10 Theorem *A space is paracompact if every interior-preserving open cover of the space has a closure-preserving closed refinement.*

Proof. Assume that every interior-preserving open cover of X has a closure-preserving closed refinement. Let $\mathcal{U}$ be an interior-preserving open cover of X. We show that $\mathcal{U}$ has a locally finite open refinement. By Lemma 9, every interior-preserving open cover of X has an interior-preserving open point-star refinement. As a consequence, we can inductively construct a sequence $\langle \mathcal{U}_n \rangle_{n \in \mathbb{N}}$ of interior-preserving open covers such that $\mathcal{U}_1 = \mathcal{U}$ and $\mathcal{U}_{n+1}$ is a point-star refinement of $\mathcal{U}_n n$ for every $n \in \mathbb{N}$. By Proposition III.1.4 and Lemma III.1.6, there exists a continuous pseudometric d of X such that we have $B_d(x, 2^{-3}) \subset \mathrm{St}(x, \mathcal{U}_2)$ for every $x \in X$. Since $\mathcal{U}_2$ is a point-star refinement of $\mathcal{U}$, the cover $\mathcal{U}$ is d-uniform, and it follows, by Corollary 1.5, that $\mathcal{U}$ has a τ_d-open and locally finite refinement $\mathcal{V}$. Since d is a continuous pseudometric of X, the family $\mathcal{V}$ is open and locally finite in X.

We have shown that every interior-preserving open cover of X has a locally finite open refinement. It follows, by Theorem 6 and Lemma 8, that X is paracompact. □

We can use the characterizations of paracompactness obtained above to study the preservation of paracompactness in topological operations. We consider preservation with subspaces and mappings below, and in the next section, we shall study preservation of paracompactness in products.

We note first that paracompactness is not a hereditary property: every compact space is paracompact and we saw in Chapter I that every Tihonov space is a subspace of some compact Hausdorff space. As a consequence, any non-normal Tihonov space, such as the "Sorgenfrey square" $S \times S$, gives an example of a non-paracompact subspace of a paracompact space.

Even though paracompactness is not hereditary, it is closed-hereditary and, for regular spaces, even F_σ-hereditary.

11 Proposition **A.** *A closed subspace of a paracompact space is paracompact.*
B. *An F_σ-subspace of a regular paracompact space is paracompact.*

Proof. A. Let X be paracompact and $F \Subset X$. To show that F is paracompact, let $\mathcal{G}$ be an open cover of F. For every $G \in \mathcal{G}$, the set $G \cup (X \smallsetminus F)$ is open in X. It follows that the family $\mathcal{U} = \{G \cup (X \smallsetminus F) : G \in \mathcal{G}\}$ is an open cover of X. Let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{U}$. Then the family $\{V \cap F : V \in \mathcal{V}\}$ is a locally finite open refinement of the cover $\mathcal{G}$ of F.

B. Let X be paracompact and regular, and let L be an F_σ-subset of X. Then there exist sets $F_n \Subset X$, for $n \in \mathbb{N}$, such that $L = \bigcup_{n\in\mathbb{N}} F_n$. We use Theorem 5 to show that L is paracompact. Let $\mathcal{G}$ be an open cover of L. For every $G \in \mathcal{G}$, since $G \Subset L$, there exists $U_G \Subset X$ such that $U_G \cap L = G$. Let $n \in \mathbb{N}$. Then the family $\mathcal{U}_n = \{U_G : G \in \mathcal{G}\} \cup \{X \smallsetminus F_n\}$ is an open cover of X. Let $\mathcal{V}_n$ be a locally finite open refinement of $\mathcal{U}_n$. Then the family $\mathcal{W}_n = \{V \cap L : V \in (\mathcal{V}_n)_{F_n}\}$ is locally finite and open in L; moreover, $\mathcal{W}_n$ is a partial refinement of $\mathcal{G}$ and $\mathcal{W}_n$ covers the set F_n. By the foregoing, the family $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a σ-locally finite open refinement of $\mathcal{G}$. We have shown that every open cover of $\mathcal{L}$ has a σ-locally finite open refinement. By Theorem 5, L is paracompact. $\square$

Unlike two other important covering properties, compactness and the Lindelöf property, paracompactness is not preserved under continuous mappings: every space is the continuous image of a discrete space. Nevertheless, there exists one important class of mappings which preserve paracompactness.

