\Omega(x) \to 0 \to x$ is a cofiber sequence. Moreover, we have $x \simeq \Omega\Sigma(x)$, since the cofiber sequence $x \to 0 \to \Sigma(x)$ is a fiber sequence. This shows that X is stable. It remains to show that X is exact. For this, it suffices to show that the conditions of 40.14 are satisfied. Let us first show that X admits surjection-mono factorisations. For this, it suffices to show that every monomorphism is invertible. If $x \to y$ is monic, then the sequence $0 \to x \to y$ is a cofiber sequence, since it is a fiber sequence. It follows that the arrow $x \to y$ is invertible. This proves that every monomorphism is invertible. Thus, every morphism is surjective. Hence the base change of a surjection is a surjection.

41.16. The opposite of an exact stable quasi-category is exact and stable.

41.17. An additive map $X \to Y$ between two exact stable quasi-categories is exact iff it is left exact iff it is right exact.

41.18. Let X be an exact stable quasi-category. Then to each arrow $f : x \to y$ in X we can associate by 40.7 a *two-sided long fiber sequence*,

$$\cdots \Omega(x) \xrightarrow{\Omega(f)} \Omega(y) \xrightarrow{\partial} z \xrightarrow{i} x \xrightarrow{f} y \xrightarrow{\partial} \Sigma(z) \xrightarrow{\Sigma(i)} \Sigma(x) \cdots .$$

where $i : z \to x$ is the fiber of f. The sequence is entirely described by a triangle

$$\begin{tikzcd} x \arrow[rr, "f"] & & y \arrow[dl, "\partial"] \\ & z \arrow[ul, "i"] & \end{tikzcd} .$$

where ∂ is now regarded as a morphism of degree -1 (ie as a morphism $y \to \Sigma(x)$).

41.19. If A and B are two stable algebraic theories then so is the quasi-category $SProd(A, B)$ of stable models $A \to B$. The 2-category **SAT** is symmetric monoidal closed. The *tensor product* $A \odot_S B$ of two stable algebraic theories is the target of a map $A \times B \to A \odot_S B$ which is a stable morphism in each variable (and which is universal with respect to that property). There is a canonical equivalence of quasi-categories

$$SProd(A \odot_S B, X) \simeq SProd(A, SProd(B, X))$$

for any cartesian quasi-category X. In particular, we have two equivalences of quasi-categories,

$$SProd(A \odot_S B) \simeq SProd(A, SProd(B)) \simeq SProd(B, Prod(A)).$$

The unit for the tensor product is the theory $SAdd$ described in **??**. The opposite of the canonical map $S \times T \to S \odot T$ can be extended along the Yoneda maps as a map cocontinuous in each variable.

$$SProd(A) \times SProd(B) \to SProd(A \odot_S B).$$

41.20. Let **CT** be the (2-)category of cartesian theories. Then the full sub(2-)category **SCT** of **CT** spanned by the stable cartesian theories is (pseudo) reflective and coreflective. The left adjoint to the inclusion **SCT** $\subset$ **CT** is the functor $T \mapsto T \odot_c Spec$ and its right adjoint is the functor $T \mapsto Mod(Spec, T) \simeq SProd(SAdd, T)$.

41.21. Let **LP** be the (2-)category of locally representable quasi-categories. Then the full sub(2-)category **SLP** of **LP** spanned by the stable locally presentable quasi-categories is (pseudo) reflective and coreflective. The left adjoint to the inclusion $\mathbf{SLP} \subset \mathbf{LP}$ is the functor $X \mapsto Mod(Spec, X) \simeq SProd(SAdd, X) \simeq X \otimes \mathbf{Spec}$ and its right adjoint is the functor $X \mapsto Map(\mathbf{Spec}, X)$.

41.22. If A is a stable quasi-category, then the map $hom_A : A^o \times A \to \mathbf{U}$ admits a factorisation

$$\begin{tikzcd} & \mathbf{Spec} \arrow[d, "U"] \\ A^o \times A \arrow[ur, "hom'_A"] \arrow[r, "hom_A"'] & \mathbf{U}, \end{tikzcd}$$

where the map hom'_A is stable in each variable, and where U is the forgetful map. The factorisation is unique up to a unique invertible 2-cell. This defines an "enrichement" of the quasi-category A over the quasi-category of spectra **Spec**. The *Yoneda map*

$$y : A^o \to \mathbf{Spec}^A$$

is obtained from hom'_A by exponential adjointness.

41.23. If T is a stable algebraic theory, then the Yoneda map $y : T^o \to \mathbf{Spec}^T$ induces a map $y : T^o \to SProd(T)$. We say that a model $f : T \to \mathbf{Spec}$ is *representable* if it belongs to the essential image of the Yoneda map.

42. Homotopoi (∞-topoi)

The notion of homotopos (∞-topos) presented here is due to Carlos Simpson and Charles Rezk.

42.1. Recall from 34.1 that a category $\mathcal{E}$ is said to be a *Grothendieck topos* if it is a left exact reflection of a presheaf category $[C^o, \mathbf{Set}]$. A *homomorphism* $\mathcal{E} \to \mathcal{F}$ between Grothendieck topoi is a cocontinuous functor $f : \mathcal{E} \to \mathcal{F}$ which preserves finite limits. The 2-category of Grothendieck topoi and homomorphism is has the structure of a 2-category, where a 2-cell is a natural transformation. Every homomorphism has a right adjoint. A *geometric morphism* $\mathcal{E} \to \mathcal{F}$ is an adjoint pair

$$g^* : \mathcal{F} \leftrightarrow \mathcal{E} : g_*$$

with g^* a homomorphism. The map g^* is called the *inverse image part of* g and the map g_* its *direct image part*. . We shall denote by **Gtop** the category of Grothendieck topoi and geometric morphisms. The category **Gtop** has the structure of a 2-category, where a 2-cell $\alpha : f \to g$ is a natural transformation $\alpha : g^* \to f^*$. The 2-category **Gtop** is equivalent to the opposite of the 2-category of Grothendieck topoi and homomophism.

42.2. Recall from 34.3 that a locally presentable quasi-category X is said to be a *homotopos*, or an ∞-*topos*, if it is a left exact reflection of a quasi-category of prestacks. $\mathbf{P}(A)$ for some simplicial set A. The quasi-category of homotopy types **Tp** is the archtype of a homotopos. If X is a homotopos, then so is the quasi-category X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A.

42.3. Recall that a cartesian quasi-category X is said to be *locally cartesian closed* if the quasi-category X/a is cartesian closed for every object $a \in X$. A cartesian quasi-category X is locally cartesian closed iff the base change map $f^* : X/b \to X/a$ has a right adjoint $f_* : X/a \to X/b$ for any morphism $f : a \to b$ in X.

42.4. A locally presentable quasi-category X is locally cartesian closed iff the base change map $f^* : X/b \to X/a$ is cocontinuous for any morphism $f : a \to b$ in X.

42.5. (Giraud's theorem)[Lu1] A locally presentable quasi-category X is a homotopos iff the following conditions are satisfied:

- X is locally cartesian closed;
- X is exact;
- the canonical map
$$X/\sqcup a_i \to \prod_i X/a_i$$
is an equivalence for any family of objects $(a_i : i \in I)$ in X.

42.6. A *homomorphism* $X \to Y$ between utopoi is a cocontinuous map $f : X \to Y$ which preserves finite limits. Every homomorphism has a right adjoint. A *geometric morphism* $X \to Y$ between utopoi is an adjoint pair

$$g^* : Y \leftrightarrow X : g_*$$

with g^* a homomorphism. The map g^* is called the *inverse image part of* g and the map g_* the *direct image part*. . We shall denote by **Utop** the category of utopoi and geometric morphisms. The category **Utop** has the structure of a 2-category, where a 2-cell $\alpha : f \to g$ between geometric morphisms is a natural transformation $\alpha : g^* \to f^*$. The opposite 2-category $\mathbf{Utop}^o$ is equivalent to the sub (2-)category of **LP** whose objects are utopoi, whose morphisms (1-cells) are the homomorphisms, and whose 2-cells are the natural transformations.

42.7. If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint maps $u^* : \mathbf{P}(B) \to \mathbf{P}(A) : u_*$ is a geometric morphism $\mathbf{P}(A) \to \mathbf{P}(B)$. If X is a homotopos, then the adjoint pair $f^* : X/b \to X/a : f_*$ is a geometric morphism $X/a \to X/b$ for any arrow $f : a \to b$ in X.

42.8. Recall that if X is a bicomplete quasi-category and A is a simplicial set, then every map $f : A \to X$ has a left Kan extension $f_! : \mathbf{P}(A) \to X$. A locally presentable quasi-category X is a homotopos iff the map $f_! : \mathbf{P}(T) \to X$ is left exact for any cartesian theory T and any cartesian map $f : T \to X$.

42.9. If X is a homotopos, we shall say that a reflexive sub quasi-category $S \subseteq X$ is a *sub-homotopos* if it is locally presentable and the reflection functor $r : X \to S$ preserves finite limits. If $i : S \subseteq X$ is a sub-homotopos and $r : X \to S$ is the reflection, then the pair (r, i) is a geometric morphism $S \to X$. In general, we say that a geometric morphism $g : X \to Y$ is an *embedding* if the map $g_* : X \to Y$ is fully faithful. We say that a geometric morphism $g : X \to Y$ is *surjective* if the map $g^* : Y \to X$ is conservative. The (2-) category **Utop** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class if surjections and $\mathcal{B}$ the class of embeddings.

42.10. If X is a homotopos, then the quasi-category $Dis(X)$ spanned by the 0-objects of X is (equivalent to) a Grothendieck topos. The inverse image part of a geometric morphism $X \to Y$ induces a homorphism $Dis(Y) \to Dis(X)$, hence also a geometric morphism $Dis(X) \to Dis(Y)$. The 2-functor

$$Dis : \mathbf{Utop} \to \mathbf{Gtop}$$

has a right adjoint constructed as follows. If $\mathcal{E}$ is a Grothendieck topos, then the category $[\Delta^o, \mathcal{E}]$ of simplicial sheaves on $\mathcal{E}$ has a simplicial model structure. The coherent nerve of the category of fibrant objects of $[\Delta^o, \mathcal{E}]$ is a homotopos $\hat{\mathcal{E}}$ and there is a canonical equivalence of categories $Dis(\hat{\mathcal{E}}) \simeq \mathcal{E}$. The 2-functor

$$(\hat{-}) : \mathbf{Gtop}(1) \to \mathbf{Utop}$$

is fully faithful and left adjoint to the functor Dis. Hence the (2-)-category $\mathbf{Gtop}$ is a reflective sub-(2)-category of $\mathbf{Utop}$.

42.11. A set Σ of arrows in a homotopos X is called a *Grothendieck topology* if the quasi-category of Σ-local objects $X^\Sigma \subseteq X$ is a sub-homotopos. Every sub-homotopos of X is of the form X^Σ for a Grothendieck topology Σ. In particular, if A is a simplicial set, every sub-homotopos of $\mathbf{P}(A)$ is of the form $\mathbf{P}(A)^\Sigma$ for a Grothendieck topology Σ on A. The pair (A, Σ) is called a *site* and a Σ-local object $f \in \mathbf{P}(A)$ is called a *stack*.

42.12. For every set Σ of arrows in a homotopos X, the sub-quasi-category X^Σ contains a largest sub-homotopos $L(X^\Sigma)$. We shall say that a Grothendieck topology Σ' is *generated* by Σ if we have $X^{\Sigma'} = L(X^\Sigma)$.

is contained in a Grothendieck topology Σ' with the property that a subtopos

then we have $f_*(X) \subseteq Y^\Sigma$ iff f^* take every arrow in Σ to a quasi-isomorphism in X.

42.13. If Σ is Grothendieck topology on Y, then we have $f_*(X) \subseteq Y^\Sigma$ iff f^* take every arrow in Σ to a quasi-isomorphism in X.

42.14. Every simplicial set A generates freely a cartesian quasi-category $A \to C(A)$. Similarly, every simplicial set A generates freely an homotopos $i : A \to UT(A)$. The universality means that every map $f : A \to X$ with values in a homotopos has an homomorphic extension $f' : UT(A) \to X$ which is unique up to a unique invertible 2-cell. By construction, $UT(A) = \mathbf{P}(C(A))$. The map $i : A \to UT(A)$ is obtained by composing the canonical map $A \to C(A)$ with the Yoneda map $C(A) \to \mathbf{P}(C(A))$.

42.15. A *geometric sketch* is a pair (A, Σ), where Σ is a set of arrows in $UT(A)$. A *geometric model* of (A, Σ) with values in a homotopos X is a map $f : A \to X$ whose homomorphic extension $f' : UT(A) \to X$ takes every arrow in Σ to an equimorphism in X. We shall denote by $Mod(A/\Sigma, X)$ the full simplicial subset of X^A spanned by the models $A \to X$.

42.16. Every geometric sktech has a *universal geometric model* $u : A \to UT(A/\Sigma)$. The universality means that for every homotopos X and every geometric model $f : A \to X$ there exists a homomorphism $f' : UT(A/\Sigma) \to X$ such that $f'u = f$, and moreover that f' is unique up to a unique invertible 2-cell. We shall say that $UT(A/\Sigma)$ is the *classifying homotopos* of (A, Σ). The homotopos $UT(A/\Sigma)$ is a sub-homotopos of the homotopos $UT(A)$. We have $UT(A/\Sigma) = UT(A)^{\Sigma'}$, where $\Sigma' \subset UT(A)$ is the Grothendieck topology generated by Σ.

43. Higher categories

We introduce the notions of n-fold category object and of n-category object in a quasi-category. We finally introduced the notion of truncated n-category object.

43.1. Let X be a quasi-category. If A is a simplicial set, we say that a map $f : A \to X$ is *essentially constant* if it belongs to the essential image of the diagonal $X \to X^A$. If A is weakly contractible, then a map $f : A \to X$ is essentially constant iff it takes every arrow in A to an isomorphism in X. A simplicial object $C : \Delta^o \to X$ in a quasi-category X is essentially constant iff the canonical morphism $sk^0(C_0) \to C$ is invertible. A category object $C : \Delta^o \to X$ is essentially constant iff it inverts the arrow $[1] \to [0]$. A n-fold category $C : (\Delta^n)^o \to X$ is essentially constant iff C inverts the arrow $[\epsilon] \to [0^n]$ for every $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{0, 1\}^n$, where $[0^n] = [0, \ldots, 0]$.

43.2. Let X be a cartesian quasi-category. We call a double category $C : \Delta^o \to Cat(X)$ a *2-category* if the simplicial object $C_0 : \Delta^o \to X$ is essentially constant. A double category $C \in Cat^2(X)$ is a 2-category iff it inverts every arrow in $[0] \times \Delta$. Let us denote by Id the set of identity arrows in Δ. Then the set of arrows

$$\Sigma_n = \bigsqcup_{i+1+j=n} Id^i \times [0] \times \Delta^j$$

is a subcategory of Δ^n. We say that a n-fold category object $C \in Cat^n(X)$ is a *n-category* if it inverts every arrow in Σ_n. The notion of n-category object in X can be defined by induction on $n \geq 0$. A category object $C : \Delta^o \to Cat_{n-1}(X)$ is a n-category iff the $(n-1)$-category C_0 is essentially constant. We denote by Cat_n the cartesian theory of n-categories and by $Cat_n(X)$ the quasi-category of n-category objects in X.

43.3. The object of *k-cells* $C(k)$ of a n-category $C : (\Delta^o)^n \to X$ is the image by C of the object $[1^k 0^{n-k}]$. The *source map* $s : C(k) \to C(k-1)$ is the image of the map $[1^{k-1}] \times d_1 \times [0^{n-k}]$ and the *target map* $t : C(k) \to C(k-1)$ is the image of the map $[1^{k-1}] \times d_0 \times [0^{n-k}]$. From the pair of arrows $(s, t) : C(k) \to C(k-1) \times C(k-1)$ we obtain an arrow $\partial : C(k) \to C(\partial k)$, where $C(\partial k)$ is defined by the following pullback square

$$\begin{tikzcd}
C(\partial k) \arrow[r] \arrow[d] & C(k-1) \arrow[d, "{(s,t)}"] \\
C(k-1) \arrow[r, "{(s,t)}"] & C(k-2) \times C(k-2).
\end{tikzcd}$$

If $n = 1$, $\partial = (s, t) : C(1) \to C(0) \times C(0)$.

43.4. There is a notion of n-fold reduced category for every $n \geq 0$. If $RCat$ denotes the cartesian theory of reduced categories, then $RCat^n$ is the theory of n-fold reduced categories. If X is a cartesian quasi-category, then we have

$$RCat^{n+1}(X) = RCat(RCat^n(X))$$

for every $n \geq 0$.

43.5. We say that a n-category $C \in Cat_n(X)$ is *reduced* if it is reduced as a n-fold category. We denote by $RCat_n$ the cartesian theory of reduced n-categories. A n-category $C : \Delta^o \to Cat_{n-1}(X)$ is reduced iff it is reduced as a category object and the $(n-1)$-category C_1 is reduced. If X is an exact quasi-category, then the inclusion $RCat_n(X) \subseteq Cat_n(X)$ has a left adjoint

$$R : Cat_n(X) \to RCat_n(X)$$

which associates to a n-category $C \in Cat_n(X)$ its *reduction* $R(C)$. We call a map $f : C \to D$ in $Cat_n(X)$ an *equivalence* if the map $R(f) : R(C) \to R(D)$ is invertible in $RCat_n(X)$. The quasi-category

$$\mathbf{Typ}_n = Mod(RCat_n)$$

is cartesian closed.

43.6. The object $[0]$ is terminal in Δ. Hence the functor $[0] : 1 \to \Delta$ is right adjoint to the functor $\Delta \to 1$. It follows that the inclusion $i_n : \Delta^n = \Delta^n \times [0] \subseteq \Delta^{n+1}$ is right adjoint to the projection $p_n : \Delta^{n+1} = \Delta^n \times \Delta \to \Delta^n$. For any cartesian quasi-category X, the pair of adjoint maps

$$p_n^* : [(\Delta^o)^n, X] \leftrightarrow [(\Delta^o)^{n+1}, X] : i_n^*$$

induces a pair of adjoint maps

$$inc : Cat_n(X) \leftrightarrow Cat_{n+1}(X) : res.$$

The "inclusion" inc is fully faithful and we can regard it as an inclusion by adopting the same notation for $C \in Cat_n(X)$ and $inc(C) \in Cat_{n+1}(X)$. The map res associates to $C \in Cat_{n+1}(X)$ its *restriction* $res(C) \in Cat_n(X)$. The adjoint pair $p_n \vdash i_n^*$ also induces an adjoint pair

$$inc : RCat_n(X) \leftrightarrow RCat_{n+1}(X) : res.$$

In particular, it induces an adjoint pair

$$inc : \mathbf{Typ}_n \leftrightarrow \mathbf{Typ}_{n+1} : res.$$

When $n = 0$, the map inc is induced by the inclusion $\mathbf{Kan} \subset \mathbf{QCat}$ and the map res by the functor $J : \mathbf{QCat} \to \mathbf{Kan}$. The inclusion $\mathbf{Typ}_n \subset \mathbf{Typ}_{n+1}$ has also a left adjoint which associates to a reduced $(n+1)$-category C the reduced n-category obtained by inverting the $(n+1)$-cells of C.

43.7. Recall from **??** that a quasi-category X is said to be *n-truncated* if the simplicial set $X(a, b)$ is a $(n-1)$-object for every pair $a, b \in X_0$. A quasi-category X has a nerve $NX : \Delta^o \to \mathbf{Typ}$ which is a (reduced) category object in $\mathbf{Typ}$ by 30.18. By construction we have $(NX)_p = J(X^{\Delta[p]})$ for every $p \geq 0$. A quasi-category X is n-truncated iff the morphism $(NX)_1 \to (NX)_0 \times (NX)_0$ is a $(n-1)$-cover.

43.8. Let X be a cartesian quasi-category. We say that a category object C in X is *n-truncated* if the morphism $C_1 \to C_0 \times C_0$ is a $(n-1)$-cover. If C is n-truncated and reduced, then C_k is a n-object for every $k \geq 0$.