12 Theorem *The image of a regular paracompact space under a closed and continuous mapping is paracompact.*

Proof. Let X be a regular paracompact space and f be a closed, continuous and onto mapping from X onto a space Y. To show that Y is paracompact, it suffices, by Theorem 10, to show that every open cover of Y has a closure-preserving closed refinement. Let $\mathcal{G}$ be an open cover of Y. Then the family $\mathcal{U} = \{f^{-1}(G) : G \in \mathcal{G}\}$ is an open cover of X. By Lemma 1.17, the open cover $\mathcal{U}$ of X has a locally finite closed refinement $\mathcal{F}$. The family $\mathcal{K} = \{f(F) : F \in \mathcal{F}\}$ is a refinement of the cover $\mathcal{G}$ of Y. We show that $\mathcal{K}$ is closure-preserving and closed. Let $\mathcal{H} \subset \mathcal{F}$. By Lemma 1.2, the family $\mathcal{F}$ is closure-preserving in X, and hence the set $\bigcup \mathcal{H}$ is closed. Since f is a closed mapping, the set $f(\bigcup \mathcal{H})$ is closed in Y. Since $f(\bigcup \mathcal{H}) = \bigcup\{f(F) : F \in \mathcal{H}\}$, we have shown that the set $\bigcup\{f(F) : F \in \mathcal{H}\}$ is closed in Y. The foregoing shows that, for every $\mathcal{L} \subset \mathcal{K}$, the set $\bigcup \mathcal{L}$ is closed in Y. As a consequence, the refinement $\mathcal{K}$ of $\mathcal{G}$ is closure-preserving and closed. $\square$

For inverse preservation, we have the following result.

13 Theorem A. *The pre-image of a paracompact space under a perfect mapping is paracompact.*

B. *A regular space is paracompact provided the space can be mapped into a paracompact space by a closed continuous mapping with Lindelöf fibers.*

Proof. We prove part B; the proof of A is similar (but simpler). Let X be a regular space, Y a paracompact space, and let $f : X \to Y$ be a closed continuous mapping such that the subspace $f^{-1}\{y\}$ of X is Lindelöf for each $y \in Y$. mapping. Let $Z = f(X)$, and note that since f is a closed mapping, we have that $Z \Subset Y$. It follows by Proposition 11 that the subspace Y of X is paracompact. We use Theorem 5 to show that Z is paracompact. Let $\mathcal{G}$ be an open cover of X. For every $y \in Z$, there exists a countable subfamily $\mathcal{G}_y$ of $\mathcal{G}$ such that the set $G_y = \bigcup \mathcal{G}_y$ covers the Lindelöf-subspace $f^{-1}\{y\}$ of X; we set $U_y = Z \setminus f(X \setminus G_y)$ and we note that U_y is an open nbhd of y in Z. The family $\mathcal{U} = \{U_y : y \in Z\}$ is an open cover of the paracompact space Z and hence $\mathcal{U}$ has a locally finite open refinement $\mathcal{V}$. Note that the family $\mathcal{W} = \{f^{-1}(V) : V \in \mathcal{V}\}$ is a locally finite open cover of X. For all $V \in \mathcal{V}$ and $y \in Z$, if $V \subset U_y$, then $f^{-1}(V) \subset G_y$. It follows that, for every $W \in \mathcal{W}$, there exists a countable $\mathcal{G}_W \subset \mathcal{G}$ such that $W \subset \bigcup \mathcal{G}_W$. We write $\mathcal{G}_W = \{G_{W,1}, G_{W,2}, ...\}$ for every $W \in \mathcal{W}$. Then it is easy to see that, for every $n \in \mathbb{N}$, the family $\mathcal{H}_n = \{W \cap G_{W,n} : W \in \mathcal{W}\}$ is a locally finite open partial refinement of $\mathcal{G}$. Moreover, the family $\bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ covers X. We have shown that every open cover of X has a σ-locally finite open refinement. Since X is regular, it follows from Theorem 5 that X is paracompact. □

It follows from Theorems 12 and 13 that if X and Y are regular spaces and there exists a closed continuous mapping with Lindelöf fibers from X onto Y, then X is paracompact iff Y is paracompact.