43.9. The notion of n-truncated category is essentially algebraic and finitary. We denotes the cartesian theory of n-truncated categories by $Cat[n]$. The notion of n-truncated reduced category is also essentially algebraic. We denotes the cartesian theory of n-truncated reduced categories by $RCat[n]$. The equivalence $N : \mathbf{Typ}_1 \simeq Mod(RCat)$ of 30.18 induces an equivalence

$$\mathbf{Typ}_1[n] \simeq Mod(RCat[n])$$

for every $n \geq 0$. In particular, an ordinary category is essentially the same thing as a 1-truncated reduced category in $\mathbf{Typ}$. Recall from 37.27 that if X is an exact quasi-category, then the inclusion $RCat(X) \subseteq Cat(X)$ has a left adjoint $R : Cat(X) \to RCat(X)$ which associates to a category $C \in Cat(X)$ its *reduction* $R(C)$. If $C \in Cat[n](X)$, then $R(C) \in RCat[n](X)$.

43.10. Let C be a n-category object in a cartesian quasi-category X. If $1 \leq k \leq n$ and $C(k)$ is the object of k-cells of C, then from the pair of arrows $(s,t) : C(k) \to C(k-1) \times C(k-1)$ we obtain an arrow $\partial : C(k) \to C(\partial k)$ by 43.3. If $m \geq n$, we say that C is *m-truncated* if the map $C(n) \to C(\partial n)$ is a $(m-n)$-cover. If $n = 1$, this means that the category C is m-truncated in the sense of 30.19. We shall denote by $Cat_n[m]$ the cartesian theory of m-truncated n-categories. We shall denote by $RCat_n[m]$ the cartesian theory of m-truncated reduced n-categories. If X is an exact quasi-category, then a n-category $C \in Cat_n(X)$ is m-truncated iff its reduction $R(C) \in RCat_n(X)$ is m-truncated. Hence the notion of m-truncated n-category in X is invariant under equivalence of n-categories. If $C \in Cat_n[m](X)$ and $n < m$, then $inc(C) \in Cat_{n+1}[m](X)$. Moreover, if $C \in RCat_n[m](X)$, then $res(C) \in RCat_{n-1}[m](X)$ and C_p is a m-object for every $p \in \Delta^n$. Hence the canonical morphism

$$RCat_n[m] \to RCat_n[m] \odot_c OB(m)$$

is an equivalence of quasi-categories for every $m \geq n$.

44. Higher monoidal categories

The stabilisation hypothesis of Breen-Baez-Dolan was proved by Simpson in [Si2]. We show that it is equivalent to a result of classical homotopy theory 40.30.

44.1. If Mon denotes the theory of monoids, then Mon^k is the theory of k-monoids and $Mon^k \odot Cat_n$ the theory of k-monoidal n-categories. For any cartesian quasi-category X we have

$$Mod(Mon^k \odot Cat_n, X) = Cat_n(Mon^k(X)).$$

If X is an exact quasi-category, then inclusion $RCat_n(Mon^k(X)) \subseteq Cat_n(Mon^k(X))$ has a left adjoint

$$R : Cat_n(Mon^k(X)) \to RCat_n(Mon^k(X)),$$

since the quasi-category $Mon^k(X)$ is exact. We call a map $f : C \to D$ between k-monoidal n-categories in X an *equivalence* if the map $R(f) : R(C) \to R(D)$ is invertible in $RCat_n(Mon^k(X))$.

44.2. An object of the quasi-category $Mod^k(Cat_n[n](X))$ is a k-fold monoidal n-truncated n-category. The stabilisation hypothesis of Baez and Dolan in [BD] can be formulated by saying that the forgetful map

$$Mon^{k+1}(Cat_n[n](X)) \to Mon^k(Cat_n[n](X))$$

is an equivalence if $k \geq n+2$ and $X = \mathbf{Typ}$. But this formulation cannot be totally correct, since it it does use the correct notion of equivalence between n-categories. In order to take this notion into account, it suffices to replace $Cat_n(X)$ by $RCat_n(X)$. If correctly formulated, the hypothesis asserts the forgetful map

$$Mon^{k+1}(RCat_n[n](X)) \to Mon^k(RCat_n[n](X))$$

is an equivalence of quasi-categories if $k \geq n+2$ and $X = \mathbf{Typ}$. A stronger statement is that it is an equivalence for any X. In other words, that the canonical map

$$Mon^k \odot_c RCat_n[n] \to Mon^{k+1} \odot_c RCat_n[n]$$

is an equivalence if $k \geq n+2$.

44.3. Let us show that the stabilisation hypothesis of Breen-Baez-Dolan is equivalent to the Generalised Suspension Conjecture in 40.30. We first prove the the implication GSC$\Rightarrow$BBD. For this it suffices to show by 40.25 that the cartesian theory $Mon^k \odot_c RCat_n[n]$ is semi-additive for $k \geq n+2$. But for this, it suffices to show that the cartesian theory $Mon^{n+2} \odot_c RCat_n[n]$ is semi-additive. Let us show more generally that the the cartesian theory $RCat_n[m] \odot Mon^{m+2}$ is semi-additive for every $m \geq n$. But we have an equivalence $RCat_n[m] \simeq RCat_n[m] \odot_c OB(m)$ by 43.10. Hence it suffices to show that the cartesian theory

$$RCat_n[m] \odot_c OB(m) \odot Mon^{m+2}$$

is semi-additive. But this is true of the cartesian theory $OB(m) \odot Mon^{m+2}$ by the GSC in 40.30. Hence the canonical map

$$RCat_n[m] \odot_c OB(m) \odot Mon^{m+2} \to RCat_n[m] \odot_c OB(m) \odot Mon^{m+3}$$

is an equivalence. The implication GSC$\Rightarrow$BBD is proved. Conversely, let us prove the implication BBD$\Rightarrow$GSC. The cartesian theory $RCat_0[m] \odot Mon^{m+2}$ is semi-additive if we put $n = 0$. But we have $RCat_0[m] = OB(m)$. Hence the cartesian theory $OB(m) \odot Mon^{m+2}$ is semi-additive.

45. Disks and duality

45.1. We begin by recalling the duality between the category Δ and the category of intervals. An *interval* I is a linearly ordered set with a first and last elements respectively denoted $\perp$ and $\top$ or 0 and 1. If $0 = 1$ the interval is *degenerate*, otherwise we say that is *strict*. A *morphism* $I \to J$ between two intervals is defined to be an order preserving map $f : I \to J$ such that $f(0) = 0$ and $f(1) = 1$. We shall denote by $\mathcal{D}(1)$ the category of finite strict intervals (it is the category of finite 1-*disk*). The category $\mathcal{D}(1)$ is the opposite of the category Δ. The duality functor $(-)^* : \Delta^o \to \mathcal{D}(1)$ associates to $[n]$ the set $[n]^* = \Delta([n], [1]) = [n+1]$ equipped with the pointwise ordering. The inverse functor $\mathcal{D}(1)^o \to \Delta$ associates to an interval $I \in \mathcal{D}(1)$ the set $I^* = \mathcal{D}(1)(I, [1])$ equipped with the pointwise ordering. A morphism $f : I \to J$ in $\mathcal{D}(1)$ is surjective (resp. injective) iff the dual morphism $f^* : J^* \to I$" is injective (resp. surjective). A simplicial set is usually defined to be a contravarint functors $\Delta^o \to \mathbf{Set}$; it can be defined to be a covariant functor $\mathcal{D}(1) \to \mathbf{Set}$.

45.2. If I is a strict interval, we shall put $\partial I = \{0,1\}$ and $int(I) = I \setminus \partial I$. We say that a morphism of strict intervals $f : I \to J$ is *proper* if $f(\partial I) \subseteq \partial J$. We shall say that $f : I \to J$ is a *contraction* if it induces a bijection $f^{-1}(int(J)) \to int(J)$. A morphism $f : I \to J$ is a contraction iff it has a unique section. If $\mathcal{A}$ is the class of contractions and $\mathcal{B}$ is the class of proper morphisms then the pair $(\mathcal{A}, \mathcal{B})$ is a factorisation system in $\mathcal{D}(1)$.

45.3. The *euclidian-ball* of dimension $n \geq 0$ $B^n = \{x \in \mathbf{R}^n :\| x \| \leq 1\}$ is the main geometric example of an n-disk. The boundary of the ball is a sphere ∂B^n of dimension $n-1$. The sphere ∂B^n is the union of two disks, the lower an upper hemispheres. In order to describe this structrure, it is convenient to use the projection $q : B^n \to B^{n-1}$ which forget the last coordinate. Each fiber $q^{-1}(x)$ is a strict interval except when $x \in \partial B^{n-1}$ in which case it is reduced to a point. There are two canonical sections $s_0, s_1 : B^{n-1} \to B^n$ obtained by selecting the bottom and the top elements in each fiber. The image of s_0 is the lower hemisphere of ∂B^n and the image of s_1 the upper hemisphere; observe that $s_0(x) = s_1(x)$ iff $x \in \partial B^{n-1}$.

45.4. A *bundle of intervals* over a set B is an interval object in the category $\mathbf{Set}/B$. More explicitly, it is a map $p : E \to B$ whose fibers $E(b) = p^{-1}(b)$ have the structure on an interval. The map p has two canonical sections $s_0, s_1 : B \to E$ obtained by selecting the bottom and the top elements in each fiber. The interval $E(b)$ is degenerated iff $s_0(b) = s_1(b)$. If $s_0(b) = s_1(b)$, we shall say that b is in the *singular set* indexAsingular set—textbf. The projection $q : B^n \to B^{n-1}$ is an example of bundle of intervals. Its singular set is the boundary ∂B^{n-1}. If we order the coordinates in R^n we obtain a sequence of bundles of intervals:

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

45.5. A *n-disk* D is defined to be a sequence of length n of bundles of intervals

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots D_{n-1} \leftarrow D_n$$

such that the singular set of the projection $p : D_{k+1} \to D_k$ is equal to the *boundary* $\partial D_k := s_0(D_{k-1}) \cup s_1(D_{k-1})$ for every $0 \leq k < n$. By convention $\partial D_0 = \emptyset$. If $k = 0$, the condition means that the interval D_1 is strict. It follows from the definition that we have $s_0 s_0 = s_1 s_0$ and $s_0 s_1 = s_1 s_1$. The *interior* of D_k is defined to be $int(D_k) = D_k \backslash \partial D_k$. There is then a decomposition

$$\partial D_n \simeq \bigsqcup_{k=0}^{n-1} 2 \cdot int(D_k).$$

We shall denote by $\mathcal{B}^n$ the n-disks defined by the sequence of projections

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

45.6. A *morphism* between two bundles of intervals $E \to B$ and $E' \to B'$ is a pair of maps (f, g) in a commutative square

$$\begin{tikzcd} B \arrow[d, "f"'] & E \arrow[l] \arrow[d, "g"] \\ B' & E' \arrow[l] \end{tikzcd}$$

such that the map $E(b) \to E'(f(b))$ induced by g is a morphism of intervals for every $b \in B$. A *morphism* $f : D \to D'$ between n-disks is defined to be a commutative diagram

$$\begin{tikzcd} 1 \arrow[d] & D_1 \arrow[l] \arrow[d, "f_1"'] & D_2 \arrow[l] \arrow[d, "f_2"'] & \cdots \arrow[l] & D_{n-1} \arrow[d, "f_{n-1}"'] & D_n \arrow[l] \arrow[d, "f_n"'] \\ 1 & D'_1 \arrow[l] & D'_2 \arrow[l] & \cdots \arrow[l] & D'_{n-1} & D'_n \arrow[l] \end{tikzcd}$$

and which the squares are morphisms of bundles of intervals. Every morphism $f : D \to D'$ can be factored as a surjection $D \to f(D)$ followed by an inclusion $f(D) \subseteq D'$.

45.7. A *planar tree* T of *height* $\leq n$, or a *n-tree*, is defined to be a sequence of maps

$$1 = T_0 \leftarrow T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow T_{n-1} \leftarrow T_n$$

with linearly ordered fibers. If D is a n-disk, then we have $p(int(D_k)) \subseteq int(D_{k-1})$ for every $1 \leq k \leq n$, where p is the projection $D_k \to D_{k-1}$. The sequence of maps

$$1 \leftarrow int(D_1) \leftarrow int(D_2) \leftarrow \cdots int(D_{n-1}) \leftarrow int(D_n)$$

has the structure of a planar tree called the *interior* of D and denoted $int(D)$. Every n-tree T is the interior of a n-disk $\bar{T}$. By construction, we have $\bar{T}_k = T_k \sqcup \partial\bar{T}_k$ for every $1 \leq k \leq n$, where

$$\partial\bar{T}_k = \bigsqcup_{i=0}^{k-1} 2 \cdot T_i.$$

We shall say that $\bar{T}$ is the *closure* of T. We have $\overline{int(D)} = D$ for every disk D. A morphism of disks $f : D \to D'$ is completely determined by its values on the sub-tree $int(D) \subseteq D$. More precisely, a *morphism of trees* $g : S \to T$ is defined to be a commutative diagram

$$\begin{tikzcd} 1 \arrow[d] & S_1 \arrow[l] \arrow[d, "g_1"'] & S_2 \arrow[l] \arrow[d, "g_2"'] & \cdots \arrow[l] & S_{n-1} \arrow[d, "g_{n-1}"'] & S_n \arrow[l] \arrow[d, "g_n"'] \\ 1 & T_1 \arrow[l] & T_2 \arrow[l] & \cdots \arrow[l] & T_{n-1} & T_n \arrow[l] \end{tikzcd}$$

in which f_k preserves the linear order on the fibers of the projections for each $1 \leq k \leq n$. If $Disk(n)$ denotes the category of n-disks and $Tree(n)$ the category of n-trees, then the forgetful functor $Disk(n) \to Tree(n)$ has a left adjoint $T \mapsto \bar{T}$. If $D \in Disk(n)$, then a morphism of trees $T \to D$ can be extended uniquely to a morphism of disks $\bar{T} \to D$. It follows that there a bijection between the morphisms of disks $D \to D'$ and the morphisms of trees $int(D) \to D'$.

45.8. We shall say that a morphism of disks $f : D \to D'$ is *proper* if we have $f(int(D_k)) \subseteq int(D'_k)$ for every $1 \leq k \leq n$. An proper morphism $f : D \to D'$ induces a morphism of trees $int(f) : int(D) \to int(D')$. The functor $T \mapsto \bar{T}$ induces an equivalence between the category $Tree(n)$ and the sub-category of proper morphisms of $Disk(n)$. We shall say that a morphism of disks $f : D \to D'$ is a *contraction* if it induces a bijection $f^{-1}(int(D)) \to int(D')$. Every contraction $f : D \to D'$ has a section and this section is unique. If $\mathcal{A}$ is the class of contractions and $\mathcal{B}$ is the class of proper morphisms then the pair $(\mathcal{A}, \mathcal{B})$ is a factorisation system in $\mathcal{D}(n)$. Every surjection $f : D \to D'$ admits a factorisation $f = up$ with p a contraction and u a proper surjection and this factorisation is essentially unique.

45.9. A *sub-tree* of a n-tree T is a sequence of subsets $S_k \subseteq T_k$ closed under the projection $T_k \to T_{k-1}$ for $1 \le k \le n$ and with $S_0 = 1$. If $T = int(D)$ then the map $C \mapsto C \cap T$ induces a bijection between the sub-disks of D and the sub-trees of T. The set of sub-disks of D is closed under non-empty unions and arbitrary intersections.

45.10. We shall say that a n-disk D is *finite* if D_n is a finite set. The *degree* $\mid D \mid$ of a finite disk D, is defined to be the number of edges of the tree $int(D)$. By definition,

$$\mid D \mid = \sum_{k=1}^{n} \mathrm{Card}(int(D_k)).$$

We have

$$2(1+ \mid D \mid) = \mathrm{Card}(D_n) + \mathrm{Card}(int(D_n)).$$

The set

$$D^\vee = hom(D, \mathcal{B}^n)$$

has the structure of a topological ball of dimension $\mid D \mid$. The space $D^\vee$ has the following description. Let us transport the order relation on the fibers of the planar tree $T = int(D)$ to its edges. Then $D^\vee$ is homeomorphic to the space of maps $f : edges(T) \to [-1, 1]$ which satisfy the following conditions

- $f(e) \le f(e')$ for any two edges $e \le e'$ with the same target;
- $\sum_{e \in C} f(e)^2 \le 1$ for every maximal chain C connecting the root to a leaf.

We can associate to f a map of n-disks $f' : D \to \mathcal{B}^n$ by putting

$$f'(x) = (f(e_1), \cdots, f(e_k))$$

where $(e_1, \cdots, e_k)$ is the chain of edges which connects the root to the vertex $x \in T_k$. The map $f' : D \to \mathcal{B}^n$ is monic iff f belongs to the interior of the ball $D^\vee$. Every finite n-disk D admits an embedding $D \to \mathcal{B}^n$.

45.11. We shall denote by $\Theta(n)$ the category opposite to $\mathcal{D}(n)$. We call an object of $\Theta(n)$ a *cell* of height $\le n$. To every disk $D \in \mathcal{D}(n)$ corresponds a *dual cell* $D^* \in \Theta(n)$ and to every cell $C \in \Theta(n)$ corresponds a *dual disk* $C^* \in \mathcal{D}(n)$. The *dimension* of C is the degree of C^*. A $\Theta(n)$-*set* is defined to be a functor

$$X : \Theta(n)^o \to \mathbf{Set},$$

or equivalently a functor $X : \mathcal{D}(n) \to \mathbf{Set}$. We shall denote by $\hat{\Theta}(n)$ the category of $\Theta(n)$-sets. If t is a finite n-tree we shall denote by $[t]$ the cell dual to the disk $\bar{t}$. The dimension of $[t]$ is the number of edges of t. We shall denote by $\Theta[t]$ the image of $[t]$ by the Yoneda functor $\Theta(n) \to \hat{\Theta}(n)$. The *realisation* of a cell C is defined to be the topological ball $R(C) = (C^*)^\vee$, This defines a functor $R : \Theta(n) \to \mathbf{Top}$, where **Top** denotes the category of compactly generated spaces. Its left Kan extension

$$R_! : \hat{\Theta}(n) \to \mathbf{Top}$$

preserves finite limits. We call $R_!(X)$ the *geometric realisation* of X.

45.12. We shall say that a map $f : C \to E$ in $\Theta(n)$ is *surjective* (resp. *injective*) if the dual map $f^* : E^* \to C^*$ is injective (resp. surjective). Every surjection admits a section and every injection admits a retraction. If $\mathcal{A}$ is the class of surjections and $\mathcal{B}$ is the class of injections, then the pair $(\mathcal{A}, \mathcal{B})$ is a factorisation system in $\Theta(n)$. If D' and D" are sub-disks of a disk $D \in \mathcal{D}(n)$, then the intersection diagram

$$\begin{tikzcd} D' \cap D" \arrow[r] \arrow[d] & D" \arrow[d] \\ D' \arrow[r] & D \end{tikzcd}$$

is *absolute*, ie it is preserved by any functor with codomain $\mathcal{D}(n)$. Dually, for every pair of surjections $f : C \to C'$ and $g : C \to C$" in the category $\Theta(n)$, we have an absolute pushout square (Eilenberg-Zilber lemma). square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & C" \arrow[d] \\ C' \arrow[r] & C'". \end{tikzcd}$$

If X is a $\Theta(n)$-set, we shall say that a cell $x : \Theta[t] \to X$ of dimension $n > 0$ is *degenerate* if it admits a factorisation $\Theta[t] \to \Theta[s] \to X$ via a cell of dimension $< n$, otherwise we shall say that x is *non-degenerate*. Every cell $x : \Theta[t] \to X$ admits a unique factorisation $x = yp\Theta[t] \to \Theta[s] \to X$ with p a surjection and y a non-degenerate cell.

45.13. For each $0 \le k \le n$, let put $b^k = \Theta[t^k]$, where t^k is the tree which consists of a unique chain of k-edges. There is a unique surjection $b^k \to b^{k-1}$ and the sequence of surjections

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk β^n in the topos $\hat{\Theta}^n$. It is the *generic* n-disk in the sense of classifying topos. The geometric realisation of β^n is the euclidian n-disk $\mathcal{B}^n$.