We close this section with examples indicating limits for preservation and inverse preservation of paracompactness under non-closed mappings.

14 Example **(a)** For every $x \in \mathbb{R}$, let $R_x = (\{x\} \times \mathbb{R}) \cup (\mathbb{R} \times \{x\})$. Denote by X the space obtained when $\mathbb{R}^2$ is equipped with topology in which points of the set $\mathbb{R}^2 \setminus \Delta_{\mathbb{R}}$ are isolated and a point (x, x) has a nbhd base by sets $R_x \setminus F$, where F is finite and $(x, x) \notin F$.

It is easy to see that the disjoint τ-closed sets $\{(x, x) : x \in \mathbb{Q}\}$ and $\{(x, x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ cannot be separated by open sets in X. Hence X is non-normal and thus non-paracompact.

Denote by Y the subspace $\bigcup_{x\in\mathbb{R}} R_x \times \{x\}$ of the product space $X \times \mathbb{R}_d$, where $\mathbb{R}_d$ is the discrete space on R. The space Y is paracompact, and the mapping $(a, b, c) \mapsto (a, b)$ is open and continuous from Y onto X, and each fiber has at most two points.

We leave the verification of the details of this example as an exercise.

(b) Let S be the Sorgenfrey line. A projection $S \times S \to S$ is an open continuous mapping, with Lindelöf fibers, from a non-paracompact space onto a paracompact space. □

3. Paracompactness and normality in products.

The familiar weak separation properties, from T_0 up to the Tihonov property, are productive. However, normality is not even finitely productive. On the other hand, compactness is productive but the Lindelöf property is not finitely productive.

The standard example to show non-productivity of normality or the Lindelöf property is the square $S \times S$ of the Sorgenfrey line S. The space S is regular and Lindelöf, hence normal, but the product $S \times S$ is neither normal nor Lindelöf. Since regular Lindelöf spaces are paracompact and regular paracompact spaces are normal, the space $S \times S$ also serves as an example of non-productivity of paracompactness. Our next example shows that even the product of a paracompact space and a separable completely metrizable space may fail to be paracompact.

1 Example Denote by $\mathbb{J}$ the subspace $\mathbb{R} \smallsetminus \mathbb{Q}$ of $\mathbb{R}$. There exists a paracompact space X such that $X \times \mathbb{J}$ is not paracompact.

Proof. Denote by τ the usual (euclidean) topology of $\mathbb{R}$, and denote by π the topology of $\mathbb{R}$ in which every $x \in \mathbb{J}$ is isolated and every $x \in \mathbb{Q}$ has the same nbhds as in τ. We denote by X the space obtained when $\mathbb{R}$ is equipped with the topology π. It is easy to see that X is zero-dimensional and T_1 and hence Tihonov. To show that X is paracompact, let $\mathcal{G}$ be an open cover of X. For every $G \in \mathcal{G}$, denote by G' the τ-interior of G. Let $\mathcal{G}' = \{G' : G \in \mathcal{G}\}$ and $O = \bigcup \mathcal{G}'$. In the relative τ-topology, the space O is metrizable and hence paracompact; as a consequence, $\mathcal{G}'$ has a refinement $\mathcal{U}$ such that $\mathcal{U}$ is locally finite and open in the relative τ-topology. It is easy to see that the family $\mathcal{U} \cup \{\{x\} : x \in \mathbb{R} \smallsetminus O\}$ is a locally finite open refinement of $\mathcal{G}$ in X. We have shown that X is paracompact.