45.14. We shall say that a map $f : C \to E$ in $\Theta(n)$ is *open* (resp. *is an inflation*) if the dual map $f^* : E^* \to C^*$ is proper (resp. is a contraction). Every inflation admits a unique retraction. If $\mathcal{A}$ is the class of open maps in $\Theta(n)$ and $\mathcal{B}$ is the class of inflations then the pair $(\mathcal{A}, \mathcal{B})$ is a factorisation system. Every monomorphism of cells $i : D \to D'$ admits a factorisation $i = qu$ with u an open monomorphism and q an inflation.

45.15. Recall that a *globular set* X is defined to be a sequence of pairs of maps $s_n, t_n : X_{n+1} \to X_n$ $(n \ge 0)$ such that we have

$$s_n s_{n+1} = s_n t_{n+1} \quad \text{and} \quad t_n s_{n+1} = t_n t_{n+1}$$

for every $n \ge 0$. An element $x \in X_n$ is called an *n-cell*; if $n > 0$ the element $s_{n-1}(x)$ is said to be the *source* and the element $t_{n-1}(x)$ to be the *target* of x. A globular set X can be defined to be a presheaf $X : \mathcal{G}^o \to \mathbf{Set}$ on a category $\mathcal{G}$ of *globes* which can be defined by generators and relations. By definition $Ob\mathcal{G} = \{G_0, G_1, \ldots\}$; there are two generating maps $i_0^n, i_1^n : G_n \to G_{n+1}$ for each $n \ge 0$; the relations

$$i_0^{n+1} i_0^n = i_1^{n+1} i_0^n \quad \text{and} \quad i_0^{n+1} i_1^n = i_1^{n+1} i_1^n.$$

is a presentation. The relations imply that there is exactly two maps $i_0, i_1 : G_m \to G_n$ for each $m < n$. A globular set X is thus equipped with two maps $s, t : X_n \to X_m$ for each $m < n$. A *reflexive* globular set is defined to be a globular set X equipped with a sequence of maps $u_n : X_n \to X_{n+1}$ such that $s_n u_n = t_n u_n = id$. By composing we obtain a map $u : X_m \to X_n$ for each $m < n$. There is also a notion of globular set of *height* $\leq n$ for each $n \geq 0$. It can be defined to be a presheaf $\mathcal{G}_n^o \to \mathbf{Set}$, where $\mathcal{G}_n$ is the full sub-category of $\mathcal{G}$ spanned by the globes G_k with $k \leq n$. Notice that a globular set of of height ≤ 0 is the same thing as a set and that globular set of of height ≤ 1 is a graph.

45.16. Recall that a (strict) *category* is a graph $s, t : X_1 \to X_0$, equipped with an associative composition operation

$$\circ : X_1 \times_{s,t} X_1 \to X_1$$

and a unit map $u : X_0 \to X_1$. A *functor* between two categories is a map of graphs $f : X \to Y$ which preserves composition and units. A (strict) ω-*category* is defined to be a reflexive globular set X equipped with a category structure

$$\circ_k : X_n \times_k X_n \to X_n$$

for each $0 \leq k < n$, where $X_n \times_k X_n$ is defined by the pullback square

$$\begin{tikzcd} X_n \times_k X_n \arrow[r] \arrow[d] & X_n \arrow[d, "t"] \\ X_n \arrow[r, "s"] & X_k. \end{tikzcd}$$

The unit map $u : X_k \to X_n$ is given by the reflexive graph structure. The operations should obey the interchange law

$$(x \circ_k y) \circ_m (u \circ_k v) = (x \circ_m u) \circ_k (y \circ_m v)$$

for each $k < m < n$. A *functor* $f : X \to Y$ between ω-categories is a map of globular sets which preserves composition and units. We shall denote by Cat_ω the category of ω-categories. The notion of (strict) n-*category* is defined similarly but by using a globular set of height $\leq n$. We shall denote by Cat_n the category of n-categories.

45.17. We saw in 45.12 that the sequence of cells

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk in the category Θ^n. The lower and upper sections $s_0, s_1 : b^k \to b^(k+1)$ give the sequence the structure of a co-globular set of height $\leq n$. This defines a functor $b : \mathcal{G}_n \to \Theta^n$ from which we obtain a functor

$$b^! : \Theta^n \to \hat{\mathcal{G}}_n.$$

Let us see that the functor $b^!$ can be lifted to Cat_n,

$$\begin{tikzcd} & Cat_n \arrow[d, "U"] \\ \Theta^n \arrow[ur, "\tilde{b}^!"] \arrow[r, "b^!"] & \hat{\mathcal{G}}_n \end{tikzcd}$$

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We shall denote by $\Theta(n)$ the category opposite to $\mathcal{D}(n)$ and by $\Theta(\infty)$ the category opposite to $\mathcal{D}(\infty)$. We call an object of $\Theta(\infty)$ a *cell*. To every disk $D \in \mathcal{D}(\infty)$ corresponds a *dual cell* $D^* \in \Theta(\infty)$ and to every cell $C \in \Theta(\infty)$ corresponds a *dual disk* $C^* \in \mathcal{D}(\infty)$. The *dimension* of C is defined to be the degree of C^* and the *height* of C to be the height of C^*. If t is a finite planar tree, we shall denote by $[t]$ the cell opposite to the disk $\bar{t}$. The dimension of $[t]$ is the number of edges of t and the height of $[t]$ is the height of t.

45.18. The *height* of a n-tree T is defined to be the largest integer $k \geq 0$ such that $T_k \neq \emptyset$. The *height* of a n-disk D is defined to be the height of its interior $int(D)$. If $m < n$, the obvious restriction functor $Disk(n) \to Disk(m)$ has a left adjoint $Ex^n : Disk(m) \to Disk(n)$. The extension functor Ex^n is fully faithful and its essential image is the full subcategory of $Disk(n)$ spanned by the disks of height $\leq n$. We shall identify the category $Disk(m)$ with a full subcategory of $Disk(n)$ by adoptiong the same notation for a disk $D \in Disk(m)$ and its extension $Ex^n(D) \in Disk(n)$. We thus obtain an increasing sequence of coreflexive subcategories,

$$Disk(1) \subset Disk(2) \subset \cdots \subset Disk(n).$$

Hence also an increasing sequence of coreflexive subcategories,

$$\mathcal{D}(1) \subset \mathcal{D}(2) \subset \cdots \subset \mathcal{D}(n).$$

The coreflection functor $\rho^k : \mathcal{D}(n) \to \mathcal{D}(k)$ takes a disk $\overline{T}$ to the sub-disk $\overline{T^k} \subset \overline{T}$, where T^k is the *k-truncation* of T. We shall denote by $\mathcal{D}(\infty)$ the union of the categories $\mathcal{D}(n)$,

$$\mathcal{D}(\infty) = \bigcup_n \mathcal{D}(n)$$

An object of $\mathcal{D}(\infty)$ is an infinite sequence of bundles of finite intervals

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow$$

such that

- the singular set of the projection $D_{n+1} \to D_n$ is the set $\partial D_n := s_0(D_{n-1}) \cup s_1(D_{n-1})$ for every $n \geq 0$;
- the projection $D_{n+1} \to D_n$ is bijective for n large enough.

We have increasing sequence of reflexive subcategories,

$$\Theta(1) \subset \Theta(2) \subset \cdots \subset \Theta(\infty),$$

where $\Theta(k)$ is the full subcategory of $\Theta(\infty)$ spanned by the cells of height $\leq k$. By 45.1, we have $\Theta^1 = \Delta$ A cell $[t]$ belongs to Δ iff the height of t is ≤ 1. If $n \geq 0$ we shall denote by n the unique planar tree height ≤ 1 with n edges. A cell $[t]$ belongs to Δ iff we have $t = n$ for some $n \geq 0$. The reflection functor $\rho^k : \Theta(\infty) \to \Theta(k)$ takes a cell $[t]$ to the cell $[t^k]$, where t^k is the *k-truncation* of t.

45.19. A Θ-*set* of height $\leq n$ is defined to be a functor

$$X : \Theta(n)^o \to \mathbf{Set},$$

or equivalently a functor $X : \mathcal{D}(n) \to \mathbf{Set}$. We shall denote by $\hat{\Theta}(n)$ the category of Θ-sets of height $\leq n$. If t is a finite tree of height $\leq n$, we shall denote by $\Theta[t]$ the image of $[t]$ by the Yoneda functor $\Theta(n) \to \hat{\Theta}(n)$. Consider the functor $R : \Theta(n) \to \mathbf{Top}$ defined by putting $R(C) = (C^*)^\vee = Hom(C^*, \mathcal{B}^n)$, where $\mathbf{Top}$ denotes the category of compactly generated spaces. Its left Kan extension $R : \hat{\Theta}(n) \to \mathbf{Top}$ preserves finite limits. We call $R(X)$ the *geometric realisation* of the Θ-set X.

The left Kan extension of the inclusion

$$\Theta^1 \subset \Theta^m$$

45.20. For each $0 \leq k \leq n$, let us denote by E^k the n-disk whose interior is a chain of k edges. The geometric realisation of dual cell $b^k = (E^k)^*$ is the euclidian k-ball B^k. There is a unique open map of disks $E^{k-1} \to E^k$, hence a map of cells $b^k \to b^{k-1}$. The sequence

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk β^n in the topos $\hat{\Theta}^n$. It is the generic n-disk in the sense of classifying topos.

45.21. Recall that a *globular set* X is defined a sequence of sets $(X_n : n \geq 0)$ equipped with a sequence of pair of maps $s_n, t_n : X_{n+1} \to X_n$ such that we have

$$s_n s_{n+1} = s_n t_{n+1} \quad \text{and} \quad t_n s_{n+1} = t_n t_{n+1}$$

for every $n \geq 0$. An element $x \in X_n$ is called an *n-cell*; if $n > 0$ the element $s_{n-1}(x)$ is said to be the *source* and the element $t_{n-1}(x)$ to be the *target* of x. A globular set X can be defined to be a presheaf $X : \mathcal{G}^o \to \mathbf{Set}$ on a category $\mathcal{G}$ of *globes* which can be defined by generators and relations. By definition $Ob\mathcal{G} = \{G_0, G_1, \ldots\}$; there are two generating maps $i_0^n, i_1^n : G_n \to G_{n+1}$ for each $n \geq 0$; the relations

$$i_0^{n+1} i_0^n = i_1^{n+1} i_0^n \quad \text{and} \quad i_0^{n+1} i_1^n = i_1^{n+1} i_1^n.$$

is a presentation. The relations imply that there is exactly two maps $i_0, i_1 : G_m \to G_n$ for each $m < n$. A globular set X is thus equipped with two maps $s, t : X_n \to X_m$ for each $m < n$. A *reflexive* globular set is defined to be a globular set X equipped with a sequence of maps $u_n : X_n \to X_{n+1}$ such that $s_n u_n = t_n u_n = id$. By composing we obtain a map $u : X_m \to X_n$ for each $m < n$. There is also a notion of globular set of *height* $\leq n$ for each $n \geq 0$. It can be defined to be a presheaf $\mathcal{G}_n^o \to \mathbf{Set}$, where $\mathcal{G}_n$ is the full sub-category of $\mathcal{G}$ spanned by the globes G_k with $k \leq n$. Notice that a globular set of of height ≤ 0 is the same thing as a set and that globular set of of height ≤ 1 is a graph.

45.22. Recall that a (strict) *category* is a graph $s, t : X_1 \to X_0$, equipped with an associative composition operation

$$\circ : X_1 \times_{s,t} X_1 \to X_1$$

and a unit map $u : X_0 \to X_1$. A *functor* between two categories is a map of graphs $f : X \to Y$ which preserves composition and units. A (strict) ω-*category* is defined to be a reflexive globular set X equipped with a category structure

$$\circ_k : X_n \times_k X_n \to X_n$$

for each $0 \leq k < n$, where $X_n \times_k X_n$ is defined by the pullback square

$$\begin{tikzcd} X_n \times_k X_n \arrow[r] \arrow[d] & X_n \arrow[d, "t"] \\ X_n \arrow[r, "s"] & X_k. \end{tikzcd}$$

The unit map $u : X_k \to X_n$ is given by the reflexive graph structure. The operations should obey the interchange law

$$(x \circ_k y) \circ_m (u \circ_k v) = (x \circ_m u) \circ_k (y \circ_m v)$$

for each $k < m < n$. A *functor* $f : X \to Y$ between ω-categories is a map of globular sets which preserves composition and units. We shall denote by Cat_ω the category of ω-categories. The notion of (strict) n-*category* is defined similarly but by using a globular set of height $\leq n$. We shall denote by Cat_n the category of n-categories.

45.23. We saw in 45.20 that the sequence of cells

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk in the category Θ^n. The lower and upper sections $s_0, s_1 : b^k \to b^(k+1)$ give the sequence the structure of a co-globular set of height $\leq n$. This defines a functor $b : \mathcal{G}_n \to \Theta^n$ from which we obtain a functor

$$b^! : \Theta^n \to \hat{\mathcal{G}}_n.$$

Let us see that the functor $b^!$ can be lifted to Cat_n,

$$\begin{tikzcd} & Cat_n \arrow[d, "U"] \\ \Theta^n \arrow[ur, "\tilde{b}^!"] \arrow[r, "b^!"'] & \hat{\mathcal{G}}_n \end{tikzcd}$$

if $0 \leq k \leq n$, let us denote by E^k the n-disk whose interior is a chain of k edges. There is a unique element $e^k \in int(E^k)_k$. The interval over e^k has exactly two points. There are two map of disks $p_0, p_1 : E^k \to E^{k-1}$. The first takes $e^k \in E^k$ to the top element of the interval over $e^{k-1} \in E^{k-1}$, and the second to the top element of the interval over $e^{k-1} \in E^{k-1}$.

There is a unique map of disks $e_{k-1} \to e_k$ and two maps of disks

let us denote by e_k the n-disk whose interior is a chain of k edges. The geometric realisation of the cell $b^k = {}^*e_k$ is the euclidian n-ball. There is a unique map of disks $e_{k-1} \to e_k$, hence also a unique map of cells $b^k \to b^{k-1}$. The sequence

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk b in the topos $\hat{\Theta}^n$. It is the generic n-disk in the sense of classifying topos.

45.24. The *composite* $D \circ E$ of a n-disk D with a m-disk E is the $m + n$ disk

$$1 = D_0 \leftarrow D_1 \leftarrow \cdots \leftarrow D_n \leftarrow (D_n, \partial D_n) \times E_1 \leftarrow \cdots \leftarrow (D_n, \partial D_n) \times E_m,$$

where $(D_n, \partial D_n) \times E_k$ is defined by the pushout square

$$\begin{tikzcd} \partial D_n \times E_k \arrow[r] \arrow[d] & D_n \times E_k \arrow[d] \\ E_k \arrow[r] & (D_n, \partial D_n) \times E_k. \end{tikzcd}$$

This composition operation is associative.

45.25. The category $\mathbf{S}^{(n)} = [(\Delta^n)^o, \mathbf{Set}]$, contains n intervals

$$I_k = 1\Box 1\Box \cdots 1\Box I\Box 1 \cdots 1\Box 1,$$

one for each $0 \le k \le n$. It thus contain a n-disk $I^{(n)} : I_1 \circ I_2 \circ \cdots \circ I_n$. Hence there is a geometric morphism

$$(\rho^*, \rho_*) : \mathbf{S}^{(n)} \to \hat{\Theta},$$

such that $\rho^*(b) = I^{(n)}$. We shall say that a map of Θ^n-sets $f : X \to Y$ is a *weak categorical equivalence* if the map $\rho^*(f) : \rho^*(X) \to \rho^*(Y)$ is a weak equivalence in the model structure for reduced Segal n-spaces. The category $\hat{\Theta}_n$ admits a model structure in which the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. We shall say that a fibrant object is a Θ^n*-category*. The model structure is cartesian closed and left proper. We call it the *model structure for* Θ^n*-categories*. We denote by $\Theta^n\mathbf{Cat}$ the category of Θ^n-categories. The pair of adjoint functors

$$\rho^* : \hat{\Theta}_n \to \mathbf{S}^{(n)} : \rho_*$$

is a Quillen equivalence between the model structure for Θ^n-categories and the model structure for reduced Segal n-spaces.

46. Higher quasi-categories

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A n*-quasi-category* can be defined to be a fibrant object with respect to a certain model structure structure on the category of presheaves on certain category Θ_n. The category Θ_n was introduced for this purpose by the author in 1998. It was first defined as the opposite of the category of finite n-disks. It was later conjectured (jointly by Batanin, Street and the author) to be isomorphic to a category T_n^* introduced by Batanin in his theory of higher operads [**?**]. The category T_n^* is a full subcategory of the category of strict n-categories. The conjecture was proved by Makkai and Zawadowski in [MZ] and by Berger in [Ber]. The model structure for n-quasi-categories can be described in various ways. In principle, the model structure for n-quasi-categories can be described by specifying the fibrant objects, since the cofibrations are supposed to be the monomorphisms. But a complete list of the filling conditions defining the n-quasi-categories is still missing (a partial list was proposed by the author in 1998). An alternative approach is find a way of specifying the class $Wcat_n$ of weak equivalences (the *weak categorical* n*-equivalences*). Let us observe that the class *Wcat* in $\mathbf{S}$ can be extracted from the canonical map $i : \Delta \to \mathbf{U}_1$, since a map of simplicial sets $u : A \to B$ is a weak categorical equivalence if the arrow $i_!(u) : i_!A \to i_!B$ is invertible in $\mathbf{U}_1$, where $i_! : \hat{\Delta} \to \mathbf{U}_1$ denotes the left Kan extension of i along the Yoneda functor. In general, it should suffices to exibit a map $i : \Theta_n \to \mathbf{U}_n$ with values in a cocomplete quasi-category chosen appropriately. The quasi-category $\mathbf{U}_1$ is equivalent to the quasi-category of reduced category object in $\mathbf{U}$. It seems reasonable to suppose that $\mathbf{U}_n$ is the quasi-category of reduced n-category object in $\mathbf{U}$. A n*-category object* in $\mathbf{U}$ is defined to be a map $C : \Theta_n^o \to \mathbf{U}$ satisfying a certain Segal condition. A n-category C is *reduced* if every invertible cell of C is a unit. The notion of reduced n-category object is essentially algebraic. Hence the quasi-category $\mathbf{U}_n$ is cocomplete, since it is locally presentable. The canonical map $i : \Theta_n \to \mathbf{U}_n$ is obtained from the inclusion of Θ_n in the category of reduced strict n-categories. A map $u : A \to B$ in $\hat{\Theta}_n$ is then defined to be a *weak categorical n-equivalence* if the arrow $i_!(u) : i_!A \to i_!B$ is invertible in $\mathbf{U}_n$, where $i_! : \hat{\Theta}_n \to \mathbf{U}_n$ denotes the left Kan extension of i along the Yoneda functor. The model category $(\hat{\Theta}_n, Wcat_n)$ is cartesian closed and its full subcategory of fibrant objects $\mathbf{QCat_n}$ has the structure of a simplicial category enriched over Kan complexes. We conjecture that the coherent nerve of $\mathbf{QCat_n}$ is equivalent to $\mathbf{U}_n$. There is another description of $Wcat_n$ which is conjectured by Cisinski and the author. It is easy to show that the localizer $Wcat$ is generated by inclusions $I[n] \subseteq \Delta[n]$ $(n \geq 0)$, where $I[n]$ is the union of the edges $(i-1, i)$ for $1 \leq i \leq n$. The simplicial set $I[n]$ is said to be the *spine* of $\Delta[n]$. The objects of Θ_n are indexed by finite planar trees of height $\leq n$. For each tree t, let us denote by $\Theta[t]$ the representable presheaf generated by the object $[t]$ of Θ_n. The *spine* $S[t] \subseteq \Theta[t]$ is the union of the generators of the n-category $[t]$ (it is the globular diagram associated to t by Batanin). It is conjectured that $Wcat_n$ is the localizer generated by the inclusions $S[t] \subseteq \Theta[t]$.

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46.1. There is a notion of *n-fold Segal space* for every $n \geq 1$. Recall that the category $[(\Delta^o)^n, \mathbf{S}] = \mathbf{S}^{(n)}\mathbf{S}$ of n-fold simplicial spaces admits a Reedy model structure in which the weak equivalences are the level wise weak homotopy equivalences and the cofibrations are the monomorphisms. A *n-fold Segal space* is defined to be a Reedy fibrant n-fold simplicial space $C : (\Delta^o)^n \to \mathbf{S}$ which satisfies the Segal condition **??** in each variable. The Reedy model structure admits a Bousfield localisation in which the fibrant objects are the n-fold Segal spaces. The model structure is simplicial. It is the *model structure for n-fold Segal spaces*. The coherent nerve of the simplicial category of n-fold Segal spaces is equivalent to the quasi-category $Cat^n(\mathbf{Typ})$.

46.2. There is a notion of *n-fold Rezk space* for every $n \geq 1$. It is a n-fold Segal space which satisfies the Rezk condition **??** in each variable. The Reedy model structure admits a Bousfield localisation in which the fibrant objects are the n-fold Rezk spaces. The model structure is simplicial. It is the *model structure for n-fold Rezk spaces*. The coherent nerve of the simplicial category of n-fold Rezk spaces is equivalent to the quasi-category $RCat^n(\mathbf{Typ})$.

46.3. There is a notion of *Segal n-space* for every $n \geq 1$. It is defined by induction on $n \geq 1$. If $n = 1$, it is a Segal space $C : \Delta^o \to \mathbf{S}$. If $n > 1$, it is a n-fold Segal space $C : \Delta^o \to \mathbf{S}^{(n-1)}\mathbf{S}$ such that

- C_k is a Segal n-space for every $k \geq 0$,
- $C_0 : (\Delta^o)^{n-1} \to \mathbf{S}$ is homotopically constant.

The model structure for n-fold Segal spaces admits a Bousfield localisation in which the fibrant objects are the Segal n-spaces. The model structure is simplicial. It is the *model structure for Segal n-spaces*. The coherent nerve of the simplicial category of Segal n-spaces is equivalent to the quasi-category $Cat_n(\mathbf{Typ})$.

46.4. There is a notion of *Rezk n-space* for every $n \geq 1$. By definition, it is a Segal n-space which satisfies the Rezk condition **??** in each variable. The model structure for Segal n-spaces admits a Bousfield localisation for which the fibrant objects are the Rezk n-spaces. It is the *model structure for Rezk n-spaces*. The coherent nerve of the simplicial category of Rezk n-spaces is equivalent to the quasi-category $RCat_n(\mathbf{Typ})$.

46.5. There is a notion of *n-fold quasi-category* for every $n \geq 1$. If $n = 1$, this is a quasi-category. The projection $p : \Delta^n \times \Delta \to \Delta^n$ is left adjoint to the functor $i : \Delta^n \to \Delta^n \times \Delta$ defined by putting $i(a) = (0, [0])$ for every $n \geq 0$. We thus obtain a pair of adjoint functors

$$p^* : \mathbf{S}^{(n)} \leftrightarrow \mathbf{S}^{(n+1)} : i^*.$$

Let us say that a map $f : X \to Y$ in $\mathbf{S}^{(n)}$ is a *weak equivalence* if the map $p^*(f)$ is a weak equivalence in the model structure for n-fold Rezk spaces. Then the category $\mathbf{S}^{(n)}$ admits a unique Cisinski model structure with these weak equivalences. We call it the *model structure for n-fold quasi-categories*, A fibrant object for this model structure is a *n-fold quasi-category*. The pair of adjoint functors (p^*, i^*) is a Quillen equivalence between the model structure for n-fold quasi-categories and the model structure for n-fold Rezk spaces.

46.6. There is box product functor

$$\Box : \mathbf{S}^{(m)} \times \mathbf{S}^{(n)} \to \mathbf{S}^{(m+n)}$$

for every $m, n \geq 0$. The functor is a left Quillen functor of two variables with respect to the model structures for p-fold quasi-categories, where $p \in \{m, n, m+n\}$.

46.7. There is a notion of *quasi-n-category* for every $n \geq 1$. Let $p^* : \mathbf{S}^{(n)} \leftrightarrow \mathbf{S}^{(n+1)} : i^*$ be the pair of adjoint functors of 46.5. Let us say that a map $f : X \to Y$ in $\mathbf{S}^{(n)}$ is a *weak equivalence* if the map $p^*(f)$ is a weak equivalence in the model structure for Rezk n-spaces. Then the category $\mathbf{S}^{(n)}$ admits a unique Cisinski model structure with these weak equivalences. We call it the *model structure for quasi-n-categories*. A fibrant object for this model structure is a *quasi-n-category*. The pair of adjoint functors (p^*, i^*) is a Quillen equivalence between the model structure for quasi-n-categories and the model structure for Rezk n-spaces.

46.8. The *composite* $D \circ E$ of a n-disk D with a m-disk E is the $m + n$ disk

$$1 = D_0 \leftarrow D_1 \leftarrow \cdots \leftarrow D_n \leftarrow (D_n, \partial D_n) \times E_1 \leftarrow \cdots \leftarrow (D_n, \partial D_n) \times E_m,$$

where $(D_n, \partial D_n) \times E_k$ is defined by the pushout square

$$\begin{tikzcd}
\partial D_n \times E_k \arrow[r] \arrow[d] & D_n \times E_k \arrow[d] \\
E_k \arrow[r] & (D_n, \partial D_n) \times E_k.
\end{tikzcd}$$

This composition operation is associative.

46.9. The category $\mathbf{S}^{(n)} = [(\Delta^n)^o, \mathbf{Set}]$, contains n intervals

$$I_k = 1\Box 1\Box \cdots 1\Box I\Box 1 \cdots 1\Box 1,$$

one for each $0 \leq k \leq n$. It thus contain a n-disk $I^{(n)} : I_1 \circ I_2 \circ \cdots \circ I_n$. Hence there is a geometric morphism

$$(\rho^*, \rho_*) : \mathbf{S}^{(n)} \to \hat{\Theta},$$

such that $\rho^*(b) = I^{(n)}$. We shall say that a map of n-cellular sets $f : X \to Y$ is a *weak categorical equivalence* if the map $\rho^*(f) : \rho^*(X) \to \rho^*(Y)$ is a weak equivalence in the model structure for quasi-n-categories. The category $\hat{\Theta}_n$ admits a model structure in which the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. We say that a fibrant object is a *n-quasi-category* The model structure is cartesian closed and left proper. We call it the *model structure for n-quasi-categories*. We denote the category of n-quasi-categories by $\mathbf{QCat}_n$. The pair of adjoint functors

$$\rho^* : \hat{\Theta}_n \to \mathbf{S}^{(n)} : \rho_*$$

is a Quillen equivalence between the model structure for n-quasi-categories and the model structure for quasi-n-categories.

47. Appendix on category theory

47.1. We fix three arbitrary Grothendieck universes $\mathbf{U}_1 \in \mathbf{U}_2 \in \mathbf{U}_3$. Sets in $\mathbf{U}_1$ are said to be *small*, sets in $\mathbf{U}_2$ are said to be *large* and sets in $\mathbf{U}_3$ are said to be *extra-large*. Beware that a small set is large and that a large set is extra-large. We denote by **Set** the category of small sets and by **SET** the category of large sets. A category is said to be *small* (resp. *large*, *extra-large*) if its set of arrows belongs to $\mathbf{U}_1$ (resp. $\mathbf{U}_2$, $\mathbf{U}_3$). The category **Set** is large and the category **SET** extra-large. We denote by **Cat** the category of small categories and by **CAT** the category of large categories. The category **Cat** is large and the category **CAT** is extra-large. A large category is *locally small* if its hom sets are small. We shall often denote small categories by ordinary capital letters and large categories by curly capital letters.

47.2. We shall denote by A^o the opposite of a category A. It can be useful to distinguish between the objects of A and A^o by writing $a^o \in A^o$ for each object $a \in A$, with the convention that $a^{oo} = a$. If $f : a \to b$ is a morphism in A, then $f^o : b^o \to a^o$ is a morphism in A^o. Beware that the *opposite* of a functor $F : A \to B$ is a functor $F^o : A^o \to B^o$. A *contravariant functor* $F : A \to B$ between two categories is defined to be a (covariant) functor $A^o \to B$; if $a \in A$, we shall often write $F(a)$ instead of $F(a^o)$. .

47.3. We say that a functor $u : A \to B$ is *biunivoque* if the map $Ob(u) : ObA \to ObB$ is bijective. Every functor $u : A \to B$ admits a factorisation $u = pq$ with q a biunivoque functor and p a fully faithful functor. The factorisation is unique up to unique isomorphism. It is called the *Gabriel factorisation* of the functor u.

47.4. The categories **Cat** and **CAT** are cartesian closed. We shall denote by B^A or $[A, B]$ the category of functors $A \to B$ between two categories. If $\mathcal{E}$ is a locally small category, then so is the category $\mathcal{E}^A = [A, \mathcal{E}]$ for any small category A. Recall that a *presheaf*

on a small category A is defined to be a functor $X : A^o \to \mathbf{Set}$. A *map* of presheaves $X \to Y$ is a natural transformation. The presheaves on A form a locally small category

$$\hat{A} = \mathbf{Set}^{A^o} = [A^o, \mathbf{Set}].$$

The category $\hat{A}$ is cartesian closed and we shall denote by Y^X. the presheaf of maps $X \to Y$ between two presheaves.

47.5. The *Yoneda functor* $y_A : A \to \hat{A}$ associates to an object $a \in A$ the presheaf $A(-,a)$. The *Yoneda lemma* asserts that for any object $a \in A$ and any presheaf $X \in \hat{A}$, the evaluation $x \mapsto x(1_a)$ induces a bijection between the set of maps $x : A(-,a) \to X$ and $X(a)$. We shall identify these two sets by adopting the same notation for a map $x : A(-,a) \to X$ and the element $x(1_a) \in X(a)$. The Yoneda functor is fully faithful. and we shall often regard it as an inclusion $A^o \subset \hat{A}$ by adopting the same notation for an object $a \in A$ and the presheaf $A(-,a)$. In this notation, the map $xu : a \to X$ obtained by composing a map $x : b \to X$ with a morphism $u : a \to b$ in A is the element $X(u)(x) = X(u)(x(1_b)) \in X(a)$. We say that a presheaf X is *represented* by an element $x \in X(a)$ if the map $x : a \to X$ is invertible in $\hat{A}$. A presheaf X is said to be *representable* if it can be represented by a pair (a,x). Recall that the *category of elements* $El(X)$ of a presheaf $X : A \to \mathbf{Set}$ is the category whose objects are the pairs (a,x), where $a \in A$ and $x \in X(a)$, and whose arrows $(a,x) \to (b,y)$ are the morphism $f : a \to b$ in A such that $X(f)(y) = x$. It follows from Yoneda lemma that we have $El(X) = A/X$, where A/X is the full subcategory of $\hat{A}/X$ whose objects are the maps $a \to X$ with $a \in A$. A presheaf X is represented by an element $x \in X(a)$ iff the object (a,x) of $El(X)$ is terminal. Thus, a presheaf X representable iff its category of elements $El(X)$ has a terminal object.

47.6. The *dual Yoneda functor* $y_A^o : A^o \to [A, \mathbf{Set}]$ associates to an object $a \in A$ the set valued functor $A(a,-)$. The *Yoneda lemma* asserts that for any object $a \in A$ and any functor $F : A \to \mathbf{Set}$, the evaluation $x \mapsto x(1_a)$ induces a bijection between the set of natural transformations $x : A(a,-) \to F$ and $F(a)$. We shall identify these two sets by adopting the same notation for a natural transformation $x : A(a,-) \to F$ and the element $x(1_a) \in F(a)$. The dual Yoneda functor is fully faithful. and we shall often regard it as an inclusion $A^o \subset [A, \mathbf{Set}]$ by adopting the same notation for an object $a^o \in A^o$ and the presheaf $A(a,-)$. We say that a functor $F : A \to \mathbf{Set}$ is *represented* by an element $x \in F(a)$ if the corresponding natural transformation $x : a^o \to X$ is invertible. The functor F is said to be *representable* if it can be represented by an element (a,x). The *category of elements* of a (covariant) functor $F : A \to \mathbf{Set}$ is the category $el(F)$ whose objects are the pairs (a,x), where $a \in A$ and $x \in F(a)$, and whose arrows $(a,x) \to (b,y)$ are the morphisms $f : a \to b$ in A such that $F(f)(x) = y$. The functor X is represented by an element $x \in F(a)$ iff (a,x) is an initial object of the category $el(X)$. Thus, F representable iff the category $el(F)$ has an initial object.

47.7. Recall that a *2-category* is a category enriched over $\mathbf{Cat}$. An object of a 2-category $\mathcal{E}$ is often called a 0-*cell*. If A and B are 0-cells, an object of the category $\mathcal{E}(A,B)$ is called a 1-*cell* and an arrow is called a 2-*cell*. We shall often write $\alpha : f \to g : A \to B$ to indicate that α is a 2-cell with source the 1-cell $f : A \to B$ and target the 1-cell $g : A \to B$. The composition law in the category $\mathcal{E}(A,B)$ is said to be *vertical* and the composition law

$$\mathcal{E}(B,C) \times \mathcal{E}(A,B) \to \mathcal{E}(A,C)$$

horizontal. The vertical composition of a 2-cell $\alpha : f \to g$ with a 2-cell $\beta : g \to h$ is a 2-cell denoted by $\beta\alpha : f \to h$. The horizontal composition of a 2-cell $\alpha : f \to g : A \to B$ with a 2-cell and $\beta : u \to v : B \to C$ is a 2-cell denoted by $\beta \circ \alpha : uf \to vg : A \to C$.

47.8. There is a notion of adjoint in any 2-category. If $u : A \to B$ and $v : B \to A$ are 1-cells in a 2-category, an *adjunction* $(\alpha, \beta) : u \dashv v$ is a pair of 2-cells $\alpha : 1_A \to vu$ and $\beta : uv \to 1_B$ for which the *adjunction identities* hold:

$$(\beta \circ u)(u \circ \alpha) = 1_u \quad \text{and} \quad (v \circ \beta)(\alpha \circ v) = 1_v.$$

The 1-cell u is the *left adjoint* and the 1-cell v the *right adjoint*. The 2-cell α is the *unit* of the adjunction and the 2-cell β the *counit*. Each of the 2-cells α and β determines the other.

47.9. In any 2-category, there is a notion of left (and right) Kan extension of 1-cell $f : A \to X$ along a 1-cell $u : A \to B$. More precisely, the *left Kan extension* of f along u is a pair (g, α) where $g : B \to X$ and $\alpha : f \to gu$ is a 2-cell which reflects the object $f \in Hom(A, X)$ along the functor $Hom(u, X) : Hom(B, X) \to Hom(A, X)$. The *right Kan extension* of f along u is a pair (g, β) where $g : B \to X$ and $\beta : gu \to f$ is a 2-cell which coreflects the object $f \in Hom(A, X)$ along the functor $Hom(u, X) : Hom(B, X) \to Hom(A, X)$.

47.10. Recall that a full subcategory $A \subseteq B$ is said to be *reflective* if the inclusion functor $A \subseteq B$ has a left adjoint $r : B \to A$ called a *reflection*. In general, the right adjoint v of an adjunction $u : A \leftrightarrow B : v$ is fully faithful iff the counit of the adjunction $\beta : uv \to 1_B$ is invertible, in which case u is said to be a *reflection* and v to be *reflective*. These notions can be defined in any 2-category. The left adjoint u of an adjunction $u : A \leftrightarrow B : v$ is said to be a *reflection* if the counit of the adjunction $\beta : uv \to 1_B$ is invertible, in which case v is said to be *reflective*. There is a dual notion of coreflection: the right adjoint v of an adjunction $u : A \leftrightarrow B : v$ is said to be a *coreflection* if the unit of the adjunction $\alpha : 1_A \to vu$ is invertible, in which case u is said to be *coreflective*.

47.11. If A and B are small categories, a functor $F : A^o \times B \to \mathbf{Set}$ is called a *distributor* $F : A \to B$. The *composite* two distributors $F : A \to B$ and $G : B \to C)$ is the distributor $G \circ F : A \to C$ obtained by putting

$$G \circ F = (F \otimes_B G)(a, c) = \int^{b \in B} F(a, b) \times G(b, c).$$

This composition operation is coherently associative, and the distributor $hom : A^o \times A \to \mathbf{S}$ is a unit. This defines a bicategory $Dist$ whose objects are the small categories. The bicategory $Dist$ is biclosed. This means that the composition functor $\circ$ is divisible on each side. See 51.25 for this notion. For every $H \in Dist(A, C)$, $F \in Dist(A, B)$ and $G \in Dist(B, C)$ we have

$$G \backslash H = Hom_C(G, H) \quad \text{and} \quad H/F = Hom_A(F, H).$$

Notice that $Dist(1, A) = [A, \mathbf{Set}]$ and that $Dist(A, 1) = \hat{A}$. To every distributor $F \in Dist(A, B)$ we can associate a cocontinuous functor $- \circ F : Dist(B, 1) \to Dist(A, 1)$. This defines an equivalence between the category $Dist(A, B)$ and the category of cocontinuous functors $\hat{B} \to \hat{A}$. Dually, to every distributor $F \in Dist(A, B)$ we can associate a cocontinuous functor $F \circ - : Dist(1, A) \to Dist(1, B)$. This defines an equivalence between $Dist(A, B)$ and the category of cocontinuous functors $[A, \mathbf{Set}] \to [B, \mathbf{Set}]$.

47.12. The bicategory *Dist* is symmetric monoidal. The *tensor product* of $F \in Dist(A, B)$ and $G \in Dist(C, D)$ is the distributor $F \otimes G \in Dist(A \times C, B \times D)$ defined by putting

$$(F \times G)((a, c), (b, d)) = F(a, b) \times G(c, d)$$

for every quadruple of objects $(a, b, c, d) \in A \times B \times C \times D$.

47.13. The symmetric monoidal bicategory *Dist* is compact closed. The *dual* of a category A is the opposite category A^o and the *adjoint* of a distributor $F \in Dist(A, B)$ is the distributor $F^* \in Dist(B^o, A^o)$ obtained by putting $F^*(b^o, a^o) = F(a, b)$. The unit of the adjunction $A \vdash A^o$ is a distributor $\eta_A \in Dist(1, A^o \times A)$ and the counit a distributor $\epsilon_A \in Dist(A \times A^o, 1)$. We have $\eta_A = \epsilon_A = Hom_A : A^o \times A \to \mathbf{Set}$. The adjunction $A \vdash A^o$ is defined by a pair of invertible 2-cells,

$$\alpha_A : I_A \simeq (\epsilon_A \otimes A) \circ (A \otimes \eta_A) \quad \text{and} \quad \beta_A : I_{A^o} \simeq (A^o \otimes \epsilon_A) \circ (\eta_A \otimes A^o).$$

each of which is defined by using fthe canonical isomorphism

$$\int_{b \in A} \int_{c \in A} A(a, b) \times A(b, c) \times A(c, d) \simeq A(a, d).$$

47.14. The *trace* of a distributor $F \in Dist(A, A)$ defined by putting

$$Tr_A(F) = \epsilon_A \circ (F \otimes A^o) \circ \eta_{A^o}$$

is isomorphic to the coend of the functor $F : A^o \times A \to \mathbf{Set}$

$$coend_A(F) = \int^{a \in A} F(a, a).$$

47.15. To every functor $u : A \to B$ in **Cat** is associated a pair of adjoint functor

$$u_! : [A^o, \mathbf{Set}] \leftrightarrow [B^o, \mathbf{Set}] : u^*.$$

We have $u^*(Y) = \Gamma(u) \otimes_B Y = Y \circ \Gamma(u)$ for every $Y \in [B^o, \mathbf{Set}]$, where the distributor $\Gamma(u) \in Dist(A, B)$ obtained by putting $\Gamma(u)(a, b) = B(ua, b)$ for every pair of objects $a \in A$ and $b \in B$. We have $u_!(X) = \Gamma^*(u) \otimes_A X = X \circ \Gamma^*(u)$ for every $X \in [A^o, \mathbf{Set}]$, where the distributor $\Gamma(u) \in Dist(B, A)$ is defined by putting $\Gamma^*(u)(b, a) = B(b, ua)$. Notice that the functor u^* has a right adjoint u_* and that we have $u_*(X) = X/\Gamma(u)$ for every $X \in [A^o, \mathbf{Set}]$.