We show that the product space $X \times \mathbb{J}$ is not paracompact. Note that, for every $x \in \mathbb{J}$, the set $V_x = \{(x,x)\} \cup (X \times \mathbb{J} \smallsetminus \Delta_{\mathbb{J}})$ is open in $X \times \mathbb{J}$. We show that the open cover $\mathcal{V} = \{V_x : x \in \mathbb{J}\}$ of $X \times \mathbb{J}$ has no locally finite open refinement. Let $\mathcal{W}$ be an open refinement of $\mathcal{V}$. For every $x \in \mathbb{J}$, let $W_x \in (\mathcal{W})_{(x,x)}$, and note that $W_x \neq W_y$ for $x \neq y$. For every $x \in \mathbb{J}$, there exists $n_x \in \mathbb{N}$ such that $\{x\} \times (x - \frac{1}{n_x}, x + \frac{1}{n_x}) \subset W_x$. By the Baire Category Theorem, there exist $m \in \mathbb{N}$ and $a, b \in \mathbb{R}$ such that we have $a < b$ and $(a,b) \subset \operatorname{Cl}_\tau\{x \in \mathbb{J} : n_x = m\}$. Let q be a rational number in the interval (a,b), and let p be an irrational number with $|p - q| < \frac{1}{2m}$. We show that $\mathcal{W}$ is not locally finite at the point (q,p). Let N be a nbhd of (q,p). Then there exists $0 < \epsilon < \frac{1}{2m}$ such that $(q-\epsilon, q+\epsilon) \times \{p\} \subset N$. Since we have that $q \in \mathrm{Cl}_{\mathcal{T}}\{x \in \mathbb{J} : n_x = m\}$, the set $A = \{x \in \mathbb{J} : n_x = m \text{ and } |x-q| < \epsilon\}$ is infinite. For every $x \in A$, we have that

$$|x-p| \le |x-q| + |q-p| < \epsilon + \frac{1}{2m} < \frac{1}{m} = \frac{1}{n_x}$$

and it follows that $(x,p) \in W_x$; we also have that $(x,p) \in N$ and hence we have that $W_x \cap N \neq \emptyset$. We have shown that every nbhd of the point (q,p) meets infinitely many sets of the family $\mathcal{W}$. As a consequence, $\mathcal{W}$ is not locally finite. □

The space X above is known as the *Michael line*.

After the above counter-examples, we state a positive result.

2 Theorem *The product of a paracompact space with a compact space is paracompact.*

Proof. The result follows from Theorem 2.13.A, since the projection map $p : X \times K \to X$ is perfect whenever K is a compact space (see Exercise 1, Problem 4). □

It follows from Theorem 2 that the product $X \times K$ of a paracompact Hausdorff space X with a compact Hausdorff space K is normal. Curiously, it turns out that this result has a converse: if X is a Tihonov space and $X \times K$ is normal for every compact Hausdorff space K, then X is paracompact. We shall obtain this result as a corollary to the following one, known as "Tamano's Theorem".

3 Theorem *The following are equivalent for a Tihonov space X:*

A. *X is paracompact.*

B. *$X \times \beta X$ is normal.*

C. *There exists a Hausdorff compactification K of X such that $X \times K$ is normal.*

Proof. A⇒B: This follows from Theorem 2 and Proposition 1.3.

C⇒A: Asume that K is a Hausdorff compactification of X and the space $X \times K$ is normal. To show that X is paracompact, let $\mathcal{G}$ be an open cover of X. For every $G \in \mathcal{G}$, there exists $G^* \Subset K$ such that $G^* \cap X = G$. Let $\mathcal{G}^* = \{G^* : G \in \mathcal{G}\}$ and $U = \bigcup \mathcal{G}^*$, and note that we have $X \subset U \Subset K$. Denote by F the closed subset $X \times (K \smallsetminus U)$ of $X \times K$. Since K is a Hausdorff space, the set $\Delta_X = \{(x,x) : x \in X\}$ is closed in X. Moreover, we have that $\Delta_X \cap F = \emptyset$. By normality of $X \times K$, there exists a continuous function $f : X \times K \to \mathbb{R}$ such that we have $f(x,x) = 0$ for every $(x,x) \in \Delta_X$ and $f(x,k) = 1$ for every $(x,k) \in F$.