47.16. Recall that a set S of objects in a category A is called a *sieve* if the implication

$$\text{target}(f) \in S \Rightarrow \text{source}(f) \in S$$

is true for every arrow $f \in A$. We shall often identify S with the full subcategory of A spanned by S. A *cosieve* in A is defined dually. The opposite of a sieve $S \subseteq A$ is a cosieve $S^o \subseteq A^o$. For any sieve $S \subseteq A$ (resp. cosieve), there exists a unique functor $p : A \to I$ such that $S = p^{-1}(0)$ (resp. $S = p^{-1}(1)$). We shall say that the sieve $p^{-1}(0)$ and the cosieve $p^{-1}(1)$ are *complementary*. Complementation defines a bijection between the sieves and the cosieves of A.

47.17. We shall say that an object of the category $\mathbf{Cat}/I$ is a *categorical cylinder*, or just a *cylinder* if the context is clear. The *cobase* of a cylinder $p : C \to I$ is defined to be the the sieve $C(0) = p^{-1}(0)$ and its *base* to be the cosieve $C(1) = p^{-1}(1)$. If i denotes the inclusion $\partial I \subset I$, then the functor

$$i^* : \mathbf{Cat}/I \to \mathbf{Cat} \times \mathbf{Cat}$$

has left adjoint $i_!$ and a right adjoint i_*. The functor i^* is a Grothendieck bifibration, since it is an isofibration and the functor $i_!$ is fully faithful. Its fiber at (A, B) is the category $Cyl(A, B)$ of cylinders with cobase A and base B. The cylinder $i_!(A, B) = A \sqcup B$ is the initial object of $Cyl(A, B)$ and the cylinder $i_*(A, B) = A \star B$ is the terminal object.

47.18. The category $\mathbf{Cat}/I$ is cartesian closed. The model structure $(\mathbf{Cat}, Eq)$ induces a cartesian closed model structure on the category $\mathbf{Cat}/I$.

47.19. To every cylinder $C \in Cyl(A, B)$ we can associate a distributor $D(C) \in Dist(A, B)$ by putting $D(C)(a, b) = C(a, b)$ for every pair of objects $a \in A$ and $b \in B$. The functor

$$D : Cyl(A, B) \to Dist(B, A)$$

so defined is an equivalence of categories. The inverse equivalence associate to a distributor $F : A^o \times B$ the *collage cylinder* $C = col(F) = A \star_F B$ constructed as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for $x, y \in Ob(A) \sqcup Ob(B)$,

$$C(x, y) = \begin{cases} A(x, y) & \text{if } x \in A \text{ and } y \in A \\ B(x, y) & \text{if } x \in B \text{ and } y \in B \\ F(x, y) & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition of arrows is obvious. The category C has the structure of a cylinder with base B and cobase A. The resulting functor

$$col : Dist(A, B) \to Cyl(A, B)$$

is an equivalence of categories. The collage of the distributor $hom : A^o \times A \to \mathbf{Set}$ is the cylinder $A \times I$; the collage of the terminal distributor $1 : A^o \times B \to \mathbf{Set}$ is the join $A \star B$; the collage of the empty distributor $\emptyset : A^o \times A \to \mathbf{Set}$ is the coproduct $A \sqcup A$.

48. Appendix on factorisation systems

In this appendix we study the notion of factorisation system. We give a few examples of factorisation systems in $\mathbf{Cat}$.

Definition 48.1. *If $\mathcal{E}$ is a category, we shall say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a* (strict) factorisation system *if the following conditions are satisfied:*

- *each class $\mathcal{A}$ and $\mathcal{B}$ is closed under composition and contains the isomorphisms;*
- *every map $f : A \to B$ admits a factorisation $f = pu : A \to E \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$, and the factorisation is unique up to unique isomorphism.*

We say that $\mathcal{A}$ is the left *class and $\mathcal{B}$ the* right *class of the weak factorisation system.*

In this definition, the uniqueness of the factorisation $f = pu : A \to E \to B$ means that for any other factorisation $f = p'u' : A \to E' \to B$ with $u' \in \mathcal{A}$ and $p' \in \mathcal{B}$, there exists a unique isomorphism $i : E \to F$ such that $iu = u'$ and $p'i = p$,

$$\begin{tikzcd} A \arrow[r, "u'"] \arrow[d, "u"'] & E' \arrow[d, "p'"] \\ E \arrow[r, "p"'] \arrow[ur, dashed, "i"] & B. \end{tikzcd}$$

Recall that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *invariant under isomorphisms* if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & A' \arrow[d, "u'"] \\ B \arrow[r] & B' \end{tikzcd}$$

in which the horizontal maps are isomorphisms we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$. It is obvious from the definition that each class of a factorisation system is invariant under isomorphism.

Definition 48.2. *We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ has the* right cancellation property *if the implication*

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \ \Rightarrow \ v \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$. Dually, we shall say that $\mathcal{M}$ has the left cancellation property *if the implication*

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \ \Rightarrow \ u \in \mathcal{M}$$

is true.

Proposition 48.3. *The intersection of the classes of a factorisation system $(\mathcal{A}, \mathcal{B})$ is the class of isomorphisms. Moreover,*

- *the class $\mathcal{A}$ has the right cancellation property;*
- *the class $\mathcal{B}$ has the left cancellation property.*

Proof: If a map $f : A \to B$ belongs to $\mathcal{A} \cap \mathcal{B}$, consider the factorisations $f = f1_A$ and $f = 1_B f$. We have $1_A \in \mathcal{A}$ and $f \in \mathcal{B}$ in the first, and we have $f \in \mathcal{A}$ and $1_B \in \mathcal{B}$ in the second. Hence there exists an isomorphism $i : B \to A$ such that $if = 1_A$ and $fi = 1_B$. This shows that f is invertible. If $u \in \mathcal{A}$ and $vu \in \mathcal{A}$, let us show that $v \in \mathcal{A}$. For this, let us choose a factorisation $v = ps : B \to E \to C$, with $s \in \mathcal{A}$ and $p \in \mathcal{B}$, and put $w = vu$. Then w admits the factorisation $w = p(su)$ with $su \in \mathcal{A}$ and $p \in \mathcal{B}$ and the factorisation $w = 1_C(vu)$ with $vu \in \mathcal{A}$ and $1_C \in \mathcal{B}$. Hence there exists an isomorphism $i : E \to C$ such that $i(su) = vu$ and $1_C i = p$. Thus, $p \in \mathcal{A}$ since $p = i$ and every isomorphism belongs to $\mathcal{A}$. It follows that $v = ps \in \mathcal{A}$, since $\mathcal{A}$ is closed under composition. ∎

Definition 48.4. *We say that a map $u : A \to B$ in a category $\mathcal{E}$ is* left orthogonal *to a map $f : X \to Y$, or that f is* right orthogonal *to u, if every commutative square*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "y"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a unique diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \perp f$.

Notice that the condition $u \perp f$ means that the square

$$\begin{tikzcd} Hom(B,X) \arrow[r, "Hom(u,X)"] \arrow[d, "Hom(B,f)"'] & Hom(A,X) \arrow[d, "Hom(A,f)"] \\ Hom(B,Y) \arrow[r, "Hom(u,Y)"'] & Hom(A,Y) \end{tikzcd}$$

is cartesian. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we shall write $\mathcal{A} \perp \mathcal{B}$ to indicate that we have $a \perp b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

If $\mathcal{M}$ is a class of maps in a category $\mathcal{E}$, we shall denote by ${}^{\perp}\mathcal{M}$ (resp. $\mathcal{M}^{\perp}$) the class of maps which are left (resp. right) orthogonal to every map in $\mathcal{M}$. Each class ${}^{\perp}\mathcal{M}$ and $\mathcal{M}^{\perp}$ is closed under composition and contains the isomorphisms. The class ${}^{\perp}\mathcal{M}$ has the right cancellation property and the class $\mathcal{M}^{\perp}$ the left cancellation property. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, then

$$\mathcal{A} \subseteq {}^{\perp}\mathcal{B} \;\Leftrightarrow\; \mathcal{A} \perp \mathcal{B} \;\Leftrightarrow\; \mathcal{A}^{\perp} \supseteq \mathcal{B}.$$

Proposition 48.5. *If $(\mathcal{A}, \mathcal{B})$ is a factorisation system then*

$$\mathcal{A} = {}^{\perp}\mathcal{B} \quad \text{and} \quad \mathcal{B} = \mathcal{A}^{\perp}.$$

Proof Let us first show that we have $\mathcal{A} \perp \mathcal{B}$. If $a : A \to A'$ is a map in $\mathcal{A}$ and $b : B \to B'$ is a map in $\mathcal{B}$, let us show that every commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "u'"'] & B' \end{tikzcd}$$

has a unique diagonal filler. Let us choose a factorisation $u = ps : A \to E \to B$ with $s \in \mathcal{A}$ and $p \in \mathcal{B}$ and a factorisation $u' = p's' : A' \to E' \to B'$ with $s' \in \mathcal{A}$ and $p' \in \mathcal{B}$. From the commutative diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "a"'] & E \arrow[r, "p"] & B \arrow[d, "b"] \\ A' \arrow[r, "s'"'] & E' \arrow[r, "p'"'] & B', \end{tikzcd}$$

we can construct a square

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "s'a"'] & E \arrow[d, "bp"] \\
E' \arrow[r, "p'"'] & B'.
\end{tikzcd}
$$

Observe that $s \in \mathcal{A}$ and $bp \in \mathcal{B}$ and also that $s'a \in \mathcal{A}$ and $p' \in \mathcal{B}$. By the uniqueness of the factorisation of a map, there is a unique isomorphism $i : E' \to E$ such that $is'a = s$ and $bpi = p'$:

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "a"'] & E \arrow[r, "p"] & B \arrow[d, "b"] \\
A' \arrow[r, "s'"'] & E' \arrow[u, "i"'] \arrow[r, "p'"'] & B'.
\end{tikzcd}
$$

The composite $d = pis'$ is then a diagonal filler of the first square

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "u'"'] \arrow[ur, dashed, "d"] & B'.
\end{tikzcd}
$$

It remains to prove the uniqueness of d. Let d' be an arrow $A' \to B$ such that $d'a = u$ and $bd' = u'$. Let us choose a factorisation $d' = qt : A' \to F \to B$ with $t \in \mathcal{A}$ and $q \in \mathcal{B}$. From the commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[dd, "a"'] & E \arrow[r, "p"] & B \arrow[dd, "b"] \\
 & F \arrow[ur, "q"] & \\
A' \arrow[ur, "t"] \arrow[r, "s'"'] & E' \arrow[r, "p'"'] & B'.
\end{tikzcd}
$$

we can construct two commutative squares

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "ta"'] & E \arrow[d, "p"] \\
F \arrow[r, "q"'] & B,
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "t"] \arrow[d, "s'"'] & F \arrow[d, "bq"] \\
E' \arrow[r, "p'"'] & B'.
\end{tikzcd}
$$

Observe that we have $ta \in \mathcal{A}$ and $q \in \mathcal{B}$. Hence there exists a unique isomorphism $j : F \to E$ such that $jta = s$ and $pj = q$. Similarly, there exists a unique isomorphism $j' : E' \to F$ such that $j's' = t$ and $bqj' = p'$. The maps fits in the following commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[dd, "a"'] & E \arrow[r, "p"] & B \arrow[dd, "b"] \\
 & F \arrow[u, "j"] \arrow[ur, "q"] & \\
A' \arrow[ur, "t"] \arrow[r, "s'"'] & E' \arrow[u, "j'"'] \arrow[r, "p'"'] & B'.
\end{tikzcd}$$

Hence the diagram

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "s'a"'] & E \arrow[d, "bp"] \\
E' \arrow[ur, "jj'"] \arrow[r, "p'"'] & B'.
\end{tikzcd}$$

commutes. It follows that we have $jj' = i$ by the uniqueness of the isomorphism between two factorisations. Thus, $d' = qt = (pj)(j's') = pis' = d$. The relation $\mathcal{A}\perp\mathcal{B}$ is proved. This shows that $\mathcal{A} \subseteq {}^{\perp}\mathcal{B}$. Let us show that ${}^{\perp}\mathcal{B} \subseteq \mathcal{A}$. If a map $f : A \to B$ is in ${}^{\perp}\mathcal{B}$. let us choose a factorisation $f = pu : A \to C \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$. Then the square

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "p"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

has a diagonal filler $s : B \to C$, since $f \in {}^{\perp}\mathcal{B}$. We have $ps = 1_B$. Let us show that $sp = 1_C$. Observe that the maps sp and 1_C are both diagonal fillers of the square

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "u"'] & C \arrow[d, "p"] \\
C \arrow[r, "p"'] & B.
\end{tikzcd}$$

This proves that $sp = 1_C$ by the uniqueness of a diagonal filler. Thus, $p \in \mathcal{A}$, since every isomorphism is in $\mathcal{A}$. Thus, $f = pu \in \mathcal{A}$. ∎

Corollary 48.6. *Each class of a factorisation system determines the other.*

48.1. We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is *closed under limits* if the full subcategory of $\mathcal{E}^I$ spanned by the maps in $\mathcal{M}$ is closed under limits. There is a dual notion of a class of maps closed under colimits.

Proposition 48.7. *The class $\mathcal{M}^{\perp}$ is closed under limits for any class of maps $\mathcal{M}$ in a category $\mathcal{E}$. Hence the right class of a factorisation system is closed under limits.*

Proof: For any pair of morphisms $u : A \to B$ and $f : X \to Y$ in $\mathcal{E}$, we have a commutative square $Sq(u, f)$:

$$\begin{tikzcd}
\mathcal{E}(B,X) \arrow[r, "\mathcal{E}(u,X)"] \arrow[d, "\mathcal{E}(B,f)"'] & \mathcal{E}(A,X) \arrow[d, "\mathcal{E}(A,f)"] \\
\mathcal{E}(B,Y) \arrow[r, "\mathcal{E}(u,Y)"'] & \mathcal{E}(A,Y).
\end{tikzcd}$$

The resulting functor

$$Sq : (\mathcal{E}^o)^I \times \mathcal{E}^I \to \mathbf{Set}^{I \times I}$$

continuous in each variable. An arrow $f \in \mathcal{E}$ belongs to $\mathcal{M}^\perp$ iff the square $Sq(u, f)$ is cartesian for every arrow $u \in \mathcal{M}$. This proves the result, since the full subcategory of $\mathbf{Set}^{I\times I}$ spanned by the cartesian squares is closed under limits. QED

Recall that a map $u : A \to B$ in a category $\mathcal{E}$ is said to be a *retract* of another map $v : C \to D$, if u is a retract of v in the category of arrows $\mathcal{E}^I$. A class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *closed under retracts* if the retract of a map in $\mathcal{M}$ belongs to $\mathcal{M}$.

Corollary 48.8. *The class $\mathcal{M}^\perp$ is closed under retracts for any class of maps $\mathcal{M}$ in a category $\mathcal{E}$. Each class of a factorisation system is closed under retracts.*

48.2. Let $(\mathcal{A}, \mathcal{B})$ be a factorisation system in a category $\mathcal{E}$. Then the full subcategory of $\mathcal{E}^I$ spanned by the elements of $\mathcal{B}$ is reflective. Dually, the full subcategory of $\mathcal{E}^I$ spanned by the elements of $\mathcal{A}$ is coreflective.
Proof: Let us denote by $\mathcal{B}'$ the full subcategory of $\mathcal{E}^I$ whose objects are the arrows in $\mathcal{B}$. Every map $u : A \to B$ admits a factorisation $u = pi : A \to E \to B$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$. The pair $(i, 1_B)$ defines an arrow $u \to p$ in $\mathcal{E}^I$. Let us show that the arrow reflects u in the subcategory $\mathcal{B}'$. For this, it suffices to show that for every arrow $f : X \to Y$ in $\mathcal{B}$ and every commutative square

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "u"'] & X \arrow[d, "f"] \\
B \arrow[r, "y"'] & Y,
\end{tikzcd}$$

there exists a unique arrow $z : E \to X$ such that $fz = yp$ and $zi = x$. But this is clear, since the square

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
E \arrow[r, "yp"'] & Y.
\end{tikzcd}$$

has a unique diagonal filler by 48.5. ■

Recall that the projection $A \times_B E \to A$ in a pullback square

$$\begin{tikzcd} A \times_B E \arrow[r] \arrow[d] & E \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is said to be the *base change* of the map $E \to B$ along the map $A \to B$. A class of maps $\mathcal{B}$ in a category $\mathcal{E}$ is said to be *closed under base changes* if the base change of a map in $\mathcal{B}$ along any map in $\mathcal{E}$ belongs to $\mathcal{B}$ when this base change exists. The class $\mathcal{M}^{\perp}$ is closed under base changes for any class of maps $\mathcal{M} \subseteq \mathcal{E}$. In particular, the right class of a factorisation system is closed under base change. Recall that the map $B \to E \sqcup_A B$ in a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ E \arrow[r] & E \sqcup_A B \end{tikzcd}$$

is said to be the *cobase change* of the map $A \to E$ along the map $A \to B$. A class of maps $\mathcal{A}$ in category $\mathcal{E}$ is said to be *closed under cobase changes* if the cobase change of a map in $\mathcal{A}$ along any map in $\mathcal{E}$ belongs to $\mathcal{A}$ when this cobase change exists. The class ${}^{\perp}\mathcal{M}$ is closed under cobase changes for any class of maps $\mathcal{M} \subseteq \mathcal{E}$. In particular, the left class of a factorisation system is closed under cobase changes.

48.3. Let us say that an arrow $f : X \to Y$ in a category with finite limits is *surjective* if it is left orthogonal to every monomorphism. The class of surjections is closed under cobase change, under colimits and it has the right cancellation property. Every surjection is an epimorphism, but the converse is not necessarly true.

We now give some examples of factorisation systems.

Proposition 48.9. *Let $p : \mathcal{E} \to \mathcal{C}$ be a Grothendieck fibration. Then the category $\mathcal{E}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of cartesian morphisms. An arrow $u \in \mathcal{E}$ belongs to $\mathcal{A}$ iff the arrow $p(u)$ is invertible.*

Dually, if $p : \mathcal{E} \to \mathcal{C}$ is a Grothendieck opfibration, then the category $\mathcal{E}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of cocartesian morphisms. A morphism $u \in \mathcal{E}$ belongs to $\mathcal{B}$ iff the morphism $p(u)$ is invertible.

If $\mathcal{E}$ is a category with pullbacks, then the target functor $t : \mathcal{E}^I \to \mathcal{E}$ is a Grothendieck fibration. A morphism $f : X \to Y$ of the category $\mathcal{E}^I$ is a commutative square in $\mathcal{E}$,

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "x"] & Y_0 \arrow[d, "y"] \\ X_1 \arrow[r, "f_1"] & Y_1. \end{tikzcd}$$

The morphism f is cartesian iff the square is a pullback (also called a *cartesian square*). Hence the category $\mathcal{E}^I$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of cartesian squares. A square $f : X \to Y$ belongs to $\mathcal{A}$ iff the morphism $f_1 : X_1 \to Y_1$ is invertible.

Corollary 48.10. *Suppose that we have a commutative diagram*

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & C_0 \arrow[d] \\
A_1 \arrow[r] & B_1 \arrow[r] & C_1
\end{tikzcd}$$

in which the right hand square is cartesian. Then the left hand square is cartesian iff the composite square is cartesian.

Proof: This follows from the left cancellation property of the right class of a factorisation system. ∎

Corollary 48.11. *Suppose that we have a commutative cube*

$$\begin{tikzcd}
A_0 \arrow[rr] \arrow[dr] \arrow[dd] & & C_0 \arrow[dr] \arrow[dd] & \\
& B_0 \arrow[rr, crossing over] & & D_0 \arrow[dd] \\
A_1 \arrow[rr] \arrow[dr] & & C_1 \arrow[dr] & \\
& B_1 \arrow[rr] \arrow[from=uu, crossing over] & & D_1.
\end{tikzcd}$$

in which the left face, the right face and front face are cartesian. Then the back face is cartesian.

We now give a few examples of factorisation systems in the category **Cat**.

Recall that a functor $p : E \to B$ is said to be a *discrete fibration* if for every object $e \in E$ and every arrow $g \in B$ with target $p(e)$, there exists a unique arrow $f \in E$ with target e such that $p(f) = e$. There is a dual notion of *discrete opfibration*. Recall that a functor between small categories $u : A \to B$ is said to be *final* (but we shall say *0-final*) if the category $b\backslash A = (b\backslash B) \times_B A$ defined by the pullback square

$$\begin{tikzcd}
b\backslash A \arrow[r, "h"] \arrow[d] & A \arrow[d, "u"] \\
b\backslash B \arrow[r] & B.
\end{tikzcd}$$

is connected for every object $b \in B$. There is a dual notion of *initial functor* (but we shall say *0-initial*).

Theorem 48.12. [Street] *The category* **Cat** *admits a factorisation system* $(\mathcal{A}, \mathcal{B})$ *in which* $\mathcal{B}$ *is the class of discrete fibrations and* $\mathcal{A}$ *the class of 0-final functors. Dually, category* **Cat** *admits a factorisation system* $(\mathcal{A}', \mathcal{B}')$ *in which* $\mathcal{B}'$ *is the class of discrete opfibrations and* $\mathcal{A}'$ *is the class of 0-initial functors.*

48.4. Recall that a functor $p : C \to D$ is said to be *conservative* if the implication

$$p(f) \text{ invertible } \Rightarrow f \text{ invertible}$$

is true for every arrow $f \in C$. The model category $(\mathbf{Cat}, Eq)$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative functors. A functor in the class $\mathcal{A}$ is an *iterated strict localisation* Let us describe the strict localisations explicitly. We say that a functor $g : A \to B$ *inverts* a set S of arrows in A if every arrow in $g(S)$ is invertible. When the category A is small. there is a functor $l_S : A \to S^{-1}A$ which inverts S universally. The universality means that for any functor $g : A \to B$ which inverts S there exists a unique functor $h : S^{-1}A \to B$ such that $hl_S = g$. The functor l_S is a strict localisation. Every functor $u : A \to B$ admits a factorisation $u = u_1 l_1 : A \to S_0^{-1}A \to B$, where S_0 is the set of arrows inverted by u and where $l_1 = l_{S_0}$. Let us put $A_1 = S_0^{-1}A$. The functor u_1 is not necessarily conservative but it admits a factorisation $u_1 = u_2 l_2 : A_1 \to S_1^{-1}A_1 \to B$, where S_1 is the set of arrows inverted by u_1. Let us put $A_2 = S_1^{-1}A_1$. By iterating this process, we obtain an infinite sequence of categories and functors,

$$\begin{tikzcd}
A = A_0 \arrow[r, "l_1"] \arrow[ddrrrr, "u=u_0"'] & A_1 \arrow[r, "l_2"] \arrow[ddrrr, "u_1"'] & A_2 \arrow[r, "l_3"] \arrow[ddrr, "u_2"'] & A_3 \arrow[r, "l_4"] \arrow[ddr, "u_4"'] & \cdots & E \arrow[ddl, "v"] \\
& & & & & \\
& & & & B. &
\end{tikzcd}$$

If the category E is the colimit of the sequence, then the functor $v : E \to B$ is conservative. and the canonical functor $l : A \to E$ is an iterated strict localisation.

48.5. For any category A, the full subcategory of $A\backslash\mathbf{Cat}$ spanned by the iterated strict localisations $A \to C$ is equivalent to a complete lattice $Loc_0(A)$. Its maximum element is defined by the strict localisation $A \to \pi_1 A$ which inverts every arrow in A. Every functor $u : A \to B$ induces a pair of adjoint maps

$$u_! : Loc_0(A) \to Loc_0(B) : u^*,$$

where $u_!$ is defined by cobase change along u.

49. Appendix on weak factorisation systems

49.1. Recall that an arrow $u : A \to B$ in a category $\mathcal{E}$ is said to have the *left lifting property* with respect to another arrow $f : X \to Y$, or that f has the *right lifting property* with respect to u, if every commutative square

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "u"'] & X \arrow[d, "f"] \\
B \arrow[r, "y"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

has a diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \pitchfork f$. If the diagonal filler is unique we shall write $u \perp f$ and say that

u is *left orthogonal* to f, ot that f is *right orthogonal* to u. For any class of maps $\mathcal{M} \subseteq \mathcal{E}$, we shall denote by ${}^{\pitchfork}\mathcal{M}$ (resp. $\mathcal{M}^{\pitchfork}$) the class of maps having the left lifting property (resp. right lifting property) with respect to every map in $\mathcal{M}$. Each class ${}^{\pitchfork}\mathcal{M}$ and $\mathcal{M}^{\pitchfork}$ contains the isomorphisms and is closed under composition. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we shall write $\mathcal{A} \pitchfork \mathcal{B}$ to indicate that we have $u \pitchfork f$ for every $u \in \mathcal{A}$ and $f \in \mathcal{B}$. Then

$$\mathcal{A} \subseteq {}^{\pitchfork}\mathcal{B} \iff \mathcal{A} \pitchfork \mathcal{B} \iff \mathcal{B} \subseteq \mathcal{A}^{\pitchfork}.$$

49.2. We say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is a *weak factorisation system* if the following two conditions are satisfied:

- every map $f \in \mathcal{E}$ admits a factorisation $f = pu$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$;
- $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$ and $\mathcal{A}^{\pitchfork} = \mathcal{B}$.

We say that $\mathcal{A}$ is the *left class* and $\mathcal{B}$ the *right class* of the weak factorisation system.

49.3. Every factorisation system is a weak factorisation system.

49.4. We say that a map in a topos is a trivial fibration if it has the right lifting property with respect to every monomorphism. This terminology is non-standard but useful. The trivial fibrations often coincide with the acyclic fibrations (which can be defined in any model category). An object X in a topos is said to be *injective* if the map $X \to 1$ is a trivial fibration. If $\mathcal{B}$ is the class of trivial fibrations in a topos and $\mathcal{A}$ is the class monomorphisms, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system. A map of simplicial sets is a trivial fibration iff it has the right lifting property with respect to the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$.

49.5. We say that a Grothendieck fibration $E \to B$ is a *1-fibration* if its fibers $E(b)$ are groupoids. We say that a category C is *1-connected* if the functor $\pi_1 C \to 1$ is an equivalence. We say that functor $u : A \to B$ is is *1-final*) if the category $b\backslash A = (b\backslash B) \times_B A$ is 1-connected for every object $b \in B$. The category **Cat** admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of 1-fibrations and $\mathcal{A}$ the class of 1-final functors.

49.6. Let $\mathcal{E}$ be a cocomplete category. If $\alpha = \{i : i < \alpha\}$ is a non-zero ordinal, we shall say that a functor $C : \alpha \to \mathcal{E}$ is an *α-chain* if the canonical map

$$\varinjlim_{i<j} C(i) \to C(j)$$

is an isomorphism for every non-zero limit ordinal $j < \alpha$. The *composite* of C is the canonical map

$$C(0) \to \varinjlim_{i<\alpha} C(i).$$

We shall say that a subcategory $\mathcal{A} \subseteq \mathcal{E}$ is closed under *transfinite composition* if the composite of any α-chain $C : \alpha \to \mathcal{E}$ with values in $\mathcal{A}$ belongs to $\mathcal{A}$.

49.7. Let $\mathcal{E}$ be a cocomplete category. We shall say that a class of maps $\mathcal{A} \subseteq \mathcal{E}$ is *saturated* if it satisfies the following conditions:

- $\mathcal{A}$ contains the isomorphisms and is closed under composition ;
- $\mathcal{A}$ is closed under transfinite composition;
- $\mathcal{A}$ is closed under cobase change and retract;

Every class of maps $\Sigma \subseteq \mathcal{E}$ is contained in a smallest saturated class $\overline{\Sigma}$ called the saturated class *generated* by Σ. We shall say that a saturated class $\mathcal{A}$ is *accessible* if it is generated by a set of maps $\Sigma \subseteq \mathcal{A}$.

49.8. [Ci1] The class $\mathcal{M}$ of monomorphisms in a Grothendieck topos is accessible.

49.9. If Σ is a set of maps in a locally presentable category, then the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system, where $\overline{\Sigma}$ denotes the saturated class generated by Σ.

50. Appendix on simplicial sets

We fix some notations about simplicial sets. The category of finite non-empty ordinals and order preserving maps is denoted Δ. It is standard to denote the ordinal $n+1 = \{0, \ldots, n\}$ by $[n]$. A map $u : [m] \to [n]$ in Δ can be specified by listing its values $(u(0), \ldots, u(m))$. The order preserving injection $[n-1] \to [n]$ which omits $i \in [n]$ is denoted d_i, and the order preserving surjection $[n] \to [n-1]$ which repeats $i \in [n-1]$ is denoted s_i.

50.1. Recall that a (small) *simplicial set* is a presheaf on the category Δ. We shall denote the category of simplicial sets by $\mathbf{S}$. If X is a simplicial set and $n \geq 0$, it is standard to denote the set $X([n])$ by X_n. An element of X_n is said to be a *simplex of dimension* n, or a n*-simplex* of X; a 0-simplex is called a *vertex* and a 1-simplex an *arrow*. If $n > 0$ and $i \in [n]$, the map $X(d_i) : X_n \to X_{n-1}$ is denoted by ∂_i, and if $i \in [n-1]$, the map $X(s_i) : X_{n-1} \to X_n$ is denoted by σ_i. The simplex $\partial_i(x) \in X_{n-1}$ is the *i-th face* of a simplex $x \in X_n$. The *source* of an arrow $f \in X_1$ is defined to be the vertex $\partial_1(f)$ and its *target* to be the vertex $\partial_0(f)$; we shall write $f : a \to b$ to indicate that $a = \partial_1(f)$ and $b = \partial_0(f)$.

50.2. The *combinatorial simplex* $\Delta[n]$ of dimension n is defined to be the simplicial set $\Delta(-, [n])$; the simplex $\Delta[1]$ is the *combinatorial interval* and we shall denote it by I; the simplex $\Delta[0]$ is the terminal object of the category $\mathbf{S}$ and we shall denote it by 1. We shall often identify a morphism $u : [m] \to [n]$ in Δ with its image $y_\Delta(u) : \Delta[m] \to \Delta[n]$ by the Yoneda functor $y_\Delta : \Delta \to \mathbf{S}$. By the Yoneda lemma, for every $X \in \mathbf{S}$ and $n \geq 0$ the evaluation map $x \mapsto x(1_{[n]})$ is a bijection between the maps $\Delta[n] \to X$ and the elements of X_n; we shall identify these two sets by adopting the same notation for a map $\Delta[n] \to X$ and the corresponding simplex in X_n. If $u : [m] \to [n]$ and $x \in X_n$, we shall denote the simplex $X(u)(x) \in X_m$ as a composite $xu : \Delta[m] \to X$. For example, $\partial_i(x) = xd_i : \Delta[n-1] \to X$ for every $x \in X_n$ and $\sigma_i(x) = xs_i$ for every $x \in X_{n-1}$. A simplex $x \in X_n$ is said to be *degenerate* if it belongs to the image of $\sigma_i : X_{n-1} \to X_n$ for some $i \in [n-1]$. To every vertex $a \in X_0$ is associated a degenerate arrow $\sigma_0(a) : a \to a$ that we shall denote as a *unit* $1_a : a \to a$.

50.3. The *cardinality* of a simplicial set X is defined to be the cardinality of the set of non-degenerate simplices of X. A simplicial set is *finite* if it has a finite number of non-degenerate simplices. A simplicial set can be large. A *large simplicial set* is defined to be a functor $\Delta^o \to \mathbf{SET}$, where $\mathbf{SET}$ is the category of large sets. We say that a large simplicial set X is *locally small* if the vertex map $X_n \to X_0^{n+1}$ has small fibers for every $n \geq 0$. If X is locally small, then so is the simplicial set X^A for any small simplicial set A.

50.4. Let $\tau : \Delta \to \Delta$ be the automorphism of the category Δ which reverses the order of each ordinal. If $u : [m] \to [n]$ is a map in Δ, then $\tau(u)$ is the map $u^o : [m] \to [n]$ obtained by putting $u^o(i) = n - f(m-i)$. The *opposite* X^o of a simplicial set X is obtained by composing the (contravariant) functor $X : \Delta \to \mathbf{Set}$ with the functor τ. We distinguish between the simplices of X and X^o by writing $x^o \in X^o$ for each $x \in X$, with the convention that $x^{oo} = x$. If $f : a \to b$ is an arrow in X, then $f^o : b^o \to a^o$ is an arrow in X^o. Beware that the *opposite* of a map of simplicial sets $u : A \to B$ is a map $u^o : A^o \to B^o$. A *contravariant map* $p : A \to B$ between two simplicial sets is defined to be a map $q : A^o \to B$; we shall often write $p(a)$ instead of $q(a^o)$ for $a \in A$. .

50.5. If X is a simplicial set, we say that a subfunctor $A \subseteq X$ is a *simplicial subset* of X. If $n > 0$ and $i \in [n]$ the image of the map $d_i : \Delta[n-1] \to \Delta[n]$ is denoted $\partial_i\Delta[n] \subset \Delta[n]$. The *simplicial sphere* $\partial\Delta[n] \subset \Delta[n]$ is the union the faces $\partial_i\Delta[n]$ for $i \in [n]$; by convention $\partial\Delta[0] = \emptyset$. If $n > 0$, a map $x : \partial\Delta[n] \to X$ is said to be a *simplicial sphere of dimension* $n-1$ *in* X; it is determined by the sequence of its faces $(x_0, \ldots, x_n) = (xd_0, \ldots, xd_n)$. A simplicial sphere $\partial\Delta[2] \to X$ is called a *triangle*. Every n-simplex $y : \Delta[n] \to X$ has a *boundary* $\partial y = (\partial_0 y, \ldots, \partial_n y) = (yd_0, \ldots, yd_n)$ obtained by restricting y to $\partial\Delta[n]$. A simplex y is said to *fill* a simplicial sphere x if we have $\partial y = x$. A simplicial sphere $x : \partial\Delta[n] \to X$ *commutes* if it can be filled.

50.6. If $n > 0$ and $k \in [n]$, the *horn* $\Lambda^k[n] \subset \Delta[n]$ is defined to be the union of the faces $\partial_i\Delta[n]$ with $i \neq k$. A map $x : \Lambda^k[n] \to X$ is called a *horn in* X; it is determined by a lacunary sequence of faces $(x_0, \ldots, x_{k-1}, *, x_{k+1}, \ldots, x_n)$. A *filler* for x is a simplex $\Delta[n] \to X$ which extends x. Recall that a simplicial set X is said to be a *Kan complex* if every horn $\Lambda^k[n] \to X$ $(n > 0, k \in [n])$ has a filler $\Delta[n] \to X$,

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta[n]. \arrow[ur, "\exists"'] & \end{tikzcd}$$

50.7. Let us denote by $\Delta(n)$ the full subcategory of Δ spanned by the objects $[k]$ for $0 \leq k \leq n$. We say that a presheaf on $\Delta(n)$ is a *n-truncated simplicial set* and we put $\mathbf{S}(n) = [\Delta(n)^o, \mathbf{Set}]$. If i_n denotes the inclusion $\Delta(n) \subset \Delta$, then the restriction functor $i_n^* : \mathbf{S} \to \mathbf{S}(n)$ has a left adjoint $(i_n)_!$ and a right adjoint $(i_n)_*$. The functor $Sk^n = (i_n)_!(i_n)^* : \mathbf{S} \to \mathbf{S}$ associates to a simplicial set X its *n-skeleton* $Sk^n X \subseteq X$; it is the simplicial subset of X generated by the simplices $x \in X_k$ of dimension $k \leq n$. The functor $Cosk^n = (i_n)_*(i_n)^* : \mathbf{S} \to \mathbf{S}$ associates to a simplicial set X its *n-coskeleton* $Cosk^n X$. A simplex $\Delta[k] \to Cosk^n X$ is the same thing as a simplex $Sk^n\Delta[k] \to X$.

50.8. We say that a map of simplicial sets $f : X \to Y$ is *biunivoque* if the map $f_0 : X_0 \to Y_0$ is bijective. We say that a map of simplicial sets $f : X \to Y$ is *n-full* if the ollowing square of canonical maps is a pullback,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ Cosk^n(X) \arrow[r, "Cosk^n(f)"] & Cosk^n(Y). \end{tikzcd}$$

The n-full maps are closed under composition and base change. Every map $f : X \to Y$ admits a factorisation $f = pq : X \to Z \to Y$ with p a 0-full map and q biunivoque. The factorisation is unique up to unique isomorphism. It is the *Gabriel factorisation* of the map. A 0-full map between quasi-categories is fully faithful. We say that a simplicial subset S of a simplicial set X is *n-full* if the inclusion of the subset $S \subseteq X$ is n-full. The inclusion of a subcategory in a category is always 1-full.

50.9. Let **Top** be the category of (small) topological spaces. Consider the functor $r : \Delta[n] \to \mathbf{Top}$ which associates to $[n]$ the geometric simplex

$$\Delta^n = \{(x_1, \ldots, x_n) : 0 \le x_1 \le \cdots \le x_n \le 1\}.$$

The *singular complex* of a topological space Y is the simplicial set $r^! Y$ defined by putting

$$(r^! Y)_n = \mathbf{Top}(\Delta^n, Y)$$

for every $n \ge 0$. The simplicial set $r^! Y$ is a Kan complex. The singular complex functor $r^! : \mathbf{Top} \to \mathbf{S}$ has a left adjoint $r_!$ which associates to a simplicial set X its *geometric realisation* $r_! X$. A map of simplicial sets $u : A \to B$ is said to be a *weak homotopy equivalence* if the map $r_!(u) : r_! A \to r_! B$ is a homotopy equivalence of topological spaces.

51. Appendix on model categories

51.1. We shall say that a class $\mathcal{W}$ of maps in a category $\mathcal{E}$ has the *"three for two"* property if the following condition is satisfied:

- If two of three maps $u : A \to B$, $v : B \to C$ and $vu : A \to C$ belong to $\mathcal{W}$, then so does the third.

51.2. Let $\mathcal{E}$ be a finitely bicomplete category. We shall say that a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of maps in $\mathcal{E}$ is a *model structure* if the following conditions are satisfied:

- $\mathcal{W}$ has the "three for two" property;
- the pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorisation systems.

A map in $\mathcal{W}$ is said to be *acyclic* or to be a *weak equivalence*. A map in $\mathcal{C}$ is called a *cofibration* and a map in $\mathcal{F}$ a *fibration* . An object $X \in \mathcal{E}$ is said to be *fibrant* if the map $X \to \top$ is a fibration, where $\top$ is the terminal object of $\mathcal{E}$. Dually, an object $A \in \mathcal{E}$ is said to be *cofibrant* if the map $\bot \to A$ is a cofibration, where $\bot$ is the initial object of $\mathcal{E}$. A *Quillen model category* is a category $\mathcal{E}$ equipped with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

51.3. We shall say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ in a cocomplete category $\mathcal{E}$ is *accessible* or *cofibrantly generated* if the saturated classes $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ are accessible.

51.4. A model structure is said to be *left proper* if the cobase change of a weak equivalence along a cofibration is a weak equivalence. Dually, a model structure is said to be *right proper* if the base change of a weak equivalence along a fibration is a weak equivalence. A model structure is *proper* if it is both left and right proper.

51.5. If $\mathcal{E}$ is a model category, then so is the slice category $\mathcal{E}/B$ for each object $B \in \mathcal{E}$. By definition, a map in $\mathcal{E}/B$ is a weak equivalence (resp. a cofibration , resp. a fibration) iff the underlying map in $\mathcal{E}$ is a weak equivalence (resp. a cofibration , resp. a fibration). Dually, each category $B\backslash\mathcal{E}$ is a model category.

51.6. Let $\mathcal{E}$ be a finitely bicomplete category equipped a class of maps $\mathcal{W}$ having the "three-for-two" property and two factorisation systems $(\mathcal{C}_W, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}_W)$. Suppose that the following two conditions are satisfied:

- $\mathcal{C}_W \subseteq \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}_W \subseteq \mathcal{F} \cap \mathcal{W}$;
- $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}_W$ or $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_W$.

Then we have $\mathcal{C}_W = \mathcal{C} \cap \mathcal{W}$, $\mathcal{F}_W = \mathcal{F} \cap \mathcal{W}$ and $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure.

51.7. The *homotopy category* of a model category $\mathcal{E}$ is defined to be the category of fractions $Ho(\mathcal{E}) = \mathcal{W}^{-1}\mathcal{E}$. We shall denote by $[u]$ the image of a map $u \in \mathcal{E}$ by the canonical functor $\mathcal{E} \to Ho(\mathcal{E})$. A map $u : A \to B$ is a weak equivalence iff $[u]$ invertible in $Ho(\mathcal{E})$ by [Q].