We define a pseudometric d of X by the formula $d(x,y) = \sup_{k \in K} |f(x,k) - f(y,k)|$. To show that d is continuous, it suffices to show that we have $B_d(x,\epsilon) \in \eta_x(X)$ for all

$x \in X$ and $\epsilon > 0$. Let $x \in X$ and $\epsilon > 0$. For every $k \in K$, since f is continuous on $X \times K$, there exist $V_k \in \eta_x(X)$ and $W_k \in \eta_k(K)$ such that we have $|f(y,\ell) - f(z,t)| \leq \frac{\epsilon}{2}$ for all $(y,\ell),(z,t) \in V_k \times W_k$; in particular, we have that $|f(y,\ell) - f(x,\ell)| \leq \frac{\epsilon}{2}$ whenever $y \in V_k$ and $\ell \in W_k$. Since K is compact, there exists a finite $A \subset K$ such that $\bigcup_{a \in A} W_a = K$. Denote by V the nbhd $\bigcap_{a \in A} V_a$ of x. For every $y \in V$, we have that

$$\{y\} \times K \subset V \times \bigcup_{a \in A} W_a = \bigcup_{a \in A} V \times W_a \subset \bigcup_{a \in A} V_a \times W_a$$

and it follows that $|f(y,\ell) - f(x,\ell)| \leq \frac{\epsilon}{2}$ for every $\ell \in K$. As a consequence, we have that $\rho(y,x) \leq \frac{\epsilon}{2} < \epsilon$ for every $y \in V$. We have shown that $B_d(x,\epsilon) \in \eta_x(X)$ for every $\epsilon > 0$.

By Corollary 1.5, there exists a τ_d-locally finite and open cover $\mathcal{H}$ of X such that each $H \in \mathcal{H}$ has d-diameter at most $\frac{1}{2}$. Since d is a continuous pseudometric of X, the family $\mathcal{H}$ is a locally finite open cover of X. We show that, for every $H \in \mathcal{H}$, there exists a finite family $\mathcal{G}_H \subset \mathcal{G}$ such that $H \subset \bigcup \mathcal{G}_H$. Let $H \in \mathcal{H}$, and let $x \in H$. For every $y \in Y$, we have that $d(y,x) \leq \frac{1}{2}$, and hence that $f(x,y) = |f(x,y) - f(y,y)| \leq \frac{1}{2}$. It follows that we have

$$\mathrm{Cl}_K(H) \subset \mathrm{Cl}_K\{y \in X : f(x,y) \leq \tfrac{1}{2}\} \subset \{k \in K : f(x,k) \leq \tfrac{1}{2}\}\,.$$

We have that $f(x,\ell) = 1$ for every $(x,\ell) \in F$. Since $F = X \times (K \setminus U))$, it follows from the foregoing that we have $\mathrm{Cl}_K(H) \subset U$. Moreover, the set $\mathrm{Cl}_K(H)$ is compact and the set U is the union of the open family $\mathcal{G}^*$. As a consequence, there exists a finite $\mathcal{G}_H \subset \mathcal{G}$ such that we have $\mathrm{Cl}_K(H) \subset \bigcup\{G^* : G \in \mathcal{G}_H\}$. Since we have that $H \subset X$ and $G^* \cap X = G$ for every $G \in \mathcal{G}$, it follows that $H \subset \bigcup \mathcal{G}_H$.

It follows from the foregoing that the family $\{H \cap G : H \in \mathcal{H} \text{ and } G \in \mathcal{G}_H\}$ is a locally finite open refinement of $\mathcal{G}$. □

4 Corollary *A Tihonov space X is paracompact iff $X \times K$ is normal for every compact Hausdorff space K.*

Let X be a non-paracompact normal space. The above results show that there exist compact Hausdorff spaces K such that the product $X \times K$ is non-normal. However, it is still possible that all products $X \times C$ are normal for some sufficiently simple compact spaces C. In particular, the above results leave the following problem open.

4 Problem *Let X be a normal space. Is the product $X \times \mathbb{I}$ normal?*