51.8. We shall denote by $\mathcal{E}_f$ (resp. $\mathcal{E}_c$) the full sub-category of fibrant (resp. cofibrant) objects of a model category $\mathcal{E}$. We shall put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. A *fibrant replacement* of an object $X \in \mathcal{E}$ is a weak equivalence $X \to RX$ with codomain a fibrant object. Dually, a *cofibrant replacement* of X is a weak equivalence $LX \to X$ with domain a cofibrant object. Let us put $Ho(\mathcal{E}_f) = \mathcal{W}_f^{-1}\mathcal{E}_f$ where $\mathcal{W}_f = \mathcal{W} \cap \mathcal{E}_f$ and similarly for $Ho(\mathcal{E}_c)$ and $Ho(\mathcal{E}_{fc})$. Then the diagram of inclusions

$$\begin{tikzcd}
\mathcal{E}_{fc} \arrow[r] \arrow[d] & \mathcal{E}_f \arrow[d] \\
\mathcal{E}_c \arrow[r] & \mathcal{E}
\end{tikzcd}$$

induces a diagram of equivalences of categories

$$\begin{tikzcd}
Ho(\mathcal{E}_{fc}) \arrow[r] \arrow[d] & Ho(\mathcal{E}_f) \arrow[d] \\
Ho(\mathcal{E}_c) \arrow[r] & Ho(\mathcal{E}).
\end{tikzcd}$$

51.9. A *path object* for an object X in a model category is obtained by factoring the diagonal map $X \to X \times X$ as weak equivalence $\delta : X \to PX$ followed by a fibration $(p_0, p_1) : PX \to X \times X$. A *right homotopy* $h : f \sim_r g$ between two maps $u, v : A \to X$ is a map $h : A \to PX$ such that $u = p_0 h$ and $v = p_1 h$. Two maps $u, v : A \to X$ are *right homotopic* if there exists a right homotopy $h : f \sim_r g$ with codomain a path object for X. The right homotopy relation on the set of maps $A \to X$ is an equivalence if X is fibrant. There is a dual notion of *cylinder object* for A obtained by factoring the codiagonal $A \sqcup A \to A$ as a cofibration

$(i_0, i_1) : A \sqcup A \to IA$ followed by a weak equivalence $p : IA \to A$. A *left homotopy* $h : u \sim_l v$ between two maps $u, v : A \to X$ is a map $h : IA \to X$ such that $u = hi_0$ and $v = hi_1$. Two maps $u, v : A \to X$ are *left homotopic* if there exists a left homotopy $h : u \sim_l v$ with domain some cylinder object for A. The left homotopy relation on the set of maps $A \to X$ is an equivalence if A is cofibrant. The left homotopy relation coincides with the right homotopy relation if A is cofibrant and X is fibrant; in which case two maps $u, v : A \to X$ are said to be homotopic if they are left (or right) homotopic; we shall denote this relation by $u \sim v$.

Proposition 51.1. [Q]. *If A is cofibrant and X is fibrant, let us denote by $\mathcal{E}(A, X)^\sim$ the quotient of the set $\mathcal{E}(A, X)$ by the homotopy relation $\sim$. Then the canonical map $u \mapsto [u]$ induces a bijection*

$$\mathcal{E}(A, X)^\sim \simeq Ho(\mathcal{E})(A, X).$$

A map $X \to Y$ in $\mathcal{E}_{cf}$ is a homotopy equivalence iff it is a weak equivalence.

51.10. A model structure $M = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ in a category $\mathcal{E}$ is determined by its class $\mathcal{C}$ of cofibrations together with its class of fibrant objects $Fib(M)$. If $M' = (\mathcal{C}, \mathcal{W}', \mathcal{F}')$ is another model structure with the same cofibrations, then the relation $\mathcal{W} \subseteq \mathcal{W}'$ is equivalent to the relation $Flb(M') \subseteq Fib(M)$.
Proof: Let us prove the first statement. It suffices to show that the class $\mathcal{W}$ is determined by $\mathcal{C}$ and $Fib(M)$. The class $\mathcal{F} \cap \mathcal{W}$ is determined by $\mathcal{C}$, since the pair $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a weak factorisation system. For any map $u : A \to B$, there exists a commutative square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r] & B \end{tikzcd}$$

in which the horizontal maps are acyclic fibrations and the objects A' and B' are cofibrants. The map u is acyclic iff the map u' is acyclic. Hence it suffices to show that the class $\mathcal{W} \cap \mathcal{E}_c$ is is determined by $\mathcal{C}$ and $Fib(M)$. If A and B are two objects of $\mathcal{E}$, let us denote by $h(A, B)$ the set of maps $A \to B$ between in the category $Ho(\mathcal{E})$. A map $u : A \to B$ in $\mathcal{E}$ is invertible in $Ho(\mathcal{E})$ iff the map $h(u, X) : h(B, X) \to h(A, X)$ is bijective for every object $X \in \mathcal{E}$ by Yoneda lemma. Hence a map $u : A \to B$ in $\mathcal{E}$ belongs to $\mathcal{W}$ iff the map $h(u, X) : h(B, X) \to h(A, X)$ is bijective for every object $X \in Fib(M)$, since every object in $Ho(\mathcal{E})$ is isomorphic to a fibrant object. If A is cofibrant and X is fibrant, let us denote by $\mathcal{E}(A, X)^\sim$ the quotient of the set $\mathcal{E}(A, X)$ by the homotopy relation. It follows from 51.1 that a map $u : A \to B$ in $\mathcal{E}_c$ belongs to $\mathcal{W}$ iff the map $\mathcal{E}(B, X)^\sim \to \mathcal{E}(A, X)^\sim$ induced by the map $\mathcal{E}(u, X)$ is bijective for every object $X \in Fib(M)$. Hence the result will be proved if we show that the homotopy relation $\sim$ on the set $\mathcal{E}(A, X)$ only depends on the class $\mathcal{C}$ if A is cofibrant and X is fibrant. But two maps $A \to X$ are homotopic iff they are left homotopic, since A is cofibrant and X is fibrant. A cylinder for A can be constructed by factoring the codiagonal $A \sqcup A \to A$ as a cofibration $(i_0, i_1) : A \sqcup A \to I(A)$ followed by an acyclic fibration $I(A) \to A$. Two maps $f, g : A \to X$ are left homotopic iff there exists a map $h : I(A) \to X$ such that $hi_0 = f$ and $hi_1 = g$. The construction of $I(A)$ only depends on $\mathcal{C}$, since it only depends on the factorisation system $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$. Hence the left homotopy relation on the set $\mathcal{E}(A, X)$ only depends on $\mathcal{C}$. The first statement of the proposition follows. The proof of the second statement is left to the reader. ∎

51.11. Recall from [Ho] that a cocontinuous functor $F : \mathcal{U} \to \mathcal{V}$ between two model categories is said to be a *left Quillen functor* if it takes a cofibration to a cofibration and an acyclic cofibration to an acyclic cofibration. A left Quillen functor takes a weak equivalence between cofibrant objects to a weak equivalence. Dually, a continuous functor $G : \mathcal{V} \to \mathcal{U}$ between two model categories is said to be a *right Quillen functor* if it takes a fibration to a fibration and an acyclic fibration to an acyclic fibration. A right Quillen functor takes a weak equivalence between fibrant objects to a weak equivalence.

51.12. A left Quillen functor $F : \mathcal{U} \to \mathcal{V}$ induces a functor $F_c : \mathcal{U}_c \to \mathcal{V}_c$ hence also a functor $Ho(F_c) : Ho(\mathcal{U}_c) \to Ho(\mathcal{V}_c)$. Its *left derived functor* is a functor

$$F^L : Ho(\mathcal{U}) \to Ho(\mathcal{V})$$

for which the following diagram of functors commutes up to isomorphism,

$$\begin{tikzcd} Ho(\mathcal{U}_c) \arrow[r, "Ho(F_c)"] \arrow[d] & Ho(\mathcal{V}_c) \arrow[d] \\ Ho(\mathcal{U}) \arrow[r, "F^L"] & Ho(\mathcal{V}), \end{tikzcd}$$

The functor F^L is unique up to a canonical isomorphism. It can be computed as follows. For each object $A \in \mathcal{U}$, we can choose a cofibrant replacement $\lambda_A : LA \to A$, with λ_A an acyclic fibration. We can then choose for each arrow $u : A \to B$ an arrow $L(u) : LA \to LB$ such that $u\lambda_A = \lambda_B L(u)$,

$$\begin{tikzcd} LA \arrow[r, "\lambda_A"] \arrow[d, "L(u)"'] & A \arrow[d, "u"] \\ LB \arrow[r, "\lambda_B"] & B. \end{tikzcd}$$

Then

$$F^L([u]) = [F(L(u))] : FLA \to FLB.$$

51.13. Dually, a right Quillen functor $G : \mathcal{V} \to \mathcal{U}$ induces a functor $G_f : \mathcal{V}_f \to \mathcal{U}_f$ hence also a functor $Ho(G_f) : Ho(\mathcal{V}_f) \to Ho(\mathcal{U}_f)$. Its *right derived functor* is a functor

$$G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$$

for which the following diagram of functors commutes up to a canonical isomorphism,

$$\begin{tikzcd} Ho(\mathcal{V}_f) \arrow[r, "Ho(G_f)"] \arrow[d] & Ho(\mathcal{U}_f) \arrow[d] \\ Ho(\mathcal{V}) \arrow[r, "G^R"] & Ho(\mathcal{U}). \end{tikzcd}$$

The functor G^R is unique up to a canonical isomorphism. It can be computed as follows. For each object $X \in \mathcal{V}$ let us choose a fibrant replacement $\rho_X : X \to RX$, with ρ_X an acyclic cofibration. We can then choose for each arrow $u : X \to Y$ an arrow $R(u) : RX \to RY$ such that $R(u)\rho_X = \rho_Y u$,

$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[d, "u"'] & RX \arrow[d, "R(u)"] \\ Y \arrow[r, "\rho_Y"] & RY. \end{tikzcd}$$

Then

$$G^R([u]) = [G(R(u))] : GRX \to GRY.$$

51.14. Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be an adjoint pair of functors between two model categories. Then the following two conditions are equivalent:

- F is a left Quillen functor;
- G is a right Quillen functor.

When these conditions are satisfied, the pair (F, G) is said to be a *Quillen pair*. In this case, we obtain an adjoint pair of functors

$$F^L : Ho(\mathcal{U}) \leftrightarrow Ho(\mathcal{V}) : G^R.$$

If $A \in \mathcal{U}$ is cofibrant, the adjunction unit $A \to G^R F^L(A)$ is obtained by composing the maps $A \to GFA \to GRFA$, where $FA \to RFA$ is a fibrant replacement of FA. If $X \in \mathcal{V}$ is fibrant, the adjunction counit $F^L G^R(X) \to X$ is obtained by composing the maps $FLGX \to FGX \to X$, where $LGX \to GX$ is a cofibrant replacement of GX.

51.15. We shall say that a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ a *homotopy reflection* of $\mathcal{U}$ into $\mathcal{V}$ if the right derived functor G^R is fully faithful. Dually, we shall say that (F, G) is a *homotopy coreflection* of $\mathcal{V}$ into $\mathcal{U}$ if the left derived functor F^L is fully faithful. We shall say that (F, G) is called a *Quillen equivalence* if the adjoint pair (F^L, G^R) is an equivalence of categories.

51.16. A Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a homotopy reflection iff the map $FLGX \to X$ is a weak equivalence for every fibrant object $X \in \mathcal{V}$, where $LGX \to GX$ denotes a cofibrant replacement of GX. A homotopy reflection $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a Quillen equivalence iff the functor F reflects weak equivalences between cofibrant objects.

51.17. Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be a homotopy reflection beween two model categories. We shall say that an object $X \in \mathcal{U}$ is *local* (with respect to the the pair (F, G)) if it belongs to the essential image of the right derived functor $G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$.

51.18. Let $\mathcal{M}_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then the identity functor $\mathcal{E} \to \mathcal{E}$ is a homotopy reflection $\mathcal{M}_1 \to \mathcal{M}_2$. The following conditions on an object A are equivalent:

- A is local;
- there exists a $\mathcal{M}_1$-equivalence $A \to A'$ with codomain a $\mathcal{M}_2$-fibrant object A';
- (every $\mathcal{M}_2$-fibrant replacement $A \to A'$ is a $\mathcal{M}_1$-fibrant replacement.

In particular, every $\mathcal{M}_2$-fibrant object is local. A map between local objects is a $\mathcal{M}_1$-equivalence iff it is a $\mathcal{M}_2$-equivalence.

51.19. Let $\mathcal{M}_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. If $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we shall say that $\mathcal{M}_2$ is a *Bousfield localisation* of $\mathcal{M}_1$. We shall say that $\mathcal{M}_1$ is the *localised model structure* and $\mathcal{M}_2$ is the *local model structure*.

51.20. Let $\mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ be a Bousfield localisation of a model structure $\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ on a category $\mathcal{E}$. A local object is $\mathcal{M}_1$-fibrant iff it is $\mathcal{M}_2$-fibrant. An object A is local iff every $\mathcal{M}_1$-fibrant replacement $i : A \to A'$ is a $\mathcal{M}_2$-fibrant replacement. A map between $\mathcal{M}_2$-fibrant objects is a $\mathcal{M}_2$-fibration iff it is a $\mathcal{M}_1$-fibration.

51.21. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables with values in a finitely cocomplete category $\mathcal{E}_3$. If $u : A \to B$ is map in $\mathcal{E}_1$ and $v : S \to T$ is a map in $\mathcal{E}_2$, we shall denote by $u \odot' v$ the map

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{tikzcd}
A \odot S \arrow[r] \arrow[d] & B \odot S \arrow[d] \\
A \odot T \arrow[r] & B \odot T.
\end{tikzcd}$$

This defines a functor of two variables

$$\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I,$$

where $\mathcal{E}^I$ denotes the category of arrows of a category $\mathcal{E}$.

51.22. [Ho] We shall say that a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ between three model categories is a *left Quillen functor* it is concontinuous in each variable and the following conditions are satisfied:

- $u \odot' v$ is a cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations;
- $u \odot' v$ is an acyclic cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations and one of the maps u or v is acyclic.

Dually, we shall say that the functor of two variables $\odot$ is a *right Quillen functor* if the opposite functor $\odot^o : \mathcal{E}_1^o \times \mathcal{E}_2^o \to \mathcal{E}_3^o$ is a left Quillen functor.

51.23. [Ho] A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on monoidal closed category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *monoidal* if the tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the unit object of the tensor product is cofibrant.

51.24. A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{E}$ is said to be *cartesian* if the cartesian product $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the terminal object 1 is cofibrant.

51.25. We say that a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is *divisible on the left* if the functor $A\odot(-) : \mathcal{E}_2 \to \mathcal{E}_3$ admits a right adjoint $A\backslash(-) : \mathcal{E}_3 \to \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. In this case we obtain a functor of two variables $(A, X) \mapsto A\backslash X$,

$$\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2,$$

called the *left division functor*. Dually, we say that $\odot$ is *divisible on the right* if the functor $(-) \odot B : \mathcal{E}_1 \to \mathcal{E}_3$ admits a right adjoint $(-)/B : \mathcal{E}_3 \to \mathcal{E}_1$ for every object $B \in \mathcal{E}_2$. In this case we obtain a functor of two variables $(X, B) \mapsto X/B$,

$$\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1,$$

called the *right division functor*.

51.26. If a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on both sides, then so is the left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ and the right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$. This is called a *tensor-hom-cotensor* situation by Gray [**?**]. There is then a bijection between the following three kinds of maps

$$A \odot B \to X, \qquad B \to A\backslash X, \qquad A \to X/B.$$

The contravariant functors $A \mapsto A\backslash X$ and $B \mapsto B\backslash X$ are mutually right adjoint for any object $X \in \mathcal{E}_3$.

51.27. Suppose the category $\mathcal{E}_2$ is finitely complete and that the functor $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on the left. If $u : A \to B$ is map in $\mathcal{E}_1$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle u\backslash\ f\rangle$ the map

$$B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$$

obtained from the commutative square

$$\begin{tikzcd} B\backslash X \arrow[r] \arrow[d] & A\backslash X \arrow[d] \\ B\backslash Y \arrow[r] & A\backslash Y. \end{tikzcd}$$

The functor $f \mapsto \langle u\backslash f\rangle$ is right adjoint to the functor $v \mapsto u \odot' v$ for every map $u \in \mathcal{E}_1$. Dually, suppose that the category $\mathcal{E}_1$ is finitely complete and that the functor $\odot$ is divisible on the right. If $v : S \to T$ is map in $\mathcal{E}_2$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle f/v\rangle$ the map

$$X/T \to Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{tikzcd} X/T \arrow[r] \arrow[d] & X/S \arrow[d] \\ Y/T \arrow[r] & Y/S. \end{tikzcd}$$

the functor $f \mapsto \langle f/v\rangle$ is right adjoint to the functor $u \mapsto u \odot' v$ for every map $v \in \mathcal{E}_2$.

51.28. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on both sides, where $\mathcal{E}_i$ is a finitely bicomplete category for $i = 1, 2, 3$. If $u \in \mathcal{E}_1$, $v \in \mathcal{E}_2$ and $f \in \mathcal{E}_3$, then

$$(u \odot' v) \pitchfork f \quad \Longleftrightarrow \quad u \pitchfork \langle f/v \rangle \quad \Longleftrightarrow \quad v \pitchfork \langle u \backslash f \rangle.$$

51.29. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on each side between three model categories. Then the functor $\odot$ is a left Quillen functor iff the corresponding left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor iff the the corresponding right division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor.

51.30. Let $\mathcal{E}$ be a *symmetric* monoidal closed category. Then the objects X/A and $A\backslash X$ are canonicaly isomorphic; we can identify them by adopting a common notation, for example $[A, X]$. Similarly, the maps $\langle f/u \rangle$ and $\langle u\backslash f \rangle$ are canonicaly isomorphic; we shall identify them by adopting a common notation, for example $\langle u, f \rangle$. A model structure on $\mathcal{E}$ is monoidal iff the following two conditions are satisfied:

- if u is a cofibration and f is a fibration, then $\langle u, f \rangle$ is a fibration which is acyclic if in addition u or f is acyclic;
- the unit object is cofibrant.

51.31. Recall that a functor $P : \mathcal{E} \to \mathcal{K}$ is said to be a *bifibration* if it is both a Grothendieck fibration and a Grothendieck opfibration. If P is a bifibration, then every arrow $f : A \to B$ in $\mathcal{E}$ admits a factorisation $f = c^f u^f$ with c^f a cartesian arrow and u^f a unit arrow (ie $P(u^f) = 1_{P(A)})$), together with a factorisation $f = u_f c_f$ with c^f a cocartesian arrow and u_f a unit. Let us denote by $\mathcal{E}(S)$ the fiber of the functor P at an object $S \in \mathcal{K}$. Then for every arrow $g : S \to T$ in $\mathcal{K}$ we can choose pair of adjoint functors

$$g_! : \mathcal{E}(S) \to \mathcal{E}(T) : g^*.$$

The *pullback functor* g^* is obtained by choosing for each object $B \in \mathcal{E}(T)$ a cartesian lift $g^*(B) \to B$ of the arrow g. The *pushforward functor* $g_!$ is obtained by choosing for each object $A \in \mathcal{E}(S)$ a cocartesian lift $A \to g_!(A)$ of the arrow g.

51.32. Let $P : \mathcal{E} \to \mathcal{K}$ be a Grothendieck bifibration where $\mathcal{K}$ is a model category. We shall say that a model structure $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{E}$ is *bifibered* by the functor P if the following conditions are satisfied:

- The intersection $\mathcal{M}(S) = (\mathcal{C} \cap \mathcal{E}(S), \mathcal{W} \cap \mathcal{E}(S), \mathcal{F} \cap \mathcal{E}(S))$ is a model structure on $\mathcal{E}(S)$ for each object $S \in \mathcal{K}$;
- The pair of adjoint functors

  $$g_! : \mathcal{E}(S) \to \mathcal{E}(T) : g^*$$

  is a Quillen pair for each arrow $g : S \to T$ in $\mathcal{K}$ and it is a Quillen equivalence if g is a weak equivalence;
- An arrow $f : A \to B$ in $\mathcal{E}$ is a cofibration iff the arrows $u_f \in \mathcal{E}(B)$ and $P(f) \in \mathcal{K}$ are cofibrations;
- An arrow $f : A \to B$ in $\mathcal{E}$ is a fibration iff the arrows $u^f \in \mathcal{E}(A)$ and $P(f) \in \mathcal{K}$ are fibrations.

It follows from these conditions that the functor P takes a fibration to a fibration, a cofibration to a cofibration and a weak equivalence to a weak equivalence. For another notion of bifibered model category, see [Ro].

51.33. Let $P : \mathcal{E} \to \mathcal{K}$ be a bifibered model category over a model category $\mathcal{K}$. Then the model structure on $\mathcal{E}$ is determined by the model structure on $\mathcal{K}$ together with the model structure on $\mathcal{E}(S)$ for each object $S \in \mathcal{K}$.

- An arrow $f : A \to B$ in $\mathcal{E}$ is an acyclic cofibration iff the arrows $u_f \in \mathcal{E}(B)$ and $P(f) \in \mathcal{K}$ are acyclic cofibrations;
- An arrow $f : A \to B$ in $\mathcal{E}$ is an acyclic fibration iff the arrows $u^f \in \mathcal{E}(A)$ and $P(f) \in \mathcal{K}$ are acyclic fibrations.

52. Appendix on Cisinski theory

We briefly describe Cisinki's theory of model structures on a Grothendieck topos. It can be used to generate the model structure for n-quasi-category for every $n \geq 1$.

52.1. We shall say that a model structure on a Grothendieck topos $\mathcal{E}$ is a *Cisinski structure* if its cofibrations are the monomorphisms.

52.2. The classical model structure $(\mathbf{S}, \mathit{Who})$ is a Cisinski model structure on the category $\mathbf{S}$. Also the model structure for quasi-categories. The model structure for Segal categories is a Cisinski model structure on $\mathbf{PCat}$. The model structure for Segal spaces is a Cisinski structure on $\mathbf{S}^{(2)}$, and also the model structure for Rezk categories.

52.3. Let $\mathcal{C}$ be the left class of a weak factorisation system in a finitely bicomplete category $\mathcal{E}$. We shall say that a class of maps $\mathcal{W} \subseteq \mathcal{E}$ is a *localizer* (with respect to $\mathcal{C}$) if the triple $M(\mathcal{W}) = (\mathcal{C}, \mathcal{W}, (\mathcal{C} \cap \mathcal{W})^{\pitchfork})$ is a model structure. A class of maps $\mathcal{W} \subseteq \mathcal{E}$ is a localizer with respect to $\mathcal{C}$ iff the following conditions are satisfied:

- $\mathcal{W}$ has the"three for two" property;
- $\mathcal{C}^{\pitchfork} \subseteq \mathcal{W}$;
- $\mathcal{C} \cap \mathcal{W}$ is the left class of a weak factoriszation system.

The map $\mathcal{W} \mapsto M(\mathcal{W})$ induces a bijection between the localizers with respect to $\mathcal{C}$ and the model structures on $\mathcal{E}$ having $\mathcal{C}$ for class of cofibrations. If $\mathcal{W}$ and $\mathcal{W}'$ are two localizers with respect to $\mathcal{C}$, then the model structure $M(\mathcal{W}')$ is a Bousfield localisation of the model structure $M(\mathcal{W})$ iff we have $\mathcal{W} \subseteq \mathcal{W}'$. This defines a partial order relation on the class of model structures having $\mathcal{C}$ for class of cofibrations.

52.4. [Ci1] We say that a class $\mathcal{W}$ of maps in a Grothendieck topos $\mathcal{E}$ is a *localizer* if it is a localizer with respect to the class $\mathcal{C}$ of monomorphisms. We shall say that a localizer $\mathcal{W}$ is *accessible* if the saturated class $\mathcal{C} \cap \mathcal{W}$ is accessible. A localizer $\mathcal{W} \subseteq \mathcal{E}$ is accessible iff the triple $M(\mathcal{W}) = (\mathcal{C}, \mathcal{W}, \mathcal{C} \cap \mathcal{W})^{\pitchfork})$ is a Cisinski model structure. The map $\mathcal{W} \mapsto M(\mathcal{W})$ induces a bijection between the accessible localizers and the Cisinski model structures.

52.5. [Ci1] If $\mathcal{E}$ is a Grothendieck topos, then every set of maps $S \subseteq \mathcal{E}$ is contained in a smallest (accessible) localizer $\mathcal{W}(S)$ called the localizer *generated* by S . In particular, there is a smallest localizer $\mathcal{W}_0 = \mathcal{W}(\emptyset)$. We say that the model structure $M(\mathcal{W}_0)$ is *minimal*. The minimal Cisinski model structure $M(\mathcal{W}_0)$ is cartesian closed and proper. Every Cisinski model structure is a Bousfield localisation of $M(\mathcal{W}_0)$.

52.6. In the category **S** of simplicial sets, the localizer *Who* is generated by the maps $\Delta[n] \to 1$, where $n \geq 0$. The localizer *Wcat* is generated by the inclusions $I[n] \subseteq \Delta[n]$, where $n \geq 0$.

52.7. [Ci2] Let L be the Lawvere object in a topos $\mathcal{E}$ and let $t_0, t_1 : 1 \to L$ be the canonical elements (the first is classifying the subobject $\emptyset \subseteq 1$ and the second the subobject $1 \subseteq 1$). Then an object $X \in \hat{C}$ is fibrant with respect to minimal Cisinski model structure $(\mathcal{C}, \mathcal{W}_0, \mathcal{F}_0)$ iff the projection $X^{t_i} : X^L \to X$ is a trivial fibration for $i = 0, 1$. A monomorphism $A \to B$ is acyclic iff the map $X^B \to X^A$ is a trivial fibration for every fibrant object X.

References

[ADR] J. Adámek, F.W. Lawvere and J. Rosický, *On the duality between varieties and algebraic theories* Algebra Universalis 49 (2003), 35-49.

[ARo] J. Adámek and J. Rosický, *On sifted colimits and generalised varieties* Theory Appl. Categ. 8 (2001), 33-53.

[Bad1] B. Badzioch, *Algebraic theories in homotopy theory*, Annals of Math. 155 (2002), 895-913.

[Bad2] B. Badzioch, *From Γ-spaces to algebraic theories*, Trans. Amer. Math. Soc. 357 (2005), 1779-1799.

[Bat1] M. Batanin, *Categorical strong shape theory*, Cahier de Topologie et de Gomtrie Diffrentielle Catgorique 38 (1997) no1, 3-66.

[Bat2] M. Batanin, *Homotopy coherent category theory and A_∞-structures in monoidal categories*,

[Bat3] M. Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Advances in Mathematics 136 (1998), 39103.

[BD] J. C. Baez, J. Dolan, *Higher-dimensional Algebra and Topological Quantum Fields Theory* J. Math. Phys. 36 (1995) 6073-6105.

[BS] M. Batanin, R. Street, *The universal property of the multitude of trees* JPAA 154(2000) 3-13.

[BPJ] H. J. Baues, M. Jibladze andT. Pirashvili, *Strengthening track theories* arXiv:math/0907185v1, 2003.

[Bec] J. Beck, *On H-spaces and infinite loop spaces*, Lect. Notes Math. 99, 139-153, Springer-Verlag 1969.

[Bek] T. Beke, *Sheafable homotopy model categories*, Math. Proc. Camb. Phil. Soc. 129 (2000), no.3, 447-475

[Ber] C. Berger, *A cellular nerve for higher categories*, Advances in Mathematics 169 (2002), 118175.

[B1] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Math. Soc., 359 (2007) 2043-2058.

[B2] J. E. Bergner, *Three models for the homotopy theory of homotopy theories*, Topology. Vol. 46, Issue 4, September 2007, 397-436.

[B3] J. E. Bergner, *A characterisation of fibrant Segal categories*, preprint available at arXiv:math.AT/0603400.

[B4] J. E. Bergner, *Rigidification of algebras over multisorted theories*, arXiv:math.AT/0508152

[B5] J. E. Bergner, *A survey of $(\infty, 1)$-categories*, preprint. July 14, 2006.

[B6] J. E. Bergner, *Adding inverses to diagrams encoding algebraic structures*, preprint. October 9, 2006.

[BJ] T. Bisson, A. Joyal *The Dyer-Lashof algebra in bordism*, C.R. Math. Rep. Acad. Sci. Vol. 17 no.4 (1995) pp 135-140.

[BV] M. Boardman and R. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lectures Notes in Math. vol. 347, Springer Verlag, 1973.

[Bo] F. Borceux, *Handbook of Categorical Algebra, I, II, III*, Encyclopedia of Mathematics and its Applications. vol 50, 51, 52. Cambridge University Press (1994).

[BC] D. Bourn and J.M. Cordier, *A general formulation of homotopy limits*, Jour. Pure Appl. Alg. 29(1983), 129-141.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localisations*, Lectures Notes in Math. vol. 304, Springer Verlag, 1972.

[CC] C. Casacuberta and B. Chorny, *The orthogonal subcategory problem in homotopy theory*, Proceedings of the Arolla conference on Algebraic Topology 2004.

[Ci1] D.C. Cisinski, *Théories homotopiques dans les topos*, Journal of Pure and Applied Algebra 174 (2002) 43-82.

[Ci2] D.C. Cisinski,*Les prfaisceaux comme modles des types d'homotopie*, Astrisque 308, Soc. Math. France (2006), xxiv+392 pages.

[Ci3] D.C. Cisinski,*Batanin higher groupoids and homotopy types*, in Categories in Algebra, Geometry and Physics, Street Festschrift, A. Davydov et al. eds., Contemp. Math. 431, Amer. Math. Soc., pp. 171-186 (2007).

[C] J. M. Cordier, *Sur la notion de diagramme homotopiquement coherent*, Proc. 3ieme Colloque sur les Catégories, Amiens (1980).

[CP1] J. M. Cordier and T. Porter, *Vogt's theorem on categories of homotopy coherent diagrams*, Math. Proc. Camb. Phil. Soc. (1986), 100, 65-90.

[CP2] J. M. Cordier and T. Porter, *Homotopy coherent category theory*, Trans. Amer. Math. Soc. 349(1997), 1-54.

[Da] B. Day,

[Du] D. Dugger, *Universal homotopy theories*, Adv. in Math. 164, (2001), no. 1, 144-176.

[DHKS] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan and J.H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs 113, American Mathematical Society, Providence, RI, 1999.

[DK1] W.G. Dwyer and D. Kan, *Functions complexes in homotopical algebra* , Topology 19 (1980), 427-440.

[DK2] W.G. Dwyer and D. Kan, *Simplicial localisations of categories*, J. of Pure and Appl. Algebra. vol 17 (1980), 427-440.

[DS] W.G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of Algebraic Topology, Elsevier, 1995.

[EM] S. Eilenberg and S. MacLane, *General theory of natural equivalences*, Trans. Amer. Math. Soc., 58, (1945)

[Eh] C. Ehresmann, *Espece de structures locales*, Séminaire de topologie et de géomtrie différentielle, Paris (1961)

[GZ] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag, New York, 1963.

[Ga] N. Gambino, *Homotopy limits in 2-categories*, to appear in the Mathematical Proceedings of the Cambridge Philosophical Society.

[GJ] P. G. Goerss and J.F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol 174, Birkhauser, 1999.

[Gon] Beatriz Rodriguez Gonzales, *Simplicial Descent Categories*, Ph Thesis, Universidad de Sevilla, 2008. arXiv:0804.2154.

[Gra] J.Gray, *Closed categories, lax limits and homotopy limits*, J. Pure and Applied Algebra 19 (1980) 127-158

[Gro] A.Grothendieck, *Pursuing Stacks*, Manuscript 1983, To appear in Documents Mathématiques, edited by Georges Maltsiniotis.

[He] A. Heller, *Homotopy theories*, Memoir of the American Math. Soc. 383, 1988.

[Hi] P. S. Hirschhorn, *Model Categories and Their Localisations*, Mathematical Surveys and Monographs, 99, American Mathematical Society, Providence, RI, 2003.

[HS] A. Hirschowitz and C. Simpson, *Descente pour les n-champs*, arXiv:math. AG/9807049.

[Ho] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, RI, 1999.

[Illu] L. Illlusie, *Complexes cotangents et dformations II*, Lectures Notes in Math. vol. 283, Springer Verlag, 1972.

[J1] A. Joyal, *Quasi-categories and Kan complexes*, J. of Pure and Appl. Algebra. vol 175 (2002), 207-222.

[J2] A. Joyal, *The theory of quasi-categories I.* in preparation.

[J3] A. Joyal, *The theory of quasi-categories II.* in preparation.

[J4] A. Joyal, *Quasi-categories vs Simplicial categories*, in preparation.

[J6] A. Joyal *A course on quasi-categories and applications.* CRM Lecture Notes, Barcelona.

[JT1] A. Joyal and M. Tierney, *Strong stacks and classifying spaces*, Lectures Notes in Math. vol. 1588, Springer Verlag, 1991.

[JT2] A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in Algebra, Geometry and Mathematical Physics. AMS Contemporary Mathematics 431. A. Davydov. M. Batanin, M. Johnson, S.Lack, A. Neeman editors.

[JT3] A. Joyal and M. Tierney, *Elements of Simplicial Homotopy Theory.* in preparation.

[Ka] D. Kan, *Adjoint functors*, Trans. Amer. Math. Soc. 87; 330-346(1958)

[Ke] M. Kelly, *Basic concepts of enriched category theory*, LMS Lecture note series. 64, Cambridge university Press, 1982.

[KP] K. H. Kamps and T. Porterf, *Abstract homotopy and simple homotopy theory*, World Scientific, 1997.

[KJBM] J. Kock, A.Joyal, M.Batanin, J. Mascari, *Polynomial functors and opetopes*, arXiv:0706.1033v1

[La] C. Lair, *Sur le genre d'esquissibilté des catégories modélables (accessibles) possédant les produits de deux*, Diagrammes **35** (1996), 25-52.

[Law] F. W Lawvere, *Equality in hyperdoctrines and comprehension scheme as an adjoint functor*, Proc. Symp. Pure Math. Vol XVII, AMS (1970).

[Law] W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad, Sc. 50(1963), 869-872.

[Le] T. Leinster, *Higher Operads, Higher Categories*, arXiv:math.CT/0305049v1, 2003.

[Lu1] J. Lurie, *Higher Topos Theory*, arXiv:math.CT/0608040, 2007.

[Lu2] J. Lurie, *Derived algebraic Geometry I: Stable ∞-Categories*, arXiv:math.CT/0608228v4, 2007.

[Lu3] J. Lurie, *Derived Algebraic Geometry II: Non-commutative Algebra*, arXiv:math.CT/0702299, 2007.

[Lu4] J. Lurie, *Derived Algebraic Geometry III: Commutative Algebra*, arXiv:math.CT/0703204, 2007.

[Lu5] J. Lurie, *Derived Algebraic Geometry IV: Deformation Theory*, arXiv:math.CT/0709.3091, 2007.

[Mac] S. MacLane, *Categories for the Working Mathematicians*, Graduate Texts in Mathematics 5, Springer-Verlag(1971).

[MZ] M. Makkai, M. Zawadowski *duality for simple ω-categories and disks*, Theory and Applications pf Categories, Vol 8, No. 5, pp. 114-243 (2001).

[Malt1] G. Maltsiniotis, *Introduction à la théorie des dérivateurs d'après Grothendieck*, unpublished manuscript available at http://www.math.jussieu.fr/ maltsin/.

[Malt2] G. Maltsiniotis, *Structure d'asphricit, foncteurs lisses, et fibrations,* Ann. Math. Blaise Pascal, 12, pp. 1-39 (2005)

[Malt3] G. Maltsiniotis, *La théorie de l'"homotopie de Grothendieck*, Asterisque 301, 2005.

[Malt3] G. Maltsiniotis, *Infini groupoides d'aprs Grothendieck*, Preprint. http://www.institut.math.jussieu.fr/ maltsin/ps/infgrart.pdf

[Malt5] G. Maltsiniotis, *Infini catgories non strictes, une nouvelle définition*, Preprint. http://www.institut.math.jussieu.fr/ maltsin/ps/infctart.pdf.

[May1] P. May, *The Geometry of Iterated loop spaces,* Lect. Notes in Math. 271, Springer Verlag, 1972.

[May2] P. May, *A concise Course in Algebraic Topology* Chicago Lect. Notes in Math., The University of Chigago Press. 1999.

[MW1] I. Moerdijk, I. Weiss *Dendroidal Sets*, arXiv:math,AT/0701293, 2007.

[MW2] I. Moerdijk, I. Weiss *On inner Kan complexes in the category of dendroidal sets* arXiv:math,AT/0701295, 2007.

[Ni] J.P. Nichols-Barrer, *On quasi-categories as a foundation for Higher Algebraic Stacks* PhD Thesis, MIT, June 2001.

[P] R. Pellissier. *Catégories enrichies faibles.* Thèse, Université de Nice-Sophia Antipolis. June 2002, arXiv:math/0308246, 2003.

[Q] D. Quillen, *Homotopical algebra*, Lectures Notes in Math. vol. 43, Springer Verlag, 1967.

[Ree] C.L. Reedy, *Homotopy theories of model categories*, unpublished manuscript.

[Rez] C. Rezk, *Toposes and homotopy toposes*, unpublished manuscript. Dec 2005.

[Rez] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math Soc., 353(2001), no. 3, 973-1007.

[Ro] A. Roig, *Model category structures in bifibered categories*, JPAA 95 (1994) 203-223.

[Ros] J. Rosicky, *On homotopy varieties*, Advances in Mathematics, Vol 214, N.2, 1 Oct 2007.

[ScVo] R. Schwnzl and R. Vogt *Homotopy homomorphisms and the hammock localisation*,

[Sch] S. Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, JPAA 120 (1997) 77-104.

[Sch] S. Schwede, *Stable homotopy of algebraic theories*, Topology 40 (2001) 1-41

[S1] G.B. Segal, *Classifying spaces and spectral sequences*, IHES Publ. Math. 34, 105-112(1968).

[S2] G. Segal, *Categories and cohomology theories*, Topology 13 (1974) 293-312.

[Si4] C. Simpson, *Some properties of the theory of n-categories* arXiv:math.CT/0110273, 2001.

[Si3] C. Simpson, *A Giraud-type characterisation of the simplicial category associated to closed model categories as ∞-pretopoi* arXiv:math.CT/990316,1999.

[Si2] C. Simpson, *On the Breen-Baez-Dolan stabilisation hypothesis for Tamsamani weak n-categories* arXiv:math.CT/9810058, 1998.

[Si1] C. Simpson, *A closed model structure for n-categories, internal Hom, n-stacks and generalised Seifert-Van Kampen* arXiv:math.AG/9704006v2, 2000.

[SGA] A. Grothendieck, J.L. Verdier *Topos theory and etale cohomology, SGA4*,

[SW] R. Street and R. Walters, *The comprehensive factorisation of a functor*, Bulletin American Math. Soc. **79** (1973) 936-941.

[Sta] J,D, Stasheff, *Homotopy associativity of H-spaces I, II,* Trans. Amer. Math, Soc. 108 (1963), 275-312.

[Ta] Z. Tamsamani *On non-strict notions of n-category and n-groupoid via multisimplicial sets*, arXiv:math.AG/9512006, 1995.

[T1] B. Töen *Homotopical and Higher Categorical Structures in Algebraic Geometry*, Habilitation thesis, May 2003. Unpublished manuscript, preprint available at arXiv:math.AG/0312262.

[TV1] B. Toen and G. Vezzosi, *Algebraic Geometry over model categories (a general approach to derived algebraic geometry* arXiv:math.AG/0110109

[TV2] B. Toen and G. Vezzosi, *Homotopical algebraic geometry I: Topos theory*, Adv. Math. 193 (2005), 257-372, arXiv:math.AG/0207028

[TV3] B. Toen and G. Vezzosi, *Segal topoi and stacks over Segal categories*, arXiv:math.AG/0212330

[Vi] E. Vitale *Localizations of algebraic categories II*, Journal of Pure and Applied Algebra 133 (1998) 317-326.

[V] R.M. Vogt, *Homotopy limits and colimits*, Math.Z. 134. 11-52 (1973)

[W] M. Weber, *Generic Morphisms, parametric representations and weakly cartesian monads*, Theory and Applications of Categories, Vol. 13, No. 14, 2004, pp. 191234.

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